Digital Communication

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CONTENTS

Chapter 1 Two Dice

1.1 SUM OF INDEPENDANT RANDOM VARIABLES

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.1.1 *The Uniform Distribution:* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.1.1.2}$$

$$\implies X \in \{1,2,\dots,12\} \tag{1.1.1.3}$$

The objective is to show that

$$p_X(n) \ne \frac{1}{11} \tag{1.1.1.4}$$

1.1.2 *Convolution:* From (1.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.1.2.1}$$

$$= \sum_k \Pr(X_1 = n-k | X_2 = k)\, p_{X_2}(k) \tag{1.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n-k|X_2 = k) = \Pr(X_1 = n-k) = p_{X_1}(n-k) \tag{1.1.2.3}$$

From (1.1.2.2) and (1.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (1.1.1.1) in (1.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.1.2.6}$$

From (1.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.1.2.7}$$

Substituting from (1.1.1.1) in (1.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.1.2.8}$$

satisfying (1.1.1.4).

1.1.3 *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.1.3.1}$$

From (1.1.1.1) and (1.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.1.3.4}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (1.1.3.3) and (1.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{1.1.3.6}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{1.1.3.8}$$

$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.9}$$

after some algebra, it can be shown that

$$\begin{aligned} &\frac{1}{36}\left[(n-1)\, u(n-1) - 2\,(n-7)\, u(n-7)\right. \\ &\qquad \left. + (n-13)\, u(n-13)\right] \\ &\overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \end{aligned} \tag{1.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.1.3.11}$$

From (1.1.3.1), (1.1.3.7) and (1.1.3.10)

$$\begin{aligned} p_X(n) = \frac{1}{36}&\left[(n-1)\, u(n-1)\right. \\ &\left. -2\,(n-7)\, u(n-7) + (n-13)\, u(n-13)\right] \end{aligned} \tag{1.1.3.12}$$

which is the same as (1.1.2.8). Note that (1.1.2.8) can be obtained from (1.1.3.10) using contour integration as well.

1.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1.1.4.1. The theoretical pmf obtained in (1.1.2.8) is plotted for comparison.

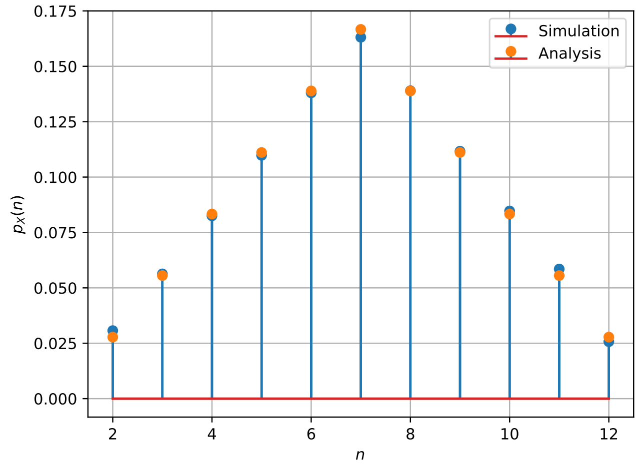

Fig. 1.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

1.1.5 The python code is available in

```
/codes/chapter1/dice.py
```

Chapter 2 Random Numbers

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

2.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/chapter2/uni_gen_stat.c
```

2.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{2.1.2.1}$$

Solution: The following code plots Fig. 2.1.2.1

```
codes/chapter2/cdf_plot_uni.py
```

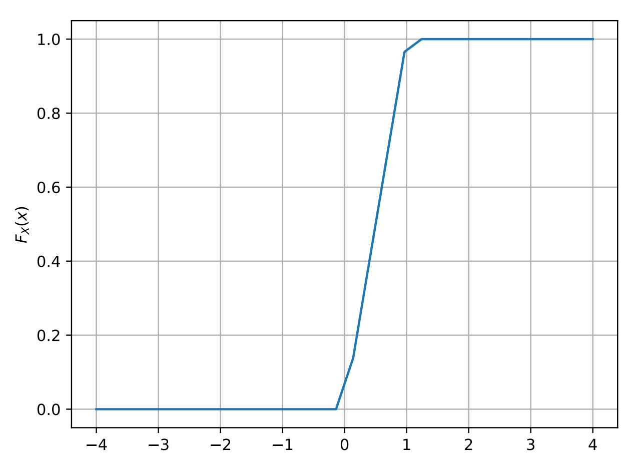

Fig. 2.1.2.1: The CDF of U

2.1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\,dx \tag{2.1.3.1}$$

For the uniform random variable U, $f_U(x)$ is given by

$$f_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases} \tag{2.1.3.2}$$

Substituting (2.1.3.2) in (2.1.3.1), $F_U(x)$ is found to be

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 0 \end{cases} \tag{2.1.3.3}$$

2.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2.1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{2.1.4.2}$$

Write a C program to find the mean and variance of U.
Solution: The following code prints the mean and variance of U

```
codes/chapter2/uni_gen_stat.c
```

The output of the program is

```
Uniform stats:
Mean: 0.500007
Variance: 0.083301
```

2.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.1.5.1}$$

Solution: For a random variable X, the mean μ_X and variance σ_X^2 are given by

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.1.5.2}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{2.1.5.3}$$

Substituting the CDF of U from (2.1.3.3) in (2.1.5.2) and (2.1.5.3), we get

$$\mu_U = \frac{1}{2} \tag{2.1.5.4}$$

$$\sigma_U^2 = \frac{1}{12} \tag{2.1.5.5}$$

which match with the values printed in problem 2.1.4

2.2 Central Limit Theorem

2.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/chapter2/gau_gen_stat.c
```

2.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2.2.1

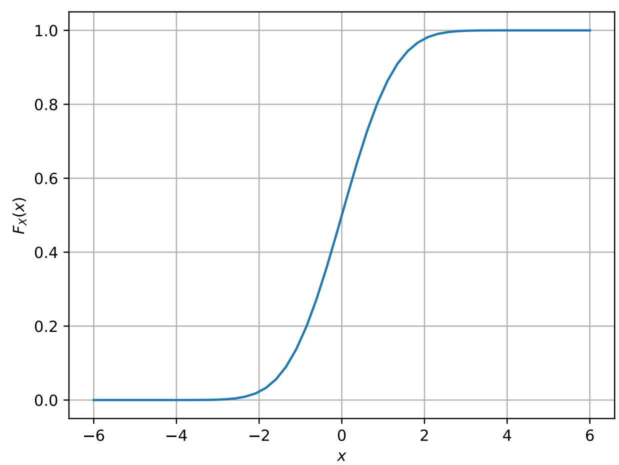

Fig. 2.2.2.1: The CDF of X

The properties of a CDF are

$$F_X(-\infty) = 0 \tag{2.2.2.1}$$

$$F_X(\infty) = 1 \tag{2.2.2.2}$$

$$\frac{dF_X(x)}{dx} \ge 0 \tag{2.2.2.3}$$

2.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2.3.1}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.2.3.1 using the code below

```
codes/chapter2/cdf_pdf_plot_gau.py
```

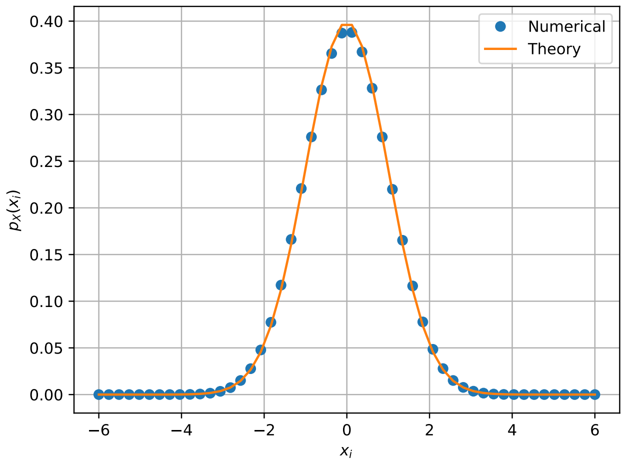

Fig. 2.2.3.1: The PDF of X

The properties of PDF are

$$f_X(x) \geq 0 \tag{2.2.3.2}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{2.2.3.3}$$

2.2.4 Find the mean and variance of X by writing a C program.
Solution: The following code prints the mean and variance of X

```
codes/chapter2/gau_gen_stat.c
```

The output of the program is

```
Gaussian stats:
Mean: 0.000294
Variance: 0.999562
```

2.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.2.5.1}$$

repeat the above exercise theoretically.
Solution: Substituting the PDF from (2.2.5.1) in (2.1.5.2),

$$\mu_X = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\, dx \tag{2.2.5.2}$$

Using

(2.2.5.3)

$$\int x \cdot \exp\left(-ax^2\right) \mathrm{d}x = -\frac{1}{2a} \cdot \exp\left(-ax^2\right) \tag{2.2.5.4}$$

$$\mu_X = \frac{1}{\sqrt{2\pi}} \left[-\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{2.2.5.5}$$

$$\mu_X = 0 \tag{2.2.5.6}$$

Substituting μ_X and the PDF in (2.1.5.3) to compute variance,

$$\sigma_X^2 = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\, dx \tag{2.2.5.7}$$

Substituting

$$t = \frac{x^2}{2}, \tag{2.2.5.8}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} \exp\left(-t\right)\, dt \tag{2.2.5.9}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}-1} \exp\left(-t\right)\, dt$$

Using the gamma function

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \cdot e^{-z}\, \mathrm{d}z \tag{2.2.5.10}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{2.2.5.11}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2}$$

$$= 1$$

2.3 From Uniform to Other

2.3.1 Generate samples of

$$V = -2\ln\left(1-U\right) \tag{2.3.1.1}$$

and plot its CDF.
Solution: The samples for U are loaded from uni.dat file generated in problem 2.1.4. The CDF of V is plotted in Fig. 2.3.1.1 using the code below,

```
codes/chapter2/function_1.py
```

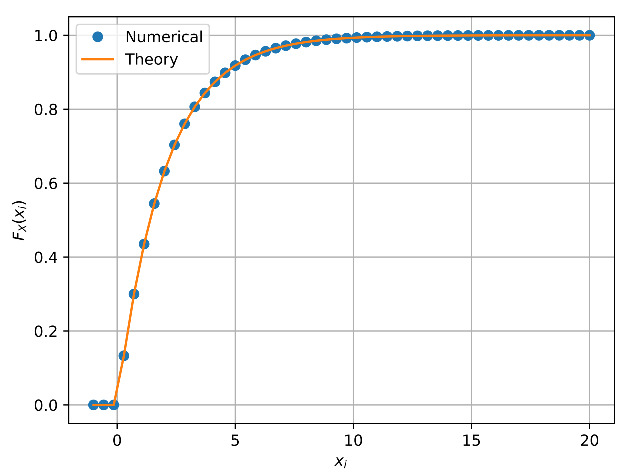

Fig. 2.3.1.1: The CDF of V

2.3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V < x) \tag{2.3.2.1}$$

$$= P(-2\ln\left(1-U\right) < x) \tag{2.3.2.2}$$

$$= P(U < 1 - e^{\frac{-x}{2}}) \tag{2.3.2.3}$$

$$= F_U(1 - e^{\frac{-x}{2}}) \tag{2.3.2.4}$$

Using $F_U(x)$ defined in (2.1.3.3),

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{2.3.2.5}$$

2.4 Triangular Distribution

2.4.1 Generate

$$T = U_1 + U_2 \tag{2.4.1.1}$$

Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/chapter2/two_uni_gen.c
```

2.4.2 Find the CDF of T.
Solution: Loading the samples from uni1.dat and uni2.dat in python, the CDF is plotted in Fig. 2.4.2.1

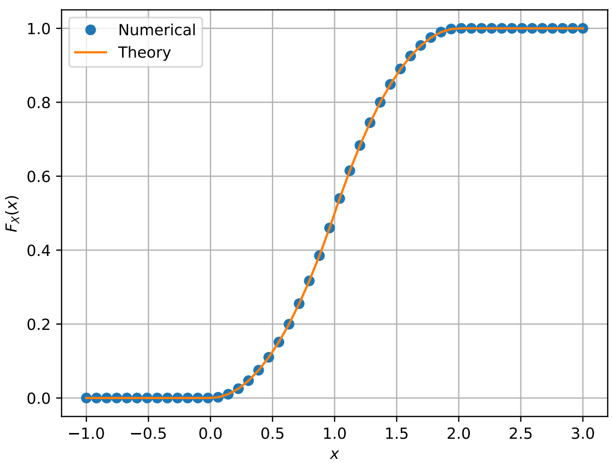

Fig. 2.4.2.1: The CDF of T

2.4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 2.4.3.1 using the code below

```
codes/chapter2/function_2.py
```

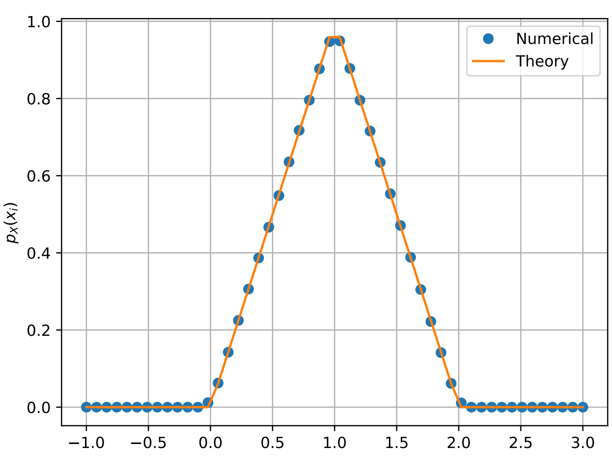

Fig. 2.4.3.1: The PDF of T

2.4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Since T is the sum of two independant random variables $U1$ and $U2$, the PDF of T is given by

$$p_T(x) = p_{U1}(x) * p_{U2}(x) \tag{2.4.4.1}$$

Using the PDF of U from (2.1.3.2), the convolution results in

$$p_T(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases} \tag{2.4.4.2}$$

The CDF of T is found using (2.1.3.1) by replacing U with T. Evaluating the integral for the piecewise function $p_T(x)$,

$$F_T(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{2.4.4.3}$$

2.4.5 Verify your results through a plot.
Solution: The theoretical and numerical plots for the CDF and PDF of T closely match in Fig. 2.4.2.1 and Fig. 2.4.3.1

Chapter 3 Maximum Likelihood Detection: BPSK

3.1 Maximum Likelihood

3.1.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: X can be generated in python using the below code section,

```
import numpy as np
num_samples = 500
x_var = np.random.binomial(1, 0.5,
    ↪ num_samples)
```

3.1.2 Generate

$$Y = AX + N, \tag{3.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Y can be generated in python using the below code section,

```
import numpy as np
num_samples = 500
x_var = np.random.binomial(1, 0.5,
    ↪ num_samples)*2-1
n_var = np.random.normal(0, 1,
    ↪ num_samples)
A_db = 5
A = 10**(0.1*A_db)
y_var = A*x_var + n_var
```

3.1.3 Plot Y using a scatter plot.
Solution: The scatter plot of Y is plotted in Fig. 3.1.3.1 using the below code,

```
codes/chapter3/bpsk_scatter.py
```

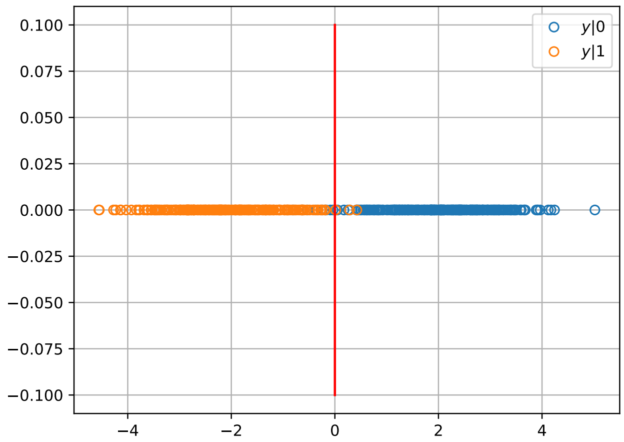

Fig. 3.1.3.1: Scatter plot of Y

3.1.4 Guess how to estimate X from Y.
Solution:

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{3.1.4.1}$$

3.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{3.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{3.1.5.2}$$

Solution: Based on the decision rule in (3.1.4.1),

$$\begin{aligned}\Pr\left(\hat{X} = -1 | X = 1\right) &= \Pr\left(Y < 0 | X = 1\right) \\ &= \Pr\left(AX + N < 0 | X = 1\right) \\ &= \Pr\left(A + N < 0\right) \\ &= \Pr\left(N < -A\right)\end{aligned}$$

Similarly,

$$\begin{aligned}\Pr\left(\hat{X} = 1 | X = -1\right) &= \Pr\left(Y > 0 | X = -1\right) \\ &= \Pr\left(N > A\right)\end{aligned}$$

Since $N \sim \mathcal{N}(0, 1)$,

$$\Pr\left(N < -A\right) = \Pr\left(N > A\right) \tag{3.1.5.3}$$

$$\implies P_{e|0} = P_{e|1} = \Pr\left(N > A\right) \tag{3.1.5.4}$$

3.1.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = \Pr\left(X = 1\right) P_{e|1} + \Pr\left(X = -1\right) P_{e|0} \tag{3.1.6.1}$$

Since X is equiprobable

(3.1.6.2)

$$P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0} \tag{3.1.6.3}$$

Substituting from (3.1.5.4)

$$P_e = \Pr\left(N > A\right) \tag{3.1.6.4}$$

Given a random varible $X \sim \mathcal{N}(0, 1)$ the Q-function is defined as

$$Q(x) = \Pr\left(X > x\right) \tag{3.1.6.5}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \tag{3.1.6.6}$$

(3.1.6.7)

Using the Q-function, P_e is rewritten as

$$P_e = Q(A) \tag{3.1.6.8}$$

3.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: The theoretical P_e is plotted in Fig. 3.1.7.1, along with numerical estimations from generated samples of Y. The below code is used for the plot,

```
codes/chapter3/bpsk_pe_vs_snr.py
```

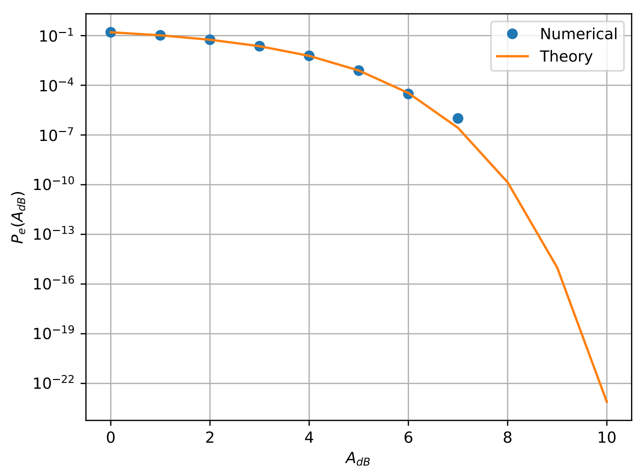

Fig. 3.1.7.1: P_e versus A plot

3.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: Given the decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{3.1.8.1}$$

$$\begin{aligned}P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) \\ &= \Pr\left(Y < \delta | X = 1\right) \\ &= \Pr\left(AX + N < \delta | X = 1\right) \\ &= \Pr\left(A + N < \delta\right) \\ &= \Pr\left(N < -A + \delta\right) \\ &= \Pr\left(N > A - \delta\right) \\ &= Q(A - \delta)\end{aligned}$$

$$\begin{aligned}P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) \\ &= \Pr\left(Y > \delta | X = -1\right) \\ &= \Pr\left(N > A + \delta\right) \\ &= Q(A + \delta)\end{aligned}$$

Using (3.1.6.3), P_e is given by

$$P_e = \frac{1}{2}Q(A+\delta) + \frac{1}{2}Q(A-\delta) \tag{3.1.8.2}$$

Using the integral for Q-function from (3.1.6.6),

$$P_e = k(\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + \int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \tag{3.1.8.3}$$

where k is a constant

Differentiating (3.1.8.3) wrt δ (using Leibniz's rule) and equating to 0, we get

$$\exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) = 0$$
$$\frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)} = 1$$
$$\exp\left(-\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) = 1$$
$$\exp(-2A\delta) = 1$$

Taking $\ln$ on both sides

$$-2A\delta = 0$$
$$\implies \delta = 0$$

P_e is maximum for $\delta = 0$

3.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{3.1.9.1}$$

Solution: Since X is not equiprobable, P_e is given by,

$$P_e = (1-p)P_{e|1} + pP_{e|0} \tag{3.1.9.2}$$
$$= (1-p)Q(A+\delta) + pQ(A-\delta) \tag{3.1.9.3}$$

Using the integral for Q-function from (3.1.6.6),

$$P_e = k((1-p)\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + p\int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \tag{3.1.9.4}$$

where k is a constant.
Following the same steps as in problem 3.1.8, δ for maximum P_e evaluates to,

$$\delta = \frac{1}{2A}\ln\left(\frac{1}{p} - 1\right) \tag{3.1.9.5}$$

3.1.10 Repeat the above exercise using the MAP criterion.
Solution: The MAP rule can be stated as

$$\text{Set } \hat{x} = x_i \text{ if} \tag{3.1.10.1}$$
$$p_X(x_k)p_Y(y|x_k) \text{ is maximum for } k = i$$

For the case of BPSK, the point of equality between $p_X(x = 1)p_Y(y|x = 1)$ and $p_X(x = -1)p_Y(y|x = -1)$ is the optimum threshold. If this threshold is δ, then

$$pp_Y(y|x=1) > (1-p)p_Y(y|x=-1) \text{ when } y > \delta$$
$$pp_Y(y|x=1) < (1-p)p_Y(y|x=-1) \text{ when } y < \delta$$

The above inequalities can be visualized in Fig. 3.1.10.1 for $p = 0.3$ and $A = 3$.

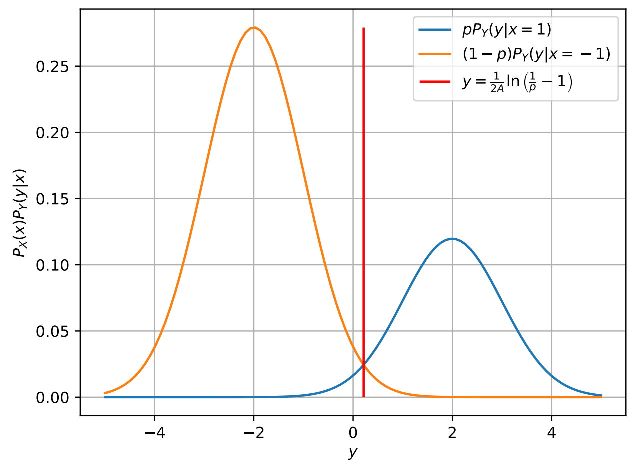

Fig. 3.1.10.1: $p_X(X = x_i)p_Y(y|x = x_i)$ versus y plot for $X \in \{-1, 1\}$

Given $Y = AX + N$ where $N \sim \mathcal{N}(0,1)$, the optimum threshold is found as solution to the below equation

$$p\exp\left(-\frac{(y_{eq}-A)^2}{2}\right) = (1-p)\exp\left(-\frac{(y_{eq}+A)^2}{2}\right) \tag{3.1.10.2}$$

Solving for y_{eq}, we get

$$y_{eq} = \delta = \frac{1}{2A}\ln\left(\frac{1}{p} - 1\right) \tag{3.1.10.3}$$

which is same as δ obtained in problem 3.1.9

Chapter 4 Transformation of Random Variables

4.1 Gaussian to Other

4.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{4.1.1.1}$$

Solution: The CDF and PDF of V are plotted in Fig. 4.1.1.1 and Fig. 4.1.1.2 respectively using the below code

```
codes/chapter3/sum_of_squares.py
```

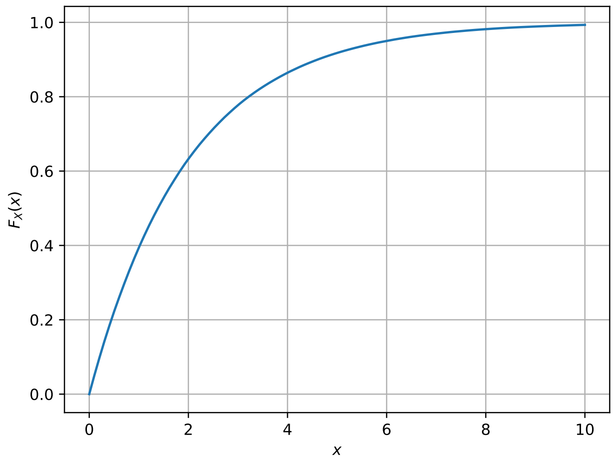

Fig. 4.1.1.1: CDF of V

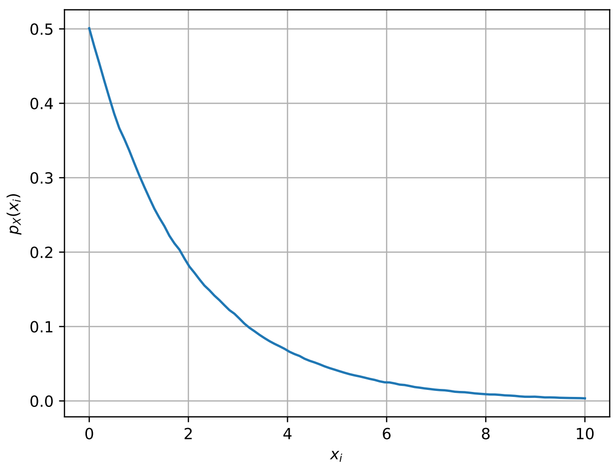

Fig. 4.1.1.2: PDF of V

4.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{4.1.2.1}$$

find α.

Solution: Let $Z = X^2$ where $X \sim \mathcal{N}(0,1)$. Defining the CDF for Z,

$$\begin{aligned} P_Z(z) &= \Pr(Z < z) \\ &= \Pr\left(X^2 < z\right) \\ &= \Pr\left(-\sqrt{z} < X < \sqrt{z}\right) \\ &= \int_{-\sqrt{z}}^{\sqrt{z}} p_X(x)\, dx \end{aligned}$$

Using (2.2.3.1), the PDF of Z is given by

$$\frac{d}{dz} P_Z(z) = p_Z(z)$$
$$= \frac{p_X(\sqrt{z}) + p_X(-\sqrt{z})}{2\sqrt{z}} \text{ (Using Lebniz's rule)} \tag{4.1.2.2}$$

Substituting the standard gaussian density function $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ in (4.1.2.2),

$$p_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{4.1.2.3}$$

The PDF of X_1^2 and X_2^2 are given by (4.1.2.3). Since V is the sum of two independant random variables,

$$\begin{aligned} p_V(v) &= p_{X_1^2}(x_1) * p_{X_2^2}(x_2) \\ &= \frac{1}{2\pi} \int_0^v \frac{e^{-\frac{x}{2}}}{\sqrt{x}} \frac{e^{-\frac{v-x}{2}}}{\sqrt{v-x}}\, dx \\ &= \frac{e^{-\frac{v}{2}}}{2\pi} \int_0^v \frac{1}{\sqrt{x(v-x)}}\, dx \\ &= \frac{e^{-\frac{v}{2}}}{2\pi} \left[-\arcsin\left(\frac{v-2x}{v}\right) \right]_0^v \\ &= \frac{e^{-\frac{v}{2}}}{2\pi} \pi \\ &= \frac{e^{-\frac{v}{2}}}{2} \text{ for } v \geq 0 \end{aligned}$$

$F_V(v)$ can be obtained from $p_V(v)$ using (2.1.3.1)

$$\begin{aligned} F_V(v) &= \frac{1}{2} \int_0^v \exp\left(-\frac{v}{2}\right) \\ &= 1 - \exp\left(-\frac{v}{2}\right) \text{ for } v \geq 0 \end{aligned} \tag{4.1.2.4}$$

Comparing (4.1.2.4) with (4.1.2.1), $\alpha = \frac{1}{2}$

4.1.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{4.1.3.1}$$

Solution: The CDF and PDF of A are plotted in Fig. 4.1.3.1 and Fig. 4.1.3.2 respectively using the below code

```
codes/chapter3/square_root.py
```

The CDF of A is given by,

$$\begin{aligned} F_A(a) &= \Pr(A < a) && (4.1.3.2) \\ &= \Pr\left(\sqrt{V} < a\right) && (4.1.3.3) \\ &= \Pr\left(V < a^2\right) && (4.1.3.4) \\ &= F_V\left(a^2\right) && (4.1.3.5) \\ &= 1 - \exp\left(-\frac{a^2}{2}\right) && (4.1.3.6) \end{aligned}$$

Using (2.2.3.1), the PDF is found to be

$$p_A(a) = a \exp\left(-\frac{a^2}{2}\right) \tag{4.1.3.7}$$

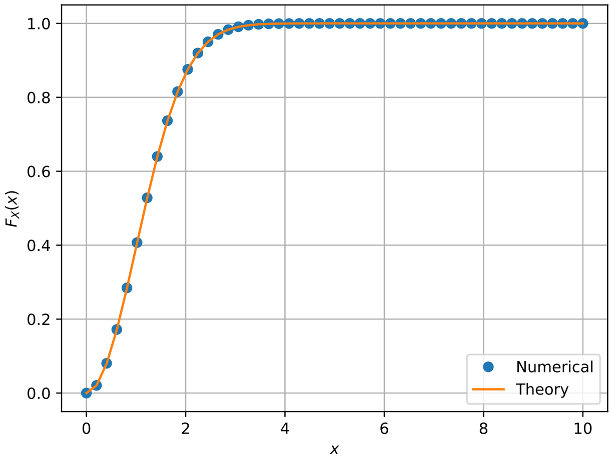

Fig. 4.1.3.1: CDF of A

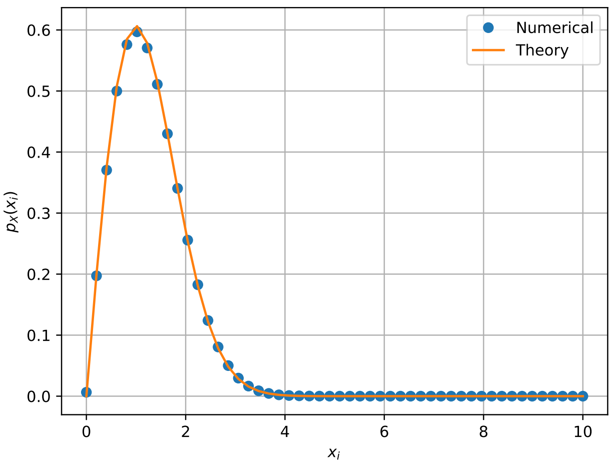

Fig. 4.1.3.2: PDF of A

4.2 Conditional Probability

4.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.2.1.1}$$

for

$$Y = AX + N, \tag{4.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

Solution: The blue dots in Fig. 4.2.4.1 is the required plot. The below code is used to generate the plot,

```
codes/chapter4/prob_error.py
```

4.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Solution: Assuming the decision rule in (3.1.4.1), when N is constant, P_e is given by

$$\begin{aligned}
P_e &= \Pr\left(\hat{X} = -1 | X = 1\right) \\
&= \Pr\left(Y < 0 | X = 1\right) \\
&= \Pr\left(AX + N < 0 | X = 1\right) \\
&= \Pr\left(A + N < 0\right) && (4.2.2.1) \\
&= \Pr\left(A < -N\right) \\
&= \begin{cases} F_A(-N) & N \ge 0 \\ 0 & N < 0 \end{cases} && (4.2.2.2)
\end{aligned}$$

For a Rayleigh random variable X with $E\left[X^2\right] = \gamma$, the PDF and CDF are given by

$$p_X(x) = \frac{2x}{\gamma}\exp\left(-\frac{x^2}{\gamma}\right) \quad \text{for } x \ge 0 \tag{4.2.2.3}$$

$$F_X(X) = 1 - \exp\left(-\frac{x^2}{\gamma}\right) \quad \text{for } x \ge 0 \tag{4.2.2.4}$$

Substituting (4.2.2.4) in (4.2.2.2),

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \ge 0 \\ 0 & N < 0 \end{cases} \tag{4.2.2.5}$$

4.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{4.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

Solution: Using $P_e(N)$ from (4.2.2.5),

$$\begin{aligned}
P_e &= \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \\
&= \int_{0}^{\infty} \left(1 - e^{-\frac{x^2}{\gamma}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx
\end{aligned}$$

$$\begin{aligned}
P_e = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}}\, dx \\
- \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-x^2\left(\frac{1}{\gamma} + \frac{1}{2}\right)\right) dx
\end{aligned}$$

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}}$$

4.2.4 Plot P_e in problems 4.2.1 and 4.2.3 on the same graph w.r.t γ. Comment.

Solution: P_e plotted in same graph in Fig. 4.2.4.1. The value of P_e is much higher when the channel gain A is Rayleigh distributed than the case where A is a constant (compare with Fig. 3.1.7.1).

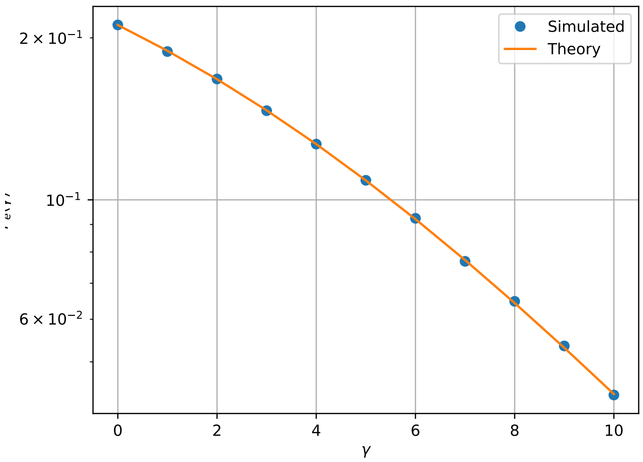

Fig. 4.2.4.1: P_e versus γ

From (4.2.2.1), P_e is given by

$$P_e = \Pr\left(A + N < 0\right) \tag{4.2.4.1}$$

One method of computing (4.2.2.1) is by finding the PDF of $Z = A + N$ (as the convolution of the individual PDFs of A and N) and then integrating $p_Z(z)$ from $-\infty$ to 0. The other method is by first computing P_e for constant N and then finding the expectation of $P_e(N)$. Both provide the same result but the computation of integrals is simpler when using the latter method.

Chapter 5 Bivariate Random Variables: FSK

5.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{5.1.0.1}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.0.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}\left(0, 1\right). \tag{5.1.0.3}$$

5.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{5.1.1.1}$$

on the same graph using a scatter plot.

Solution: The scatter plot in Fig. 5.1.1.1 is generated using the below code,

```
codes/chapter5/biv_scatter.py
```

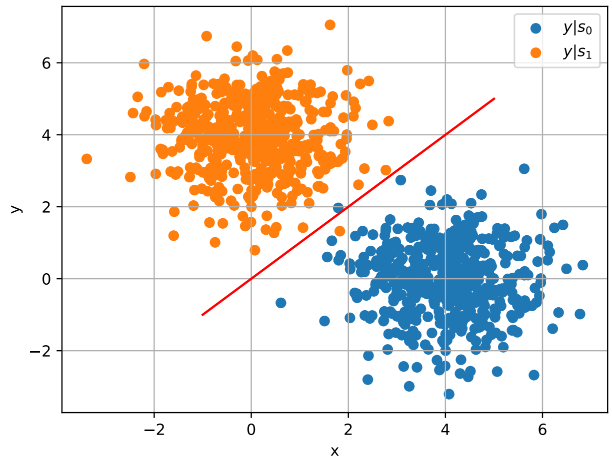

Fig. 5.1.1.1: Scatter plot of $\mathbf{y}|\mathbf{s}_0$ and $\mathbf{y}|\mathbf{s}_1$

5.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution: Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T$. Then the decision rule is

$$y_1 \underset{1}{\overset{0}{\gtrless}} y_2 \tag{5.1.2.1}$$

$\mathbf{y}|\mathbf{s}_i$ is a random vector with each of its components normally distributed. The PDF of $\mathbf{y}|\mathbf{s}_i$ is given by,

$$p_{\mathbf{y}|\mathbf{s}_i}\left(\mathbf{y}\right) = \frac{1}{2\pi\sqrt{\left|\mathbf{\Sigma}\right|}} \exp\left(-\frac{1}{2}\left(\mathbf{y} - \mathbf{s}_i\right)^\top \mathbf{\Sigma}^{-1}\left(\mathbf{y} - \mathbf{s}_i\right)\right) \tag{5.1.2.2}$$

Where $\mathbf{\Sigma}$ is the covariance matrix. Substituting $\mathbf{\Sigma} = \sigma\mathbf{I}$,

$$p_{\mathbf{y}|\mathbf{s}_i}\left(\mathbf{y}\right) = \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y} - \mathbf{s}_i\right)^\top \mathbf{I}\left(\mathbf{y} - \mathbf{s}_i\right)\right) \tag{5.1.2.3}$$

$$= \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y} - \mathbf{s}_i\right)^\top \left(\mathbf{y} - \mathbf{s}_i\right)\right) \tag{5.1.2.4}$$

Assuming equiprobable symbols, use MAP rule in (3.1.10.1) to find optimum decision. Since there are only two possible symbols $\mathbf{s}_0$ and $\mathbf{s}_1$, the optimal decision criterion is found by equating $p_{\mathbf{y}|\mathbf{s}_0}$ and $p_{\mathbf{y}|\mathbf{s}_1}$.

$$p_{\mathbf{y}|\mathbf{s}_0} = p_{\mathbf{y}|\mathbf{s}_1}$$

$$\implies \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y} - \mathbf{s}_0\right)^\top \left(\mathbf{y} - \mathbf{s}_0\right)\right) = \exp\left(-\frac{1}{2\sigma}\left(\mathbf{y} - \mathbf{s}_1\right)^\top \left(\mathbf{y} - \mathbf{s}_1\right)\right)$$

$$\implies \left(\mathbf{y} - \mathbf{s}_0\right)^\top \left(\mathbf{y} - \mathbf{s}_0\right) = \left(\mathbf{y} - \mathbf{s}_1\right)^\top \left(\mathbf{y} - \mathbf{s}_1\right)$$
$$\implies \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_0^\top\mathbf{y} + \mathbf{s}_0^T\mathbf{s}_0 = \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_1^\top\mathbf{y} + \mathbf{s}_1^T\mathbf{s}_1$$
$$\implies 2\left(\mathbf{s}_1 - \mathbf{s}_0\right)^\top\mathbf{y} = \left\|\mathbf{s}_1\right\|^2 - \left\|\mathbf{s}_0\right\|^2$$
$$\implies \left(\mathbf{s}_1 - \mathbf{s}_0\right)^\top\mathbf{y} = 0$$
$$\implies \begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{y} = 0$$

5.1.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{5.1.3.1}$$

with respect to the SNR from 0 to 10 dB.
Solution: The blue dots in Fig. 5.1.4.1 are the P_e versus SNR plot. It is generated using the below code,

```
codes/chapter5/biv_pe_snr.py
```

5.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution: Using the decision rule from (5.1.2.1),

$$\begin{aligned} P_e &= \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \\ &= \Pr\left(y_1 < y_2 | \mathbf{x} = \mathbf{s}_0\right) \\ &= \Pr\left(A + n_1 < n_2\right) \\ &= \Pr\left(n_1 - n_2 < -A\right) \end{aligned} \tag{5.1.4.1}$$

Let $Z = n_1 - n_2$ where $n_1, n_2 \sim \mathcal{N}\left(0, \sigma^2\right)$. The PDF of X is given by,

$$\begin{aligned} p_Z(z) &= p_{n_1}(n_1) * p_{-n_2}(n_2) \\ &= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-\frac{(t-z)^2}{2\sigma^2}} \, dt \\ &= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2 + t^2}{2\sigma^2}} \, dt \\ &= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(2t-z)^2 + z^2}{2(\sqrt{2}\sigma)^2}} \, dt \\ &= \frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}} \int_{-\infty}^{\infty} e^{-\frac{(2t-z)^2}{2(\sqrt{2}\sigma)^2}} \, dt \\ &= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2(\sqrt{2}\sigma)^2}} \, dk \\ &= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \end{aligned} \tag{5.1.4.2}$$

From (5.1.4.2), $Z \sim \mathcal{N}\left(0, 2\sigma^2\right)$. Substituting $\sigma = 1$, $Z \sim \mathcal{N}(0, 2)$. (5.1.4.1) can be further simplified as,

$$\begin{aligned} P_e &= \Pr\left(Z < -A\right) \\ &= \Pr\left(Z > A\right) \\ &= Q\left(\frac{A}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sqrt{2}}}^{\infty} \exp\left(-\frac{x^2}{2}\right) \, dx \end{aligned}$$

Fig. 5.1.4.1 compares the theoretical and simulation plots.

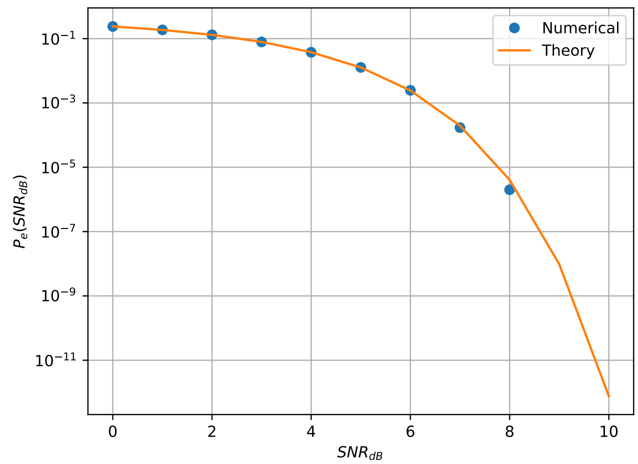

Fig. 5.1.4.1: P_e versus SNR plot for FSK

Chapter 6 Exercises

6.1 BPSK

6.1.1 The *signal constellation diagram* for BPSK is given by Fig. 6.1.1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

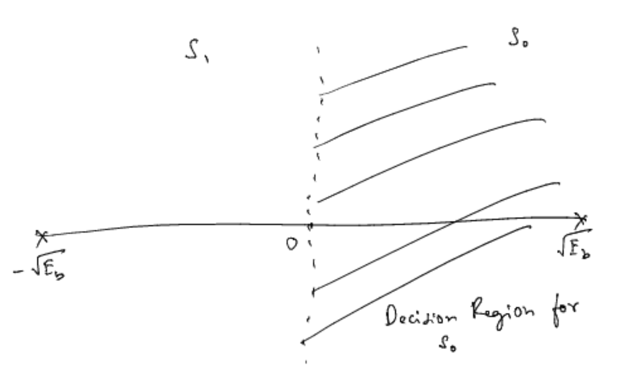

Fig. 6.1.1.1

Solution: The possible received symbols are

$$y | s_0 = \sqrt{E_b} + n \tag{6.1.1.1}$$
$$y | s_1 = -\sqrt{E_b} + n \tag{6.1.1.2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

6.1.2 From Fig. 6.1.1.1 obtain a decision rule for BPSK
Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{6.1.2.1}$$

6.1.3 Repeat the previous exercise using the MAP criterion.
Solution: When the symbols are equiprobable, the MAP rule stated in (3.1.10.1) simplifies to finding the symbol s_i that maximizes the conditional PDF $p_Y(y|s_i)$ i.e.

$$\begin{aligned} &\text{Set } \hat{x} = x_i \text{ if} \\ &p_Y(y|x_k) \text{ is maximum for } k = i \end{aligned} \tag{6.1.3.1}$$

In the case of BPSK, $y|s_0 \sim \mathcal{N}\left(\sqrt{E_b}, \frac{N_0}{2}\right)$ and $y|s_1 \sim \mathcal{N}\left(-\sqrt{E_b}, \frac{N_0}{2}\right)$. The two PDFs meet at $Y = 0$. So,

$$p_Y(y|s_0) > p_Y(y|s_1) \text{ when } y > 0$$
$$p_Y(y|s_0) < p_Y(y|s_1) \text{ when } y < 0$$

The optimum threshold is therefore $Y = 0$

6.1.4 Using the decision rule in Problem 6.1.2, obtain an expression for the probability of error for BPSK. **Solution:** Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0|s_0) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{6.1.4.1}$$
$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{6.1.4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (6.1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6.1.4.3}$$
$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{6.1.4.4}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

6.1.5 The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{6.1.5.1}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{6.1.5.2}$$

Show that

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{6.1.5.3}$$

Solution: From the definition of Q-function in (3.1.6.6),

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t}{2}}\, dt$$

Substitute $u = \frac{t}{\sqrt{2}}$

$$Q(x) = \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2}}}^\infty e^{-u^2}\, du$$
$$= \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

6.1.6 Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.
Solution: The following code

```
codes/chapter6/bpsk_ber.py
```

yields Fig. 6.1.6.1

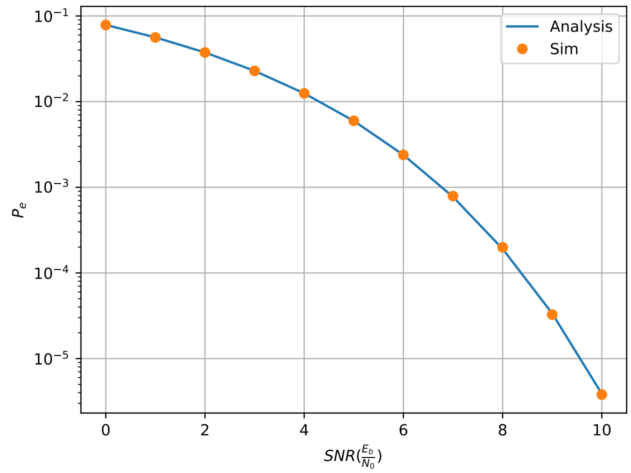

Fig. 6.1.6.1

6.1.7 Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{6.1.7.1}$$

Solution: Consider the bivariate gaussian distribution of $X, Y \sim \mathcal{N}(0,1)$,

$$p_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right) \tag{6.1.7.2}$$

Using $p_{X,Y}(x,y)$, the Q-function can be expressed as,

$$Q(z) = \int_z^\infty \int_{-\infty}^\infty p_{X,Y}(x,y)\, dx\, dy \tag{6.1.7.3}$$
$$= \frac{1}{2\pi} \int_z^\infty \int_{-\infty}^\infty \exp\left(-\frac{x^2+y^2}{2}\right) dx\, dy \tag{6.1.7.4}$$

Transforming the integral in (6.1.7.4) to polar coordinates (r, θ) for $z > 0$,

$$Q(z) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{z}{\sin\theta}}^\infty \exp\left(-\frac{r^2}{2}\right) r\, dr\, d\theta$$
$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-\frac{z^2}{2\sin^2\theta}\right) d\theta$$
$$= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(-\frac{z^2}{2\sin^2\theta}\right) d\theta \text{ , for } z > 0$$

6.2 Coherent BFSK

6.2.1 The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 6.2.1.1. Obtain the equations for the received symbols.

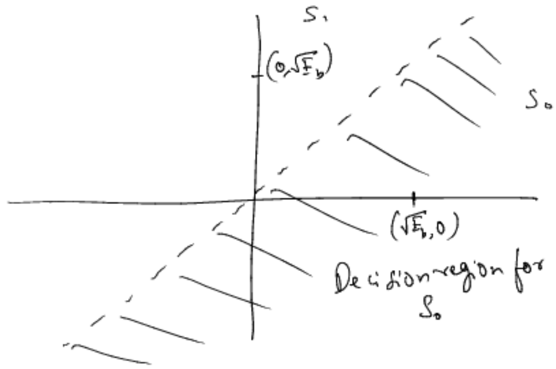

Fig. 6.2.1.1

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{6.2.1.1}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{6.2.1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

6.2.2 Obtain a decision rule for BFSK from Fig. 6.2.1.1.
Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{6.2.2.1}$$

6.2.3 Repeat the previous exercise using the MAP criterion.
Solution: Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}^T$ be the received signal. For an AWGN channel, given that symbol $\mathbf{s_k}$ was sent, every component y_i is normally distributed i.e.

$$y_i|\mathbf{s_k} \sim \mathcal{N}\left(s_{ki}, \frac{N_0}{2}\right) \quad ,i = 1, 2, ..., n \tag{6.2.3.1}$$

where s_{ki} is the ith component of symbol $\mathbf{s_k}$. Since all components of $\mathbf{y}$ are independent, the joint PDF of $\mathbf{y}$ given $\mathbf{s_k}$ was sent is,

$$p_{\mathbf{y}}(\mathbf{y}|\mathbf{s_k}) = \prod_{i=1}^{n} p_{y_i}(y_i|\mathbf{s_k}) \tag{6.2.3.2}$$

$$= \left(\frac{2}{N_0\sqrt{2\pi}}\right)^n \exp\left(-\frac{2}{N_0^2}\sum_{i=1}^{n}(y_i - s_{ki})^2\right) \tag{6.2.3.3}$$

$$= \left(\frac{2}{N_0\sqrt{2\pi}}\right)^n \exp\left(-\frac{2}{N_0^2}\left\|\mathbf{y} - \mathbf{s_k}\right\|^2\right) \tag{6.2.3.4}$$

Ignoring the constants in (6.2.3.4),

$$p_{\mathbf{y}}(\mathbf{y}|\mathbf{s_k}) \propto \exp\left(-\left\|\mathbf{y} - \mathbf{s_k}\right\|^2\right) \tag{6.2.3.5}$$

According to the MAP rule in (6.1.3.1), the optimal criterion to estimate $\mathbf{s}$ is to choose $\hat{\mathbf{s}} = \mathbf{s_k}$ whose distance from $\mathbf{y}$ is minimum.

$$\text{Set } \hat{s} = s_k \text{ if} \tag{6.2.3.6}$$

$$\left\|\mathbf{y} - \mathbf{s_i}\right\|^2 \text{ is minimum for } i = k$$

For the case of coherent BFSK, the locus for point equidistant from $\mathbf{s_0}$ and $\mathbf{s_1}$ is the line $y_1 - y_2 = 0$. So, the optimum decision is found as

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{6.2.3.7}$$

6.2.4 Derive and plot the probability of error. Verify through simulation.
Solution:

$$P_e = \Pr\left(y_1 < y_2|s_0\right) \tag{6.2.4.1}$$

$$= \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{6.2.4.2}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{6.2.4.3}$$

$$= \Pr\left(z > \sqrt{E_b}\right), \text{where } z \sim \mathcal{N}\left(0, N_0\right)$$

$$= Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{6.2.4.4}$$

The following code,

```
codes/chapter6/bfsk_coherent_ber.py
```

yields Fig. 6.2.4.1

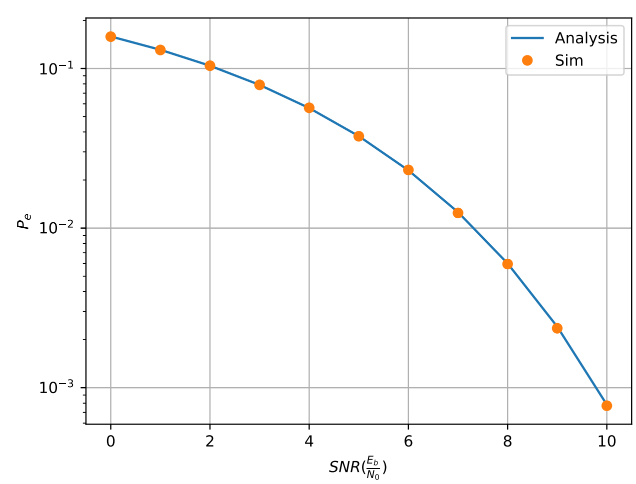

Fig. 6.2.4.1: P_e versus SNR for coherent BFSK

6.3 QPSK

6.3.1 Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{6.3.1.1}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{6.3.1.2}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{6.3.1.3}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{6.3.1.4}$$

(a) Show that the MAP decision for detecting s_0 results in

$$|r|_2 < r_1 \tag{6.3.1.5}$$

Solution: For choosing $\hat{\mathbf{s}} = \mathbf{s_0}$, the below inequalities have to be satisfied (from (6.2.3.6)).

$$\left\|\mathbf{r}-\mathbf{s_0}\right\|^2 < \left\|\mathbf{r}-\mathbf{s_1}\right\|^2 \tag{6.3.1.6}$$
$$\left\|\mathbf{r}-\mathbf{s_0}\right\|^2 < \left\|\mathbf{r}-\mathbf{s_2}\right\|^2 \tag{6.3.1.7}$$
$$\left\|\mathbf{r}-\mathbf{s_0}\right\|^2 < \left\|\mathbf{r}-\mathbf{s_3}\right\|^2 \tag{6.3.1.8}$$

Simplifying (6.3.1.6),

$$\begin{aligned}
(\mathbf{r}-\mathbf{s_0})^\top(\mathbf{r}-\mathbf{s_0}) &< (\mathbf{r}-\mathbf{s_1})^\top(\mathbf{r}-\mathbf{s_1})\\
\|r\|^2 - 2\mathbf{s_0}^\top\mathbf{r} + \|s_0\|^2 &< \|r\|^2 - 2\mathbf{s_1}^\top\mathbf{r} + \|s_1\|^2\\
(\mathbf{s_1}-\mathbf{s_0})^\top\mathbf{r} &< 0, \; [\|s_0\| = \|s_1\| = \sqrt{E_b}]\\
\begin{pmatrix}-\sqrt{E_b}\\ \sqrt{E_b}\end{pmatrix}^\top \mathbf{r} &< 0\\
\begin{pmatrix}-1\\ 1\end{pmatrix}^\top \mathbf{r} &< 0\\
r_2 &< r_1
\end{aligned}$$

Similarly, simplifying (6.3.1.7) and (6.3.1.8) we obtain

$$\begin{aligned}
r_2 &< r_1\\
-r_2 &< r_1\\
0 &< r_1\\
\implies |r|_2 &< r_1
\end{aligned}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[\left(X-\mu_x\right)\left(Y-\mu_y\right)\right]}{\sigma_x\sigma_y} \tag{6.3.1.9}$$

X and Y are said to be uncorrelated if $\rho = 0$
Solution:

$$\begin{aligned}
\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) &= \Pr\left(|r|_2 < r_1 | \mathbf{s} = \mathbf{s}_0\right)\\
&= \Pr\left(r_2 < r_1, -r_2 < r_1 | \mathbf{s} = \mathbf{s}_0\right)
\end{aligned} \tag{6.3.1.10}$$

(c) Show that if X and Y are uncorrelated Verify this numerically.
Solution: Since n_1 and n_2 are independent,

$$p_{n_1,n_2}\left(n_1,n_2\right) = p_{n_1}\left(n_1\right)p_{n_2}\left(n_2\right) \tag{6.3.1.11}$$
$$= \frac{1}{2\pi\sigma}e^{-\frac{n_1^2+n_2^2}{2\sigma^2}} \tag{6.3.1.12}$$

Finding $\mu_x = E\left[X\right]$,

$$\begin{aligned}
E\left[X\right] &= E\left[n_2 - n_1\right]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(n_2-n_1\right)p_{n_1,n_2}\left(n_1,n_2\right)\,dn_1\,dn_2\\
&= \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(n_2-n_1\right)e^{-\frac{n_1^2+n_2^2}{2\sigma^2}}\,dn_1\,dn_2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{\infty} e^{-\frac{n_2^2}{2\sigma^2}}\,dn_2 \right.\\
&\qquad \left. \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\left(n_2-n_1\right)e^{-\frac{n_1^2}{2\sigma^2}}\,dn_1\right]\\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} n_2 e^{-\frac{n_2^2}{2\sigma^2}}\,dn_2\\
&= 0
\end{aligned}$$

Similarly $\mu_y = 0$. Substituting X, Y, μ_x and μ_y in ρ,

$$\begin{aligned}
\rho &= \frac{E\left[\left(n_2-n_1\right)\left(-n_2-n_1\right)\right]}{\sigma_X\sigma_Y}\\
&= \frac{1}{\sigma_X\sigma_Y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(n_1^2-n_2^2\right)p_{n_1,n_2}\left(n_1,n_2\right)\,dn_1\,dn_2\\
&= \frac{1}{\sigma_X\sigma_Y}\frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(n_1^2-n_2^2\right)e^{-\frac{n_1^2+n_2^2}{2\sigma^2}}\,dn_1\,dn_2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma_X\sigma_Y}\frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{\infty} e^{-\frac{n_2^2}{2\sigma^2}}\,dn_2 \right.\\
&\qquad \left. \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\left(n_1^2-n_2^2\right)e^{-\frac{n_1^2}{2\sigma^2}}\,dn_1\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sigma_x\sigma_y}\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\left(\sigma^2-n_2^2\right)e^{-\frac{n_2^2}{2\sigma^2}}\,dn_2\\
&= \frac{1}{\sigma_X\sigma_Y}\left(\sigma^2-\sigma^2\right)\\
&= 0
\end{aligned}$$

The code below can be used to verify numerically that $\rho = 0$,

```
codes/chapter6/zero_corr_verify.py
```

The output of the code is,

```
Correlation coefficient is: 0.0002998
```

The scatter plot in Fig. 6.3.1.1 visually shows that there is no significant correlation between X and Y.

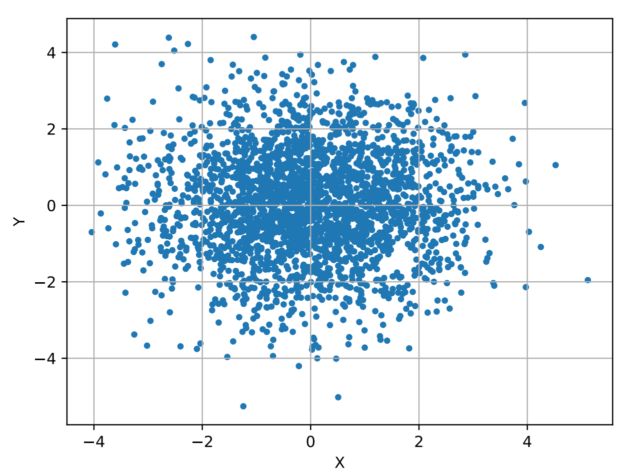

Fig. 6.3.1.1: Scatter plot of X and Y

(d) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.
Solution: Since X and Y are linear combinations of

Gaussian variables, X and Y are normally distributed. The joint PDF of two Gaussian variables is given by

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(Y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\} \tag{6.3.1.13}$$

Substituting $\rho = 0$,

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X^2} - \frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\} \tag{6.3.1.14}$$

$$= \frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{(x-\mu_x)^2}{2\sigma_X^2}}\frac{1}{\sigma_Y\sqrt{2\pi}}e^{-\frac{(y-\mu_y)^2}{2\sigma_Y^2}} \tag{6.3.1.15}$$

$$= p_X(x)\,p_Y(y) \tag{6.3.1.16}$$

(e) Show that $X, Y \sim \mathcal{N}(0, 2\sigma^2)$.
Solution: $X = n_2 - n_1$ where $n_2, -n_1 \sim \mathcal{N}(0, \sigma^2)$. The PDF of X is given by,

$$\begin{aligned}
p_X(x) &= p_{n_2}(n_2) * p_{-n_1}(n_1) \\
&= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}e^{-\frac{(x-t)^2}{2\sigma^2}}\,dt \\
&= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-\frac{(x-t)^2+t^2}{2\sigma^2}}\,dt \\
&= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-\frac{(2t-x)^2+x^2}{2(\sqrt{2}\sigma)^2}}\,dt \\
&= \frac{1}{2\pi\sigma^2}e^{-\frac{x^2}{2(\sqrt{2}\sigma)^2}}\int_{-\infty}^{\infty} e^{-\frac{(2t-x)^2}{2(\sqrt{2}\sigma)^2}}\,dt \\
&= \frac{e^{-\frac{x^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma}\frac{1}{\sqrt{2\pi}\sqrt{2}\sigma}\int_{-\infty}^{\infty} e^{-\frac{k^2}{2(\sqrt{2}\sigma)^2}}\,dk \\
&= \frac{e^{-\frac{x^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \qquad (6.3.1.17)
\end{aligned}$$

From (6.3.1.17), $X \sim \mathcal{N}(0, 2\sigma^2)$. Since n_2 and $-n_2$ are identically distributed (due to zero mean),

$$\begin{aligned}
p_Y(y) &= p_{-n_2}(n_2) * p_{-n_1}(n_1) \\
&= p_{n_2}(n_2) * p_{-n_1}(n_1) \\
&= p_X(x)
\end{aligned}$$

So, $X, Y \sim \mathcal{N}(0, 2\sigma^2)$.

(f) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.
Solution: From (6.3.1.10),

$$\begin{aligned}
\Pr(\hat{\mathbf{s}} = \mathbf{s}_0|\mathbf{s} = \mathbf{s}_0) &= \Pr(r_2 < r_1, -r_2 < r_1 | \mathbf{s} = \mathbf{s}_0) \\
&= \Pr\left(n_2 < n_1 + \sqrt{E_b}, -n_2 < n_1 + \sqrt{E_b}\right) \\
&= \Pr\left(n_2 - n_1 < \sqrt{E_b}, -n_2 - n_1 < \sqrt{E_b}\right) \\
&= \Pr\left(X < \sqrt{E_b}, Y < \sqrt{E_b}\right) \\
&= \Pr(X < A, Y < A)
\end{aligned}$$

(g) Find $\Pr(X < A, Y < A)$.
Solution: Since X and Y are independant,

$$\Pr(X < A, Y < A) = \Pr(X < A)\Pr(Y < A)$$

$$\begin{aligned}
&= F_X(A)F_Y(A) \\
&= \left(1 - Q\left(\frac{A}{\sqrt{2}\sigma}\right)\right)\left(1 - Q\left(\frac{A}{\sqrt{2}\sigma}\right)\right) \\
&= \left(1 - Q\left(\frac{A}{\sqrt{2}\sigma}\right)\right)^2 \\
&= \left(1 - Q\left(\sqrt{\frac{E_b}{N_0}}\right)\right)^2
\end{aligned}$$

since $A = \sqrt{E_b}, \sigma = \sqrt{\frac{N_0}{2}}$

$$\begin{aligned}
&= 1 - 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) + Q\left(\sqrt{\frac{E_b}{N_0}}\right)^2 \\
&\approx 1 - 2Q\left(\sqrt{\frac{E_b}{N_0}}\right) \\
&= 1 - \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)
\end{aligned}$$

(h) Verify the above through simulation.
Solution: The following code,

```
codes/chapter6/qfsk_ber.py
```

yields Fig. 6.3.1.2. $P_e = 1 - \Pr(X < A, Y < A)$ plotted for SNR values from 0 to 10.

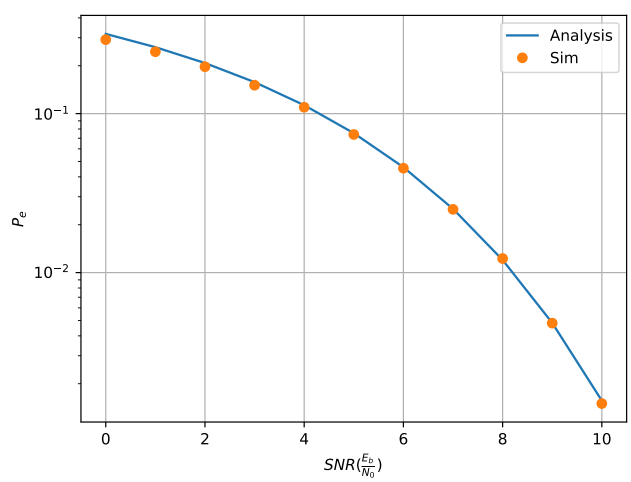

Fig. 6.3.1.2

6.4 M-PSK

6.4.1 Consider a system where $\mathbf{s}_i = \begin{pmatrix}\cos\left(\frac{2\pi i}{M}\right)\\ \sin\left(\frac{2\pi i}{M}\right)\end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix}\sqrt{E_s} + n_1 \\ n_2\end{pmatrix} \tag{6.4.1.1}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{6.4.1.2}$$

$$r_2 = R\sin\theta \tag{6.4.1.3}$$

show that the joint pdf of R, θ is

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{6.4.1.4}$$

Solution: $r_1 \sim \mathcal{N}\left(\sqrt{E_s}, \frac{N_0}{2}\right)$ and $r_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. Since r_1 and r_2 are independent,

$$p_{r_1,r_2}(r_1,r_2) = p_{r_1}(r_1)p_{r_2}(r_2)$$

$$= \frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(r_1-\sqrt{E_s})^2}{N_0}\right)\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{r_2^2}{N_0}\right)$$

$$= \frac{1}{\pi N_0}\exp\left(-\frac{(r_1-\sqrt{E_s})^2 + r_2^2}{N_0}\right)$$

Substituting for r_1 and r_2 in terms of R and θ

$$p(R,\theta) = \frac{R}{\pi N_0}\exp\left(-\frac{(R\cos\theta - \sqrt{E_s})^2 + R\sin^2\theta}{N_0}\right)$$

$$= \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right)$$

(b) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\,e^{-(V-\alpha)^2}\,dV = 0 \tag{6.4.1.5}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\,dV = \sqrt{\pi} \tag{6.4.1.6}$$

Solution: For (6.4.1.5), let $(V-\alpha)^2 = t$. Then

$$2(V-\alpha)dV = dt$$

Changing the integral in terms of t,

$$\lim_{\alpha\to\infty}\frac{1}{2}\int_{-\alpha^2}^\infty e^{-t}\,dt = \lim_{\alpha\to\infty}\frac{1}{2}\left[-e^{-t}\right]_{\alpha^2}^\infty$$

$$= \lim_{\alpha\to\infty}\frac{1}{2}e^{-\alpha^2}$$

$$= 0$$

For (6.4.1.6), let $(V-\alpha) = \frac{k}{\sqrt{2}}$. Then

$$dV = \frac{dk}{\sqrt{2}}$$

Changing the integral in terms of k,

$$\lim_{\alpha\to\infty}\frac{1}{\sqrt{2}}\int_{-\sqrt{2}\alpha}^\infty e^{-\frac{k^2}{2}}\,dk = \sqrt{\pi}\lim_{\alpha\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{2}\alpha}^\infty e^{-\frac{k^2}{2}}\,dk$$

$$= \sqrt{\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{k^2}{2}}\,dk$$

$$= \sqrt{\pi}$$

(c) Using the above, evaluate

$$\int_0^\infty V\exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}\,dV \tag{6.4.1.7}$$

for large values of γ.
Solution: By completing the square in the exponent, we get

$$\int_0^\infty Ve^{-\left((V-\sqrt{\gamma}\cos\theta)^2 + \gamma\sin^2\theta\right)}\,dV$$

$$= e^{-\gamma\sin^2\theta}\int_0^\infty Ve^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}\,dV$$

$$= e^{-\gamma\sin^2\theta}\left[\int_0^\infty \left(V-\sqrt{\gamma}\cos\theta\right)e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}\right.$$

$$\left. + \sqrt{\gamma}\cos\theta\int_0^\infty e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}\,dV\right]$$

Applying the limit $\gamma\to\infty$ and using the results from (6.4.1.5) and (6.4.1.6),

$$= \lim_{\gamma\to\infty} e^{-\gamma\sin^2\theta}\int_0^\infty \left(V-\sqrt{\gamma}\cos\theta\right)e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}$$

$$+ \lim_{\gamma\to\infty} e^{-\gamma\sin^2\theta}\sqrt{\gamma}\cos\theta\int_0^\infty e^{-\left(V-\sqrt{\gamma}\cos\theta\right)^2}\,dV$$

$$= 0 + \lim_{\gamma\to\infty} e^{-\gamma\sin^2\theta}\sqrt{\gamma}\cos\theta\sqrt{\pi} \tag{6.4.1.8}$$

The value of (6.4.1.8) is mainly determined by the product $e^{-\gamma\sin^2\theta}\sqrt{\gamma}$ which has a $0\cdot\infty$ indeterminate form in the limit. Keeping the product without applying the limits,

$$\int_0^\infty Ve^{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)}\,dV = e^{-\gamma\sin^2\theta}\sqrt{\gamma\pi}\cos\theta \tag{6.4.1.9}$$

For large γ.

(d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\,d\theta \tag{6.4.1.10}$$

Solution: Substituting $\sqrt{\gamma}\sin\theta = \frac{k}{\sqrt{2}}$,

$$\sqrt{\gamma}\cos\theta d\theta = \frac{dk}{\sqrt{2}}$$

$$\implies I = 1 - \frac{1}{\sqrt{2\pi}}\int_{-\sqrt{2\gamma}\sin\frac{\pi}{M}}^{\sqrt{2\gamma}\sin\frac{\pi}{M}} e^{-\frac{k^2}{2}}\,dk$$

$$= 1 - \left(1 - 2Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right)\right)$$

$$= 2Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right) \tag{6.4.1.11}$$

$$= \operatorname{erfc}\left(\sqrt{\gamma}\sin\frac{\pi}{M}\right) \tag{6.4.1.12}$$

(e) Find $P_{e|\mathbf{s}_0}$.
Solution: The optimal decision to detect $\mathbf{s}_0$ is,

$$\frac{|r|_2}{\alpha} < r_1 \tag{6.4.1.13}$$

$$\text{where } \alpha = \tan\frac{\pi}{M}$$

$P_{e|\mathbf{s}_0}$ can then be found as

$$P_{e|\mathbf{s}_0} = 1 - \Pr\left(\frac{|r|_2}{\alpha} < r_1\right) \tag{6.4.1.14}$$

Appying the transformation from problem (a) and using the joint pdf from (6.4.1.4),

$$P_{e|\mathbf{s}_0} = 1 - \Pr\left(R < \infty, -\frac{\pi}{M} < \theta < \frac{\pi}{M}\right)$$
$$= 1 - \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \int_0^\infty p\left(R, \theta\right)\, dR\, d\theta$$
$$= 1 - \frac{1}{\pi N_0} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \int_0^\infty R e^{-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}}\, dR\, d\theta$$

Substituting $V = \frac{R}{\sqrt{N_0}}$ and taking $\gamma = \frac{E_b}{N_0}$,

$$P_{e|\mathbf{s}_0} = 1 - \frac{1}{\pi} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} \int_0^\infty V e^{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)}\, dV\, d\theta$$

Using the result from (6.4.1.9),

$$P_{e|\mathbf{s}_0} = 1 - \frac{1}{\pi} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2\theta} \sqrt{\gamma\pi}\cos\theta\, d\theta$$
$$= 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\, d\theta$$

Using the result from (6.4.1.11),

$$P_{e|\mathbf{s}_0} = 2Q\left(\sqrt{2\gamma}\sin\frac{\pi}{M}\right) \tag{6.4.1.15}$$
$$= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\sin\frac{\pi}{M}\right) \tag{6.4.1.16}$$

6.5 Noncoherent BFSK

6.5.1 Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta \tag{6.5.1.1}$$
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\, d\theta \tag{6.5.1.2}$$
$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta + m_2\sin\theta}\, d\theta = I_0\left(\sqrt{m_1^2 + m_2^2}\right) \tag{6.5.1.3}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta\, d\theta \tag{6.5.1.4}$$

Solution: From (6.5.1.4)

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\, d\theta \tag{6.5.1.5}$$

Splitting the integral in (6.5.1.1),

$$\frac{1}{2\pi}\left[\int_0^\pi e^{x\cos\theta}\, d\theta + \int_\pi^{2\pi} e^{x\cos\theta}\, d\theta\right] \tag{6.5.1.6}$$
$$\frac{1}{2\pi}\left[\pi I_0(x) + \int_0^\pi e^{-x\cos\theta}\, d\theta\right] \tag{6.5.1.7}$$

Since $-\cos\theta$ is monotonic and lies between $[-1, 1]$ for $\theta \in [0, \pi]$, $e^{x\cos\theta}$ and $e^{-x\cos\theta}$ are symmetric for $\theta \in [0, \pi]$ about $\theta = \frac{\pi}{2}$. So,

$$\int_0^\pi e^{x\cos\theta}\, d\theta = \int_0^\pi e^{-x\cos\theta}\, d\theta \tag{6.5.1.8}$$

Substituting (6.5.1.8) in (6.5.1.7),

$$\frac{1}{2\pi}\left[\pi I_0(x) + \int_0^\pi e^{x\cos\theta}\, d\theta\right]$$
$$= \frac{1}{2\pi}\left[\pi I_0(x) + \pi I_0(x)\right]$$
$$= I_0(x)$$

Substituting $\theta - \phi = \omega$ in (6.5.1.2),

$$I_0(x) = \frac{1}{2\pi}\int_{-\phi}^{2\pi-\phi} e^{x\cos(\omega)}\, d\omega \tag{6.5.1.9}$$

Since $\cos\omega$ is periodic with period 2π,

$$\int_0^{2\pi} \cos\omega\, d\omega = \int_\alpha^{2\pi+\alpha} \cos\omega\, d\omega \tag{6.5.1.10}$$

Using (6.5.1.10) in (6.5.1.9),

$$\frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\omega)}\, d\omega$$
$$= I_0(x) \qquad \text{from (6.5.1.1)}$$

Using the trigonometric identity,

$$m_1\cos\theta + m_2\sin\theta = \sqrt{m_1^2 + m_2^2}\cos\left(\theta - \phi\right) \tag{6.5.1.11}$$
$$\text{where } \tan\phi = \frac{m_2}{m_1}$$

the integral in (6.5.1.3) is given by,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{m_1^2+m_2^2}\cos(\theta-\phi)}\, d\theta$$
$$= I_0\left(\sqrt{m_1^2 + m_2^2}\right) \qquad \text{from (6.5.1.2)}$$

6.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0\\ 0\\ \cos\phi_1\\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{6.5.2.1}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = \left(r_1, r_2, r_3, r_4\right)^T$,, find the pdf $p\left(\mathbf{r}|0, \phi_0\right)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

Solution: Since r_1, r_2, r_3, r_4 are independant,

$$p\left(\mathbf{r}|0, \phi_0\right) = \prod_{i=1}^4 p\left(r_i|0, \phi_0\right) \tag{6.5.2.2}$$

Substituting the PDFs for

$$\mathbf{r}|0, \phi_0 \sim \mathcal{N}\left(\sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix}, \frac{N_0}{2}\mathbf{I}\right) \tag{6.5.2.3}$$

in (6.5.2.2), we get

$$p\left(\mathbf{r}|0, \phi_0\right)$$
$$= \frac{1}{N_0^2\pi^2} e^{-\frac{\left(r_1 - \sqrt{E_b}\cos\phi_0\right)^2 + \left(r_2 - \sqrt{E_b}\sin\phi_0\right)^2 + r_3^2 + r_4^2}{N_0}} \tag{6.5.2.4}$$

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p(\mathbf{r}|0)$. Note that this expression will no longer contain ϕ_0.
Solution:

$$p(\mathbf{r}|0) = \int_0^{2\pi} p(\mathbf{r}|0,\phi_0)\, d\phi_0$$

$$= \frac{1}{N_0^2\pi^2}\int_0^{2\pi} e^{-\frac{(r_1-\sqrt{E_b}\cos\phi_0)^2+(r_2-\sqrt{E_b}\sin\phi_0)^2+r_3^2+r_4^2}{N_0}}\, d\phi_0$$
$$= \frac{1}{N_0^2\pi^2}e^{-\frac{r_1^2+r_2^2+r_3^2+r_4^2+E_b}{N_0}}\int_0^{2\pi} e^{\frac{2\sqrt{E_b}}{N_0}(r_1\cos\phi_0+r_2\sin\phi_0)}\, d\phi_0$$

$$= \frac{1}{N_0^2\pi^2}e^{-\frac{r_1^2+r_2^2+r_3^2+r_4^2+E_b}{N_0}} I_0\left(\frac{2\sqrt{E_b}}{N_0}\sqrt{r_1^2+r_2^2}\right) \tag{6.5.2.5}$$

from (6.5.1.3)

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2+r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2+r_4^2}\right) \tag{6.5.2.6}$$

where k is a constant.
Solution: The PDF of $\mathbf{r}|1$ has a similar form to (6.5.2.5),

$$p(\mathbf{r}|1) = \frac{1}{N_0^2\pi^2}e^{-\frac{r_1^2+r_2^2+r_3^2+r_4^2+E_b}{N_0}} I_0\left(\frac{2\sqrt{E_b}}{N_0}\sqrt{r_3^2+r_4^2}\right) \tag{6.5.2.7}$$

From (6.5.2.5) and (6.5.2.7),

$$p(\mathbf{r}|0) = f(\mathbf{r})I_0\left(k\sqrt{r_1^2+r_2^2}\right) \tag{6.5.2.8}$$
$$p(\mathbf{r}|1) = f(\mathbf{r})I_0\left(k\sqrt{r_3^2+r_4^2}\right) \tag{6.5.2.9}$$

Using the MLE rule from (6.1.3.1), the optimum decision is given by

$$I_0\left(k\sqrt{r_1^2+r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2+r_4^2}\right) \tag{6.5.2.10}$$

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.
Solution: Since $I_0(x)$ is monotonic for $x \geq 0$, and since the arguements to the Bessel function in (6.5.2.6) are positive, the decision simplifies to

$$\sqrt{r_1^2+r_2^2} \underset{1}{\overset{0}{\gtrless}} \sqrt{r_3^2+r_4^2}$$
$$\implies \left(r_1^2+r_2^2\right) \underset{1}{\overset{0}{\gtrless}} \left(r_3^2+r_4^2\right) \tag{6.5.2.11}$$

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2+r_2^2 < r_3^2+r_4^2|0\right) \tag{6.5.2.12}$$

Solution: From (6.5.2.11),

$$P_{e|0} = \Pr(\hat{\mathbf{s}} = \mathbf{s}_1|\mathbf{s} = \mathbf{s}_0) \tag{6.5.2.13}$$
$$= \Pr\left(r_1^2+r_2^2 < r_3^2+r_4^2|0\right) \tag{6.5.2.14}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0}e^{-\frac{y}{N_0}}, y > 0 \tag{6.5.2.15}$$

Solution: Using (4.1.2.2), the PDF of $Z = r_3^2$ is given by

$$p_Z(z) = \frac{1}{2\sqrt{z}}p_{r_1}\left(\sqrt{z}\right) + p_{r_1}\left(-\sqrt{z}\right)$$
$$= \frac{1}{2\sqrt{\pi N_0 z}}\left(e^{-\frac{r_1}{N_0}} + e^{-\frac{r_1}{N_0}}\right)$$
$$= \frac{1}{\sqrt{\pi N_0 z}}e^{-\frac{r_1}{N_0}} \tag{6.5.2.16}$$

Since r_3 and r_4 are identically distributed, r_4^2 has the same PDF as Z. Since Y is the sum of two independant random variables,

$$p_Y(y) = p_{r_3^2}(r_3) * p_{r_4^2}(r_4)$$
$$= \frac{1}{\pi N_0}\int_0^y \frac{e^{-\frac{x}{N_0}}}{\sqrt{x}}\frac{e^{-\frac{y-x}{N_0}}}{\sqrt{y-x}}\, dx$$
$$= \frac{e^{-\frac{y}{N_0}}}{\pi N_0}\int_0^y \frac{1}{\sqrt{x(y-x)}}\, dx$$
$$= \frac{e^{-\frac{y}{N_0}}}{\pi N_0}\left[-\arcsin\left(\frac{y-2x}{v}\right)\right]_0^v$$
$$= \frac{e^{-\frac{y}{N_0}}}{\pi N_0}\pi$$
$$= \frac{e^{-\frac{y}{N_0}}}{N_0} \text{ for } y \geq 0$$

(g) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y|0, r_1, r_2\right). \tag{6.5.2.17}$$

Solution: Using the PDF of Y from (6.5.2.15),

$$\Pr\left(r_1^2+r_2^2 < Y|0,r_1,r_2\right) = \int_{r_1^2+r_2^2}^{\infty} \frac{e^{-\frac{y}{N_0}}}{N_0}\, dy \tag{6.5.2.18}$$
$$= \left[-e^{-\frac{y}{N_0}}\right]_{r_1^2+r_2^2}^{\infty} \tag{6.5.2.19}$$
$$= e^{-\frac{r_1^2+r_2^2}{N_0}} \tag{6.5.2.20}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}}e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.
Solution:

$$E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2+x^2}{2\sigma^2}}\, dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(2x-\mu)^2+\mu^2}{2(\sqrt{2}\sigma)^2}}\, dx$$
$$= e^{-\frac{\mu^2}{4\sigma^2}}\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(2x-\mu)^2}{2(\sqrt{2}\sigma)^2}}\, dx$$
$$= \frac{e^{-\frac{\mu^2}{4\sigma^2}}}{\sqrt{2}}\frac{1}{\sqrt{2}\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{k^2}{2(\sqrt{2}\sigma)^2}}\, dk$$
$$E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}}e^{-\frac{\mu^2}{4\sigma^2}} \tag{6.5.2.21}$$

(i) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2}e^{-\frac{E_b}{2N_0}}. \tag{6.5.2.22}$$

Solution: Since r_1 and r_2 are independent,

$$E\left[g\left(r_1, r_2\right)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(r_1, r_2\right) p_{r_1}\left(r_1\right) p_{r_2}\left(r_2\right)\, dr_1\, dr_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{r_1^2+r_2^2}{N_0}} p_{r_1}\left(r_1\right) p_{r_2}\left(r_2\right)\, dr_1\, dr_2$$

$$= \int_{-\infty}^{\infty} e^{-\frac{r_1^2}{N_0}} p_{r_1}\left(r_1\right)\, dr_1 \int_{-\infty}^{\infty} e^{-\frac{r_2^2}{N_0}} p_{r_2}\left(r_2\right)\, dr_2 \tag{6.5.2.23}$$

Since $r_1, r_2 \sim \mathcal{N}\left(\sqrt{E_b}\begin{pmatrix}\cos\phi_0 & \sin\phi_0\end{pmatrix}^\top, \frac{N_0}{2}\mathbf{I}\right)$, (6.5.2.21) can be used to evaluate the integrals in (6.5.2.23),

$$E\left[g\left(r_1, r_2\right)\right] = \frac{1}{\sqrt{2}} e^{-\frac{E_b\cos^2\phi_0}{2N_0}} \frac{1}{\sqrt{2}} e^{-\frac{E_b\sin^2\phi_0}{2N_0}} \tag{6.5.2.24}$$

$$= \frac{1}{2} e^{\frac{E_b\left(\cos^2\phi_0+\sin^2\phi_0\right)}{2N_0}} \tag{6.5.2.25}$$

$$= \frac{1}{2} e^{-\frac{E_b}{2N_0}} \tag{6.5.2.26}$$

6.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R}\cos\Theta \tag{6.5.3.1}$$

$$V = \sqrt{R}\sin\Theta \tag{6.5.3.2}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det\begin{pmatrix}\frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta}\\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta}\end{pmatrix} \tag{6.5.3.3}$$

Solution:

$$J = \det\begin{pmatrix}\frac{\cos\Theta}{2\sqrt{R}} & -\sqrt{R}\sin\Theta\\ \frac{\sin\Theta}{2\sqrt{R}} & \sqrt{R}\cos\Theta\end{pmatrix} \tag{6.5.3.4}$$

$$= \frac{1}{2}\left(\cos^2\Theta + \sin^2\Theta\right) \tag{6.5.3.5}$$

$$= \frac{1}{2} \tag{6.5.3.6}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}\left(r, \theta\right) = p_{U,V}\left(u, v\right) J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{6.5.3.7}$$

Show that

$$p_R(r) = \begin{cases}\frac{1}{k}e^{-\frac{r}{k}} & r > 0,\\ 0 & r < 0,\end{cases} \tag{6.5.3.8}$$

assuming that Θ is uniformly distributed between 0 to 2π.
Solution: Since $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ are i.i.d,

$$p_{U,V}\left(u, v\right) = \frac{1}{k\pi} e^{-\frac{u^2+v^2}{k}} \tag{6.5.3.9}$$

Transforming from (U, V) to (R, Θ),

$$p_{R,\Theta}\left(r, \theta\right) = \frac{J}{k\pi} e^{-\frac{r\cos^2\Theta + r\sin^2\Theta}{k}} \tag{6.5.3.10}$$

$$= \frac{1}{2k\pi} e^{-\frac{r}{k}} \text{ for } r > 0 \tag{6.5.3.11}$$

Finding marginal PDF w.r.t R,

$$p_R\left(r\right) = \int_0^{2\pi} p_{R,\Theta}\left(r, \theta\right)\, d\Theta \tag{6.5.3.12}$$

$$= \frac{1}{k2\pi} e^{-\frac{r}{k}} \int_0^{2\pi} d\Theta \tag{6.5.3.13}$$

$$= \frac{1}{k} e^{-\frac{r}{k}} \text{ for } r > 0 \tag{6.5.3.14}$$

$$p_R\left(r\right) = \begin{cases}\frac{1}{k}e^{-\frac{r}{k}} & r > 0\\ 0 & r < 0\end{cases} \tag{6.5.3.15}$$

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k} e^{-\frac{|y|}{k}} \tag{6.5.3.16}$$

Solution: Given the PDF of Y is given by

$$p_Y\left(y\right) = p_{R_1}\left(r_1\right) * p_{R_2}\left(-r_2\right) \tag{6.5.3.17}$$

$$= \begin{cases}\int_0^{\infty} p_{R_1}\left(t\right) p_{R_2}\left(t - y\right)\, dt & y < 0\\ \int_y^{\infty} p_{R_1}\left(t\right) p_{R_2}\left(t - y\right)\, dt & y > 0\end{cases} \tag{6.5.3.18}$$

Evaluating the integral for $y < 0$,

$$\int_0^{\infty} p_{R_1}\left(t\right) p_{R_2}\left(t - y\right)\, dt = \frac{1}{k^2}\int_0^{\infty} e^{-\frac{t}{k}} e^{-\frac{t-y}{k}}\, dt \tag{6.5.3.19}$$

$$= \frac{e^{\frac{y}{k}}}{k^2}\int_0^{\infty} e^{-\frac{2t}{k}}\, dt \tag{6.5.3.20}$$

$$= \frac{e^{\frac{y}{k}}}{k^2}\left[-\frac{k}{2}e^{-\frac{2t}{k}}\right]_0^{\infty} \tag{6.5.3.21}$$

$$= \frac{1}{2k} e^{\frac{y}{k}} \text{ , for } y < 0 \tag{6.5.3.22}$$

Substituting limits of the integral for $y > 0$ in (6.5.3.21),

$$= \frac{e^{\frac{y}{k}}}{k^2}\left[-\frac{k}{2}e^{-\frac{2t}{k}}\right]_y^{\infty} \tag{6.5.3.23}$$

$$= \frac{e^{\frac{y}{k}}}{k^2}\frac{k}{2}e^{-\frac{2y}{k}} \tag{6.5.3.24}$$

$$= \frac{1}{2k} e^{\frac{-y}{k}} \text{ , for } y > 0 \tag{6.5.3.25}$$

Combining (6.5.3.22) and (6.5.3.25),

$$p_Y(y) = \frac{1}{2k} e^{-\frac{|y|}{k}} \tag{6.5.3.26}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{6.5.3.27}$$

where ϕ is a constant.
Solution: Let $A = \left(\sqrt{p}\cos\phi\right)U$ and $B = \left(\sqrt{p}\sin\phi\right)V$. Then,

$$A \sim \mathcal{N}\left(0, \frac{kp\cos^2\phi}{2}\right) \tag{6.5.3.28}$$

$$B \sim \mathcal{N}\left(0, \frac{kp\sin^2\phi}{2}\right) \tag{6.5.3.29}$$

If $C = A + B$, then

$$C \sim \mathcal{N}\left(0, \frac{kp}{2}\right) \tag{6.5.3.30}$$

Since $Z = p + C$, $Z \sim \mathcal{N}\left(p, \frac{kp}{2}\right)$. The PDF of Z is given by,

$$p_Z(z) = \frac{1}{\sqrt{kp\pi}} e^{-\frac{(z-p)^2}{kp}} \tag{6.5.3.31}$$

(e) Find $\Pr(Y > Z)$.
Let $g(z) = \Pr(Y > z | Z = z)$. Evaluating $g(z)$,

$$\Pr(Y > z | Z = z) = \int_z^\infty p_Y(y)\, dy \tag{6.5.3.32}$$

$$= \int_z^\infty \frac{1}{2k} e^{-\frac{|y|}{k}}\, dy \tag{6.5.3.33}$$

$$= \begin{cases} \int_z^\infty \frac{1}{2k} e^{-\frac{y}{k}}\, dy & z > 0 \\ \int_z^0 \frac{1}{2k} e^{\frac{y}{k}}\, dy + \int_0^\infty \frac{1}{2k} e^{-\frac{y}{k}}\, dy & z < 0 \end{cases} \tag{6.5.3.34}$$

$$g(z) = \begin{cases} \frac{1}{2} e^{-\frac{z}{k}} & z > 0 \\ 1 - \frac{1}{2} e^{\frac{z}{k}} & z < 0 \end{cases} \tag{6.5.3.35}$$

$\Pr(Y > Z)$ is calculated as $E[g(z)]$,

$$E[g(z)] = \int_{-\infty}^\infty g(z)\, p_Z(z)\, dz \tag{6.5.3.36}$$

$$= \int_{-\infty}^0 \left(1 - \frac{1}{2} e^{\frac{z}{k}}\right) p_Z(z)\, dz + \int_0^\infty \frac{1}{2} e^{-\frac{z}{k}} p_Z(z)\, dz \tag{6.5.3.37}$$

$$= I_1 + I_2 \tag{6.5.3.38}$$

Substituting $p_Z(z)$ in I_1,

$$I_1 = \int_{-\infty}^0 \left(1 - \frac{1}{2} e^{\frac{z}{k}}\right) \frac{1}{\sqrt{kp\pi}} e^{-\frac{(z-p)^2}{kp}}\, dz \tag{6.5.3.39}$$

$$= \frac{1}{\sqrt{kp\pi}} \left(\int_{-\infty}^0 e^{-\frac{(z-p)^2}{kp}}\, dz - \int_{-\infty}^0 \frac{1}{2} e^{\frac{z}{k}} e^{-\frac{(z-p)^2}{kp}}\right) dz \tag{6.5.3.40}$$

Substituting $t = \frac{z}{\sqrt{kp}}$ and $\gamma = \frac{p}{k}$,

$$I_1 = \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^0 e^{-\left(t - \sqrt{\gamma}\right)^2}\, dt \right.$$
$$\left. - \frac{1}{2} \int_{-\infty}^0 e^{t\sqrt{\gamma}} e^{-\left(t - \sqrt{\gamma}\right)^2}\, dt\right) \tag{6.5.3.41}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-\left(t - \sqrt{\gamma}\right)^2}\, dt - \frac{e^{\frac{5\gamma}{4}}}{2\sqrt{\pi}} \int_{-\infty}^0 e^{-\left(t - \frac{\sqrt{3\gamma}}{2}\right)^2}\, dt \tag{6.5.3.42}$$

Since area under a Gaussian curve is unity, from (6.4.1.6) it can concluded that,

$$\lim_{\alpha \to \infty} \int_{-\infty}^0 e^{-(V-\alpha)^2}\, dV = 0 \tag{6.5.3.43}$$

Let $\sqrt{\gamma} = \alpha$. Computing I_1 for large values of α,

$$\lim_{\alpha \to \infty} I_1 = \lim_{\alpha \to \infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-(t-\alpha)^2}\, dt$$
$$- \lim_{\alpha \to \infty} \frac{e^{\frac{5\alpha^2}{4}}}{2\sqrt{\pi}} \int_{-\infty}^0 e^{-\left(t - \frac{3\alpha}{2}\right)^2}\, dt \tag{6.5.3.44}$$

Using (6.5.3.43),

$$\lim_{\alpha \to \infty} I_1 = 0 - \lim_{\alpha \to \infty} \frac{e^{\frac{5\alpha^2}{4}}}{2\sqrt{\pi}} \int_{-\infty}^0 e^{-\left(t - \frac{3\alpha}{2}\right)^2}\, dt \tag{6.5.3.45}$$

$$= -\frac{1}{2\sqrt{\pi}} \lim_{\alpha \to \infty} \frac{\int_{-\infty}^0 e^{-\left(t - \frac{3\alpha}{2}\right)^2}\, dt}{e^{-\frac{5\alpha^2}{4}}} \tag{6.5.3.46}$$

Using L'Hopital's rule,

$$\lim_{\alpha \to \infty} I_1 = \frac{1}{2\sqrt{\pi}} \lim_{\alpha \to \infty} \frac{6}{5} \frac{\int_{-\infty}^0 \left(t - \frac{3\alpha}{2}\right) e^{-\left(t - \frac{3\alpha}{2}\right)^2}\, dt}{\alpha e^{-\frac{5\alpha^2}{4}}} \tag{6.5.3.47}$$

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{6.5.3.48}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{6.5.3.49}$$

Solution: The CDF of R is given by.

$$F_R(r) = \Pr(R < r)\ \text{, for } r > 0 \tag{6.5.3.50}$$

$$= \Pr\left(\sqrt{U^2 + V^2} < r\right) \tag{6.5.3.51}$$

$$= \Pr\left(U^2 + V^2 < r^2\right) \tag{6.5.3.52}$$

The area of integration to obtain $F_R(r)$ is the circle with radius r.

$$F_R(r) = \int_{-r}^r \int_{-\sqrt{r^2-v^2}}^{\sqrt{r^2-v^2}} p_{U,V}(u, v)\, du\, dv \tag{6.5.3.53}$$

$$= \int_{-r}^r \int_{-\sqrt{r^2-v^2}}^{\sqrt{r^2-v^2}} p_U(u)\, p_V(v)\, du\, dv \tag{6.5.3.54}$$

(since U and V are independent)

$$= \int_{-r}^r \int_{-\sqrt{r^2-v^2}}^{\sqrt{r^2-v^2}} \frac{1}{k\pi} e^{-\frac{(u-m_1)^2 + (v-m_2)^2}{k}}\, du\, dv \tag{6.5.3.55}$$

Transforming the integral to polar coordinates with $u = R\cos\theta$ and $v = R\sin\theta$,

$$F_R(r) = \frac{1}{k\pi} \int_0^r \int_0^{2\pi} R e^{-\frac{(R\cos\theta - m_1)^+ (R\sin\theta - m_2)^2}{k}}\, dR\, d\theta \tag{6.5.3.56}$$

To find $p_R(r)$, differentiate $F_R(r)$ w.r.t r using Leibniz's rule.

$$p_R(r) = \frac{1}{k\pi} \int_0^{2\pi} r e^{-\frac{(r\cos\theta - m_1)^2 + (r\sin\theta - m_2)^2}{k}}\, d\theta \tag{6.5.3.57}$$

$$= \frac{1}{k\pi} r e^{-\frac{r^2 + m_1^2 + m_2^2}{k}} \int_0^{2\pi} e^{\frac{2rm_1}{k}\cos\theta + \frac{2rm_2}{k}\sin\theta}\, d\theta \tag{6.5.3.58}$$

Using (6.5.1.3),

$$p_R(r) = \frac{2}{k} r e^{-\frac{r^2+m_1^2+m_2^2}{k}} I_0\left(\frac{2r}{k}\sqrt{m_1^2+m_2^2}\right) \tag{6.5.3.59}$$

$$= \frac{2}{k} r e^{-\frac{r^2+m^2}{k}} I_0\left(\frac{2rm}{k}\right), \ m = \sqrt{m_1^2+m_2^2} \tag{6.5.3.60}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \tag{6.5.3.61}$$

Solution: The modified bessel function of first kind is given by

$$I_n(x) = \left(\frac{1}{2}x\right)^n \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{k!\Gamma(n+k+1)} \tag{6.5.3.62}$$

Substituting $n = 0$,

$$I_0(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{k!\Gamma(k+1)} \tag{6.5.3.63}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{4^k (k!)^2} \tag{6.5.3.64}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{6.5.3.65}$$

find $\Pr(R < Z)$.

6.6 Craig's Formula and MGF

6.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{6.6.1.1}$$

where X is a random variable and $E[\cdot]$ is the expectation.

(a) Let $Y \sim \mathcal{N}(0,1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \tag{6.6.1.2}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{6.6.1.3}$$

Solution: Refer solution to problem 6.1.7.

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.
Solution: $|h|^2 = \Re\{h\}^2 + \Im\{h\}^2$ where $\Re\{h\}, \Im\{h\} \sim \mathcal{N}\left(0, \frac{\Omega}{4}\right)$. Let $R = |h|^2$. Using (6.5.3.49), $p_R(r)$ is given by

$$p_R(r) = \frac{2}{\Omega} e^{-\frac{2r}{\Omega}} \tag{6.6.1.4}$$

(c) Let

$$P_e = \Pr\left(\Re\{h^* y\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{6.6.1.5}$$

Show that

$$P_e = \int_0^{\infty} Q\left(\sqrt{2x}\right) p_A(x) \, dx \tag{6.6.1.6}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{6.6.1.7}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.