25

FADING, DIVERSITY, AND COMBINING

- Derive and simulate the bit error rate (BER) in the Rayleigh fading environment and compare it with the BER in the additive white Gaussian noise environment (AWGN).
- Understand the concept of diversity and the diversity combining.
- Compare the performances of various diversity-combining methods.

25.1 RAYLEIGH FADING CHANNEL MODEL AND THE AVERAGE BER

Consider a two-dimensional modulation such as quadrature amplitude modulation (QAM) and MPSK, and denote s as the coordinate of the modulated symbol in the complex plane. Then, the real and imaginary parts of s correspond to the in-phase and quadrature components, respectively. The symbol energy E_s is equal to $\mathrm{E}[|s|^2]$.

With this signal model, the receiver's matched filter output in a frequency nonselective fading channel can be expressed as

$$r = hs + n, \tag{25.1}$$

where h is the signal scaling coefficient due to fading [1–3], which is often called the "fading coefficient," and n is the AWGN term, which follows the complex Gaussian distribution with zero mean and variance $N_0/2$.

1.A For Rayleigh fading, the fading coefficient h follows a complex Gaussian distribution, expressed as $h = z + jy$, where z and y are real-valued independent

Problem-Based Learning in Communication Systems Using MATLAB and Simulink, First Edition.
Kwonhue Choi and Huaping Liu.

Companion website: www.wiley.com/go/choi_problembasedlearning

JWBS174-c25 JWBS174-Choi December 16, 2015 20:56 Printer Name: Trim: 6.125in × 9.25in

Gaussian variables with zero mean and variance 1/2 [3–5]. The command line below generates a sample of h. Complete the quantities marked by **'?'**.

```
>>h=?*randn + ?*?*?
```

1.B The fading coefficient can be expressed in polar coordinate form as

$$h = |h|e^{j\angle h}. \tag{25.2}$$

1.B-1 [T]Express the fading magnitude square, that is, $|h|^2$, as a function of z and y.

1.B-2 [T]From the result in 1.B-1 and the fact that z and y are independent Gaussian random variables with zero mean and variance 1/2, determine $\mathrm{E}[|h|^2]$.

1.C To create the decision variable D, the receiver derotates the received signal r by using the phase of h as

$$\begin{aligned} D &= re^{-j\angle h} \\ &= (hs + n)e^{-j\angle h} \\ &= |h|e^{j\angle h}e^{-j\angle h}s + ne^{-j\angle h} \\ &= |h|s + ne^{-j\angle h}. \end{aligned} \tag{25.3}$$

As a result, the signal term in the decision variable, $|h|s$, has the same phase as that of the original symbol s.

1.C-1 The noise element $ne^{-j\angle h}$ of the decision variable D is still a complex Gaussian random variable, which has the same distribution as n. Therefore, rotating the phase due to fading does not affect the statistical characteristics of the noise in the decision variable D. On the contrary, fading affects the signal term by a scaling factor $|h|$. This scaling factor is the magnitude of a complex Gaussian random variable, which has the Rayleigh distribution [4, 5] given as

$$f_{|h|}(x) = \begin{cases} 2x\exp(-x^2) & x \geq 0 \\ 0 & x < 0. \end{cases} \tag{25.4}$$

Execute the following lines of command to plot equation (25.4). Capture the result.

```
>>x=0:0.01:5;
>>f=2*x.*exp(-x.^2);
>>figure
>>plot(x,f)
```

1.C-2 Define $c \triangleq |h|^2$. Then c is the magnitude square of a complex Gaussian variable and follows the exponential distribution expressed as [4, 5]

$$f_c(x) = \exp(-x), \quad x \geq 0. \tag{25.5}$$

Execute the following lines of commands to plot equation (25.5). Capture the result.

```
>>x=0:0.01:5;
>>f_c=exp(-x);
>>figure
>>plot(x,f_c)
```

1.C-3 [T]We can find the expected value of c as $\mathrm{E}[c] = \int_0^{\infty} (?) \times f_c(x)dx$. Determine the quantity marked by **'?'**.

1.C-4 (a) [T]Calculate the average of c using the completed equation in C-3. (b) Is the calculated result equal to the answer in 1.B-2? Also use symbolic math (discussed in Section 1.2 of Chapter 1) to verify the result in (a).

1.C-5 [T]In the absence of fading (i.e., $h = 1$), the energy (mean square) of the signal term in the decision variable D, $E[||h|s|^2]$, equals E_s. For Rayleigh fading for which h follows the distribution given by equation (25.4), calculate the energy of the signal term, $E[||h|s|^2]$, and prove that it is still equal to E_s.

1.C-6 [T]The result in 1.C-5 shows that the average symbol energies of the received signals over a Rayleigh fading channel and a Gaussian channel are same. This is based on the assumption that the power of Rayleigh fading channel coefficients is normalized to unity. Provide a discussion on the relative performances (a conjecture on what you expect to see) of the same signaling scheme over a Rayleigh fading channel and over an AWGN channel.

1.C-7 [A]Prove that the noise term $ne^{-j\angle h}$ in the decision variable D given in equation (25.3) is a complex Gaussian random variable and it has the same distribution as n.

1.D In a Rayleigh fading channel, $|h|$ in equation (25.3) is a random variable following the distribution in equation (25.4). The instantaneous symbol energy received over the fading channel is $|h|^2 E_s$. Let $c = |h|^2$. Therefore, for BPSK, the instantaneous BER is given as $Q(\sqrt{2cE_b/N_0})$, where $E_b = E$. The average BER can be obtained by taking the expected value of the instantaneous BER over c as

$$\begin{aligned} BER_{\text{fading}} &= \mathrm{E}_c\left[Q\left(\sqrt{\frac{2cE_b}{N_0}}\right)\right] \\ &= \mathrm{E}_c\left[0.5\left\{1 - \operatorname{erf}\left(\sqrt{\frac{cE_b}{N_0}}\right)\right\}\right]. \end{aligned} \tag{25.6}$$

1.D-1 [WWW]The following m-file calculates equation (25.6) using symbolic math. Add a comment to each line to explain what that line does. Capture the completed m-file with the comments.

```
clear
syms c EbN0
instantBER=0.5*(1-erf(sqrt(c*EbN0)));
BER_fading=int(instantBER*exp(-c),c,0,inf);
pretty((BER_fading))
```

1.D-2 The closed-form expression for equation (25.6) is given as

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{E_b/N_0}{1 + E_b/N_0}}\right). \tag{25.7}$$

Execute the m-file in 1.D-1 and capture the result. Is the symbolic math result the same as equation (25.7)?

1.D-3 Modify the second line of the m-file in D-1 into **'syms c EbN0 positive'**. Here the argument **positive** is not a symbolic variable but a MATLAB argument to specify that the symbolic variables **c** and **EbN0** are positive (and real, of course). Execute the modified m-file and capture the results. Is the symbolic math result the same as equation (25.7) now?

NOTE: If the type of the symbolic variables is not specified, then symbolic math will try to derive a general solution assuming that the symbolic variables are complex. In such a case, it often fails to find the solution.

25.2 BER SIMULATION IN THE RAYLEIGH FADING ENVIRONMENT

2.A [WWW]The following m-file simulates the BER of BPSK over a Rayleigh fading channel.

```
clear
EbN0dB_vector=0:2:20;

Eb=1;

for snr_i=1:length(EbN0dB_vector)
   EbN0dB=EbN0dB_vector(snr_i);
   EbN0=10.^(EbN0dB/10);
   N0=Eb/EbN0;
   sym_cnt=0;
   err_cnt=0;
   while err_cnt<500
      s=sqrt(Eb)*sign(rand-0.5);
      h=sqrt(1/2)*(randn+j*randn);
```

```
        n=sqrt(N0/2)*(randn+j*randn);

        r=?*s + ?;

        D=r*exp(-j*angle(h));
        s_hat=sign(D);
        if ?
            err_cnt=err_cnt+1;
        end
        sym_cnt=sym_cnt+1;
    end
    BER(snr_i)=err_cnt/sym_cnt
end
figure
semilogy(EbN0dB_vector, BER)
xlabel('E_b/N_0 [dB]')
ylabel('BER')
grid
```

2.A-1 (a) Explain why the three parts in bold in the m-file are set as they are. (b) Complete the places marked by **'?'**, and add a comment to each line of the m-file. For lines that involve the operator **'='**, use the comment to explain what the variable on the left-hand side represents and justify how the right-hand side expression is properly formulated accordingly. Capture the completed m-file.

2.A-2 Explain why the line **'h=sqrt(1/2)*(randn+j*randn);'** should not be placed before the '**while**' loop.

2.A-3 If the line **'h=sqrt(1/2)*(randn+j*randn);'** is placed before the '**while**' loop, what will happen to the resulting BER.

2.B Execute the completed m-file in 2.A and capture the resulting BER graph.

2.C We should find that the BER values in 2.B are all equal to 1. This is because the line **'s_hat=sign(D);'** is incorrect. It should be corrected as **'s_hat = sign(real(D));'**. Explain why it is necessary to take the real part of the decision variable **D** although the signal term of **D** (**sqrt(Eb)** or **-sqrt(Eb)**) is already real-valued.

2.D Modify the line **'s_hat=sign(D);'** into **'s_hat=sign(real(D));'** and execute the m-file. Capture the BER result.

2.E The theoretical BER of BPSK signaling over a Rayleigh fading channel was given in equation (25.7). Execute the following commands in the command window to overlay the theoretical BER on top of the simulated BER curve obtained in 2.D. Capture the resulting figure.

```
>>EbN0_vector=10.^(EbN0dB_vector/10);
>>BER_theory=0.5*(1-sqrt((EbN0_vector)./(1+EbN0_vector)));
>>hold on;
>>semilogy(EbN0dB_vector, BER_theory,'r')
>>legend('Rayleigh fading, Simulation', 'Rayleigh fading, Theory')
```

2.F Do the simulated and theoretical BER values match each other?

2.G In this section we compare the BER performances of BPSK over Rayleigh fading and Gaussian channels.

2.G-1 Rewrite the BER expression of BPSK in an AWGN channel as a function of E_b/N_0.

2.G-2 Execute the commands below to overlay the BPSK BER curve in an AWGN channel on top of the BER curve obtained in 2.D or 2.E. Capture the result.

```
>>BER_AWGN=0.5*erfc(sqrt(EbN0_vector));
>>hold on;
>>semilogy(EbN0dB_vector, BER_AWGN,'g')
>>legend('Rayleigh fading, Sim', 'Rayleigh fading, Theory', 'AWGN')
>>axis([0 20 1e-6 1])
```

2.H Summarize the characteristics of the BER curves of BPSK over Rayleigh fading and Gaussian channels in terms of the changing rate of BER and performance gap as E_b/N_0 increases and decreases.

2.I Are the observations made in 2.H from simulation consistent with what was discussed in 1.C-6?

2.J [A]Next we modify the m-file in 2.D to simulate the BER performance of QPSK over a Rayleigh fading channel. The modified m-file is given below.

2.J-1 The parts in bold are the modified parts from the previous m-file for BPSK BER simulation. Add a comment to each of these lines to justify the modifications made. Capture the completed m-file.

```
clear
EbN0dB_vector=0:2:20;
Eb=1;
for snr_i=1:length(EbN0dB_vector)
   EbN0dB=EbN0dB_vector(snr_i);
   EbN0=10.^(EbN0dB/10);
   N0=Eb/EbN0;
   sym_cnt=0;
```

```
    err_cnt=0;
    while err_cnt<500
        s=sqrt(Eb)*sign(rand-0.5)+j*sqrt(Eb)*sign(rand-0.5);
        h=sqrt(1/2)*(randn+j*randn);
        n=sqrt(N0/2)*(randn+j*randn);
        r=h*s+n;
        D=r*exp(-j*angle(h));
        s_hat=sign(real(D));
        s_hat2=sign(imag(D));
        if sign(real(s)) ~= s_hat
            err_cnt=err_cnt+1;
        end
        if sign(imag(s)) ~= s_hat2
            err_cnt=err_cnt+1;
        end
        sym_cnt=sym_cnt+2;
    end
    BER(snr_i)=err_cnt/sym_cnt
end
figure
semilogy(EbN0dB_vector, BER)
xlabel('E_b/N_0 [dB]')
ylabel('BER')
grid
```

2.J-2 Execute the m-file in 2.J-1 and capture the simulated BER plot.

2.J-3 Compare the simulated BER curve of QPSK with the BER curve of BPSK obtained in 2.E. They should be identical (ignore the small simulation errors). Justify why they should be the same.

25.3 DIVERSITY

In 1.D we defined the variable c, which determines the instantaneous symbol energy in a Rayleigh fading channel. From the distribution of c, we can easily explain why the BER performance of BPSK in a Rayleigh fading channel is significantly worse than that in an AWGN channel (no fading). The instantaneous symbol energy cE_b could change from 0 to infinity due to fading, and the instantaneous BER given as $Q(\sqrt{2cE_b/N_0})$ almost exponentially decreases as c increases. If c becomes larger than 1, then the BER is lower than the BER over an AWGN channel. However, the error rate will be dominated by the bit errors when c is smaller than 1. For a mathematical proof, refer to Jensen's inequality [4, 5], which is covered in most of the existing textbooks.

If there is a method that will drastically reduce the probability of encountering a small instantaneous symbol energy, then the BER performance will improve significantly. One such method is the diversity technique [5]. Diversity here refers to the mechanism that multiple independently faded copies of the same signal are available for use in the detection process. Common diversity methods include spatial diversity (refer to Chapter 28), temporal/time diversity, frequency diversity, and multipath diversity. Spatial diversity could be achieved by transmitting the same data through multiple transmit antennas or receiving the same data through multiple receive antennas. For such schemes to be effective, the transmit antennas or the receive antennas must operate independently or at least have sufficiently low correlations. Multipath diversity is unique [5]: if multiple received signal paths are resolvable, then this case may be considered a form of time diversity; however, the resolvable paths are closely related to the fact that the channel is frequency selective, that is, the different frequency components of the desired signal are faded differently (refer to Chapter 27).

In this section we consider the scenario that L independently received copies, $r(1), r(2), \ldots, r(L)$, all carrying the same transmitted signal s, are available for use in the detection process. These copies could be exploited to achieve a maximum diversity order of L. The received signal copies are expressed as

$$r(1) = h(1)s + n(1),\ r(2) = h(2)s + n(2), \ldots,\ r(L) = h(L)s + n(L), \qquad (25.8)$$

where $h(1), h(2), \ldots, h(L)$ are assumed to be independent and identically distributed (i.i.d.), all having the same distribution as h created in 1.A; the noise terms $n(1), n(2), \ldots, n(L)$ are also i.i.d. with the same distribution as n in equation (25.1), and the transmitted signal s is a BPSK signal, taking on the values of $\sqrt{E_s/L}$ or $-\sqrt{E_s/L}$ with equal probability.

3.A [T]Assuming a noiseless and the nonfading condition ($h(1) = \ldots = h(L) = 1$ and $n(1) = \ldots = n(L) = 0$), prove that the total received symbol energy from all L branches, that is, $|r(1)|^2 + |r(2)|^2 + \ldots + |r(L)|^2$, equals E_s.

25.4 COMBINING METHODS

This section investigates techniques for combining the L branches of the received signals to form the decision variable [5,6]. Three commonly used combining methods are as follows:

1. Selection diversity combining (SDC)
2. Equal gain combining (EGC)
3. Maximum ratio combining (MRC)

The fading coefficients $h(1), h(2), \ldots, h(L)$ are assumed to be known in the receiver. In practice, most communications channels can be classified as "slow" fading channels, for which the fading coefficients will be nearly constant over many symbol periods. This allows the receiver to estimate the fading coefficients.

4.A Selection diversity combining.

In SDC, the branch with the largest fading coefficient is selected for detection and the rest are not used. The decision variable D is generated as

$$\begin{aligned} &\text{Step 1. Set } k_{\text{best}} = \arg\max_k |h(k)|, \\ &\text{Step 2. Set } D = r(k_{\text{best}})e^{-j\angle h(k_{\text{best}})}. \end{aligned} \tag{25.9}$$

4.A-1 [WWW]The following m-file simulates the BER of BPSK over a Rayleigh fading channel with three SDC branches, that is, $L = 3$. Complete the places marked by **'?'**. Add a comment to each of the lines in bold to explain what the line does. Especially for the lines with '=', explain what the variable on the left-hand side represents and justify how the right-hand side expression is properly formulated accordingly.

Capture the completed m-file.

```
clear
EbN0dB_vector=0:3:15;
Eb=1;
L=3;
for snr_i=1:length(EbN0dB_vector)
   EbN0dB=EbN0dB_vector(snr_i);
   EbN0=10.^(EbN0dB/10);
   N0=Eb/EbN0;
   sym_cnt=0;
   err_cnt=0;
   while err_cnt<100 % If you increase err_cnt (currently 100), the accuracy increases
and time also increase.
      b=sign(rand-0.5);  %BPSK symbol {1,-1}
      s=sqrt(Eb/L)*b;
      for k=1:L
         h(k)=sqrt(1/2)*(randn+j*randn);
         n(k)=sqrt(N0/2)*(randn+j*randn);
         r(k)=?*s+?;
      end

       [T1 T2]=max(?); % Refer to (25.9). To see how to use max( ), execute '>>help
max' in the command window.
      D=r(?)*exp(-j*angle(h(?))); %Refer to (25.9).

      b_hat=sign(real(D));
      if b_hat~=b;
         err_cnt=err_cnt+1;
      end
      sym_cnt=sym_cnt+1;
```

```
    end
    BER(snr_i)=err_cnt/sym_cnt
end
figure
semilogy(EbN0dB_vector, BER)
xlabel('E_b/N_0 [dB]')
ylabel('BER')
grid
```

4.A-2 Execute the completed m-file and capture the simulated BER.

4.A-3 Execute the m-file separately for each of two other cases: $L = 1$ and 5. Then plot the three BER curves in a single figure. Methods to overlay curves from different figures in a single plot were discussed in Section 4.C-1 of Chapter 24. Finally, execute **legend('L=1','L=3','L=5')** in the command window and capture the resulting plot. Save this figure in .fig format and name it **Ch25_4A_3.fig**, as it will be needed later in this chapter.

4.A-4 (a) Analyze how the slope of the BER curves changes as the diversity order L increases, and intuitively explain the reason that causes such characteristics. (b) Explain why the BER becomes slightly worse in the low-SNR region as the diversity order L increases.

4.B Equal gain combining.

In EGC, $r(1), r(2), \ldots, r(L)$ are combined with the same weight regardless of the magnitudes of the L instantaneous fading coefficients. For coherent combining, the phase rotation due to fading at each branch is compensated first and the decision variable D is written as

$$D = \sum_{k=1}^{L} r(k) e^{-j\angle h(k)}. \tag{25.10}$$

4.B-1 In the m-file completed in 4.A-1, modify the right-hand side of the line **'D=r(?)*exp(-j*angle(h(?)));'** into **'D=sum(r.*exp(-j*angle(h)));'** and complete this line properly to implement the right-hand side of equation (25.10). Capture the modified m-file.

4.B-2 In addition, modify the line **'EbN0dB_vector=0:3:15'** into **'EbN0dB_vector=0:3:12'**. Execute the modified m-file for each of the following cases: $L = 1$, $L = 3$, and $L = 5$. Then overlay the three BER curves in a single figure. After this, execute **legend('L=1', 'L=3', 'L=5')** in the command window and capture the figure. Save this figure in .fig format and name it **Ch25_4B_2.fig**, as it will be needed later in this chapter.

4.B-3 Analyze how the BER and the slope of the BER curves change as the diversity order L changes.

4.B-4 Compare the BER results with EGC in 4.B-2 and with SDC in 4.A-3. This can be done best by overlaying all six curves in a single figure. Focus on the relative BER values and the slopes of the BER curves of the same diversity order obtained by using the two combining techniques.

4.C Maximum ratio combining.

Maximum ratio combining is similar to EGC in the sense that all branches are combined in both schemes. Unlike EGC, MRC employs different combining weights that are proportional to the fading magnitude of each branch. The decision variable is expressed as

$$D = \sum_{k=1}^{L} |h(k)| r(k) e^{-j\angle h(k)}. \tag{25.11}$$

4.C-1 [T]Show that equation (25.11) can be simplified as

$$D = \sum_{k=1}^{L} h^*(k) r(k). \tag{25.12}$$

4.C-2 In the m-file completed in 4.A-1, modify the right-hand side of the line **'D=r(?)*exp(-j*angle(h(?)));'** into **'D=sum(conj(?).*?);'**, and complete this line properly to implement the right-hand side of equation (25.12). Capture the completed line.

4.C-3 In addition, modify the line **'EbN0dB_vector=0:3:15'** into **'EbN0dB_vector=0:3:12'** as done in 4.B-2. Execute the modified m-file for each of the following cases: $L = 1$, $L = 3$, and $L = 5$. Then overlay the three BER curves in a single figure. After this, execute **legend('L=1', 'L=3', 'L=5')** in the command window and capture the resulting figure. Save this figure in .fig format and name it **Ch25_4C_3.fig**, which will be needed later in this chapter.

4.C-4 Analyze how the BER and the slope of the BER curves change as the diversity order L changes.

4.C-5 Compare the BER results with MRC in 4.C-3 and with EGC in 4.B-2. As in 4.B-4, this can be done best by overlaying all six curves in a single figure. Focus on the relative BER values and the slopes of the BER curves of the same diversity order obtained by using the two combining techniques.

4.C-6 [A,T]Prove that MRC maximizes the SNR among all three combining methods using the Cauchy Schwarz inequality [7].

4.D Now we compare the performances all three combining methods.

4.D-1 Open the figure files (.fig files) saved in 4.A-3, 4.B-2, and 4.C-3 if you have closed these figures. Overlay all of the nine BER curves in one figure. Change the line color of the BER curves for SDC to blue, for EGC to green, and for MRC to red. Add a legend for all nine curves. Then capture the completed figure.

(a) For the same diversity order L, which combining scheme performs the best?
(b) For $L = 5$, analyze the BER gaps among the three schemes as SNR changes and summarize the observations.

4.D-2 Based on the decision variables in equations (25.9), (25.10), and (25.12), which combining method has the lowest implementation complexity and which one is most difficult to implement? Why?

4.E In this problem we investigate the diversity gain.

4.E-1 Summarize the common trend in the change of BER values with the three methods as L increases and explain the reason that causes this trend.

4.E-2 Revisit the m-file completed in 4.C for MRC. Set **L=5** and modify the '**for**' loop inside the '**while**' loop as shown below. The distribution of **h(k)** after this modification remains the same as that in the original version. Explain why.

```
hk=sqrt(1/2)*(randn+j*randn);
for k=1:L
   h(k)=hk;
   n(k)=sqrt(N0/2)*(randn+j*randn);
   r(k)=?*s+?;
end
```

4.E-3 Execute the modified m-file and capture the BER graph.

4.E-4 (a) Compare the BER curve in 4.E-3 with the BER curve obtained from the original m-file. (b) Observe that these two BER curves are significantly different. Justify it; that is, explain why the modified m-file cannot achieve any diversity gain although the distribution of **h(k)** remains the same as before.

4.E-5 Based on the results in 4.E-3 and 4.E-4, determine the conditions on the statistical properties of the fading coefficients **h(1)**, **h(2)**, …, **h(L)**, so that the diversity gain is maximized and the BER is minimized accordingly.

REFERENCES

[1] T. S. Rappaport, *Wireless Communications: Principles and Practice*, 2nd ed., Upper Saddle River, NJ: Prentice Hall, 2002.

[2] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge, UK: Cambridge University Press, 2005.

[3] B. Sklar, "Rayleigh Fading Channels in Mobile Digital Communication Systems Part I: Characterization," *IEEE Communications Magazine*, Vol. 35, No. 7, 1997, pp. 90–100.

[4] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, New York: McGraw-Hill, 1965.

[5] J. G. Proakis, *Digital Communications*, 5th ed., New York: McGraw-Hill, 2008.

[6] D. G. Brennan, "Linear Diversity Combining Techniques," *Proceedings IRE.*, Vol. 47, 1959, pp. 1075–1102.

[7] G. Strang, *Linear Algebra and Its Applications*, 4th ed., Belmont, CA: Brooks Cole, 2005.