Normal(s) to a conic from a point

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Problem Statement **- To find the equation of all possible normals to a conic from a point**

Let the point from which normals are drawn be $\mathbf{h}$. Then, the equation of the normal can be written as

$$\mathbf{x} = \mathbf{h} + \lambda\mathbf{m} \tag{1}$$

Say the point of intersection of (1) with the conic is $\mathbf{q}$. A tangent drawn at $\mathbf{q}$ satisfies the equation

$$\mathbf{n}^\top(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{2}$$

Where $\mathbf{n}$ is the direction vector of the tangent and is perpendicular to $\mathbf{m}$ in (1).

In general, the parameter values for points of intersection of a line given by (1) with a conic is given by

$$\lambda_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{3}$$

Using (3) and (1), the intersection point $\mathbf{q}$ can be written as

$$\mathbf{q} = \mathbf{h} + \lambda_i\mathbf{m} \tag{4}$$

Substituting (4) in (2),

$$\mathbf{n}^\top(\mathbf{V}(\mathbf{h} + \lambda_i\mathbf{m}) + \mathbf{u}) = 0 \tag{5}$$

$$\implies \lambda_i\mathbf{n}^\top\mathbf{V}\mathbf{m} = -\mathbf{n}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \tag{6}$$

Substituting value of λ_i from (3) in (6)

$$\frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right)\mathbf{n}^\top\mathbf{V}\mathbf{m} = -\mathbf{n}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right) \tag{7}$$

Rearranging the terms,

$$\pm\sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right) = \left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top\left(\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)\mathbf{m} - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathbf{n}\right) \tag{8}$$

Squaring on both sides

$$\left[\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\right]\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)^2 = \left[\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top\left(\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)\mathbf{m} - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathbf{n}\right)\right]^2 \tag{9}$$

If $\mathbf{n}$ is take as $\begin{pmatrix}1\\ \mu\end{pmatrix}$, then $\mathbf{m}$ is $\begin{pmatrix}\mu\\ -1\end{pmatrix}$. Substituting these values in (9) and solving for μ, the different possible normals passing through $\mathbf{h}$ are obtained.