Conic section Assignment

Mohamed Hamdan

September 2022

***Problem Statement* - Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum**

Solution

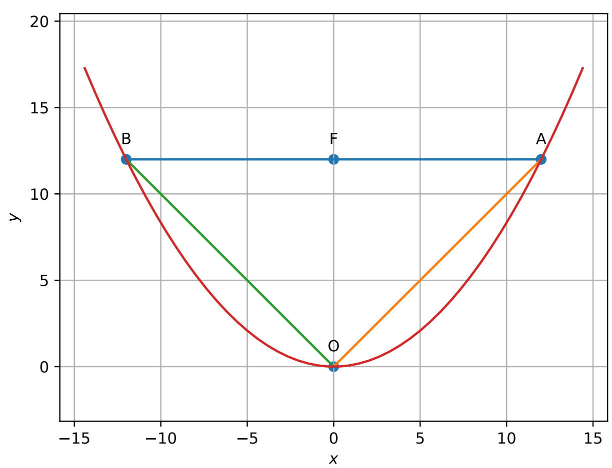

Figure 1: Triangle formed by vertex and ends of latus rectum of parabola $x^2 = 12y$

The given equation of parabola $x^2 = 12y$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

The parabola in (1) can be expressed in standard form (center/vertex at origin, major-axis - x axis) as

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -2\eta \mathbf{e}_1^\top \mathbf{y} \qquad |V| = 0 \tag{5}$$

where

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{6}$$

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{9}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{10}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

To find $\mathbf{c}$ which is the center of the parabola in (1), substitute (6) in (1)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{12}$$

yielding

$$\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{13}$$

From (13) and (7),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{14}$$

For a parabola $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{15}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (7)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{16}$$

Substituting (16) in (14),

$$\begin{aligned}
&\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} \\
&\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \\
&\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\
&\quad + 2\left(\left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_1 \left(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y} \\
&\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \\
&\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\
&\quad + 2\left(\mathbf{u}^T\mathbf{p}_1 \quad \left(\lambda_2\mathbf{c}^T + \mathbf{u}^T\right)\mathbf{p}_2\right)\mathbf{y} \\
&\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \text{ from (15)}
\end{aligned}$$

$$\begin{aligned}
&\implies \lambda_2 y_2^2 + 2\left(\mathbf{u}^T\mathbf{p}_1\right)y_1 + 2y_2(\lambda_2\mathbf{c} + \mathbf{u})^T\mathbf{p}_2 \\
&\quad + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0
\end{aligned}$$

which is the equation of a parabola. Thus, (17) can be expressed as (5) by choosing

$$\eta = \mathbf{u}^T \mathbf{p}_1 \tag{17}$$

and $\mathbf{c}$ in (14) such that

$$\mathbf{P}^T \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{18}$$

$$\mathbf{c}^T \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T \mathbf{c} + f = 0 \tag{19}$$

Multiplying (18) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \mathbf{p}_1, \tag{20}$$

which, upon substituting in (19) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \tag{21}$$

(20) and (21) can be clubbed together to obtain (22).

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |V| = 0 \tag{22}$$