Digital Communication

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CONTENTS

Chapter 1 Two Dice

1.1 Sum of Independant Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.1.1 *The Uniform Distribution:* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr\left(X_i = n\right) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.1.1.2}$$
$$\implies X \in \{1, 2, \dots, 12\} \tag{1.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.1.1.4}$$

1.1.2 *Convolution:* From (1.1.1.2),

$$p_X(n) = \Pr\left(X_1 + X_2 = n\right) = \Pr\left(X_1 = n - X_2\right) \tag{1.1.2.1}$$
$$= \sum_k \Pr\left(X_1 = n-k | X_2 = k\right) p_{X_2}(k) \tag{1.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr\left(X_1 = n-k | X_2 = k\right) = \Pr\left(X_1 = n-k\right) = p_{X_1}(n-k) \tag{1.1.2.3}$$

From (1.1.2.2) and (1.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (1.1.1.1) in (1.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.1.2.6}$$

From (1.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.1.2.7}$$

Substituting from (1.1.1.1) in (1.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.1.2.8}$$

satisfying (1.1.1.4).

1.1.3 *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.1.3.1}$$

From (1.1.1.1) and (1.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.1.3.2}$$
$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.1.3.4}$$
$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (1.1.3.3) and (1.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{1.1.3.6}$$
$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{1.1.3.8}$$
$$n u(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.1.3.11}$$

From (1.1.3.1), (1.1.3.7) and (1.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{1.1.3.12}$$

which is the same as (1.1.2.8). Note that (1.1.2.8) can be obtained from (1.1.3.10) using contour integration as well.

1.1.4 The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1.1.4.1. The theoretical pmf obtained in (1.1.2.8) is plotted for comparison.

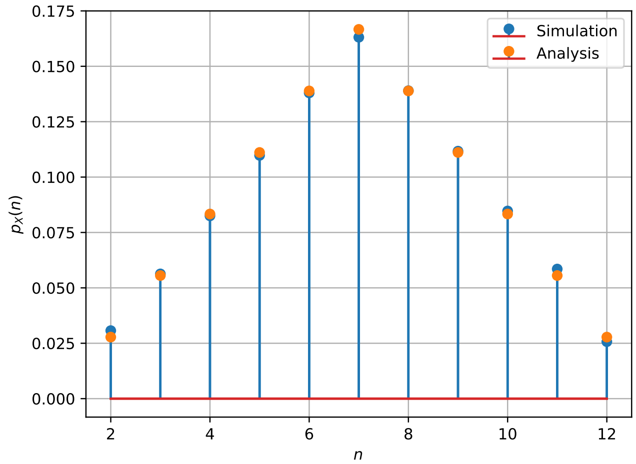

Fig. 1.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

1.1.5 The python code is available in

```
/codes/chapter1/dice.py
```

Chapter 2 Random Numbers

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

2.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/chapter2/uni_gen_stat.c
```

2.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \leq x\right) \tag{2.1.2.1}$$

Solution: The following code plots Fig. 2.1.2.1

```
codes/chapter2/cdf_plot_uni.py
```

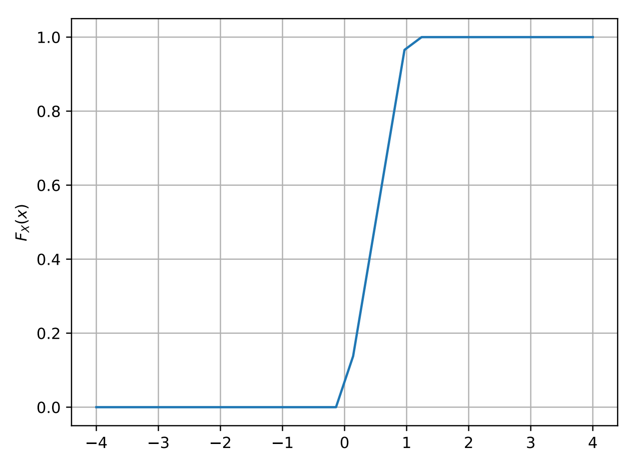

Fig. 2.1.2.1: The CDF of U

2.1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{2.1.3.1}$$

For the uniform random variable U, $f_U(x)$ is given by

$$f_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases} \tag{2.1.3.2}$$

Substituting (2.1.3.2) in (2.1.3.1), $F_U(x)$ is found to be

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 0 \end{cases} \tag{2.1.3.3}$$

2.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2.1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{2.1.4.2}$$

Write a C program to find the mean and variance of U.
Solution: The following code prints the mean and variance of U

```
codes/chapter2/uni_gen_stat.c
```

The output of the program is

```
Uniform stats:
Mean: 0.500007
Variance: 0.083301
```

2.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.1.5.1}$$

Solution: For a random variable X, the mean μ_X and variance σ_X^2 are given by

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.1.5.2}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{2.1.5.3}$$

Substituting the CDF of U from (2.1.3.3) in (2.1.5.2) and (2.1.5.3), we get

$$\mu_U = \frac{1}{2} \tag{2.1.5.4}$$

$$\sigma_U^2 = \frac{1}{12} \tag{2.1.5.5}$$

which match with the values printed in problem 2.1.4

2.2 Central Limit Theorem

2.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/chapter2/gau_gen_stat.c
```

2.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2.2.1

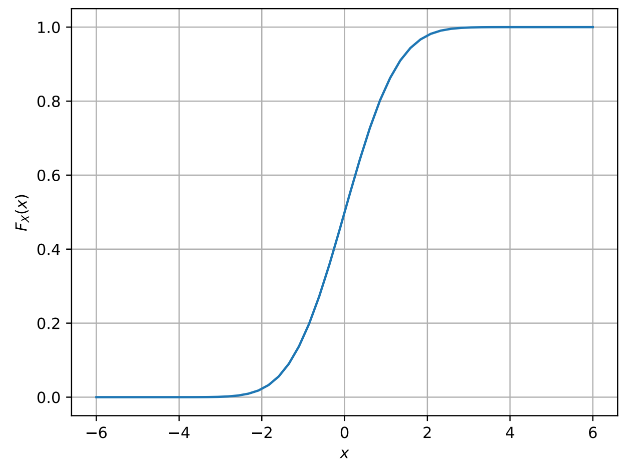

Fig. 2.2.2.1: The CDF of X

The properties of a CDF are

$$F_X(-\infty) = 0 \tag{2.2.2.1}$$

$$F_X(\infty) = 1 \tag{2.2.2.2}$$

$$\frac{dF_X(x)}{dx} \ge 0 \tag{2.2.2.3}$$

2.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2.3.1}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.2.3.1 using the code below

```
codes/chapter2/cdf_pdf_plot_gau.py
```

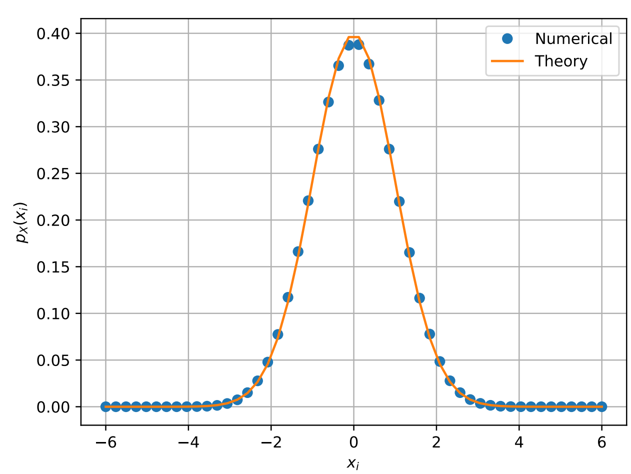

Fig. 2.2.3.1: The PDF of X

The properties of PDF are

$$f_X(x) \geq 0 \tag{2.2.3.2}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{2.2.3.3}$$

2.2.4 Find the mean and variance of X by writing a C program.
Solution: The following code prints the mean and variance of X

```
codes/chapter2/gau_gen_stat.c
```

The output of the program is

```
Gaussian stats:
Mean: 0.000294
Variance: 0.999562
```

2.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.2.5.1}$$

repeat the above exercise theoretically.
Solution: Substituting the PDF from (2.2.5.1) in (2.1.5.2),

$$\mu_X = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.2.5.2}$$

Using

(2.2.5.3)

$$\int x \cdot \exp\left(-ax^2\right) \mathrm{d}x = -\frac{1}{2a} \cdot \exp\left(-ax^2\right) \tag{2.2.5.4}$$

$$\mu_X = \frac{1}{\sqrt{2\pi}} \left[-\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{2.2.5.5}$$

$$\mu_X = 0 \tag{2.2.5.6}$$

Substituting μ_X and the PDF in (2.1.5.3) to compute variance,

$$\sigma_X^2 = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.2.5.7}$$

Substituting

$$t = \frac{x^2}{2}, \tag{2.2.5.8}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} \exp(-t)\, dt \tag{2.2.5.9}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}-1} \exp(-t)\, dt$$

Using the gamma function

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \cdot e^{-z}\, \mathrm{d}z \tag{2.2.5.10}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \tag{2.2.5.11}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2}$$

$$= 1$$

2.3 From Uniform to Other

2.3.1 Generate samples of

$$V = -2\ln(1-U) \tag{2.3.1.1}$$

and plot its CDF.
Solution: The samples for U are loaded from uni.dat file generated in problem 2.1.4. The CDF of V is plotted in Fig. 2.3.1.1 using the code below,

```
codes/chapter2/function_1.py
```

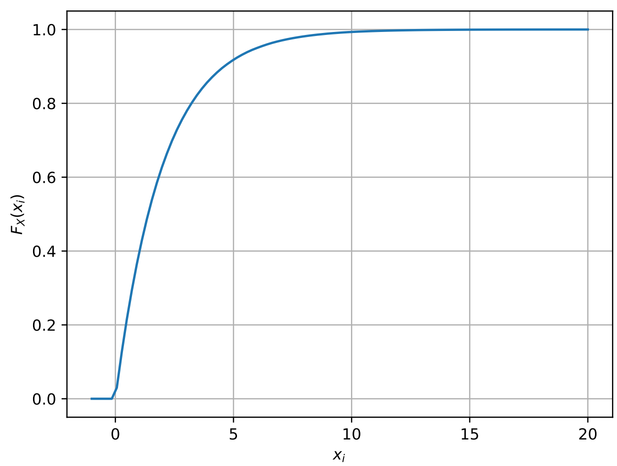

Fig. 2.3.1.1: The CDF of V

2.3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V < x) \tag{2.3.2.1}$$

$$= P(-2\ln(1-U) < x) \tag{2.3.2.2}$$

$$= P(U < 1 - e^{\frac{-x}{2}}) \tag{2.3.2.3}$$

$$= F_U(1 - e^{\frac{-x}{2}}) \tag{2.3.2.4}$$

Using $F_U(x)$ defined in (2.1.3.3),

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{2.3.2.5}$$

2.4 Triangular Distribution

2.4.1 Generate

$$T = U_1 + U_2 \tag{2.4.1.1}$$

Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/chapter2/two_uni_gen.c
```

2.4.2 Find the CDF of T.
Solution: Loading the samples from uni1.dat and uni2.dat in python, the CDF is plotted in Fig. 2.4.2.1

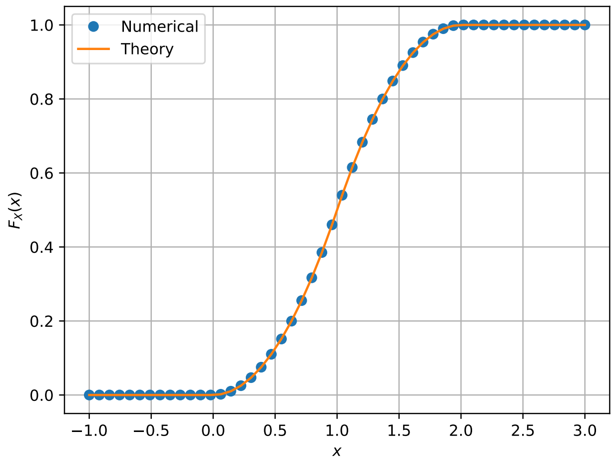

Fig. 2.4.2.1: The CDF of T

2.4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 2.4.3.1 using the code below

```
codes/chapter2/function_2.py
```

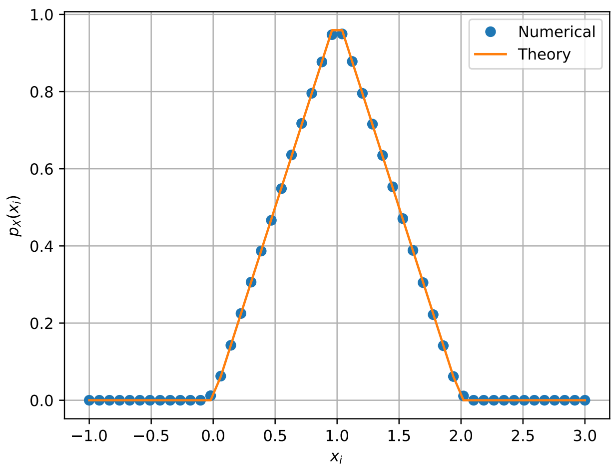

Fig. 2.4.3.1: The PDF of T

2.4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Since T is the sum of two independant random variables $U1$ and $U2$, the PDF of T is given by

$$p_T(x) = p_{U1}(x) * p_{U2}(x) \tag{2.4.4.1}$$

Using the PDF of U from (2.1.3.2), the convolution results in

$$p_T(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases} \tag{2.4.4.2}$$

The CDF of T is found using (2.1.3.1) by replacing U with T. Evaluating the integral for the piecewise function $p_T(x)$,

$$F_T(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{2.4.4.3}$$

2.4.5 Verify your results through a plot.
Solution: The theoretical and numerical plots for the CDF and PDF of T closely match in Fig. 2.4.2.1 and Fig. 2.4.3.1

Chapter 3 Maximum Likelihood Detection: BPSK

3.1 Maximum Likelihood

3.1.1 Generate equiprobable $X \in \{1, -1\}$.
3.1.2 Generate

$$Y = AX + N, \tag{3.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
3.1.3 Plot Y using a scatter plot.
3.1.4 Guess how to estimate X from Y.
3.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{3.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{3.1.5.2}$$

3.1.6 Find P_e assuming that X has equiprobable symbols.
3.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
3.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
3.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{3.1.9.1}$$

3.1.10 Repeat the above exercise using the MAP criterion.

Chapter 4 Transformation of Random Variables

4.1 Gaussian to Other

4.1.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{4.1.1.1}$$

4.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ 0 & x < 0, \end{cases} \tag{4.1.2.1}$$

find α.
4.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{4.1.3.1}$$

4.2 Conditional Probability

4.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.2.1.1}$$

for

$$Y = AX + N, \tag{4.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0,1\right), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

4.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

4.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{4.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

4.2.4 Plot P_e in problems 4.2.1 and 4.2.3 on the same graph w.r.t γ. Comment.

Chapter 5 Bivariate Random Variables: FSK

5.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{5.1.0.1}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.0.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}\left(0,1\right). \tag{5.1.0.3}$$

5.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{5.1.1.1}$$

on the same graph using a scatter plot.

5.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

5.1.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{5.1.3.1}$$

with respect to the SNR from 0 to 10 dB.

5.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Chapter 6 Exercises

6.1 BPSK

6.1.1 The *signal constellation diagram* for BPSK is given by Fig. 6.1.1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

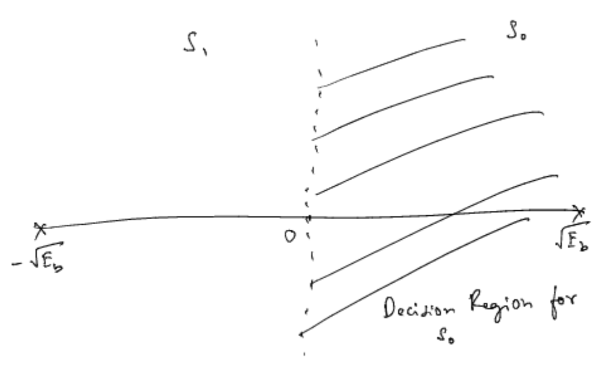

Fig. 6.1.1.1

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{6.1.1.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{6.1.1.2}$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

6.1.2 From Fig. 6.1.1.1 obtain a decision rule for BPSK
Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{6.1.2.1}$$

6.1.3 Repeat the previous exercise using the MAP criterion.

6.1.4 Using the decision rule in Problem 6.1.2, obtain an expression for the probability of error for BPSK.
Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{6.1.4.1}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{6.1.4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}\left(0,1\right)$. Then $n = \sqrt{\frac{N_0}{2}} w$. Substituting this in (6.1.4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}} w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{6.1.4.3}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{6.1.4.4}$$

where $Q\left(x\right) \triangleq \Pr\left(w > x\right), x \ge 0$.

6.1.5 The PDF of $w \sim \mathcal{N}\left(0,1\right)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{6.1.5.1}$$

and the complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt. \tag{6.1.5.2}$$

Show that

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{6.1.5.3}$$

6.1.6 Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.
Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 6.1.6.1

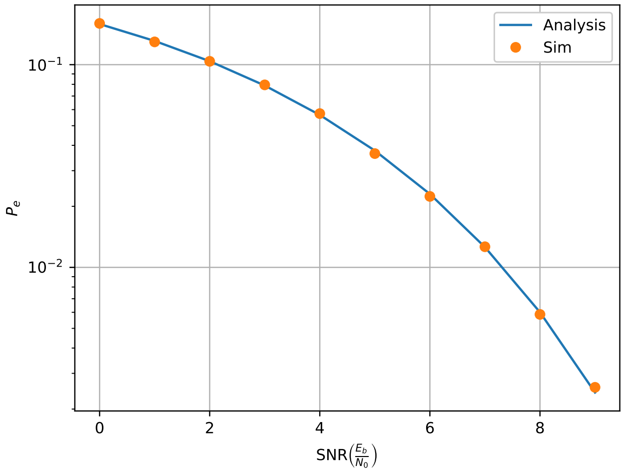

Fig. 6.1.6.1

6.1.7 Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{6.1.7.1}$$

6.2 Coherent BFSK

6.2.1 The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 6.2.1.1. Obtain the equations for the received symbols.

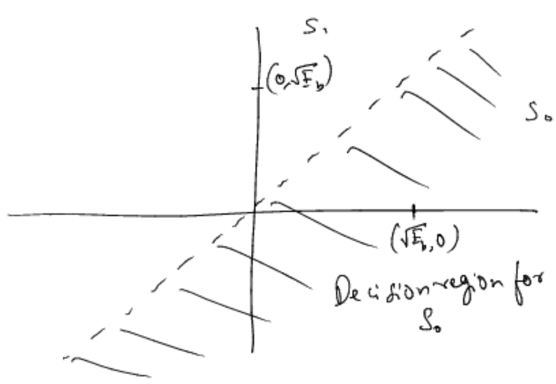

Fig. 6.2.1.1

Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{6.2.1.1}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{6.2.1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

6.2.2 Obtain a decision rule for BFSK from Fig. 6.2.1.1.
Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{6.2.2.1}$$

6.2.3 Repeat the previous exercise using the MAP criterion.

6.2.4 Derive and plot the probability of error. Verify through simulation.

6.3 QPSK

6.3.1 Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{6.3.1.1}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{6.3.1.2}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{6.3.1.3}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2 \mathbf{I} \tag{6.3.1.4}$$

(a) Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{6.3.1.5}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[(X - \mu_x)(Y - \mu_y)\right]}{\sigma_x \sigma_y} \tag{6.3.1.6}$$

X and Y are said to be uncorrelated if $\rho = 0$

(c) Show that if X and Y are uncorrelated Verify this numerically.

(d) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.

(e) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

(f) Show that $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right)$.

(g) Find $\Pr\left(X < A, Y < A\right)$.

(h) Verify the above through simulation.

6.4 M-PSK

6.4.1 Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{6.4.1.1}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{6.4.1.2}$$

$$r_2 = R\sin\theta \tag{6.4.1.3}$$

show that the joint pdf of R, θ is

$$p\left(R, \theta\right) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{6.4.1.4}$$

(b) Show that

$$\lim_{\alpha \to \infty} \int_0^\infty (V - \alpha)\, e^{-(V-\alpha)^2} \, dV = 0 \tag{6.4.1.5}$$

$$\lim_{\alpha \to \infty} \int_0^\infty e^{-(V-\alpha)^2} \, dV = \sqrt{\pi} \tag{6.4.1.6}$$

(c) Using the above, evaluate

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \tag{6.4.1.7}$$

for large values of γ.

(d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2 \theta} \cos\theta \, d\theta \tag{6.4.1.8}$$

(e) Find $P_{e|\mathbf{s}_0}$.

6.5 Noncoherent BFSK

6.5.1 Show that

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos\theta} \, d\theta \tag{6.5.1.1}$$

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x\cos(\theta-\phi)} \, d\theta \tag{6.5.1.2}$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{m_1 \cos\theta + m_2 \sin\theta} \, d\theta = I_0\left(\sqrt{m_1^2 + m_2^2}\right) \tag{6.5.1.3}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi} \int_0^{\pi} e^{x\cos\theta} \cos n\theta \, d\theta \tag{6.5.1.4}$$

6.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b} \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b} \begin{pmatrix} 0 \\ 0 \\ \cos\phi_1 \\ \sin\phi_1 \end{pmatrix} + \mathbf{n}_1 \tag{6.5.2.1}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p\left(\mathbf{r}|0, \phi_0\right)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p\left(\mathbf{r}|0\right)$. Note that this expression will no longer contain ϕ_0.

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{6.5.2.2}$$

where k is a constant.

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{6.5.2.3}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \tag{6.5.2.4}$$

(g) Find

$$g\left(r_1, r_2\right) = \Pr\left(r_1^2 + r_2^2 < Y | 0, r_1, r_2\right). \tag{6.5.2.5}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(i) Now show that

$$E\left[g\left(r_1, r_2\right)\right] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \tag{6.5.2.6}$$

6.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R}\cos\Theta \tag{6.5.3.1}$$

$$V = \sqrt{R}\sin\Theta \tag{6.5.3.2}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det\begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \tag{6.5.3.3}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}\left(r, \theta\right) = p_{U,V}\left(u, v\right) J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{6.5.3.4}$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k} e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \tag{6.5.3.5}$$

assuming that Θ is uniformly distributed between 0 to 2π.

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k} e^{-\frac{|y|}{k}} \tag{6.5.3.6}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{6.5.3.7}$$

where ϕ is a constant.

(e) Find $\Pr\left(Y > Z\right)$.

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{6.5.3.8}$$

is

$$p_R\left(r\right) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{6.5.3.9}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n \left(n!\right)^2} \tag{6.5.3.10}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \ge 0 \\ 0 & z < 0 \end{cases} \tag{6.5.3.11}$$

find $\Pr\left(R < Z\right)$.

6.6 Craig's Formula and MGF

6.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{6.6.1.1}$$

where X is a random variable and $E\left[\cdot\right]$ is the expectation.

(a) Let $Y \sim \mathcal{N}\left(0, 1\right)$. Define

$$Q(x) = \Pr\left(Y > x\right), x > 0 \tag{6.6.1.2}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{6.6.1.3}$$

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{6.6.1.4}$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{6.6.1.5}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{6.6.1.6}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.