Circle Assignment

Mohamed Hamdan

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Problem Statement **- Let ABC be a right triangle in which AB = 6 cm, BC = 8 cm and ∠B = 90°. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.**

Solution

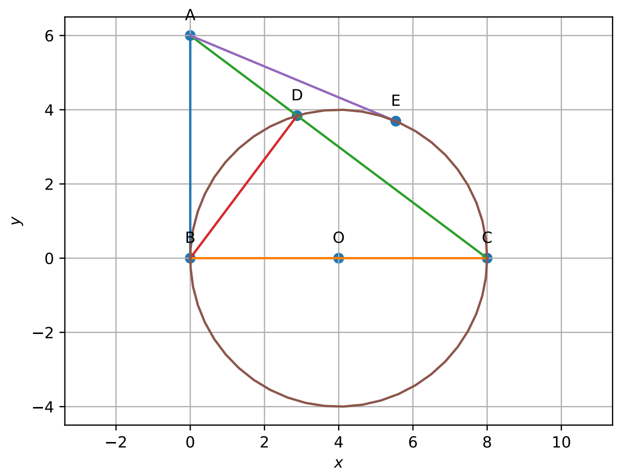

Figure 1: Tangents from A to circle through B, C and D

Given that BD ⊥ AC, which implies

$$\angle D = 90°. \tag{1}$$

So, D can be found as the foot of the perpendicular from B on line AC. This is given by

$$\mathbf{D} = \mathbf{A} + \frac{\mathbf{m}^T(\mathbf{B}-\mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{2}$$

where $\mathbf{m}$ is the direction vector for line AC.

The chord BC of the circle subtends 90° at D. By the inclusive angle theorem, BC is the diameter of the circle with center O given by

$$\mathbf{O} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{3}$$

In order to find the intersection points E and B of tangents from A, the origin is shifted from B to O. The equation of the circle in the new frame is

$$\mathbf{x}^T\mathbf{x} = r^2 \tag{4}$$

Let the the point of intersection between the tangent from A and the circle be P. Since P lies on the circle given by (4) it is of the form

$$\mathbf{P} = \begin{pmatrix} t \\ \sqrt{r^2 - t^2} \end{pmatrix} \tag{5}$$

Since AP is a tangent to the circle, OP ⊥ AP. This implies that

$$(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{O}) = 0 \tag{6}$$

Since O is the origin in the new frame, $\mathbf{O} = \mathbf{0}$. Expanding (6), we get

$$\mathbf{A}^T\mathbf{P} = \mathbf{P}^T\mathbf{P} \tag{7}$$

Substituting value of $\mathbf{P}$ from (5) in (7), we get

$$\mathbf{A}^T\begin{pmatrix} t \\ \sqrt{r^2 - t^2} \end{pmatrix} = r^2 \tag{8}$$

Expanding and rearranging terms in (8)

$$\|\mathbf{A}\|^2 t^2 - 2r^2\mathbf{e}_1^\top\mathbf{A}t + r^2(r^2 - (\mathbf{e}_2^\top\mathbf{A})^2) = 0 \tag{9}$$

Which is a quadratic equation in t with roots given by

$$t = \frac{r^2\mathbf{e}_1^\top\mathbf{A} \pm \sqrt{r^4\mathbf{e}_1^\top\mathbf{A}^2 - r^2\|\mathbf{A}\|^2(r^2 - \mathbf{e}_2^\top\mathbf{A}^2)}}{\|\mathbf{A}\|^2} \tag{10}$$

Substituting the values of t from (10) in (5) the two points of contact are obtained say $\mathbf{E_O}$ and $\mathbf{B_O}$.

The coordinates for position vectors $\mathbf{E_O}$ and $\mathbf{B_O}$ are with respect to origin O. The actual coordinates with respect to origin B is given by

$$\mathbf{E} = \mathbf{E_O} + \mathbf{O} \tag{11}$$
$$\mathbf{B} = \mathbf{B_O} + \mathbf{O} \tag{12}$$

Construction

The input parameters are the lengths

$$AB = a = 6$$
$$BC = b = 8$$

Symbol	Value	Description
a	6	AB
b	8	BC
r	$\frac{b}{2}$	Radius
$\mathbf{m}$	$\mathbf{A}-\mathbf{C}$	Direction vector of line AC
$\mathbf{D}$	$\mathbf{A}+\frac{\mathbf{m}^T(\mathbf{B}-\mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m}$	Point D
$\mathbf{A_O}$	$\mathbf{A}-\mathbf{O}$	$\mathbf{A}$ when origin shifted to $\mathbf{O}$
t_1, t_2	evaluate (10)	solution of (9)
$\mathbf{E}$	$\begin{pmatrix} t_1 \\ \sqrt{r^2-t_1^2} \end{pmatrix}+\mathbf{O}$	Point E
$\mathbf{B}$	$\begin{pmatrix} t_2 \\ \sqrt{r^2-t_2^2} \end{pmatrix}+\mathbf{O}$	Point B

Proofs

Foot of perpendicular from point P on line $\mathbf{A}+\lambda\mathbf{m}$

Let the intersection point be $\mathbf{X}$. Since X is foot of perpendicular from point P to line with direction vector $\mathbf{m}$,

$$\mathbf{m}^T(\mathbf{X}-\mathbf{P})=0 \tag{13}$$

Since X lies on the line with direction vector $\mathbf{m}$,

$$\mathbf{X}=\mathbf{A}+\lambda\mathbf{m} \tag{14}$$

Substituting (14) in (13) and solving for λ,

$$\lambda=\frac{\mathbf{m}^T(\mathbf{P}-\mathbf{A})}{\|\mathbf{m}\|^2} \tag{15}$$

Substituting (15) in (14),

$$\mathbf{X}=\mathbf{A}+\frac{\mathbf{m}^T(\mathbf{P}-\mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{16}$$

Inclusive angle theorem

The inclusive angle theorem states that the angle subtended by any chord at the center of a circle is twice the angle angle subtended by the same chord at any other point on the major segment. Take three points A, B, and C on a unit circle at angles θ, ϕ and ψ. Then,

$$\mathbf{A}=\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}, \mathbf{B}=\begin{pmatrix} cos\phi \\ sin\phi \end{pmatrix}, \mathbf{C}=\begin{pmatrix} cos\psi \\ sin\psi \end{pmatrix} \tag{17}$$

Let AB be the chord that subtends angles at the center O and at point C. The cosine of the angle subtended at point C is given by

$$cos(\angle ACB)=\frac{\langle A-C,B-C\rangle}{|A-C||B-C|} \tag{18}$$

Where

$$\langle A-C,B-C\rangle=\langle(\cos\theta-\cos\psi,\sin\theta-\sin\psi),(\cos\phi-\cos\psi, \sin\phi-\sin\psi)\rangle$$

$$=(\cos\theta-\cos\psi)(\cos\phi-\cos\psi)+(\sin\theta-\sin\psi)(\sin\phi-\sin\psi)$$

$$=-2\sin\frac{\theta-\psi}{2}\sin\frac{\theta+\psi}{2}\cdot(-2)\sin\frac{\phi-\psi}{2}\sin\frac{\phi+\psi}{2} + 2\cos\frac{\theta+\psi}{2}\sin\frac{\theta-\psi}{2}\cdot 2\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2}$$

$$=4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}(\sin\frac{\theta+\psi}{2}\sin\frac{\phi+\psi}{2}+ \cos\frac{\theta+\psi}{2}\cos\frac{\phi+\psi}{2})$$

$$=4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\left(\frac{\theta+\psi}{2}-\frac{\phi+\psi}{2}\right)$$

$$=4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\frac{\theta-\phi}{2} \tag{19}$$

$$|A-C|^2|B-C|^2=((\cos\theta-\cos\psi)^2+(\sin\theta-\sin\psi)^2)((\cos\phi-\cos\psi)^2+(\sin\phi-\sin\psi)^2)$$

$$=(2-2\cos\theta\cos\psi-2\sin\theta\sin\psi)(2- 2\cos\phi\cos\psi-2\sin\phi\sin\psi)$$

$$=4(1-\cos(\theta-\psi))(1-\cos(\phi-\psi))$$
$$=4\cdot 2\sin^2\frac{\theta-\psi}{2}\cdot 2\sin^2\frac{\phi-\psi}{2}$$

$$=16\sin^2\frac{\theta-\psi}{2}\sin^2\frac{\phi-\psi}{2} \tag{20}$$

Substituting (19) and (20) in (18),

$$cos(\angle ACB)=cos(\frac{\theta-\phi}{2}) \tag{21}$$

Hence $\angle ACB=\frac{\theta-\phi}{2}=\frac{\angle AOB}{2}$