

Circle Assignment

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Problem Statement - Let ABC be a right triangle in which $AB = 6$ cm, $BC = 8$ cm and $\angle B = 90^\circ$. BD is the perpendicular from B on AC . The circle through B, C, D is drawn. Construct the tangents from A to this circle.

Solution

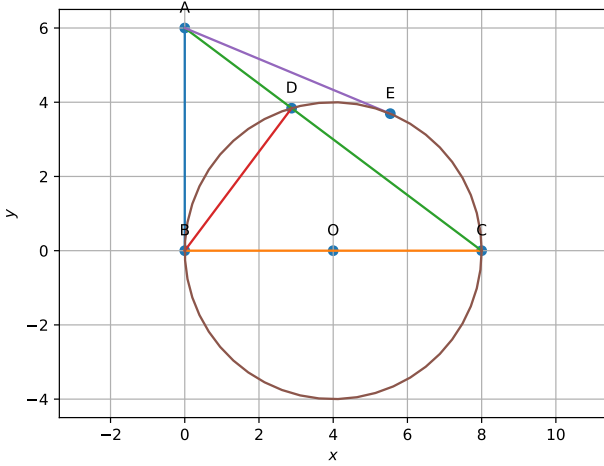


Figure 1: Tangents from A to circle through B, C and D

Given that $BD \perp AC$, which implies

$$\angle D = 90^\circ. \quad (1)$$

So, D can be found as the foot of the perpendicular from B on line AC . This is given by

$$\mathbf{D} = \mathbf{A} + \frac{\mathbf{m}^T(\mathbf{B} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \quad (2)$$

where \mathbf{m} is the direction vector for line AC .

The chord BC of the circle subtends 90° at D . By the inclusive angle theorem, BC is the diameter of the circle with center O given by

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (3)$$

In order to find the intersection points E and B of tangents from A , the origin is shifted from B to O . The equation of the circle in the new frame is

$$\mathbf{x}^T \mathbf{x} = r^2 \quad (4)$$

Let the the point of intersection between the tangent from A and the circle be P . Since P lies on the circle given by (4) it is of the form

$$\mathbf{P} = \begin{pmatrix} t \\ \sqrt{r^2 - t^2} \end{pmatrix} \quad (5)$$

Since AP is a tangent to the circle, $OP \perp AP$. This implies that

$$(\mathbf{A} - \mathbf{P})^T (\mathbf{P} - \mathbf{O}) = 0 \quad (6)$$

Since O is the origin in the new frame, $\mathbf{O} = \mathbf{0}$. Expanding (6), we get

$$\mathbf{A}^T \mathbf{P} = \mathbf{P}^T \mathbf{P} \quad (7)$$

Substituting value of \mathbf{P} from (5) in (7), we get

$$\mathbf{A}^T \begin{pmatrix} t \\ \sqrt{r^2 - t^2} \end{pmatrix} = r^2 \quad (8)$$

Expanding and rearranging terms in (8)

$$\|\mathbf{A}\|^2 t^2 - 2r^2 \mathbf{e}_1^T \mathbf{A} t + r^2(r^2 - (\mathbf{e}_2^T \mathbf{A})^2) = 0 \quad (9)$$

Which is a quadratic equation in t with roots given by

$$t = \frac{r^2 \mathbf{e}_1^T \mathbf{A} \pm \sqrt{r^4 \mathbf{e}_1^T \mathbf{A}^2 - r^2 \|\mathbf{A}\|^2 (r^2 - \mathbf{e}_2^T \mathbf{A}^2)}}{\|\mathbf{A}\|^2} \quad (10)$$

Substituting the values of t from (10) in (5) the two points of contact are obtained say \mathbf{E}_O and \mathbf{B}_O .

The coordinates for position vectors \mathbf{E}_O and \mathbf{B}_O are with respect to origin O . The actual coordinates with respect to origin B is given by

$$\mathbf{E} = \mathbf{E}_O + \mathbf{O} \quad (11)$$

$$\mathbf{B} = \mathbf{B}_O + \mathbf{O} \quad (12)$$

Construction

The input parameters are the lengths

$$AB = a = 6$$

$$BC = b = 8$$

Symbol	Value	Description
a	6	AB
b	8	BC
r	$\frac{b}{2}$	Radius
\mathbf{m}	$\mathbf{A} - \mathbf{C}$	Direction vector of line AC
\mathbf{D}	$\mathbf{A} + \frac{\mathbf{m}^T(\mathbf{B}-\mathbf{A})}{\ \mathbf{m}\ ^2}\mathbf{m}$	Point D
\mathbf{A}_O	$\mathbf{A} - \mathbf{O}$	\mathbf{A} when origin shifted to \mathbf{O}
t_1, t_2	evaluate (10)	solution of (9)
\mathbf{E}	$\begin{pmatrix} t_1 \\ \sqrt{r^2 - t_1^2} \end{pmatrix} + \mathbf{O}$	Point E
\mathbf{B}	$\begin{pmatrix} t_2 \\ \sqrt{r^2 - t_2^2} \end{pmatrix} + \mathbf{O}$	Point B

Proofs

Foot of perpendicular from point \mathbf{P} on line $\mathbf{A} + \lambda\mathbf{m}$

Let the intersection point be \mathbf{X} . Since \mathbf{X} is foot of perpendicular from point \mathbf{P} to line with direction vector \mathbf{m} ,

$$\mathbf{m}^T(\mathbf{X} - \mathbf{P}) = 0 \quad (13)$$

Since \mathbf{X} lies on the line with direction vector \mathbf{m} ,

$$\mathbf{X} = \mathbf{A} + \lambda\mathbf{m} \quad (14)$$

Substituting (14) in (13) and solving for λ ,

$$\lambda = \frac{\mathbf{m}^T(\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \quad (15)$$

Substituting (15) in (14),

$$\mathbf{X} = \mathbf{A} + \frac{\mathbf{m}^T(\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \quad (16)$$

Inclusive angle theorem

The inclusive angle theorem states that the angle subtended by any chord at the center of a circle is twice the angle subtended by the same chord at any other point on the major segment. Take three points \mathbf{A} , \mathbf{B} , and \mathbf{C} on a unit circle at angles θ , ϕ and ψ . Then,

$$\mathbf{A} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \quad (17)$$

Let \mathbf{AB} be the chord that subtends angles at the center \mathbf{O} and at point \mathbf{C} . The cosine of the angle subtended at point \mathbf{C} is given by

$$\cos(\angle ACB) = \frac{\langle \mathbf{A} - \mathbf{C}, \mathbf{B} - \mathbf{C} \rangle}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{B} - \mathbf{C}\|} \quad (18)$$

Where

$$\langle \mathbf{A} - \mathbf{C}, \mathbf{B} - \mathbf{C} \rangle = \langle (\cos\theta - \cos\psi, \sin\theta - \sin\psi), (\cos\phi - \cos\psi, \sin\phi - \sin\psi) \rangle$$

$$\begin{aligned}
&= (\cos\theta - \cos\psi)(\cos\phi - \cos\psi) + (\sin\theta - \sin\psi)(\sin\phi - \sin\psi) \\
&= -2\sin\frac{\theta-\psi}{2}\sin\frac{\theta+\psi}{2} \cdot (-2)\sin\frac{\phi-\psi}{2}\sin\frac{\phi+\psi}{2} \\
&\quad + 2\cos\frac{\theta+\psi}{2}\sin\frac{\theta-\psi}{2} \cdot 2\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2} \\
&= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}(\sin\frac{\theta+\psi}{2}\sin\frac{\phi+\psi}{2} + \cos\frac{\theta+\psi}{2}\cos\frac{\phi+\psi}{2}) \\
&= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\left(\frac{\theta+\psi}{2} - \frac{\phi+\psi}{2}\right) \\
&= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\frac{\theta-\phi}{2} \quad (19)
\end{aligned}$$

$$\begin{aligned}
|A - C|^2 |B - C|^2 &= ((\cos\theta - \cos\psi)^2 + (\sin\theta - \sin\psi)^2) \\
&\quad ((\cos\phi - \cos\psi)^2 + (\sin\phi - \sin\psi)^2) \\
&= (2 - 2\cos\theta\cos\psi - 2\sin\theta\sin\psi)(2 - 2\cos\phi\cos\psi - 2\sin\phi\sin\psi) \\
&= 4(1 - \cos(\theta - \psi))(1 - \cos(\phi - \psi)) \\
&= 4 \cdot 2\sin^2\frac{\theta-\psi}{2} \cdot 2\sin^2\frac{\phi-\psi}{2} \\
&= 16\sin^2\frac{\theta-\psi}{2}\sin^2\frac{\phi-\psi}{2} \quad (20)
\end{aligned}$$

Substituting (19) and (20) in (18),

$$\cos(\angle ACB) = \cos\left(\frac{\theta - \phi}{2}\right) \quad (21)$$

$$\text{Hence } \angle ACB = \frac{\theta - \phi}{2} = \frac{\angle AOB}{2}$$