CHAPTER 13 Inverse Trigonometric Functions and Trigonometric Equations

Related Definitions and Formulae

Inverse Circular (OR) Trigonometrical Functions:

The inverse trigonometrical functions $Sin^{-1}x$, $Cos^{-1}x$, $tan^{-1}x$, $Cot^{-1}x$, $Sec^{-1}x$, $Cosec^{-1}x$ are generally defined as the inverse of the corresponding trigonometrical functions for instance $Sin^{-1}x$ is defined as the angle whose sine is x. This definition as it stands is incomplete and ambiguous as will be clear from the following discussion.

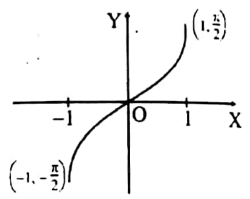
If $y = Sin^{-1}x$ ——— (1) then $x = Sin y$ ——— (2)

Where y is independent and x is dependent. Now to each value of y there corresponds just one value of x in (2). On the other hand, the same value of x corresponds to an unlimited number of values of the angle y so that to any given value of x in $[-1 \le x \le 1]$. There corresponds an Unlimited number of values of the angle y whose sine is x. Thus $Sin^{-1}x$ as define above is not unique. In other words, $Sin^{-1}x$ is an inverse relation and not inverse function.

The same remark applies to other remaining functions also. We now proceed to modify the definition of the inverse trigonometrical function. So as to remove the ambiguity referred to above.

$y = sin^{-1} x$:

Consider the function equation $x = \sin y$. We know that as y increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, then x increases monotonically, taking up every real value in $[-1, 1]$, so that to each value of x in this interval there corresponds one and only one value of y in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Thus, there is one and only one angle with a given sine.



Accordingly we define $sin^{-1} x$ as follows:

$Sin^{-1} x$ is the angle in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, whose sine is x.

In others word, the domain of $Sin^{-1} x$ is $[-1, 1]$ and the range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

To Draw the graph of $y = sin^{-1} x$:

We note that

(i) y increases monotonically from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ as x increases from -1 to 1.

(ii) $Sin^{-1} x = 0$, $Sin^{-1}(1) = \frac{\pi}{2}$, $Sin^{-1}(-1) = -\frac{\pi}{2}$

(iii) $Sin^{-1} x$ is defined in the interval $[-1, 1]$ only.

We thus have the graph as drawn in Fig.

Note 1: We know that if $x = \sin y$, then x varies monotionically taking up every value in $[-1, 1]$ as y increases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, and so on. Thus, the definition could also have been equally suitably modified by restricting $Sin^{-1} x$ to any of the intervals $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$, etc. instead of $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, but what we have done here is however, more usual or conventional.

Note 2. The graph of $y = Sin^{-1} x$ is the reflection of $y = \sin x$, in the line $y = x$, as by interchanging x and y in $y = \sin x$ we get $x = \sin y$, i.e. $y = \sin^{-1} x$.

$y = cos^{-1} x$:

Consider the functional equation $x = \cos y$.

We know that as y increases from 0 to π, then x decreases monotonically taking up every real value between 1 and −1. Thus, there is one and only one angle, lying between 0 to π, with a given cosine.

Accordingly we define $cos^{-1} x$ as follows:

$cos^{-1} x$ is the angle is the interval $[0, \pi]$, whose cosine is x.

In other words, the domain of $\text{Cos}^{-1}x$ is $[-1, 1]$ and the range is $[0, \pi]$.

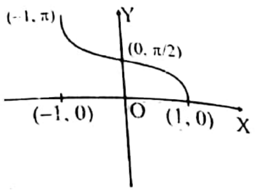
To draw the graph of $y = \cos^{-1}x$:

We note that

(i) y decreases monotonically from π to 0 as x increases from -1 to 1.

(ii) $\cos^{-1}(-1) = \pi$, $\cos^{-1} 0 = \frac{\pi}{2}$, $\cos^{-1} 1 = 0$

(iii) $\cos^{-1}x$ is defined in the interval $[-1, 1]$ only.



Note 1: We know that x varies monotonically taking up every value in $[-1, 1]$, as y increases from π to 2π, 2π to 3π, and so on. Thus, the definition could also have been equally suitably modified by restricting $\cos^{-1}x$ to any of the intervals $[\pi, 2\pi]$ $[2\pi, 3\pi]$, etc., instead of $[0, \pi]$, but what we have done here is, however, more usual or conventional.

Note 2: The graph of $y = \cos^{-1}x$ is the reflection of $y = \cos x$ in the line $y = x$.

$y = \tan^{-1}x$:

Consider the functional equation $x = \tan y$.

We know that a y increases from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$, then x increases monotonically taking up every value in $(-\infty, \infty)$ so that to each value of x in this interval there corresponds one and only one value of y the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Accordingly we have the following definition of $\tan^{-1}x$.

$\tan^{-1}x$ is the angle in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, whose tangent is x. In other words, the domain of $\tan^{-1}x$ is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

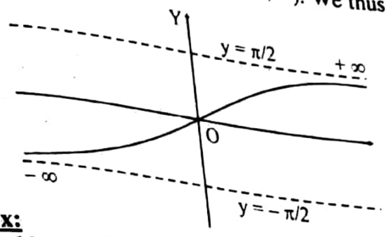
To draw the graph of $y = \tan^{-1}x$:

We note that

(i) y increases monotonically from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ as x increases from $-\infty$ to ∞.

(ii) $\tan^{-1} 0$ is 0.

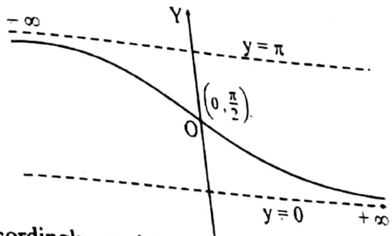
(iii) $\tan^{-1}x$ is defined in the interval $(-\infty, \infty)$. We thus have the graph as is drawn.



$y = \cot^{-1}x$:

Consider the functional equation $x = \cot y$. We know that as y increases from 0 to π, then x decreases monotonically from $+\infty$ to $-\infty$, taking up every real value between $-\infty$ and $+\infty$.

Thus, there is one and only one angle, lying between 0 and π, with a given contangent.



Accordingly we define $\cot^{-1}x$ as follows: $(\cot^{-1}x)$ is the angle in the interval $(0, \pi)$ whose cotangent in x.

In other words the domain of $\cot^{-1}x$ is $(-\infty, \infty)$ and the range is $(0, \pi)$.

To draw the graph of $y = \cot^{-1}x$.

We note that

(i) y decreases monotonically from π to 0 as x increases from $-\infty$ to ∞.

(ii) $\cot^{-1} 0 = \frac{\pi}{2}$

(iii) $\cot^{-1}x$ is defined in the interval $(-\infty, \infty)$.

We thus have the graph as is drawn.

$y = \sec^{-1} x$:

Consider the functional equation $x = \sec y$. We know that as y increases from 0 to $\frac{\pi}{2}$, then x increases monotonically from 1 to $+\infty$. Also as y increases from $\frac{\pi}{2}$ to π, then x increases monotonically from $-\infty$ to -1. Thus, there is one and only one value of the angle, lying between 0 and π, whose secant is any given number, not lying between -1 and 1.

Accordingly we define $\sec^{-1} x$ as follows:

$\text{Sec}^{-1} x$ is the angle, lying between 0 and π, whose secant is x, excluding $\frac{\pi}{2}$, but including 0 and π.

In other words, the domain is $x \le -1$ and $x \ge 1$, and the range is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

To draw the graph of $\text{Sec}^{-1} x$:

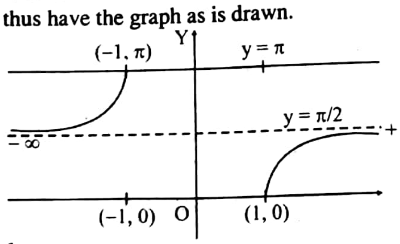
We note that

(i) y increases from 0 to $\frac{\pi}{2}$ as x increases from 1 to $+\infty$ and y increases from $\frac{\pi}{2}$ to π as x increases from $-\infty$ to -1.

(ii) $\sec^{-1}(-1) = \pi$; $\sec^{-1} 1 = 0$

(iii) $\sec^{-1} x$ is defined for $x \le -1$ and $x \ge 1$.

We thus have the graph as is drawn.



$y = \text{cosec}^{-1} x$:

Consider the functional equation $x = \text{cosec } y$. We know that as y increases from $-\frac{\pi}{2}$ to 0, then x decreases monotonically

from -1 to $-\infty$, and as y increases from 0 to $\frac{\pi}{2}$, x decreases monotonically from $+\infty$ to 1.

Thus, there is one and only one value of the angle lying between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, whose cosecant is any given number, not lying between -1 and 1.

Accordingly we define $\text{cosec}^{-1} x$ as follows:

$\text{Cosec}^{-1} x$ is the angle, lying between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, whose cosecant is x, excluding 0, but including $\frac{-\pi}{2}$ and $\frac{\pi}{2}$.

In other words the domain of $\text{cosec}^{-1} x$ is $x \le -1$ and $x \ge 1$, and the range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

To draw the graph of $y = \text{cosec}^{-1} x$:

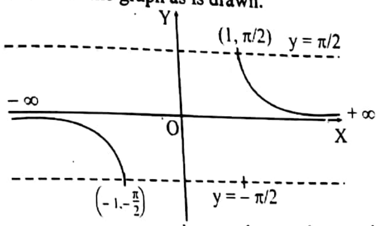
We note that

(i) y decreases from 0 to $\frac{-\pi}{2}$ as x increases from $-\infty$ to -1, and y decreases from $\frac{\pi}{2}$ to 0 as x increases from 1 to $+\infty$.

(ii) $\text{Cosec}^{-1}\left(\frac{-\pi}{2}\right) = -1$; $\text{Cosec}^{-1}\left(\frac{\pi}{2}\right) = 1$.

(iii) $\text{Cosec}^{-1} x$ is defined.
for $x \le -1$ and $x \ge 1$.

We thus have the graph as is drawn.



Note 1: The graph of $y = \sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$ are the reflections respectively of the graphs of y = sinx, cos x, tan x, cot x, sec x, cosec x in the line y = x.

Note 2: The symbol $\sin^{-1} x$ is also written as "arc sin x" in certain books. Similarly, $\cos^{-1} x$ is written as "are cos x" while $\tan^{-1} x$ is written as "arc tan x", and so on.

Principal value of an inverse function:

The ranges of the inverse trigonometric functions found in the definitions, given above are called the ranges of the principal value of the inverse functions. (The inverse trigonometric functions are also called inverse circular functions.)

Thus, the principal value of $\sin^{-1} x$, $\tan^{-1} x$, $\operatorname{cosec}^{-1} x$, are the angles that lie between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, and the principal value of $\cos^{-1} x$, $\cot^{-1}$, $\sec^{-1} x$ are the angles that lie between 0 and π.

Note: Unless otherwise stated, by any inverse value of a circular function is meant its principal value.

The student should carefully remember the domain and range of all inverse trigonometric functions as given below:

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R or $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$x \le -1$ or $x \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$x \le -1$ or $x \ge 1$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R or $(-\infty, \infty)$	$(0, \pi)$

PROPERTIES OF INVERSE CIRCULAR FUNCTIONS:

Self adjusting property of trigonometric functions:

(i) $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1} x) = x$;
(ii) $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1} x) = x$;
(iii) $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1} x) = x$;
(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ and $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$;
(v) $\sec^{-1}(\sec\theta) = \theta$ and $\sec(\sec^{-1} x) = x$;
(vi) $\cot^{-1}(\cot\theta) = \theta$ and $\cot(\cot^{-1} x) = x$;

Also $\tan^{-1}(\tan\theta) = \theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\tan(\tan^{-1} x) = x$ where $x \in (-\infty, \infty)$.

Similarly other results can be obtained.

Principal values for $x \ge 0$:

$y = \sin^{-1} x$	$(0 \le x \le 1)$	$0 \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$(0 \le x \le 1)$	$0 \le y \le \frac{\pi}{2}$
$y = \tan^{-1} x$	$(0 \le x \le \infty)$	$0 \le y < \frac{\pi}{2}$
$y = \operatorname{cosec}^{-1} x$	$(1 \le x \le \infty)$	$0 < y \le \frac{\pi}{2}$
$y = \sec^{-1} x$	$(1 \le x \le \infty)$	$0 \le y < \frac{\pi}{2}$
$y = \cot^{-1} x$	$(0 \le x < \infty)$	$0 < y \le \frac{\pi}{2}$

Principal values of $x < 0$:

$y = \sin^{-1} x$	$(-1 \le x < 0)$	$-\frac{\pi}{2} \le y < 0$
$y = \cos^{-1} x$	$(-1 \le x < 0)$	$\frac{\pi}{2} < y \le \pi$
$y = \tan^{-1} x$	$(-\infty < x < 0)$	$\frac{-\pi}{2} < y < 0$
$y = \operatorname{cosec}^{-1} x$	$(-\infty < x \le -1)$	$\frac{-\pi}{2} \le y < 0$
$y = \sec^{-1} x$	$(-\infty < x \le -1)$	$\frac{\pi}{2} < y \le \pi$
$y = \cot^{-1} x$	$(-\infty < x < 0)$	$\frac{\pi}{2} < y < \pi$

Ex. 1. Write down the values of

(i) $\text{Sin}^{-1}\frac{1}{2}$ (ii) $\text{Cos}^{-1}\left(\frac{-1}{2}\right)$ (iii) $\tan^{-1}(-1)$

(iv) $\text{Sec}^{-1} 2$ (v) $\text{Cot}^{-1}\sqrt{3}$ (vi) $\text{Cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

Solution:

(i) Let $\text{Sin}^{-1}\frac{1}{2} = x \Rightarrow \text{Sin}\, x = \frac{1}{2} = \text{Sin}\frac{\pi}{6} = \text{Sin}\left(\pi - \frac{\pi}{6}\right) \therefore x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Since the principal value lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, therefore $x = \frac{\pi}{6}$ is the answer.

(ii) Let $x = \cos^{-1}\left(\frac{-1}{2}\right) \Rightarrow \cos x = \frac{-1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right)$

$\therefore \quad x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Since the principal value lies between 0 and π, therefore $x = \frac{2\pi}{3}$ is the answer.

(iii) Let $x = \tan^{-1}(-1) \Rightarrow \tan x = -1 = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right) = \tan\left(2\pi - \frac{\pi}{4}\right)$

$\therefore \quad x = \frac{-\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

Since the principal value lies between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, therefore $x = -\frac{\pi}{4}$ is the answer.

(iv) Let $x = \sec^{-1} 2 \Rightarrow \sec x = 2 = \sec\frac{\pi}{3} = \sec\left(2\pi - \frac{\pi}{3}\right)$

$\therefore \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$

Since the principal value lies between 0 and π, therefore $x = \frac{\pi}{3}$ is the answer.

(v) Let $x = \cot^{-1}\sqrt{3} \Rightarrow \cot x = \sqrt{3} = \cot\frac{\pi}{6} = \cot\left(\pi + \frac{\pi}{6}\right)$

$\therefore \quad x = \frac{\pi}{6}, \frac{7\pi}{6}$

Since the principal value lies between 0 and π, therefore $x = \frac{\pi}{6}$ is the answer.

(vi) Let $\text{cosec}^{-1}\left(\frac{-2}{\sqrt{3}}\right) = x \Rightarrow \text{cosec}\, x = \frac{-2}{\sqrt{3}} = -\text{cosec}\frac{\pi}{3} = \text{cosec}\left(\frac{-\pi}{3}\right) = \text{cosec}\left(\pi + \frac{\pi}{3}\right) = \text{cosec}\left(2\pi - \frac{\pi}{3}\right)$

$\therefore \quad x = \frac{-\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Since the principal value lies between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$, therefore $x = \frac{-\pi}{3}$ is the answer.

Ex. 2. Evaluate the following

(i) $\text{Sin}^{-1}\left(\text{Sin}\frac{5\pi}{6}\right)$ **(ii)** $\text{Cos}^{-1}\left(\text{Cos}\frac{9\pi}{8}\right)$

(iii) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution:

(i) $\text{Sin}^{-1}(\text{Sin}\theta) = \theta$ if $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$\text{Sin}\frac{5\pi}{6} = \text{Sin}\left(\pi - \frac{5\pi}{6}\right) = \text{Sin}\frac{\pi}{6}$

$\therefore \quad \text{Sin}^{-1}\left(\text{Sin}\frac{5\pi}{6}\right) = \text{Sin}^{-1}\left(\text{Sin}\frac{\pi}{6}\right) = \frac{\pi}{6}$

(ii) $\text{Cos}^{-1}(\text{Cos}\theta) = \theta$ if $0 \le \theta \le \pi$

$$\text{Cos}\frac{9\pi}{8} = \text{Cos}\left(2\pi - \frac{9\pi}{8}\right) = \text{Cos}\frac{7\pi}{8}$$

$$\text{Cos}^{-1}\left(\text{Cos}\frac{9\pi}{8}\right) = \text{Cos}^{-1}\left(\text{Cos}\frac{7\pi}{8}\right) = \frac{7\pi}{8}$$

(iii) $\tan^{-1}(\tan\theta) = \theta$ if $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$$\tan\frac{3\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)$$

$$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] = -\frac{\pi}{4}$$

General Values of inverse trigonometric functions:

Inverse trigonometric functions have an infinite number of values. For example.

if $\quad \text{Sin}\,\theta = \frac{1}{\sqrt{2}}$, then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{5}, \frac{7\pi}{4}$

$\Rightarrow \quad \text{Sin}^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

If f denotes any one of the trigonometric functions and $f^{-1}(x) = \alpha$, then the principal value of $f^{-1}(x)$ is the smallest numerical value of α.

In case of sine, if α is the smallest angle (principal value) whose sine is x, then all the angles whose sine is x can be written as $n\pi + (-1)^n\alpha$, $n \in I$.

Thus the general value of $\sin^{-1} x = n\pi + (-1)^n \alpha$.

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

We give below general values of all inverse trigonometric functions.

If $\sin\alpha = x$	$\sin^{-1} = n\pi + (-1)^n\alpha$	$\lvert x\rvert \le 1$	and $\frac{-\pi}{2} \le \alpha \le \frac{\pi}{2}$
If $\cos\alpha = x$	$\cos^{-1}x = 2n\pi \pm \alpha$	$\lvert x\rvert \le 1$	and $0 \le \alpha \le \pi$
If $\tan\alpha = x$	$\tan^{-1}x = n\pi + \alpha$	$x \in R$	and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$
If $\text{cosec}\,\alpha = x$	$\text{cosec}^{-1}x = n\pi + (-1)^n\alpha$	$\lvert x\rvert \ge 1$	and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}, \alpha \ne 0$
If $\sec\alpha = x$	$\sec^{-1}x = 2n\pi \pm \alpha$	$\lvert x\rvert \ge 1$	and $0 \le \alpha \le \pi, \alpha \ne \frac{\pi}{2}$
If $\cot\alpha = x$	$\cot^{-1}x = n\pi + \alpha$	$x \in R$	and $0 < \alpha < \pi$

What Inverse Trigonometric Functions do?

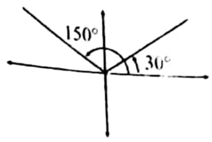
They take the value as input and give the angle as output.

$\text{Sin}30° = 0.5$

$30° = \text{Sin}^{-1}(0.5)$

$\text{Sin}30° = \text{Sin}150° = 0.5$

i-e $\boxed{\text{Sin}(\pi - \theta) = \text{Sin}\theta}$



$\text{Sin}\,\boxed{\text{ANGLE}} = \text{Value}$

↓

lies in a certain quadrant

Inverse Trigonometric Functions:

let $y = f(\theta)$ be a trigonometric function define in the set or real number R. i.e

$y = \{(\theta, y) \mid (\theta, y) \in R\}$ then

$f^{-1}(\theta) = \{(y, \theta) \mid (y, \theta) \in R \times R\}$

is called the inverse functions.

In other words in an inverse function f.

(i) The domain of f becomes the range of f^{-1}.

(ii) The range of f becomes the domain of f^{-1}.

Note: * $\text{Sin}^{-1}x$ = The inverse Sine f x

* arc Sinx = The arc Sine of x
* In $\text{Sin}^{-1}x$ (−1) is not the exponent
* $\text{Sin}^{-1}x \rightarrow$ means an angle whose sine is x
* It must be remembered that $\text{Sin}^{-1}x \ne (\text{Sinx})^{-1}$

Sum and Difference Formulae:

S.No.	FORMULAE
1	$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$
2	$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$
3	$\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Sin}^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$
4	$\text{Sin}^{-1}x - \text{Sin}^{-1}y = \text{Sin}^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$
5	$\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Cos}^{-1}[xy - \sqrt{(1-x^2)(1-y^2)}]$

6	$\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{(1-x^2)(1-y^2)}]$
7	$\sin^{-1}x + \sin^{-1}(-x) = 0$
8	$\text{Cosec}^{-1}x + \text{Cosec}^{-1}(-x) = 0$
9	$\cos^{-1}x + \cos^{-1}(-x) = \pi$
10	$\sec^{-1}x + \sec^{-1}(-x) = \pi$
11	$\tan^{-1}x + \tan^{-1}(-x) = 0$
12	$\cot^{-1}x + \cot^{-1}(-x) = \pi$
13	$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
14	$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
15	$\text{Cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

Domain and Range of Inverse Trigonometric Functions

Function	Domain	Range
arc Sin	$-1 \le \sin\theta \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
arc Cos	$-1 \le \cos\theta \le 1$	$0 \le \theta \le \pi$
arc tan	$\mathbb{R}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

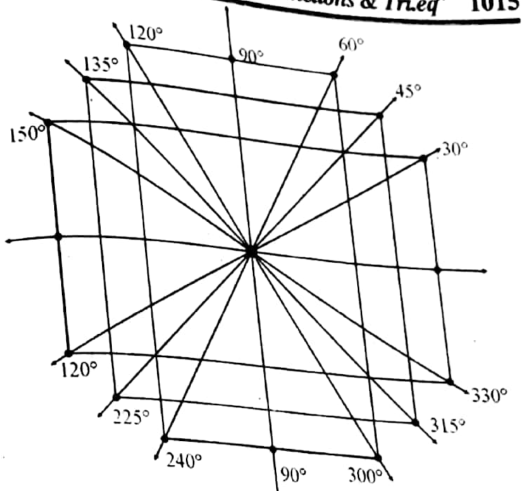
Trigonometric Equations: Equations involving trigonometric functions are called trigonometric equations.

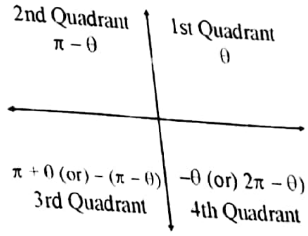
If the given equation can be readily expressed in terms of a Single function express it in terms of that function and solve the resulting equation considering this function as the unknown (نامعلوم) when the values of the function that satisfy the equation have been found. Find all the angles (positive) or zero less than 360°) for which the function has the prescribed values. After that we shall take general values by adding $2n\pi$ to each angle. The result will be Union of all these Sets of values.

Note:(1) $\sin^{-1}x = \theta, \pi - \theta$

(2) $\cos^{-1}x = \theta, -\theta$ (or) $\theta, 2\pi - \theta$

(3) $\tan^{-1}x = \theta, \pi + \theta$



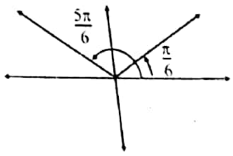


Example # 1: Solve the equation $\sin\theta = \frac{1}{2}$

Solution: $\sin\theta = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\boxed{\theta = \frac{\pi}{6}}$ and $\theta = \pi - \frac{\pi}{6}$

$\boxed{\theta = \frac{5\pi}{6}}$



$\therefore$ $\sin\theta$ is positive in I and II quadrants with the reference angle $\theta = \frac{\pi}{6}$

$$S.S = \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\} n \in Z$$

Example # 2: Solve the equation $1 + \cos\theta = 0$

Solution: $1 + \cos\theta = 0 \Rightarrow \cos\theta = -1$

$\theta = \cos^{-1}(-1) \Rightarrow \boxed{\theta = \pi}$

Since $\cos\theta$ is $-$ve there is only one solution $\theta = \pi$ in $[0, 2\pi]$

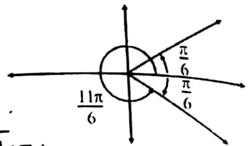
$S.S = \{\pi + 2n\pi\}\ n \in Z$

Example # 3: Solve the equation $4\cos^2\theta - 3 = 0$

Solution: $4\cos^2\theta - 3 = 0$

$\cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

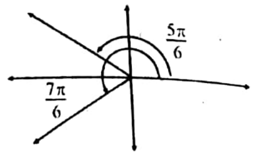
If $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$



$\boxed{\theta = \frac{\pi}{6}}$ and $\theta = 2\pi - \frac{\pi}{6} \Rightarrow \boxed{\theta = \frac{11\pi}{6}}$

Since $\cos\theta$ is +ve in I and IV quadrants with the reference angle $\theta = \frac{\pi}{6}$

If $\cos\theta = -\frac{\sqrt{3}}{2}$.



$\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \boxed{\theta = \frac{-\pi}{6}}$

$\theta = \pi - \frac{\pi}{6} \Rightarrow \boxed{\theta = \frac{5\pi}{6}}$ and $\theta = \pi + \frac{\pi}{6} \Rightarrow \boxed{\theta = \frac{7\pi}{6}}$

Since $\cos\theta$ is $-$ve in II and III quadrants with reference angle $\theta = \frac{\pi}{6}$

$$S.S = \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{11\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\} \cup \left\{\frac{7\pi}{6} + 2n\pi\right\}\ n \in Z \quad \text{Ans.}$$

Example # 4: Solve $\sin\theta + \cos\theta = 0$

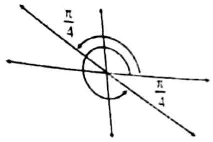
Solution: $\sin\theta + \cos\theta = 0$ ($\div$ b.s by $\cos\theta$)

$\frac{\sin\theta}{\cos\theta} + \frac{\cancel{\cos\theta}}{\cancel{\cos\theta}} = 0 \Rightarrow \tan\theta + 1 = 0$

$\tan\theta = -1 \Rightarrow \theta = \tan^{-1}(-1) \Rightarrow \boxed{\theta = \frac{-\pi}{4}}$

Since $\tan\theta$ is $-$ve in II and IV quadrants with the reference angle $\theta = \frac{\pi}{4}$

$\theta = \pi - \frac{\pi}{4} \Rightarrow \boxed{\theta = \frac{3\pi}{4}}$



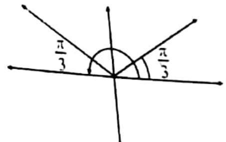
$\theta = 2\pi - \frac{\pi}{4} \Rightarrow \boxed{\theta = \frac{7\pi}{4}}$

$$S.S = \left\{\frac{3\pi}{4} + n\pi\right\} \cup \left\{\frac{7\pi}{4} + n\pi\right\} n \in Z$$

Example # 5: Solve $\sin\theta\cos\theta = \frac{\sqrt{3}}{4}$

Solution: $\sin\theta\cos\theta = \frac{\sqrt{3}}{4}$

$\frac{1}{2}(2\sin\theta\cos\theta) = \frac{\sqrt{3}}{4}$



$\sin 2\theta = \frac{\sqrt{3}}{2} \Rightarrow 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\because$ $\sin 2\theta$ is + ve in I and II quadrants with the reference angle $2\theta = \frac{\pi}{3}$

$2\theta = \frac{\pi}{3} \Rightarrow \boxed{\theta = \frac{\pi}{6}}$ and $2\theta = \pi - \frac{\pi}{3} \Rightarrow 2\theta = \frac{2\pi}{3} \Rightarrow \boxed{\theta = \frac{\pi}{3}}$

General values of 2θ are $\frac{\pi}{3} + 2n\pi$ and $\frac{2\pi}{3} + 2n\pi$, $n \in Z$.

General values of θ are $\frac{\pi}{6} + n\pi$ and $\frac{\pi}{3} + n\pi$, $n \in Z$.

$$S.S = \left\{\frac{\pi}{6} + n\pi\right\} \cup \left\{\frac{\pi}{3} + n\pi\right\}, n \in Z$$

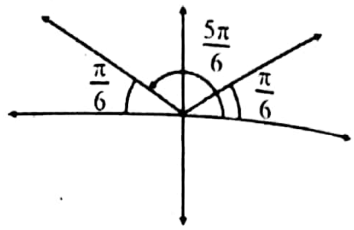
Example # 6: **Solve Sin2x = Cosx**

Solution: $\text{Sin}2x = \text{Cos}x \Rightarrow 2\text{Sin}x\ \text{Cos}x - \text{Cos}x = 0$

$\text{Cos}x\,(2\text{Sin}x - 1) = 0$

Either

$\text{Cos}x = 0 \Rightarrow x = \text{Cos}^{-1}(0)$



$\boxed{x = \frac{\pi}{2}}$ and $\boxed{x = \frac{3\pi}{2}}$

$2\text{Sin}x - 1 = 0 \Rightarrow \text{Sin}x = \frac{1}{2} \Rightarrow x = \text{Sin}^{-1}\left(\frac{1}{2}\right)$

$\boxed{x = \frac{\pi}{6}}$

Since Sinx is +ve in I and II quadrants with the reference angle $x = \frac{\pi}{6} \Rightarrow x = \pi - \frac{\pi}{6} \Rightarrow \boxed{x = \frac{5\pi}{6}}$

$$\textbf{S.S} = \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi\right\}\ n \in z$$

Ans.

Example # 7: **Solve $\text{Sin}^2x + \text{Cos}x = 1$**

Solution: $\text{Sin}^2x + \text{Cos}x = 1$

$\cancel{1} - \text{Cos}^2x + \text{Cos}x = \cancel{1}$

$-\text{Cos}^2x + \text{Cos}x = 0 \Rightarrow -\text{Cos}x\,(\text{Cos}x - 1) = 0$

Either

$-\text{Cos}x = 0 \Rightarrow \text{Cos}x = 0 \Rightarrow x = \text{Cos}^{-1}(0)$

$\boxed{x = \frac{\pi}{2}}$ and $\boxed{x = \frac{3\pi}{2}}$

$\text{Cos}x - 1 = 0 \Rightarrow \text{Cos}x = 1 \Rightarrow x = \text{Cos}^{-1}(1)$

$\boxed{x = 0}$ and $\boxed{x = 2\pi}$

$$\textbf{S.S} = \{2n\pi\} \cup \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \{2\pi + 2n\pi\}\ n \in Z$$

Ans.

Exercise No.=13.1

Q5. $Sin(arcCos\frac{\sqrt{3}}{2}+arcSin\frac{1}{2})$

Let $arcCos\frac{\sqrt{3}}{2}=\alpha$ $;Cos\alpha=\frac{\sqrt{3}}{2},Cos^2\alpha=\frac{3}{4}$

$Sin^2\alpha=1-Cos^2\alpha=1-\frac{3}{4}=\frac{1}{4}$ $;Sin^2\alpha=\frac{1}{4},Sin\alpha=\pm\frac{1}{2}$

$Sin\alpha=\frac{1}{2}$ **Let** $arcSin\frac{1}{2}=\beta$

$Sin\beta=\frac{1}{2};Sin^2\beta=\frac{1}{4};Cos^2\beta=1-\frac{1}{4}=\frac{3}{4};Cos\beta=\frac{\sqrt{3}}{2}$

$\left(Sinarc\frac{\sqrt{3}}{2}+\text{arc }Sin\frac{1}{2}\right)=Sin(\alpha+\beta)$

$Sin(\alpha+\beta)=Sin\alpha Cos\beta+Cos\alpha Sin\beta$

$=\frac{1}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\times\frac{1}{2}=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}$

$=\frac{\sqrt{3}+\sqrt{3}}{4}=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$

Q6. $Sin(arcCos\theta+arcSin\theta)$

Let $arcCos\theta=\alpha,Cos\alpha=\theta$

$Sin^2\theta=1-Cos^2\theta=1-\theta^2$

$Sin\alpha=\sqrt{1-\theta^2}$

Let $arcSin\theta=\beta,Sin\beta=\theta$

$Cos^2\beta=1-Sin^2\beta=1-\theta^2$ $;Cos\beta=\sqrt{1-\theta^2}$

$Sin(arcCos\theta+arcSin\theta)=Sin(\alpha+\beta)$

$Sin(\alpha+\beta)=Sin\alpha Cos\beta+Cos\alpha Sin\beta$

$=(\sqrt{1-\theta^2})(\sqrt{1-\theta^2})+(\theta)(\theta)=1-\theta^2+\theta^2=1.$

Q10. $\tan^{-1}\theta+Cot^{-1}\theta=\frac{\pi}{2}$

Let $\tan^{-1}\theta=\alpha,\tan\alpha=\theta$

Let $Cot^{-1}\theta=\beta,Cot\beta=\theta,\frac{1}{\tan\beta}=\theta,\tan\beta=\frac{1}{\theta}$

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\theta+\frac{1}{\theta}}{1-\theta.\frac{1}{\theta}}$

$=\frac{\theta+\frac{1}{\theta}}{0}=\infty$ $,\alpha+\beta=\tan^{-1}\infty=\frac{\pi}{2}$ *So* $\tan^{-1}\theta+Cot^{-1}\theta=\frac{\pi}{2}$

Q12. $Sin^{-1}\theta=Cos^{-1}\sqrt{1-\theta^2}$

$Sin^{-1}\theta=\alpha;Sin\alpha=\theta;Sin^2\alpha=\theta^2$ $;1-Sin^2\alpha=1-\theta^2$

$;Cos^2\alpha=1-\theta^2$ $;Cos\alpha=\sqrt{1-\theta^2}$

$;Cos^{-1}\sqrt{1-\theta^2}=\alpha$ $;Sin^{-1}\alpha=Cos^{-1}\sqrt{1-\theta^2}$

Q11. $\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{1}{3}$

Let $\tan^{-1}\frac{1}{13}=\alpha,\tan\alpha=\frac{1}{13}$

Let $\tan^{-1}\frac{1}{4}=\beta,\tan\beta=\frac{1}{4}$

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{13}+\frac{1}{4}}{1-\frac{1}{13}\times\frac{1}{4}}=\frac{\frac{4+13}{52}}{\frac{51}{52}}=\frac{17}{51}=\frac{1}{3}$

$\tan(\alpha+\beta)=\frac{1}{3},\alpha+\beta=\tan^{-1}\frac{1}{3};$ $\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{1}{3}$

Q13. $\tan^{-1}\frac{1}{3}+\frac{1}{2}\tan^{-1}\frac{1}{7}=\frac{\pi}{8}$

Let $\tan^{-1}\frac{1}{3}=\alpha,\tan\alpha=\frac{1}{3}$ $;\frac{1}{2}\tan^{-1}\frac{1}{7}=\beta$ $;\tan^{-1}\frac{1}{7}=2\beta$

$\tan2\beta=\frac{1}{7};\tan\beta=?$ $;\tan2\beta=\frac{2\tan\beta}{1-\tan^2\beta};\frac{1}{7}=\frac{2\tan\beta}{1-\tan^2\beta}$

$1-\tan^2\beta=14\tan\beta(CM)$

$\tan^2\beta+14\tan\beta-1=0$

$a=1,b=14,c=-1;$ $\tan\beta=\frac{-14\pm\sqrt{14^2-4\times1(-1)}}{2(1)}=\frac{-14\pm\sqrt{196+4}}{2}=\frac{-14\pm\sqrt{200}}{2}$

$=\frac{-14\pm10\sqrt{2}}{2}=2\left(\frac{-7\pm5\sqrt{2}}{2}\right)=-7\pm5\sqrt{2}=.071;\tan\beta=.071$

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{3}+.071}{1-\frac{1}{3}(.071)}=\frac{.33+.071}{1-.023}$

$\tan(\alpha+\beta)=.4137;\tan^{-1}.4137=22.47^\circ=22.5^\circ;\frac{225}{10}\times\frac{\pi}{180}=\frac{\pi}{8}$

Q14. $\tan^{-1}\theta=Sin^{-1}\frac{\theta}{\sqrt{1-\theta^2}}$

Let $\tan^{-1}\theta=\alpha,\tan\alpha=\theta,\tan^2\alpha=\theta^2$ $;1+\tan^2\alpha=1+\theta^2$

$;Sec^2\alpha=1+\theta^2;\frac{1}{Cos^2\alpha}=1+\theta^2;Cos^2\alpha=\frac{1}{1+\theta^2}$ $;1-Cos^2\alpha=1-\frac{1}{1+\theta^2}$

$=Sin^2\alpha=\frac{1+\theta^2-1}{1+\theta^2}=\frac{\theta^2}{1+\theta^2}$

$;Sin\alpha=\sqrt{\frac{\theta^2}{1+\theta^2}}=\frac{\theta}{\sqrt{1+\theta^2}}$ $;\alpha=Sin^{-1}\frac{\theta}{\sqrt{1+\theta^2}}$

But $\alpha=\tan^{-1}\theta\therefore\tan^{-1}\theta=Sin^{-1}\frac{\theta}{\sqrt{1+\theta^2}}$

Exercise No.=13.2

Q1. $Sin\theta + Cos\theta = 1$

$Cos\theta = 1 - Sin\theta$

$Cos^2\theta = (1 - Sin\theta)^2$

$1 - Sin^2\theta = 1 - 2Sin\theta + Sin^2\theta$

$= 2Sin^2\theta - 2Sin\theta + 1 - 1 = 0$

$= 2Sin^2\theta - 2Sin\theta = 0$

$= 2Sin\theta(Sin\theta - 1) = 0$

Either $Sin\theta = 0, (Sin\theta - 1) = 0, Sin\theta = 1$

If $Sin\theta = 0 \Rightarrow \theta = 0, \pi$

If $Sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

Solution Set: $\{2n\pi\} \cup \left\{\frac{\pi}{2} + 2n\pi\right\}$.

Q3. $\tan^2\theta + \tan\theta = 2$

$\tan^2\theta + \tan\theta - 2 = 0$

$\tan^2\theta + 2\tan\theta - \tan\theta - 2 = 0$

$= \tan\theta(\tan\theta + 2) - 1(\tan\theta + 2) = 0$

$= (\tan\theta + 2)(\tan\theta - 1) = 0$

$\tan\theta + 2 = 0$ or $\tan\theta - 1 = 0$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$, \tan\theta = -2; \theta = \tan^{-1}(-2)$

$\left(n\pi + \frac{\pi}{4}\right) \cup \left(n\pi + \tan^{-1}(-2)\right)$

Q4. Cos [illegible] $\theta + 1 = 0$

$(2Cos^2\theta - 1) + Cos\theta + 1 = 0$

$2Cos^2\theta - 1 + Cos\theta + 1 = 0$

$= 2Cos^2\theta + Cos\theta = 0$

$Cos\theta(2Cos\theta + 1) = 0$

Either $Cos\theta = 0, 2Cos\theta + 1 = 0$

$Cos\theta = 0, Cos\theta = \frac{-1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

$\left(n\pi + \frac{\pi}{2}\right) \cup \left(2n\pi \pm \frac{2\pi}{3}\right)$.

$Q5).\ 3Sin^2\theta - Sin\theta = \frac{1}{4}$

$12Sin^2\theta - 4Sin\theta = 1; 12Sin^2\theta - 4Sin\theta - 1 = 0$

$12Sin^2\theta - 6Sin\theta + 2Sin\theta - 1 = 0$

$6Sin\theta(2Sin\theta - 1) + 1(2Sin\theta - 1) = 0$

$(2Sin\theta - 1)(6Sin\theta + 1) = 0$

Either $2Sin\theta - 1 = 0, 6Sin\theta + 1 = 0$

$2Sin\theta = 1, 6Sin\theta = -1$; $Sin\theta = \frac{1}{2}, Sin\theta = -\frac{1}{6}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$; $\theta = Sin^{-1}\left(-\frac{1}{6}\right)$ **So** $\left(2n\pi + \frac{\pi}{6}\right) \cup \left(2n\pi + \frac{5\pi}{6}\right) \cup \left(2n\pi + Sin^{-1}(-\frac{1}{6})\right)$.

Q6. $Sin3\theta - Sin\theta = 0$

Formula: $-SinA - SinB = 2Cos\frac{A+B}{2}Sin\frac{A-B}{2}$

$= 2Cos\frac{3\theta+\theta}{2}Sin\frac{3\theta-\theta}{2} = 2Cos\frac{4\theta}{2}Sin\frac{2\theta}{2} = 0$

$= 2Cos2\theta Sin\theta = 0$; *Formula*: $-Cos2\theta = 1 - 2Sin^2\theta$

$= (1 - 2Sin^2\theta)Sin\theta = 0$; $1 - 2Sin^2\theta = 0$ or $Sin\theta = 0$

$Sin\theta = 0 \Rightarrow \theta = 0, \pi$; $Sin^2\theta = \frac{1}{2}, Sin\theta = \pm\frac{1}{\sqrt{2}}$; $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

$n\pi \cup \left(n\pi + \frac{\pi}{4}\right) \cup \left(n\pi + \frac{3\pi}{4}\right)$.

Q7. $Sin^2\theta - 1 = Cos\theta - Cos\theta Sin^2\theta$

$Sin^2\theta + Cos\theta Sin^2\theta - 1 - Cos\theta = 0$

$Sin^2\theta(1 + Cos\theta) - 1(1 + Cos\theta) = 0$

$(1 + Cos\theta)(Sin^2\theta - 1) = 0$

Either $1 + Cos\theta = 0, Sin^2\theta - 1 = 0$

$Cos\theta = -1, Sin^2\theta = 1$

$Cos\theta = -1, Sin\theta = \pm 1$ $\quad \theta = 2n\pi + \pi = (2n+1)\pi, \theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$\left\{2n\pi \pm \frac{\pi}{2}\right\} \cup \{(2n+1)\pi\}$

$\theta = \frac{\pi}{2} \longrightarrow \left(\sin\frac{\pi}{2}\right)^2 - 1 = \cos\frac{\pi}{2} - \cos\frac{\pi}{2}\left(\sin\frac{\pi}{2}\right)^2$

$(1)^2 - 1 = 0 - (0)(1)^2$

$\cancel{1} - \cancel{1} = 0 \Rightarrow \boxed{0 = 0}$ (Verified)

$\theta = \frac{3\pi}{2} \longrightarrow \left(\sin\frac{3\pi}{2}\right)^2 - 1 = \cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{3\pi}{2}\right)\left(\sin\frac{3\pi}{2}\right)^2$

$(-1)^2 - 1 = 0 - (0)(-1)^2$

$\cancel{1} - \cancel{1} = 0 \Rightarrow \boxed{0 = 0}$ (Verified)

S.S = $\{\pi + 2n\pi\} \cup \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\}$ n∈z Ans.

(8) $\sec^4\theta - \tan^4\theta = 3$

Solution:

$\sec^4\theta - \tan^4\theta = 3$

$(\sec^2\theta)^2 - (\tan^2\theta)^2 = 3$

$\boxed{\therefore\ a^2 - b^2 = (a-b)(a+b)}$

$(\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta) = 3$

$\boxed{\therefore 1 + \tan^2\theta = \sec^2\theta}$

$(1 + \tan^2\theta + \tan^2\theta)(1 + \cancel{\tan^2\theta} - \cancel{\tan^2\theta}) = 3$

$1 + 2\tan^2\theta = 3 \Rightarrow 2\tan^2\theta = 3 - 1 \Rightarrow \cancel{2}\tan^2\theta = \cancel{2}$

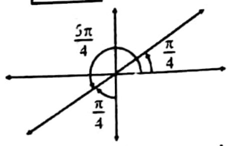
$\tan^2\theta = 1 \Rightarrow \tan\theta = \pm 1$

Either

$\tan\theta = 1$

$\theta = \tan^{-1}(1)$

$\boxed{\theta = \frac{\pi}{4}}$

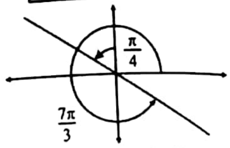


Since $\tan\theta$ is +ve in I and III quadrants with reference angle $\theta = \frac{\pi}{4}$,

$\tan\theta = -1$

$\theta = \tan^{-1}(-1)$

$\boxed{\theta = -\frac{\pi}{4}}$



Since $\tan\theta$ is −ve in II and IV quadrants with reference angle $\theta = \frac{\pi}{4}$

$\theta = \pi + \frac{\pi}{4} \Rightarrow \boxed{\theta = \frac{5\pi}{4}}$ | $\theta = 2\pi - \frac{\pi}{4} \Rightarrow \boxed{\theta = \frac{7\pi}{4}}$

Verification:

$\theta = \frac{\pi}{4} \longrightarrow \left(\sec\frac{\pi}{4}\right)^4 - \left(\tan\frac{\pi}{4}\right)^4 = 3$

$(\sqrt{2})^4 - (1)^4 = 3 \Rightarrow 4 - 1 = 3$

$\boxed{3 = 3}$ (Verified)

$\theta = \frac{7\pi}{4} \longrightarrow \left\{\sec\left(\frac{7\pi}{4}\right)\right\}^4 - \left\{\tan\left(\frac{7\pi}{4}\right)\right\}^4 = 3$

$(\sqrt{2})^4 - (-1)^4 = 3 \Rightarrow 4 - 1 = 3$

$\boxed{3 = 3}$ (Verified)

$\theta = \frac{5\pi}{4} \longrightarrow \left\{\sec\frac{5\pi}{4}\right\}^4 - \left\{\tan\frac{5\pi}{4}\right\}^4 = 3$

$\{-\sqrt{2}\}^4 - \{1\}^4 = 3 \Rightarrow 4 - 1 = 3$

$\boxed{3 = 3}$ (Verified)

S.S = $\left\{\frac{\pi}{4} + n\pi\right\} \cup \left\{\frac{7\pi}{4} + n\pi\right\} \cup \left\{\frac{5\pi}{4} + n\pi\right\}$ n∈Z Ans.

(9) $2^{\cos\theta} = 1$

Solution:

$2^{\cos\theta} = 1 \Rightarrow 2^{\cos\theta} = 2^\circ$

Since base are same powers are equal

$\cos\theta = 0 \Rightarrow \theta = \cos^{-1}(0)$

$\boxed{\theta = \frac{\pi}{2}}$ (or) $\boxed{\theta = \frac{3\pi}{2}}$

Verification:

$\theta = \frac{\pi}{2} \Rightarrow 2^{\cos\frac{\pi}{2}} = 1 \Rightarrow 2^\circ = 1$

$\boxed{1 = 1}$ (Verified)

$\theta = \frac{3\pi}{2} \Rightarrow 2^{\cos\left(\frac{3\pi}{2}\right)} = 1 \Rightarrow 2^\circ = 1$

$\boxed{1 = 1}$ (Verified)

$$S.S=\left\{\frac{\pi}{2}+2n\pi\right\} \cup \left\{\frac{3\pi}{2}+2n\pi\right\} \quad n\in Z \text{ Ans.}$$

(10) $\dfrac{\text{Sin}\theta+\text{Cos}\theta}{\text{Sin}\theta-\text{Cos}\theta}=1$

Solution:

$$\frac{\text{Sin}\theta+\text{Cos}\theta}{\text{Sin}\theta-\text{Cos}\theta}=1 \Rightarrow \cancel{\text{Sin}\theta}+\text{Cos}\theta=\cancel{\text{Sin}\theta}-\text{Cos}\theta$$

$$\text{Cos}\theta+\text{Cos}\theta=0 \Rightarrow 2\text{Cos}\theta=0$$

$$\text{Cos}\theta=0 \Rightarrow \theta=\text{Cos}^{-1}(0)$$

$$\boxed{\theta=\frac{\pi}{2}} \text{ (or) } \boxed{\theta=\frac{3\pi}{2}}$$

Verification:

$$\theta=\frac{\pi}{2} \longrightarrow \frac{\text{Sin}\left(\frac{\pi}{2}\right)+\text{Cos}\left(\frac{\pi}{2}\right)}{\text{Sin}\left(\frac{\pi}{2}\right)-\text{Cos}\left(\frac{\pi}{2}\right)}=1$$

$$\boxed{1=1} \quad \text{(Verified)}$$

$$\theta=\frac{3\pi}{2} \longrightarrow \frac{\text{Sin}\left(\frac{3\pi}{2}\right)+\text{Cos}\left(\frac{3\pi}{2}\right)}{\text{Sin}\left(\frac{3\pi}{2}\right)-\text{Cos}\left(\frac{3\pi}{2}\right)}=1$$

$$\boxed{1=1} \quad \text{(Verified)}$$

$$S.S=\left\{\frac{\pi}{2}+2n\pi\right\} \cup \left\{\frac{3\pi}{2}+2n\pi\right\} \quad n\in Z \text{ Ans.}$$

(11) $\sqrt{3}\,\text{tanx}-\text{Secx}-1=0$

Solution:

$$\sqrt{3}\,\text{tanx}-\text{Secx}-1=0$$

$$\sqrt{3}\left(\frac{\text{Sinx}}{\text{Cosx}}\right)-\frac{1}{\text{Cosx}}=1 \Rightarrow \frac{\sqrt{3}\,\text{Sinx}-1}{\text{Cosx}}=1$$

$$\sqrt{3}\,\text{Sinx}-1=\text{Cosx}$$

$$\sqrt{3}\,\text{Sinx}=1+\text{Cosx}$$

Squaring on b.s

$$(\sqrt{3}\,\text{Sinx})^2=(\text{Cosx}+1)^2$$

$$3\text{Sin}^2x=1+2\text{Cosx}+\text{Cos}^2x$$

$$3(1-\text{Cos}^2x)=1+2\text{Cosx}+\text{Cos}^2x$$

$$3-3\text{Cos}^2x=1+2\text{Cosx}+\text{Cos}^2x$$

$$3\text{Cos}^2x+\text{Cos}^2x+2\text{Cosx}+1-3=0$$

$$4\text{Cos}^2x+2\text{Cosx}-2=0$$

$$2(2\text{Cos}^2x+\text{Cosx}-1)=0$$

$$2\text{Cos}^2x+\text{Cosx}-1=0$$

$$2\text{Cos}^2x+2\text{Cosx}-\text{Cosx}-1=0$$

$$2\text{Cosx}(\text{Cosx}+1)-1(\text{Cosx}+1)=0$$

$$(\text{Cosx}+1)(2\text{Cosx}-1)=0$$

Either

$$\text{Cosx}+1=0$$

$$\text{Cosx}=-1$$

$$x=\text{Cos}^{-1}(-1)$$

$$\boxed{x=\pi}$$

Since Cosx is −ve, there is only one solution

$$2\text{Cosx}-1=0$$

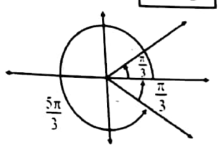
$$2\text{Cosx}=1$$

$$\text{Cosx}=\frac{1}{2}$$

$$x=\text{Cos}^{-1}\left(\frac{1}{2}\right)$$

Since Cosx is +ve in I & IV quadrants

$$x=2\pi-\frac{\pi}{3} \Rightarrow \boxed{x=\frac{5\pi}{3}}$$



Verification:

$$x=\pi \longrightarrow \sqrt{3}\,\text{tan}\pi-\text{Sec}\,\pi-1=0$$

$$\sqrt{3}(0)-(-1)-1=0 \Rightarrow \cancel{1}-\cancel{1}=0$$

$$\boxed{0=0} \quad \text{(Verified)}$$

$$x=\frac{\pi}{3} \longrightarrow \sqrt{3}\,\text{tan}\frac{\pi}{3}-\text{Sec}\frac{\pi}{3}-1=0$$

$$3-2-1=0 \Rightarrow \cancel{1}-\cancel{1}=0$$

$$\boxed{0=0} \quad \text{(Verified)}$$

Q8. $Sec^4\theta - \tan^4\theta = 3$

$= (Sec^2\theta)^2 - \tan^4\theta = 3$

$= (1+\tan^2\theta)^2 - \tan^4\theta = 3$

$= 1 + 2\tan^2\theta + \tan^4\theta - \tan^4\theta = 3$

$= 1 + 2\tan^2\theta - 3 = 0, = 2\tan^2\theta - 2 = 0$

$= 2(\tan^2\theta - 1) = 0; \tan^2\theta = 1; \tan\theta = \pm 1$

$\therefore \theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$ *(Particular Solution)*

General Solution $\left(n\pi + \frac{\pi}{4}\right)\left(n\pi - \frac{\pi}{4}\right)$

Q9. $2^{Cos\theta} = 1 \quad = 2^{Cos\theta} = 2^0 \; ; Cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = \frac{\pi}{2}, -\frac{\pi}{2} \quad ; \left(2n\pi \pm \frac{\pi}{2}\right)$

Q10. $\frac{Sin\theta + Cos\theta}{Sin\theta - Cos\theta} = 1$

$Sin\theta + Cos\theta = Sin\theta - Cos\theta (CM)$

$Sin\theta - Sin\theta + Cos\theta + Cos\theta = 0$

$2Cos\theta = 0; Cos\theta = 0; \theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$\left(2n\pi \pm \frac{\pi}{2}\right)$

Q11. $\sqrt{3}\tan x - Secx - 1 = 0$

$\sqrt{3}\frac{Sinx}{Cosx} - \frac{1}{Cosx} - 1 = 0$

$\frac{\sqrt{3}Sinx - 1 - Cosx}{Cosx} = 0; \sqrt{3}Sinx - 1 - Cosx = 0$

$\sqrt{3}Sinx = Cosx + 1 ; \left(\sqrt{3}Sinx\right)^2 = (Cosx+1)^2$

$= 3Sin^2x = 1 + 2Cosx + Cos^2x$

$= 3(1 - Cos^2x) = 1 + 2Cosx + Cos^2x$

$= 3 - 3Cos^2x = 1 + 2Cosx + Cos^2x$

$= 1 + 2Cosx + Cos^2x - 3 + 3Cos^2x = 0$

$4Cos^2x + 2Cosx - 2 = 0 ; 2Cos^2x + Cosx - 1 = 0$

$= 2Cos^2x + 2Cosx - Cosx - 1 = 0$

$= 2Cosx(Cosx+1) - 1(Cosx+1) = 0$

$= (Cosx+1)(2Cosx-1) = 0$

Either $Cosx + 1 = 0$ or $2Cosx - 1 = 0$

$Cosx = -1$ or $Cosx = \frac{1}{2}; x = \pi$ or $x = \frac{\pi}{3}, -\frac{\pi}{3}$

$x = -\frac{\pi}{3}$ *Does not satisfy the equation*

So $(2n\pi + \pi) \cup \left(2n\pi + \frac{\pi}{3}\right)$.

Q12. $\sqrt{Cos\theta\sqrt{Cos\theta}\sqrt{Cos\theta}\sqrt{Cos\theta}\ldots\ldots\ldots} = 1$

After repeated squaring $Cos\theta = 1$

$\theta = 0; (2n\pi + 0)$

Q13. $Cos\theta - 2Sin\theta = 0 \quad ; Cos\theta = 2Sin\theta$

$\frac{Sin\theta}{Cos\theta} = \frac{1}{2}; \tan\theta = \frac{1}{2}; \theta = \tan^{-1}\frac{1}{2}; \left\{n\pi + \tan^{-1}\frac{1}{2}\right\}$

Q15. $4Sin^2\theta\tan\theta + 4Sin^2\theta - 3\tan\theta - 3 = 0$

$= 4Sin^2\theta(\tan\theta + 1) - 3(\tan\theta + 1) = 0$

$= (\tan\theta + 1)(4Sin^2\theta - 3) = 0$

Either $\tan\theta + 1 = 0; 4Sin^2\theta - 3 = 0$

$\tan\theta = -1; Sin^2\theta = \frac{3}{4} = \tan\theta = -1; Sin\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = -\frac{\pi}{4}, \frac{3\pi}{4}; Sin\theta = \frac{\sqrt{3}}{2}; \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

$Sin\theta = \frac{-\sqrt{3}}{2}; \theta = \frac{-\pi}{3}, \frac{-2\pi}{3}, \frac{5\pi}{3}, \frac{4\pi}{3}$

$\left\{n\pi - \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{3\pi}{4}\right\} \cup \left\{2n\pi + \frac{\pi}{3}\right\} \cup \left\{2n\pi + \frac{2\pi}{3}\right\} \cup \left\{2n\pi + \frac{4\pi}{3}\right\} \cup \left\{2n\pi + \frac{5\pi}{3}\right\}$.

Q14. $Sin3\theta - Sin2\theta - Sin\theta = 0$

$[Sin3\theta - Sin\theta] - Sin2\theta = 0$

Formula :$-SinA - SinB = 2Cos\frac{A+B}{2}Sin\frac{A-B}{2}$

$= 2Cos\frac{3\theta+\theta}{2}Sin\frac{3\theta-\theta}{2} - Sin2\theta = 0$

$= 2Cos\frac{4\theta}{2}Sin\frac{2\theta}{2} - Sin2\theta = 0$

$= 2Cos2\theta Sin\theta - 2Sin\theta Cos\theta = 0$

$= 2Sin\theta(Cos2\theta - Cos\theta) = 0$

$= Sin\theta(Cos2\theta - Cos\theta) = 0$

Either $Sin\theta = 0; Cos2\theta - Cos\theta = 0$

$\theta = 0, \pi$; $Cos2\theta - Cos\theta = 0$

$2Cos^2\theta - 1 - Cos\theta = 0$; $2Cos^2\theta - Cos\theta - 1 = 0$

$= 2Cos^2\theta - 2Cos\theta + Cos\theta - 1 = 0$

$= 2Cos\theta(Cos\theta - 1) + 1(Cos\theta - 1) = 0$

$= (Cos\theta - 1)(2Cos\theta + 1) = 0$

Either $Cos\theta - 1 = 0$ *or* $2Cos\theta + 1 = 0$

$Cos\theta = 1$ *or* $Cos\theta = -\frac{1}{2}$

$\theta = 0, \pi$ $\theta = 0^\circ$ $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\{n\pi\} \cup \left\{2n\pi + \frac{2\pi}{3}\right\} \cup \left\{2n\pi + \frac{4\pi}{3}\right\}$.

Q16. $\tan 2\theta Cot\theta = 3$; $\tan 2\theta \times \frac{1}{\tan\theta} = 3$

$\tan 2\theta = 3\tan\theta$; $\frac{2\tan\theta}{1-\tan^2\theta} = 3\tan\theta$

$= 2\tan\theta = 3\tan\theta(1-\tan^2\theta)$ (CM)

$= 2\tan\theta = 3\tan\theta - 3\tan^3\theta$

$= 3\tan^3\theta + 2\tan\theta - 3\tan\theta = 0$; $3\tan^3\theta - \tan\theta = 0$

$= \tan\theta(3\tan^2\theta - 1)$ **Either** $\tan\theta = 0$ or $3\tan^2\theta - 1 = 0$

$\theta = 0, \pi$; $\tan^2\theta = \frac{1}{3} = \tan\theta = \pm\frac{1}{\sqrt{3}}$

Does not satisfy ; $\theta = \frac{\pi}{6}, \frac{-\pi}{6}, \pm\frac{5\pi}{6}$

So $\left\{2n\pi \pm \frac{\pi}{6}\right\} \cup \left\{2n\pi \pm \frac{5\pi}{6}\right\}$

Q2. $\sqrt{3}Cos\theta + Sin\theta - 2 = 0$; $\sqrt{3}Cos\theta = 2 - Sin\theta$

$(\sqrt{3}Cos\theta)^2 = (2 - Sin\theta)^2$

$= 3Cos^2\theta = 4 - 4Sin\theta + Sin^2\theta$

$= 3(1 - Sin^2\theta) = 4 - 4Sin\theta + Sin^2\theta$

$= 3 - 3Sin^2\theta = 4 - 4Sin\theta + Sin^2\theta$; $4Sin^2\theta - 4Sin\theta + 1 = 0$

$(2Sin\theta - 1)(2Sin\theta - 1) = 0; (2Sin\theta - 1)^2 = 0$

$= 2Sin\theta - 1 = 0; Sin\theta = \frac{1}{2}$; $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

But $\frac{5\pi}{6}$ *does not satisfy* *So* $\left\{2n\pi + \frac{\pi}{6}\right\}$

EXAMPLES FROM THE TEXT BOOK FOR TRIGNOMETRIC SECTION.

(1)If $Sin\theta = \frac{4}{5}$ **and** $\rho(\theta)$ **is in 1st quadrant, find the remaining trigonometric functions.**

$Sin\theta = y = \frac{4}{5}$ *Now* $x^2 + y^2 = 1, x^2 + \left(\frac{4}{5}\right)^2 = 1, x^2 = 1 - \frac{16}{25} = \frac{9}{25}$ *so* $x = \pm\frac{3}{5}$.

Therefore $\rho(\theta)$ **is in 1st quadrant, So x>0 i.e** $x = \frac{3}{5}$ **thus** $Cos\theta = x = \frac{3}{5}$.

$\tan\theta = \frac{y}{x} = \frac{4}{5} \times \frac{5}{3} = \frac{4}{3}, Cot\theta = \frac{1}{\tan\theta} = \frac{3}{4}. Sec\theta = \frac{1}{Cos\theta} = \frac{5}{3}$, *and* $Co\sec\theta = \frac{1}{Sin\theta} = \frac{5}{4}$.

(2)If $\tan\theta = -\frac{3}{4}$ **and** $p(\theta)$ **is in 2nd quadrant, Find remaining trigonometric functions.**

$\tan\theta = \frac{y}{x} = -\frac{3}{4} \Rightarrow y = -\frac{3}{4}x$ **Since** $x^2 + y^2 = 1$.

$x^2 + \left(\frac{-3}{4}x\right)^2 = 1, x^2 + \frac{9}{16}x^2 = 1, 25x^2 = 16, x^2 = \frac{16}{25}, x = \pm\frac{4}{5}$

Since $p(\theta)$ **is in 2nd quadrant** $x = -\frac{4}{5}, y = -\frac{3}{4}x = \frac{3}{5}$.

Now $Sin\theta = y = \frac{3}{5}, Cos\theta = x = -\frac{4}{5}, Cot\theta = \frac{1}{\tan\theta} = -\frac{4}{3}$

$Co\sec\theta = \frac{1}{Sin\theta} = \frac{5}{3}, Sec\theta = \frac{1}{Cos\theta} = -\frac{5}{4}$.

(3)Prove that $Cos^4\theta - Sin^4\theta = 1 - 2Sin^2\theta$

L.H.S= $Cos^4\theta - Sin^4\theta$

$= (Cos^2\theta + Sin^2\theta)(Cos^2\theta - Sin^2\theta)$

$= (1)(Cos^2\theta - Sin^2\theta) \because Cos^2\theta + Sin^2\theta = 1$

$= 1 - Sin^2\theta - Sin^2\theta, \because Cos^2\theta = 1 - Sin^2\theta$

$= 1 - 2Sin^2\theta = R.H.S$ So, $L.H.S = R.H.S$

(4)Show that $\frac{1+Sec\theta}{1-Sec\theta} = \frac{\tan\theta + Sin\theta}{Sin\theta - \tan\theta}, Cos\theta \neq 1$

$R.H.S = \frac{\tan\theta + Sin\theta}{Sin\theta - \tan\theta} = \frac{\frac{Sin\theta}{Cos\theta} + Sin\theta}{Sin\theta - \frac{Sin\theta}{Cos\theta}}$

$= \frac{Sin\theta + Sin\theta Cos\theta}{Sin\theta Cos\theta - Sin\theta} = \frac{Sin\theta(1+Cos\theta)}{Sin\theta(Cos\theta - 1)} = \frac{1+Cos\theta}{Cos\theta - 1}$

$= \frac{1 + \frac{1}{Sec\theta}}{\frac{1}{Sec\theta} - 1} = \frac{Sec\theta + 1}{1 - Sec\theta} = L.H.S, So\ L.H.S = R.H.S$.

(5)Prove that the points P(1,1), Q(4,5), R(7,9) form an isosceles triangle.

Sol: $|\overline{PQ}| = \sqrt{(4-1)^2 + (5-1)^2}$

$|\overline{PQ}| = \sqrt{9+16} = \sqrt{25} = 5$

$|\overline{QR}| = \sqrt{(7-4)^2 + (9-5)^2} = \sqrt{9+16} = \sqrt{25} = 5$

So $|\overline{PQ}| = |\overline{QR}|$ **Hence PQR is an isosceles triangle.**

(6)Express $4Sin\theta + 3Cos\theta$ **in the form** $rSin(\theta+\phi)$ **where** $p(\theta)$ **is in 1st quadrant.**

Sol:We can write $4Sin\theta + 3Cos\theta = 5\left(\frac{4}{5}Sin\theta + \frac{3}{5}Cos\theta\right)$. **Comparing with** $5Sin(\theta+\phi) = 5(Sin\theta Cos\phi + Cos\theta Sin\phi)$ **we get** $Cos\theta = \frac{4}{5}, Sin\theta = \frac{3}{4}$ **so we have** $5\left(\frac{4}{5}Sin\theta + \frac{3}{5}Cos\theta\right) = 5Sin(\theta+\phi)$ **or** $5(Sin\theta Cos\phi + Cos\theta Sin\phi) = 5Sin(\theta+\phi)$ **where** $Cos\theta = 4/5, Sin\theta = 3/4$.