Sets and Indices

- **Sets:**
  - V: Set of vertices, including the supplier (0) and customers $\{1, 2, \ldots, n\}$.
  - $V' \subset V$: Set of customer nodes, excluding the depot.
  - $E \subseteq V \times V$: Set of edges representing the possible routes between vertices ($E = \{(i, j) : i, j \in V, i < j\}$).
  - K: Set of vehicles.
  - $S = \{0, 1, \ldots, s\}$: Set of ages of the products.
  - $T = \{1, 2, \ldots, H\}$: Set of periods in the planning horizon.
- **Indices:**
  - $i, j \in V$: Indices for vertices.
  - $i, j \in V'$: Indices for customer nodes.
  - $k \in K$: Index for vehicles.
  - $g \in S$: Index for the age of the product.
  - $t \in T$: Index for periods.

Parameters

- C_{ij}: Cost of traveling from node i to j (routing cost for traveling edge (i, j)).
- d_i^{gt}: Demand of customer i for products of age g in period t.
- d_i^t: Demand of customer (node) i in period t.
- Q_k: Capacity of vehicle k.
- C_i: Maximum inventory capacity at node i.
- r^t: Quantity of fresh products produced by the supplier in period t.
- u_i^g: Selling revenue of one unit of product of age g.
- h_i^g: Inventory holding cost for one unit of product of age g at node i.
- I_i^{g0}: Initial inventory level of product of age g at customer i.
- I_0^{g0}: Initial inventory level of product of age g at the supplier.

Variables

- x_{ij}^{kt}: Binary variable indicating if vehicle k travels from node i to node j in period t.
- y_i^{kt}: Binary variable indicating if customer i is visited by vehicle k in period t.
- I_i^t: Total inventory level at node i in period t.
- I_i^{gt}: Inventory level of product of age g at node i in period t.
- q_i^{gkt}: Quantity of product of age g delivered to customer i by vehicle k in period t.

Objective Function

$$\text{(PIRP) maximize} \quad \sum_{g\in S}\sum_{t\in T} u_i^g d_i^{gt} - \sum_{i\in V}\sum_{g\in S}\sum_{t\in T} h_i^g I_i^{gt} - \sum_{(i,j)\in E}\sum_{k\in K}\sum_{t\in T} C_{ij} x_{ij}^{kt}$$

Explanation

- **Revenue from Sales:**
$$\sum_{g\in S}\sum_{t\in T} u_i^g d_i^{gt}$$
Summing the revenue from selling products of different ages g to customers i over all periods t.
- **Inventory Holding Costs:**
$$\sum_{i\in V}\sum_{g\in S}\sum_{t\in T} h_i^g I_i^{gt}$$
Summing the costs of holding inventory of different ages g at different nodes i over all periods t.
- **Transportation Costs:**
$$\sum_{(i,j)\in E}\sum_{k\in K}\sum_{t\in T} C_{ij} x_{ij}^{kt}$$
Summing the costs of transportation for each vehicle k traveling from node i to node j over all periods t.

The objective is to maximize the total profit, which is the revenue from sales minus the inventory holding costs and transportation costs.

Constraints

1. Inventory Conservation at Supplier

$$I_0^{gt} = I_0^{g-1,t-1} - \sum_{i \in V'} \sum_{k \in K} q_i^{gkt}, \quad g \in S \setminus \{0\}, \quad t \in T$$

Keeps track of the inventory at the supplier, aging the product by one unit in each period.

2. Supplier Production

$$I_0^{0t} = r^t, \quad t \in T$$

Ensures that the supplier always produces or receives top fresh products in each period.

3. Inventory Conservation at Customers

$$I_i^{gt} = I_i^{g-1,t-1} + \sum_{k \in K} q_i^{gkt} - d_i^{gt}, \quad i \in V', \quad g \in S \setminus \{0\}, \quad t \in T$$

Keeps track of inventory at customers, including deliveries and aging.

4. Fresh Inventory Conservation at Customers

$$I_i^{0t} = \sum_{k \in K} q_i^{0kt} - d_i^{0t}, \quad i \in V', \quad t \in T$$

Ensures fresh product deliveries and consumption are balanced.

5. Maximum Inventory Capacity

$$\sum_{g \in S} I_i^{gt} \leq C_i, \quad i \in V', \quad t \in T$$

Ensures customers' inventory does not exceed capacity.

6. Demand Satisfaction

$$d_i^t = \sum_{g \in S} d_i^{gt}, \quad i \in V', \quad t \in T$$

Ensures demand is met by products of different ages. The constraint states that the demand of each customer in each period is the sum of product quantities of different ages. Note that by design, any product whose age g is higher than s is spoiled, i.e., it no longer appears in the inventory nor can it be used to satisfy the demand.

7. Delivery Quantity Limits

$$\sum_{g \in S} \sum_{k \in K} q_i^{gkt} \leq C_i - \sum_{g \in S} I_i^{g,t-1}, \quad i \in V', \quad t \in T$$

Limits the quantity delivered to customers based on their inventory capacity.

8. Linking Deliveries to Visits

$$q_i^{gkt} \leq C_i y_i^{kt}, \quad i \in V', \quad g \in S, \quad k \in K, \quad t \in T$$

Allows product deliveries only if the vehicle visits the customer. Constraints 7 and 8 link the quantities delivered to the routing variables. In particular, they only allow a vehicle to deliver products to a customer if a vehicle has been assigned to visit that customer.

9. Vehicle Capacity

$$\sum_{i \in V'} \sum_{g \in S} q_i^{gkt} \leq Q_k y_0^{kt}, \quad k \in K, \quad t \in T$$

Ensures vehicle capacities are not exceeded.

10. Degree Constraints

$$\sum_{j \in V, j<i} x_{ij}^{kt} + \sum_{j \in V, j<i} x_{ji}^{kt} = 2y_i^{kt}, \quad i \in V, \quad k \in K, \quad t \in T$$

Ensures vehicles enter and leave each visited node exactly once.

11. Subtour Elimination

$$\sum_{i \in S} \sum_{j \in S, i<j} x_{ij}^{kt} \leq \sum_{i \in S} y_i^{kt} - y_m^{kt}, \quad S \subseteq V', \quad k \in K, \quad t \in T, \quad m \in S$$

Prevents the formation of subtours within the set of customer nodes S.

12. No Split Deliveries

$$\sum_{k \in K} y_i^{kt} \leq 1, \quad i \in V', \quad t \in T$$

Ensures that at most one vehicle visits each customer in each period, thus forbidding split deliveries.

Variable Domains

1. Non-negative Continuous Variables

$$I_i^{gt}, d_i^{gt}, q_i^{gkt} \geq 0, \quad i \in V', \quad g \in S, \quad k \in K, \quad t \in T$$

Inventory levels, demand quantities, and delivery quantities cannot be negative.

2. Binary and Integer Variables

$$x_{0i}^{kt} \in \{0, 1, 2\}, \quad i \in V', \quad k \in K, \quad t \in T$$

$$x_{ij}^{kt} \in \{0, 1\}, \quad (i, j) \in E, \quad k \in K, \quad t \in T$$

$$y_i^{kt} \in \{0, 1\}, \quad i \in V, \quad k \in K, \quad t \in T$$

Routing variables x_{ij}^{kt} indicate whether an edge is used in a route and y_i^{kt} indicates if a node is visited by a vehicle.