Circle Assignment

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Problem Statement **- Let ABC be a right triangle in which AB = 6 cm, BC = 8 cm and ∠B = 90°. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.**

Solution

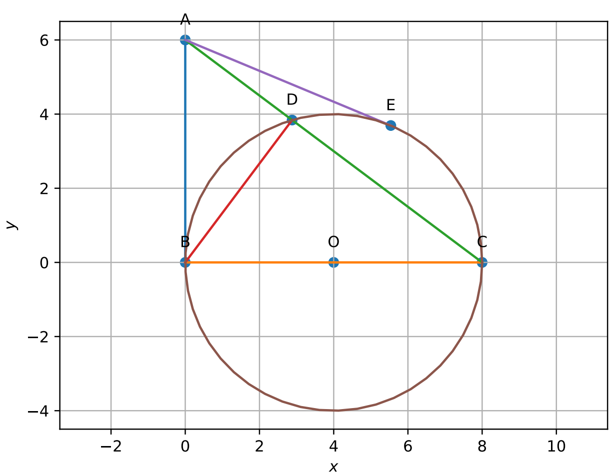

Figure 1: Tangents from A to circle through B, C and D

Given that BD ⊥ AC, which implies

$$\angle D = 90^\circ. \tag{1}$$

So, D can be found as the foot of the perpendicular from B on line AC. This is given by

$$\mathbf{D} = \mathbf{A} + \frac{\mathbf{m}^T(\mathbf{B} - \mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{2}$$

where $\mathbf{m}$ is the direction vector for line AC.

The chord BC of the circle subtends 90° at D. By the inclusive angle theorem, BC is the diameter of the circle with center O given by

$$\mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \tag{3}$$

In order to find the intersection points E and B of tangents from A, the origin is shifted from B to O. The equation of the circle in the new frame is

$$\mathbf{x}^T\mathbf{x} = r^2 \tag{4}$$

Let the the point of intersection between the tangent from A and the circle be P. Since P lies on the circle given by (4) it is of the form

$$\mathbf{P} = r\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \tag{5}$$

Since AP is a tangent to the circle, OP ⊥ AP. This implies that

$$(\mathbf{A} - \mathbf{P})^T(\mathbf{P} - \mathbf{O}) = 0 \tag{6}$$

Since O is the origin in the new frame, $\mathbf{O} = \mathbf{0}$. Expanding (6), we get

$$\mathbf{A}^T\mathbf{P} = \mathbf{P}^T\mathbf{P} \tag{7}$$

Substituting value of $\mathbf{P}$ from (5) in (7), we get

$$\mathbf{A}^T\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} = r \tag{8}$$

Which is a standard trigonometric equation with the solution given by

$$\theta = n\pi + (-1)^n\alpha - \phi \tag{9}$$

where

$$\alpha = arcsin(\frac{r}{\|\mathbf{A}\|})$$
$$\phi = arctan(\frac{\mathbf{e_1}^T\mathbf{A}}{\mathbf{e_2}^T\mathbf{A}})$$
$$n \in \mathbf{Z}$$

Since two tangents are possible from any point outside a circle, let the two solutions be θ_1 and θ_2. The value of n for both the solutions is chosen such that

$$0 \leq \theta_1, \theta_2 < 2\pi \tag{10}$$
$$|\theta_1 - \theta_2| < \pi \tag{11}$$

Using θ_1 and θ_2, the intersection points are

$$\mathbf{E_O} = r\begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix} \tag{12}$$
$$\mathbf{B_O} = r\begin{pmatrix} cos\theta_2 \\ sin\theta_2 \end{pmatrix} \tag{13}$$

The coordinates for position vectors $\mathbf{E_O}$ and $\mathbf{B_O}$ are with respect to origin O. The actual coordinates with respect to origin B is given by

$$\mathbf{E} = \mathbf{E_O} + \mathbf{O} \tag{14}$$
$$\mathbf{B} = \mathbf{B_O} + \mathbf{O} \tag{15}$$

Construction

The input parameters are the lengths

$$AB = a = 6$$
$$BC = b = 8$$

Symbol	Value	Description
a	6	AB
b	8	BC
r	$\frac{b}{2}$	Radius
$\mathbf{m}$	$\mathbf{A}-\mathbf{C}$	Direction vector of line AC
$\mathbf{D}$	$\mathbf{A}+\frac{\mathbf{m}^T(\mathbf{B}-\mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m}$	Point D
$\mathbf{A_O}$	$\mathbf{A}-\mathbf{O}$	$\mathbf{A}$ when origin shifted to $\mathbf{O}$
α	$arcsin(\frac{r}{\|\mathbf{A_O}\|})$	from (9)
ϕ	$arctan(\frac{\mathbf{e_1}^T\mathbf{A_O}}{\mathbf{e_2}^T\mathbf{A_O}})$	from (9)
θ_1	$\alpha-\phi$	$n=0$ in (9)
θ_2	$\pi-\alpha-\phi$	$n=1$ in (9)
$\mathbf{E}$	$r\begin{pmatrix}cos\theta_1\\ sin\theta_1\end{pmatrix}+\mathbf{O}$	Point E
$\mathbf{B}$	$r\begin{pmatrix}cos\theta_2\\ sin\theta_2\end{pmatrix}+\mathbf{O}$	Point B

Proofs

Foot of perpendicular from point P on line $\mathbf{A}+\lambda\mathbf{m}$

Let the intersection point be $\mathbf{X}$. Since X is foot of perpendicular from point P to line with direction vector $\mathbf{m}$,

$$\mathbf{m}^T(\mathbf{X}-\mathbf{P}) = 0 \tag{16}$$

Since X lies on the line with direction vector $\mathbf{m}$,

$$\mathbf{X} = \mathbf{A}+\lambda\mathbf{m} \tag{17}$$

Substituting (17) in (16) and solving for λ,

$$\lambda = \frac{\mathbf{m}^T(\mathbf{P}-\mathbf{A})}{\|\mathbf{m}\|^2} \tag{18}$$

Substituting (18) in (17),

$$\mathbf{X} = \mathbf{A}+\frac{\mathbf{m}^T(\mathbf{P}-\mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{19}$$

Inclusive angle theorem

The inclusive angle theorem states that the angle subtended by any chord at the center of a circle is twice the angle angle subtended by the same chord at any other point on the major segment. Take three points A, B, and C on a unit circle at angles θ, ϕ and ψ. Then,

$$\mathbf{A} = \begin{pmatrix}cos\theta\\ sin\theta\end{pmatrix}, \mathbf{B} = \begin{pmatrix}cos\phi\\ sin\phi\end{pmatrix}, \mathbf{C} = \begin{pmatrix}cos\psi\\ sin\psi\end{pmatrix} \tag{20}$$

Let AB be the chord that subtends angles at the center O and at point C. The cosine of the angle subtended at point C is given by

$$cos(\angle ACB) = \frac{\langle A-C, B-C\rangle}{|A-C||B-C|} \tag{21}$$

Where

$$\langle A-C, B-C\rangle = \langle(\cos\theta-\cos\psi, \sin\theta-\sin\psi),(\cos\phi-\cos\psi, \sin\phi-\sin\psi)\rangle$$

$$= (\cos\theta-\cos\psi)(\cos\phi-\cos\psi)+(\sin\theta-\sin\psi)(\sin\phi-\sin\psi)$$

$$= -2\sin\frac{\theta-\psi}{2}\sin\frac{\theta+\psi}{2}\cdot(-2)\sin\frac{\phi-\psi}{2}\sin\frac{\phi+\psi}{2} + 2\cos\frac{\theta+\psi}{2}\sin\frac{\theta-\psi}{2}\cdot 2\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2}$$

$$= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}(\sin\frac{\theta+\psi}{2}\sin\frac{\phi+\psi}{2}+\cos\frac{\theta+\psi}{2}\cos\frac{\phi+\psi}{2})$$

$$= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\left(\frac{\theta+\psi}{2}-\frac{\phi+\psi}{2}\right)$$

$$= 4\sin\frac{\theta-\psi}{2}\sin\frac{\phi-\psi}{2}\cos\frac{\theta-\phi}{2} \tag{22}$$

$$|A-C|^2|B-C|^2 = ((\cos\theta-\cos\psi)^2+(\sin\theta-\sin\psi)^2)((\cos\phi-\cos\psi)^2+(\sin\phi-\sin\psi)^2)$$

$$= (2-2\cos\theta\cos\psi-2\sin\theta\sin\psi)(2-2\cos\phi\cos\psi-2\sin\phi\sin\psi)$$

$$= 4(1-\cos(\theta-\psi))(1-\cos(\phi-\psi))$$
$$= 4\cdot 2\sin^2\frac{\theta-\psi}{2}\cdot 2\sin^2\frac{\phi-\psi}{2}$$

$$= 16\sin^2\frac{\theta-\psi}{2}\sin^2\frac{\phi-\psi}{2} \tag{23}$$

Substituting (22) and (23) in (21),

$$cos(\angle ACB) = cos(\frac{\theta-\phi}{2}) \tag{24}$$

Hence $\angle ACB = \frac{\theta-\phi}{2} = \frac{\angle AOB}{2}$