UPC-UPF
ALGEBRA AND GEOMETRY, GETIAE

Year 2019/20. Instructor: Jaume Amorós

Matlab assignment: Constrained linear optimization

The problem:

We will solve the following *constrained linear optimization* problem: given a linear system of equations

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ and A is a $m \times n$ matrix, we want the best possible solution that complies the constraint $\|\mathbf{x}\| \leq M$, where M is an upper bound set by the user to the size of the solution $\mathbf{x}$. The best possible solution here means that if we cannot get an exact solution we will choose a solution that minimizes the *residue* $\|A\mathbf{x} - \mathbf{b}\|$. Also, if there is more than one solution satisfying the bound and producing the same residue, we will choose the one with smallest size $\|\mathbf{x}\|$.

The tool:

We will solve this problem with an algorithm based on the Singular Value Decomposition (SVD) of the matrix A: if we regard A as the matrix of a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ in canonical basis for start and arrival spaces, there exist bases $v_1, \dots, v_n$ for start and $u_1, \dots, u_m$ for arrival such that they are orthonormal (vectors of length 1 and pairwise orthogonal), and the matrix of f in these bases is (empty positions mean 0):

$$D = \begin{pmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & \end{pmatrix}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values of f (or of A), and r is its rank.

In matrix terms, if we denote U the matrix that has as columns the u-basis for $\mathbb{R}^m$ and V the matrix that has as columns the v-basis for $\mathbb{R}^n$, the relation between A and D is

$$A = UDV^t \tag{2}$$

This equality is just the change of basis formula for the matrix of a linear map, because as the v-basis is orthonormal it turns out that $V^{-1} = V^t$.

The algorithm:

We can compute the SVD decomposition (2) by just applying the `svd` function. If we see the equation system (1) as looking for a vector $\mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = \mathbf{b}$ now we can look for this antiimage working in the SVD bases: Denoting $\tilde{\mathbf{x}}, \tilde{\mathbf{b}}$ the vectors written in basis v, resp. u, the system of equations to solve is now

$$D\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \tag{3}$$

and the constraint is still $\|\tilde{\mathbf{x}}\| \le M$ because the v basis is orthonormal.

The exact, or at least residue-minimizing i.e. least square, solution to system (3) is to set

$$\tilde{x}_1 = \frac{\tilde{b}_1}{\sigma_1}, \tilde{x}_2 = \frac{\tilde{b}_2}{\sigma_2}, \ldots, \tilde{x}_r = \frac{\tilde{b}_r}{\sigma_r}$$

and, if $\tilde{\mathbf{x}}$ has any more components, setting them to zero minimizes the norm of the solution while leaving the residue unchanged.

After finding this solution $\tilde{\mathbf{x}}$, we must impose the constraint: if already $\|\tilde{\mathbf{x}}\| \le M$, nothing has to be done. If the norm is bigger than M, we must remove components of the vector $\tilde{\mathbf{x}}$ *starting from the end of the vector*. This is because if we eliminate (set to zero) a component $\tilde{x}_i$ the squared norm $\|\tilde{\mathbf{x}}\|^2$ will decrease by $\tilde{x}_i^2$, while the square of the residue will increase by $(\sigma_i \tilde{x}_i)^2$. Therefore, the smaller the singular value σ_i is, the less residue we add for the same decrease in $\|\tilde{\mathbf{x}}\|$.

So we remove components of the vector $\tilde{\mathbf{x}}$ starting from the end, until its norm is below the maximum M. If removing the final component makes the norm of the vector drop strictly below M, this means that we can remove only part of that component so that the resulting norm is exactly M and the residue is smaller than if we removed all of it.

The above steps yield the optimal, constrained solution $\tilde{\mathbf{x}}$ to system (3). All that remains to get the corresponding solution to system (1) is to apply the change from v to canonical basis to $\tilde{\mathbf{x}}$. The residue of this solution can be already found with the u basis.

The programming specifications:

The concrete instructions for the assignment are: you must create a Matlab funcion with header line

```
function [x,residue]=constr_linear_opt(A,b,M)
```

The input variables `A,b` are respectively the coefficient matrix and the independent term (as a column vector) of the linear system to be solved. The variable `M` is the maximal allowable size for the solution.

The function must return variables `x`, which is the solution to the constrained problem as described above (column vector), and `residue`, which is the residue for this solution.

Verification and handing in:

Hand in your assignment by sending an e-mail to Jaume Amorós attaching to it the Matlab file of the function. Write the name of the two group members in a comment line at the bottom of the file. There will be a penalty of 1 point (out of total score 10) in case of single member groups. Groups with more than 2 members, or multiple group membership, will not be accepted.

The deadline for handing in the assignment is 24:00, December 5. Late hand-ins will have a penalty of 2 points (out of total score 10) for every 24 hours or fraction of delay.

Verify that your function works correctly before handing it in. A Matlab script with examples will be provided in Atenea, but these do not cover all types of cases. Any function that works incorrectly (breaks down, or gives incorrect solutions) in any of the cases tested when grading will receive a fail score.

You can discuss freely how to do this assignment with other groups, except that you are not allowed neither to read other groups' code nor to show your code to anyone outside your group. Hand ins with outstanding coincidence of part of the code will be considered unoriginal work, and will get total score 0.