Line Assignment

Mohamed Hamdan

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Problem Statement **- ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that**

1. **D is the mid-point of AC**
2. **MD $\perp$ AC**
3. **CM = MA = $\frac{1}{2}$AB**

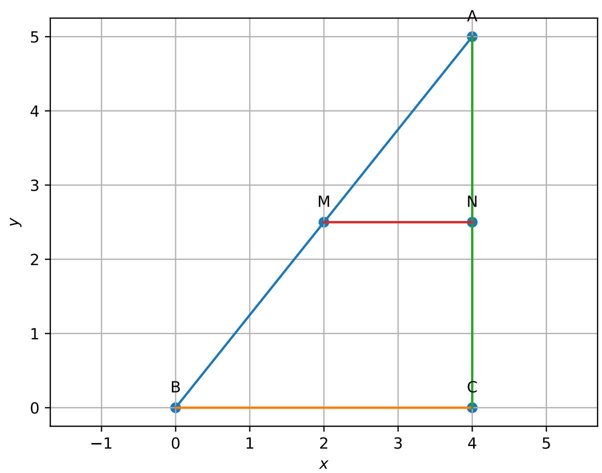

Figure 1: Triangle ABC right angled at C

Solution

Part 1

Given MN is $\|$ to BC, hence

$$\frac{AD}{DC} = \frac{AM}{MB} \tag{1}$$

Since M is the mid-point of AB, $\frac{AM}{MB} = 1$. Substituting in (1), we get

$$AD = DC \tag{2}$$

Therefore, D is the midpoint of AC.

Part 2

Since ABC is right angled at C,

$$(\boldsymbol{C} - \boldsymbol{A})^T(\boldsymbol{C} - \boldsymbol{B}) = 0 \tag{3}$$

Given that MD is parallel to BC, so

$$(\boldsymbol{C} - \boldsymbol{B}) = \lambda(\boldsymbol{M} - \boldsymbol{D}) \tag{4}$$

Substituting (4) in (3) and dividing by λ, we get

$$(\boldsymbol{C} - \boldsymbol{A})^T(\boldsymbol{M} - \boldsymbol{B}) = 0 \tag{5}$$

From (5) it can be concluded that MD $\perp$ AC.

Part 3

Let

$$\boldsymbol{M} - \boldsymbol{D} = \boldsymbol{p} \tag{6}$$
$$\boldsymbol{C} - \boldsymbol{D} = \boldsymbol{q} \tag{7}$$
$$\boldsymbol{A} - \boldsymbol{D} = \boldsymbol{r} \tag{8}$$

Then vectors along CM and AM can be written as

$$\boldsymbol{C} - \boldsymbol{M} = \boldsymbol{q} - \boldsymbol{p} \tag{9}$$
$$\boldsymbol{A} - \boldsymbol{M} = \boldsymbol{r} - \boldsymbol{p} \tag{10}$$

The magnitudes of CM and AM are therefore

$$||\boldsymbol{C} - \boldsymbol{M}|| = ||\boldsymbol{q} - \boldsymbol{p}|| = (\boldsymbol{q} - \boldsymbol{p})^T(\boldsymbol{q} - \boldsymbol{p}) \tag{11}$$
$$||\boldsymbol{A} - \boldsymbol{M}|| = ||\boldsymbol{r} - \boldsymbol{p}|| = (\boldsymbol{r} - \boldsymbol{p})^T(\boldsymbol{r} - \boldsymbol{p}) \tag{12}$$
$$\tag{13}$$

Upon expanding the vector products, the terms $\boldsymbol{q}^T\boldsymbol{p}$ and $\boldsymbol{r}^T\boldsymbol{p}$ evaluate to 0 (from eq.5). From eq.2,

$$||\boldsymbol{q}|| = ||\boldsymbol{r}||$$

. Therefore,

$$||\boldsymbol{C} - \boldsymbol{M}|| = ||\boldsymbol{q}|| + ||\boldsymbol{p}|| \tag{14}$$

$$||\boldsymbol{A} - \boldsymbol{M}|| = ||\boldsymbol{r}|| + ||\boldsymbol{p}|| = ||\boldsymbol{q}|| + ||\boldsymbol{p}|| \tag{15}$$

Equating (14) and (15), we get

$$AM = CM \tag{16}$$

Since given that M is midpoint of AB,

$$AM = \frac{1}{2}AB \tag{17}$$

Equating (16) and (17), the result is proved.

Construction

The input parameters are the lengths a and c.

Symbol	Value	Description
a	4	BC
c	5	AC
k	1	$\frac{AM}{MB} = \frac{AD}{DC}$
θ	$\arctan(\frac{c}{a})$	∠B
A	$\sqrt{a^2+c^2}\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$	Point A

Proofs

Let the vertices of the triangle be $\mathbf{A}, \mathbf{B}, \mathbf{C}$ such that

$$\mathbf{B} = \mathbf{0} \tag{18}$$

and $\mathbf{A}, \mathbf{C}$ are known. Let $\mathbf{P}$ be a known point on AB such that PQ is parallel to BC. Let

$$\begin{aligned} \mathbf{P} &= \lambda\left(\mathbf{A} - \mathbf{B}\right) && (19) \\ &= \lambda\mathbf{A} && (20) \\ \text{and },\ \frac{\|\mathbf{P}\|}{\|\mathbf{A}\|} &= \frac{BP}{AB} = |\lambda| && (21) \end{aligned}$$

Since

$$\begin{aligned} PQ &\parallel BC, && (22) \\ \mathbf{Q} &= P + \mu\mathbf{B} - \mathbf{C} && (23) \\ &= \lambda\mathbf{A} - \mu\mathbf{C} && (24) \end{aligned}$$

using the equation of the line PQ and substituting from (20)
Also, since $\mathbf{Q}$ lies on the line AC,

$$\begin{aligned} \mathbf{Q} &= \mathbf{A} + k\left(\mathbf{A} - \mathbf{C}\right) && (25) \\ &= \left(1+k\right)\mathbf{A} - k\mathbf{C} && (26) \\ \text{and } \frac{\|\mathbf{A} - \mathbf{Q}\|}{\|\mathbf{A} - \mathbf{C}\|} &= \frac{AQ}{AC} = |k| && (27) \end{aligned}$$

From (24) and (26)

$$\begin{aligned} \lambda\mathbf{A} - \mu\mathbf{C} &= \left(1+k\right)\mathbf{A} - k\mathbf{C} && (28) \\ \implies \left(1+k+\lambda\right)\mathbf{A}\left(k-\mu\right)\mathbf{C} &= 0 && (29) \\ \implies k = \mu, \lambda &= -1-\mu && (30) \\ \text{or, } |\lambda| &= 1+k && (31) \end{aligned}$$

From (21), (27) and (31),

$$\frac{AQ}{AC} = \frac{AP}{AB} \tag{32}$$