Conic section Assignment

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***Problem Statement* - Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum**

Solution

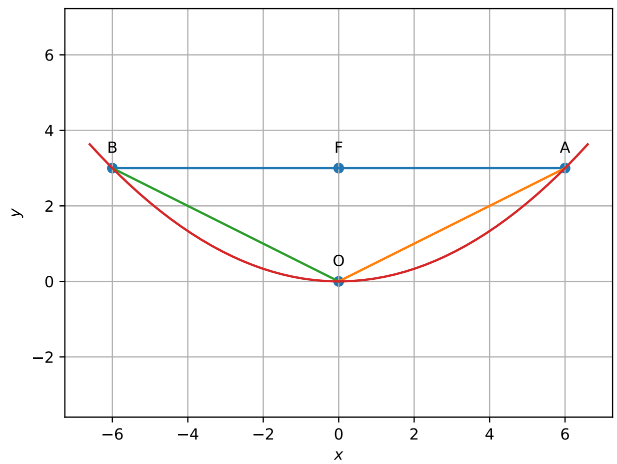

Figure 1: Triangle formed by vertex and ends of latus rectum of parabola $x^2 = 12y$

The given equation of parabola $x^2 = 12y$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

The parabola in (1) can be expressed in standard form (center/vertex at origin, major-axis - x axis) as

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -2\eta \mathbf{e}_1^\top \mathbf{y} \qquad |V| = 0 \tag{5}$$

where

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{6}$$

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{9}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{10}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

To find $\mathbf{c}$ which is the center of the parabola in (1), substitute (6) in (1)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{12}$$

yielding

$$\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{13}$$

From (13) and (7),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{14}$$

For a parabola $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{15}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (7)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{16}$$

Substituting (16) in (14),

$$\begin{aligned} &\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} \\ &\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\ &\quad + 2\left(\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 \ \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y} \\ &\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\ &\quad + 2\begin{pmatrix} \mathbf{u}^T\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} \\ &\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \text{ from (15)} \end{aligned}$$

$$\begin{aligned} \implies &\lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2(\lambda_2\mathbf{c} + \mathbf{u})^T\mathbf{p}_2 \\ &+ \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \end{aligned}$$

which is the equation of a parabola. Thus, (17) can be expressed as (5) by choosing

$$\eta = \mathbf{u}^T\mathbf{p}_1 \tag{17}$$

and $\mathbf{c}$ in (14) such that

$$\mathbf{P}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) = \eta\begin{pmatrix}1\\0\end{pmatrix} \tag{18}$$

$$\mathbf{c}^T\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)+\mathbf{u}^T\mathbf{c}+f = 0 \tag{19}$$

Multiplying (18) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) = \eta\mathbf{p}_1, \tag{20}$$

which, upon substituting in (19) results in

$$\eta\mathbf{c}^T\mathbf{p}_1+\mathbf{u}^T\mathbf{c}+f = 0 \tag{21}$$

(20) and (21) can be clubbed together to obtain (22).

$$\begin{pmatrix}\mathbf{u}^\top+\eta\mathbf{p}_1^\top\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p}_1-\mathbf{u}\end{pmatrix} \qquad |V| = 0 \tag{22}$$

Substituting appropriate values from (2), (3), (4), (9), and (10) into (22), the below matrix equation is obtained

$$\begin{pmatrix}0 & -12\\ 1 & 0\\ 0 & 0\end{pmatrix}\mathbf{c} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{23}$$

$$\tag{24}$$

The augmented matrix for (23) can be expressed as

$$\left(\begin{array}{cc|c}0 & -12 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{array}\right) \tag{25}$$

$$\xleftrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & -12 & 0\\ 0 & 0 & 0\end{array}\right) \tag{26}$$

$$\xleftrightarrow{-\frac{R_2}{12}\leftarrow R_2}\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{array}\right) \tag{27}$$

$$\implies \mathbf{c} = \begin{pmatrix}0\\0\end{pmatrix} \tag{28}$$

From (2)

$$\begin{aligned}\mathbf{V}^\top\mathbf{V} &= \left(\|\mathbf{n}\|^2\mathbf{I}-e^2\mathbf{n}\mathbf{n}^\top\right)^\top\\ &\qquad\left(\|\mathbf{n}\|^2\mathbf{I}-e^2\mathbf{n}\mathbf{n}^\top\right)\\ \implies \mathbf{V}^2 &= \|\mathbf{n}\|^4\mathbf{I}+e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top\\ &\quad-2e^2\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top\\ &= \|\mathbf{n}\|^4\mathbf{I}+e^4\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top-2e^2\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top\\ &= \|\mathbf{n}\|^4\mathbf{I}+e^2\left(e^2-2\right)\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top\\ &= \|\mathbf{n}\|^4\mathbf{I}+\left(e^2-2\right)\|\mathbf{n}\|^2\left(\|\mathbf{n}\|^2\mathbf{I}-\mathbf{V}\right)\end{aligned} \tag{29}$$

which can be expressed as

$$\mathbf{V}^2+\left(e^2-2\right)\|\mathbf{n}\|^2\mathbf{V}-\left(e^2-1\right)\|\mathbf{n}\|^4\mathbf{I} = 0 \tag{30}$$

Using the Cayley-Hamilton theorem, (30) results in the characteristic equation,

$$\lambda^2-\left(2-e^2\right)\|\mathbf{n}\|^2\lambda+\left(1-e^2\right)\|\mathbf{n}\|^4 = 0 \tag{31}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2-\left(2-e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)+\left(1-e^2\right) = 0 \tag{32}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1-e^2, 1 \tag{33}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1-e^2\right)\lambda_2 \tag{34}$$

From (34), the eccentricity of (1) is given by

$$e = \sqrt{1-\frac{\lambda_1}{\lambda_2}}. \tag{35}$$

Multiplying both sides of (2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2\mathbf{n}-e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} \tag{36}$$

$$= \|\mathbf{n}\|^2\left(1-e^2\right)\mathbf{n} \tag{37}$$

$$= \lambda_1\mathbf{n} \tag{38}$$

from (34) Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (9), (34) and (38),

$$\mathbf{n} = \|\mathbf{n}\|\,\mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{39}$$

From (3) and (34),

$$\mathbf{F} = \frac{ce^2\mathbf{n}-\mathbf{u}}{\lambda_2} \tag{40}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n}-\mathbf{u}\right)^\top\left(ce^2\mathbf{n}-\mathbf{u}\right)}{\lambda_2^2} \tag{41}$$

$$\implies \lambda_2^2\|\mathbf{F}\|^2 = c^2e^4\lambda_2-2ce^2\mathbf{u}^\top\mathbf{n}+\|\mathbf{u}\|^2 \tag{42}$$

Also, (4) can be expressed as

$$\lambda_2\|\mathbf{F}\|^2 = f+c^2e^2 \tag{43}$$

From (42) and (43),

$$c^2e^4\lambda_2-2ce^2\mathbf{u}^\top\mathbf{n}+\|\mathbf{u}\|^2 = \lambda_2\left(f+c^2e^2\right) \tag{44}$$

$$\begin{aligned}\implies \lambda_2e^2\left(e^2-1\right)c^2-2ce^2\mathbf{u}^\top\mathbf{n}&\\ +\|\mathbf{u}\|^2-\lambda_2f &= 0\end{aligned} \tag{45}$$

yielding

$$c = \begin{cases}\frac{e\mathbf{u}^\top\mathbf{n}\pm\sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2-\lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2-\lambda_2f\right)}}{\lambda_2e\left(e^2-1\right)} & e\neq 1\\ \frac{\|\mathbf{u}\|^2-\lambda_2f}{2e^2\mathbf{u}^\top\mathbf{n}} & e = 1\end{cases}. \tag{46}$$

The points of intersection of the line

$$L:\quad \mathbf{x} = \mathbf{q}+\mu\mathbf{m}\quad \mu\in\mathbf{R} \tag{47}$$

with the conic section in (1) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{48}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right.$$
$$\left.\pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2-\left(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{49}$$

Substituting (47) in (1),

$$\left(\mathbf{q}+\mu\mathbf{m}\right)^T\mathbf{V}\left(\mathbf{q}+\mu\mathbf{m}\right) \tag{50}$$
$$+2\mathbf{u}^T\left(\mathbf{q}+\mu\mathbf{m}\right)+f=0 \tag{51}$$
$$\implies \mu^2\mathbf{m}^T\mathbf{V}\mathbf{m}+2\mu\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right) \tag{52}$$
$$+\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f=0 \tag{53}$$

Solving the above quadratic in (53) yields (49).

The area of a triangle whose vertices are $\mathbf{A}, \mathbf{B}$ and $\mathbf{O}$ is given by

$$ar(AOB) = \frac{1}{2}\left\|\left(\mathbf{A}-\mathbf{O}\right)\times\left(\mathbf{B}-\mathbf{O}\right)\right\| \tag{54}$$

Construction

The input parameters are $\mathbf{V}$ from (2), $\mathbf{u}$ from (3) and f from (4)

Symbol	Value	Description
$\mathbf{P}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	eigenvectors of $\mathbf{V}$
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	center of parabola
η	$\mathbf{u}^\top\mathbf{p}_1 = -6$	from (10)
λ_2	$\mathbf{e}_2^\top D\mathbf{e}_2 = 1$	from (8)
$\mathbf{n}$	$\sqrt{\lambda_2}\mathbf{p_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	normal to directrix
c	from (46)	$\mathbf{n}^\top\mathbf{x} = c$
$\mathbf{F}$	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	focus from (40)
$\mathbf{m}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	latus rectum direction vector
$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$	$\begin{pmatrix} \mathbf{F}+\mu_1\mathbf{m} \\ \mathbf{F}+\mu_2\mathbf{m} \end{pmatrix}$	μ_1, μ_2 from (49)
ar(AOB)	18 sq units	from (54)