Double Selection Circuit For Entanglement Purification

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Abstract

Entanglement Purification is an important problem in Quantum Communication. In this report, I have analyzed double selection circuit for entanglement purification. The probability of coincidence measurements happening is calculated mathematically and experimentally with qiskit.

Contents

Chapter 1

Introduction

1.1 Entanglement

Quantum Entanglement is a phenomenon in which states of quantum particles are dependent on each other. Entanglement is a really important concept in the study of Quantum Communication. It is a key element in effects such as quantum teleportaion, fast quantum algorithms and quantum error correction [4].

1.2 Bell States

Depending on the concept of quantum entanglement, we have another ingredient in quantum teleportation and super dense coding called Bell States [4]. These are based on pairs of entangled particles. The four basic bell states on 2-Qubits are following:

$$|A\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{1.1}$$

$$|B\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{1.2}$$

$$|C\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \tag{1.3}$$

$$|D\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \tag{1.4}$$

These correlations are the reason of debate in physics community since the famous paper of Einstein, Podolsky and Rosen came out in which they pointed out the strange properties of states like Bell States. [1]

1.3 Fidelity

Fidelity is defined as following:

> "Fidelity measures the average distortion introduced by the compression scheme. The idea of quantum data compression is that the compressed data should be recovered with very good fidelity. Think of the fidelity as being analogous to the probability of doing the decompression correctly." [4]

Loosely Speaking, Fidelity is the degree of exactness or measure of closeness of two quantum states.

1.4 Entanglement Purification

Quantum communication highly depends on the purity of entanglement. The problem with entangled states is that due to unavoidable noise in quantum channel, the entangled states are disturbed. This will create problems where we won't be sure about the correctness of the information we are receiving on a quantum channel. This problem can be solved by entanglement purification protocols. Suppose there are two parties Alice and Bob where Alice wants to send classical information through noisy quantum channel to Bob. She wants to make sure that the entangled states are purified and not disturbed by noise while sending the information over a larger distance. It can be described as starting with a larger number of pure states ψ and repeating the process again and again to get the maximum copies of pure Bell state $|A\rangle$ from 1.1 using local circuits and classical communication [4]. It is also known as entanglement distillation in literature.

1.5 Local Circuits and Classical Communication

Local circuits and classical communication is described in quantum information theory as a method in which local operations are performed on a circuit and the result is communicated to another part of the circuit classically where another local operation is conditioned on the information received.

1.6 Quantum Circuits for Entanglement Purification

Quantum Circuits are used for Entanglement Purification. The idea is to break the circuit in two parts where half of the circuit is with Bob while the other half is with Alice [2]. They can only communicate classically. The Bell States are provided with given probabilities. The circuit is run different times to get the pure $|A\rangle$ state as many times as possible in it's purified form. There are a lot of possibilities to generate circuits for entanglement purification. But finding one with the best performance can be think of as a discrete combinatorial optimization problem.

1.7 Double Selection Circuit

Double Selection Circuit is known as the optimized version for 3 Qubits to be used as entanglement purification protocol [2]. The circuit in Fig 1.1 shows

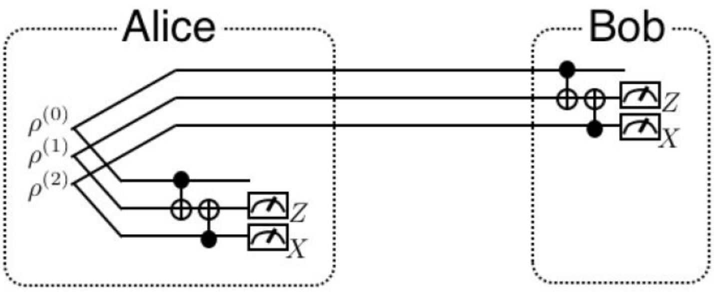

Figure 1.1: Double Selection Circuit for Entanglement Purification [2]

a double selection circuit. Three entangled pairs are introduced in this and one qubit from each pair is given to both, Alice and Bob. They perform two CNOT circuits on respective wires and then coincidence measurements in Z and X Basis are performed. One pair is left unmeasured which is the required pair to be purified.

1.8 Coincidence Measurement

Coincidence measurement is nothing new but a normal measurement we do in quantum information but with the condition that we will compare the results from two similar measurements. If the results stored in classical bits are equal (i.e 00 or 11) then we proceed with it.

Chapter 2

Theory and Mathematical Model

2.1 Theory

The imperfect Bell pairs in the input state is defined as

$$|\psi\rangle = F\,|A\rangle\,\langle A| + \frac{1-F}{3}(|B\rangle\,\langle B| + |C\rangle\,\langle C| + |D\rangle\,\langle D|) \tag{2.1}$$

where F represents the probability of the state being in A while $\frac{1-F}{3}$ represents the probability of state being in B, C or D. We will later define $q = \frac{1-F}{3}$ and use q instead of $\frac{1-F}{3}$ for simplicity.
Since we are dealing with 3-Qubit states, we will write states as $|ABC\rangle$ where A represents the pair to be purified while the pairs B and C are sacrificial which will be measured.

2.2 Mathematical Analysis of Circuit

The circuit shown in Fig 2.1 is the Bob's side of the double selection circuit. The Alice side is just the mirror of this as shown in Fig 1.1. We will go through this circuit while solving an input as an example. Let's suppose we have the state $|AAA\rangle$ as input. Let's go through this example step by step to understand how this circuit works.
First we have the following state:

$$|AAA\rangle = \frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle)(|00\rangle + |11\rangle)(|00\rangle + |11\rangle) \tag{2.2}$$

which is equal to

$$\frac{1}{2\sqrt{2}}((|00\rangle\,(|00\rangle + |11\rangle) + |11\rangle\,(|00\rangle + |11\rangle))(|00\rangle + |11\rangle)) \tag{2.3}$$

After taking the first CNOT which acts on second qubit controlled by the first qubit of both parties, we will have

$$\frac{1}{2\sqrt{2}}((|00\rangle\,(|00\rangle + |11\rangle) + |11\rangle\,(|11\rangle + |00\rangle))(|00\rangle + |11\rangle)) \tag{2.4}$$

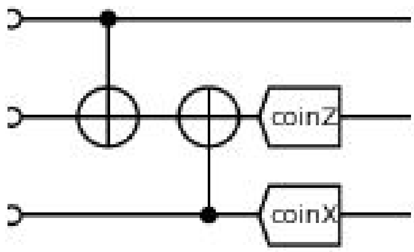

Figure 2.1: Bob's side of the circuit [3]

which is equal to

$$\frac{1}{2\sqrt{2}}((|00\rangle + |11\rangle)(|00\rangle + |11\rangle)(|00\rangle + |11\rangle)) = |AAA\rangle \tag{2.5}$$

This can also be written as

$$\frac{1}{2\sqrt{2}}(|A\rangle\,(|00\rangle + |11\rangle)\,|00\rangle + (|00\rangle + |11\rangle)\,|11\rangle) \tag{2.6}$$

Now, we will apply second CNOT operation which acts on second qubit controlled by third qubit for both parties. This will give us

$$\frac{1}{2\sqrt{2}}(|A\rangle\,(|00\rangle + |11\rangle)\,|00\rangle + (|11\rangle + |00\rangle)\,|11\rangle) \tag{2.7}$$

which is again equal to

$$\frac{1}{2\sqrt{2}}((|00\rangle + |11\rangle)(|00\rangle + |11\rangle)(|00\rangle + |11\rangle)) = |AAA\rangle \tag{2.8}$$

Hence we can see that the state $|AAA\rangle$ maps to $|AAA\rangle$ after performing both CNOT operations. We can perform the same procedure for all the possibilities of the states to analyze the circuit. The Table 2.1 and 2.2 shows all the possible states and their mapping with the final states as well as the probability of initial state. The probabilities are also showed for all the events happening in terms of F and q in the fourth column. The fifth column shows the probabilities of the event happening when there is a coincidence measurement which means either a $|A\rangle$ or $|D\rangle$ in sacrificial pairs while all other rows are blank. We will be calculating the probability of getting coincidence measurement.

2.3 Mathematical Prediction

The sum of column 5 of Tables 2.1 and 2.2 gives us the equation $F^3 + 3F^2q + 7Fq^2 + 5q^3$. The sum of column 4 sums to 1 because that's the total probability of anything happening. Now the probability of coincidence measurement happening will be given by the equation

$$F^3 + 3F^2q + 7Fq^2 + 5q^3 \tag{2.9}$$

Initial State	After 1st CNOT	After 2nd CNOT	Probability	Probability
AAA	AAA	AAA	F^3	F^3
AAB	AAB	ACB	F^2q	
AAC	AAC	ACC	F^2q	
AAD	AAD	AAD	F^2q	F^2q
ABA	DBA	DBD	F^2q	
ABB	DBB	DDC	Fq^2	
ABC	DBC	DDB	Fq^2	
ABD	DBD	DBA	Fq^2	
ACA	ACA	ACA	F^2q	
ACB	ACB	AAB	Fq^2	
ACC	ACC	AAC	Fq^2	
ACD	ACD	ACD	Fq^2	
ADA	DDA	DDD	F^2q	F^2q
ADB	DDB	DBC	Fq^2	
ADC	DDC	DBB	Fq^2	
ADD	DDD	DDA	Fq^2	Fq^2
BAA	BCA	BCA	F^2q	
BAB	BCB	BAB	Fq^2	
BAC	BCC	BAC	Fq^2	
BAD	BCD	BCD	Fq^2	
BBA	CDA	CDD	Fq^2	Fq^2
BBB	CDB	CCB	q^3	
BBC	CDC	CBB	q^3	
BBD	CDD	CDA	q^3	q^3
BCA	BAA	BAA	Fq^2	Fq^2
BCB	BAB	BCB	q^3	
BCC	BAC	BCC	q^3	
BCD	BAD	BAD	q^3	q^3
BDA	CBA	CBD	Fq^2	
BDB	CBB	CDC	q^3	
BDC	CBC	CDB	q^3	
BDD	CBD	CBA	q^3	
CAA	CCA	CCA	F^2q	
CAB	CCB	CAB	Fq^2	
CAC	CCC	CAC	Fq^2	
CAD	CCD	CCD	Fq^2	
CBA	BDA	BDD	Fq^2	Fq^2
CBB	BDB	BBC	q^3	
CBC	BDC	BBB	q^3	
CBD	BDD	BDA	q^3	q^3

Table 2.1: To be continued in Table 2.2

Initial State	After 1st CNOT	After 2nd CNOT	Probability	Probability
CCA	CAA	CAA	Fq^2	Fq^2
CCB	CAB	CCB	q^3	
CCC	CAC	CCC	q^3	
CCD	CAD	CAD	q^3	q^3
CDA	BBA	BBD	Fq^2	
CDB	BBB	BDC	q^3	
CDC	BBC	BDB	q^3	
CDD	BBD	BBA	q^3	
DAA	DAA	DAA	F^2q	F^2q
DAB	DAB	DCB	Fq^2	
DAC	DAC	DCC	Fq^2	
DAD	DAD	DAD	Fq^2	Fq^2
DBA	ABA	ABD	Fq^2	
DBB	ABB	ADC	q^3	
DBC	ABC	ADB	q^3	
DBD	ABD	ABA	q^3	
DCA	DCA	DCA	Fq^2	
DCB	DCB	DAB	q^3	
DCC	DCC	DAC	q^3	
DCD	DCD	DCD	q^3	
DDA	ADA	ADD	Fq^2	Fq^2
DDB	ADB	ABC	q^3	
DDC	ADC	ABB	q^3	
DDD	ADD	ADA	q^3	q^3

Table 2.2: This table shows the mapping of all the initial states to the states after CNOT's are performed and the probability of every event happening in terms of F and q

Chapter 3

Experimentation and Results

3.1 Procedure

The circuit is given the raw Bell states with the probabilities defined in Eq 2.1. The circuit is run 1000 times and the probability of getting coincidence measurement is calculated by counting the total number of times we get coincidence measurement and dividing it by 1000. This will give the numeric value of probability. This numeric value can be compared to the expected theoretical value by plugging in the value of F and q in Eq 2.9. The F is hyper parameter and can be changed in the code. So I iterated over F starting from 0.5 to 0.95 with a step of 0.05. The circuit which is used in qiskit is shown in Fig 3.1. This circuit has H and X gates in the beginning which is due to the creation of Bell States which needs to be done in qiskit manually. It shows the circuit which has state $|AAA\rangle$ as input.

3.2 Results

The results can be seen in the Table 3.2. This table represents the expected and calculated probabilities at different values of F. We can see that there is a slight error of 0.01-0.09 in the expected and calculated probabilities in some values. The values in the table 3.2 can be viewed and analyzed by a graph which is shown below in Fig 3.2. We can see the closeness of our expectation and results within a small range of error

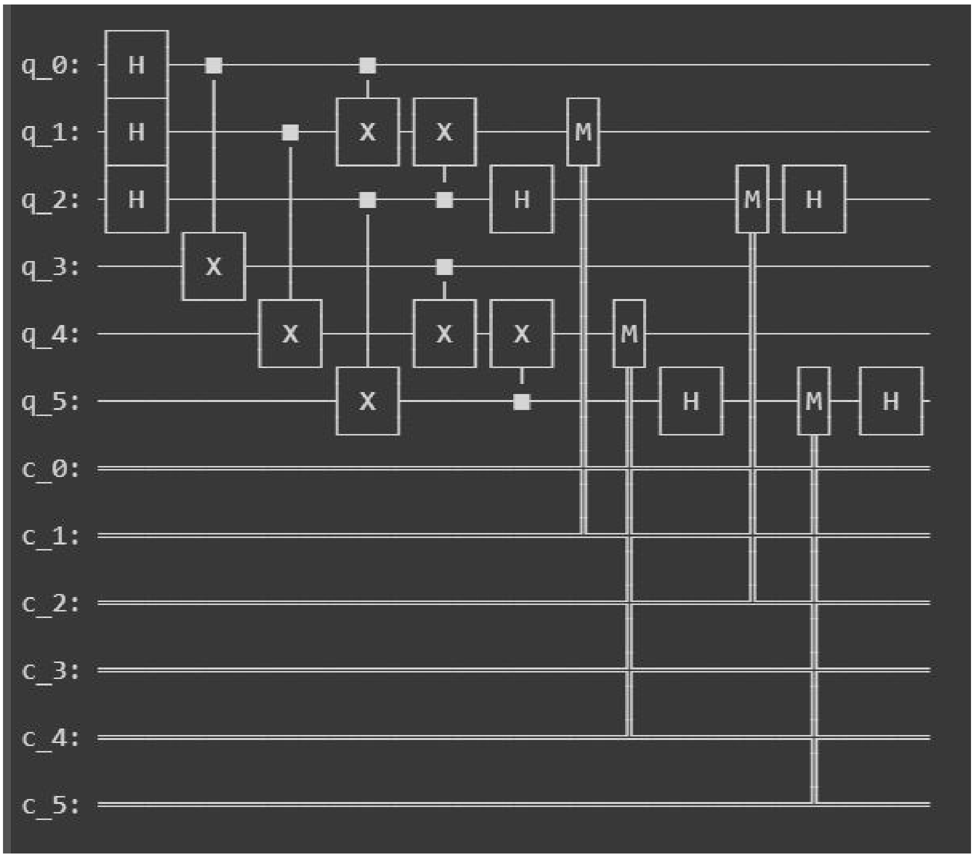

Figure 3.1: Circuit used in qiskit having $|AAA\rangle$ as initial state

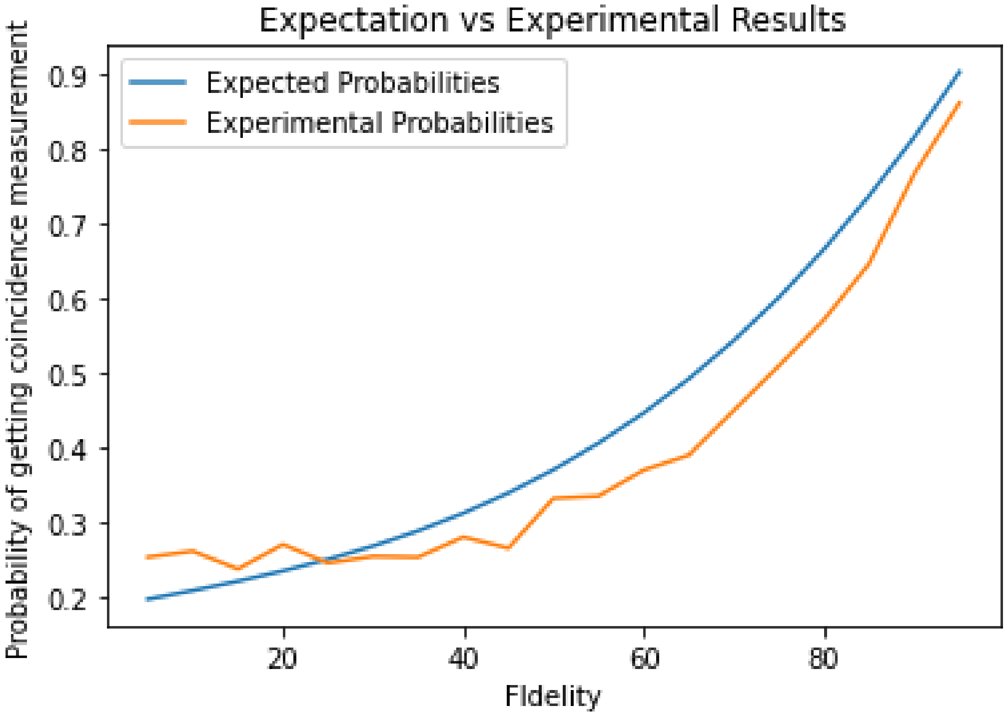

Figure 3.2: Graph of Result

F	Expected Probability	Calculated Probability
0.5	0.196	0.253
0.10	0.208	0.261
0.15	0.220	0.237
0.20	0.234	0.27
0.25	0.25	0.245
0.30	0.268	0.254
0.35	0.288	0.253
0.40	0.312	0.28
0.45	0.339	0.265
0.50	0.370	0.332
0.55	0.406	0.335
0.60	0.446	0.37
0.65	0.492	0.39
0.70	0.544	0.45
0.75	0.602	0.51
0.80	0.666	0.572
0.85	0.738	0.647
0.90	0.817	0.768
0.95	0.904	0.863

Table 3.1: This table shows the results

Chapter 4

Supplementary Material

4.1 Acknowledgments

Special thanks to Dr. Abdullah Khalid for his help and constant support during the project.

4.2 Code

The following code was used in the process:

```
# these three import lines are essential for running a quantum circuit
import qiskit
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircuit
from qiskit import Aer, execute
from qiskit.providers.aer import QasmSimulator
# also import a visualization tool from qiskit
from qiskit.tools.visualization import plot_histogram
# import numpy incase we need it
import numpy as np
import random
def measurementX(circ, qbit, cbit):
  '''It will measure in X Basis'''
  circ.h(qbit)
  circ.measure(qbit, cbit)
  circ.h(qbit)
def measurementZ(circ, qbit, cbit):
  '''Measurement in usual z basis which is default'''
  circ.measure(qbit, cbit)
def bellStates(circ, qbit1, qbit2):
  '''Will take circuit and qubits to prepare Bell States'''
  circ.h(qbit1)
  circ.cx(qbit1,qbit2)
def initializeCircuit(n):
```

```
  qreg = QuantumRegister(n,'q')
  creg = ClassicalRegister(n,'c')
  circ = QuantumCircuit(qreg, creg)
  return circ,qreg,creg
def fixedPurificationCircuit(circ, qreg, creg, inputs)
   :
  ''' This Function makes a single selection Circuit
     for Entanglement Purification '''
  # To put x gates according to the input
  for i in range(len(inputs)):
    if inputs[i] == 1:
      circ.x(qreg[i])
  n = len(inputs)
  for i in range(0,n//2):
    bellStates(circ, qreg[i], qreg[i+n//2])
  #The following circuit is for Alice
  circ.cx(qreg[0], qreg[1])
  circ.cx(qreg[2], qreg[1])
  measurementZ(circ,qreg[1], creg[1])
  measurementX(circ,qreg[2], creg[2])
  #The following circuit is for Bob
  circ.cx(qreg[3], qreg[4])
  circ.cx(qreg[5], qreg[4])
  measurementZ(circ,qreg[4], creg[4])
  measurementX(circ,qreg[5], creg[5])
  #print(qiskit.circuit.Measure.c_if(,1))
  return circ
def checkCoincidence(counts):
  '''This Function checks if measurements on
     respective sacrificial pairs are coincidence or
     not'''
  for i in counts:
    if not (i[0] == i[3] and i[1] == i[4]):
      return False
  return True
def showResults(circ):
  simulator = Aer.get_backend('qasm_simulator')
  job = execute(circ,simulator,shots=1000)
  result = job.result()
  counts = result.get_counts()
  return counts
def prob(F,q):
  '''This is the function to calculate the theoretical
      probability depending on F'''
  return ((F)**3 + 3 * (F)**2 * (q) + 7 * (F) * (q)**2
      + 5 * (q)**3)
n = 6
dic = {'A': (0,0), 'B': (1,1), 'C': (0,1), 'D': (1,0)}
inputs = [0,0,0,0,0,0]
expectedProbabilites = []
```

```
calculatedProbabilities = []
for F in range(5,100,5): #The loop to iterate over
   different probabities of A i.e F
  q = (100-F)/3
  weights = [F, q, q, q]
  coincidenceCount = 0 #This will count the
     coincidence measurements
  for _ in range(1000): #This loop will run circuit
     1000 times
    for i in range(n//2):
      arg = random.choices(["A", "B", "C", "D"],
         weights = weights, k = 1)
      inputs[i],inputs[i+3] = dic[arg[0]]
    circ, qreg, creg = initializeCircuit(n)
    fixedPurificationCircuit(circ, qreg, creg, inputs)
    counts = showResults(circ)
    if checkCoincidence(counts):
      coincidenceCount += 1
  expectedProbabilites.append(prob(F/100,q/100))
  calculatedProbabilities.append(coincidenceCount
     /1000)
```

Bibliography

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