3-§. Qisuvchi akslantirishlar prinsipi va uning tadbiqlari

Berilgan shartlarda tenglama yechimining mavjudligi va yagonaligi bilan bog‘liq masalalarni mos metrik fazolardagi biror akslantirishning qo‘zg‘almas nuqtasi mavjudligi va yagonaligi haqidagi masala ko‘rinishida ifodalash mumkin. Qo‘zg‘almas nuqta mavjudligi va yagonaligi belgilari ichida eng sodda va shu bilan birga juda muhim belgi - bu «qisuvchi akslantirishlar prinsipi» deb nomlanuvchi belgidir.

4.1-ta’rif. X *metrik fazo va uni o‘zini-o‘ziga akslantiruvchi* A *akslantirish berilgan bo‘lsin. Agar shunday* $\alpha \in (0, 1)$ *son mavjud bo‘lib, barcha* $x, y \in X$ *nuqtalar uchun*

$$\rho(Ax, Ay) \leq \alpha\, \rho(x, y) \tag{4.1}$$

tengsizlik bajarilsa, A *qisuvchi akslantirish deb ataladi.*

Har bir qisuvchi akslantirish uzluksizdir. Haqiqatan ham, agar $x_n \to x$ $(\rho(x_n, x) \to 0)$ bo‘lsa, u holda

$$\rho(Ax_n, Ax) \leq \alpha\, \rho(x_n, x)$$

bo‘lgani uchun $Ax_n \to Ax$.

Agar $A: X \to X$ akslantirish uchun shunday $x \in X$ nuqta mavjud bo‘lib, $Ax = x$ tenglik bajarilsa, x nuqta A akslantirishning *qo‘zg‘almas nuqtasi* deyiladi.

4.1-teorema. *(Qisuvchi akslantirishlar prinsipi). To‘la metrik fazoda aniqlangan har qanday qisuvchi akslantirish yagona qo‘zg‘almas nuqtaga ega.*

Isbot. X metrik fazodan ixtiyoriy x_0 nuqtani olamiz. Keyin

$$x_1 = Ax_0,\ x_2 = Ax_1 = A^2 x_0,\ x_3 = Ax_2 = A^3 x_0, \ldots,\ x_n = Ax_{n-1} = A^n x_0, \ldots$$

nuqtalar ketma-ketligini qaraymiz. Ixtiyoriy $n < m$ natural sonlar uchun

$$\rho(x_n, x_m) = \rho\left(A^n x_0, A^m x_0\right) \leq \alpha^n \rho(x_0, x_{m-n}) \leq$$

$$\leq \alpha^n \left(\rho(x_0, x_1) + \rho(x_1, x_2) + \ldots + \rho(x_{m-n-1}, x_{m-n})\right) \leq$$

$$\leq \alpha^n \rho(x_0, x_1)\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{m-n-1}\right) \leq \alpha^n \frac{1}{1-\alpha} \rho(x_0, x_1)$$

tengsizlik o‘rinli. $\alpha \in (0, 1)$ bo‘lgani uchun

$$\lim_{n\to\infty} \alpha^n (1-\alpha)^{-1} \rho(x_0, x_1) = 0.$$

Shuning uchun $\{x_n\}$ fundamental ketma-ketlikdir. X to‘la metrik fazo va $\{x_n\}$ fundamental ketma-ketlik bo‘lgani uchun u yaqinlashuvchi. Aytaylik,

$$x = \lim_{n\to\infty} x_n$$

bo‘lsin. U holda A akslantirishning uzluksizligiga ko‘ra

$$Ax = A\lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = x.$$

Shunday qilib, A akslantirish uchun qo‘zg‘almas nuqta mavjud ekan. Uning yagonaligini isbotlaymiz. Agar

$$Ax = x, \quad Ay = y$$

desak, (4.1) tengsizlikka ko‘ra

$$\rho(x,y) = \rho(Ax,\, Ay) \le \alpha\,\rho(x,y).$$

Bundan $\alpha \in (0,1)$ bo‘lgani uchun

$$\rho(x,y)(1-\alpha) \le 0 \Rightarrow \rho(x,y) = 0$$

ya’ni $x = y$ bo‘lishi kelib chiqadi. Qo‘zg‘almas nuqta yagona ekan. Δ

3.1. Qisuvchi akslantirishlar prinsipining tadbiqlari

Qisuvchi akslantirishlar prinsipini har xil turdagi tenglamalar yechimlari mavjudligi va yagonaligi haqidagi teoremalarni isbotlashda qo‘llash mumkin. Qisuvchi akslantirishlar prinsipi $Ax = x$ tenglama yechimi mavjudligi va yagonaligini isbotlash uchungina qo‘llanib qolmay, bu tenglama yechimini topish usulini ham beradi.

Qisuvchi akslantirishlar prinsipining tadbig‘iga doir misollar qaraymiz.

4.1-misol. R^n fazoni o‘zini-o‘ziga akslantiruvchi va

$$(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j + b_i, \quad i = 1,2,\ldots,n$$

formulalar orqali aniqlangan A akslantirishning qisuvchilik shartlarini toping.

Yechish. Qanday shartlarda A qisuvchi akslantirish bo‘ladi? Bu savolga javob fazoda qanday metrika berilishiga bog‘liq. Biz quyida uch xil variantni qaraymiz:

a) R_∞^n fazo, ya’ni $\rho(x,y) = \max\limits_{1\le i\le n}|x_i - y_i|$ bo‘lsin.

$$\rho(y', y'') = \max_{1\le i\le n}|y_i' - y_i''| = \max_{1\le i\le n}\left|\sum_{j=1}^{n} a_{ij}\left(x_j' - x_j''\right)\right| \le \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|\,|x_j' - x_j''| \le$$

$$\le \max_{1\le i\le n}\sum_{j=1}^{n}\left| a_{ij}\right| \cdot \max_{1\le j\le n}\left| x'_j - x''_j\right| = \left(\max_{1\le i\le n}\sum_{j=1}^{n}\left| a_{ij}\right|\right)\rho\left(x', x''\right).$$

Bu yerdan kelib chiqadiki, A qisuvchi akslantirish bo‘lishi uchun

$$\max_{1\le i\le n}\sum_{j=1}^{n}\left| a_{ij}\right| = \alpha < 1 \tag{4.2}$$

shartning bajarilishi yetarli. Shuning uchun R_∞^n fazoda (4.2) shartni A akslantirishning qisuvchilik sharti sifatida qabul qilamiz.

b) R_1^n fazo, ya’ni $\rho(x,y)=\sum_{i=1}^{n}\left|x_i - y_i\right|$ bo‘lsin. U holda

$$\rho(y', y'') = \sum_{i=1}^{n}\left| y'_i - y''_i\right| = \sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\left(x'_j - x''_j\right)\right| \le \sum_{i=1}^{n}\sum_{j=1}^{n}\left| a_{ij}\right|\left| x'_j - x''_j\right| =$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\left| a_{ij}\right|\right)\cdot\left|x'_j - x''_j\right| \le \left(\max_{1\le j\le n}\sum_{i=1}^{n}\left| a_{ij}\right|\right)\cdot\sum_{j=1}^{n}\left|x'_j - x''_j\right| \le \left(\max_{1\le j\le n}\sum_{i=1}^{n}\left| a_{ij}\right|\right)\cdot\rho\left(x', x''\right).$$

Bu yerdan ko‘rinadiki, A akslantirish uchun qisuvchilik sharti R_1^n fazoda

$$\max_{1\le j\le n}\sum_{i=1}^{n}\left| a_{ij}\right| = \alpha < 1 \tag{4.3}$$

ko‘rinishga ega.

c) R^n fazo, ya’ni

$$\rho(x,y) = \sqrt{\sum_{i=1}^{n}\left(x_i - y_i\right)^2}$$

bo‘lsin. U holda

$$\rho^2(y', y'') = \sum_{i=1}^{n}\left(y'_i - y''_i\right)^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\left(x'_j - x''_j\right)\right)^2 \le$$

$$\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left| a_{ij}\right|^2\right)\cdot\sum_{j=1}^{n}\left(x'_j - x''_j\right)^2 \le \left(\sum_{j=1}^{n}\sum_{i=1}^{n}\left| a_{ij}\right|^2\right)\cdot\rho^2\left(x', x''\right).$$

Yuqorida keltirilgan tenglik va tengsizliklarga ko‘ra R^n fazoda A akslantirishning qisuvchilik sharti

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\left|a_{ij}\right|^{2}\leq\alpha<1 \tag{4.4}$$

ko‘rinishga ega.

Shunday qilib, agar (4.2)-(4.4) shartlardan birortasi bajarilsa, u holda yagona $x=\left(x_1, x_2, \ldots, x_n\right)$ nuqta mavjud bo‘lib,

$$x_i=\sum_{j=1}^{n}a_{ij}x_j+b_i,\quad i=1,2,\ldots,n$$

bo‘ladi. Bundan tashqari bu nuqtada ketma-ket yaqinlashishlar quyidagi ko‘rinishga ega

$$x_i^{(k)}=\sum_{j=1}^{n}a_{ij}x_j^{(k-1)}+b_i,\quad x^{(k)}=\left(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)}\right),\ k=1,2,3\ldots.$$

Bu yerda $x^{(0)}=\left(x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)}\right)$ sifatida R^n dagi ixtiyoriy nuqtani qabul qilish mumkin.

Qaralayotgan $y=Ax$ akslantirish qisuvchi bo‘lishi uchun (4.2)-(4.4) shartlarning ixtiyoriy birining bajarilishi yetarli. Isbotlash mumkinki, (4.2) va (4.3) shartlar mos ravishda R_∞^n va R_1^n fazolarda $y=Ax$ akslantirish qisuvchi bo‘lishi uchun zarur ham bo‘ladi.

Ta’kidlash lozimki, (4.2) - (4.4) shartlarning birortasi ham ketma-ket yaqinlashishlar usulining tadbig‘i uchun zarur emas.

Agar $\left|a_{ij}\right|<n^{-1}$ bo‘lsa, u holda (4.2) - (4.4) shartlarning hammasi bajariladi va ketma-ket yaqinlashishlar usulini qo‘llash mumkin.

Agar $\left|a_{ij}\right|\geq n^{-1}$ bo‘lsa, u holda (4.2) - (4.4) shartlarning birortasi ham bajarilmaydi.