

Conic section Assignment

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Problem Statement - Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum

Solution

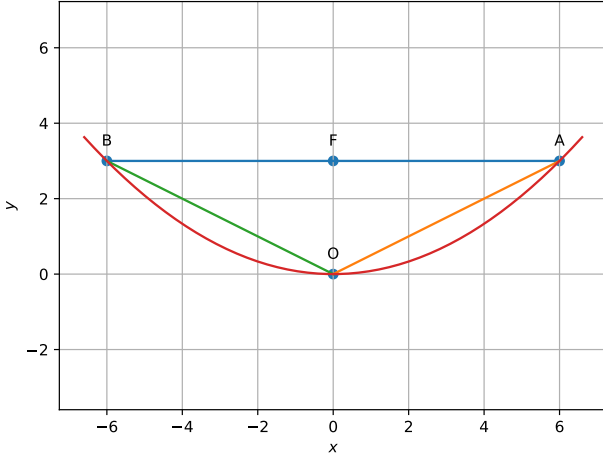


Figure 1: Triangle formed by vertex and ends of latus rectum of parabola $x^2 = 12y$

The given equation of parabola $x^2 = 12y$ can be written in the general quadratic form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}, \quad (3)$$

$$f = 0 \quad (4)$$

The parabola in (1) can be expressed in standard form (center/vertex at origin, major-axis - x axis) as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \mathbf{e}_1^T \mathbf{y} \quad |V| = 0 \quad (5)$$

where

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (\text{Affine Transformation}) \quad (6)$$

$$\mathbf{P}^T \mathbf{V} \mathbf{P} = \mathbf{D}. \quad (\text{Eigenvalue Decomposition}) \quad (7)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (8)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad \mathbf{P}^T = \mathbf{P}^{-1}, \quad (9)$$

$$\eta = \mathbf{u}^T \mathbf{p}_1 \quad (10)$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (11)$$

To find \mathbf{c} which is the center of the parabola in (1), substitute (6) in (1)

$$(\mathbf{P} \mathbf{y} + \mathbf{c})^T \mathbf{V} (\mathbf{P} \mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T (\mathbf{P} \mathbf{y} + \mathbf{c}) + f = 0, \quad (12)$$

yielding

$$\mathbf{y}^T \mathbf{P}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V} \mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \quad (13)$$

From (13) and (7),

$$\mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{V} \mathbf{c} + \mathbf{u})^T \mathbf{P} \mathbf{y} + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (14)$$

For a parabola $|V| = 0, \lambda_1 = 0$ and

$$\mathbf{V} \mathbf{p}_1 = 0, \mathbf{V} \mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad (15)$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of \mathbf{V} such that (7)

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad (16)$$

Substituting (16) in (14),

$$\begin{aligned} & \mathbf{y}^T \mathbf{D} \mathbf{y} + 2(\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) (\mathbf{p}_1 \quad \mathbf{p}_2) \mathbf{y} \\ & + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \implies \mathbf{y}^T \mathbf{D} \mathbf{y} \\ & + 2((\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_1 (\mathbf{c}^T \mathbf{V} + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} \\ & + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \\ & \implies \mathbf{y}^T \mathbf{D} \mathbf{y} \\ & + 2(\mathbf{u}^T \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^T + \mathbf{u}^T) \mathbf{p}_2) \mathbf{y} \\ & + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \text{ from (15)} \end{aligned}$$

$$\begin{aligned} & \implies \lambda_2 y_2^2 + 2(\mathbf{u}^T \mathbf{p}_1) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^T \mathbf{p}_2 \\ & + \mathbf{c}^T (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \end{aligned}$$

which is the equation of a parabola. Thus, (17) can be expressed as (5) by choosing

$$\eta = \mathbf{u}^T \mathbf{p}_1 \quad (17)$$

and \mathbf{c} in (14) such that

$$\mathbf{P}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (18)$$

$$\mathbf{c}^T (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T \mathbf{c} + f = 0 \quad (19)$$

Multiplying (18) by \mathbf{P} yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \mathbf{p}_1, \quad (20)$$

which, upon substituting in (19) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \quad (21)$$

(20) and (21) can be clubbed together to obtain (22).

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad |V| = 0 \quad (22)$$

Substituting appropriate values from (2), (3), (4), (9), and (10) into (22), the below matrix equation is obtained

$$\begin{pmatrix} 0 & -12 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (23)$$

The augmented matrix for (23) can be expressed as

$$\begin{aligned} & \begin{pmatrix} 0 & -12 & | & 0 \\ 1 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \\ & \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & | & 0 \\ 0 & -12 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \\ & \xleftrightarrow{-\frac{R_2}{12} \leftarrow R_2} \begin{pmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \\ & \Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \quad (25)$$

From (2)

$$\begin{aligned} \mathbf{V}^T \mathbf{V} &= \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{nn}^T \right)^T \\ &= \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{nn}^T \right) \\ \Rightarrow \mathbf{V}^2 &= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{nn}^T \mathbf{nn}^T \\ &\quad - 2e^2 \|\mathbf{n}\|^2 \mathbf{nn}^T \\ &= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{nn}^T - 2e^2 \|\mathbf{n}\|^2 \mathbf{nn}^T \\ &= \|\mathbf{n}\|^4 \mathbf{I} + e^2 (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{nn}^T \\ &= \|\mathbf{n}\|^4 \mathbf{I} + (e^2 - 2) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V} \right) \end{aligned} \quad (29)$$

which can be expressed as

$$\mathbf{V}^2 + (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{V} - (e^2 - 1) \|\mathbf{n}\|^4 \mathbf{I} = 0 \quad (30)$$

Using the Cayley-Hamilton theorem, (30) results in the characteristic equation,

$$\lambda^2 - (2 - e^2) \|\mathbf{n}\|^2 \lambda + (1 - e^2) \|\mathbf{n}\|^4 = 0 \quad (31)$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2} \right)^2 - (2 - e^2) \left(\frac{\lambda}{\|\mathbf{n}\|^2} \right) + (1 - e^2) = 0 \quad (32)$$

$$\Rightarrow \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \quad (33)$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = (1 - e^2) \lambda_2 \quad (34)$$

From (34), the eccentricity of (1) is given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}}. \quad (35)$$

Multiplying both sides of (2) by \mathbf{n} ,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{nn}^T \mathbf{n} \quad (36)$$

$$= \|\mathbf{n}\|^2 (1 - e^2) \mathbf{n} \quad (37)$$

$$= \lambda_1 \mathbf{n} \quad (38)$$

from (34) Thus, λ_1 is the corresponding eigenvalue for \mathbf{n} .

(24) From (9), (34) and (38),

$$\mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \quad (39)$$

From (3) and (34),

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (40)$$

$$\Rightarrow \|\mathbf{F}\|^2 = \frac{(ce^2 \mathbf{n} - \mathbf{u})^T (ce^2 \mathbf{n} - \mathbf{u})}{\lambda_2^2} \quad (41)$$

$$\Rightarrow \lambda_2^2 \|\mathbf{F}\|^2 = c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^T \mathbf{n} + \|\mathbf{u}\|^2 \quad (42)$$

Also, (4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2 e^2 \quad (43)$$

From (42) and (43),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^T \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 (f + c^2 e^2) \quad (44)$$

$$\Rightarrow \lambda_2 e^2 (e^2 - 1) c^2 - 2ce^2 \mathbf{u}^T \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \quad (45)$$

yielding

$$c = \begin{cases} \frac{e \mathbf{u}^T \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^T \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2e^2 \mathbf{u}^T \mathbf{n}} & e = 1 \end{cases} \quad (46)$$

The points of intersection of the line

$$L: \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbf{R} \quad (47)$$

with the conic section in (1) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (48)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f \right) \left(\mathbf{m}^T \mathbf{V} \mathbf{m} \right)} \right) \quad (49)$$

Substituting (47) in (1),

$$(\mathbf{q} + \mu \mathbf{m})^T \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) \quad (50)$$

$$+ 2\mathbf{u}^T (\mathbf{q} + \mu \mathbf{m}) + f = 0 \quad (51)$$

$$\implies \mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \quad (52)$$

$$+ \mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \quad (53)$$

Solving the above quadratic in (53) yields (49).

The area of a triangle whose vertices are \mathbf{A}, \mathbf{B} and \mathbf{O} is given by

$$ar(AOB) = \frac{1}{2} \left\| \begin{pmatrix} \mathbf{A}^\top & 1 \\ \mathbf{B}^\top & 1 \\ \mathbf{C}^\top & 1 \end{pmatrix} \right\| \quad (54)$$

Construction

The input parameters are \mathbf{V} from (2), \mathbf{u} from (3) and f from (4)

Symbol	Value	Description
\mathbf{P}	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	eigenvectors of \mathbf{V}
\mathbf{O}	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	center of parabola
η	$\mathbf{u}^\top \mathbf{p}_1 = -6$	from (10)
λ_2	$\mathbf{e}_2^\top D \mathbf{e}_2 = 1$	from (8)
\mathbf{n}	$\sqrt{\lambda_2} \mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	normal to directrix
c	from (46)	$\mathbf{n}^\top \mathbf{x} = c$
\mathbf{F}	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	focus from (40)
\mathbf{m}	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	latus rectum direction vector
$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$	$\begin{pmatrix} \mathbf{F} + \mu_1 \mathbf{m} \\ \mathbf{F} + \mu_2 \mathbf{m} \end{pmatrix}$	μ_1, μ_2 from (49)
$ar(AOB)$	18 sq units	from (54)