EXERCISE NO. = 5.1

Qno:01) Show that multiplication is binary option on $S=\{1,-1\}$ but not on $T=\{1,2\}$

$S=\{1,-1\}$. $1\times 1\in S$, $(-1)(-1)=1\in S$, $(1)(-1)=-1\in S$, $(-1)(1)=-1\in S$.

$\therefore$ Multiplication is binary operation on S.

$T=\{1,2\}$. $1\times 1=1\in T$, $(1)(2)=2\in T$, $(2)(2)=4\notin T$

$\therefore$ Multiplication is not binary operation on T.

Q.No. (02) Show that addition is binary operation on $S=\{a/a\in I, a<0\}$. but not multiplication.

$I=\left\{x/x\neq\frac{p}{q}: p+q\in z, q\neq 0\right\}$ As a<0 i.e. –ve no: Let $-a\in S, -b\in S$

For addition $-a+(-b)<0$:. $-(a+b)\in S$ hence addition is binary operation on S.

For multiplication $-a\in S, -b\in S$. $(-a)(-b)=ab>0$:. $ab\notin S$. Multiplication is not binary operation on S.

Q.No. (03) Let S = {A,B,C,D}, where A = {a,b}, B = {a,c}, C = {a,b,c} and $D=\phi$. Show that U is binary operation on S but not $\cap$.

S = {A,B,C,D}. A = {a,b}, B = {a,c}, C = {a,b,c}, $D=\phi$

$A\cup B=\{a,b,c\}\in S$, $A\cup C=\{a,b,c\}=B\in S$.

$B\cup C=\{a,b,c\}=B\in S$, $A\cup D=A\in S$, $B\cup D=B, D\cup D=D\in S$, $C\cup D=C\in S$

$\cup$ is binary operation on S.

$A\cap B=\{a\}\cap\{a,b\}=\{a\}\notin S$:. $\cap$ is not binary operation on S.

Q.No. (05) Show that multiplication is binary operation on $S=\{1,-1,i,-i\}$. Is multiplication commutative and associative?

$1(-1)=-1\in S$, $-1.i=-i\in S, 1(i)=i\in S, -1(-i)=-i\in S,$

$i(-i)=-1=-i^2=-1(-1)=1\in S$ $\therefore$ Multiplication is binary operation on S.

$1(-1)=(-1)(1)=-1\in S, i(-i)=(-i)(i)=1\in S, -1(-i)=(-1)(-i)=i\in S$ $\therefore$ Multiplication is commutative. $1(-1i)=(1.-1)i=-i\in S, -1(i.-i)=(-1i)(-i)$,

$(-1)(-i^2)=(-i)=i^2.i^2=-1\in S$. $\therefore$ Multiplication is associative.

Q.No.(04) Let S={A,B,C,D}.A={a},B={a,b},C={a,b,c},D=ϕ. Construct multiplication table to show that $\cup$ and $\cap$ are binary operation on S.

$\cup$	A	B	C	D
A	A	B	C	A
B	B	B	C	B
C	C	C	C	C
D	A	B	C	D

$\cup$ is binary operation on S.

$\cap$	A	B	C	D
A	A	A	A	D
B	A	B	B	D
C	A	B	B	D
D	D	D	D	D

$\cap$ is binary operation on S.

Q.No. (06) On R, Define a binary operation $a\odot b=|a-b|\forall a,b\in R$. Show that $\odot$ is commutative but not associative.

$a\odot b=|a-b|, b\odot a=|b-a|=|-a(a-b)|=|a-b|$ $\therefore a\odot b=b\odot a$. $\odot$ is commutative.

Let $a,b,c\in R=|a-b|\odot c=(a\odot b)\odot c$ $(a\odot b)\odot c=||a-b|-c|\rightarrow(1)$

$(a\odot b)\odot c=a\odot|b-c|=|a-b|b-c||\rightarrow(2)$ $(1)\neq(2)\therefore(a\odot b)\odot c\neq a\odot(b\odot c)$

Q.No. (07) The binary operation $\odot$ on R is defined by $a\odot b=\max(a,b)$. Verify $\odot$ is both associative and commutative in R.

If a>b $a\odot b=a$, $b\odot a=a$ If a <b

$a\odot b=b$, $b\odot a=b$ $\therefore b\odot a=b$ $\therefore \odot$ *is commutative.*

If a<b<c $(a\odot b)\odot c = b\odot c = \odot$ $a\odot(b\odot c)=a\odot c=c$

If a>b>c $(a\odot b)\odot c=a\odot c=a$ $a\odot(b\odot c)=a\odot b=a$:. $\odot$ is associative in R.

QNo. 08 (i)

$a\odot b=\sqrt{a^2+b^2}$ $\forall a,b\in\mathbb{R}$ (ii)$a\odot b=\max(a,b),\forall a,b\in\mathbb{R}$.*Find identity element.*

(i) Pr*operty of identity is* $a*e=a$ *if* $e=0$ $a\odot 0=\sqrt{a^2+0^2}=\sqrt{a^2}=a$

(ii) $a\odot b=\max(a,b), a*e=a$(*property of* e)*If* $a>e$ $a*e=a$ *but if* $a<e$, $a*e=e$

:. 'E' does not exist.

Q.No.(09) define a binary operation * in $\mathbb{Q}$ by:

(I) $a*b =$ [illegible] $4ab, b*a = 4ba$ [illegible] $a*b = b*a$. * is commutative.

(II) $(a*b)*c = 4ab*c = 4(4ab)c = 16abc$ [illegible] $= a*(4b.c) = 4a.4bc = 16abc$

$\therefore$ * is associative.

(III) $a*e = a$ (Property of e) $4ae = a, e = \frac{a}{4a} = \frac{1}{4}$ [illegible] $e = \frac{1}{4}$.

(IV) $a*a^{-1} = e$ $4.a.a^{-1} = 4\left(\frac{1}{12}\right)\left(\frac{3}{4}\right) = \frac{3}{12} = \frac{1}{4}$ also $\therefore e = \frac{1}{4}$.

Hence $\frac{1}{12}$ and $\frac{3}{4}$ are inverse of each other.

Q.No. (10) Let $S = \{1, \omega, \omega^2\}$ ω being complex cube root of unity. Construct a composition table w.r.t multiplication on $\mathbb{C}$ and show that, (i) associative law holds, (ii) 1 is the identity element, (iii) each element of S has an inverse.

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\omega^3 = 1$

$\omega^4 = \omega^3.\omega$

$1.\omega = \omega$

(i) $1, \omega, \omega^2 \in S$

$(1.\omega).\omega^2 = 1(\omega.\omega^2)$

$(\omega).\omega^2 = \omega^3 = 1$

$= 1 \in S$ $\quad 1 \in S$

(ii) $a*e = a*e = a$

$\omega.1 = 1.\omega = \omega$

$\omega^2.1 = 1.\omega^2 = \omega^2$

$\therefore 1$ is identity element.

(iii) $a*a^{-1} = e$

But e = 1

$\omega.\omega^2 = \omega^3 = 1 = identity$

Hence $a*a^{-1} = e$ holds

ω is inverse of ω^2

As well ω^2 is inverse of ω.

Q.No. (11) A = {0,1,2,3}; Define $a*b = a$

Construct multiplication table

*	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

Q.No (12) Let * be defined in $\mathbb{Z}$ by $m*n = m+n+2$ (a) * is associative + commutative. (b)Identity w.r.t * exists in $\mathbb{Z}$. (c)Every element of $\mathbb{Z}$ has an inverse.

(a) $l*(m*n) = l*(m+n+2) = l+m+n+2+2 = l+m+n+4$

$(l*m)*n = (l+m+2)*n = l+m+2+n+2 = l+m+n+4$

$l*(m*n) = (l*m)*n$ [illegible] associative.

(b)Identity $a*e = a, a+e+2 = a, e = a-a-2,$ [illegible] $\in \mathbb{Z}$

(c) $a*a^{-1} = e, \quad a+a^{-1}+2 = e, \quad a+a^{-1}+2 = -2, \quad a^{-1} = -a-2-2 = -a-4 = -(a+4) \in \mathbb{Z}$

QNO(13) Prove that set operations $\cup$ and $\cap$ of set are commutative and associative binary operations.

Solution:- Let S = {a,b} , P(S) = {{},{a},{b},{a,b}}, A = {}, B = {a}, C = {b}, D = {a,b}

Composition table for $\cup$

$\cup$	A	B	C	D
A	A	B	C	D
B	B	B	D	D
C	C	D	C	D
D	D	D	D	D

As the table is symmetric $\cup$ is commutative.
To prove associativity. Use table.

Composition table for $\cap$

$\cap$	A	B	C	D
A	A	A	A	A
B	A	B	A	B
C	A	A	C	C
D	A	B	C	D

As the above table is symmetric $\cap$ is commutative.
To prove the associativity use table.

$A\cup(B\cup C)$ $= A\cup D = D$ $A\cup(C\cup D)$ $= A\cup D = D$ $B\cup(C\cup D)$ $= B\cup D = D$	$(A\cup B)\cup C$ $= B\cup C = D$ $(A\cup C)\cup D$ $= C\cup D = D$ $(B\cup C)\cup D$ $= D\cup D = D$	$(i) A\cap(B\cap C)$ $A\cap A = A$ $(ii) A\cap(C\cap D)$ $= A\cap C = A$ $(iii) B\cap(C\cap B)$ $= B\cap C = A$ $(iv) A\cap(B\cap D)$ $= A\cap B = A$	$(A\cap B)\cap C$ $= A\cap C = A$ $(A\cap C)\cap D$ $= A\cap D = A$ $(B\cap C)\cap D$ $= A\cap D = A$ $(A\cap B)\cap D$ $= A\cap D = A$

Exercise No. = 5.2

QNo(1) Are the following groupoids.

(i) $(\mathbb{Q}, \div)$ $\mathbb{Q} = \{x / x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\}$ $\frac{3}{4} \div \frac{4}{3} = 1$ .is an integer. $\div$ is not groupoid

(ii) $(\mathbb{R}, -)$ $\mathbb{R} = \{x / x = \frac{p}{q} or x \neq \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\}$ $\frac{2}{5}, \frac{3}{7} \in \mathbb{R}$ $\frac{2}{5} - \frac{3}{7} = \frac{14-15}{35} = \frac{-1}{35} \in \mathbb{R}$

$\mathbb{R}$ is groupoid and - is defined in $\mathbb{R}$.

(iii) $(M_3, \cdot)$ is groupoid. M_3 is 3×3 matrix.Let A,B are 3×3 matrices. $A\times B$ is also 3×3 matrix. $\therefore M_3$ is groupoid.

(iv)V is the set of vectors. Let $\bar{a}, \bar{b} \in V$. $\bar{a}\bar{b} = ab \notin V$ $\therefore$ ab is scalar. $(V, \cdot)$ is not groupoid. .

QNo(2)Is (S,*) a groupoid.If S={1,2,3,4}.defined on S by $x * y = 2$.

$1*2=2\in S$, $3*4=2\in S$, $1*4=2\in S$, $2*4=2\in S$ .Hence (S,*) is groupoid.

QNo(3)Show that (S,*) a groupoid if $S = \{\triangle, \square\}$

$a \odot b = \triangle$, $\triangle \odot \square = \triangle \in S$ $\therefore$ (S,*) is groupoid.

QNo(4)Show that for any non-empty set S={P(S),U} is an abelian semi-group with identity.

Let S={a,b,c},P(S)= $\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

{a}$\cup${c}={a,c}$\in$P(S). {c}$\cup${a}={a,c}$\in$P(S) So $\cup$ is commutative.

[{a}$\cup${a,b}]$\cup${a,c}=[{a,b}]$\cup${a,c}={a,b,c}=>(1)

{a}$\cup$[{a,b}$\cup${a,c}]={a}$\cup$[{a,b,c}]={a,b,c}=>(2)$\therefore$ $\cup$ is associative $\therefore$ $\cup$ is semi abelian group.

Identity.

{a,b}$\cup${ }={a,b}

{a,b,c}$\cup\phi$={a,b,c}

$\therefore$ ϕ is identity $\in P(S)$.

QNo(5)Show that $(M_3, +)$ is commutative semi-group with identity.

M_3 is set of all 3×3 matrices.

Let A,B,C are 3×3 matrices. Then A+(B+C) is 3×3 matrix ,also (A+B)+C is 3×3 matrix. + is associative.

A+B=B+A $\in M_3$. $A_3 + I_3 = A_3$ Hence $(M_3, +)$ is commutative semi-group with identity.

(ii) $(M_3, \times)$ is semi group with identity.

Let A,B,C$\in M_3$ $(A\times B)\times C \in M_3$ also $A\times(B\times C) \in M_3$

$A_3 I_3 = A_3$

$(M_3, \times)$ is semi-group with identity.

Exercise No.= 5.3

QNo(2)Is $(\mathbb{Q}^*,*)$ a group if $*$ is defined by $a*b=\frac{ab}{3}$, $\forall a\in\mathbb{Q}$.

$(a*b)*c$; $a*(b*c)$

$=(\frac{ab}{3})*c$; $=a*\frac{bc}{3}$

$=\frac{abc}{3\times3}=\frac{abc}{9}$; $=\frac{abc}{3\times3}=\frac{abc}{9}$

$*$ is associative. $*$ is semi group.

Let e is identity.

$=a+b+c+bc+a(b+c+bc)$

$a*e=a$

$\frac{ae}{3}=a \Rightarrow e=\frac{a}{a}\times3=e=3$

Identity exists in $\mathbb{Q}$.

$a*a^{-1}=e$

$=\frac{aa^{-1}}{3}=e=3$

$a^{-1}=3\times\frac{3}{a}=\frac{9}{a}$ in $\mathbb{Q}$.

Inverse element exists.

$(\mathbb{Q}^*,*)$ is group.

QNo(3)Show that $(\mathbb{R}-\{-1\},*)$ is group. Where $*$ is given by. $a*b=a+b+ab$

$(a*b)*c=(a+b+ab)*c$

$=a+b+ab+c+(a+b+ab)c$

$=a+b+ab+c+ac+bc+abc \rightarrow (1)$

$=a+b+c+ab+ac+bc+abc$

$a*(b*c)=a*(b+c+bc)$

$=a+b+c+bc+ab+ac+abc$ -----(2)

(1)=(2) $\therefore$ $*$ is semi-group.

Let e be identity element.

$a*e=e*a=a$,a+e+ae=a =e(1+a)=a-a=0

$=e=\frac{0}{1+a}=0\in\mathbb{R}-1$ Identity element exists

Let a^{-1} is inverse of a

$=a*a^{-1}=e=a+a^{-1}+aa^{-1}=0=a^{-1}(1+a)=-a$

$=a^{-1}=\frac{-a}{1+a}\in\mathbb{R}-\{-1\}$

Inverse of each element exists.

So $(\mathbb{R}-\{-1\},*)$ is group.

QNo(4)Show that four fourth roots of unity form a finite abelian group w.r.to the usual multiplication of complex no.

$x^4=1 \Rightarrow x^4-1=0$ or $(x^2-1)(x^2+1)=0$

Either $x^2-1=0$, or $x^2+1=0$

$x^2=1, x=\pm1$; $x^2=-1=i^2, x=\pm i$

Fourth four roots are $x=1,-1,i,-i$

i)1(-1.i)=1(-i)= -i ,(1.-1)i= -1.i= -I

ii)-1(i.-i)= $-1(-i^2)=i^2=-1$.

(-1.i)(-i)=(-i)(-i)= $i^2=-1$.

iii)1(i.-i)=1.$(-i^2)=-i^2=-(-1)=1$

(1.i)(-i)=i(-i)= $-i^2=-(-1)=1$.

'.' is associative.1 is identity element.

a.1=a,(-1).= -1,i(1)=I, -i(1)= -i

Identity element exists $a.a^{-1}=a^{-1}.a=e$

Under '.' Inverse of a is 1/a.

Inverse of 1=1/1=1 exists.

Inverse of -1=1/-1= -1 exists.

Inverse of $i=\frac{1}{i}\times\frac{i}{i}=\frac{i}{i^2}=\frac{i}{-1}=-i$ exists.

Inverse of $-i=\frac{1}{-i}\times\frac{i}{i}=\frac{i}{-i^2}=\frac{i}{-1(-1)}=i$ ext.

Inverse of each element exists.

So 1,-1,i,-I form a group.

(1)(-1)=(-1)(1). Commutative.

(-1)i=i(-1)= -i

i(-i)=(-i)i= $-i^2=1$

-1(-i)= -i(-1)=i

Hence 1,-1,i,-I form a finite abelian group.

QNo(5)Show that set of all non-singular matrices of order 2 form an infinite non-abelian group w.r.t matrix multiplication.

Let A,B,C $\in M_2$

(A.B).C=A(B.C)

Multiplication is associative.

A.B$\neq$B.A

$I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\in M_2$ exists.

Inverse of A i.e A^{-1} is also $2\times2\in M_2$

Hence (M₂, .) is non abelian group.

QNo(6)Is $(\mathbb{R},*)$ a commutative group.If $*$ is defined in $\mathbb{R}$ by $a*b=7ab$

$a*b=7a$, $b*a=7ba$ $*$ is commutative.

$a*e=a$ $7ac=a$ =e= $\frac{a}{7a}=\frac{1}{7}$, $a*a^{-1}=e$, $a.a$, $7a.a^{-1}=\frac{1}{7}=a^{-1}=\frac{1}{49a}$

Invers of each element exists. e)=>(2

$a*(b c)$	,$(a*b)*c$
$=a (7bc)$	$=(7ab)*c$
$=7 7bc$	$=7.7abc$
$= 9abc$	$=49abc$

$*$ is associative.

$\therefore (\mathbb{R},*)$ is abelian group.

QNo(9) $a*a=a\rightarrow(1)$ $a*e=a$ (property of

Comparing (1) and (2) a=e.

QNo(8)Let a binary operation $*$ be defined on $\mathbb{Z}$ by $a*b=a+b-ab$ is a group?

$a*(b*c)$	$(a*b)*c$	$a*e=a$
$=a*(b+c-bc)$	$=(a+b-ab)*c$	a+e+ac=a
$=(a+b+c-bc)-a(b+c-bc)$	$=a+b-ab+c-(a+b-ab)c$	e(1+a)=0
$=a+b+c-bc-ab-ac+abc\rightarrow(1)$	$=a+b+c-ab-ac-bc+abc\rightarrow(2)$	$e=\frac{0}{1+a}=0\in\mathbb{Z}$
(1)=(2) $*$ is associative.		

$a*a^{-1}=e$, $a+a^{-1}+aa^{-1}=0$, $a^{-1}+aa^{-1}=0-a$, $a^{-1}(1+a)=-a$, $a^{-1}=\frac{-a}{1+a}\notin\mathbb{Z}$, $(\mathbb{Z},*)$ is not group.

EXAMPLES FROM THE TEXT BOOK.

Composition table of S={1,-1,i,-i}.

$\cdot$	1	−1	i	$-i$
1	1	−1	i	$-i$
−1	−1	1	$-i$	i
i	i	$-i$	−1	1
$-i$	$-i$	i	1	−1

(1) Using multiplication table, find whether $\times$ is a binary operation on $S=\{1,\omega,\omega^2\}$

Solution:

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

As the elements in each and every square belong to S, so $\times$ is a binary operation on S

Ex : Three cube roots of unity form a group w.r.t the usual multiplication of complex no:s denoted by "$\cdot$".

Sol: Here G= $\{1,\omega,\omega^2\}$ .From the multiplication table .It is easy to verify that all the axioms for a group are satisfied.

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Ex: Semi group (M_3 ,+) is also a group.

Solution: The null matrix O_3 is the identity w.r.t "+".Also corresponding to every matrix $A\in M_3 \exists$ a matrix $-A\in M_3$, such that A+(-A)=(-A)+A=O_3, i.e –A is the additive inverse of $A\in M_3$. Hence $(M_3,+)$ satisfies all the axioms of a group.