

Advanced Statistics

Dr. Syed Faisal Bukhari

Associate Professor

Department of Data Science

Faculty of Computing and Information Technology

University of the Punjab

Textbooks

- ❑ **Probability & Statistics for Engineers & Scientists,**
Ninth Edition, Ronald E. Walpole, Raymond H.
Myer

Joint Probability Distributions

- Given random variables X, Y, \dots , that are defined on a probability space, the **joint probability distribution** for X, Y, \dots , is a **probability distribution** that gives the probability that each of X, Y, \dots , falls in any particular range or discrete set of values specified for that variable.
- In the case of only **two random variables**, this is called a **bivariate distribution**, but the concept generalizes to **any number of random variables**, giving a **multivariate distribution**.

Joint Probability Distributions

- The **joint probability distribution** can be expressed either in terms of a **joint cumulative distribution function** or in terms of a **joint probability density function** (in the case of continuous variables) or **joint probability mass function** (in the case of discrete variables).

Joint Probability Distributions cont.

- These in turn can be used to find two other types of distributions: the **marginal distribution** giving the probabilities for any one of the variables with **no reference to any specific ranges of values for the other variables**, and the **conditional probability distribution** giving the probabilities for any subset of the variables conditional on particular values of the remaining variables

Joint Probability Distributions

- ❑ Our study of random variables and their probability distributions in previous lectures is restricted to **one-dimensional sample spaces**, in that we recorded outcomes of an experiment as values assumed by a **single random variable**.
- ❑ There will be situations, however, where we may find it desirable to record the **simultaneous outcomes** of **several random variables**.

Joint Probability Distributions

- ❑ **For example**, we might measure the amount of precipitate P and volume V of gas released from a controlled chemical experiment, giving rise to a **two-dimensional sample space** consisting of the **outcomes (p, v)** , or
- ❑ we might be interested in the **hardness H** and **tensile strength T** of cold-drawn copper, resulting in the **outcomes (h, t)** .

Joint Probability Distributions

- ❑ In a study to determine the likelihood of **success in college** based on high school data, we might use a **three dimensional sample space** and record for **each individual** his or her **aptitude test score**, **high school class rank**, **and grade-point average** at the end of freshman year in college.

Joint Probability Distributions

- If X and Y are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values $f(x, y)$ for any pair of values (x, y) within the range of the random variables X and Y .
- It is customary to refer to this function as the joint probability distribution of X and Y . Hence, in the discrete case,
$$f(x, y) = P(X = x, Y = y);$$
that is, the values $f(x, y)$ give the probability that outcomes x and y occur at the same time.

Joint Probability Distributions

The function $f(x, y)$ is a **joint probability distribution or probability mass function** of the **discrete random variables X and Y** if

1. $f(x, y) \geq 0$ for all (x, y) ,
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane,

$$P[(X, Y) \in A] = \sum \sum_A f(x, y).$$

Example : Two ballpoint pens are selected at random from a box that contains **3 blue pens**, **2 red pens**, and **3 green pens**. If **X** is the **number of blue pens** selected and **Y** is the **number of red pens** selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) / x + y \leq 1\}$.

Solution

a) The possible pairs of values (x, y) are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, and $(2, 0)$.

The joint probability distribution of

$$f(x, y) = \frac{({}_3C_x)({}_2C_y)({}_3C_{2-x-y})}{{}_8C_2}, \text{ for } x = 0, 1, 2; y = 0, 1, 2; \text{ and } 0 \leq x + y \leq 2.$$

$$f(0, 0) = \frac{({}_3C_0)({}_2C_0)({}_3C_{2-0-0})}{{}_8C_2} = \frac{3}{28}$$

$$f(0, 1) = \frac{({}_3C_0)({}_2C_1)({}_3C_{2-0-1})}{{}_8C_2} = \frac{6}{28}$$

$$f(\mathbf{1}, \mathbf{0}) = \frac{({}_3C_1)({}_2C_0)({}_3C_{2-1-0})}{{}_8C_2} = \frac{\mathbf{9}}{\mathbf{28}}$$

$$f(\mathbf{1}, \mathbf{1}) = \frac{({}_3C_1)({}_2C_1)({}_3C_{2-1-1})}{{}_8C_2} = \frac{\mathbf{6}}{\mathbf{28}}$$

$$f(\mathbf{0}, \mathbf{2}) = \frac{({}_3C_0)({}_2C_2)({}_3C_{2-0-2})}{{}_8C_2} = \frac{\mathbf{1}}{\mathbf{28}}$$

$$f(\mathbf{2}, \mathbf{0}) = \frac{({}_3C_2)({}_2C_0)({}_3C_{2-2-0})}{{}_8C_2} = \frac{\mathbf{3}}{\mathbf{28}}$$

f(x, y)		x			Row totals
		0	1	2	
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{6}{28}$	$\frac{6}{28}$	$\frac{0}{28}$	$\frac{12}{28}$
	2	$\frac{1}{28}$	$\frac{0}{28}$	$\frac{0}{28}$	$\frac{1}{28}$
Column totals		$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	$\frac{28}{28} = 1$

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) / x + y \leq 1\}$.

The probability that (X, Y) fall in the region A is

$$\begin{aligned} P[(X, Y) \in A] &= P(X + Y \leq 1) \\ &= f(0, 0) + f(0, 1) + f(1, 0) \end{aligned}$$

$$= \frac{3}{28} + \frac{6}{28} + \frac{9}{28}$$

$$= \frac{18}{28}$$

$$= \frac{9}{14}$$

Joint Density Function

The function $f(x, y)$ is a **joint density function** of the **continuous random variables X and Y** if

1. $f(x, y) \geq 0$, for all (x, y) ,

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$,

3. $P[(X, Y) \in A] = \int \int_A f(x, y) dx dy$, for any region A in the xy plane.

Example: A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y , respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{5} (2x + 3y), & 0 \leq x \leq 1, 0 \leq y \leq 1. \\ 0, & \text{elsewhere} \end{cases}$$

(a) Verify $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.

(b) $P[(X, Y) \in A]$, where $A = \{(x, y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= \int_{y=0}^1 \left\{ \int_{x=0}^1 \frac{2}{5} (2x + 3y) dx \right\} dy \\&= \frac{2}{5} \int_{y=0}^1 \left| \frac{(2)x^2}{2} + 3xy \right|_{x=0}^1 dy \\&= \frac{2}{5} \int_0^1 \{1^2 + 3(1)y\} dy - 0 \\&= \frac{2}{5} \left| y + 3 \frac{y^2}{2} \right|_0^1 \\&= \frac{2}{5} \left(1 + \frac{3}{2} \right) - 0 \\&= 1\end{aligned}$$

$$\begin{aligned}
P[(X, Y) \in A] &= P(0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}) \\
&= \int_{y=\frac{1}{4}}^{\frac{1}{2}} \left\{ \int_{x=0}^{\frac{1}{2}} \frac{2}{5} (2x + 3y) dx \right\} dy \\
&= \frac{2}{5} \left\{ \int_{y=\frac{1}{4}}^{\frac{1}{2}} \left[\frac{2x^2}{2} + 3xy \right]_{x=0}^{\frac{1}{2}} dy \right\} \\
&= \frac{2}{5} \int_{y=\frac{1}{4}}^{\frac{1}{2}} \left[x^2 + 3xy \right]_{x=0}^{\frac{1}{2}} dy \\
&= \frac{2}{5} \int_{\frac{1}{4}}^{\frac{1}{2}} \left\{ \frac{1}{4} + 3 \left(\frac{1}{2} \right) y \right\} dy \\
&= \frac{2}{5} \int_{\frac{1}{4}}^{\frac{1}{2}} \left\{ \frac{1}{4} + \frac{3}{2} y \right\} dy \\
&= \frac{2}{5} \left[\frac{1}{4} y + \frac{3}{4} y^2 \right]_{\frac{1}{4}}^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{5} \left| \frac{1}{4} y + \frac{3}{4} y^2 \right|^{\frac{1}{2}}_{\frac{1}{4}} \\
&= \frac{2}{5} \left\{ \frac{1}{8} + \left(\frac{3}{4}\right) \left(\frac{1}{2}\right)^2 - \frac{1}{16} - \left(\frac{3}{4}\right) \left(\frac{1}{4}\right)^2 \right\} \\
&= \frac{2}{5} \left(\frac{1}{8} + \frac{3}{16} - \frac{1}{16} - \frac{3}{64} \right) \\
&= \frac{2}{5} \left(\frac{2+12-4+3}{64} \right) \\
&= \frac{(2)(13)}{(5)(64)} \\
&= \frac{13}{160}
\end{aligned}$$

The marginal distributions of X alone and of Y alone are

- Given the **joint probability distribution** $f(x, y)$ of the discrete random variables X and Y , the **probability distribution** $g(x)$ of X alone is obtained by **summing** $f(x, y)$ over the **values of** Y .
- Similarly, the probability distribution $h(y)$ of Y alone is obtained by **summing** $f(x, y)$ over the **values of** X . We define $g(x)$ and $h(y)$ to be the **marginal distributions** of X and Y , respectively.

The marginal distributions of X alone and of Y alone are

- When X and Y are continuous random variables, summations are replaced by integrals.

The marginal distributions of X alone and of Y alone are

$g(x) = \sum_y f(x, y)$ and $h(y) = \sum_x f(x, y)$ for discrete case
, and

$$g(x) = \int_{y=-\infty}^{\infty} f(x, y) dy \text{ and } h(y) = \int_{x=-\infty}^{\infty} f(x, y) dx$$

for the continuous case.

The marginal distributions of X alone and of Y alone are

□ **Note:** The term *marginal* is used here because, in the **discrete case**, the values of $g(x)$ and $h(y)$ are just the **marginal totals** of the respective **columns** and **rows** when the values of $f(x, y)$ are displayed in a **rectangular table**.

□ **Example** : Show that the column and row totals of the table in the coming slide give the **marginal distribution** of **X** alone and of **Y** alone.

f(x, y)		x			Row totals
		0	1	2	
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{6}{28}$	$\frac{6}{28}$	$\frac{0}{28}$	$\frac{12}{28}$
	2	$\frac{1}{28}$	$\frac{0}{28}$	$\frac{0}{28}$	$\frac{1}{28}$
Column totals		$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	$\frac{28}{28} = 1$

Solution : For the random variable X , we see that

$$g(x) = \sum_y f(x, y)$$

$$g(0) = f(0, 0) + f(0, 1) + f(0, 2)$$

$$= \frac{3}{28} + \frac{6}{28} + \frac{1}{28} = \frac{10}{28} = \frac{5}{14}$$

$$g(1) = f(1, 0) + f(1, 1) + f(1, 2)$$

$$= \frac{9}{28} + \frac{6}{28} + 0 = \frac{15}{28}$$

$$g(2) = f(2, 0) + f(2, 1) + f(2, 2)$$

$$= \frac{3}{28} + 0 + 0 = \frac{3}{28}$$

Marginal Distribution of x

$x = 0$	0	1	2	Total
$g(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	$\frac{28}{28} = 1$

For the random variable y , we see that

$$h(y) = \sum_x f(x, y)$$

$$h(0) = f(0, 0) + f(1, 0) + f(2, 0)$$

$$= \frac{3}{28} + \frac{9}{28} + \frac{3}{28} = \frac{15}{28}$$

$$h(1) = f(0, 1) + f(1, 1) + f(2, 1)$$

$$= \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28}$$

$$h(2) = f(0, 2) + f(1, 2) + f(2, 2)$$

$$= \frac{1}{28} + 0 + 0 = \frac{1}{28}$$

Marginal Distribution of y

$y = 0$	0	1	2	Total
$h(y)$	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$	$\frac{28}{28} = 1$

Example : Find $g(x)$ and $h(y)$ for the joint density function

$$f(x, y) = \begin{cases} \frac{2}{5} (2x + 3y), & 0 \leq x \leq 1, 0 \leq y \leq 1. \\ 0, & \text{elsewhere} \end{cases}$$

Marginal Density of x

Solution

$$\begin{aligned}g(x) &= \int_{y=-\infty}^{\infty} f(x, y) dy \\&= \int_{y=0}^1 \frac{2}{5} (2x + 3y) dy \\&= \left| \frac{2}{5} (2xy + \frac{3y^2}{2}) \right|_{y=0}^1 \\&= \frac{2}{5} \left\{ 2x(1) + \frac{3(1)^2}{2} \right\} - 0 \\&= \frac{2}{5} \left(\frac{4x + 3}{2} \right) \\&= \frac{4x + 3}{5}\end{aligned}$$

Marginal Density of y

$$\begin{aligned}h(y) &= \int_{x=-\infty}^{\infty} f(x, y) dx \\&= \int_{x=0}^1 \frac{2}{5} (2x + 3y) dx \\&= \frac{2}{5} \left| \frac{2x^2}{2} + 3xy \right|_{x=0}^1 \\&= \frac{2}{5} \{1 + 3(1)y\} - 0 \\&= \frac{2}{5} (1 + 3y)\end{aligned}$$