Probability and Statstics

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber
- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

References

Readings for these lecture notes:

- ❑ Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer
- ❑ Probability Demystified, Allan G. Bluman
- ❑ Elementary Statistics, 10th Edition, Mario F. Triola

These notes contain material from the above three books.

Subjective Probability

A third type of probability is called **subjective probability**. Subjective probability is based upon an **educated guess**, **estimate**, **opinion**, or **inexact information**.

Sample Spaces

There are two **specific devices** that will be used to find sample spaces for probability experiments. They are **tree diagrams** and **tables**.

A **tree diagram** consists of **branches** corresponding to the outcomes of two or more probability experiments that are done in sequence.

Sample Spaces

- In order to construct a tree diagram, use branches corresponding to the outcomes of the **first experiment**. These branches will emanate from a single point.
- Then from each branch of the first experiment draw branches that represent the outcomes of the **second experiment**.
- You can continue the process for further experiments of the sequence if necessary.

Tree Diagram [1]

Example: A coin is tossed and a die is rolled. Draw a tree diagram and find the sample space.

Solution:

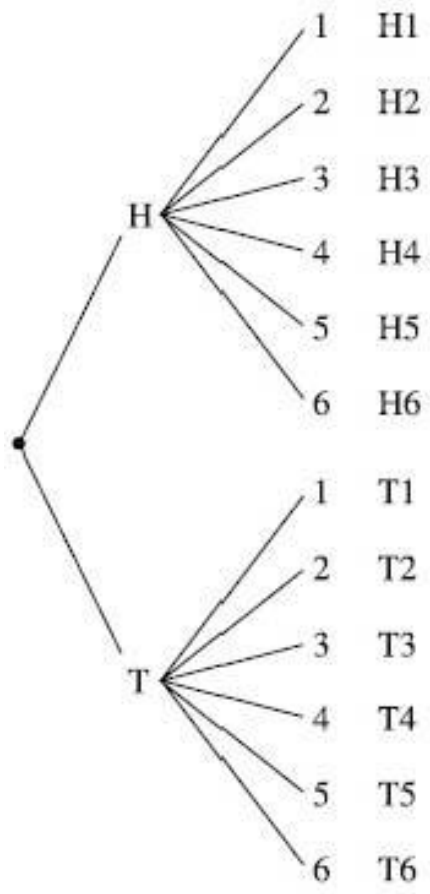
Since there are two outcomes (heads and tails for the coin), draw two branches from a single point and label one **H** for head and the other one **T** for tail.

From each one of these outcomes, draw and label six branches representing the outcomes **1, 2, 3, 4, 5,** and **6** for the die.

Trace through each branch to find the outcomes of the experiment.

Tree Diagram [2]

Example: A coin is tossed and a die is rolled. Draw a tree diagram and find the sample space.



Tree Diagram [3]

Example: A coin is tossed and a die is rolled. Find the probability of getting

a. A head on the coin and a 3 on the die.

b. A head on the coin.

c. A 4 on the die.

Solution:

a. $P(H3) = {}^{1}/_{12}$ = **0.0833 (or 8.33%)**

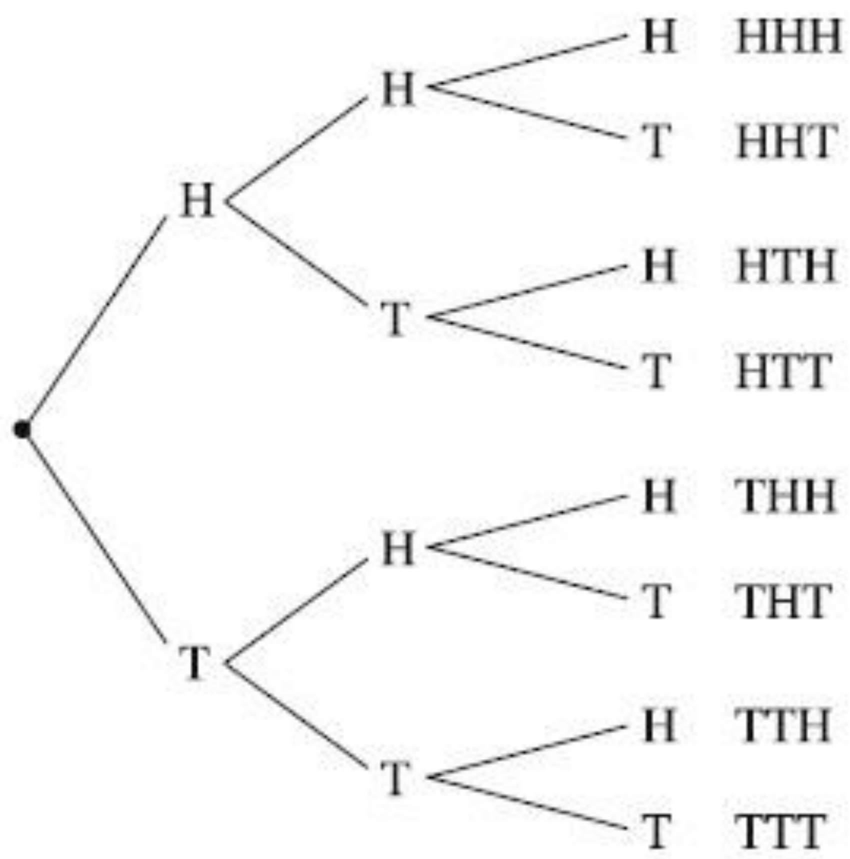
b. $P(\text{head on the coin}) = {}^{6}/_{12} = {}^{1}/_{2}$ = **0.5 (or 50%)**

c. $P(\text{4 on the die}) = {}^{2}/_{12} = {}^{1}/_{6}$ = **0.1667 (16.67%)**

Tree Diagram [4]

Example: Three coins are tossed. Draw a tree diagram and find the sample space.

Solution



Tree Diagram [5]

Example: Three coins are tossed. Find the probability of getting

a. Two heads and a tail in any order.

b. Three heads.

c. No heads.

d. **At least** two tails.

e. **At most** two tails.

Solution:

a. P(2 heads and a tail) = 3/8 = **0.375 (or 37.5 %)**

b. P(HHH) = $^{1}/_{8}$ = **0.125 (or 12.5 %)**

c. P(TTT) = $^{1}/_{8}$ = **0.125 (or 12.5 %)**

d. P(at least two tails) = $^{4}/_{8}$ = $^{1}/_{2}$ = **0.5 (or 50 %)**

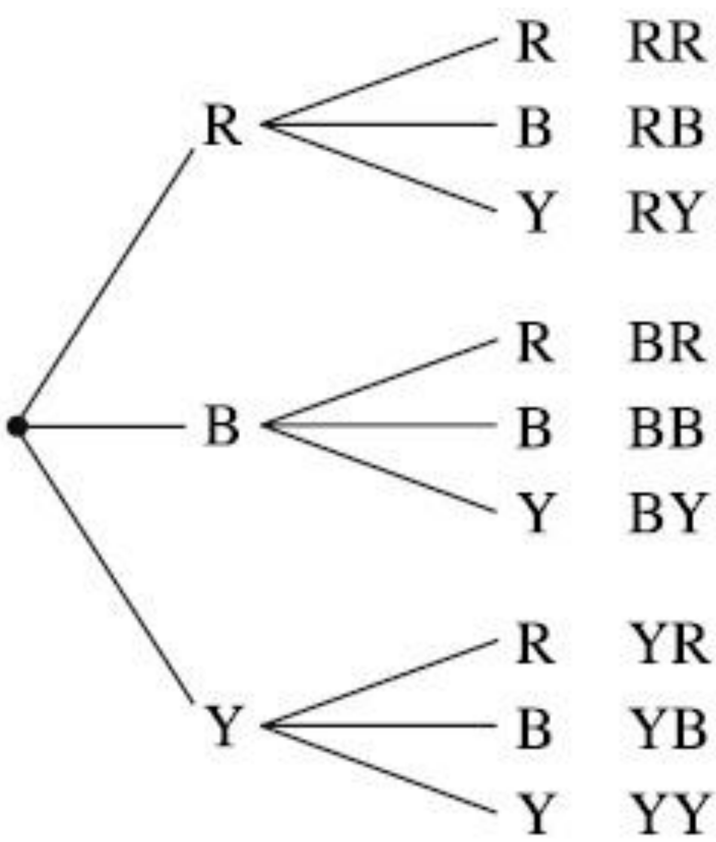
e. P(at most two tails) = $^{7}/_{8}$ = **0.875 (or 87.5 %)**

$\Rightarrow \boldsymbol{At\ most\ two\ tails\ mean\ no\ three\ tails}$

Tree Diagram [6]

Example: A box contains a **red ball (R),** a **blue ball (B),** and a **yellow ball (Y**). **Two balls** are selected at random in succession. Draw a **tree diagram** and find the sample space if the first ball is **replaced** before the second ball is selected.

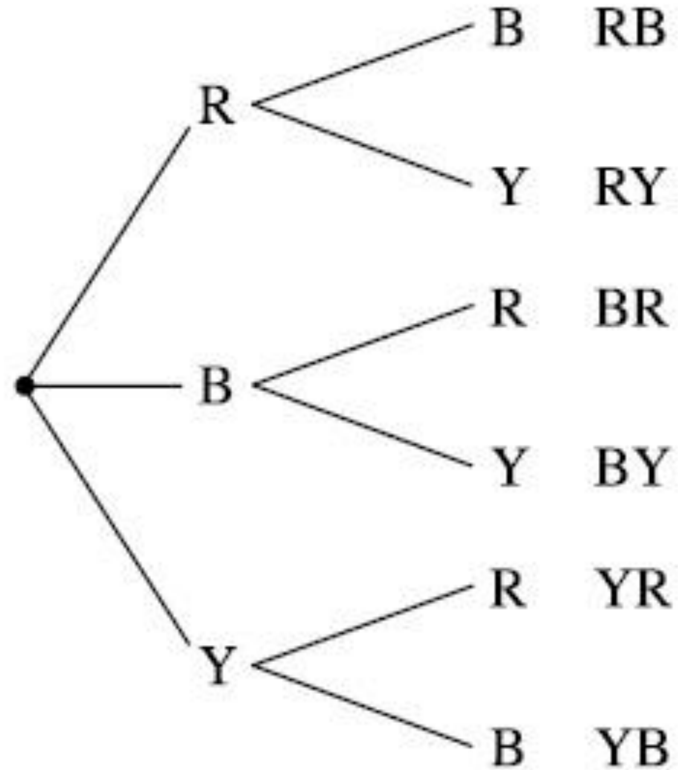
Solution



Tree Diagram [7]

Example: A box contains a **red ball (R),** a **blue ball (B)**, and a **yellow ball (Y)**. Two balls are selected at random in succession. Draw a **tree diagram** and find the sample space if the first ball is **not replaced** before the second ball is selected.

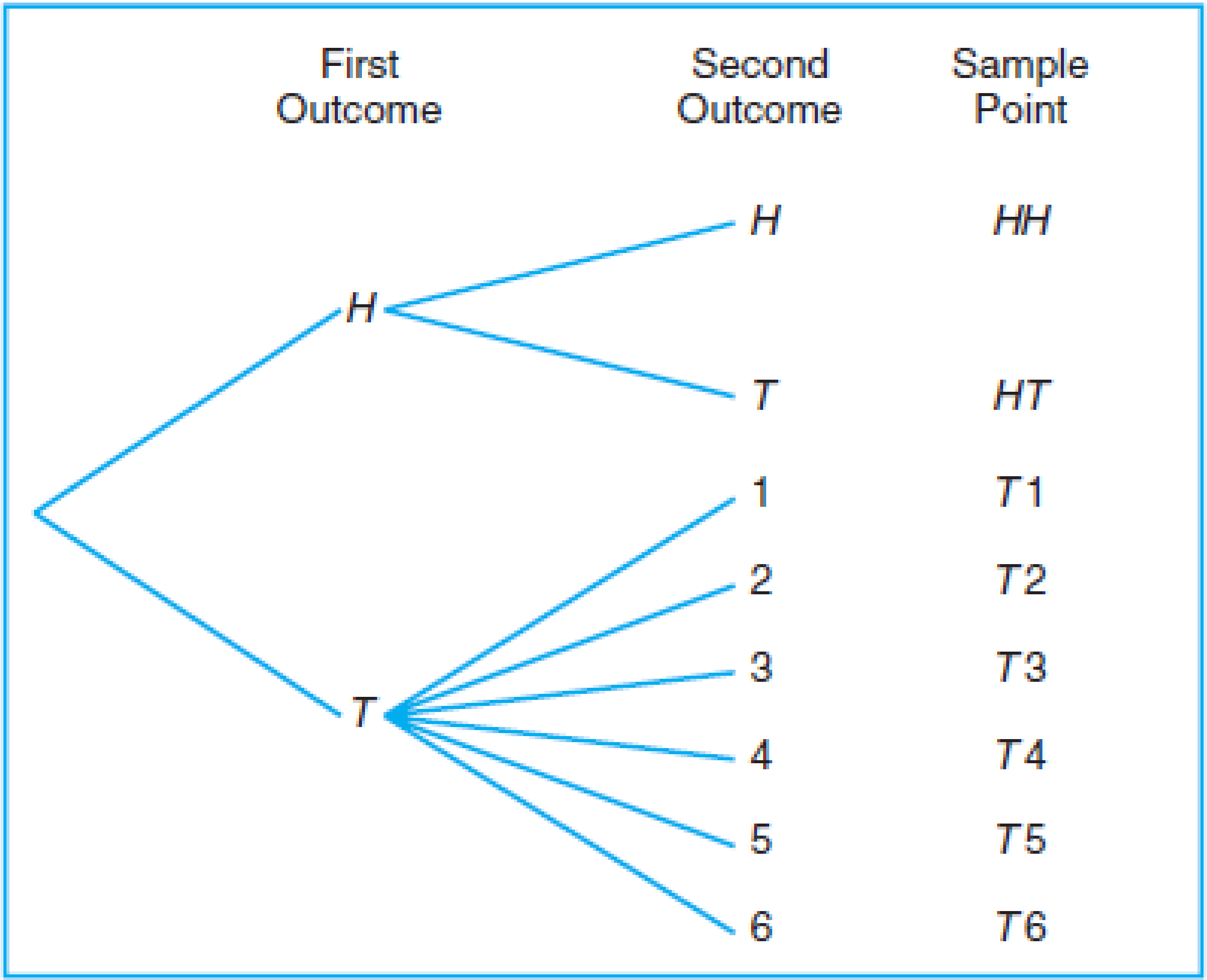
Solutions



Tree Diagram [9]

Example An experiment consists of **flipping a coin** and then flipping it a **second time** if a **head occurs**. If a **tail** occurs on the **first flip**, then a **die is tossed** once. To list the elements of the sample space providing the most information, construct the **tree diagram**

Solution

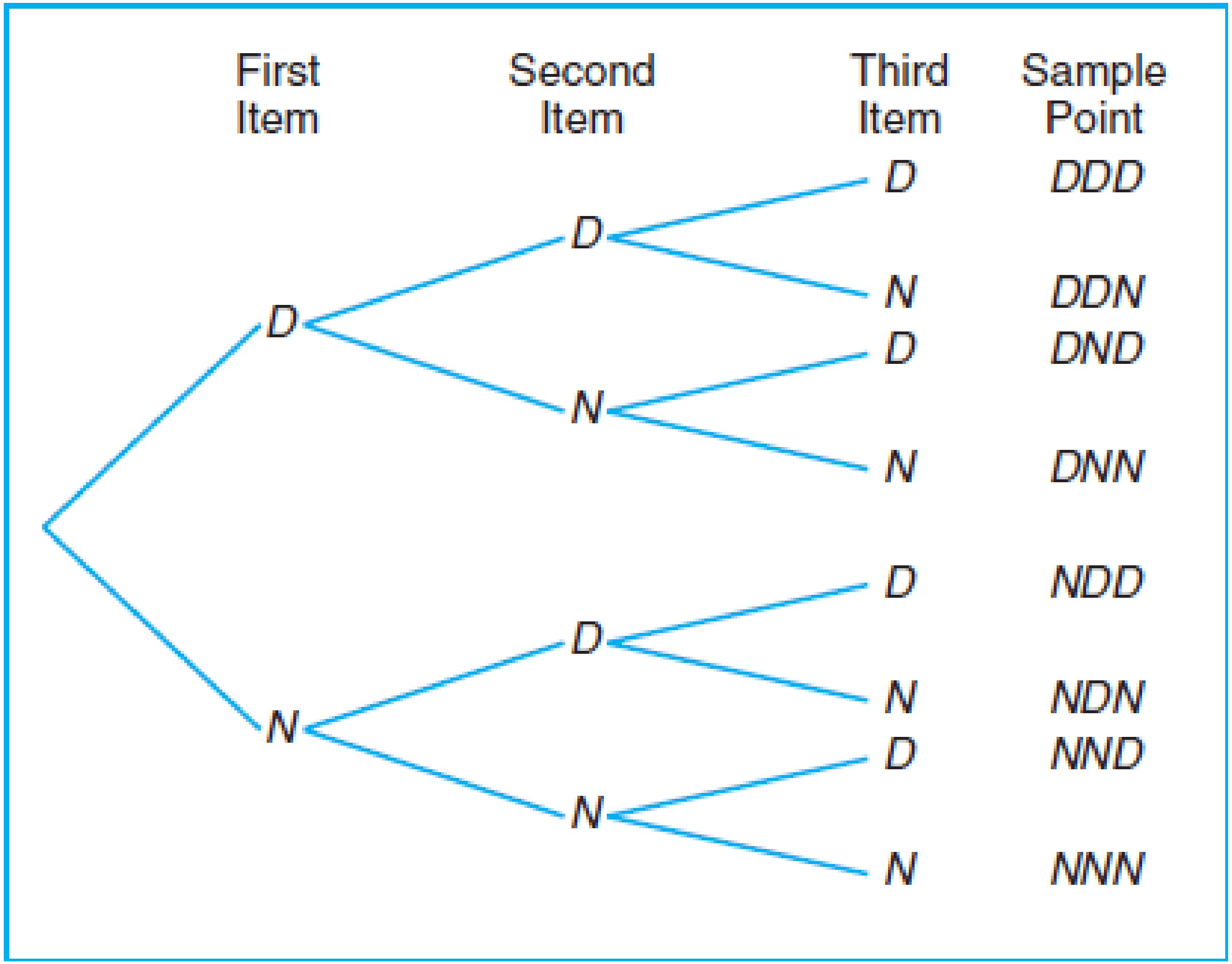

First
Outcome
Second
Outcome
Sample
Point
H
HH
H
T
HT
1
T1
2
T2
3
T3
T
4
T4
5
T5
6
T6

Tree Diagram [10]

Example Suppose that **three items** are selected at random from a manufacturing process. Each item is inspected and classified **defective**, **D**, or **nondefective**, **N**. To list the elements of the sample space providing the most information, construct the **tree diagram.**

Solution

- S = {HH, HT, T1, T2, T3, T4, T5, T6}.



Tables [1]

Another way to find a sample space is to use a **table.**

Example: Find the sample space for selecting a card from a standard deck of 52 cards.

- There are four suits—**hearts** and **diamonds**, which are **red**, and **spades** and **clubs**, which are **black**.

- Each suit consists of 13 cards—ace through king. **Face cards** are kings, queens, and jacks.

A standard deck of 52 cards

Heart	A ♥	2 ♥	3 ♥	4 ♥	5 ♥	6 ♥	7 ♥	8 ♥	9 ♥	10 ♥	J ♥	Q ♥	K ♥
Diamond	A ♦	2 ♦	3 ♦	4 ♦	5 ♦	6 ♦	7 ♦	8 ♦	9 ♦	10 ♦	J ♦	Q ♦	K ♦
Spade	A ♠	2 ♠	3 ♠	4 ♠	5 ♠	6 ♠	7 ♠	8 ♠	9 ♠	10 ♠	J ♠	Q ♠	K ♠
Club	A ♣	2 ♣	3 ♣	4 ♣	5 ♣	6 ♣	7 ♣	8 ♣	9 ♣	10 ♣	J ♣	Q ♣	K ♣

Tables [2]

Example: A single card is drawn at random from a standard deck of cards. Find the probability that it is

a. The 4 of diamonds.

b. A queen.

Solution:

a. P(The 4 of diamonds) = $^{1}/_{52}$ = **0.0192 (or 1.9231%)**

b. P(A queen) = $^{4}/_{52}$ = $^{1}/_{13}$ = **0.0769 (or 7.6923 %)**

Tables [3]

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Tables [4]

Example: When two dice are rolled, find the probability of getting a sum of nine.

Solution:

Let A be the event of getting a "sum of 9"

P(A) = ${}^{4}/_{36}$ = ${}^{1}/_{9}$ = **0.1111 (or 11.11 %)**

Intersection [1]

Intersection: The **intersection** of two events A and B, denoted by the symbol **$A \cap B$**, is the event containing all elements that are **common to A and B**.

Example: Let **E** be the event that a person selected at random in a classroom is majoring in **engineering**, and let **F** be the event that the person is **female**. Then **$E \cap F$** is the event of all female engineering students in the classroom.

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Intersection [2]

Example: Let **V = {a, e, i, o, u}** and **C = {l, r, s, t};** then it follows that **$V \cap C = \emptyset$**. That is, V and C have no elements in common and, therefore, **cannot both simultaneously occur**.

Union

Union: The **union** of the two events A and B, denoted by the symbol **AUB**, is the event containing all the elements that belong to A or B or both.

Notation for Addition Rule

P(A or B) = **P**(in a single trial, event *A* occurs or event *B* occurs or they both occur)

Mutually Exclusive or Disjoint

Two events A and B are **mutually exclusive, or disjoint,** if **$A \cap B = \{\ \}$** or **$\emptyset$**

OR

Two events A and B are **mutually exclusive**, **or disjoint,** if **$A \cap B = \emptyset$**, that is, if A and B have no elements in common.

OR

Events ***A*** and ***B*** are **disjoint** (or **mutually exclusive**) if they **cannot occur at the same time**. (That is, disjoint events do not overlap.)

Addition Rule I

Addition Rule I: When two events are **mutually exclusive** or **disjoint events**

P(A or B) = P(A) + P(B)

OR

$$P(A \cup B) = P(A) + P(B)$$

Example: When a die is rolled, find the probability of getting a **2** or a **3**.

Solution:

S = { 1, 2, 3, 4, 5, 6}; n(S) = 6

Let **A** be the event of getting a **"2"**

A = {2}; n(A) = 1

P(A) = n(A)/n(S) = $\frac{1}{6}$ = **0.1667 (or 16.67%)**

Let **B** be the event of getting a **"3"**

B = {3}; n(B) = 1

P(B) = n(B)/n(S) = $\frac{1}{6}$ = **0.1667 (or 16.67%)**

Since events **A** and **B** are **mutually exclusive**, so

$P(A \cup B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} = 0.3333$ **(or 33.33%)**

Addition Rule I [2]

A cable television company offers programs on **eight** different channels, **three** of which are affiliated with **ABC**, **two** with NBC, and **one** with **CBS**. The other **two** are an **educational channel** and the **ESPN sports** channel. Suppose that a person subscribing to this service turns on a television set without first selecting the channel. Let ***A*** be the event that the program belongs to the **NBC network** and ***B*** the event that it belongs to the **CBS network**. Since a television program cannot belong to more than one network, the events *A* and *B* have no programs in common.

Therefore, the intersection **$A \cap B$** contains no programs, and consequently the events *A* and *B* are **mutually exclusive**.

Addition Rule I [3]

Example: In a committee meeting, there were **5** freshmen, **6** sophomores, **3** juniors, and **2** seniors. If a student is selected at random to be the chairperson, find the probability that the chairperson is a **sophomore** or a **junior**.

Addition Rule I [4]

Solution:

Let A be the event of selecting a chairperson as a **"sophomore"**

$$P(A) = \frac{6}{16} = \frac{3}{8} = \mathbf{0.3750}$$

Let B be the event of selecting a chairperson as a **"junior"**

$$P(B) = \frac{3}{16} = \mathbf{0.1875}$$

Since **A** and **B** are **mutually exclusive or disjoint events**

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{16} + \frac{3}{16} = \frac{9}{16} = \mathbf{0.5625\ (or\ 56.25\%)}$$

Addition Rule I [5]

Example: A card is selected at random from a deck. Find the probability that the card is an **ace or** a **king**.

Heart	A ♥	2 ♥	3 ♥	4 ♥	5 ♥	6 ♥	7 ♥	8 ♥	9 ♥	10 ♥	J ♥	Q ♥	K ♥
Diamond	A ♦	2 ♦	3 ♦	4 ♦	5 ♦	6 ♦	7 ♦	8 ♦	9 ♦	10 ♦	J ♦	Q ♦	K ♦
Spade	A ♠	2 ♠	3 ♠	4 ♠	5 ♠	6 ♠	7 ♠	8 ♠	9 ♠	10 ♠	J ♠	Q ♠	K ♠
Club	A ♣	2 ♣	3 ♣	4 ♣	5 ♣	6 ♣	7 ♣	8 ♣	9 ♣	10 ♣	J ♣	Q ♣	K ♣

Solution:

Let **A** be the event of selecting an **ace**

P(A) = $\frac{4}{52} = \frac{1}{13}$ = **0.0769 (or 7.69%)**

Let **B** be the even of selecting a **king**

P(B) = $\frac{4}{52} = \frac{1}{13}$ = **0.0769 (or 7.69%)**

Since **A** and **B** are **mutually exclusive or disjoint events**

P(AUB) = P(A) + P(B)

$= \frac{4}{52} + \frac{4}{52}$ = 0.1538 **(or 15.38%)**

Addition Rule II [1]

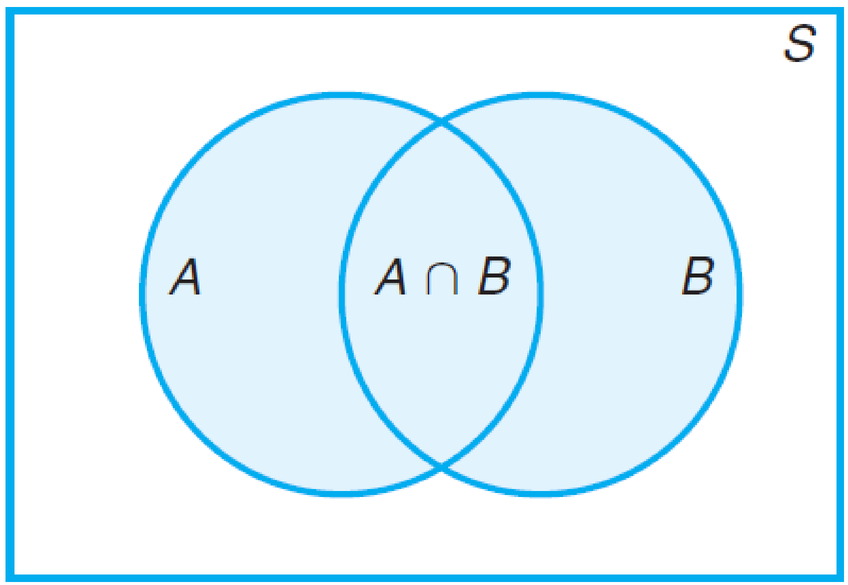
- When two events are **not mutually exclusive**, you need to add the probabilities of each of the two events and **subtract the probability of the outcomes that are common** to both events. In this case, addition **rule II** can be used.

- **Addition Rule II:** If A and B are two events that are not mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

When A and B are two events that are not mutually exclusive

If A and B are two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



Additive rule of probability

Addition Rule II [2]

Example: A card is selected at random from a deck of 52 cards. Find the probability that it is a **6** or a **diamond**.

Addition Rule II [3]

Solution:

Let A be the event of getting a **“6”.**

P(A) = $\frac{4}{52} = \frac{1}{13} =$ **0.0769 (or 7.69%)**

Let B be the event of getting a **“diamond”.**

P(B) = $\frac{13}{52} = \frac{1}{4} =$ **0.2500 (or 25%)**

Addition Rule II [4]

Let **A ∩ B** be the event of getting a **"6"** and a **"diamond"**

P(A∩B) = $\frac{1}{52}$ = **0.0192 (or 1.9231%)**

Since A and B are **not mutually exclusive**, so

P(AUB) = P(A) + P(B) – P(A ∩ B)

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$$

$$= \frac{4}{13} = 0.3077 \text{ (or } 30.77\%)$$

Addition Rule II [5]

Example: A die is rolled. Find the probability of getting an even number or a number less than 4.

Addition Rule II [6]

Solution:

S = {1, 2, 3, 4, 5, 6}

n(S) = 6

Let A be the event of getting an even number

A = { 2, 4, 6}, n(A) = 3

P(A) = $\frac{3}{6} = \frac{1}{2}$ = 0.50 (or 50%)

Let B be the event of getting a number less than 4

B = {1, 2, 3}, n(B) = 3

P(B) = $\frac{3}{6} = \frac{1}{2}$ = 0.50 (or 50%)

Addition Rule II [7]

Let A ∩ B be the event of getting an **"even number"** and a **"number less than 4"**

A ∩ B = {2}

P(A∩B) = $\frac{1}{6}$ = 0.1667 or (16.67%)

Since A and B are **not mutually exclusive**, so

P(AUB) = P(A) + P(B) – P(A ∩ B)

$$= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \mathbf{\frac{5}{6} = 0.8333\ (or\ 83.3333\ \%)}$$

Table

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	**1**	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Addition Rule II [8]

Example: Two dice are rolled; find the probability of getting **doubles** or a **sum of 8**.

Addition Rule II [9]

Solution:

Let A be the event of getting doubles

A = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}, n(A) = 6

P(A) = $\frac{6}{36}$ = $\frac{1}{6}$ = **0.1667 (or 16.67%)**

Let B be the event of getting a sum of 8

B = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}, n(A) = 5

P(B) = $\frac{5}{36}$ = **0.1389 (or 13.89%)**

Addition Rule II [10]

Let **A∩B** be the event of getting a **'doubles'** and a **'sum of 8'**

A∩B = {(4, 4)}

P(A∩B) = $\frac{1}{36}$ = 0.0277 (or 2.7777 %)

Since A and B are **not mutually exclusive**, so

P(AUB) = P(A) + P(B) – P(A ∩ B)

$$= \frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{10}{36}$$

$= \frac{5}{18} =$ **0.2777or (27.7777%)**

Addition Rule II [11]

Let **P** be the event that an employee selected at random from an oil drilling company **smokes cigarettes**.

Let **Q** be the event that the employee selected drinks **alcoholic beverages**.

Then the event **P U Q** is the set of all employees who either **drink** or **smoke** or do **both**.

Example : A coin is tossed twice. What is the probability that at **least 1 head** occurs?

Solution : The sample space for this experiment is

S = {HH, HT, TH, TT}

n(S) = 4

Let **A** be the event of getting at **least 1 head**

A = {HH, HT, TH}

n(A) = 3

$$\therefore P(A) = \frac{n(A)}{n(s)}$$

$$= \frac{3}{4} = (0.75 \text{ or } 75\%)$$

Example : A die is loaded in such a way that **an even number** is **twice** as likely to occur as an **odd number**. If ***E*** is the event that a **number less than 4** occurs on a single toss of the die, find ***P*(*E*).**

Solution

P(Even number) = 2p

P(Odd number) = p

S = { 1, 2, 3, 4, 5 ,6}

$\because$ Sum of probability $= 1$

$\therefore$ **p + 2p + p + 2p + p + 2p** = 1

$\Rightarrow 9\text{p} = 1$

$\Rightarrow$ **p = $\frac{1}{9}$**

$\Rightarrow$ P(Even number) = $\frac{2}{9}$

$\Rightarrow$ P(Odd number) = $\frac{1}{9}$

E = {1, 2, 3}

P(E) = P(1) + P(2) + P(3)

= $\frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$ (or 0.4444 or 44%)

Example A die is loaded in such a way that **an even number** is **twice** as likely to occur as an odd number.

Let ***A*** be the event that an **even number** turns up and let ***B*** be the event that a **number divisible** by **3** occurs. Find **$P(A \cup B)$** and **$P(A \cap B)$.**

Solution

S = {1, 2, 3, 4, 5, 6}

P(Even number) = 2p

P(Odd number) = p

S = { 1, 2, 3, 4, 5 ,6}

$\because$ Sum of probability $= 1$

$\therefore$**p + 2p + p + 2p + p +2p** = 1

$\Rightarrow 9\text{p} = 1$

$\Rightarrow \mathbf{p = \frac{1}{9}}$

$\Rightarrow$ P(Even number) = $\mathbf{\frac{2}{9}}$

$\Rightarrow$ P(Odd number) = $\mathbf{\frac{1}{9}}$

A = { 2, 4, 6}

P(A) = $\frac{2}{9} + \frac{2}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}$

B = {3, 6}

P(B) = $\frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$

$A \cap B = \{6\}$

$P(A \cap B) = \frac{2}{9}$ **$\because P(\text{Even number}) = \frac{2}{9}$**

Since A and B are **not mutually exclusive**, so

$P(A \cup B) = P(A) + P(B) – P(A \cap B)$

$= \frac{6}{9} + \frac{3}{9} - \frac{2}{9} = \frac{7}{9}$ (or 0.7778 or 77.7778%)

- To find ***P*(*A* or *B*),** begin by associating use of the word **"or"** with addition.

- Consider whether **events *A*** and ***B*** are **disjoint**; that is, can they happen at the same time?

- If they are **not disjoint** (that is, they can happen at the same time), be sure to avoid (or at least compensate for) **double-counting** when adding the relevant probabilities.

- If you understand the importance of not double counting when you find *P*(*A* or *B*), you don't necessarily have to calculate the value of **$P(A) + P(B) – P(A \cap B)$**

Errors made when applying the addition rule

- Errors made when applying the addition rule often involve **double-counting**; that is, events that **are not disjoint** are treated as if they were. One indication of such an error is a total probability that **exceeds 1.**

- However, errors involving the addition rule do not **always cause** the total probability to **exceed 1**.

Suggested Readings

Probability & Statistics for Engineers & Scientists, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

2.1 Sample space

2.2 Events

2.3 Counting Sample Points

2.4 Probability of an Event

2.5 Additive Rules