CHAPTER 04 Matrices and Determinants

Related Definitions and Formulae

The Concept of a matrix: It is often desirable to present a set of numbers (or other elements) in a rectangular array of rows and columns. The table of values of trigonometric functions is an example of such an arrangement; in it the column have the heading sine, cosine, tangent and cotangent, and the rows are designated by angles, expressed in degrees. It is conventional to call the vertical lines columns and the horizontal lines rows.

e.g#1 Here is a bowling analysis in cricket.

	Overs	Maidens	Runs	Wickets
Wasim Akram	15	7	70	5
Waqar Younis	18	6	55	4
Shoaib Akther	10	3	21	1

e.g#2 Here is an example of simultaneous linear equations

$2x - 3y = 7$

$\frac{1}{2}x + 5y = 9$

These may be set down as below

Cofficient of x	Coefficient of y	Constant term
2	−3	7
$\frac{1}{2}$	5	9

Tables are a concise method of presenting a mass of information. When we construct a table from a collection of data, we generally arrange the data in rows and columns. We extract the information form the table by reading the entry corresponding to a row and column intersection. Any table is a matrix.

Applications of Matrices: Matrices and determinants are important mathematical tools which have wide applications in certain branches of physics, chemistry, statistics, biology, economics, psychology and various branches of engineering.

Tons of applications including.

- Solving System of linear equations.
- Computer graphics, Image processing
- Models within many areas of computational science of Engineering
- Quantum Mechanics, Quantum Computing.
- Many, many more ....

Meanings:

- Matrix is a latin word which means a place in which something develops or orginates.
- A common device for summarizing and displaying numbers (or) data.

DEVELOPERS: English Mathematicians

(1) James Joseph Sylvester (1814 — 1897)

(2) Arther Cayley (1821 — 1895)

Notations: (1) [ ] (2) ( ) (3) || ||

Basic Purpose to use: To reduce the amount of writing.

Matrix: A matrix is a rectangular array of numbers enclosed in square brackets (or parentheses)

or

"A Matrix is a rectangular array in shape, whose elements are written within square bracket in a definite order, in rows and columns".

* Matrix can never be expanded.
* It is always represented by Capital letters. (A, B, C...... etc)

Row: The horizontal arrangement of numbers is called row.

Column: The vertical arrangement of numbers is called column.

Entries: The numbers used in rows or columns are said to be the entries or elements of the matrix.

Order (or) Dimension of a Matrix: If a matrix A has m rows and n columns, then the oder of the matrix is $m \times n$.

Types of Matrices:

(1) Row Matrix (or) Row Vector: A matrix having only one row is called row matrix (or) Row vector.

e.g# $A = [\sqrt{5} \quad 1]$ $B = [2 \quad 0 \quad 1]$

(2) Column Matrix (or) Column Vector: A matrix having only one column is called column matrix (or) column vector.

e.g# $C = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$; $D = \begin{bmatrix} \sqrt{2} \\ \sqrt{5} \end{bmatrix}$

Exercise No.=4.1

Qno.(1)Specify the type of each of the following matrices.

(i) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$ **(ii)** $\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$ **(iii)** $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ **(iv)** $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ **(v)** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(i)Diagonal Matrix.(ii)Scalar Matrix.(iii)Scalar Matrix.(iv)Diagonal(v)Unit Matrix.

Qno.(2)Write down the matrices of the coefficents of x,y and z in the following system of linear equations.

$(i)3x+2y+8z-5=0$. $\begin{bmatrix} 3 & 2 & 8 \\ 5 & 8 & -4 \end{bmatrix}$ **(ii)** $8x+5y+2z-3=0$

$5x+8y-4z+2=0$ $\quad 6x+4y+3z+2=0$

$7x-3y+5z-9=0$

$\begin{pmatrix} 8 & 5 & 2 \\ 6 & 4 & 3 \\ 7 & -3 & 5 \end{pmatrix}$ **Ans**

Qno.(3)Write down in tabular form.

$(i)X=[x_{ij}]_{(3,4)}$ $\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$

Number of rows=3

Number of columns=4

$(ii)Y=[y_{lk}]_{(4,2)}$ $\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \\ y_{41} & y_{42} \end{bmatrix}$

Number of rows=4

Number of columns=2

QNo.04 $A = \begin{pmatrix} & & \\ 5 & 6 & 7 \\ -3 & 1 & 9 \end{pmatrix}$ $B = \begin{pmatrix} & & \\ 1 & 1 & 1 \end{pmatrix}$ $C = \begin{pmatrix} -2 & -3 \\ -4 & 0 \\ & \end{pmatrix}$

Then wh[illegible]e ever possible compute the following.

(i)AB no[illegible] possible.

$$(ii)\ A\times[illegible] = \begin{pmatrix} 4 & 2 & 0 \\ 5 & 6 & 7 \\ -3 & 1 & 9 \end{pmatrix} \times \begin{pmatrix} -2 & -3 \\ -4 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4(-2)+2(-4)+0 & 4(-3)+0+0 \\ 5(-2)+6(-4)+7(1) & 5(-3)+0+7(3) \\ -3(-2)+1(-4)+9(1) & -3(-3)+0+9(3) \end{pmatrix} = \begin{pmatrix} -16 & -12 \\ -27 & 6 \\ 11 & 33 \end{pmatrix}$$

(iii)C[illegible] not possible.

$$(iv[illegible]B\times C = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ -4 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -4-12+1 & -6+0+3 \\ -2-4+1 & -3+0+3 \end{pmatrix} = \begin{pmatrix} -15 & -3 \\ -5 & 0 \end{pmatrix}$$

$$[illegible])CB = \begin{pmatrix} -2 & -3 \\ -4 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -4-3 & -6-3 & -2-3 \\ -8+0 & -12+0 & -4+0 \\ 2+3 & 3+3 & 1+3 \end{pmatrix} = \begin{pmatrix} -7 & -9 & -5 \\ -8 & -12 & -4 \\ 5 & 6 & 4 \end{pmatrix}$$

(vi)(AB)C not possible. **(viii)B^2 not possible.**

(vii)

$$A^2 = \begin{pmatrix} 4 & 2 & 0 \\ 5 & 6 & 7 \\ -3 & 1 & 9 \end{pmatrix} \begin{pmatrix} 4 & 2 & 0 \\ 5 & 6 & 7 \\ -3 & 1 & 9 \end{pmatrix} = \begin{pmatrix} 16+10+0 & 8+12+0 & 0+14+0 \\ 20+30-21 & 10+36+7 & 0+42+63 \\ -12+5-27 & -6+6+9 & 0+7+81 \end{pmatrix} = \begin{pmatrix} 26 & 20 & 14 \\ 29 & 53 & 10[illegible] \\ -34 & 9 & 8[illegible] \end{pmatrix}$$

$$(ix)C'A' = \begin{pmatrix} -2 & -4 & 1 \\ -3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 5 & -3 \\ 2 & 6 & 1 \\ 0 & 7 & 9 \end{pmatrix} = \begin{pmatrix} -8-8+0 & -10-24+7 & 6-4+9 \\ -12+0+0 & 5+0+21 & 9+0+27 \end{pmatrix} = \begin{pmatrix} -16 & -27 & 11 \\ -12 & 26 & 36 \end{pmatrix}$$

$QNo.05\ IF\ A = \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{pmatrix}; B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -2 & -1 \\ 2 & -5 & -1 \end{pmatrix}; C = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$ **then show that AB=AC.**

$$A\times B = \begin{pmatrix} 2-9+4 & 1+6-10 & -1+3-2 \\ 4+3-6 & 2-2+15 & -2-1+3 \\ 8-9-2 & 4+6+5 & -4+3+1 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 0 \\ 1 & 15 & 0 \\ -3 & 15 & 0 \end{pmatrix}$$

$$A\times C = \begin{pmatrix} 1-6+2 & 4-3-4 & 1-3+2 \\ 2+2-3 & 8+1+6 & 2+1-3 \\ 4-6-1 & 16-3+2 & 4-3-1 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 0 \\ 1 & 15 & 0 \\ -3 & 15 & 0 \end{pmatrix} Proved$$

$$QNo.06 \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \sin\theta\sin\theta+\cos\theta\cos\theta & \sin\theta\cos\theta-\cos\theta\sin\theta \\ \cos\theta\sin\theta-\sin\theta\cos\theta & \cos\theta\cos\theta+\sin\theta\sin\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta+\sin^2\theta & 0 \\ 0 & \cos^2\theta+\sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If A+B matrices; then explain why in general.

$QNo7(i)\ (A+B)^2 \neq A^2+2AB+B^2\ ;(A+B)(A+B) = A^2+AB+BA+B^2$

$In\quad general\quad AB \neq BA\quad \therefore (A+B)^2 \neq A^2+2AB+B^2$

$QNo7(ii)\ (A-B)^2 \neq A^2-2AB+B^2\ ;(A-B)(A-B) = A^2-AB-BA+B^2$

$In\quad general\quad AB \neq BA\quad \therefore (A-B)^2 \neq A^2-2AB+B^2$

$QNo7(iii)\ (A+B)(A-B) \neq A^2-B^2\ ;(A+B)(A-B) = A^2-AB+BA-B^2$

$In\quad general\quad AB \neq BA\quad \therefore (A+B)(A-B) \neq A^2-B^2$

QNo08 Let [illegible] $A=\begin{pmatrix} [illegible] & 1 & 0 \\ -1 & 4 & 3 \\ 0 & 8 & 5 \end{pmatrix}$ and [illegible] $\begin{pmatrix} 2 & 1 & -1 \\ [illegible] & 2 & -4 \\ 5 & 4 & [illegible] \end{pmatrix}$

(i)Find B [illegible] **A-2B=3X or A-3X=2B or B=**$\frac{A-3X}{2}=\frac{1}{2}(A-3X)$

$$=\frac{1}{2}\left\{\begin{pmatrix} [illegible] & 1 & 0 \\ [illegible] & 4 & 3 \\ [illegible] & 8 & 5 \end{pmatrix}-3\begin{pmatrix} 2 & 1 & -1 \\ -3 & 2 & -4 \\ 5 & 4 & 0 \end{pmatrix}\right\}=\frac{1}{2}\begin{pmatrix} -2-6 & 1-3 & 0+3 \\ -1+9 & 4-6 & 3+12 \\ 0-15 & 8-12 & 5-0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} -8 & -2 & 3 \\ 8 & -2 & 15 \\ -15 & -4 & 5 \end{pmatrix}=\begin{pmatrix} -4 & -1 & \frac{3}{2} \\ 4 & -1 & \frac{15}{2} \\ -\frac{15}{2} & -2 & [illegible] \end{pmatrix}$$

(ii) [illegible]$-A=\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\begin{pmatrix} -2 & 1 & 0 \\ -1 & 4 & 3 \\ 0 & 8 & 5 \end{pmatrix}=\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}-\begin{pmatrix} -2 & 1 & 0 \\ -1 & 4 & 3 \\ 0 & 8 & 5 \end{pmatrix}=\begin{pmatrix} \lambda+2 & -1 & 0 \\ 1 & \lambda-4 & -3 \\ 0 & -8 & \lambda-5 \end{pmatrix}$

Exercise No.=4.2

Evaluate the following determinants.

1. $\begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix}=1\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}-1\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}+1\begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix}=1(-1\times1-0)-1(-1\times1-0)+1(0-(-1)(-1)$

$=1(-1)-1(-1)+(-1)=-1+1-1=-1$

2. $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}=1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}-1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}+0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}=1(0-1)-1(1-0)+0=-1-1=-2$

3. $\begin{vmatrix} -i & 1 & 0 \\ 1 & i & 1+i \\ 1-i & 0 & 1 \end{vmatrix}=-i\begin{vmatrix} i & 1+i \\ 0 & 1 \end{vmatrix}-1\begin{vmatrix} 1 & 1+i \\ 1-i & 1 \end{vmatrix}+0=-i(i+0)-1\{1-(1-i)(1+i)\}$

$=-i^2-1\{1-(1-i^2)\}=-i^2-1[\{1-(1-(-1)\}]=-i^2-1\{1-2\}=-i^2-1(-1)=-(-1)+1=2$

4. $\begin{vmatrix} a & a & a \\ x & -a & a \\ x & -a & -a \end{vmatrix}=a\begin{vmatrix} -a & a \\ -a & -a \end{vmatrix}-a\begin{vmatrix} x & a \\ x & -a \end{vmatrix}+a\begin{vmatrix} x & -a \\ x & -a \end{vmatrix}=a\{a^2-(-a^2)-a(-ax-ax)+a(-ax+ax)\}$

$=a(a^2+a^2)-a(-2ax)+a(0)=2a^3+2a^2x=2a^2(a+x)$

5. $\begin{vmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix}=1\begin{vmatrix} \omega^2 & 1 \\ \omega & 1 \end{vmatrix}-1\begin{vmatrix} \omega & 1 \\ \omega^2 & 1 \end{vmatrix}+1\begin{vmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{vmatrix}=1(\omega^2-\omega)-1(\omega-\omega^2)+1(\omega^2-\omega^4)$

$=\omega^2-\omega-\omega+\omega^2+\omega^2-\omega,\therefore\omega^4=\omega=3\omega^2-3\omega=3(\omega^2-\omega)$

6. $\begin{vmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix}=1\begin{vmatrix} 1 & \omega^2 \\ \omega^2 & 1 \end{vmatrix}-1\begin{vmatrix} 1 & \omega^2 \\ \omega & 1 \end{vmatrix}+\omega\begin{vmatrix} 1 & 1 \\ \omega & \omega^2 \end{vmatrix}=1(1-\omega^4)-1(1-\omega^3)+\omega(\omega^2-\omega)$

$=1-\omega^4-1+\omega^3+\omega^3-\omega^2=1-\omega-1+1+1-\omega^2\ \therefore\omega^4=\omega,\omega^3=1=2-\omega-\omega^2$

$=2-(\omega+\omega^2)=2-(-1)=2+1=3$

7. $\begin{vmatrix} a+b+2c & \text{[illegible]} & a \\ c & b & b+c+2a \\ c & c+a+2b & a \end{vmatrix}$

$= \begin{vmatrix} a+b+2c-c & b-b & a-b-c-2a \\ c-\text{[illegible]} & b-c-a-2b & b+c+2a-a \\ \text{[illegible]} & c+a+2b & a \end{vmatrix} R_1-R_2$ **and** R_2-R_1.

$= \begin{vmatrix} a+b+c & 0 & -a-b-c \\ c-\text{[illegible]} & -a-b-c & a+b+c \\ \text{[illegible]} & c+a+2b & a \end{vmatrix}$

$= \begin{vmatrix} a+b+c & 0 & -(a+b+c) \\ 0 & -(a+b+c) & a+b+c \\ c & c+a+2b & a \end{vmatrix}$

$= (a+b+c)(a+b+c)\begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ c & c+a+2b & a \end{vmatrix} = (a+b+c)^2[1(-a-c-a-2b)+0-1(0+c)]$

$= (a+b+c)^2(-a-c-a-2b-c) = (a+b+c)^2(-2a-2b-2c)$

$= -2(a+b+c)^2(a+b+c) = -2(a+b+c)^3$

8. $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ y & z & x \end{vmatrix} R_1+R_2+R_3$

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix} = (x+y+z)\begin{vmatrix} 1-1 & 1-1 & 1 \\ z-x & x-y & y \\ y-z & z-x & x \end{vmatrix} C_1-C_3, C_2-C_3$

$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ z-x & x-y & y \\ y-z & z-x & x \end{vmatrix} = (x+y+z)\{1(z-x)(z-x)-(y-z)(x-y)\}$

$= (x+y\text{[illegible]}-2xz+x^2-xy+y^2+xz-yz\} = (x+y+z)\{x^2+y^2+z^2-xy-xz-\text{[illegible]}z\} =$
$x^3+y^3+z^3-3xyz$

9. $\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} = \begin{vmatrix} 1-1 & x^2-y^2 & x-y \\ 1-1 & y^2-z^2 & y-z \\ 1 & z^2 & z \end{vmatrix} R_1-R_2, R_2-R_3$

$= \begin{vmatrix} 0 & (x-y)(x+y) & x-y \\ 0 & (y-z)(y+z) & y-z \\ 1 & z^2 & z \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} 0 & x+y & 1 \\ 0 & y+z & 1 \\ 1 & z^2 & z \end{vmatrix} = (x-y)(y-z)\begin{vmatrix} 0 & x-z & 1 \\ 0 & y+z & 1 \\ 1 & z^2 & z \end{vmatrix}$

$= (x-y)(y-z)\{0-(x+z)(0-1)+0\}, = (x-y)(y-z)\{-(x-z)(-1)\}$

$= (x-y)(y-z)(x-z) = -(x-y)(y-z)(z-x)$

10. $\begin{vmatrix} x+y & x & y \\ x & y & x+y \\ y & x+y & x \end{vmatrix} = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ x & y & x+y \\ y & x+y & x \end{vmatrix} R_1+R_2+R_3$

$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ x & y & x+y \\ y & x+y & x \end{vmatrix} = 2(x+y)\begin{vmatrix} 1-1 & 1-1 & 1 \\ x-y & y-(x+y) & x+y \\ y-x-y & x+y-x & x \end{vmatrix} = 2(x+y)\begin{vmatrix} 0 & 0 & 1 \\ x-y & -x & x+y \\ -x & y & x \end{vmatrix}$

$= 2(x+y)[y(x-y)-x^2] = 2(x+y)[yx-y^2-x^2] = 2(x+y)[-(x^2-xy+y^2)]$

$= -2(x+y)(x^2-xy+y^2) = -2(x^3+y^3)$

QNo11 $\begin{vmatrix} x+2y & x+6y & x+4y \\ x+3y & x+7y & x+5y \\ x+4y & x+8y & x+6y \end{vmatrix} = 0$

$= \begin{vmatrix} x+2y-x-3y & x+6y-x-7y & x+4y-x-5y \\ x+3y-x-4y & x+7y-x-8y & x+5y-x-6y \\ x+4y & x+8y & x+6y \end{vmatrix} R_1-R_2, R_2-R_3$

$= \begin{vmatrix} -y & -y & -y \\ -y & -y & -y \\ x+4y & x+8y & x+6y \end{vmatrix} = 0 \quad \therefore R_1, R_2$ *are identical*

12. *Show that* $\begin{vmatrix}1 & 1 & 1\\ \alpha & \beta & \gamma\\ \beta\gamma & \gamma\alpha & \alpha\beta\end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

$$= \begin{vmatrix}1-1 & 1-1 & 1\\ \alpha-\beta & \beta-\gamma & \gamma\\ \beta\gamma-\gamma\alpha & \gamma\alpha-\alpha\beta & \alpha\beta\end{vmatrix} C_1-C_2, C_2-C_3 \quad = \begin{vmatrix}0 & 0 & 1\\ \alpha-\beta & \beta-\gamma & \gamma\\ -\gamma(\alpha-\beta) & -\alpha(\beta-\gamma) & \alpha\beta\end{vmatrix}$$

$$= (\alpha-\beta)(\beta-\gamma)\begin{vmatrix}0 & 0 & 1\\ 1 & 1 & \gamma\\ -\gamma & -\alpha & \alpha\beta\end{vmatrix} \quad = (\alpha-\beta)(\beta-\gamma)[1(-\alpha+\gamma)]$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

14. $\begin{vmatrix}p+1 & p+3 & p+5\\ p+4 & p+6 & p+8\\ p+7 & p+9 & p+11\end{vmatrix}$

$$\begin{vmatrix}p+1-p-4 & p+3-p-6 & p+5-p-8\\ p+4+p-7 & p+6+p-9 & p+8-p-11\\ p+7 & p+9 & p+11\end{vmatrix}$$

$$= \begin{vmatrix}-3 & -3 & -3\\ -3 & -3 & -3\\ p+7 & p+9 & p+11\end{vmatrix} = 0 \quad \therefore R_1, R_2 \text{ are identical}$$

13. *Show that* $\begin{vmatrix}1 & x & yz\\ 1 & y & zx\\ 1 & z & xy\end{vmatrix} = \begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix}x & x^2 & xyz\\ y & y^2 & xyz\\ z & z^2 & xyz\end{vmatrix} x.R_1, y.R_2, z.R_3 \quad = \frac{-1}{xyz}\begin{vmatrix}x & xyz & y^2\\ y & xyz & y^2\\ z & xyz & z^2\end{vmatrix} = \frac{-(-1)}{xyz}\begin{vmatrix}xyz & x & x^2\\ xyz & y & y^2\\ xyz & z & z^2\end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix} = \begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix}$$

15. $\begin{vmatrix}a+x & a & a\\ a & a+x & a\\ a & a & a+x\end{vmatrix} = \begin{vmatrix}a+x-a & a-a-x & a-a\\ a-a & a+x-a & a-a-x\\ a & a & a+x\end{vmatrix} = \begin{vmatrix}x & -x & 0\\ 0 & x & -x\\ a & a & a+x\end{vmatrix}$

$$= x^2\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ a & a & a+x\end{vmatrix} = x^2\{1(a+x+a)+1(a)\} = x^2(2a+x+a) = x^2(3a+x)$$

16. $\begin{vmatrix}4 & a & b+c\\ 4 & b & c+a\\ 4 & c & a+b\end{vmatrix} = \begin{vmatrix}4-4 & a-b & b+c-c-a\\ 4-4 & b-c & c+a-a-b\\ 4 & c & a+b\end{vmatrix} R_1-R_2, R_1-R_3 = \begin{vmatrix}0 & a-b & b-a\\ 0 & b-c & c-b\\ 4 & c & a+b\end{vmatrix}$

$$= \begin{vmatrix}0 & a-b & -(a-b)\\ 0 & b-c & -(b-c)\\ 4 & c & a+b\end{vmatrix} \quad (b-c)(a-b)\begin{vmatrix}0 & 1 & -1\\ 0 & 1 & -1\\ 4 & c & a+b\end{vmatrix} = 0 \; R_1, R_2 \text{ are identical}$$

17. Prove *that* $\begin{vmatrix}pq & y & y^2\\ qy & p & p^2\\ py & q & q^2\end{vmatrix} = \begin{vmatrix}1 & y^2 & y^3\\ 1 & p^2 & p^3\\ 1 & q^2 & q^3\end{vmatrix}$

$$= \frac{1}{pqy}\begin{vmatrix}pqy & y^2 & y^3\\ pqy & p^2 & p^3\\ pqy & q^2 & q^3\end{vmatrix} = \frac{pqy}{pqy}\begin{vmatrix}1 & y^2 & y^3\\ 1 & p^2 & p^3\\ 1 & q^2 & q^3\end{vmatrix} = \begin{vmatrix}1 & y^2 & y^3\\ 1 & p^2 & p^3\\ 1 & q^2 & q^3\end{vmatrix}$$

Hence proved.

18. $\begin{vmatrix}a^2 & a & \frac{a}{bc}\\ b^2 & b & \frac{b}{ac}\\ c^2 & c & \frac{c}{ab}\end{vmatrix} = \begin{vmatrix}a^2 & a & \frac{a}{bc}\times abc\\ b^2 & b & \frac{b}{ac}\times abc\\ c^2 & c & \frac{c}{ab}\times abc\end{vmatrix} abc\times C_3 = \begin{vmatrix}a^2 & a & a^2\\ b^2 & b & b^2\\ c^2 & c & c^2\end{vmatrix} = 0 \quad \therefore C_1, C_3 \text{ are identical.}$

19. $\begin{vmatrix} a-b & m-n & x-y \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix} = 0$

$\begin{vmatrix} a-b+b-c+c-a & m-n+n-p+p-m & x-y+y-z+z-x \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix}$

$\begin{vmatrix} 0 & 0 & 0 \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix} = 0$

20. $\begin{vmatrix} 1 & a & b \\ 1 & x & b \\ 1 & c & x \end{vmatrix} = 0$ *Solve for x.*

$\begin{vmatrix} 1-1 & a-x & b-b \\ 1 & x & b \\ 1 & c & x \end{vmatrix} = \begin{vmatrix} 0 & a-x & 0 \\ 1 & x & b \\ 1 & c & x \end{vmatrix}$

$= (0-(a-x)(x-b)+0) = 0$

$= -(a-x)(x-b) = 0 \quad (x-a)(x-b) = 0$

$(x-a) = 0, (x-b) = 0$

$x = a, x = b$

So {a,b} Ans.

EXERCISE NO. = 4.3

QNO:05(iii) $2x+y+z-1=0 \qquad 2x+y+z=1$

$x-2y-3z-1=0 \qquad x-2y-3z=1$

$3x+2y+4z-5=0 \qquad 3x+2y+4z=5$

$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}; AX = B$

$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{vmatrix} = 2\begin{vmatrix} -2 & -3 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix}$

$= 2(-8+6)-1(4+9)+1(2+6) = 2(-2)-1(13)+1(8) = -4-13+8 = -17+8 = -9$ So $|A| = -9 \neq 0$

A is non-singular.

$\lvert A_1\rvert \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 5 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} -2 & -3 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 5 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 5 & 4 \end{vmatrix}$ $= 1(-8+6)-1(4+15)+1(2+10)$ $= 1(-2)-1(19)+1(12) = -2-19+12$ $-21+12 = -9$ **So** $\lvert A_1\rvert = -9$	$\lvert A_2\rvert \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -3 \\ 3 & 5 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 \\ 5 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix}$ $= 2(4+15)-1(4+9)+1(5-3)$ $= 2(19)-1(13)+1(2) = 38-13+2 = 40-13 = 27$ **So 27.**
$\lvert A_3\rvert \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix}$ $2(-10-2)-1(5-3)+1(-2+6)$ $= 2(-12)-1(2)+1(8)$ $= -24-2+8 = -26+8 = -18 = \lvert A_3\rvert = -18$	$x = \frac{\lvert A_1\rvert}{\lvert A\rvert} = \frac{-9}{-9} = 1$ $y = \frac{\lvert A_2\rvert}{\lvert A\rvert} = \frac{27}{-9} = -3,\ z = \frac{\lvert A_3\rvert}{\lvert A\rvert} = \frac{-18}{-9} = 2$ **Check:-** $2x+y+z=1=2(1)+(-3)+2$ **=2-3+2=4-3=1**

QNO:05(iv) [illegible] $\qquad y+z-t=0$

$2y+t=5$ $\qquad$ [illegible] $t=5$

$2y-z-3t+$[illegible]$=0$ $\qquad 2y-z-3t=-2$

$$\begin{pmatrix}1&1&\text{[illegible]}\\2&0&\text{[illegible]}\\2&-1&\text{[illegible]}3\end{pmatrix}\begin{bmatrix}x\\z\\t\end{bmatrix}=\begin{bmatrix}0\\5\\-2\end{bmatrix};AX=B$$

$$|A|=\begin{pmatrix}1&1&-1\\2&0&1\\\text{[illegible]}&-1&-3\end{pmatrix}=1\begin{vmatrix}0&1\\-1&-3\end{vmatrix}-1\begin{vmatrix}2&1\\2&-3\end{vmatrix}+(-1)\begin{vmatrix}2&0\\2&-1\end{vmatrix}$$

$=1(-$[illegible]$+1)-1(-6-2)-1(-2-0)=1(1)-1(-8)-1(-2)$

$=1+$[illegible]$+2=11;|A|=11\neq 0$

$$|A_1|\text{[illegible]}\begin{vmatrix}0&1&-1\\5&0&1\\-2&-1&-3\end{vmatrix}=0\begin{vmatrix}0&1\\-1&-3\end{vmatrix}-1\begin{vmatrix}5&1\\-2&-3\end{vmatrix}+(-1)\begin{vmatrix}5&0\\-2&-1\end{vmatrix}$$

$=$[illegible]$(0+1)-1(-15+2)-1(-5-0)=0-1(-13)-1(-5)=13+5=18;|A_1|=18$

$$|A_2|=\begin{vmatrix}1&0&-1\\2&5&1\\2&-2&-3\end{vmatrix}=1\begin{vmatrix}5&1\\-2&-3\end{vmatrix}-0\begin{vmatrix}2&1\\2&-3\end{vmatrix}+(-1)\begin{vmatrix}2&5\\2&-2\end{vmatrix}$$

$=1(-15+2)-0(-6-2)-1(-4-10)=-13-0-1(-14)=-13+14=1;|A_2|=1$

$$|A_3|=\begin{vmatrix}1&1&0\\2&0&5\\2&-1&-2\end{vmatrix}=1\begin{vmatrix}0&5\\-1&-2\end{vmatrix}-1\begin{vmatrix}2&5\\2&-2\end{vmatrix}+0\begin{vmatrix}2&0\\2&-1\end{vmatrix}$$

$=1(0+5)-1(-4-10)+0=1(5)-1(-14)=5+14=19;|A_3|=19$

$$y=\frac{|A_1|}{|A|}=\frac{18}{11},\ z=\frac{|A_2|}{|A|}=\frac{1}{11},\ t=\frac{|A_3|}{|A|}=\frac{19}{11}$$

Check:- $2y-z-3t+2=0$

$$2\left(\frac{18}{11}\right)-\left(\frac{1}{11}\right)-3\left(\frac{19}{11}\right)+2=\frac{36-1-57+22}{11}=\frac{58-58}{11}=0$$

Q5(v) $x+2y+z=8$

$2x-y+z=3$

$x+y-z=0$

$$\begin{pmatrix}1&2&1\\2&-1&1\\1&1&-1\end{pmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\3\\0\end{bmatrix};AX=B$$

$$|A|=\begin{vmatrix}1&2&1\\2&-1&1\\1&1&-1\end{vmatrix}=1\begin{vmatrix}-1&1\\1&-1\end{vmatrix}-2\begin{vmatrix}2&1\\1&-1\end{vmatrix}+1\begin{vmatrix}2&-1\\1&1\end{vmatrix}$$

$=1(1-1)-2(-2-1)+1(2+1)=(0)-2(-3)+1(3)=0+6+3=9;|A|=9\neq 0$

$$|A_1|=\begin{vmatrix}8&2&1\\3&-1&1\\0&1&-1\end{vmatrix}=8\begin{vmatrix}-1&1\\1&-1\end{vmatrix}-2\begin{vmatrix}3&1\\0&1\end{vmatrix}+1\begin{vmatrix}3&1-1\\0&1\end{vmatrix}$$

$=8(1-1)-2(-3)+1(3-0)=8(0)+6+1(3)=0+6+3=9;A_1=9$

$$|A_2|=\begin{vmatrix}1&8&1\\2&3&1\\1&0&-1\end{vmatrix}=1\begin{vmatrix}3&1\\0&-1\end{vmatrix}-8\begin{vmatrix}2&1\\1&-1\end{vmatrix}+1\begin{vmatrix}2&3\\1&0\end{vmatrix}$$

$=1(-3-0)-8(-2-1)+1(0-3)=1(-3)-8(-3)+1(-3)=-3+24-3=18;|A_2|=18$

$$|A_3|=\begin{vmatrix}1&2&8\\2&-1&3\\1&1&0\end{vmatrix}=1\begin{vmatrix}-1&3\\1&0\end{vmatrix}-2\begin{vmatrix}2&3\\1&0\end{vmatrix}+8\begin{vmatrix}2&-1\\1&1\end{vmatrix}$$

$=1(0-3)-2(0-3)+8(2+1)=1(-3)-2(-3)+8(3)=-3+6+24=27|A_3|=27$

$$x=\frac{|A_1|}{|A|}=\frac{9}{9}=1,\ y=\frac{|A_2|}{|A|}=\frac{18}{9}=2,z=\frac{|A_3|}{|A|}=\frac{27}{9}=3 \qquad x=1,\ y=2,\ z=3$$

Check:- $x+2y+z=8=1+2(2)+3=1+4+3=8.$

Q5(vii) $x+y=5$ $\qquad x+y+0=5$

$y+z=7$ $\qquad 0+y+z=7$

$z+x=6$ $\qquad x+0+z=6$

$$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\7\\6\end{bmatrix};AX=B \qquad ;|A|=\begin{vmatrix}1&1&0\\0&1&1\\1&0&1\end{vmatrix}=1\begin{vmatrix}1&1\\0&1\end{vmatrix}-1\begin{vmatrix}0&1\\1&1\end{vmatrix}+0\begin{vmatrix}0&1\\1&0\end{vmatrix}$$

$=1(1-0)-1(0-1)+0=1(1)-1(-1)=1+1=2;|A|=2\neq 0$

$$|A_1|=\begin{vmatrix}5&1&0\\7&1&1\\6&0&1\end{vmatrix}=5\begin{vmatrix}1&1\\0&1\end{vmatrix}-1\begin{vmatrix}7&1\\6&1\end{vmatrix}+0\begin{vmatrix}7&1\\6&0\end{vmatrix}$$

$=5(1-0)-1(7-6)+0=5-1(1)=4;|A_1|=4$

$$|A_2|=\begin{vmatrix}1&5&0\\0&7&1\\1&6&1\end{vmatrix}=1\begin{vmatrix}7&1\\6&1\end{vmatrix}-5\begin{vmatrix}0&1\\1&1\end{vmatrix}+0\begin{vmatrix}0&7\\6&0\end{vmatrix}$$

$=1(7-6)-5(0-1)+0=1(1)-5(-1)=1+5=6;|A_2|=6$

$$|A_3|=\begin{vmatrix}1&1&5\\0&1&7\\1&6&1\end{vmatrix}=1\begin{vmatrix}1&7\\6&1\end{vmatrix}-1\begin{vmatrix}0&7\\1&6\end{vmatrix}+5\begin{vmatrix}0&1\\1&0\end{vmatrix}$$

$=1(6-0)-1(0-7)+5(0-1)=1(6)-(-7)+5(-1)=6+7-5=8;|A_3|=8$

$x=\dfrac{|A_1|}{|A|}=\dfrac{4}{2}=2,\ y=\dfrac{|A_2|}{|A|}=\dfrac{6}{2}=3,\ z=\dfrac{|A_3|}{|A|}=\dfrac{8}{2}=4 \qquad x=2,\ y=3,\ z=4$

Check:- $x+y=5;\ 2+3=5\rightarrow(1)$, $y+z=7;\ 3+4=7\rightarrow(2)$, $z+x=6;\ 4+2=6\rightarrow(3)$

Q5(viii) $2y-z=1$ $\qquad 0+2y-z=1$

$4x+2y-z=1$ $\qquad 4x+2y-z=1$

$8x-y+3z=2$ $\qquad 8x-y+3z=2$

$$\begin{pmatrix}0&2&-1\\4&2&-1\\8&-1&3\end{pmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix};AX=B \qquad ;|A|=\begin{vmatrix}0&2&-1\\4&2&-1\\8&-1&3\end{vmatrix}=0\begin{vmatrix}2&-1\\-1&3\end{vmatrix}-2\begin{vmatrix}4&-1\\8&3\end{vmatrix}+(-1)\begin{vmatrix}4&2\\8&-1\end{vmatrix}$$

$=0(6-1)-2(12+8)-1(-4-16)=0-2(20)-1(-20)=0-40+20=-20;|A|=20\neq 0$

$$|A_1|=\begin{vmatrix}1&2&-1\\1&2&-1\\2&-1&3\end{vmatrix}=1\begin{vmatrix}2&-1\\-1&3\end{vmatrix}-2\begin{vmatrix}1&-1\\2&3\end{vmatrix}+(-1)\begin{vmatrix}1&2\\2&-1\end{vmatrix}$$

$=1(6-1)-2(3+2)-1(-1-4)=1(5)-2(5)-1(-5)=5-10+5=0;|A_1|=0$

$$|A_2|=\begin{vmatrix}0&1&-1\\4&1&-1\\8&2&3\end{vmatrix}=0\begin{vmatrix}1&-1\\2&3\end{vmatrix}-1\begin{vmatrix}4&-1\\8&3\end{vmatrix}+(-1)\begin{vmatrix}4&1\\8&2\end{vmatrix}$$

$=0(3+2)-1(12+8)-1(8-8)=0-1(20)-1(0)=0-20-0=-20;|A_2|=-20$

$$|A_3|=\begin{vmatrix}0&2&1\\4&2&1\\8&-1&2\end{vmatrix}=0\begin{vmatrix}2&1\\-1&2\end{vmatrix}-2\begin{vmatrix}4&1\\8&2\end{vmatrix}+1\begin{vmatrix}4&2\\8&-1\end{vmatrix}$$

$=0(4+1)-2(8-8)+1(-4-16)=0-2(0)+1(-20)=0-0-20;|A_3|=-20$

$x=\dfrac{|A_1|}{|A|}=\dfrac{0}{-20}=0,\ y=\dfrac{|A_2|}{|A|}=\dfrac{-20}{-20}=1,\ z=\dfrac{|A_3|}{|A|}=\dfrac{-20}{-20}=1 \qquad x=0,\ y=1,\ z=1$

Check:- $4x+2y-z=1=4(0)+2(1)-1=0+2-1=1$

Q5(ix) $9x +$ [illegible]

$5x - y + 4t = 1$

$6x + 8y + 2t = 4$

$$\begin{pmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}; AX = B \qquad |A| = \begin{vmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{vmatrix} = 9\begin{vmatrix} -1 & 4 \\ 8 & 2 \end{vmatrix} - 7\begin{vmatrix} 5 & 4 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & -1 \\ 6 & 8 \end{vmatrix}$$

$= 9(-2-32) - 7(10-24) + 3(40+6) = 9(-34) - 7(-14) + 3(46) = -306 + 98 + 138 = -306 + 236 = -70. So |A| = -70$

$$|A_1| = \begin{vmatrix} 6 & 7 & 3 \\ 1 & -1 & 4 \\ 4 & 8 & 2 \end{vmatrix} = 6\begin{vmatrix} -1 & 4 \\ 8 & 2 \end{vmatrix} - 7\begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 4 & 8 \end{vmatrix}$$

$= 6(-2-32) - 7(2-16) + 3(8+4) = 6(-34) - 7(-14) + 3(12) = -204 + 134 = -70; A_1 = -70$

$$|A_2| = \begin{vmatrix} 9 & 6 & 3 \\ 5 & 1 & 4 \\ 6 & 4 & 2 \end{vmatrix} = 9\begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} - 6\begin{vmatrix} 5 & 4 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 6 & 4 \end{vmatrix}$$

$= 9(2-16) - 6(10-24) + 3(20-6) = 9(-14) - 6(-14) + 3(14) = -126 + 84 + 42 = -126 + 126 = 0; |A_2| = 0$

$$|A_3| = \begin{vmatrix} 9 & 7 & 6 \\ 5 & -1 & 1 \\ 6 & 8 & 4 \end{vmatrix} = 9\begin{vmatrix} -1 & 1 \\ 8 & 2 \end{vmatrix} - 7\begin{vmatrix} 5 & 1 \\ 6 & 4 \end{vmatrix} + 6\begin{vmatrix} 5 & -1 \\ 6 & 8 \end{vmatrix}$$

$= 9(-4-8) - 7(20-6) + 6(40+6) = 9(-12) - 7(14) + 6(46) = -108 - 98 + 276 = -206 - 276 = 70; |A_3| = -70$

$x = \dfrac{|A_1|}{|A|} = \dfrac{-70}{-70} = 1,\ y = \dfrac{|A_2|}{|A|} = \dfrac{0}{-70} = 0,\ t = \dfrac{|A_3|}{|A|} = \dfrac{70}{-70} = -1 \qquad x = 1, y = 0, t = -1$

$x = 1, y = 0, z = -1$

Check:- $9x + 7y + 3t = 6 = 9(1) + 7(0) + 3(-1) = 9 - 3 = 6$

Q5(x) $x + y + z = d$; $x + (1+d)y + z = 2d$; $x + y + (1+d)z = 0$

$$;\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1+d & 1 \\ 1 & 1 & 1+d \end{pmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ 2d \\ 0 \end{bmatrix}; AX = B$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+d & 1 \\ 1 & 1 & 1+d \end{vmatrix} = 1\begin{vmatrix} 1+d & 1 \\ 1 & 1+d \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1+d \end{vmatrix} + 1\begin{vmatrix} 1 & 1+d \\ 1 & 1 \end{vmatrix}$$

$= 1\{(1+d)^2 - 1\} - 1(1+d-1) + 1\{1-(1+d)\} = 1(1+2d+d^2-1) - 1(d) + 1(1-1-d)$

$= 2d + d^2 - 1(d) + 1(-d) = 2d + d^2 - d - d = 2d + d^2 - 2d = d^2; |A| = d^2$

$$|A_1| = \begin{vmatrix} d & 1 & 1 \\ 2d & 1+d & 1 \\ 0 & 1 & 1+d \end{vmatrix} = d\begin{vmatrix} 1+d & 1 \\ 1 & 1+d \end{vmatrix} - 1\begin{vmatrix} 2d & 1 \\ 0 & 1+d \end{vmatrix} + 1\begin{vmatrix} 2d & 1+d \\ 0 & 1 \end{vmatrix}$$

$= d\{(1+d)^2 - 1\} - 1\{2d(1+d) - 0\} + 1\{2d - 0\} = d\{1 + 2d + d^2 - 1\} - 1\{2d + 2d^2\} + 1(2d)$

$= d\{2d + d^2\} - 2d - 2d^2 + 2d = d\{2d + d^2\} - 2d^2$

$= d\{2d + d^2 - 2d\} = d\{2d + d^2 - 2d\} = d(d^2) = d^3$

$$|A_2| = \begin{vmatrix} 1 & d & 1 \\ 1 & 2d & 1 \\ 1 & 0 & 1+d \end{vmatrix} = 1\begin{vmatrix} 2d & 1 \\ 0 & 1+d \end{vmatrix} - d\begin{vmatrix} 1 & 1 \\ 1 & 1+d \end{vmatrix} + 1\begin{vmatrix} 1 & 2d \\ 1 & 0 \end{vmatrix}$$

$= 1\{(2d(1+d) - 0\} - d(1+d-1) + 1(0-2d) = 1\{2d + 2d^2\} - d(d) + 1(-2d) = 2d + 2d^2 - d^2 - 2d$

$= d^2 \qquad ; |A_2| = d^2$

$$|A_3| = \begin{vmatrix} 1 & 1 & d \\ 1 & 1+d & 2d \\ 1 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 1+d & 2d \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2d \\ 1 & 0 \end{vmatrix} + d\begin{vmatrix} 1 & 1+d \\ 1 & 1 \end{vmatrix}$$

$= 1(0-2d) - 1(0-2d) + d\{1-(1+d)\} = -2d + 2d + d(-d) = -d^2 \qquad ; |A_3| = -d^2$

$x = \dfrac{|A_1|}{|A|} = \dfrac{d^3}{d^2}, \quad y = \dfrac{|A_2|}{|A|} = \dfrac{d^2}{d^2} = 1, \quad z = \dfrac{|A_3|}{|A|} = \dfrac{-d^2}{d^2} = -1$

Check:- $x + (1+d)y + z = 2d \rightarrow (2) = d + (1+d)(1) + (-1) = d + 1 + d - 1 = 2d$

$x + y + (1+d)z = 0 = d + 1 + (1+d)(-1) = d + 1 - 1 - d = 0$

EXAMPLES FROM THE TEXT BOOK.

(1)Find the value of $|A| = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$.

<u>Solution</u>:-Adding R_2 and R_3 to R_1 ,we get $|A| = \begin{vmatrix} x+2a & x+2a & x+2a \\ a & x & a \\ a & a & x \end{vmatrix}$

$= (x+2a)\begin{vmatrix} 1 & 1 & 1 \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)\begin{vmatrix} 1 & 0 & 0 \\ a & x-a & 0 \\ a & 0 & x-a \end{vmatrix}$ *Subtracting C_1 from C_2 and C_3*

$= (x+2a)\{1(x-a)^2 - 0 + 0\}$ *Expanding by R_1.*

$= (x+2a)(x-a)^2$.

Solve $2x^2 + xy + 3y^2 - 6 = 0 \rightarrow (1)$

$4x^2 + xy + 3y^2 - 9 = 0 \rightarrow (2)$

Multiplying (1) by 3 and (2) by 2, we get

$6x^2 + 3xy + 9y^2 - 18 = 0 \rightarrow (3)$

$8x^2 + 2xy + 6y^2 - 18 = 0 \rightarrow (4)$

Subtracting (4) from (3) we get

$-2x^2 + xy + 3y^2 = 0 \rightarrow (5) \Rightarrow (x+y)(2x-3y) = 0$

$\Rightarrow x = -y \rightarrow (6)$ *or* $x = \frac{3}{2}y \rightarrow (7)$

The linear factors (6) and (7) are used with the original quadratic eq: (1) as follows.
Substituting x= -y in (1), we get

$2(-y)^2 + (-y)y + 3y^2 - 6 = 0 \quad \Rightarrow 2y^2 - y^2 + 3y^2 - 6 = 0; = 4y^2 = 6, y = \pm\sqrt{\frac{3}{2}}$

and $x = -y = -\left(\pm\sqrt{\frac{3}{2}}\right) = \mp\sqrt{\frac{3}{2}}$.

Substituting $x = \frac{3}{2}y$ *in (1), we get* $2\left(\frac{3}{2}y\right)^2 + \left(\frac{3}{2}y\right)(y) + 3y^2 - 6 = 0$

$\Rightarrow \frac{9}{2}y^2 + \frac{3}{2}y^2 + 3y^2 - 6 = 0; = 9y^2 = 6, y = \pm\sqrt{\frac{2}{3}}$ **and** $x = \frac{3}{2}y = \frac{3}{2}\left(\pm\sqrt{\frac{2}{3}}\right) = \pm\sqrt{\frac{3}{2}}$.

Thus the solution set is $S = \left\{\left(\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}\right), \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right), \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{2}{3}}\right), \left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{2}{3}}\right)\right\}$.

(2)*Evaluate* $|A| = \begin{vmatrix} 1+a & 1 & 1 \\ 2 & 2+a & 2 \\ 3 & 3 & 3+a \end{vmatrix}$. **<u>Solution:-</u>** *Adding R_2 and R_3 to R_1, we get*

$|A| = \begin{vmatrix} 6+a & 6+a & 6+a \\ 2 & 2+a & 2 \\ 3 & 3 & 3+a \end{vmatrix} = (6+a)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2+a & 2 \\ 3 & 3 & 3+a \end{vmatrix} = (6+a)\begin{vmatrix} 1 & 0 & 0 \\ 2 & a & 0 \\ 3 & 0 & a \end{vmatrix}$ *Subtracting C_1 from C_2 and C_3*

$= (6+a)\{1(a^2 - 0)\} = a^2(a+6)$.

(3) *Evaluate* $|A| = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

Solution: Adding C_1 and C_2 to C_3, we get

$$|A| = \begin{vmatrix} 1 & \omega & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1+\omega+\omega \\ \omega^2 & 1 & 1+\omega+\omega^2 \end{vmatrix} = (1+\omega+\omega^2)\begin{vmatrix} 1 & \omega & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & 1 \end{vmatrix}$$

$= 0 \times \begin{vmatrix} 1 & \omega & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & 1 \end{vmatrix} (\because 1+\omega+\omega^2 = 0)$. **So =0.Ans**

(4) *Solve* $\begin{vmatrix} y^2 & y & 1 \\ 8 & 4 & 10 \\ 9 & 3 & 6 \end{vmatrix} = 60$

Taking 2 and 3 common from R_2 and R_3.

$$2 \times 3 \begin{vmatrix} y^2 & y & 1 \\ 4 & 2 & 5 \\ 3 & 1 & 2 \end{vmatrix} = 60; = \begin{vmatrix} y^2 & y & 1 \\ 4 & 2 & 5 \\ 3 & 1 & 2 \end{vmatrix} = 10$$

$= y^2(4-5) - y(8-15) + 1(4-6) = 10$

$= y^2(-1) - y(-7) + 1(-2) = 10$

$= y^2 - 7y + 12 = 0; y = 3 \text{ or } y = 4$

Thus the solution set is {3,4}.

(5) $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 9 & 8 & 6 \end{bmatrix}$. *Find adj A.*

Solution: $A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 7 \\ 8 & 6 \end{vmatrix} = (-1)^2(-32) = -32$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 7 \\ 9 & 6 \end{vmatrix} = (-1)^{1+2}(-51) = 51$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 9 & 8 \end{vmatrix} = (-1)^4(-20) = -20$. $A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 5 \\ 8 & 6 \end{vmatrix} = (-1)^3(-22) = 22$.

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ 9 & 6 \end{vmatrix} = (-1)^4(-39) = -39$. $A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 9 & 8 \end{vmatrix} = (-1)^5(-19) = 19$.

$A_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} = (-1)^4(1) = 1$. $A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} = (-1)^5(-3) = 3$.

$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (-1)^6(-2) = -2$. **Hence , adjA=** $\begin{bmatrix} -32 & 51 & -20 \\ 22 & -39 & 19 \\ 1 & 3 & -2 \end{bmatrix}' = \begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix}$.

(6)Find the inverse of $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 9 & 8 & 6 \end{bmatrix}$ **by the adjoint method.**

Solution: *Here* $|A| = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 9 & 8 & 6 \end{vmatrix} = 21 \neq 0$. *Thus A is a non-singular matrix.*

For adj A See page 218 Q(5). $AdjA = \begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix}$ $\therefore A^{-1} = \frac{AdjA}{|A|} = \frac{1}{21}\begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix}$.

Verification: $AA^{-1} = \frac{1}{21}\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 9 & 8 & 6 \end{bmatrix}\begin{bmatrix} -32 & 22 & 1 \\ 51 & -39 & 3 \\ -20 & 19 & -2 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} = I_3$, *So* $AA^{-1} = I_3$.

(7)Apply Cramer's rule to solve the system of equations.

$2x_1 - x_2 + 2x_3 = 4$
$x_1 + 10x_2 - 3x_3 = 10$. *Determinant of co-efficents of system is* $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & -3 \\ -1 & 1 & 1 \end{vmatrix} = 46 \neq 0$
$-x_1 + x_2 + x_3 = -6$

The determinants in the numerators in (ii) are $|A_1| = \begin{vmatrix} 4 & -1 & 2 \\ 10 & 10 & -3 \\ -6 & 1 & 1 \end{vmatrix} = 184$.

$|A_2| = \begin{vmatrix} 2 & 4 & 2 \\ 1 & 10 & -3 \\ -1 & -6 & 1 \end{vmatrix} = 0$. **and** $|A_3| = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 10 & 10 \\ -1 & 1 & -6 \end{vmatrix} = -92$.

$\therefore x_1 = \frac{184}{46} = 4, x_2 = \frac{0}{46} = 0, x_3 = \frac{-92}{46} = -2$. **Hence the solution is** $x_1 = 4, x_2 = 0, x_3 = -2$.