Linear Algebra (MTH231)

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Part I :

- Systems of Linear Equations, Row Reduction and Echelon Forms.
- Vector Equations, The Matrix Equation.
- Solution Sets of Linear Systems, Linear Independence.
- Introduction to Linear Transformations, The Matrix of a Linear Transformation, Applications.

Part II :

- Matrix Operations
- The Inverse of a Matrix, Characterization of Invertible Matrices
- Matrix Factorizations, Applications.
- Introduction to Determinants and Properties of Determinants.

Part III :

- Vector Spaces and Subspaces, Bases, Null Spaces, Column Spaces.
- Coordinate Systems.
- The Dimension of a Vector Space Rank, Applications.
- Eigenvectors and Eigenvalues, The Characteristic Equation, Cayley Hamilton Theorem.
- Diagonalization, Applications, Inner Product, Length and Orthogonality.
- Orthogonal sets, Orthogonal Projections
- The Gram-Schmidt Process Applications

The text book :

David C. Lay, Linear Algebra and Its Applications, Fourth Edition, Addison-Wesley, ISBN-13 : 978-1408280560.

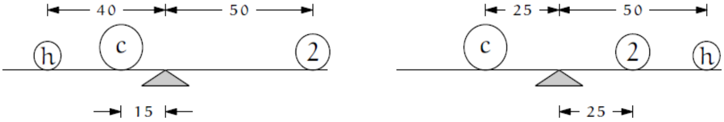
Reference books :

- **Gilbert Strang**, Introduction to Linear Algebra, Fourth Edition, Wellesley-Cambridge Press, ISBN : 9780980232714.
- **Lee W. Johnson, R. Dean Riess and Jimmy T. Arnold**, Introduction to Linear Algebra, Fifth Edition, Addison-Wesley, ISBN-13 : 9780201658590.

Assessment Plan for the Course :

- Four Assignments 10%.
- Four Quiz 15%.
- First Sessional Exam 10%.
- Second Sessional Exam 15%.
- Final Exam 50%.

Mass Balance :



$$40h + 15c = 100$$
$$25c = 50 + 50h$$

Linear Programming : The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.

Electrical Networks : Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

Example : The equations

$$3x_1 - 5x_2 = 4x_1 \quad \text{and} \quad x_1 - \sqrt{5}x_2 = 4x_2 + 5\sqrt{5}$$

are linear equations and can be simplified to

$$x_1 + 5x_2 = 0 \quad \text{and} \quad x_1 - (4 + \sqrt{5})x_2 = 5\sqrt{5}.$$

The equations

$$x_1 - x_2 + x_2x_1 = 0 \quad \text{and} \quad x_1 - \sqrt{5}x_2 = 4x_2 + \sqrt{x_2}$$

are not linear equations.

A linear equation in the variables x_1, . . . , x_n has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$$

where $a_1, \cdots, a_n$ are real or complex numbers (**usually known**) $d \in \mathbb{R}$ is the constant.

Examples :

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 10 \\ -x_1 + 5x_2 + x_3 &= 5 \end{aligned}$$

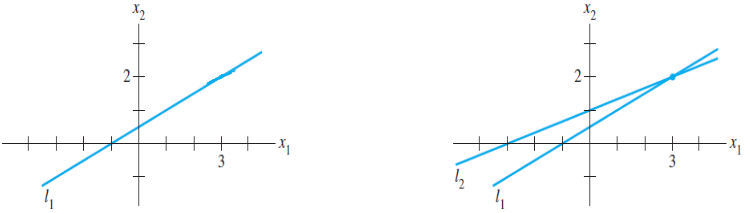
$$\begin{aligned} -x_1 + 5x_2 + 3x_3 + x_4 &= 10 \\ 2x_1 + 5x_2 + 2x_3 - 2x_4 &= 5 \\ 9x_1 - 10x_2 + x_3 - 3x_4 &= 5 \end{aligned}$$

A system of linear equations with m equations and n variables

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & d_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & d_2 \\ & & & & & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & d_m \end{array}$$

Example : Unique Solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \qquad l_1 \\ -x_1 + 3x_2 &= 3 \qquad l_2 \end{aligned}$$

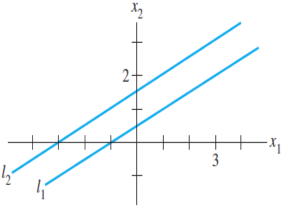


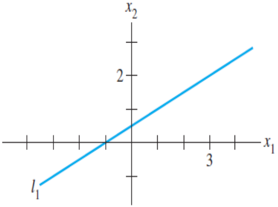
Example : No solution and Infinite many solutions

$$(a) \qquad \begin{aligned} x_1 - 2x_2 &= -1 \qquad l_1 \\ -x_1 + 2x_2 &= 3 \qquad l_2 \end{aligned}$$

$$(b) \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

Remark : For a system of linear equation with two variables and two unknown we have three possibilities, (i) system has unique solution, (ii) Infinite many solution, (iii) No solution.





Example :
The ordered pair $(-1, 5)$ is a solution of this system. In contrast, $(5, -1)$ is not a solution.

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & 7 \\ -x_1 & + & x_2 & = & 6 \end{array}$$

Example : Is $(3, 4, -2)$ a solution of the following system ?

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 2x_3 & = & 7 \\ -2x_1 & + & 6x_2 & + & 9x_3 & = & 0 \\ -7x_1 & + & 5x_2 & - & 3x_3 & = & -7 \end{array}$$

A system of linear equations with m equations and n variables

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & d_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & d_2 \\ & & & & & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & d_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.

Recall : For a system of linear equation with two variables and two unknown we have three possibilities ;

- System has a unique solution,
- Infinite many solution,
- No solution.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions ; a system is **inconsistent** if it has no solution.

Question : Can a system of linear equations has only two solutions or only three solution or only 100 solutions ?

How to find all solutions of a given system of linear equations ?

Matrix : A matrix is a rectangular array of numbers.

For example

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 6 & 1 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

is a matrix having three row and three columns.

The order of a matrix is defined as

order $=$ The number of rows $\times$ the number of columns.

The order of the above matrix is 3×3.

Examples : $\begin{bmatrix} 1 \\ 1/3 \\ 1 \end{bmatrix}_{3\times 1}$ is called a columns matrix or vector.

$\begin{bmatrix} -4 & 12 & 4 \\ 2 & -6 & -7 \end{bmatrix}_{2\times 3}$

For the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ is known as matrix of coefficients of the system of linear equations.

The matrix $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$ is called augmented matrix.

Remark : The size of a matrix tells how many rows and columns it has.

Example :

$$\begin{array}{ccccccc} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

For the above system the matrix of coefficients is $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ \frac{1}{3} & 2 & 0 \end{bmatrix}$.

The matrix of augmented matrix is $\begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{bmatrix}$.

Example : If the matrix is the augmented matrix of a system of linear equations write down the system of linear equations.

$$\begin{bmatrix} 2 & 0 & -2 & 5 \\ 7 & 2 & 5 & 0 \\ 1 & 4 & 5 & 10 \end{bmatrix}.$$

$$\begin{array}{ccccccc} 2x_1 & & & - & 2x_3 & = & 5 \\ 7x_1 & + & 2x_2 & + & 5x_3 & = & 0 \\ x_1 & + & 4x_2 & + & 5x_3 & = & 10 \end{array}$$

Solve the system of linear equations

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array} \qquad \begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{bmatrix}.$$

The first transformation rewrites the system by interchanging the first and third row.

swap row 1 with row 3

$$\begin{array}{rcrcrcl} \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array} \qquad \begin{bmatrix} \frac{1}{3} & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}.$$

The second transformation rescales the first row by multiplying both sides of the equation by 3.

multiply row 1 by 3

$$\begin{array}{rcrcrcl} x_1 & + & 6x_2 & & & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ & & & & 3x_3 & = & 9 \end{array} \qquad \begin{bmatrix} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}.$$

We multiply both sides of the first row by -1, and add that to the second row, and write the result in as the new second row.

$$\xrightarrow{\text{add } -1 \text{ times row 1 to row 2}} \quad \begin{array}{rcrcrcr} x_1 & + & 6x_2 & & & = & 9 \\ & & -x_2 & - & 2x_3 & = & -7 \\ & & & & 3x_3 & = & 9 \end{array} \quad \begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$.

Substituting those two into the top equation gives that $x_1 = 3$.

Thus the system has a unique solution ; the solution set is $\{(3, 1, 3)\}$.

Verify that the vector $\{(3,1,3)\}$ is a solution set for the system of linear equations

$$\begin{array}{rcrcrcr} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

$$\begin{array}{rcrcrcr} & & & + & 3(3) & = & 9 \\ 3 & + & 5(1) & - & 2(3) & = & 2 \\ \frac{1}{3}(3) & + & 2(1) & & & = & 3 \end{array}$$

All equations of the system of linear equations are satisfied, hence the set $\{(3,1,3)\}$ is a solution set of the system of linear equations.

Solve the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}.$$

Keep x_1 in the first equation and eliminate it from the other equations.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{4[Equation 1] +[Equation 3]} & & & 2x_2 & - & 8x_3 & = & 8 \\ & & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Multiply equation 2 by $1/2$ in order to obtain 1 as the coefficient for x_2.

$$\begin{array}{lrcrcrcr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{4[Equation 1] +[Equation 3]} & & & x_2 & - & 4x_3 & = & 4 \\ & & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3.

$$\begin{array}{rrrrrrrr} & x_1 & - & 2x_2 & + & x_3 & = & 0 \\ \text{3[Equation 2] +[Equation 3]} & & & x_2 & - & 4x_3 & = & 4 \\ & & & & & x_3 & = & 3 \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

$4[Equation3] + [Equation2]$ and $-1[Equation3] + [Equation1]$

$$\begin{array}{rrrrr} x_1 & - & 2x_2 & = & -3 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \quad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

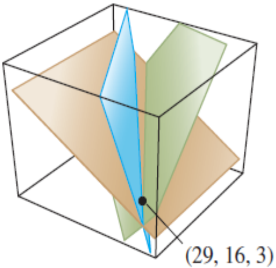
2[Equation 2]+[Equation 1] leads to

$$\begin{array}{rrrrr} x_1 & - & & = & 29 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Solution of the system is $29, 16, 3$.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcrcrcr} (29) & - & 2(16) & + & 3 & = & 0 \\ & & 2(16) & - & 8(3) & = & 8 \\ -4(29) & + & 5(16) & + & 9(3) & = & -9 \end{array}$$



Example : Determine if the following system is consistent.

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}.$$

To obtain an x_1 in the first equation, interchange rows 1 and 2 :

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3 :

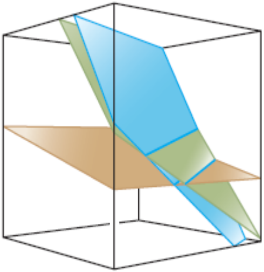
$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

To eliminate the $-1/2/x_2$ term from the third equation. Add 1/2 times row 2 to row 3 : :

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

The augmented matrix is in triangular form and we transform into equation notation

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5/2 \end{array}$$



Example : For what values of h and $k$is the following system consistent ?

$$\begin{array}{rcrl} 2x_1 & - & x_2 = & h \\ -6x_1 & + & 3x_2 = & k \end{array}$$

Solution : The augmented matrix of the system is $\left[\begin{array}{ccc} 2 & -1 & h \\ -6 & 3 & k \end{array}\right]$.

3[Equation 1] + [Equation 2] or 3[Row 1] + [Row 2]

$$\left[\begin{array}{ccc} 2 & -1 & h \\ 0 & 0 & k+3h \end{array}\right].$$

If $k + 3h \neq 0$ then we have $0 = k + 3h \neq 0$ implies the system is inconsistent.

So the system will be consistent if we have $k + 3h = 0$ or $k = -3h$.

For example : take $h = 2$ then $k = -9$ is one possibility. There are infinite many values of h and k satisfying $k + 3h = 0$.

Question : Do the lines $2x_1 + 3x_2 = -1$, $6x_1 + 5x_2 = 0$, and $2x_1 - 5x_2 = 7$ have a common point of intersection ?Justify your answer.

Question : Determine the value(s) of h such that the matrix is the augmented matrix of a consistent system

(a) $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix}$ $\quad$ (b) $\begin{bmatrix} 1 & 4 & -2 \\ 2 & h & -6 \end{bmatrix}$

Question : Determine whether the given system of linear equations are consistent or inconsistent.

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 3x_3 & = & -8 \\ & & x_2 & - & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array} \qquad \begin{array}{rcrcrcr} x_1 & - & 5x_2 & + & 4x_3 & = & -3 \\ 2x_1 & - & 7x_2 & + & 3x_3 & = & -2 \\ -2x_1 & + & x_2 & + & 7x_3 & = & -1 \end{array}$$

Question : Find an equation involving g, h, and k that makes the augmented matrix correspond to a consistent system

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$