Circle Assignment

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***Problem Statement* - Let ABC be a right triangle in which AB = 6 cm, BC = 8 cm and ∠B = 90°. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.**

Solution

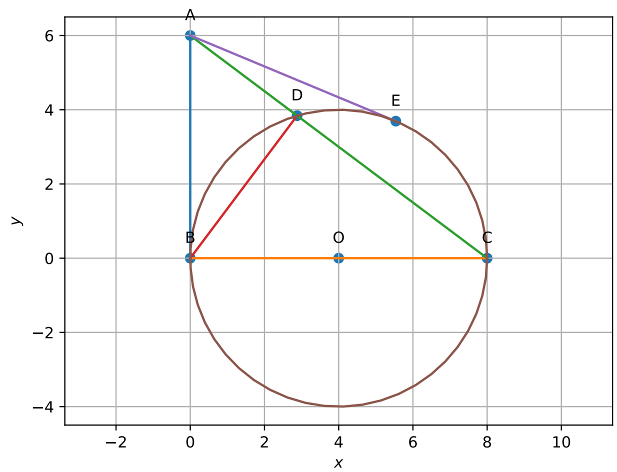

Figure 1: Tangents from A to circle through B, C and D

Given that BD ⊥ AC, which implies

$$\angle D = 90^\circ. \tag{1}$$

So, D can be found as the foot of the perpendicular from B on line AC. This is given by

$$\mathbf{D} = \mathbf{A} + \frac{\mathbf{m}^T(\mathbf{B}-\mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{2}$$

where $\mathbf{m}$ is the direction vector for line AC.

The chord BC of the circle subtends 90° at D. By the inclusive angle theorem, BC is the diameter of the circle with center O given by

$$\mathbf{O} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{3}$$

In order to find the intersection points E and B of tangents from A, the origin is shifted from B to O. The equation of the circle in the new frame is

$$\mathbf{x}^T\mathbf{x} = r^2 \tag{4}$$

Let the the point of intersection between the tangent from A and the circle be P. Since P lies on the circle given by (4) it is of the form

$$\mathbf{P} = r\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \tag{5}$$

Since AP is a tangent to the circle, OP ⊥ AP. This implies that

$$(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{O}) = 0 \tag{6}$$

Since O is the origin in the new frame, $\mathbf{O} = \mathbf{0}$. Expanding (6), we get

$$\mathbf{A}^T\mathbf{P} = \mathbf{P}^T\mathbf{P} \tag{7}$$

Substituting value of $\mathbf{P}$ from (5) in (7), we get

$$\mathbf{A}^T\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} = r \tag{8}$$

Which is a standard trigonometric equation with the solution given by

$$\theta = n\pi + (-1)^n\alpha - \phi \tag{9}$$

where

$$\alpha = arcsin(\frac{r}{\|\mathbf{A}\|})$$
$$\phi = arctan(\frac{\mathbf{e_1}^T\mathbf{A}}{\mathbf{e_2}^T\mathbf{A}})$$
$$n \in \mathbf{Z}$$

Since two tangents are possible from any point outside a circle, let the two solutions be θ_1 and θ_2. The value of n for both the solutions is chosen such that

$$0 \le \theta_1, \theta_2 < 2\pi \tag{10}$$
$$|\theta_1 - \theta_2| < \pi \tag{11}$$

Using θ_1 and θ_2, the intersection points are

$$\mathbf{E_O} = r\begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix} \tag{12}$$

$$\mathbf{B_O} = r\begin{pmatrix} cos\theta_2 \\ sin\theta_2 \end{pmatrix} \tag{13}$$

The coordinates for position vectors $\mathbf{E_O}$ and $\mathbf{B_O}$ are with respect to origin O. The actual coordinates with respect to origin B is given by

$$\mathbf{E} = \mathbf{E_O} + \mathbf{O} \tag{14}$$
$$\mathbf{B} = \mathbf{B_O} + \mathbf{O} \tag{15}$$

Construction

The input parameters are the lengths

$$AB = a = 6$$
$$BC = b = 8$$

Symbol	Value	Description
a	6	AB
b	8	BC
r	$\frac{b}{2}$	Radius
$\mathbf{m}$	$\mathbf{A} - \mathbf{C}$	Direction vector of line AC
$\mathbf{D}$	$\mathbf{A} + \frac{\mathbf{m}^T(\mathbf{B}-\mathbf{A})}{\lVert\mathbf{m}\rVert^2}\mathbf{m}$	Point D
$\mathbf{A_O}$	$\mathbf{A} - \mathbf{O}$	$\mathbf{A}$ when origin shifted to $\mathbf{O}$
α	$arcsin(\frac{r}{\lVert\mathbf{A_O}\rVert})$	from (9)
ϕ	$arctan(\frac{\mathbf{e_1}^T\mathbf{A_O}}{\mathbf{e_2}^T\mathbf{A_O}})$	from (9)
θ_1	$\alpha - \phi$	$n = 0$ in (9)
θ_2	$\pi - \alpha - \phi$	$n = 1$ in (9)
$\mathbf{E}$	$r\begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix} + \mathbf{O}$	Point E
$\mathbf{B}$	$r\begin{pmatrix} cos\theta_2 \\ sin\theta_2 \end{pmatrix} + \mathbf{O}$	Point B

Proofs

Foot of perpendicular from point P on line $\mathbf{A} + \lambda\mathbf{m}$

Let the intersection point be $\mathbf{X}$. Since X is foot of perpendicular from point P to line with direction vector $\mathbf{m}$,

$$\mathbf{m}^T(\mathbf{X} - \mathbf{P}) = 0 \tag{16}$$

Since X lies on the line with direction vector $\mathbf{m}$,

$$\mathbf{X} = \mathbf{A} + \lambda\mathbf{m} \tag{17}$$

Substituting (17) in (16) and solving for λ,

$$\lambda = \frac{\mathbf{m}^T(\mathbf{P} - \mathbf{A})}{\lVert\mathbf{m}\rVert^2} \tag{18}$$

Substituting (18) in (17),

$$\mathbf{X} = \mathbf{A} + \frac{\mathbf{m}^T(\mathbf{P} - \mathbf{A})}{\lVert\mathbf{m}\rVert^2}\mathbf{m} \tag{19}$$

Inclusive angle theorem

The inclusive angle theorem states that the angle subtended by any chord at the center of a circle is twice the angle angle subtended by the same chord at any other point on the major segment. Take three points A, B, and C on a unit circle at angles θ, ϕ and ψ. Then,

$$\mathbf{A} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} cos\phi \\ sin\phi \end{pmatrix}, \mathbf{C} = \begin{pmatrix} cos\psi \\ sin\psi \end{pmatrix} \tag{20}$$

Let AB be the chord that subtends angles at the center O and at point C. The cosine of the angle subtended at point C is given by

$$cos(\angle ACB) = \frac{\langle A - C, B - C\rangle}{|A - C||B - C|} \tag{21}$$

Where

$$\begin{aligned}
\langle A - C, B - C\rangle &= \langle(\cos\theta - \cos\psi, \sin\theta - \sin\psi), \\
&\quad (\cos\phi - \cos\psi, \sin\phi - \sin\psi)\rangle \\
&= (\cos\theta - \cos\psi)(\cos\phi - \cos\psi) + (\sin\theta - \sin\psi) \\
&\quad (\sin\phi - \sin\psi) \\
&= -2\sin\frac{\theta-\psi}{2}\sin\frac{\theta+\psi}{2}\cdot(-2)\sin\frac{\phi-\psi}{2}\sin\frac{\phi+\psi}{2} \\
&\quad + 2\cos\frac{\theta+\psi}{2}\sin\frac{\theta-\psi}{2}\cdot 2\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2} \\
&= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}(\sin\frac{\theta+\psi}{2}\sin\frac{\phi+\psi}{2} + \\
&\quad \cos\frac{\theta+\psi}{2}\cos\frac{\phi+\psi}{2}) \\
&= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\left(\frac{\theta+\psi}{2} - \frac{\phi+\psi}{2}\right) \\
&= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\frac{\theta-\phi}{2}
\end{aligned} \tag{22}$$

$$\begin{aligned}
|A - C|^2|B - C|^2 &= ((\cos\theta - \cos\psi)^2 + (\sin\theta - \sin\psi)^2)((\cos\phi - \\
&\quad \cos\psi)^2 + (\sin\phi - \sin\psi)^2) \\
&= (2 - 2\cos\theta\cos\psi - 2\sin\theta\sin\psi)(2 - \\
&\quad 2\cos\phi\cos\psi - 2\sin\phi\sin\psi) \\
&= 4(1 - \cos(\theta - \psi))(1 - \cos(\phi - \psi)) \\
&= 4\cdot 2\sin^2\frac{\theta-\psi}{2}\cdot 2\sin^2\frac{\phi-\psi}{2} \\
&= 16\sin^2\frac{\theta-\psi}{2}\sin^2\frac{\phi-\psi}{2}
\end{aligned} \tag{23}$$

Substituting (22) and (23) in (21),

$$cos(\angle ACB) = cos(\frac{\theta - \phi}{2}) \tag{24}$$

Hence $\angle ACB = \frac{\theta-\phi}{2} = \frac{\angle AOB}{2}$