Construction assignment

Mohamed Hamdan

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***Problem Statement* - Construct a quadilateral circumscribing a circle**

Solution

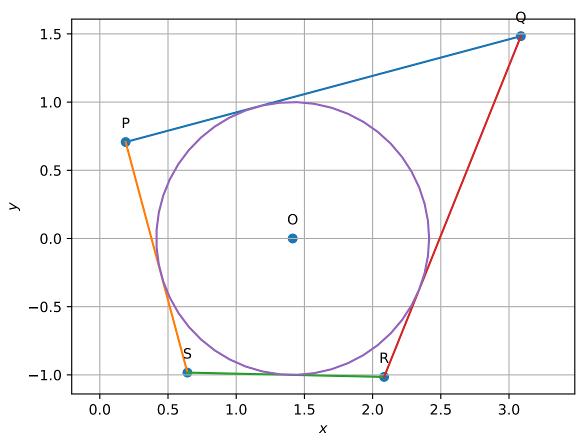

Figure 1: Quadilateral circumscribing a circle

Select a point $\mathbf{P}$ from which two tangents are drawn to the circle with center $\mathbf{O}$ and radius r. The direction vectors $\mathbf{m_i}$ satisfy the equation

$$\mathbf{m}_i^\top \mathbf{\Sigma} \mathbf{m}_i = 0 \tag{1}$$

Assuming the external point as $\mathbf{h}$, $\mathbf{\Sigma}$ is given by

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\mathbf{V} \tag{2}$$

$\mathbf{\Sigma}$ can be orthogonally diagonalized as

$$\mathbf{\Sigma} = \mathbf{\Gamma}^\top \mathbf{D} \mathbf{\Gamma} \tag{3}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{4}$$

$$\mathbf{\Gamma} = \begin{pmatrix} \mathbf{y}_1 & \mathbf{y}_2 \end{pmatrix}, \quad \mathbf{\Gamma}^\top = \mathbf{\Gamma}^{-1} \tag{5}$$

Using (4) and (5) and substituting $\mathbf{h}$ as $\mathbf{P}$, the normal vectors $\mathbf{n}_i$ of the tangents drawn from $\mathbf{P}$ can be written as

$$\mathbf{n}_i = \mathbf{\Gamma} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \pm\sqrt{|\lambda_2|} \end{pmatrix} \tag{6}$$

Using the vectors $\mathbf{n}_i$ in (6), the direction vectors $\mathbf{m}_i$ can be found in 2-dimensional space since they are orthogonal. The points of contact of the tangents are then given by

$$\mathbf{T}_i = \mathbf{P} - \frac{\mathbf{m}_i^\top (\mathbf{V}\mathbf{P} + \mathbf{u})}{\mathbf{m}_i^\top \mathbf{V} \mathbf{m}_i} \tag{7}$$

Where $\mathbf{T}_i$ are the points of contact. Consider two direction vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ as

$$\mathbf{t}_1 = \mathbf{T}_1 - \mathbf{P} \tag{8}$$

$$\mathbf{t}_2 = \mathbf{T}_2 - \mathbf{P} \tag{9}$$

The points $\mathbf{Q}$ and $\mathbf{S}$ can then be found as

$$\mathbf{Q} = \mathbf{P} + \mu_1 \mathbf{t}_1 \tag{10}$$

$$\mathbf{S} = \mathbf{P} + \mu_2 \mathbf{t}_2 \tag{11}$$

Where μ_i are the parameters for the distance of points $\mathbf{Q}$ and $\mathbf{S}$ from point $\mathbf{P}$.

Each of the vectors $\mathbf{n}_i$ are also normal vectors to one of the tangents from $\mathbf{Q}$ and $\mathbf{S}$ from (10) and (11). By using (3) and (6), the normal vectors for tangents from $\mathbf{Q}$ and $\mathbf{S}$ and not passing through $\mathbf{P}$ can be found by using the condition

$$\|\mathbf{k}_i \times \mathbf{n}_i\| \neq 0 \tag{12}$$

where $\mathbf{k}_i$ are the normal vectors of interest. The point $\mathbf{R}$ can then be found as the intersection of lines given by

$$\mathbf{k}_1^\top (\mathbf{x} - \mathbf{Q}) = 0 \tag{13}$$

$$\mathbf{k}_2^\top (\mathbf{x} - \mathbf{S}) = 0 \tag{14}$$

Solving (13) and (14) with $\mathbf{x}$ as point $\mathbf{R}$, we get

$$\mathbf{R} = \begin{pmatrix} \mathbf{k}_1^\top \\ \mathbf{k}_2^\top \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{k}_1^\top \mathbf{Q} \\ \mathbf{k}_2^\top \mathbf{S} \end{pmatrix} \tag{15}$$