INTRODUCTION:

- By stochastic processes, we mean a set of r.v $\{X(t)\}$ depending on a real parameter t which ~~defines~~ varies in a certain set of natural no.
- A stochastic process is denoted by $\{X(t), t \in T\}$.

Examples of Stochastic Processes

i) Random walk where we are concerned with the motion of a particle in a straight line which moves in a discrete jumps with certain probabilities ie. at each stage, it moves one unit right to left with probability P or $1-P$.
If initially the particle is at the origin, then its position $X(n)$ after n jumps is a discrete r.v depending on the discrete parameter n.
Thus $\{X(n)\ t \in T\}$ is a stochastic process.

ii) The no. of telephone calls $X(t)$ during the arbitrary period of the time t is a discrete r.v depending on the continuous parameter t.
Thus $\{X(t)\ t \in T\}$ is a stochastic process

iii) The water level of a river under a certain bridge measured daily at any moment t is a continuous r.V depending on the continuous parameter t. Thus $\{X(t)\ t \in T\}$ is a stochastic process

iv) The no. of particles exculting a Brownian motion with small, rapid random steps is a discontinuous r.v depending on the continuous parameter t. Thus $\{X(t), t \in T\}$ constitutes a stochastic process

GENERATING FUNCTIONS:

- Let $a_0\, a_1\, a_2 \ldots$ be a sequence of real numbers. Using a variable S we may define a function,

$$A(s) = a_0 + a_1 s + a_2 s^2 + \cdots + = \sum_{k=0}^{\infty} a_k s^k \quad \text{--- (i)}$$

If the power series $A(s)$ converges in some interval $-S_0 < S < S_0$, then $A(s)$ is called the generating function of the sequence $a_0 a_1 \ldots$

Examples

1. If $a_K = 1$ for all K. ie $\{a_K\} = \{1, 1 \ldots\}$

The generating function $\{a_K\}$ is

$$A(s) = a_0 + a_1 s + a_2 s^2 + \ldots$$

$$1 + s + s^2 + \ldots$$

$$= \frac{1}{1-s} \quad |s| < 1$$

2. If $\{a_K\} = \{\frac{1}{K!}\}$

The gf of $\{\frac{1}{K!}\}$ is

$$A(s) = a_0 + a_1 s + a_2 s^2 + \ldots$$

$$= \frac{1}{0!} + \frac{1}{1!} s + \frac{1}{2!} s^2 \ldots$$

$$= e^s$$

3. Let $\{a_K\} = \{\binom{n}{K}\}$ for fixed n.

The gf of $\{a_K\}$ is $A(s) = \sum_{K=0}^{\infty} a_K s^K = \sum_{K=0}^{\infty} \binom{n}{K} s^K$ (any value $> n$ is zero)

$$= (1+s)^{n}$$

4. Let $a_0\, a_1\, a_2 \ldots a_n$ be a sequence of numbers satisfying the recurrence relation $a_n + a_{n-1} - 16a_{n-2} + 20a_{n-3} = 0$ for $n \geq 3$ Subject to the initial Values $a_0 = 0$ $a_1 = 1$ & $a_2 = -1$. Obtain the generating function of the above sequence $\{a_n\}$

Let $A(s) = \sum_{n=0}^{\infty} a_n s^n$ be the gf of $\{a_n\}$ — Multiplying by s^n & summing over n

$$\sum_{n=3}^{\infty} a_n s^n + \sum_{n=3}^{\infty} a_{n-1} s^n - 16 \sum_{n=3}^{\infty} a_{n-2} s^n + 20 \sum_{n=3} a_{n-3} s^n = 0$$

Applying Conditions

$$(A(s) - a_0 - a_1 s - a_2 s^2) + s \sum_{n=3} a_{n-1} s^{n-1} - 16 s^2 \sum_{n=3} a_{n-2} s^{n-2} + 20 s^3 \sum_{n=3} a_{n-3} s^{n-3} = 0.$$

$$(A(s) - s + s^2) + s(A(s) - a_0 - a_1 s^1) - 16 s^2 (A(s) - a_0) + 20 s^3 A(s) = 0$$

$$A(s) - s + s^2 + sA(s) - s^2 - 16 s^2 A(s) + 20 s^3 A(s) = 0$$

Making $A(s)$ Subject

$$A(s)(1 + s - 16 s^2 + 20 s^3) = s$$

$$A(s) = \frac{s}{1 + s - 16 s^2 + 20 s^3} \text{ is gf of } \{a_n\}.$$

Probability Generating Function

- Suppose that x is a r.v which assumes non-negative integral values $0,1,2\ldots$ & that $Pr(X=K)=P_K$ $K=0,1,2\ldots$ $\sum P_K = 1$ then the probability generating function $P(s)$ of the sequence of probabilities $\{P_K\}$ is given by;

$$P(s)=\sum_{K=0}^{\infty} P_K S^K = P_0 + P_1 S + P_2 S^2 + \ldots + P_K S^K + \ldots$$

$$=\sum Pr(X=K)S^K = E(S^X)$$

Where $E(S^X)$ is the expectation of the function S^X (a r.v) of the r.v X. The series $P(s)$ converges for at least $-1 \leq S \leq 1$.

Clearly $P(1)=1$.

NB/ It is possible to have a sequence of probabilities which does not form a pgf.

eg. If $P_0 = 0$, $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{2}$, $P_3 = \frac{1}{3}$, $P_4 = \frac{3}{4}$.

$$A(s) = P_0 + P_1 S + P_2 S^2 + P_3 S^3 + P_4 S^4$$

$$= \frac{1}{4}S + \frac{1}{2}S^2 + \frac{1}{3}S^3 + \frac{3}{4}S^4$$

$A(s)$ is a gf of $\{P_K\}$ but not a pgf $P_0 + P_1 + P_2 + P_3 + P_4 \neq 1$.

Determining mean & Variance using pgf

$$P(s) = \sum_{K=0} P_K S^K$$

$$P'(s) = \sum_{K=} K P_K S^{K-1} \quad -1 < S < 1.$$

$$\& \quad P''(s) = \sum K(K-1) P_K S^{K-2}$$

The Expectation $E(x)$ is given by

$$E(x) = P'(1) = \sum_K K P_K \quad \&$$

$$E(x^2) = E[x(x-1)+x]$$

$$E[x(x-1)] + E(x)$$

But

$$E[x(x-1)] = P''(1) = \sum_K K(K-1)P_K$$

$$E(x^2) = P''(1) + P'(1)$$

Hence

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= P''(1) + P'(1) - [P'(1)]^2.$$

The pgf & moments of Std Distributions.

i) Bernoulli Distribution.

- Let x have a Bernoulli distr with p. i.e.

$$Pr\{X=K\} = P_K = p^K q^{1-K} \quad q = 1-p \quad K = 0,1.$$

The pdf of x is given by

$$P(s) = \sum P_K S^K = \sum_0^1 p^K q^{1-K} S^K \qquad \sum_0^1 (ps)^K q^{1-K}$$

$$= q + ps$$

wrt s

$$P'(s) = p \qquad P''(s) = 0 \qquad Var(x) = P''(1) + P'(1) - [P'(1)]^2$$

$$E(x) = P'(1) = p \qquad P''(1) = 0 \qquad 0 + p - p^2$$

$$p(1-p)$$

$$= pq$$

Since $p+q=1$.

ii) Binomial Distribution-

$$Pr\{x=k\} = P_K = \binom{n}{K} p^K q^{n-K} \quad K = 0,1 \dots n \quad q+p = 1.$$

The pgf of x is given by

$$P(s) = \sum P_K S^K = \sum^n \binom{n}{K} p^K q^{n-K} S^K$$

$$= (q+ps)^n$$

$$P'(s) = n(q+ps)^{n-1} p$$

$$np(q+ps)^{n-1}$$

$$E(x) = P'(1) = np(q+p)^{n-1}$$

$$= np \quad \text{since } q+p = 1.$$

$$P''(s) = np(n-1)(q+ps)^{n-2}p$$
$$P''(1) = n(n-1)p^2(q+p)^{n-2}$$
$$= n(n-1)p^2 \quad \text{since } q+p=1.$$
$$Var(x) = P''(1) + P'(1) - [P'(1)]^2$$
$$= n(n-1)p^2 + np - (np)^2$$
$$= n^2p^2 - np^2 + np - n^2p^2$$
$$= np(1-p) = npq$$

iii) Geometric Distribution.

$$Pr\{X=k\} = P_K = q^k p \quad p+q=1 \quad K=0,1,2\cdots$$

The pgf of X is given by

$$P(s) = \sum_{K=0}^{\infty} P_K s^k = \sum_{K=0}^{\infty} q^k p s^k$$
$$= p\sum_{K=0}(qs)^k \quad \rightarrow \text{Geometric Series}$$
$$= p\left(1+qs+(qs)^2+(qs)^3+\dots\right)$$
$$= p\left(\frac{1}{1-qs}\right) = \frac{p}{1-qs}$$

$$P'(s) = \frac{pq}{(1-qs)^2}. \qquad P''(s) = \frac{2pq^2}{(1-qs)^3}.$$

$$E(x) = P'(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p} \qquad P''(1) = \frac{2pq^2}{(1-q)^3} = \frac{2q^2}{(1-q)^3} = \frac{2pq^2}{p^3}$$

$$Var(x) = P''(1) + P'(1) - [P'(1)]^2$$
$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$
$$= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q^2+pq}{p^2} = \frac{q(q+p)}{p^2} = \frac{q}{p^2} = \frac{q}{p^2} \quad (p+q)=1.$$

iv) Poisson Distr.

$$Pr(X=k) = P_K = \frac{e^{-\lambda}\cdot\lambda^k}{k!} \quad K=0,1,2.$$

The pgf of X is given by;

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$$\sum_{k=0} P_k s^k = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} s^k$$

Has nothing to do with $e^{-\lambda}$

$$P(s) = e^{-\lambda} \sum_{k=0} \frac{(\lambda s)^k}{k!}$$

$$= e^{-\lambda}\left[1 + \lambda s + \frac{(\lambda s)^2}{2!} + \dots\right]$$ → Maclaurin

$$= e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}$$ → factoring $-\lambda$

$$P'(s) = \lambda e^{-\lambda(1-s)} \qquad P''(s) = \lambda^2 e^{-\lambda(1-s)}$$

$$E(x) = P'(1) = \lambda e^{-\lambda(1-1)} = \lambda e^0 = \lambda \qquad P''(1) = \lambda^2 e^{-\lambda(1-1)} = \lambda^2 e^0 = \lambda^2$$

$$Var(x) = P''(1) + P'(1) - [P'(1)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda.$$

Remarks

i) When $s = e^t$ then

$P(s) = P(e^t) = E(e^{tx})$ which is the mgf of x.

ii) $\int_0^1 P(s)ds = \int_0^1 \left(\sum P_k s^k\right) ds.$

$$\int_0^1 (P_0 + P_1 s + P_2 s^2 + P_3 s^3 + P_4 s^4 + \dots) ds$$

$$\int_0^1 P_0 ds + \int_0^1 P_1 s ds + \int_0^1 P_2 s^2 ds + \int_0^1 P_3 s^3 ds + \int_0^1 P_4 s^4 ds + \dots$$

$$= P_0 + \frac{P_1}{2} + \frac{P_2}{3} + \frac{P_3}{4} + \frac{P_4}{5} + \frac{P_5}{6} + \dots$$

$$= \sum_{k=0} \frac{P_k}{k+1} = E\left(\frac{1}{x+1}\right)$$

Example.

1. Let X be a r.v with pgf $P(s)$. Find the pgf of $Y = mX + n$, where m, n are integers & $m \neq 0$.

$P_x(s) = E(s^x)$

Let $P_y(s)$ be the pgf of Y. Then

$$P_y(s) = E(s^y) = E\left[s^{mX+n}\right]$$

$$= E\left[s^{mx} s^n\right]$$

$$= s^n E\left[(s^m)^x\right] = s^n P(s^m)$$

Remark i) We can obtain probabilities P_k from the pgf. P_k can be found from $P(s)$ i.e.

$$P_k = \frac{1}{k!}\left[\frac{d^k P(s)}{ds^k}\Big|_{s=0}\right].$$

ii) P_k is also given by the coefficient of s^k in the expansion of $P(s)$ as a power series eg

If $P(s) = e^{-\lambda(1-s)}$

$$= e^{-\lambda} e^{\lambda s}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda s}{1!} + \frac{(\lambda s)^2}{2!} + \dots \frac{(\lambda s)^k}{k!} + \dots\right]$$

P_k = Coefficient of s^k

$= \frac{e^{-\lambda}\lambda^k}{k!}$ Which is a poisson distr. with parameter λ.

Sum of Random Variables.

i) Fixed Number.

- Let X & Y be two independent r.v's (non-negative integral r.v's) with prob distributions

$$a_k = P(X=k) \text{ \& } b_j = P(Y=j)$$

- Let $P_x(s)$ be the pgf of X & let $P_y(s)$ be the pgf of Y. If the pgf of $Z = X+Y$ is $P_z(s)$

$$P_z(s) = E[s^z] = E[s^{x+y}]$$

$$E[s^x s^y]$$

$$= E[s^x]E[s^y] \text{ Since } X \text{ \& } Y \text{ are independent}$$

$$= P_x(s) \cdot P_y(s)$$

- This result says that the pgf of the sum of 2 independent r.v X & Y is the product of the pgf of X & that of Y. The result expands to the case when we have n independent r.vs $X_1, X_2 \dots X_n$ i.e. The pgf of $Z = X_1 + X_2 + \dots X_n$ is the product of pgfs of the individual r.v's

$$P_z(s) = E[s^z] = E(s^{x_1+x_2+\cdots x_n})$$

$$E(s^{x_1})\cancel{E(x)}\cdot E(s^{x_2}) \cdot \ldots E(s^{x_n}) \rightarrow \text{Since } x_1, x_2 \cdots x_n \text{ are independent}$$

$$= P_{x_1}(s)\, P_{x_2}(s) \ldots P_{x_n}(s)$$

Product

- If $X_1, X_2 \cdots X_n$ are also identically distributed (iid) each with pgf $P(s)$ then the pgf of $Z = X_1 + X_2 \cdots X_n$ is $\cancel{P(s)}$ $[P(s)]^n$.

Example:

1. Let X_1 & X_2 be independent poisson r.v's with parameters λ_1 & λ_2. Find the pgf of $Z = X_1 + X_2$.

Let $P_{x_1}(s)$ & $P_{x_2}(s)$ be the pgf's of X_1 & X_2. Then

$$P_{x_1}(s) = \sum_{K=0} P_K S^K = \sum_{K=0}^{\infty} e^{-\lambda_1} \frac{\lambda_1^K}{K!} S^K = e^{-\lambda_1(1-s)}$$

Similarly $P_{x_2}(s) = e^{-\lambda_2(1-s)}$

The pgf of Z is $P_z(s) = P_{x_1}(s) \cdot P_{x_2}(s)$ since x_1 & x_2 are indep.

Product

$$= e^{-\lambda_1(1-s)} \cdot e^{-\lambda_2(1-s)}$$

$$= e^{-(\lambda_1+\lambda_2)(1-s)}$$

This is the pgf of a poisson r.v with parameters $\lambda_1 + \lambda_2$. Therefore Z has a poisson distr. with parameter $\lambda_1 + \lambda_2$

ii) Random Number

Theorem:

Let X_i, $i = 1, 2, \ldots$ be iid rv's with $P(X_i = x) = P_x$ & pgf $P(s) = \sum_{K=0}^{\infty} P_K S^K$ $i = 1, 2, 3 \ldots$. Let $S_N = X_1 + X_2 + \ldots + X_N$ where N is a r.v independent of the X_i's. Let the distribution of N be given by $P(N=n) = g_n$ & the pgf of N be $G(s) = \sum_{n=0}^{\infty} g_n s^n$.

Then the pgf $H(s)$ of S_N is given by the compound function $G[P(s)]$ ie.

$$H(s) = \sum_{j=0}^{n} P(S_N = j) s^j = G[P(s)]$$

$$H(s) = G[P(s)] \qquad P(s) = \sum P_k s^k$$
$$P(1) = P_0 + P_1 + P_2 + \dots = 1$$
$$H'(s) = G'[P(s)]P'(s)$$
$$E(S_N) = H'(1) = G'[P(1)]P'(1)$$
$$= G'(1)P'(1)$$
$$= E(N)E(X_i)$$

Example.

1. Suppose N has a poisson distr. with parameter λ & X_i $i = 1, 2 \dots N$ are iid with pgf $P(s)$. Find the pgf of $S_N = X_1 + X_2 + \dots X_N$. Hence find mean of S_N

The pgf of S_N is
$$H(s) = G[P(s)]$$
But pgf of a poisson r.v with mean λ is given by $G(s) = e^{-\lambda(1-s)}$
$$H(s) = e^{-\lambda(1-P(s))}$$
$$E(S_N) = E(N)E(X_i)$$
$$= \lambda E(X_i)$$

Tailed or Cumulative Probabilities.

i) Upper Tail

- Let X have a pdf $P_r\{X = k\} = P_k$, $k = 0, 1, 2$ with pgf $P(s) = \sum_{k=0} P_k s^k$ & $q_k = P_r\{X > k\} = P_{k+1} + P_{k+2} \dots$ $k = 0, 1, 2$ with generating function $\phi(s) = \sum_{k=0}^{\infty} q_k s^k$ $(\sum q_k \neq 1)$
- Express $\phi(s)$ in terms of $P(s)$
$$\phi(s) = \sum_{k=0}^{\infty} q_k s^k$$
$$= q_0 + q_1 s + q_2 s^2 + q_3 s^3 + \dots$$

Substituting

$$\phi(s) = (P_1 + P_2 + P_3 + \dots) + (P_2 + P_3 + P_4 + \dots)s + (P_3 + P_4 + P_5 + \dots)s^2 + (P_4 + P_5 + P_6 + \dots)s^3 + \dots = 1.$$

Collecting like terms

$$= P_1 + P_2(1+s) + P_3(1+s+s^2) + P_4(1+s+s^2+s^3) + \dots$$
$$= P_1 + P_2\frac{(1-s^2)}{1-s} + P_3\frac{(1-s^3)}{1-s} + P_4\frac{(1-s^4)}{1-s}$$

$$\frac{P_1(1-S) + P_2(1-S^2) + P_3(1-S^3) + P_4(1-S^4) + \dots}{(1-S)}$$

Common den

$$\frac{(P_1+P_2+P_3+P_4+\dots) - (P_1 S + P_2 S^2 + P_3 S^3 + P_4 S^4 + \dots)}{1-S}$$

→ Introducing P_0

$$\frac{(P_0+P_1+P_2+P_3+P_4+\dots) - (P_0 + P_1 S + P_2 S^2 + P_3 S^3 + P_4 S^4 + \dots)}{1-S}$$

↳ pgf

$$= \frac{1-P(S)}{1-S} \quad |S|<1.$$

OR

$$q_K = P_{K+1} + P_{K+2} + P_{K+3} + P_{K+4} + \dots$$

Substituting K with K−1

$$q_{K-1} = P_K + P_{K+1} + P_{K+2} + P_{K+3} + \dots$$

$$q_{K-1} - q_K = P_K$$

Multiplying by S^K & summing

$$\sum_{K=1} q_{K-1} S^K - \sum_{K=1} q_K S^K = \sum_{K=1} P_K S^K$$

$$S \cdot \sum_{K=1} q_{K-1} S^{K-1} - \sum_{K=1} q_K S^K = \sum_{K=1} P_K S^K$$

$$\phi(S) = \sum_{K=0} q_K S^K$$

$$S\phi(S) - [\phi(S) - q_0] = P(S) - P_0$$

$$P(S) = \sum_{K=0} P_K S^K$$

$$S\phi(S) - \phi(S) + q_0 = P(S) - P_0$$

$$\phi(S)[S-1] = P(S) - (P_0 + q_0)^{\to 1}$$

$$q_0 = P_1 + P_2 + \dots$$

$$\phi(S) = \frac{P(S)-1}{S-1}$$

$$q_0 + P_0 = [P_1 + P_2 + \dots] + P_0$$

$$q_0 + P_0 = 1$$

$$\phi(S) = \frac{1-P(S)}{1-S}.$$

ii) Lower Tail.

- Let $q_K = Pr\{X \le K\}$ $K = 0, 1, 2 \dots$ & $P_K = \sum P_K S^K = P(S)$ pgf of $\{P_K\}$ &

$\phi(S) = \sum_{K=0} q_K S^K \to$ gf of $\{q_K\}$

$$\cancel{\phi(S)=}\ q_K = P_K + P_{K-1} + P_{K-2} + \dots + P_1 + P_0$$

$$q_{K-1} = P_{K-1} + P_{K-2} + \dots + P_1 + P_0.$$

$$q_K - q_{K-1} = P_K \quad \text{--- (1)}.$$

Multiplying both sides of eq. (1) by S^K & summing over K we have

$$\sum_{K=1}^{\infty} q_K S^K - \sum_{K=1} q_{K-1} S^K = \sum P_K S^K$$

$$\phi(s) - q_0 - S \sum q_{K-1} S^{K-1} = P(S) - P_0$$

$$\phi(s) - q_0 - S\phi(s) = P(S) - P_0.$$

$$\phi(s) - S\phi(s) = P(s) \text{ since } q_0 = P_0$$

$q_K - q_{K-1} = P_K$, $K=0$, $q_0 - q_{-1} = P_0$

$$\phi(s)(1-s) = P(s)$$

$$\phi(s) = \frac{P(s)}{1-s} \quad -1 < s < 1.$$

BIVARIATE PGFs.

- Suppose X & Y are integral valued r.v with joint prob distr

$$\Pr\{X=j, Y=K\} = P_{JK} \quad j=0,1,2\ldots \quad K=0,1,2\ldots$$

$$\& \sum_J \sum_K P_{JK} = 1.$$

Then the joint pgf of X & Y is given by:

$$P(S_1 S_2) = \sum_{J=0} \sum_{K=0} P_{JK} S_1^J S_2^K$$

- Assuming convergence for the values of S_1 & S_2 ie $|S_1| < 1$, $|S_2| < 1$

$$P(S_1 S_2) = E(S_1^X S_2^Y) \quad \ldots (1)$$

- If $S_1 = S_2 = S$ then 1 becomes

$$P(S,S) = E[S^X S^Y] = E[S^{X+Y}]$$

- If X & Y are mutually indep r.v then (1) becomes

$$P(S_1 S_2) = E(S_1^X) E(S_2^Y)$$

If X & Y are iid with a common pgf G(s) then we have

$$P(S_1 S_2) = E(S_1^X) E(S_2^Y) = G(s) \cdot G(s) = [G(s)]^2$$

Pgf of Marginal Distributions.

$P(S_1 S_2) = E(S_1^X S_2^Y) \rightarrow$ joint pgf of X & Y

The pgf of X is

$$P(S_1, 1) = E(S_1^X)$$

re

Similarly pgf of Y is

$$P(1,S_2) = E(S_2^{Y})$$

Calculations of Means, Variances, Covariances & Correlation of Bivariate Distr. Functions using joint pdf.

The joint Pgf of X

$$P(S_1) = P(S_1, 1)$$

$$P'(S_1) = \frac{\delta P(S_1, 1)}{\delta S_1}$$

$$E(X) = P'(1) = \frac{\partial P(S_1, 1)}{\partial S_1}\Big|_{S_1=1.}$$

$$E(Y) = \frac{\delta P(1, S_2)}{\delta S_2}\Big|_{S_2=1}$$

$$E(X(X-1)) = \frac{\delta^2 P(S_1, 1)}{\delta S_1^2}\Big|_{S_1=1}$$

$$E[Y(Y-1)] = \frac{\delta^2 P(1, S_2)}{\delta S_2^2}\Big|_{S_2=1}$$

Also

$$E(XY) = \frac{\delta^2 P(S_1 S_2)}{\partial S_1 \partial S_2}\Big|_{S_1=S_2=1}$$

$$\delta_x^2 = E[X(X-1)] + E(X) - [E(X)]^2 = P_x''(1) + P_x'(1) - [P_x'(1)]^2$$

$$\delta_Y^2 = E[Y(Y-1)] + E(Y) - [E(Y)]^2 = P_Y''(1) + P_Y'(1) - [P_Y'(1)]^2$$

$$Cov(XY) = \delta_{XY} = E(XY) - E(X)E(Y)$$

$$\rho_{xy} = \frac{Cov(X,Y)}{\sqrt{\delta_X^2 \delta_Y^2}}$$

Example.

1. Consider a series of Bernoulli trials with prob of success p. Suppose that X denotes the no. of failures proceeding the first success & Y the no. of failures preceeding the ~~first~~ second success ~~& of the~~ & no of failures following the first success. Then the joint prob distr. is given by

$P_{jk} = Pr\{X=j, Y=k\} = q^{j+k}p^2 \quad j,k = 0,1,2 \ldots q+p=1$

Obtain the joint pdf of X & Y then use it to determine means & Variances of X & Y, Cov btwn X & Y & corr. btwn X & Y.

The joint pgf is $P(s_1 s_2) = \sum_{j=0}\sum_{k=0} P_{jk} s_1^j s_2^k$

$$P(s_1 s_2) = \sum_{j=0}\sum_{k=0} q^{j+k} p^2 s_1^j s_2^k \quad \to \text{Substituting}$$

$$p^2 \sum\sum (qs_1)^j (qs_2)^k$$

$$p^2 \left(\sum_{j=0}^{k} (qs_1)^j\right)\left(\sum_{k=0} (qs_2)^k\right)$$

$$p^2\left[1+(qs_1)+(qs_1)^2+\ldots\right]\left[1+(qs_2)+(qs_2)^2+\ldots\right] \quad \to \text{Geometric}$$

$$= p^2\left(\frac{1}{1-qs_1}\right)\left(\frac{1}{1-qs_2}\right)$$

$$= \frac{p^2}{(1-qs_1)(1-qs_2)}$$

$p+q=1$

$$P(s_1) = P(s_1, 1) = \frac{p^2}{(1-qs_1)(1-q)} = \frac{p}{(1-qs_1)} \quad \text{pgf of X.}$$

$$P(s_2) = P(1, s_2) = \frac{p^2}{(1-q)(1-qs_2)} = \frac{p}{(1-qs_2)} \quad \text{pgf of Y.}$$

$$p(1-qs_1)^{-1}, \quad -p(1-qs_1), \quad \frac{pq}{1-qs_1}$$

$$P'(s_1) = \frac{pq}{(1-qs_1)^2} \Rightarrow E(X) = P'(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$P'(s_2) = \frac{pq}{(1-qs_2)^2} \Rightarrow E(Y) = P'(1) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

$$P''(s_1) = \frac{2pq^2}{(1-qs_1)^3} \Rightarrow P''(1) = \frac{2pq^2}{(1-q)^3} = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

$$P''(s_2) = \frac{2pq^2}{(1-qs_2)^3} \Rightarrow P''(1) = \frac{2pq^2}{(1-q)^3} = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

$$Var(X) = P''(1) + P'(1) - [P'(1)]^2$$

$$\frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$

$$\frac{q^2}{p^2} + \frac{q}{p} = \frac{q^2+qp}{p^2} \Rightarrow \frac{q(q+p)}{p^2} = \frac{q}{p^2}$$

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Similarly $var(Y) = \frac{q}{p^2}$

$$\frac{\delta^2 P(S_1S_2)}{\delta S_1 \delta S_2} = \frac{\delta}{\delta S_1}\left(\frac{\delta P(S_1S_2)}{\delta S_2}\right) = \frac{\delta}{\delta S_1}\left(\frac{p^2 q}{(1-qS_1)(1-qS_2)^2}\right)$$

$$= \frac{p^2q^2}{(1-qS_1)^2(1-qS_2)^2}$$

$$E(XY) = \frac{\delta^2 P(S_1 S_2)}{\delta S_1 \delta S_2}\bigg|_{S_1 = S_2 = 1}$$

substitute

$$\frac{p^2q^2}{(1-q)^2(1-q^2)} = \frac{p^2q^2}{(p^2)(p^2)} = \frac{q^2}{p^2}.$$

$$Cov(XY) = E(XY) - E(X)E(Y)$$

$$\frac{q^2}{p^2} - \left(\frac{q}{p}\right)\left(\frac{q}{p}\right) = 0.$$

$$\rho_x = \frac{Cov(XY)}{\sqrt{\delta_x^2 \delta_y^2}} = 0$$

BIRTH-DEATH PROCESS

- Let $z(t)$ be the pop size of time t & $P_n(t)$ be the prob that a pop is of z size n at time t.

Therefore $P_n(t) = Prob\{z(t) = n\}$

- Let us make the following assumption for the time ~~int~~ interval Δt,

1) The prob that from a pop of size n a birth occurs at is $\lambda_n \Delta t + o(\Delta t)$

ii) The prob that from a pop of size n a death occurs is $\mu_n \Delta t + o(\Delta t)$.

iii) The prob that there is no change in the pop size within the time interval Δt is $1 - [\lambda_n + \mu_n]\Delta t + o(\Delta t)$.

iv) The prob that there is more than a birth or a death within the time interval Δt is negotiable.

- We now wish to build up a model for $(P_n(t+\Delta t))$. Therefore;

$$P_n(t+\Delta t) = \text{Prob}\{(z(t)=n-1, z(\Delta t)=1) \text{ or } (z(t)=n+1, z(\Delta t)=-1) \text{ or } \cancel{z(t)=n} \; z(t)=n, z(\Delta t)=0)\}$$

$$= \text{Prob}[z(t)=n-1, z(\Delta t)=1] + \text{Prob}[z(t)=n+1, z(\Delta t)=-1] + \text{Prob}[z(t)=n, z(\Delta t)=0]$$

$P(A|B) = \frac{P(A,B)}{P(B)}$

& $P(B|A) = \frac{P(A,B)}{P(A)}$

$P(B,A) = P(B|A)P(A)$

$$= \text{Prob}[z(\Delta t)=1 / z(t)=n-1]\,\text{Prob}[z(t)=n-1] + \text{Prob}[z(\Delta t=0) / z(t)=n]\,\text{Prob}[z(t)=n]$$

$$= [\lambda_n \Delta t + o(\Delta t)]P_{n-1}(t) + [\mu_{n+1}\Delta t + o(\Delta t)]P_{n+1}(t) + [1-(\lambda_n+\mu_n)\Delta t + o(\Delta t)]P_n(t)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

But; $$P_n'(t) = \frac{dP_n(t)}{dt} = \lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t}$$

Therefore

$$P_n'(t) = \lim_{\Delta t\to 0}\left\{\left[\lambda_{n-1} + \frac{o(\Delta t)}{\Delta t}\right]P_{n-1}(t) + \left[\mu_{n+1} + \frac{o(\Delta t)}{\Delta t}\right]P_{n+1}(t) - \left[\lambda_n+\mu_n\right] + \frac{o(\Delta t)}{\Delta t}\right]P_n(t)\right\}.$$

Since $\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$ we have

$$P_n'(t) = -(\lambda_n+\mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \quad n\geq 1 \quad \cdots (1)$$

and

$$P_0'(t) = -(\lambda_0+\mu_0)P_0(t) + \mu P_1(t), \quad n=0$$

which are called Difference differential Eq.

- The problem is to solve these difference-differential eq (1) for some special cases of λ_n & μ_n along with some initial conditions.
- One approach is to use the pgf technique ~~using~~ along with the ordinary partial differencial equations technique

Solns of Linear Partial Differential Eqns

- Suppose we have the eqs

$$P\frac{\delta z}{\delta x} + Q\frac{\delta z}{\delta y} = R \quad \cdots (1)$$

Subject to some boundary conditions where P, Q, R are functions of x, y, z. It can be shown that eq (1) is equated to

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \cdots (2)$$

which is known as Auxiliary or Subsidiary set of eq.

- Eq (1) is known as Langrange's Linear Eq. For eq (2) we can have two equations eg

$$\frac{dx}{P} = \frac{dy}{Q} \quad \& \quad \frac{dy}{Q} = \frac{dz}{R}$$

- Let $u(x,y,z) =$ Constant & $v(x,y,z) =$ Constant by any 2 solns of auxilliary eq. A general soln to eq (1) becomes:

$$\phi(u,v) = 0 \quad \text{or} \quad u = \phi(v)$$

- In most of our problems we shall be using $u = \phi(v)$ given the appropriate conditions

THE PURE BIRTH PROCESS

- The general pure birth process is obtained from birth-death process by putting $\mu_n = 0$ in the diff-differential eq, thus we have

$$P_n'(t) = -\lambda_n P(t) + \lambda_{n-1} P_{n-1}(t) \quad n \geq 1$$

$$\& \quad P_0'(t) = -\lambda_0 P_0(t), \; n = 0$$

- For $\lambda_n = \lambda$ we have Poisson Process

For $\lambda_n = n\lambda$ we have a Single Birth Process (Yule-Process or Yule Furry Process)

For $\lambda_n = \lambda\left(\frac{1+an}{1+\lambda at}\right)$ we have Polya Process.

a) Poisson Process

$\lambda_n = \lambda$ & $\mu_n = 0$

- Thus the diff differential eq is given by

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \quad n \geq 1$$

$$\& \ P_0'(t) = -\lambda P_0(t) \quad n = 0$$

- Define

$$G(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$$

to be the pgf of $\{P_n(t)\}$

$$\frac{\delta G(s,t)}{\delta s} = \sum_{n=0}^{\infty} n P_n(t) s^{n-1} \quad \& \ \frac{\delta G'(s,t)}{\delta t} = \sum_{n=0}^{\infty} P_n'(t) s^n$$

- To introduce $G(s,t)$ & its derivatives in the diff differential eq, multiply the eq by s^n & sum the result over n

$$\sum_{n=1}^{\infty} P_n'(t) s^n = -\lambda \sum_{n=1} P_n(t) s^n + \lambda \sum P_{n-1}(t) s^n$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -\lambda \left[G(s,t) - P_0(t) \right] + \lambda s \sum_{n=1}^{\infty} P_{n-1}(t) s^{n-1}$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -\lambda G(s,t) + \lambda P_0(t) + \lambda s G(s,t)$$

$$\frac{\delta G(s,t)}{\delta t} = -\lambda G(s,t) + \lambda s G(s,t) \quad \text{Since } P_0'(t) = -\lambda P_0(t)$$

$$\frac{\delta G(s,t)}{\delta t} = -\lambda(1-s) G(s,t) \qquad -\lambda G(s,t)[1-s]$$

$$\frac{\delta G(s,t)}{\delta t} + (0)\frac{G(s,t)}{\delta s} = -\lambda(1-s) G(s,t) \rightarrow \text{Largrange's Linear eq.}$$

- The Corresponding set of auxilliary eq is

$$\frac{dt}{1} = \frac{ds}{0} = \frac{dG(s,t)}{-\lambda(1-s)G(s,t)}$$

From $\dfrac{dt}{1} = \dfrac{dG(s,t)}{-\lambda(1-s)G(s,t)}$

$\rightarrow$ Separating variable

$$\int \frac{dG(s,t)}{G(s,t)} = \int -\lambda(1-s)\,dt$$

$$\ln G(s,t) = -\lambda(1-s)t + c$$

$$G(s,t) = e^{-\lambda(1-s)t + c}$$

Suppose the initial conditions are $P_0($

$P_0(0) = 1$

$P_n(0) = 0 \quad n \neq 0$

$G(s,0) = e^{c}$

But $G(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n = P_0(t) + P_1(t)s + P_2(t)s^2 + \dots$

$$G(s,0) = P_0(0) + P_1(0)s + \dots$$

$$= P_0(0) = 1$$

$e^{c} = 1$. $\quad c = 0$

$\therefore \; G(s,t) = e^{-\lambda(1-s)t}$

$$= e^{-\lambda t} \cdot e^{\lambda st}$$

$\rightarrow$ Maclaurin

$$= e^{-\lambda t}\left[1 + \lambda st + \frac{(\lambda st)^2}{2!} + \dots \frac{(\lambda st)^n}{n!}\right] \qquad e^{-\lambda t} \frac{(\lambda ts)^k}{k!}$$

$P_n(t) =$ Coefficient of S_n in the expansion of $G(s,t)$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Which is a Poisson Distr. with parameters λt

Mean.

$G(s,t) = e^{-\lambda t} e^{\lambda st}$

wrt s

$G'(s,t) = e^{-\lambda t} \lambda t e^{\lambda st}$

$E(n) = G'(1,t) = e^{-\lambda t} \lambda t e^{\lambda t} = \lambda t$.

Variance

$Var(n) = G''(1,t) + G'(1,t) - [G'(1,t)]^2$

But $G''(s,t) = e^{-\lambda t}(\lambda t)^2 e^{\lambda st}$

$G''(1,t) = e^{-\lambda t}(\lambda t)^2 e^{\lambda t} = (\lambda t)^2$

$$Var(n) = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t$$

$\lambda_n = n\lambda$

$\lambda_{n-1} = (n-1)\lambda$

$\mu_n = 0$

b) The Simple Birth Process (Yule Process)

- In this case, $\lambda_n = n\lambda \Rightarrow \lambda_{n-1} = (n-1)\lambda$, $U_n = 0$ Therefore the diff - differential eq becomes

$$P_n'(t) = -n\lambda P_n(t) + \lambda(n-1)P_{n-1}(t) \quad n \geq 1$$

$$\& \quad P_0(t) = 0$$

$G(s,t) = \sum P_n(t) s^n$

$\frac{\delta G(s,t)}{\delta s} = \sum_{n=1} n P_n(t) s^{n-1}$

$\frac{\delta G(s,t)}{\delta t} = \sum_{n=0} P_n'(t) s^n$

Multiplying by s^n & Summing over n

$$\sum_{n=1} P_n'(t) s^n = -\lambda \sum_{n=1} n P_n(t) s^n + \lambda \sum_{n=1}^{\infty} (n-1) P_{n-1}(t) s^n$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -\lambda s \sum n P_n(t) s^{n-1} + \lambda s^2 \sum_{n=2} (n-1) P_{n-1}(t) s^{n-2}$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -\lambda s \frac{\delta G(s,t)}{\delta s} + \lambda s^2 \frac{\delta G(s,t)}{\delta s}$$

$$\frac{\delta G(s,t)}{\delta t} = -\lambda s(1-s) \frac{\delta G(s,t)}{\delta s} \quad \text{since } P_0'(t) = 0$$

- Therefore the Largrange's linear eq is

$$\frac{\delta G(s,t)}{\delta t} + \lambda s(\cancel{s-1}\ 1-s) \frac{\delta G(s,t)}{\delta s} = 0$$

- The Corresponding set of auxilliary eq are,

$$\frac{dt}{1} = \frac{ds}{\lambda s(1-s)} = \frac{dG(s,t)}{0}$$

From $\frac{dt}{1} = \frac{dG(s,t)}{0}$ we have

$$\int dG(s,t) = \int 0\, dt$$

$$G(s,t) = C_1$$

From $\frac{dt}{1} = \frac{ds}{\lambda s(1-s)}$ we have

$$\int \frac{ds}{s(1-s)} = \int \lambda \, dt$$

$\frac{1}{s} + \frac{1}{1-s} = \frac{A}{s} + \frac{B}{1-s}$

$\ln s - \ln(1-s)$

$$\ln\left(\frac{s}{1-s}\right) = \lambda t + c$$

$$\frac{s}{1-s} = e^{\lambda t + c}$$

$$\left(\frac{s}{1-s}\right)e^{-\lambda t} = e^{c} = C_2$$

Therefore the general soln is;

$$C_1 = \phi C_2$$

$G(s,t) = C_1 = \phi C_2$

i.e. $G(s,t) = \phi\left[\left(\frac{s}{1-s}\right)e^{-\lambda t}\right]$

Therefore

$$G(s,0) = \phi\left(\frac{s}{1-s}\right)$$

Suppose the initial conditions are

$$P_n(0) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$G(s,t) = P_0(t) + P_1(t)s + P_2(t)s^2 + \dots$$

$$G(s,0) = P_0(0) + P_1(0)s + P_2(0)s^2 + \dots$$

$$= P_1(0)s = s \quad \text{Since } P_1(0) = 1$$

$$\phi\left(\frac{s}{1-s}\right) = s$$

Let $W = \frac{s}{1-s} \iff s = \frac{W}{1+W}$.

$W - Ws = s$

$W = s + Ws$

$\frac{W}{1+W} = s$

$$\phi(W) = \frac{W}{1+W}$$

$$G(s,t) = \phi\left[\left(\frac{s}{1-s}\right)e^{-\lambda t}\right] = \phi\left[We^{-\lambda t}\right]$$

$\phi(W)e^{-\lambda t}$

$$\frac{We^{-\lambda t}}{1+We^{-\lambda t}} = \frac{\frac{s}{1-s}e^{-\lambda t}}{1+\frac{s}{1-s}e^{-\lambda t}}$$

$se^{-\lambda t}$

$-s + se^{-\lambda t}$

$$\frac{se^{-\lambda t}}{1-s+se^{-\lambda t}} = \frac{se^{-\lambda t}}{1-(1-e^{-\lambda t})s}$$

$$= se^{-\lambda t}\left[1-(1-e^{-\lambda t})s\right]^{-1}$$

$$G(s,t) = se^{-\lambda t}\left[1+(1-e^{-\lambda t})s+[(1-e^{-\lambda t})s]^2+\dots+[(1-e^{-\lambda t})s]^{n-1}+\dots\right]$$

$P_n(t)$ = Coefficient of s^n to in the expansion of $G(s,t)$

$$= e^{-\lambda t}(1-e^{\lambda t})^{n-1} \quad n=1,2\dots$$

Whish is a Geometric Distr

Mean

$$G(s,t) = se^{-\lambda t}\left[1-(1-e^{-\lambda t})s\right]^{-1}$$

wrt s → Product rule

$$G'(s,t) = e^{-\lambda t}\left[1-(1-e^{-\lambda t})s\right]^{-1} + se^{-\lambda t}\left[1-e^{-\lambda t}\right]\left[1-(1-e^{-\lambda t})s\right]^{-2}$$

$$E(n) = G'(1,t) = e^{-\lambda t}\cdot e^{\lambda t} + e^{-\lambda t}(1-e^{-\lambda t})\cdot e^{2\lambda t}$$

$$= 1 + e^{\lambda t}(1-e^{-\lambda t})$$

$$= 1 + e^{\lambda t} - 1$$

$$= e^{\lambda t}$$

$$G''(s,t) = e^{-\lambda t}(1-e^{-\lambda t})\left[1-(1-e^{-\lambda t})s\right]^{-2} + e^{-\lambda t}(1-e^{-\lambda t})\left\{\left[1-(1-e^{-\lambda t})s\right]^{-2} + (s)(2)(1-e^{-\lambda t})\left[1-(1-e^{-\lambda t})s\right]^{-3}\right\}$$

$$G''(s,t) = e^{-\lambda t}(1-e^{-\lambda t})e^{2\lambda t} + e^{-\lambda t}(1-e^{-\lambda t})\left\{e^{2\lambda t}+2(1-e^{-\lambda t})e^{3\lambda t}\right\}$$

$$= e^{\lambda t}(1-e^{-\lambda t}) + e^{-\lambda t}(1-e^{-\lambda t})e^{2\lambda t} + 2e^{-\lambda t}(1-e^{-\lambda t})^2 e^{3\lambda t}$$

$$= e^{\lambda t} - 1 + e^{\lambda t} - 1 + 2e^{2\lambda t} - 4e^{\lambda t} + 2$$

$$= 2e^{2\lambda t} - 2e^{\lambda t}$$

$$Var(n) = G''(1,t) + G'(1,t) - [G'(1,t)]^2$$

$$= 2e^{2\lambda t} - 2e^{\lambda t} + e^{\lambda t} - e^{2\lambda t}$$

$$= e^{2\lambda t} - e^{\lambda t}$$

$$= e^{\lambda t}(e^{\lambda t}-1)$$

c) The Polya Process

$$\lambda_n = \lambda\left(\frac{1+an}{1+\lambda at}\right), \mu_n = 0$$

- The diff. differential eq. are;

$$P_n'(t) = -\lambda\left(\frac{1+an}{1+\lambda at}\right)P_n(t) + \lambda\left(\frac{1+a(n-1)}{1+\lambda at}\right)P_{n-1}(t) \quad n \geq 1$$

&

$$P_0'(t) = -\frac{\lambda}{1+\lambda at}P_0(t) \quad n=0$$

Multiplying by s^n & summing over n

$$\sum_{n=1}^{\infty} P_n'(t)s^n = \frac{-\lambda}{1+\lambda at}\left\{\sum_{n=1} P_n(t)s^n + a\sum_{n=1} n P_n(t)s^n - \sum_{n=1} P_{n-1}(t)s^n - a\sum_{n=1}(n-1)P_{n-1}(t)s^n\right\}$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = \frac{-\lambda}{1+\lambda at}\left\{G(s,t) - P_0(t) + as\frac{\delta G(s,t)}{\delta s} - sG(s,t) - as^2\frac{\delta G(s,t)}{\delta s}\right\}$$

$$\frac{\delta G(s,t)}{\delta t} = \frac{-\lambda}{1+\lambda at}\left\{G(s,t)(1-s) + as(1-s)\frac{\delta G(s,t)}{\delta s}\right\}$$

Since $P_0'(t) = \frac{-\lambda}{1+\lambda at}$.

- The langrage's linear eq is;

$$(1+\lambda at)\frac{\delta G(s,t)}{\delta t} + \lambda as(1-s)\frac{\delta G(s,t)}{\delta s} = -\lambda(1-s)G(s,t)$$

- The corresponding set of auxilliary eq. is;

$$\frac{dt}{1+\lambda at} = \frac{ds}{\lambda as(1-s)} = \frac{dG(s,t)}{-\lambda(1-s)G(s,t)}.$$

From $\frac{dt}{1+\lambda at} = \frac{ds}{\lambda as(1-s)}$ We have

$$\int\frac{dt}{1+\lambda at} = \int\frac{ds}{\lambda as(1-s)}.$$

$$\frac{1}{\lambda a}\ln(1+\lambda at) + \frac{1}{\lambda a}\ln c_1 = \frac{1}{\lambda a}\ln\left(\frac{s}{1-s}\right)$$

$$c_1 = \left(\frac{s}{1-s}\right)(1+\lambda at)^{-1}$$

Model for a series of discrete event where average time btwn events is known but exact timing of events is random.

From $\frac{ds}{\lambda a s(1-s)} = \frac{dG(s,t)}{-\lambda(1-s)G(s,t)}$ We have

Multiplying by $-\lambda(1-s)$ both sides

$$\int -\frac{ds}{as} = \int \frac{dG(s,t)}{G(s,t)}$$

$$-\frac{1}{a}\ln s + \ln C_2 = \ln G(s,t)$$

$\ln s^{-1/a} + \ln C_2 = \ln G(s,t)$

$$C_2 = s^{1/a} G(s,t)$$

Thus the general soln is;

$$C_2 = \varphi(C_1)$$

i.e. $s^{1/a} G(s,t) = \phi\left[\left(\frac{s}{1-s}\right)(1+\lambda a t)^{-1}\right]$ $t=0$ $i=1$

$$G(s,0) = s^{-1/a} \phi\left(\frac{s}{1-s}\right)$$

Suppose the initial conditions are;

$$P_n(0) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$G(s,0) = P_0(0) + P_1(0)s + P_2(0)s^2 + \ldots$ $t=0$

$= P_0(0) = 1$ $G(s,0) = P_0(0) = 1$

$$s^{-1/a}\phi\left(\frac{s}{1-s}\right) = 1$$

$$\phi\left(\frac{s}{1-s}\right) = s^{1/a}$$

Let $w = \frac{s}{1-s} \Leftrightarrow s = \frac{w}{1+w}$

$w - ws = s$
$w = s + ws$
$w = s(1+w)$
$\frac{w}{1+w} = s$

$$\phi(w) = \left(\frac{w}{1+w}\right)^{1/a}$$

$$s^{1/a} G(s,t) = \phi\left[w(1+\lambda a t)^{-1}\right].$$

$\phi(w)(1+\lambda a t)^{-1}$

$$\left[\frac{w(1+\lambda a t)^{-1}}{1+w(1+\lambda a t)^{-1}}\right]^{1/a}.$$

$$\left[\frac{\frac{s}{1-s}(1-\lambda at)^{-1}}{1+\frac{s}{1-s}(1+\lambda at)^{-1}}\right]^{1/a}$$ Substituting $w = \frac{s}{1-s}$

$$G(s,t) = s^{-1/a}\,\frac{s^{1/a}}{\left[(1-s)(1+\lambda at)+s\right]^{1/a}}$$ Dividing both sides by $s^{1/a}$

$1+\lambda at - s - s\lambda at$

$$G(s,t) = \left[\frac{1}{1-s+\lambda at-\lambda ats+s}\right]^{1/a}$$

$$\left[\frac{1}{1+\lambda at-\lambda ats}\right]^{1/a}$$

$$\left[\frac{\frac{1}{1+\lambda at}}{1-\frac{\lambda at s}{1+\lambda at}}\right]^{1/a}$$

$$\left[\frac{1}{1-\frac{\lambda at}{1+\lambda at}s}\right]^{1/a}(1+\lambda at)^{-1/a}$$

$$(1+\lambda at)^{-1/a}\left[1-\frac{\lambda at}{1+\lambda at}s\right]^{-1/a}$$ → Expanding term with s

→ Binomial expansion

$$(1+\lambda at)^{-1/a}\left[1+\binom{-1/a}{1}\left(\frac{-\lambda at}{1+\lambda at}s\right)+\ldots+\binom{-1/a}{n}\left(\frac{-\lambda at}{1+\lambda at}s\right)^{n}+\ldots\right]$$

$P_n(t)$ = Coefficient of s^n in the expansion of $G(s,t)$

$$(1+\lambda at)^{-1/a}\binom{-1/a}{n}\left(\frac{-\lambda at}{1+\lambda at}\right)^{n}$$

$4C_2 = \frac{4!}{2!(4-2)!}$

$$(1+\lambda at)^{-1/a-n}\binom{-1/a}{n}(-\lambda at)^{n}$$

$$(1+\lambda at)^{-1/a-n}\,\frac{(1+a)(1+2a)\ldots\{1+(n-1)a\}}{n!}(\lambda at)^{n}$$

$$\frac{(1+a)(1+2a)\ldots\{1+(n-1)a\}}{n!}(\lambda at)^{n}(1+\lambda at)^{-1/a-n}$$

Mean & Variance

$$G(s,t) = \left[\frac{1}{1+\lambda at-\lambda ats}\right]^{1/a}$$

$$G(s,t) = (1+\lambda at-\lambda ats)^{-1/a}$$

$$G'(s,t) = \left(-\tfrac{1}{a}\right)(-\lambda at)(1+\lambda at-\lambda ats)^{-1/a-1}$$ → Product rule

$$= \lambda t(1+\lambda at-\lambda ats)^{-1/a-1}$$

$$E(n)= G'(1,t)= \lambda t(1+\lambda at-\lambda at)^{-1/a-1} \qquad s=1$$

$$= \lambda t$$

Vary

$$G''(s,t)= \lambda t(-1/a-1)(-\lambda at)(1+\lambda at-\lambda ats)^{-1/a-2}$$

$$= \lambda t(1+a)(\lambda t)(1+\lambda at-\lambda ats)^{-1/a-2} \qquad \frac{1}{a}-1=\frac{1-a}{a}$$

$$G''(1,t)= \lambda t(1+a)(\lambda t)(1+\lambda at-\lambda at)^{-1/a-2}$$

$$=(1+a)(\lambda t)^2$$

$$Var(n)= G''(1,t)+G'(1,t)-[G'(1,t)]^2$$

$$(1+a)(\lambda t)^2-(\lambda t)-(\lambda t)^2$$

$$(\lambda t)^2+\lambda^2at^2+\lambda t-\lambda^2t^2$$

$$= \lambda t(1+\lambda at)$$

$$\lambda^2t^2+\lambda^2at^2-(\lambda t)-\lambda^2t^2$$

$$\lambda t+\lambda^2at$$

$$\lambda t(1+\lambda at)$$

Finding Mean & Variance of Distr.

i) Using $G(s,t)$ ie. Pgf

Mean is $G'(1,t)$

Variance is $G''(1,t)+G'(1,t)-[G'(1,t)]^2$

ii) Using $P_n(t)$ ie Pdf.

$$E(n)= \sum_{n=0} nP_n(t)$$

$$E(n^2)= \sum_{n=0} n^2P_n(t)$$

$$Var(n)= E(n^2)-[E(n)]^2$$

iii) Fellers Method.

Define;

$$M_1(t)= \sum_{n=0} nP_n(t) \Rightarrow M_1'(t)= \sum_{n=0} nP_n'(t)$$

$$M_2(t)= \sum_{n=0} n^2P_n(t) \Rightarrow M_2'(t)= \sum_{n=0} n^2P_n'(t)$$

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$$M_k(t) = \sum_{n=0} n^k P_n(t)$$

$$M_3(t) = \sum_{n=0} n^3 P_n(t) \Rightarrow M_3'(t) = \sum_{n=0} n^3 P_n'(t).$$

Poisson Process.

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \quad n \geq 1$$

Multiplying by n & Summing over n

$$\sum_{n=1} n P_n'(t) = -\lambda \sum_{n=1} n P_n(t) + \lambda \sum_{n=1} n P_{n-1}(t)$$

Null operation

$$M_1'(t) = -\lambda M_1(t) + \lambda \sum_{n=1} [(n-1)+1] P_{n-1}(t)$$

$$M_1'(t) = -\lambda M_1(t) + \lambda \sum_{n=1} (n-1) P_{n-1}(t) + \lambda \sum_{n=1} P_{n-1}(t)$$

→ pgf.

$P_0 + P_1 + P_2$

$$M_1'(t) = -\lambda M_1(t) + \lambda M_1(t) + \lambda$$

$$M_1'(t) = \lambda.$$

$$\int M_1'(t)\,dt = \int \lambda\,dt$$

$$M_1(t) = \lambda t + C$$

$$M_1(0) = C.$$

$$P_n(0) = \begin{cases} 1 & n=0 \\ 0 & \end{cases}$$

$$M_1(t) = \sum n P_n(t)$$

$$P_0(0) = 1$$

$$M_1(t) = 0P_0(t) + 1P_1(t) + 2P_2(t) + \dots$$

$$M_1(0) = 0P_0(0) + 1P_1(0) + 2P_2(0) + \dots$$

$$(0)(1) = 0$$

$$\Rightarrow C = 0 \quad \Rightarrow M_1(t) = \lambda t.$$

$$n = (n-1)+1$$

$$n^2 = (n-1)^2 + 2(n-1) + 1$$

$$(n+1)^2 - 2(n+1) + 1$$

Mean.

$$M_1(t) = \lambda t$$

$\boxed{n^2 = (n-1)^2 + 2(n-1) + 1}$

$n^2 = (n-1)^2 + 2(n-1) + 1$

Variance

$$\sum_{n=1} n^2 P_n'(t) = -\lambda \sum_{n=1} n^2 P_n(t) + \lambda \sum_{n=1} n^2 P_{n-1}(t)$$ → Multiplying by n^2 & summing over n

$$M_2'(t) = -\lambda M_2(t) + \lambda \sum_{n=1} \left[(n-1)^2 + 2(n-1) + 1\right] P_{n-1}(t).$$

$$M_2'(t) = -\lambda M_2(t) + \lambda \sum_{n=1} (n-1)^2 P_{n-1}(t) + 2\lambda \sum (n-1) P_{n-1}(t) + \lambda \sum_{n=1} P_{n-1}(t).$$ ↳ pgf

$$M_2'(t) = -\lambda M_2(t) + \lambda M_2(t) + 2\lambda M_1(t) + \lambda$$ $M_1(t) = \lambda t$

$$M_2'(t) = 2\lambda M_2(t) + 2\lambda(\lambda t) + \lambda.$$

$$M_2'(t) - 2\lambda M_2(t) = 2\lambda^2 t + \lambda.$$

Integrating factor $I = e^{\int -2\lambda dt} = e^{-2\lambda t + c_1}.$

$$I M_2'(t) - 2\lambda I M_2(t) = 2\lambda^2 t I + \lambda I$$

$$\frac{d}{dt}(I M_2(t)) = 2\lambda^2 t e^{-2\lambda t + c_1} + \lambda e^{-2\lambda t + c_1}$$

$$I M_2(t) = \int 2\lambda^2 t e^{-2\lambda t + c_1} dt + \int \lambda e^{-2\lambda t + c_1} dt.$$

$$= -\lambda e^{-2\lambda t + c_1} - \frac{\lambda}{2} e^{-2\lambda t + c_1} + c_2.$$

$$M_2(t) = -\lambda - \frac{\lambda}{2} + e^{2\lambda t} \frac{c_2}{e^{c_1}}$$

$$M_2(0) = -\lambda - \frac{\lambda}{2} + c_3$$

$$M_2(0) = 0$$

$$-\lambda - \frac{\lambda}{2} + c_3 = 0.$$

$$c_3 = \lambda + \lambda/2.$$

$$Var(n) = M_2(t) - [M_1(t)]^2$$

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THE LINEAR BIRTH-DEATH PROCESS

a) The Simple Birth-Death Process

$-(n\lambda + n\mu)+$

$\lambda_n = n\lambda$ & $\mu_n = n\mu$

- The diff-differential eq. are:

$$P_n'(t) = -n(\lambda+\mu)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \quad n \geq 1$$

& $$P_0'(t) = \mu P_1(t) \quad n = 0$$

→ Multiplying by s^n & summing over n

$$\sum_{n=1} P_n'(t)s^n = -(\lambda+\mu)\sum_{n=1} nP_n(t)s^n + \lambda \sum_{n=1}(n-1)P_{n-1}(t)s^n + \mu\sum_{n=1}(n+1)P_{n+1}(t)s^n$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -(\lambda+\mu)s\frac{\delta G(s,t)}{\delta s} + \lambda s^2 \frac{\delta G(s,t)}{\delta s} + \mu\left[\frac{\delta G(s,t)}{\delta s} - P_1(t)\right]$$

$$\frac{\delta G(s,t)}{\delta t} = \left[-(\lambda+\mu)s + \lambda s^2 + \mu\right]\frac{\delta G(s,t)}{\delta s}$$

$-\lambda s - \mu s + \lambda s^2 + \mu$

$\mu - \mu s + (\lambda s^2 - \lambda s)$

$\mu(1-s)\ \lambda s(\lambda s-1)$

$(\mu - \lambda s)(1-s)$

Since $P_0'(t) = \mu P_1(t)$

$$\frac{\delta G(s,t)}{\delta t} = (1-s)(\mu - \lambda s)\frac{\delta G(s,t)}{\delta s}$$

- The Langrage's linear eq is;

$$\frac{\delta G(s,t)}{\delta t} - (1-s)(\mu-\lambda s)\frac{\delta G(s,t)}{\delta s} = 0$$

- The set of auxilliary eq is;

$$\frac{dt}{1} = \frac{ds}{-(1-s)(\mu-\lambda s)} = \frac{dG(s,t)}{0}$$

From $\frac{dt}{1} = \frac{dG(s,t)}{0}$ we have

$$\int dG(s,t) = \int 0\, dt$$

$$G(s,t) = c_1$$

From $\frac{dt}{1} = \frac{ds}{-(1-s)(\mu-\lambda s)}$ we have

$$\int dt = \int \frac{ds}{-(1-s)(\mu-\lambda s)}$$

$$t + C = \frac{1}{\mu-\lambda} \ln\left(\frac{\mu-\lambda s}{1-s}\right)$$

$$\frac{\mu-\lambda s}{1-s} = e^{(\mu-\lambda)t + (\mu-\lambda)C}$$ → Multiplying by $(\mu-\lambda)$

$$\left(\frac{\mu-\lambda s}{1-s}\right) e^{-(\mu-\lambda)t} = e^{(\mu-\lambda)C} = C_3.$$

$C_1 = \phi C_2$

- The general soln. is;

$$G(s,t) = \phi\left[\left(\frac{\mu-\lambda s}{1-s}\right) e^{-(\mu-\lambda)t}\right] \quad --(1)$$

Suppose the initial conditions are;

$$P_n(0) = \begin{cases} 1 & n=1 \\ 0 & n \neq 1 \end{cases}$$

$$G(s,0) = P_1(0)s = s.$$

- From 1 $G(s,0) = \phi\left(\frac{\mu-\lambda s}{1-s}\right)$.

$$\phi\left(\frac{\mu-\lambda s}{1-s}\right) = s.$$

Let $w = \frac{\mu-\lambda s}{1-s} \Leftrightarrow s = \frac{\mu-w}{\lambda-w}$

$w - ws = \mu - \lambda s$

$w - \mu = -\lambda s + ws$

$w - \mu = s(w-\lambda)$

$s = \frac{w-\mu}{w-\lambda}$

$$\phi(w) = \frac{\mu-w}{\lambda-w}$$

$$G(s,t) = \phi\left(we^{-(\mu-\lambda)t}\right) \quad \phi(w)e^{-(\mu-\lambda)t}$$

$$\frac{\mu - we^{-(\mu-\lambda)t}}{\lambda - we^{-(\mu-\lambda)t}} = \frac{\mu - \frac{\mu-\lambda s}{1-s} e^{-(\mu-\lambda)t}}{\lambda - \frac{\mu-\lambda s}{1-s} e^{-(\mu-\lambda)t}} \quad \times(1-s)$$

$$= \frac{\mu(1-s) - (\mu-\lambda s)e^{-(\mu-\lambda)t}}{\lambda(1-s) - (\mu-\lambda s)e^{-(\mu-\lambda)t}}$$

$\mu - \mu e^{\cdot} - \mu s + \lambda s e$
$\mu(1-e^{\cdot}) - s(\mu - \lambda s)e^{\cdot\cdot}$ $(\lambda - \mu e^{\cdot}) - (1 - e^{\cdot})\lambda$

$$G(s,t) = \frac{\mu\left[1 - e^{-(\mu-\lambda)t}\right] - \left[\mu - \lambda e^{-(\mu-\lambda)t}\right]s}{\left[\lambda - \mu e^{-(\mu-\lambda)t}\right] - \lambda\left[1 - e^{-(\mu-\lambda)t}\right]s}$$

– Dividing by $\lambda - \mu e^{-(\mu-\lambda)t}$

$$= \frac{\mu\left[\frac{1 - e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right] - \left[\frac{\mu - \lambda e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right]s}{\left[1 - \lambda\left(\frac{1 - e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right)s\right]}$$

$P_n(t) =$ Coefficient of s^n

$$\mu\left[\frac{1 - e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right]\lambda^n\left[\frac{1 - e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right]^n$$

$$= -\lambda^{n-1}\left[\frac{\mu - \lambda e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right]\left[\frac{1 - e^{-(\mu-\lambda)t}}{\lambda - \mu e^{-(\mu-\lambda)t}}\right]^{n-1}$$

Mean $\rightarrow$ $e^{(\lambda-\mu)t}$

Variance $\rightarrow$ $\left(\frac{\lambda+\mu}{\lambda-\mu}\right)\left[e^{(\lambda-\mu)t} - 1\right]e^{(\lambda-\mu)t}$

b) The Zero Growth Rate

$\lambda_n = n\lambda$, $\mu_n = n\mu$ & $\lambda = \mu$

– The diff.-differential eq. are,

$$P_n'(t) = -2\lambda n P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\lambda P_{n+1}(t) \quad n \geq 1$$

& $P_0'(t) = \lambda P_1(t)$ $n = 0$

Multiplying by s^n and summing over

$$\sum_{n=1} P_n'(t)s^n = -2\lambda \sum_{n=1} nP_n(t)s^n + \lambda\sum_{n=1}(n-1)P_{n-1}(t)s^n + \lambda\sum_{n=1}(n+1)P_{n+1}(t)s^n$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -2\lambda s\frac{\delta G(s,t)}{\delta s} + \lambda s^2\frac{\delta G(s,t)}{\delta s} + \lambda\left[\frac{\delta G(s,t)}{\delta s} - P_1'(t)\right]$$

$$\frac{\delta G(s,t)}{\delta t} = \left[-2\lambda s + \lambda s^2 + \lambda\right]\frac{\delta G(s,t)}{\delta s}$$

Since $P_0'(t) = \lambda P_1(t)$

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$$\frac{\delta G(s,t)}{\delta t} = \lambda(1-s)^2 \frac{\delta G(s,t)}{\delta s}.$$

- The langranges linear eq is

$$\frac{\delta G(s,t)}{\delta t} - \lambda(1-s)^2 \frac{\delta G(s,t)}{\delta s} = 0.$$

- The set of auxilliary eq is

$$\frac{dt}{1} = \frac{ds}{-\lambda(1-s)^2} = \frac{dG(s,t)}{0}.$$

From $\frac{dt}{1} = \frac{dG(s,t)}{0}$ we have

$$\int dG(s,t) = \int 0\, dt$$

$$G(s,t) = C_1$$

From $\frac{dt}{1} = \frac{ds}{-\lambda(1-s)^2}$ we have

$$\int -\lambda\, dt = \int \frac{ds}{(1-s)^2}.$$

$$-\lambda t + C_2 = \frac{1}{1-s}.$$

$$C_2 = \frac{1}{1-s} + \lambda t$$

The general soln. is

$$C_1 = \phi C_2.$$

$$G(s,t) = \phi\left[\frac{1}{1-s} + \lambda t\right]$$

$$G(s,0) = \phi\left(\frac{1}{1-s}\right)$$

Suppose the initial conditions are

$$P_n(0) = \begin{cases} 1 & n=1 \\ 0 \end{cases}$$

$$G(s,0) = P_1(0)s = s$$

$$\phi\left(\frac{1}{1-s}\right) = s$$

Let $w = \frac{1}{1-s} \iff w - ws = 1$

$$s = \frac{w-1}{w}$$

$$\phi(w) = \frac{w-1}{w}$$

$$G(s,t) = \phi(w+\lambda t)$$

$$= \frac{(w+\lambda t)-1}{w+\lambda t}$$

$$G(s,t) = \frac{\left(\frac{1}{1-s}+\lambda t\right) - 1}{\frac{1}{1-s}+\lambda t}$$

$$\frac{1+\lambda t-\lambda ts-1+s}{1+\lambda t-\lambda ts}$$

Use to get var Stuff

$$\frac{\lambda t+(1-\lambda t)s}{1+\lambda t-\lambda ts}$$

Dividing by $(1+\lambda t)$

$$\left[\frac{\lambda t}{1+\lambda t}+\left(\frac{1-\lambda t}{1+\lambda t}\right)s\right]\left[1-\frac{\lambda t}{1+\lambda t}s\right]^{-1}$$

Expanding using binomial

$$\left[\frac{\lambda t}{1+\lambda t}+\left(\frac{1-\lambda t}{1+\lambda t}\right)s\right]\left[1+\frac{\lambda ts}{1+\lambda t}+\left(\frac{\lambda ts}{1+\lambda t}\right)^2+\dots+\left(\frac{\lambda ts}{1+\lambda t}\right)^{n-1}+\left(\frac{\lambda ts}{1+\lambda t}\right)^{n}+\dots+\dots\right]$$

$P_n(t) =$ Coefficient of s^n

$$\left(\frac{\lambda t}{1+\lambda t}\right)^{n+1}+\left[\frac{1-\lambda t}{1+\lambda t}\right]\left[\frac{\lambda t}{1+\lambda t}\right]^{n-1}$$

$$\frac{(\lambda t)^{n-1}}{(1+\lambda t)^n}\left[\frac{(\lambda t)^2}{(1+\lambda t)}+(1-\lambda t)\right]$$

$$\frac{(\lambda t)^{n-1}}{(1+\lambda t)^n}\left[\frac{(\lambda t)^2+1-(\lambda t)^2}{1+\lambda t}\right]$$

$$\frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}} \quad n \geq 1$$

Mean $\frac{1}{2\lambda t}$

THE SIMPLEST TRUNKING PROBLEM (QUEUEING PROCESS)

Here $\lambda_n = \lambda$, $\mu_n = n\mu$

- Thus the ~~diff~~ differential equations are;

$$P_n'(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \quad n \geq 1$$

& $P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \quad n=0.$

Multiplying by s^n & summing over n

$$\sum_{n=1} P_n'(t)s^n = -\lambda \sum_{n=1} P_n(t)s^n - \mu \sum_{n=1} nP_n(t)s^n + \lambda \sum_{n=1} P_{n-1}(t)s^n + \mu \sum_{n=1} (n+1)P_{n+1}(t)s^n$$

$$\frac{\delta G(s,t)}{\delta t} - P_0'(t) = -\lambda\left[G(s,t) - P_0(t)\right] - \mu s\frac{\delta G(s,t)}{\delta s} + \lambda s G(s,t) + \mu\left[\frac{\delta G(s,t)}{\delta s} - P_1(t)\right]$$

$$\frac{\delta G(s,t)}{\delta t} = \mu(1-s)\frac{\delta G(s,t)}{\delta s} - \lambda(1-s)G(s,t)$$

$-\lambda G(s,t) - \mu s \frac{\delta G(s,t)}{\delta s} + \lambda s G(s,t) + \mu \frac{\delta G(s,t)}{\delta s}$

Since $P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$

- The langranges linear eq. is;

$$\frac{\delta G(s,t)}{\delta t} - \mu(1-s)\frac{\delta G(s,t)}{\delta s} = -\lambda(1-s)G(s,t)$$

- The Corresponding set of auxilliary eq is;

$$\frac{dt}{1} = \frac{ds}{-\mu(1-s)} = \frac{dG(s,t)}{-\lambda(1-s)G(s,t)}$$

From $\frac{dt}{1} = \frac{ds}{-\mu(1-s)}$ we have

$$\int \mu\, dt = \int \frac{ds}{-(1-s)}$$

$$\mu t + c = \ln(1-s).$$

$$(1-s) = e^{\mu t + c} = e^{\mu t}e^{c}.$$ Making e^c subject

$$(1-s)e^{-\mu t} = e^c = c_1$$

From $\frac{ds}{-\mu(1-s)} = \frac{dG(s,t)}{-\lambda(1-s)G(s,t)}$ Multiplying by $-\lambda(1-s)$

$$\int \frac{\lambda}{\mu} ds = \int \frac{dG(s,t)}{G(s,t)}$$

$$\frac{\lambda}{\mu}s + c = \ln G(s,t)$$

$$G(s,t) = e^{\lambda/\mu s + c} = e^{\frac{\lambda}{\mu}s} \cdot e^{c}$$

$$G(s,t)e^{-\frac{\lambda}{\mu}s} = e^{c} = c_2.$$

- Thus the general soln is;

$$c_2 = \phi(c_1)$$

$$G(s,t)e^{-\frac{\lambda}{\mu}s} = \phi\left[(1-s)e^{-\mu t}\right]$$

$$G(s,0)e^{-\frac{\lambda}{\mu}s} = \phi(1-s)$$

- Suppose the initial conditions are;

$$P_n(0) = \begin{cases} 1 & n=0 \\ 0 & \end{cases}$$

$$G(s,0) = P_0(0) = 1$$

$$\phi(1-s) = e^{-\lambda/\mu s}$$

Let $W = 1-s \Rightarrow s = 1-W$

$$\phi(W) = e^{-\lambda/\mu(1-W)}$$

$$G(s,t)e^{-\frac{\lambda s}{\mu}} = \phi(We^{-\mu t})$$

$$e^{-\frac{\lambda}{\mu}(1-We^{-\mu t})}$$

$$e^{-\frac{\lambda}{\mu}[1-(1-s)e^{-\mu t}]}$$

$$1 - e^{-\mu t} + se^{-\mu t} - s$$

$$1 - e^{-\mu t} - s(1-e^{-\mu t})$$

$$(1-s)(1-e^{-\mu t})$$

$$G(s,t) = e^{\frac{\lambda}{\mu}s} e^{-\frac{\lambda}{\mu}[1-(1-s)e^{-\mu t}]}$$

$$= e^{-\frac{\lambda}{\mu}[(1-e^{-\mu t})(1-s)]}$$

$$G(s,t) = e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} e^{\frac{\lambda}{\mu}(1-e^{-\mu t})s}.$$

Expanding

$$e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\left[1 + \frac{\frac{\lambda}{\mu}(1-e^{-\mu t})s}{1!} + \frac{\left[\frac{\lambda}{\mu}(1-e^{-\mu t})s\right]^2}{2!} + \dots + \frac{\left[\frac{\lambda}{\mu}(1-e^{-\mu t})s\right]^n}{n!} + \dots\right]$$

$P_n(t)$ = Coefficient of s^n in the expansion of $G(s,t)$

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$$= \frac{e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^n}{n!} \quad n \geq 0.$$

Which is a poisson distr with parameter $\frac{\lambda}{\mu}(1-e^{-\mu t})$

$$G'(s,t) = e^{-\lambda/\mu(1-e^{-\mu t})} \frac{\lambda}{\mu}(1-e^{-\mu t}) e^{\frac{\lambda}{\mu}(1-e^{-\mu t})s}$$ → Differentiating wrt s

$$E(n) = G'(1,t) = e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \cdot \frac{\lambda}{\mu}(1-e^{-\mu t}) e^{\lambda/\mu(1-e^{-\mu t})}$$

$$= \frac{\lambda}{\mu}(1-e^{-\mu t}).$$

$$G''(s,t) = e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^2 e^{\frac{\lambda}{\mu}(1-e^{-\mu t})s}.$$

$$G''(1,t) = e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^2 e^{\frac{\lambda}{\mu}(1-e^{-\mu t})}.$$

$$= \left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^2$$

$$Var(n) = G''(1,t) + G'(1,t) - [G'(1,t)]^2.$$

$$\left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^2 + \frac{\lambda}{\mu}(1-e^{-\mu t}) - \left[\frac{\lambda}{\mu}(1-e^{-\mu t})\right]^2$$

$$= \frac{\lambda}{\mu}(1-e^{-\mu t})$$

$$\lim_{\substack{n\to\infty \\ t}} P_n(t) = \frac{e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^n}{n!} \quad n \geq 0$$

MARKOV CHAIN

- Let E_j & E_k be 2 events. Then

$$P[E_j, E_k] = P[E_k|E_j]\,P(E_j)$$
$$= P(E_j)\,P(E_k|E_j)$$
$$= a_j P_{jk}$$

Where $a_j = P(E_j)$ & $P_{jk} = P(E_k|E_j)$

- Extending the notation to 3 events say E_j E_k & E_r we have;

$$P[E_j, E_k, E_r] = P(E_j)\,P(E_k|E_j)\,P(E_r|E_k).$$
$$= a_j P_{jk} P_{kr}$$

Markov Chain → Sequence of trials with possible outcomes $E_{j_0}, E_{j_1}, E_{j_2} \ldots E_{j_n}$

$$P[E_{j_0}, E_{j_1}, E_{j_2} \ldots E_{j_{n-1}}, E_{j_n}] = a_{j_0} P_{j_0 j_1} P_{j_1 j_2} \cdots P_{j_{n-1} j_n}.$$

- Where $a_{j_0} = P(E_{j_0})$ is called the Prob at the initial or zeroth trial & P_{jk} is the Conditional prob of E_k given E_j

Terminology.

- The event E_j shall be called State E_j or Simply state j. The Conditional prob P_{jk} Shall be called Transitional prob from state E_j to State E_k.

$$P_{jk} = P[E_k|E_j] = P[E_j \rightarrow E_k].$$
$$= P[\text{Moving from State } E_j \text{ to State } E_k.]$$

- Thus we can express the transitional probs in a matrix form as follows;

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & \cdots \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ P_{31} & P_{32} & P_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \end{array}$$

- The transitional matrix is either finite or infinite

$$\sum_K P_{JK} = 1 \quad P_{JK} \geq 0.$$

- Any matrix whose elements $P_{JK} \geq 0$ & $\sum_K P_{JK} = 1$ is called a Stochastic matrix.

$P_{11} + P_{12} + P_{13} + \cdots = 1$

eg. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{bmatrix}$

- If each column adds up to unity, then we have a double Stochastic matrix.

eg $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$E_1 \xrightarrow{1/2} E_3 \xrightarrow{1/3} E_1$

Higher Orders of Transitional Probs.

- $P_{JK}^{(n)}$ is the prob of moving from state E_j to E_K in n steps

In particular $P_{JK}^{(2)}$ is the mvt from E_j to E_K in 2 steps.

ie.

$$P_{JK}^{(2)} = \sum_{\lambda} P[E_j \to E_\lambda \to E_K]$$

P_{12} $P_{\cdot\cdot}$ ...

- In general

$$P_{JK}^{(n)} = \sum_{\lambda} P_{J\lambda}^{(n-1)} P_{\lambda K}$$

2.

- More generally
$$P_{JK}^{(m+n)} = \sum_{\lambda} P_{J\lambda}^{(m)} P_{\lambda K}^{(n)}$$

- Since $P_{JK}^{(n)} \geq 0$, it is not always possible to move from state to state
The state E_K is said to be reachable or accessible from state E_j
if there exists some positive $n \ni P_{JK}^{(n)} > 0$.
- For $n=0$, $P_{jj}^{(0)} = 1$ $P_{JK}^{(0)} = 0$ for $j \neq K$.
If $E_i \to E_j$ & $E_j \to E_K$ then $E_i \to E_K$
$\exists$ Some positive integers m & $n \ni$

$$P_{ij}^{(m)} > 0 \quad \& \quad P_{JK}^{n} > 0$$

then $P_{iK}^{(m+n)} > 0 \Rightarrow E_K$ is reachable from E_i
- $P_{JK}^{(n)}$ is the prob of moving from E_j to E_K in n steps & regardless of the no. of entrances into E_K prior to n.
- Let us now define $f_{jK}^{(n)}$ to be the Prob of entering E_K from E_j in n steps for the first time.

$$P_{12}^{(4)} = P[E_1 \to E_3 \to E_2 \to E_5 \to E_2]$$

$$P_{13}^{4} = P[E_1 \to E_2 \to E_4 \to E_5 \to E_3]$$

Persistent & Transient States
- A State E_j is said to be persistent / recurrent if,
$$f_{jj} = \sum_{n} f_{jj}^{(n)} = 1.$$
- A State E_j is said to be transient / non-recurrent if,
$$f_{jj} = \sum_{n} f_{jj}^{(n)} < 1.$$
- If $f_{jj} = 1$, then $f_{jj}^{(n)}$ is a prob distr known as the recurrent time distr.
- In general if
$$f_{jK} = \sum_{n} f_{jK}^{(n)} = 1.$$

3.

then $f_{jj}^{(n)}$ is a prob distr called first page distr.

- The mean recurrence time for E_j is given by:

$$\mu_j = \sum_n n f_{jj}^{(n)}$$

If $\mu_j = \infty$ then E_{ij} is null

If $\mu_j < \infty$ " " " non-null.

Periodicity.

- A state is said to be of period t if t is the greatest common divisor of $\{n : p_{jj}^{(n)} > 0\}$. If $t = 1$, then the state is said to be aperiodic.

Ergodicity.

- A state that is persistent, non-null & aperiodic is said to be ergodic

Absorbing State.

- A state is absorbing if once entered it cannot be left.

ie: $P_{KK} = 1$.

$$\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$P_{22} = 1$.

$E_1 \to E_1$, $E_1 \to E_3$; $E_3 \to E_1$, $E_3 \to E_3$.

E_2 is an absorbing state.

Irreducible Markov Chain.

- A markov chain is irreducible if there exists no closed subset other than the set of all states.

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$E_1 \to E_1$, $E_1 \to E_2$, $E_1 \to E_3$

$E_2 \to E_2$, $E_2 \to E_3$

$E_3 \to E_1$, $E_3 \to E_2$.

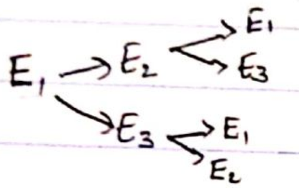
Theorem 1: All states of an irreducible Markov chain are of the same type.

Theorem 2: For a persistent state E_j there exists a unique irreducible closed set C containing E_j & such that for every pair E_i, E_k of states in C, $f_{ik} = 1$ & $f_{ki} = 1$

Example

1) Classify the states of the following transition prob matrix

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array}\begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$



All the states are reachable from one another, so the chain is irreducible. Since the chain is ~~or~~ irreducible, all the states have the same properties.

Therefore we consider only one state to be representative of the others say E_1

$$f_{11}^{(2)} = P[E_1 \to E_2 \to E_1] + P[E_1 \to E_3 \to E_1]$$

$$(1/2)(1/2) + (1/2)(1/2) = 1/4 + 1/4 = 1/2.$$

$$f_{11}^{(3)} = P[E_1 \to E_2 \to E_3 \to E_1] + P[E_1 \to E_3 \to E_2 \to E_1].$$

$$(1/2)(1/2)(1/2) + (1/2)(1/2)(1/2)$$

$$1/8 + 1/8 = 1/4 = (1/2)^2$$

$$f_{11}^{(4)} = (1/2)^3 \quad f_{11}^{(5)} = (1/2)^4 \; \ldots \; f_{11}^{(n)} = (1/2)^{n-1}$$

5

$$f_{11} = \sum_n f_{11}^{(n)} = \sum_{n=2} (1/2)^{n-1} = 1/2 + (1/2)^2 + (1/2)^3 + \dots$$

$$\frac{1/2}{1 - 1/2} = 1$$

E_1 is persistent, hence E_2 & E_3 are also persistent.
H.C.F of $\{2, 3, 4, \dots\} = 1 \Rightarrow E_1$ is aperiodic.
Hence E_2 & E_3 are aperiodic.

$$\mu_1 = \sum_n n f_{11}^{(n)} = \sum_{n=2} (n)(1/2)^{n-1}$$

$$2(1/2) + 3(1/2)^2 + 4(1/2)^3 + \dots$$

$$\left(1 + 2(1/2) + 3(1/2)^2 + 4(1/2)^3 + \dots\right) - 1$$

$\frac{a}{1-r}$

Let $f'(x) = 1 + 2x + 3x^2 + 4x^3 + \dots$

$$f(x) = x + x^2 + x^3 + x^4 + \dots + c.$$

$$= \frac{x}{1-x} + c.$$

$$f'(x) = \frac{(1)(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

$$\mu_1 = \frac{1}{(1-1/2)^2} - 1 = 4 - 1 = 3 < \infty$$

E_1 is non-null hence E_2 & E_3 are also non-null.
E_1 & E_2 & E_3 are ergodic states since they are persistent, aperiodic & non-null hence the chain is irreducible & ergodic Markov chain.

Example

1. Classify the states for this infinite chain

$$P = \begin{array}{c|c} & E_1 \quad E_2 \quad E_3 \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ \vdots \\ \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 1/2 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 1/2 & 0 & \dots \\ \vdots & & & & \vdots & \\ 1/2 & 0 & 0 & \dots & & 1/2 \end{bmatrix} \end{array}$$

6.

$$E_1 < \begin{matrix} \to E_1 \\ \to E_2 < \begin{matrix} \to E_1 - \cdots \\ E_3 < \begin{matrix} E_1 - \cdots \\ E_4 - \cdots \end{matrix} \end{matrix} \end{matrix}$$

The chain is irreducible since each of the states $E_1 E_2 E_3 \cdots$ can be reached from every other state (ie R is irreducible).
Hence all the states are of the same type So all we have to do is to examine only one state E_1 say

$$f_{11}^{(1)} = P[E_1 \to E_1] = \tfrac{1}{2} \qquad f_{11}^{2} = P[E_1 \to E_2 \to E_1] = (\tfrac{1}{2})^2$$

$$f_{11}^{3} = P[E_1 \to E_2 \to E_3 \to E_1] = (\tfrac{1}{2})^3 \cdots \quad f_{11}^{n} = (\tfrac{1}{2})^n.$$

$$f_{11} = \sum_n f_{11}^{(n)} = \sum_{n=1} (\tfrac{1}{2})^n = \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \cdots$$

$$= \frac{1/2}{1 - 1/2} = 1.$$

Thus E_1 is persistent $\Rightarrow$ The rest of the states are persistent.
HCF of $\{1, 2, 3 \cdots\} = 1 \Rightarrow E_1$ is aperiodic

$$\mu_1 = \sum_n n f_{11}^{n} = \sum (n)(\tfrac{1}{2})^n =$$

$$\tfrac{1}{2} + 2(\tfrac{1}{2})^2 + 3(\tfrac{1}{2})^3 + \cdots$$

$$\tfrac{1}{2}\left(1 + 2(\tfrac{1}{2}) + 3(\tfrac{1}{2})^2 + \cdots\right)$$

$$\tfrac{1}{2}\left(\frac{1}{(1 - \frac{1}{2})^2}\right) = 2 < \infty$$

E_1 is non-null.

Let $f'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$$f(x) = x^2 + x^3 + x^4 + \cdots + c.$$

$$= \frac{x}{1 - x} + c.$$

$$f'(x) = \frac{(1)(1-x) - (x)(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

7.

Therefore E_1 is ergodic since it is persistent, aperiodic & non-null.
Thus is irreducible & ergodic Markov Chain.

Exercise.

a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ b) $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

State Matrix

- The State matrix for a pop is a row matrix whose entries represent the portions of the total pop that are in each of the Various states.

Consider a Pop of 10,000 with the following 3 States

State	S_1: Under 21yrs	S_2: 21-65yrs	S_3: Over 65yrs
No in State	2500	4500	3000
Portion in State	0.25	0.45	0.3

- The state matrix for this pop is as follows:

$$\begin{array}{cccc} & S_1 & S_2 & S_3 \\ X = & [0.25 & 0.45 & 0.30] \end{array}.$$

- In general the state matrix for a pop with n states $\{S_1 S_2 \dots S_n\}$ having positions $x_1, x_2 \dots x_n$ is given by

$$X = [x_1, x_2 \dots x_n] \rightarrow \text{State matrix}$$

where $x_1 + x_2 + \dots x_n = 1$

- By multiplying a state matrix by a transition matrix, we obtain a new state matrix. This new state matrix gives the portions of the total pop that will be in each of the various states after one transition as follows,

State Matrix x Transition Matrix = New State Matrix

$$\underline{X}\,\underline{P} = [x_1 x_2 \dots x_n] \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \dots & P_{nn} \end{bmatrix} = [y_1 y_2 \dots y_n].$$

state matrix — Transition Matrix — new matrix

Example

1. Two grocery stores in a certain city compete for the same customers. Although a few of each store's customers are loyal most shop in both stores. After conducting a survey, the management of one of the stores has determined that 40% of the pple who shop at store A will return to shop A for their next weekly shopping trip while 60% will go to store B. At store B, 80% of the pple will return to store B while 20% will go to store ~~B~~ A. If 3000 pple shop at store A & 5000 pple shop at store B this week. How many ~~store~~ will shop at each store next week?

The matrix representing the transition probabilities for this problem is

$$\underline{P} = \begin{array}{c|cc} & A & B \\ \hline A & 0.4 & 0.6 \\ B & 0.2 & 0.8 \end{array}$$

& the initial state matrix is

$$\underline{X_0} = \left[\frac{3000}{8000} \quad \frac{5000}{8000}\right] = [0.375 \quad 0.675].$$

0.625

Thus the state matrix after one week is

$$\underline{X}_1 = \underline{X_0 P} = [0.375 \quad 0.675] \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} = \begin{array}{cc} A & B \\ [0.275 & 0.725] \end{array}$$

$(0.275 \times 8000) = 2200$ customers will shop at shop A

$(0.725 \times 8000) = 5800$ customers will shop at shop B.

9.

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Regular Matrix.

- A Stochastic matrix P is called Regular if some power of P has only Positive entries

eg i) $P = \begin{bmatrix} 0.50 & 0.50 \\ 0.25 & 0.75 \end{bmatrix}$

is regular coz P is stochastic & P' has only positive entries

ii) $P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is not regular coz every power of P has zeros in its 2^{nd} row.

iii) $P = \begin{bmatrix} 0.50 & 0.25 \\ 0.50 & 0.75 \end{bmatrix}$ is not regular coz its not Stochastic.

Regular Markov Chain.

- If P is a regular matrix, then the sequence $P\ P^2\ P^3\ P^4 \ldots$ approaches a Stable Matrix
- The Corresponding Markov Chain is called a regular Markov Chain & the Sequence

$$X P,\ X P^2,\ X P^3,\ X P^4 \ldots$$

approaches $\bar{X}$ which we call the Stable / Stationary distr. matrix.

- Furthermore, entries in each row $\bar{P}$ are equal to the corresponding entries in the row matrix $\bar{X}$

Example

1. Find the stable matrix for the regular matrix

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix}$$

Let $\bar{X} = [x_1\ x_2]$ where $x_1 + x_2 = 1$.

$$\bar{X} P = \bar{X}$$

$$[x_1\ x_2] \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix} = [x_1\ x_2]$$

$$[0.4x_1 + 0.7x_2 \quad 0.6x_1 + 0.3x_2] = [x_1\ x_2]$$

10.

Comparing coefficients

$0.4x_1 + 0.7x_2 = x_1 \Rightarrow 0.6x_1 = 0.7x_2 \Rightarrow x_1 = 7/6\, x_2.$

$0.6x_1 + 0.3x_2 = x_2 \Rightarrow 0.6x_1 = 0.7x_2 \Rightarrow x_1 = 7/6\, x_2.$

But $x_1 + x_2 = 1.$

$$\frac{7}{6}x_2 + x_2 = 1$$

$$\frac{13}{6}x_2 = 1.$$

$$x_2 = \frac{6}{13} \quad x_1 = \frac{7}{13}.$$

$\bar{X} = \begin{bmatrix} \frac{7}{13} & \frac{6}{13} \end{bmatrix} \rightarrow$ Stable distr. matrix (Stationary distr).

The Stable Matrix $\bar{P} = \begin{bmatrix} 7/13 & 6/13 \\ 7/13 & 6/13 \end{bmatrix}$

Absorbing Markov Chain.

- Consider a Markov Chain with n diff states $\{S_1, S_2 \ldots S_n\}$. The i^{th} state S_i is called absorbing if $P_{ii} = 1$. Moreover, the Markov chain is called Absorbing if it has at least one absorbing state & it is ~~less~~ possible for a member of the pop to move from one absorbing state to an absorbing one in a finite no. of transitions.

Example.

1 Rewrite the following transition matrix in std form.

$$\begin{array}{c|cccc} & S_1 & S_2 & S_3 & S_4 \\ \hline S_1 & 0.5 & 0.2 & 0.1 & 0.2 \\ S_2 & 0 & 1 & 0 & 0 \\ S_3 & 0.2 & 0.3 & 0.4 & 0.1 \\ S_4 & 0 & 0 & 0 & 1 \end{array}$$

S_2 & S_4 are absorbing states.

11.

$$\begin{array}{c|c} & \begin{array}{cccc} S_2 & S_4 & S_1 & S_3 \end{array} \\ \begin{array}{c} S_2 \\ S_4 \\ S_1 \\ S_3 \end{array} & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0.2 & 0.2 & 0.5 & 0.1 \\ 0.3 & 0.1 & 0.2 & 0.4 \end{array}\right] \end{array}$$

Partitioning

The Stable Matrix for an Absorbing Markov Chain.

- Consider an absorbing Markov Chain whose transition matrix has the form,

$$P = \left[\begin{array}{c|c} I & O \\ \hline S & R \end{array}\right]$$

eg $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$S = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}$ $R = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}$

- The stable matrix for this Markov Chain is

$$\bar{P} = \left[\begin{array}{c|c} I & O \\ \hline (I-R)^{-1}S & O \end{array}\right]$$

Using the previous example we have,

$$I - R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.1 \\ -0.2 & 0.6 \end{bmatrix}$$

$$(I-R)^{-1} = \frac{1}{(0.5)(0.6) - (-0.2)(-0.1)} \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$$

$\frac{1}{\det} \cdot Adj$

$$= \frac{1}{0.28} \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 60/28 & 10/28 \\ 20/28 & 50/28 \end{bmatrix}$$

12:

$$(I-R)^{-1}S = \begin{bmatrix} \frac{60}{28} & 10/28 \\ 20/28 & 50/28 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.5357 & 0.4643 \\ 0.6786 & 0.3214 \end{bmatrix}.$$

The stable matrix for P is:

$$\bar{P} = \left[\begin{array}{c|c} I_2 & 0 \\ \hline (I_2-R)^{-1}S & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0.5354 & 0.46413 & 0 & 0 \\ 0.6786 & 0.3214 & 0 & 0 \end{array}\right].$$

Exercise

1 Rewrite the following matrix in std form, Hence obtain the stable matrix for this absorbing Markov chain

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

STATIONARY PROCESSES.

- If for the arbitrary $t_1, t_2 \cdots t_n$ the joint distn of the vector r.v $(X(t_1), X(t_2) \cdots X(t_n))$ & $(X(t_1+h), X(t_2+h), \cdots X(t_n+h))$ are the same for all $h>0$ then the stochastic process $\{X(t), t \in T\}$ is said to be stationary of order n.
- If the mean of the process ~~exists~~ $\{X(t)\}$ exists then $E\{X(t)\}$ must be equal to $E\{X(t+h)\}$ for any h so that $E\{X(t)\}$ must be a constant μ independent of t
- Without loss of generality we shall assume that μ is zero. If Moreover the covariance function $c(s,t)$ exists then,

$$c(s,t) = Cov\{X(t), X(s)\}$$
$$E\{X(t)\bar{X}(s)\}.$$
$$E\{X(t+h)\, X(s+h)\} \text{ for any } h.$$
$$E\{X(t-s)\, X(0)\}.$$

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- This shows that $c(s,t)$ is a function of time diff $|t-s|$
- A process $\{x(t)\}$ with finite 2^{nd} order moments is called **Covariance stationary** if its mean $E\{x(t)\} = \mu$ is independent of t & its Covariance function

$$c(s,t) = E\{x(t)x(s)\}$$

depends only on the diff $|t-s|$ for all t, s.
- A process which is not stationary is said to be **evolutionary**.

Example

1. Poisson process. Consider the process $\{x(t), t \in T\}$ with

$$P_r\{x(t)=n\} = \frac{e^{-at}(at)^n}{n!} \quad a>0, \; n=0,1,2$$

→ Poisson

We see that $E\{x(t)\} = at$ & $Var\{x(t)\} = at$ are functions of t. The process is evolutionary.

2. Consider the process

$$x(t) = A\cos\lambda t + B\sin\lambda t$$

Where A, B are uncorrelated r.v each with mean 0 & Var 1 & λ is a positive constant. Show that the process $\{x(t)\}$ is covariance stationary

$$E\{x(t)\} = E\{A\cos\lambda t + \cancel{B\cos\lambda} B\sin\lambda t\}$$

$$E(A)\cos\lambda t + E(B)\sin\lambda t$$

$E(A) = E(B) = 0$

$$= 0$$

$$Cov\{x(t), x(s)\} = E\{x(t)x(s)\} - E\{x(t)\}E\{x(s)\}$$

$$E\{x(t)x(s)\}$$

$$= E\{x(t)x(s)\}$$

$$= E\{(A\cos\lambda t + B\sin\lambda t)(A\cos\lambda s + B\sin\lambda s)\}$$

$$= E\{A^2\cos\lambda t\cos\lambda s + AB\cos\lambda t\sin\lambda s + AB\sin\lambda t\cos\lambda s + B^2\sin\lambda t\sin\lambda s\}$$

$$E(A^2)\cos\lambda t\cos\lambda s + E(B^2)\sin\lambda t\sin\lambda s.$$

But $Var(A) = E(A^2) - [E(A)]^2$

$$1 = E(A^2) - 0$$

$$E(A^2) = 1$$

Similarly $E(B^2) = 1$.

14.

$$\text{Cov}\{x(t), x(s)\} = \cos\lambda t\cos\lambda s + \sin\lambda t\sin\lambda s$$
$$\cos(\lambda t - \lambda s)$$
$$\cos(t-s)\lambda$$

Hence the first 2 moments are finite & the covariance function is a function of $(t-s)$. Thus the process is covariance stationary.

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Fellers Method. Zero Growth

$\lambda = \mu \quad \lambda_n = n\lambda \quad \mu = n\mu.$

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$$

$$P_n'(t) = -(n\lambda + n\lambda)P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\lambda P_{n+1}(t).$$

$$P_n'(t) = -2n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t) + (n+1)\lambda P_{n+1}(t).$$

$$P_0'(t) = \lambda P_1(t).$$

$\sum (n^2-n)P_{n-1}(t)$

Multiplying by n & summing over n

$$\sum nP_n'(t) = -2\lambda \sum n^2 P_n(t) + \lambda \sum n(n-1)P_{n-1}(t) + \lambda \sum n(n+1)P_{n+1}(t).$$

$$\sum nP_n'(t) = -2\lambda \sum n^2 P_n(t) + \lambda\left[\sum n^2 P_{n-1}(t) - \sum nP_{n-1}(t)\right] + \lambda\left[\sum n^2 P_{n+1}(t) + \sum nP_{n+1}(t)\right]$$

$n^2 = (n-1)^2 + 2(n-1) + 1 \qquad n = (n-1) + 1$

$n^2 = (n+1)^2 - 2(n+1) + 1 \qquad n = (n+1) - 1$

$$M_1'(t) = -2\lambda M_2(t) + \lambda\left\{\left[\sum (n-1)^2 P_{n-1}(t) + 2\sum (n-1)P_{n-1}(t) + \sum P_{n-1}(t)\right] - \sum (n-1)P_{n-1}(t) - \sum P_{n-1}(t)\right\} + \lambda\left\{\left[\sum (n+1)^2 P_{n+1}(t) - 2\sum (n+1)P_{n+1}(t) + \sum P_{n+1}(t)\right] + \sum (n+1)P_{n+1}(t) - \sum P_{n+1}(t)\right\}$$

$$M_1'(t) = -2\lambda M_2(t) + \lambda\left[\sum (n-1)^2 P_{n-1}(t) + \sum (n-1)P_{n-1}(t)\right] + \lambda\left\{\sum (n+1)^2 P_{n+1}(t) - \sum (n+1)P_{n+1}(t)\right\}$$

$$M_1'(t) = -2\lambda M_2(t) + \lambda\left[M_2(t) + M_1(t)\right] + \lambda\left[M_2(t) - M_1(t)\right].$$

$$M_1'(t) = -2\lambda M_2(t) + \lambda M_2(t) + \lambda M_1(t) + \lambda M_2(t) - \lambda M_1(t).$$

$$M_1'(t) = 0.$$

$$M_1(t) = c.$$

$$P_n(0) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1. \end{cases}$$

$$M_1(0) = 0P_0(0) + 1P_1(0) + \cdots$$

$$M_1(0) = 1.$$

$$c = 1.$$

$M_1(t) = 1 \Leftarrow$ Mean.

Answer ALL the questions

1. Consider a Markov chain with only three states and transition matrix

$$\begin{bmatrix} 0 & \frac{3}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Find the stable(stationary) distribution.

2. Consider the process $X(t) = A_1 + A_2 t$ where A_1, A_2 are independent random variables with $E(B_i)=a_i$, $Var(A_i)=\sigma_i^2$ i=1,2 . Show that the above process is evolutionary.

3. Classify the states of the following Markov chain

$$\begin{bmatrix} 0 & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

4. The difference-differential equations of an immigration rate of birth-death process are given by

$$p_n'(t) = -kp_n(t) + kp_{n-1}(t) \quad ,n \geq 1$$

and

$$p_0'(t) = -kp_0(t) \quad ,n = 0$$

Where k is a constant.

$M_1(0)=\sum n P_n(0)=1$

Given the initial conditions are $p_n(0) = \begin{cases} 1 & ,n = 0 \\ 0 & otherwise \end{cases}$

Obtain the mean of the above process by using Feller`s method.

$nP_n'(t)s^n$

$nP_i(t) \to 0 \; P_i(t) \; 2P_i(t)$

SECTION A

QUESTION ONE (30 MARKS)

a) Define the following terms

i) Stochastic matrix (2 Marks)

ii) Ergodic state (2 Marks)

iii) Irreducible Markov chain (2 Marks)

b) Let X be a random variable such that

$$\Pr\{X=k\}=p_k,\ \Pr\{X>k\}=q_k=\sum_{r=k+1}^{\infty}p_r\ ,\ k\geq 0.\quad \textit{If}\ P(s)=\sum_{k=0}^{\infty}p_k s^k\ \textit{and}\ Q(s)=\sum_{k=0}^{\infty}q_k s^k,$$

Show that (1- s)Q(s)=1-P(s) and that E(X)=Q(1). (7 Marks)

c) Given that Fibonacci numbers are given by $a_0=0, a_1=1, \quad a_n=a_{n-1}+a_{n-2}, \quad n\geq 2$.
Find the generating function of the sequence $\{a_n\}$ (3 marks)

d) The probability generating function of a certain process is given by

$$G(s,t)=\frac{se^{-\lambda t}}{\left[1-\left(1-e^{-\lambda t}\right)s\right]}$$

Determine the mean and variance of this process. (5 Marks)

e) Consider the three-state Markov chain having transitional probability matrix given by

$$\begin{bmatrix} 0 & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Obtain the stationary distribution for this Markov chain. (6 Marks)

f) Given that the probability generating function of a process is $G(s)=\frac{1}{2}+\frac{1}{4}s+\frac{1}{4}s^2$ Find the probability of extinction of the process. (6 Marks)

SECTION B

QUESTION TWO (20 MARKS)

a) The difference-differential equation for a Poisson is given by

$$p_n'(t) = -\lambda\, p_n(t) + \lambda\, p_{n-1}(t) \quad , \quad n \geq 1 \; and$$
$$p_0'(t) = -\lambda\, p_0(t), \quad n = 0$$

Suppose that initial conditions are $p_n(0) = 1$ when n=0 and 0 otherwise. Obtain the mean and variance of this process by using Feller`s method. (13 Marks)

b) Consider the process $\{X(t) \quad ,t \in T\}$ whose probability distribution, under a certain condition, is given by

$$\Pr\{X(t) = n\} = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & , \quad n = 1,2,3,\ldots \\[2ex] \dfrac{at}{1+at} & , \quad n = 0 \end{cases}$$

Show that the process $\{X(t)\}$ is not stationary. (7 Marks)

QUESTION THREE (20 MARKS)

(a) Define the following terms

(i) Persistent state (2 Marks)
(ii) Absorbing state (2 Marks)
(iii) Non-recurrent state (2 Marks)

(b) Classify the states of the following Markov chain

$$\begin{array}{c c} & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} & \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}.$$

(14 Marks)

QUESTION FOUR (20 MARKS)

Consider the differential-difference equation of the zero-growth rate of birth-death process where $\lambda_n = n\lambda,\ \mu_n = n\mu$ and $\lambda = \mu$. Suppose the initial conditions are $p_n(0)=1$ for $n=1$ and 0 otherwise.

a) Obtain the distribution $p_n(t)$ of this process. (15 Marks)

b) Using the probability generating function of this process, determine the mean and variance of this process. (5 Marks)

$1-\lambda t-2(\lambda t)^2$

$\left.\begin{array}{l} p=-2 \\ s=-1 \end{array}\right\} 2,-1$

$1+2\lambda t-\lambda t-2(\lambda t)^2$

$1(1-2\lambda t)+\lambda t(1$

$1(1+2\lambda t)-\lambda t(1+2\lambda t)$

$(1-\lambda t)(1+2\lambda t)$

$\frac{(1-\lambda t)}{\lambda t+s(1-\lambda t)} + \lambda t\frac{[\lambda t+s(1-\lambda t)]}{(1+\lambda t-\lambda ts)^2}$

$\frac{(1-\lambda t)(1+\lambda ts-\lambda ts)+\lambda t[\lambda t+s(1-\lambda}{(1+\lambda t-\lambda ts)^2}$

$\{(1-\lambda t)(1+\lambda t-\lambda ts)+\lambda t[\lambda t+s(1-\lambda t)]\}/(1+$

$(\lambda t)(1-\lambda t)[\quad]$

$e^{\lambda t}$

$1(1-\lambda t)-\lambda t(1-\lambda t)$

$2\lambda t(\lambda t)$

$1-\lambda t-\lambda t+(\lambda t)^2$

$1-2\lambda t+(\lambda t)^2$

W1-2-60-1-6

JOMO KENYATTA UNIVERSITY
OF
AGRICULTURE AND TECHNOLOGY

University Examinations 2017/2018

FOUTH YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE IN FINANCIAL ENGINEERING AND BIOSTATISTICS

STA 2391/STA 2406: STOCHASTIC PROCESSES FOR ACTUARIAL AND FINANCE/STOCHASTIC PROCESSES

DATE: JANUARY 2018 TIME: 2 HOURS

INSTRUCTIONS: Answer question ONE and any other TWO questions .

QUESTION ONE (30 MARKS)

(a) Define the following terms;

(i) State of a process state field of the process descriptor.

(ii) Discrete time Stochastic process - If the state $\{X(t)\}$ is discrete then $X(t)$ is called a discrete state s.p

(iii) Poisson process - Type of random mathematical object whose points are randomly located in mathematical space.

(iv) A Markov chain stochastic process whose state space is discrete and discrete time

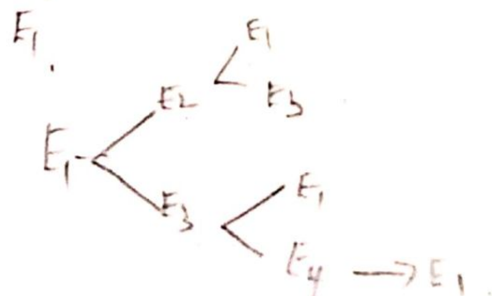
[9 marks]

(b) Obtain the generating function of the sequence $\{Y_t\} = \{n^2\}$, *for* $n = 0, \quad 1, \quad 2, ...$

[6 marks]

(c) Determine whether the following Markov chains is irreducible.

$$\begin{array}{cc} & \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} & \underline{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



[2 marks]

(d) Let X be a random variable such that $pr(X = k) = p_k$ and $pr(X > k) = q_k = \sum_{r=k+1}^{\infty} p_k$ for $k \geq 0$. Obtain the generating function of $pr(X > k)$.

[5 marks]

(e) Let $\{Y_t\}_{t\epsilon T}$ be a stochastic process with probability distribution

$$P(Y_t = x) = \begin{cases} pq^{x-1}, & x = 0, 1, 2, 3, 4, 5, ... \\ 0 & elsewhere \end{cases}$$

. Find the probability generating function of the process and hence use the p.g.f to obtain the mean and the variance of the process..

[8 marks]

QUESTION TWO (20 MARKS)

(a) Consider a pure birth process with difference differential equations:

$$p_n'(t) = -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t), \qquad n \geq 1$$

and

$$p_0'(t) = 0, \quad n = 0$$

with initial conditions : $p_n(0) = 1 \; for \quad n = 1$ and $p_n(0) = 0 \; for \quad n \neq 1$. Use Feller's method to find the mean and variance of the process.

[13 marks]].

(b) Use the probability generating function technique to investigate covariance stationarity of the process

$$pr\{X_t = k\} = \begin{cases} \frac{e^{-\lambda t}(\lambda t)^k}{k!}, & k = 0, 1, 2, \ldots \\ 0 & elsewhere \end{cases}$$

[7 marks]

QUESTION THREE (20 MARKS)

(a) Define the following terms;

(i) Stochastic matrix - square matrix that describes transition of a markov chain.

(ii) Period of a state - state is said to be a period of t if it is the GCD of $\{n : P_{jj}^{(n)} > 0\}$ if t=1 the state is a periodic

(iii) Irreducible Markov chain $\{n : P_{jj}^{(n)} > 0\}$

(iv) Recurrent time distribution

If $F_{jj} = \sum_n f_{jj}^{(n)} = 1$ the $f_{jj}^{(n)}$ is the probability distribution called recurrent time distribution.

[4 marks]

(b) Classify the states of the following Markov chain: $\underline{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

[16 marks]]

QUESTION FOUR (20 MARKS)

Consider a birth-death process with difference differential equations:

$$p_n'(t) = -2n\mu p_n(t) + (n-1)\mu p_{n-1}(t) + (n+1)\mu p_{n+1}(t) \qquad n \geq 1$$

and

$$p_0'(t) = \mu p_1(t), \quad n = 0$$

Suppose the initial conditions : $p_n(0) = 1, \quad n = 1$ and $p_n(0) = 0,\ n \neq 1$. Obtain the solution of the process

[20 marks]

COURSE: Biostatistics YEAR OF STUDY: 3.2 SHEET NO.

UNIT CODE: STA 2305 TITLE: Stochastic Processes DATE.

NOTE: This stationery will be used for Continuous Assessment work only.

It will be a breach of examination regulations to use it otherwise.

1. $\Pr\{Y=K\} = P_K = q^{K-2}p$.

The probability generating function is given by

$$\sum_{K=0} P_K S^K.$$

Substituting for P_K

$$\sum_{K=2} q^{K-2} p S^K.$$

$$= p\sum_{K=2} q^{K-2} S^K.$$

$$= pS^2 \sum_{K=2} q^{K-2}\cdot S^{K-2}.$$

$$= pS^2 \sum_{K=2} (qs)^{K-2}.$$

$$\sum_{K=2}(qs)^{K-2} = 1 + qs + (qs)^2 + \cdots$$

Hence the probability generating function becomes

$$pS^2 \cdot \frac{1}{1-qs}$$

$$= \frac{pS^2}{1-qs}$$

Hence Shown.

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$E(Y) = P'(1)$.

Differentiating with respect to S.

$$\frac{pS^2}{1-qs} = pS^2(1-qs)^{-1}$$

Using Product rule;

$$2pS(1-qs)^{-1} + pS^2(-1)(1-qs)^{-2}(-q).$$

$$2ps(1-qs)^{-1} + pS^2(1-qs)^{-2}\cdot q$$

$$= \frac{2ps}{(1-qs)} + \frac{pS^2q}{(1-qs)^2}$$

$P'(1) =$

$$\frac{2p}{(1-q)} + \frac{pq}{(1-q)^2}.$$

but $q+p=1$ hence $p = 1-q$.

$$\frac{2p}{p} + \frac{pq}{p^2}.$$

$$2 + \frac{q}{p} = \frac{2p+q}{p}$$

$q = 1-p$.

$$\frac{2p+(1-p)}{p} = \frac{2p+1-p}{p} = \frac{1+p}{p}$$

$$= 1 + \frac{1}{p}$$

Hence Shown.

4

$$Var(Y) = P''(1) + P'(1) - [P'(1)]^2$$

$P''(1) =$

$$2ps(1-qs)^{-1} + pqs^2(1-qs)^{-2}$$

Differentiating using Product rule.

$$2p(1-qs)^{-1} + 2ps(1-qs)^{-2}(-1)(-q)$$

$$\frac{2p}{1-qs} + \frac{2pqs}{(1-qs)^2} \quad \ldots (1)$$

$pqs^2(1-qs)^{-2} =$

$$2pqs(1-qs)^{-2} + pqs^2(-2)(1-qs)^{-3}(-q)$$

$$\frac{2pqs}{(1-qs)^2} + \frac{2pq^2s^2}{(1-qs)^3} \quad \ldots (2)$$

$P''(1) =$

$$\frac{2p}{(1-q)} + \frac{2pq}{(1-q)^2} + \frac{2pq}{(1-q)^2} + \frac{2pq^2}{(1-q)^3}$$

$$\frac{2p}{p} + \frac{2pq}{p^2} + \frac{2pq}{p^2} + \frac{2pq^2}{p^3}$$

$$2 + \frac{2q}{p} + \frac{2q}{p} + \frac{2q^2}{p^2}$$

$$\frac{2p^2+2pq+2pq+2q^2}{p^2} = \frac{2p^2+4pq+2q^2}{p^2}$$

$$[P'(1)]^2 = \left(\frac{1+p}{p}\right)^2$$

$$1(1+p)+p(1+p) = \frac{1+2p+p^2}{p^2}$$

$$\frac{2p^2+4pq+2q^2-1-2p-p^2}{p^2}$$

$$\frac{p^2+4pq+2q^2-1-2p}{p^2} + \frac{1+p}{p}$$

$$\frac{p^2+4pq+2q^2-1-2p+p+p^2}{p^2}$$

$$\frac{2p^2+4pq+2q^2-1-p}{p^2}$$

$p = 1-q \cdot (1-q)^2 = 1(1-q) - q(1-q)$

$$1-2q+q^2$$

$$2(1-2q+q^2)+4q(1-q)+2q^2-1-(1-q)$$

$$\frac{2-4q+2q^2+4q-4q^2+2q^2-1-1+q}{p^2}$$

$$= \frac{q}{p^2}$$

Hence Shown.

Question 2.

2. $G(s,t) = e^{-\frac{\lambda}{\mu}[(1-e^{-\mu t})(1-s)]}$

$P_n(t) =$ Coefficient of s^n.

Expanding yields

$$1(1-e^{-\mu t}) - s(1-e^{-\mu t})$$

$$[1-e^{-\mu t} - s + se^{-\mu t}] \cdot -\frac{\lambda}{\mu}$$

$$-\frac{\lambda}{\mu} + \frac{\lambda}{\mu}e^{-\mu t} + \frac{\lambda}{\mu}s - \frac{\lambda}{\mu}se^{-\mu t}$$

$$-\frac{\lambda}{\mu}(1-e^{-\mu t}) + s\left(\frac{\lambda}{\mu} - \frac{\lambda}{\mu}e^{-\mu t}\right)$$

Hence,

$$e^{-\frac{\lambda}{\mu}(1-e^{-\mu t}) + s\left(\frac{\lambda}{\mu} - \frac{\lambda}{\mu}e^{-\mu t}\right)}$$

$$= e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \cdot e^{s\left(\frac{\lambda}{\mu} - \frac{\lambda}{\mu}e^{-\mu t}\right)}$$

Expanding the term with s yields

$$1 + \frac{s\left(\frac{\lambda}{\mu} - \frac{\lambda}{\mu}e^{-\mu t}\right)}{1!} + \ldots \frac{\left[s\left(\lambda/\mu - \frac{\lambda}{\mu}e^{-\mu t}\right)\right]^n}{n!}$$

Hence, Since $P_n(t)$ is the Coefficient of s^n in the sequence,

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$P_n(t) =$

$$e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})}\left[\frac{\left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right)^n}{n!}\right].$$

For the Mean and Variance

$E(x) = P'(1)$.

Differentiating with respect to s.

$$= e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \cdot \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right) e^{s\left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right)}.$$

$P'(1)$; $s = 1$.

$$e^{-\frac{\lambda}{\mu}(1-e^{-\mu t})} \cdot \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right) \cdot e^{\lambda/\mu(1-e^{-\mu t})}$$

Hence Mean is

$$= \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right) = \frac{\lambda}{\mu}(1-e^{-\mu t}).$$

For the Variance

$Var(x) = P''(1) + P'(1) - [P'(1)]^2$

$P''(1) =$

$$e^{-\lambda/\mu(1-e^{-\mu t})} \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right)^2 e^{\frac{\lambda}{\mu}(1-e^{-\mu t})}.$$

$P''(1)$; $s = 1$.

$$e^{-\lambda/\mu(1-e^{-\mu t})} \cdot \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right)^2 e^{\frac{\lambda}{\mu}(1-e^{-\mu t})}. \qquad e^0 = 1$$

$$= \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right)^2.$$

Variance =

$$\left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right)^2 + \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right) - \left[\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right]^2.$$

Hence the Variance is

$$= \left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu}e^{-\mu t}\right) = \frac{\lambda}{\mu}(1-e^{-\mu t})$$

Question 3.

3. $a_n = 5a_{n-1} - 6a_{n-2} \quad n \geq 2$.

$a_0 = 1 \quad a_1 = -2$.

$a_n - 5a_{n-1} + 6a_{n-2} = 0$.

Multiplying by s^n and summing over n yields

$$\sum_{n=2} a_n s^n - 5\sum_{n=2} a_{n-1}s^n + 6\sum_{n=2} a_{n-2}s^n = 0.$$

$$\sum_{n=2} a_n s^n = A(s) - a_0 - a_1 s \quad \text{.. (a)}$$

$$\sum_{n=2} a_{n-1}s^n = s\left[\sum_{n=2} a_{n-1}s^{n-1}\right]$$

$$= s\left[A(s) - a_0\right] \quad \text{.. (b)}$$

$$\sum_{n=2} a_{n-2}s^n = s^2\left[\sum_{n=2} a_{n-2}s^{n-2}\right].$$

$$= s^2 A(s) \quad \text{... (c)}.$$

Substituting (a) (b) and (c) in the equation

$$(A(s) - a_0 - a_1 s) - 5s(A(s) - a_0) + 6s^2A(s) = 0$$

Applying the Conditions

$a_0 = 1, a_1 = -2$.

$$(A(s) - 1 + 2s) - 5s(A(s) - 1) + 6s^2A(s) = 0$$

$$A(s) - 1 + 2s - 5sA(s) + 5s + 6s^2A(s)$$

Making $A(s)$ the subject

$$A(s) - 1 + 7s - 5sA(s) + 6s^2A(s) = 0.$$

$$A(s) - 5sA(s) + 6s^2A(s) = 1 - 7s$$

$$A(s)\left[1 - 5s + 6s^2\right] = 1 - 7s$$

Dividing both sides by $(1 - 5s + 6s^2)$ to obtain $A(s)$.

$$A(s) = \frac{1 - 7s}{1 - 5s + 6s^2}$$

UNIT CODE: STA 2305 TITLE: Stochastic Processes DATE: 8/4/2019

NOTE: This stationery will be used for Continuous Assessment work only.

It will be a breach of examination regulations to use it otherwise.

1 $\begin{bmatrix} 0 & 3/5 & 2/5 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

Let $\bar{x} = [x_1\ x_2\ x_3]$

Where $x_1 + x_2 + x_3 = 1$.

$\bar{x}P = \bar{x}$

$$[x_1\ x_2\ x_3]\begin{bmatrix} 0 & 3/5 & 2/5 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x_1\ x_2\ x_3]$$

$$\left[\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3,\ \tfrac{3}{5}x_1 + \tfrac{1}{2}x_3,\ \tfrac{2}{5}x_1 + \tfrac{1}{2}x_2\right] = [x_1\ x_2\ x_3]$$

$\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = x_1$

$\tfrac{3}{5}x_1 + \tfrac{1}{2}x_3 = x_2$.

$\tfrac{2}{5}x_1 + \tfrac{1}{2}x_2 = x_3$

Substituting x_1 in equation 2

$\tfrac{3}{5}\left[\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3\right] + \tfrac{1}{2}x_3 = x_2$.

$\tfrac{3}{10}x_2 + \tfrac{3}{10}x_3 + \tfrac{1}{2}x_3 = x_2$.

$\tfrac{3}{10}x_2 + \tfrac{4}{5}x_3 = x_2$

$\tfrac{4}{5}x_3 = x_2 - \tfrac{3}{10}x_2$

$\tfrac{4}{5}x_3 = \tfrac{7}{10}x_2$

$x_3 = \tfrac{7}{8}x_2$ ... (i)

~~Substituting x_1 in equation 3~~

$\cancel{\tfrac{2}{5}\left[\tfrac{1}{2}x_2 + \tfrac{1}{2}x_3\right] + \tfrac{1}{2}x_2 = x_3}$

$\cancel{\tfrac{2}{10}x_2}$

Substituting x_2 in equation 1.

$\tfrac{1}{2}x_2 + \tfrac{1}{2}\left(\tfrac{7}{8}x_2\right) = x_1$

$\tfrac{1}{2}x_2 + \tfrac{7}{16}x_2 = x_1$

$\tfrac{15}{16}x_2 = x_1$ - (ii)

$x_1 + x_2 + x_3 = 1$

$\tfrac{15}{16}x_2 + x_2 + \tfrac{7}{8}x_2 = 1$

9

$\tfrac{45}{16}x_2 = 1$

$x_2 = \tfrac{16}{45}$ $x_3 = \tfrac{7}{8}x_2$ $x_1 = \tfrac{15}{16}x_2$

$x_3 = \tfrac{7}{8}\left(\tfrac{16}{45}\right) = \tfrac{14}{45}$

$x_1 = \tfrac{15}{16}\left(\tfrac{16}{45}\right) = \tfrac{1}{3}$

$\left[\tfrac{1}{3}\ \tfrac{16}{45}\ \tfrac{14}{45}\right]$

$$= \begin{bmatrix} 1/3 & 16/45 & 14/45 \\ 1/3 & 16/45 & 14/45 \\ 1/3 & 16/45 & 14/45 \end{bmatrix}$$

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2. $X(t) = A_1 + A_2 t$

$E\{X(t)\} = E\{A_1 + A_2 t\}$

$= E(A_1) + E(A_2)t$

but $E(A_1) = a_1$

hence substituting yields

$= a_1 + a_2 t$

Hence the mean is a function containing t.

$Cov\{X(t)X(s)\} = E[X(t)X(s)] - E[X(t)]E[X(s)]$

$E[X(t)X(s)] =$

$E[(A_1 + A_2 t)(A_1 + A_2 s)]$

$A_1(A_1 + A_2 s) + A_2 t(A_1 + A_2 s)$

$E[A_1^2 + A_1A_2 s + A_1A_2 t + A_2^2 st]$ ...(a)

$E[X(t)]E[X(s)] =$

$(a_1 + a_2 t)(a_1 + a_2 s)$

$a_1(a_1 + a_2 s) + a_2 t(a_1 + a_2 s)$

$a_1^2 + a_1a_2 s + a_1a_2 t + a_2^2 ts$ ...(b)

but

$Var(A_1) = E(A_1^2) - \{E[A_1]\}^2$

$\delta_1^2 = E(A_1^2) - a_1^2$

$\left.\begin{array}{l} E(A_1^2) = \delta_1^2 + a_1^2 \\ E(A_2^2) = \delta_2^2 + a_2^2 \end{array}\right\}$ ...(c)

Expanding equation (a)

$E(A_1^2) + E(A_1A_2)s + E(A_1A_2)t + E(A_2^2)st$

Substituting (c)

$(\delta_1^2 + a_1^2) + a_1a_2 s + a_1a_2 t + (\delta_2^2 + a_2^2)st$

~~Subtracting equation (b) and (d)~~

$\cancel{(\delta_1^2 + a_1^2 + a_1a_2 s + a_1a_2 t + \delta_2^2 + a_2^2)} -$

$\cancel{(a_1^2 + a_1a_2 s + a_1a_2 t + a_1}$

$(\delta_1^2 + a_1^2 + a_1a_2 s + a_1a_2 t + \delta_2^2 st + a_2^2 st)$ ...(d)

Subtracting equations (d) and (b)

$(\delta_1^2 + a_1^2 + a_1a_2 s + a_1a_2 t + \delta_2^2 st + a_2^2 st) -$

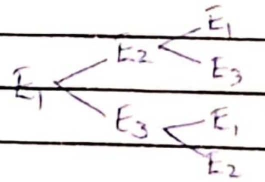
$(a_1^2 + a_1a_2 s + a_1a_2 t + a_2^2 ts)$

$= \delta_1^2 + \delta_2^2 st$

Since both the mean and the covariance contain t ~~and~~ thus the process is evolutionary hence shown.

Question 3

3.
$$\begin{array}{c|ccc} & E_1 & E_2 & E_3 \\ \hline E_1 & 0 & 5/6 & 1/6 \\ E_2 & 1/2 & 0 & 1/2 \\ E_3 & 1/2 & 1/2 & 0 \end{array}$$



Since all the states are reachable, then the Markov Chain is irreducible thus all the states have the same properties

Considering E_1

$f_{11}^{(2)} = P[E_1 \to E_2 \to E_1] + P[E_1 \to E_3 \to E_1]$

$\left(\frac{5}{6}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = 1/2$

$f_{11}^{(3)} = P[E_1 \to E_2 \to E_3 \to E_1] + P[E_1 \to E_3 \to E_2 \to E_1]$

$\left(\frac{5}{6}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$= 1/4 = (1/2)^2$

$f_{11}^{3}=(1/2)^3 \quad f_{11}^{n}=(1/2)^{n-1}$

$f_{11}=\sum_n f_{11}^{n}=\sum_n (1/2)^{n-1}$

$= 1/2+(1/2)^2+(1/2)^3+\cdots$

$= \frac{1/2}{1-1/2} = 1$

Hence the state is Persistent

$\mu_{jj}=\sum_n n f_{jj} = \sum_n n(1/2)^{n-1}$

$= 2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)+\cdots$

$\left[1+2(1/2)+3(1/2)^2+\cdots\right]-1$

Let $x = 1/2$

$f'(x)= 1+2x+3x^2+\cdots$

$f(x)= x+x^2+x^3+\cdots$

$= \frac{x}{1-x} \Rightarrow x(1-x)^{-1}$

Differentiating using Product rule

$-x(1-x)^{-2}(-1)+(1-x)^{-1}$

$\frac{x}{(1-x)^2}+\frac{1}{(1-x)} = \frac{x+1-x}{(1-x)^2} = \frac{1}{(1-x)^2}$

$f'(x)=\frac{1}{(1-x)^2} = \frac{1}{(1-1/2)^2} = \frac{1}{1/4} = 4$

$4-1 = 3 < \infty$

Hence the state is non-null

$[n: 2,3,4\ldots]$ the greatest common divisor is 1. $t=1$. Since the period is 1 the state is aperiodic

Since E_1 is aperiodic, persistent and non-null, then E_1 is said to be Ergodic. Since all the states have the same properties, E_2 and E_3 are also ergodic.

4. $P_n'(t) = -KP_n(t)+KP_{n-1}(t) \quad n \geq 1$

$P_0'(t) = -KP_0(t) \quad n=0$

$P_n(0)=\begin{cases}1 & n=0\\ 0 & \end{cases}$

Multiplying the equation by n and summing over n

$\sum_n nP_n'(t) = -K\sum_n nP_n(t) + K\sum_n nP_{n-1}(t)$

$M_1(t)=\sum nP_n(t) \quad M_1'(t)=\sum nP_n'(t) \quad n=(n-1)+1$

Substituting in the equation

$M_1'(t) = -KM_1(t)+K\left[\sum_n (n-1)P_{n-1}(t)+\sum_n P_{n-1}(t)\right]$

$\boxed{P_0'(t) = -KP_0(t)}$

~~$M_1'(t)=-KM_1(t)+K[\ldots]$~~

$M_1'(t) = -KM_1(t)+KM_1(t)+K\sum P_{n-1}(t) \quad \sum P_{n-1}(t)=1$

$M_1'(t) = K$

Integrating yields

~~$\int M_1'(t) = \int K$~~

$M_1(t) = Kt + C$

Applying Condition

$P_n(0) = \begin{cases} 1 & n=0 \\ 0 \end{cases}$

~~$M_1(t) =$~~ $C = 0$

$M_1(t) = Kt$

Thus the mean is Kt

$M_1(t) = \sum_{n=0} n P_n(t) =$ ~~[illegible]~~

$(1)0 + P_1(t) + 2P_2(t) + \cdots$

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$M_1(0) = K(0) + C$

$M_1(0) = 0$

Thus $C = 0$

~~[illegible]~~

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