Sayısal Metotlar

Kaynak Kitaplar :

Numerical Analysis
Richard L.Burden, J.Douglas Faires
International Thomson Publishing Company

Numerical Methods Using Matlab
John H. Mathews, Kurtis D. Fink
Prentice Hall

Mühendisler için Sayısal Yöntemler
Steven C. Chapra, Raymond P.Canale
Çeviri: Hasan Heperkan, Uğur Kesgin
Literatür Yayınevi

İÇERİK

-Tek değişkenli fonksiyonların köklerinin bulunması

-Interpolasyon(ara nokta tayini)

-Sayısal integral

-Sayısal türev

-Adi Diferansiyel denklemlerin köklerinin bulunması

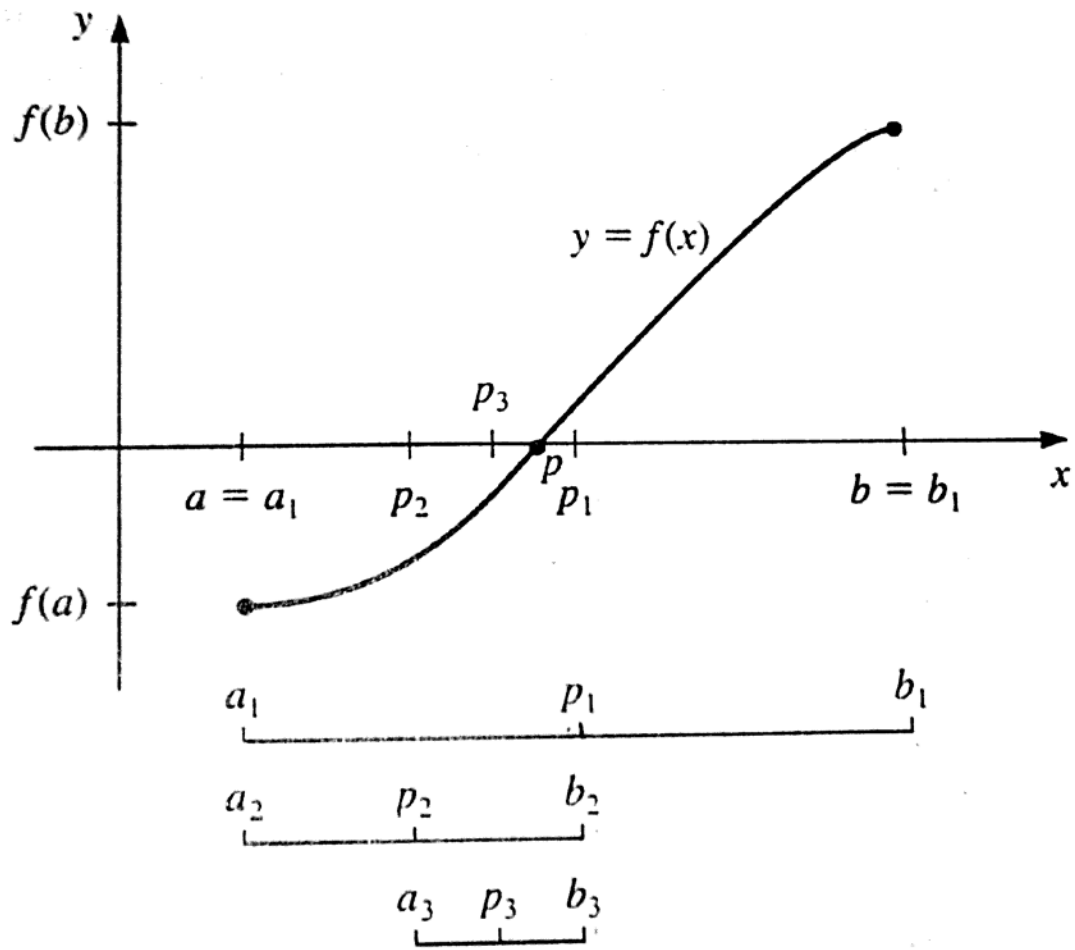
-LU ayrıştırması

-Öz değer, Öz vektör

-Lineer olamayan denklemlerin köklerinin bulunması

Tek değişkenli fonksiyonlar

Yarılama Metodu :



[a,b] aralığında köke bakılır p_1 bu aralığın orta noktasıdır.

$f(p_1)=0$ ise $p=p_1$ dir.

$f(p_1)$ ve $f(a_1)$ aynı işaretliyse kök $[p_1,b_1]$, $a_2=p_1$, $b_2=b_1$ dir.

$f(p_1)$ ve $f(a_1)$ farklı işaretliyse kök $[a_1,p_1]$, $a_2=a_1$, $b_2=p_1$ dir.

Algoritması:

```
Adım 1:   i=1
          FA=f(a)
Adım 2:   while i<=N0 do steps 3-6
          Adım 3:   p=(a+b)/2
                    FP=f(p)
          Adım 4:   if FP=0 or (b-a)/2<TOL then
                    OUTPUT(p)
                    STOP
Adım 5:   i=i+1
Adım 6:   if FA.FP>0 then a=p; FA=FP
                    else b=p
Adım 7:   OUTPUT(‘Method failed after N0 iterations, N0=‘,N0)
          STOP
```

Örnek:

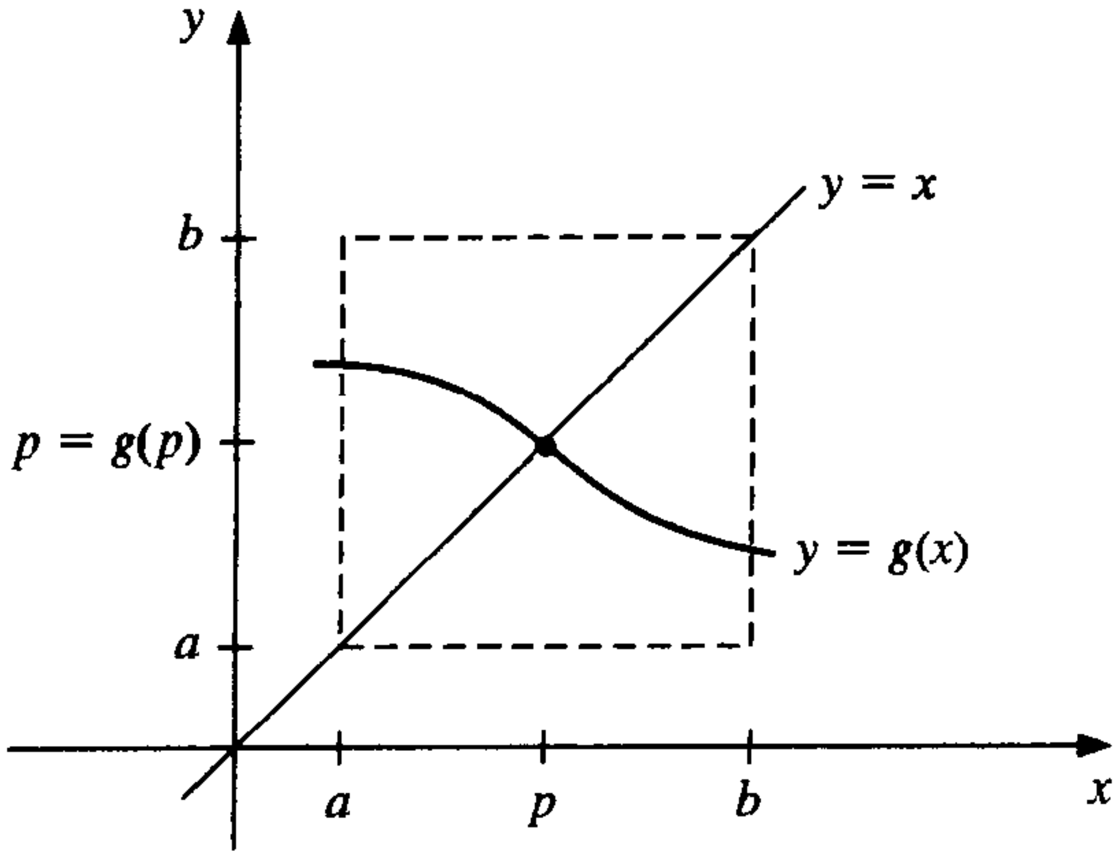
$$f(x) = x^3 + 4x^2 - 10$$

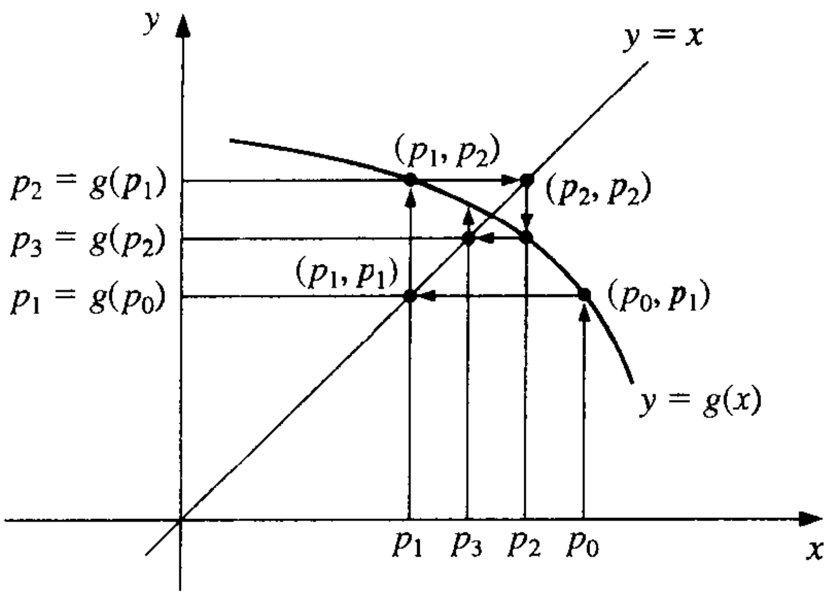
fonksiyonunun [1,2] aralığında kökünün olup olmadığına bakılacak:

f(1)= - 5 , f(2)= 14

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

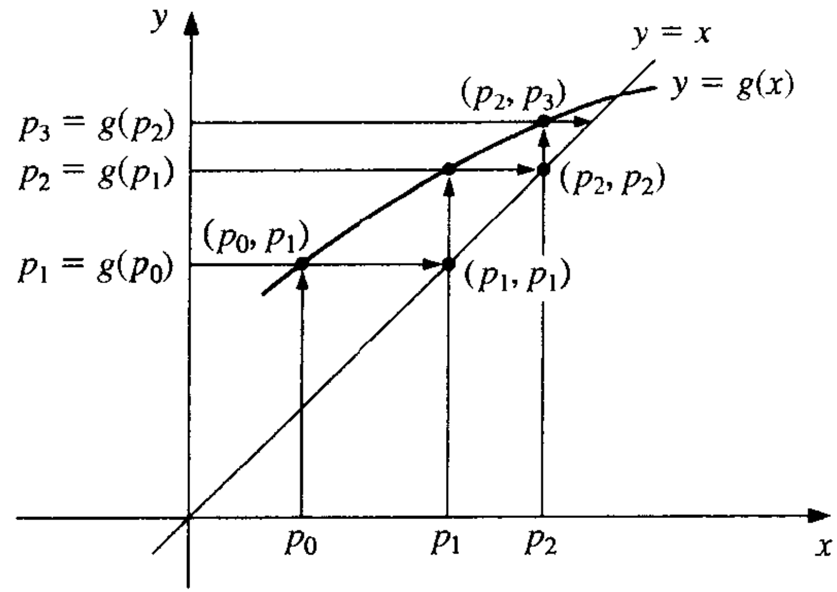
Sabit Nokta Metodu :





y
y = x
(p_1, p_2)
p_2 = g(p_1)
(p_2, p_2)
p_3 = g(p_2)
(p_1, p_1)
p_1 = g(p_0)
(p_0, p_1)
y = g(x)
p_1
p_3
p_2
p_0
x

(a)



y
y = x
(p_2, p_3)
y = g(x)
p_3 = g(p_2)
p_2 = g(p_1)
(p_2, p_2)
(p_0, p_1)
p_1 = g(p_0)
(p_1, p_1)
p_0
p_1
p_2
x

(b)

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. (*Compute p_i.*)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)

STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update p_0.*)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Örnek:

$$f(x) = x^3 + 4x^2 - 10$$

fonksiyonunun [1,2] aralığında kökünün olup olmadığına bakılacak:
f(1)= - 5 , f(2)= 14

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3),$$

$$x = \pm\frac{1}{2}(10 - x^3)^{1/2}.$$

$$f(x) = x^3 + 4x^2 - 10$$

a. $x = g_1(x) = x - x^3 - 4x^2 + 10$

b. $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

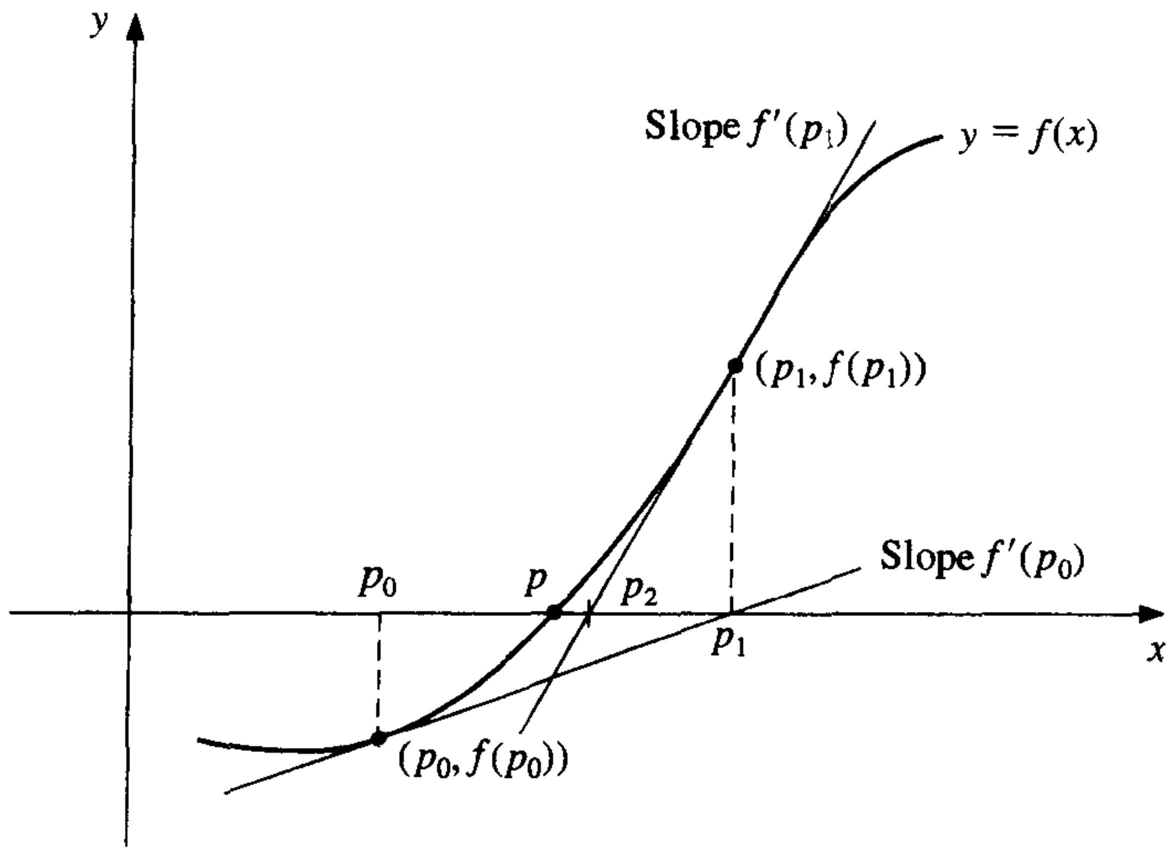
c. $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

d. $x = g_4(x) = \left(\frac{10}{4 + x}\right)^{1/2}$

e. $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

Newton-Raphson Metodu :



$$f(x) = f(\overline{x}) + (x - \overline{x})f'(\overline{x}) + \frac{(x - \overline{x})^2}{2} f''(\xi(x)),$$

where $\xi(x)$ lies between x and $\overline{x}$. Since $f(p) = 0$, this equation with $x = p$ gives

$$0 = f(\overline{x}) + (p - \overline{x})f'(\overline{x}) + \frac{(p - \overline{x})^2}{2} f''(\xi(p)).$$

Newton's method is derived by assuming that since $|p - \overline{x}|$ is small, the term involving $(p - \overline{x})^2$ is much smaller, so

$$0 \approx f(\overline{x}) + (p - \overline{x})f'(\overline{x}).$$

Solving for p gives

$$p \approx \overline{x} - \frac{f(\overline{x})}{f'(\overline{x})}.$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

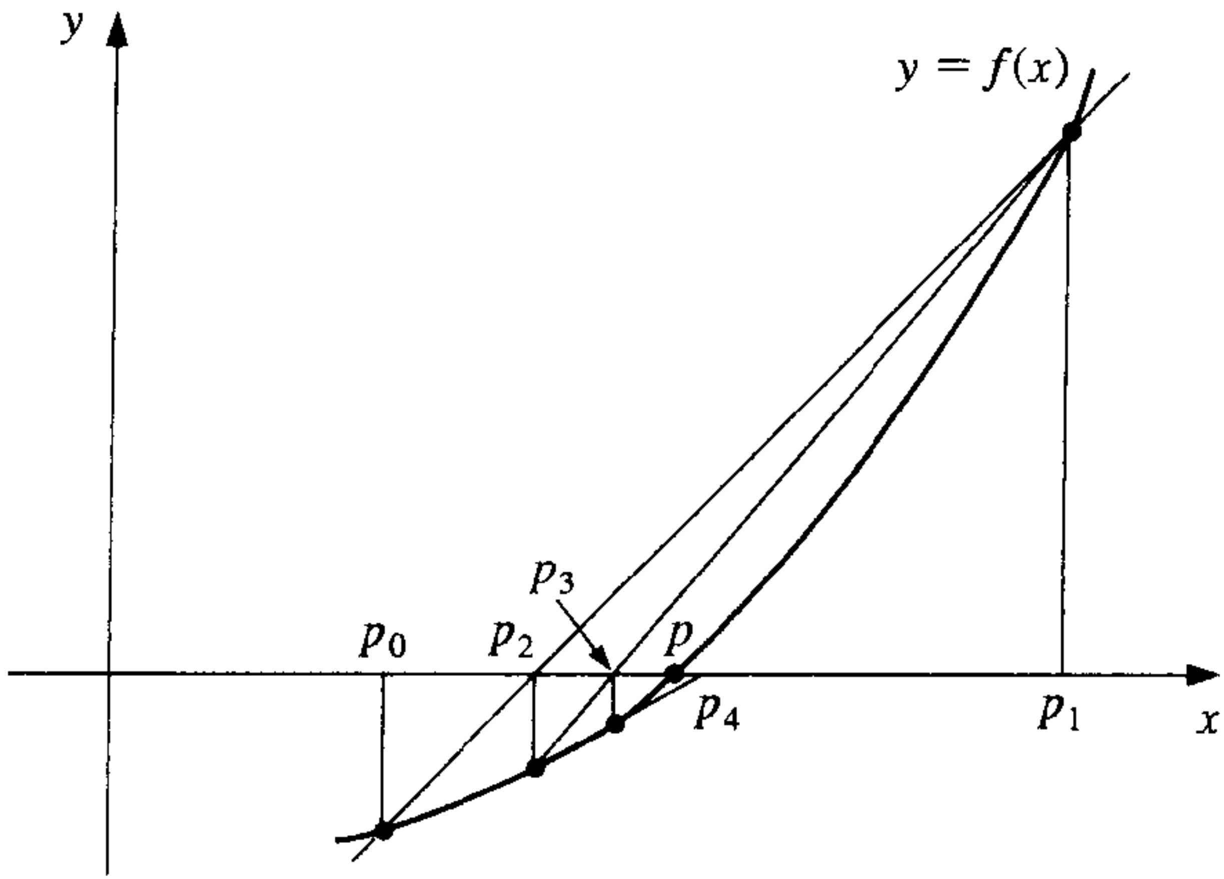
Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

To approach this problem differently, define $f(x) = \cos x - x$ and apply Newton's method. Since $f'(x) = -\sin x - 1$, the sequence is generated by

$$p_n = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}, \quad \text{for } n \geq 1.$$

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Secant Metodu :



INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute p_i.*)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update p_0, q_0, p_1, q_1.*)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

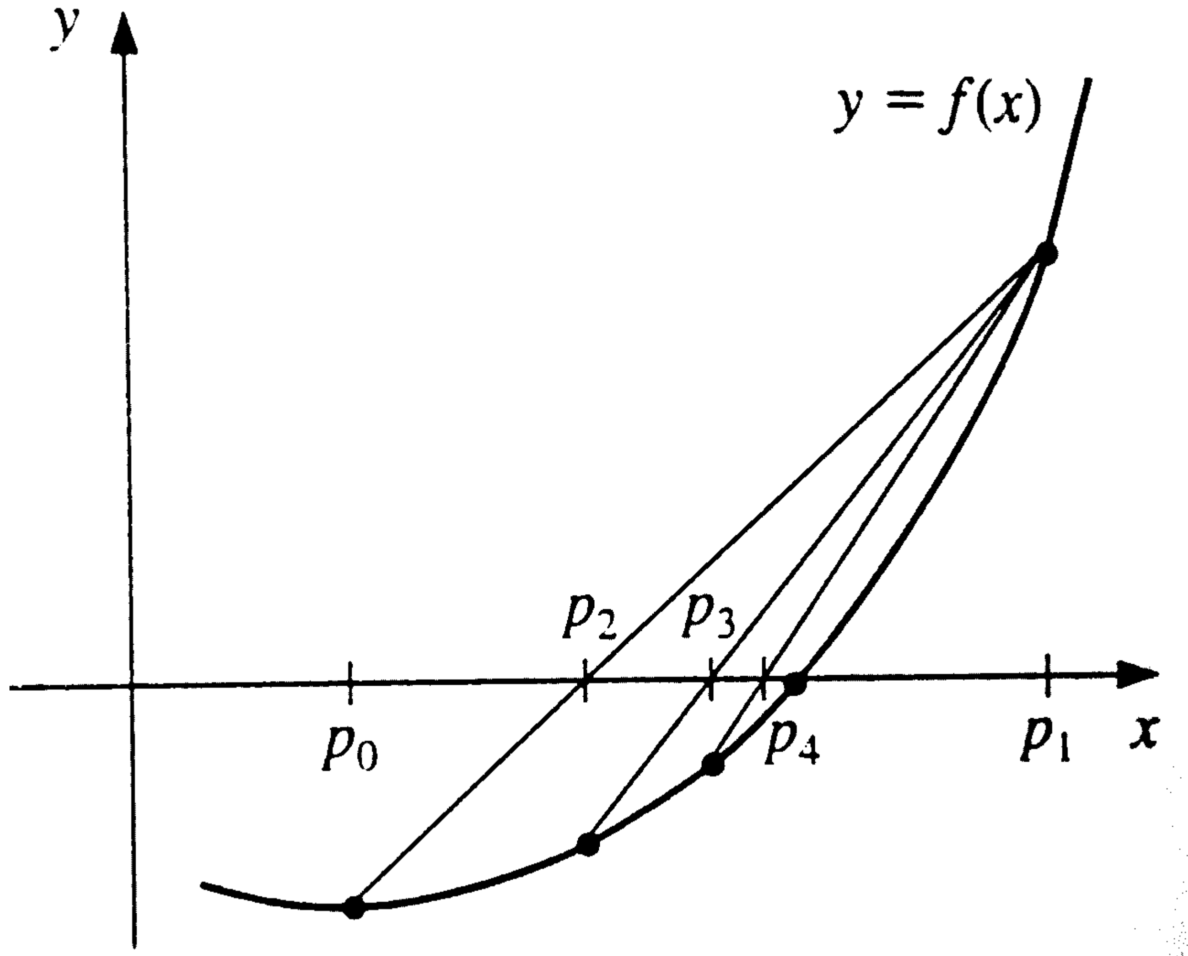
Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP. ■

Use the Secant method to find a solution to $x = \cos x$. In Example 1 we compared functional iteration and Newton's method with the initial approximation $p_0 = \pi/4$. Here we need two initial approximations. Table 2.5 lists the calculations with $p_0 = 0.5$, $p_1 = \pi/4$, and the formula

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2,$$

n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

Regula-Falsi Metodu :



INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–7.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$;
$q = f(p)$.

Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

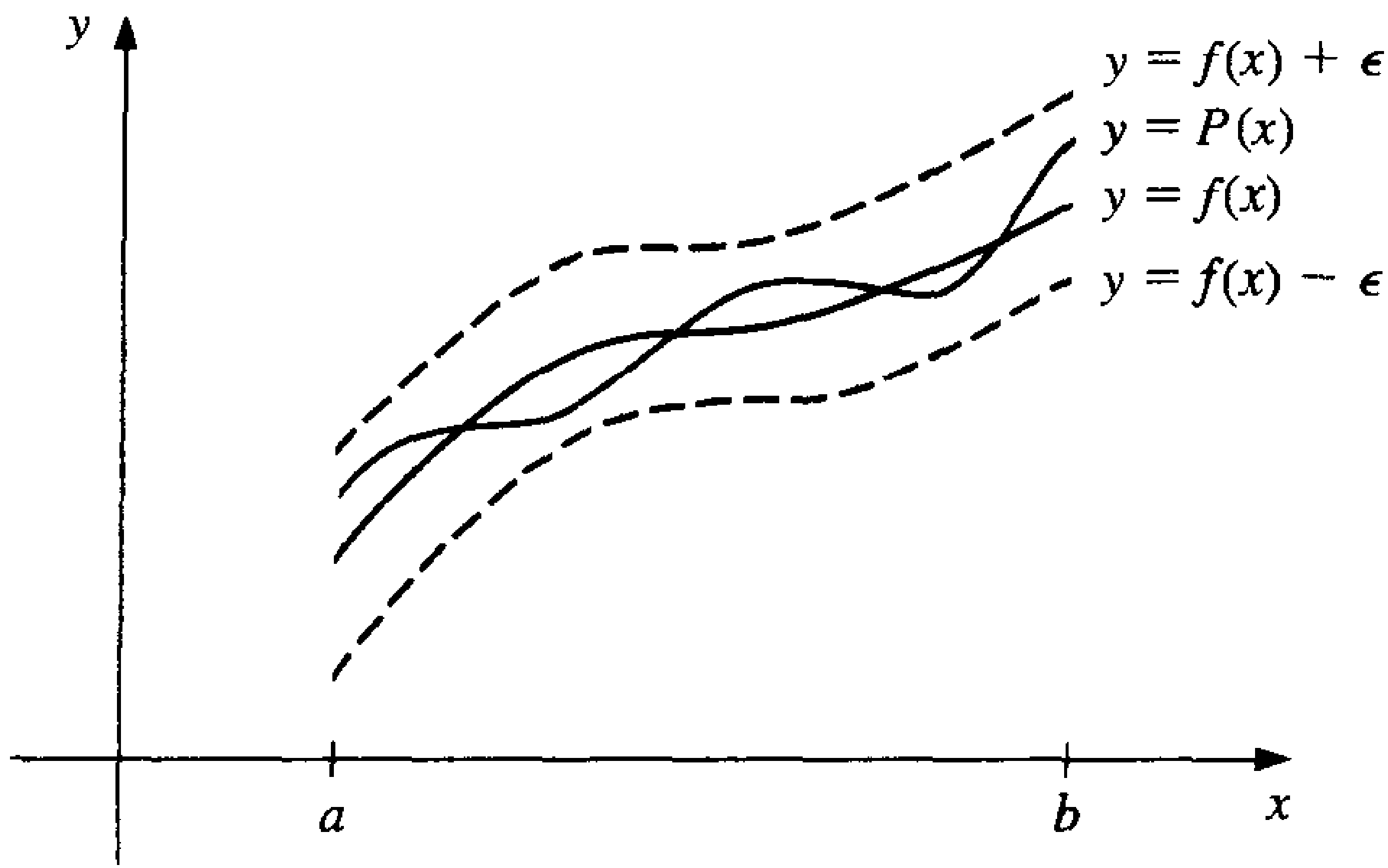
Step 7 Set $p_1 = p$;
$q_1 = q$.

Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure unsuccessful.*)
STOP.

$$f(x) = \cos x - x$$

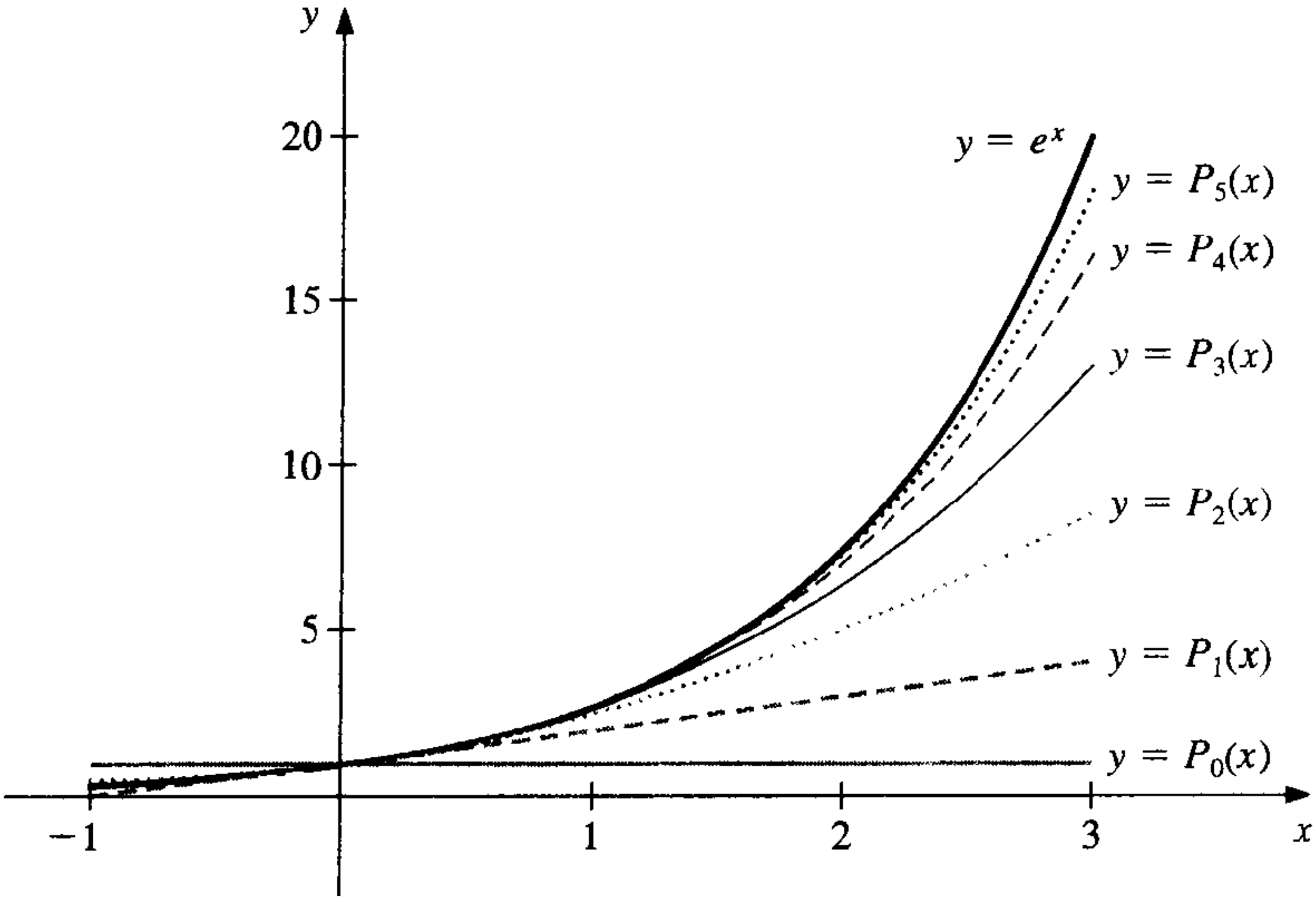
n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390848638
5	0.7390851305
6	0.7390851332

Interpolasyon



$$P_0(x) = 1, \quad P_1(x) = 1 + x, \quad P_2(x) = 1 + x + \frac{x^2}{2}, \quad P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6},$$

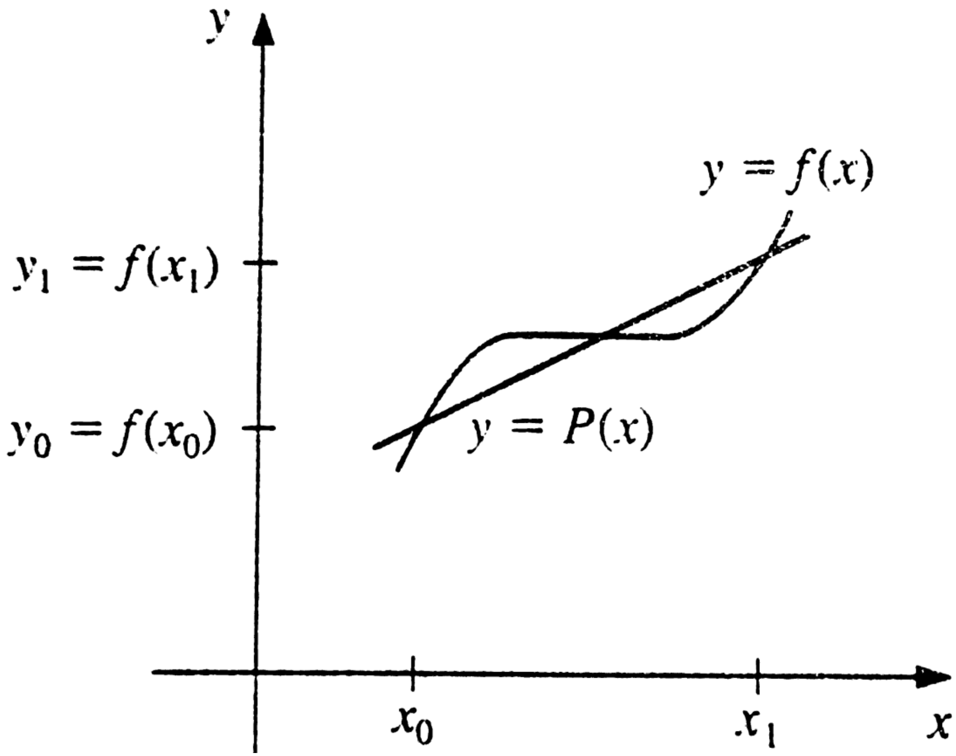
$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}, \quad \text{and} \quad P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$



y
20
15
10
5
−1
1
2
3
x
$y = e^x$
$y = P_5(x)$
$y = P_4(x)$
$y = P_3(x)$
$y = P_2(x)$
$y = P_1(x)$
$y = P_0(x)$

Interpolasyon

Lagrange Interpolasyonu :



$$P(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1.$$

$$P(x_0) = 1 \cdot y_0 + 0 \cdot y_1 = y_0 = f(x_0),$$

$$P(x_1) = 0 \cdot y_0 + 1 \cdot y_1 = y_1 = f(x_1),$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1).$$

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

x	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

$$
\begin{aligned}
P_2(1.5) &= \frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)}(0.6200860) + \frac{(1.5-1.3)(1.5-1.9)}{(1.6-1.3)(1.6-1.9)}(0.4554022) \\
&\quad + \frac{(1.5-1.3)(1.5-1.6)}{(1.9-1.3)(1.9-1.6)}(0.2818186) \\
&= 0.5112857,
\end{aligned}
$$

Newton'un Bölünmüş Farklar Metodu :

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots$$
$$+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

$$a_0 = P_n(x_0) = f(x_0).$$

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1); \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

$$f[x_i] = f(x_i).$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

$$f[x_i, x_{i+1}, \ldots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1)$$
$$+ \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

$$a_k = f[x_0, x_1, x_2, \ldots, x_k],$$

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1}).$$

Newton'un Bölünmüş Farklar Metodu :

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
			$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
x_1	$f[x_1]$			
		$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
			$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
x_2	$f[x_2]$			
		$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
			$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
x_3	$f[x_3]$			
		$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
			$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
x_4	$f[x_4]$			
		$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
x_5	$f[x_5]$			

Örnek : Aşağıdaki nokta değerleri verilmiş olsun.

x_i	$f[x_i]$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \ldots, x_i]$	$f[x_{i-4}, \ldots, x_i]$
0	1.0	0.7651977				
			−0.4837057			
1	1.3	0.6200860		−0.1087339		
			−0.5489460		0.0658784	
2	1.6	0.4554022		−0.0494433		0.0018251
			−0.5786120		0.0680685	
3	1.9	0.2818186		0.0118183		
			−0.5715210			
4	2.2	0.1103623				

$$
\begin{aligned}
P_4(x) = {} & 0.7651977 - 0.4837057(x - 1.0) - 0.1087339(x - 1.0)(x - 1.3) \\
& + 0.0658784(x - 1.0)(x - 1.3)(x - 1.6) \\
& + 0.0018251(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9).
\end{aligned}
$$

Newton’un İleri Bölünmüş Farklar Metodu :

		First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

Newton’un geri Bölünmüş Farklar Metodu :

		First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

Newton’un orta Bölünmüş Farklar Metodu :

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
x_{-2}	$f[x_{-2}]$				
		$f[x_{-2}, x_{-1}]$			
x_{-1}	$f[x_{-1}]$		$f[x_{-2}, x_{-1}, x_0]$		
		$f[x_{-1}, x_0]$		$f[x_{-2}, x_{-1}, x_0, x_1]$	
x_0	$f[x_0]$		$f[x_{-1}, x_0, x_1]$		$f[x_{-2}, x_{-1}, x_0, x_1, x_2]$
		$f[x_0, x_1]$		$f[x_{-1}, x_0, x_1, x_2]$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$			
x_2	$f[x_2]$				

x	$f(x)$	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

İleri Fark Formülü :

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$
$$+ \cdots + s(s-1)(s-n+1)h^n f[x_0, x_1, \ldots, x_n]$$
$$= \sum_{k=0}^{n} s(s-1)\cdots(s-k+1)h^k f[x_0, x_1, \ldots, x_k].$$

Geri Fark Formülü :

If the nodes are equally spaced with $x = x_n + sh$ and $x = x_i + (s+n-i)h$, then

$$P_n(x) = P_n(x_n + sh)$$
$$= f[x_n] + shf[x_n, x_{n-1}] + s(s+1)h^2 f[x_n, x_{n-1}, x_{n-2}] + \cdots$$
$$+ s(s+1)\cdots(s+n-1)h^n f[x_n, \ldots, x_0].$$

Orta Fark Formülü :

$$P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \qquad (3.14)$$

$$+ \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2])$$

$$+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m]$$

$$+ \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \ldots, x_m] + f[x_{-m}, \ldots . x_{m+1}]),$$

$$
\begin{aligned}
P_4(1.1) &= P_4\left(1.0 + \frac{1}{3}(0.3)\right) \\
&= 0.7651997 + \frac{1}{3}(0.3)(-0.4837057) + \frac{1}{3}\left(-\frac{2}{3}\right)(0.3)^2(-0.1087339) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(0.3)^3(0.0658784) \\
&\quad + \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(0.3)^4(0.0018251) \\
&= 0.7196480.
\end{aligned}
$$

$$
\begin{aligned}
P_4(2.0) &= P_4\left(2.2 - \frac{2}{3}(0.3)\right) \\
&= 0.1103623 - \frac{2}{3}(0.3)(-0.5715210) - \frac{2}{3}\left(\frac{1}{3}\right)(0.3)^2(0.0118183) \\
&\quad - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)(0.3)^3(0.0680685) - \frac{2}{3}\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)(0.3)^4(0.0018251) \\
&= 0.2238754.
\end{aligned}
$$

$$
\begin{aligned}
f(1.5) \approx P_4\left(1.6+\left(-\frac{1}{3}\right)(0.3)\right)
&= 0.4554022+\left(-\frac{1}{3}\right)\left(\frac{0.3}{2}\right)((-0.5489460)+(-0.5786120)) \\
&\quad +\left(-\frac{1}{3}\right)^2(0.3)^2(-0.0494433) \\
&\quad +\frac{1}{2}\left(-\frac{1}{3}\right)\left(\left(-\frac{1}{3}\right)^2-1\right)(0.3)^3(0.0658784+0.0680685) \\
&\quad +\left(-\frac{1}{3}\right)^2\left(\left(-\frac{1}{3}\right)^2-1\right)(0.3)^4(0.0018251) \\
&= 0.5118200.
\end{aligned}
$$

Hermit Metodu :

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x),$$

$$H_{n,j}(x) = [1 - 2(x - x_j) L'_{n,j}(x_j)] L^2_{n,j}(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j) L^2_{n,j}(x).$$

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \ldots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi), \qquad a < \xi < b.$$

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	−0.5220232
1	1.6	0.4554022	−0.5698959
2	1.9	0.2818186	−0.5811571

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}, \qquad L'_{2,0}(x) = \frac{100}{9}x - \frac{175}{9};$$

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}, \qquad L'_{2,1}(x) = \frac{-200}{9}x + \frac{320}{9};$$

$$L_{2,2} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}, \qquad L'_{2,2}(x) = \frac{100}{9}x - \frac{145}{9}.$$

$$H_{2,0}(x) = [1 - 2(x - 1.3)(-5)]\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

$$= (10x - 12)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2,$$

$$H_{2,1}(x) = 1 \cdot \left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right)^2,$$

$$H_{2,2}(x) = 10(2 - x)\left(\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}\right)^2,$$

$$\hat{H}_{2,0}(x) = (x - 1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2,$$

$$\hat{H}_{2,1}(x) = (x - 1.6)\left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right)^2,$$

$$\hat{H}_{2,2}(x) = (x - 1.9)\left(\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}\right)^2$$

$$
\begin{aligned}
H_5(x) = {} & 0.6200860 H_{2,0}(x) + 0.4554022 H_{2,1}(x) + 0.2818186 H_{2,2}(x) \\
& - 0.5220232 \hat{H}_{2,0}(x) - 0.5698959 \hat{H}_{2,1}(x) - 0.5811571 \hat{H}_{2,2}(x)
\end{aligned}
$$

$$
\begin{aligned}
H_5(1.5) &= 0.6200860 \left(\frac{4}{27}\right) + 0.4554022 \left(\frac{64}{81}\right) + 0.2818186 \left(\frac{5}{81}\right) \\
&\quad - 0.5220232 \left(\frac{4}{405}\right) - 0.5698959 \left(\frac{-32}{405}\right) - 0.5811571 \left(\frac{-2}{405}\right) \\
&= 0.5118277,
\end{aligned}
$$

INPUT numbers $x_0, x_1, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$ and $f'(x_0), \ldots, f'(x_n)$.

OUTPUT the numbers $Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1}$ where

$$\begin{aligned} H(x) = Q_{0,0} &+ Q_{1,1}(x - x_0) + Q_{2,2}(x - x_0)^2 + Q_{3,3}(x - x_0)^2(x - x_1) \\ &+ Q_{4,4}(x - x_0)^2(x - x_1)^2 + \cdots \\ &+ Q_{2n+1,2n+1}(x - x_0)^2(x - x_1)^2 \cdots (x - x_{n-1})^2(x - x_n). \end{aligned}$$

Step 1 For $i = 0, 1, \ldots, n$ do Steps 2 and 3.

Step 2 Set $z_{2i} = x_i$;

$z_{2i+1} = x_i$;

$Q_{2i,0} = f(x_i)$;

$Q_{2i+1,0} = f(x_i)$;

$Q_{2i+1,1} = f'(x_i)$.

Step 3 If $i \neq 0$ then set

$$Q_{2i,1} = \frac{Q_{2i,0} - Q_{2i-1,0}}{z_{2i} - z_{2i-1}}.$$

Step 4 For $i = 2, 3, \ldots, 2n + 1$

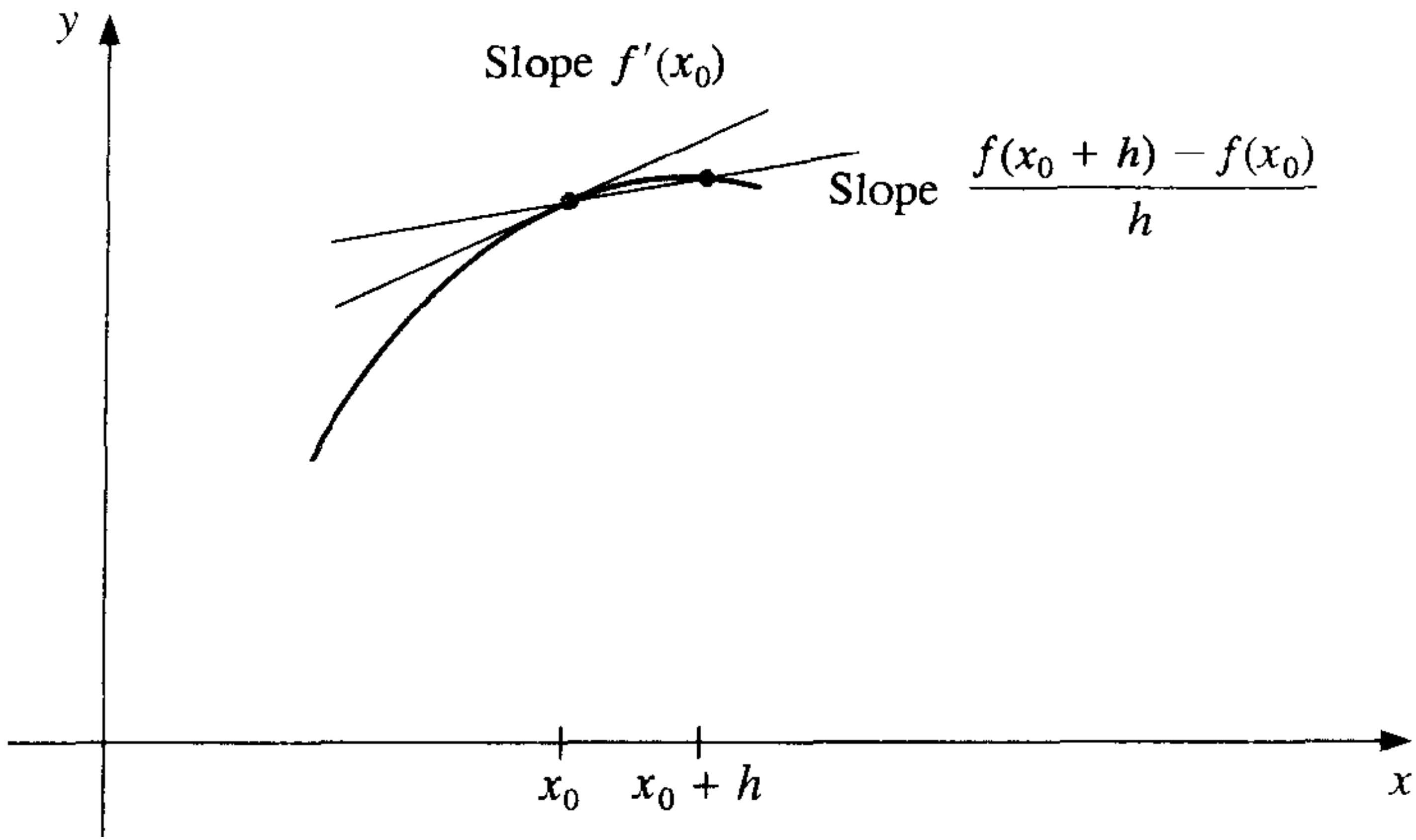
for $j = 2, 3, \ldots, i$ set $Q_{i,j} = \dfrac{Q_{i,j-1} - Q_{i-1,j-1}}{z_i - z_{i-j}}$.

Step 5 OUTPUT $(Q_{0,0}, Q_{1,1}, \ldots, Q_{2n+1,2n+1})$;

STOP

Sayısal Türev

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi).$$



Let $f(x) = \ln x$ and $x_0 = 1.8$. The forward-difference formula

$$\frac{f(1.8+h) - f(1.8)}{h}$$

is used to approximate $f'(1.8)$ with error

$$\frac{|hf''(\xi)|}{2} = \frac{|h|}{2\xi^2} \leq \frac{|h|}{2(1.8)^2}, \quad \text{where} \quad 1.8 < \xi < 1.8 + h.$$

h	$f(1.8+h)$	$\dfrac{f(1.8+h) - f(1.8)}{h}$	$\dfrac{\lvert h \rvert}{2(1.8)^2}$
0.1	0.64185389	0.5406722	0.0154321
0.01	0.59332685	0.5540180	0.0015432
0.001	0.58834207	0.5554013	0.0001543

$f'(x) = 1/x$, $\qquad$ $f'(1.8)$ is $0.55\overline{5}$, Using Eq. (4.3) with $x_j = x_0$, $x_1 = x_0 + h$, and $x_2 = x_0 + 2h$ gives

$$f'(x_0) = \frac{1}{h}\left[-\frac{3}{2}f(x_0) + 2f(x_1) - \frac{1}{2}f(x_2)\right] + \frac{h^2}{3}f^{(3)}(\xi_0).$$

Doing the same for $x_j = x_1$ gives

$$f'(x_1) = \frac{1}{h}\left[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_2)\right] - \frac{h^2}{6}f^{(3)}(\xi_1),$$

and for $x_j = x_2$,

$$f'(x_2) = \frac{1}{h}\left[\frac{1}{2}f(x_0) - 2f(x_1) + \frac{3}{2}f(x_2)\right] + \frac{h^2}{3}f^{(3)}(\xi_2).$$

Since $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$, these formulas can also be expressed as

$$f'(x_0) = \frac{1}{h}\left[-\frac{3}{2}f(x_0) + 2f(x_0+h) - \frac{1}{2}f(x_0+2h)\right] + \frac{h^2}{3}f^{(3)}(\xi_0),$$

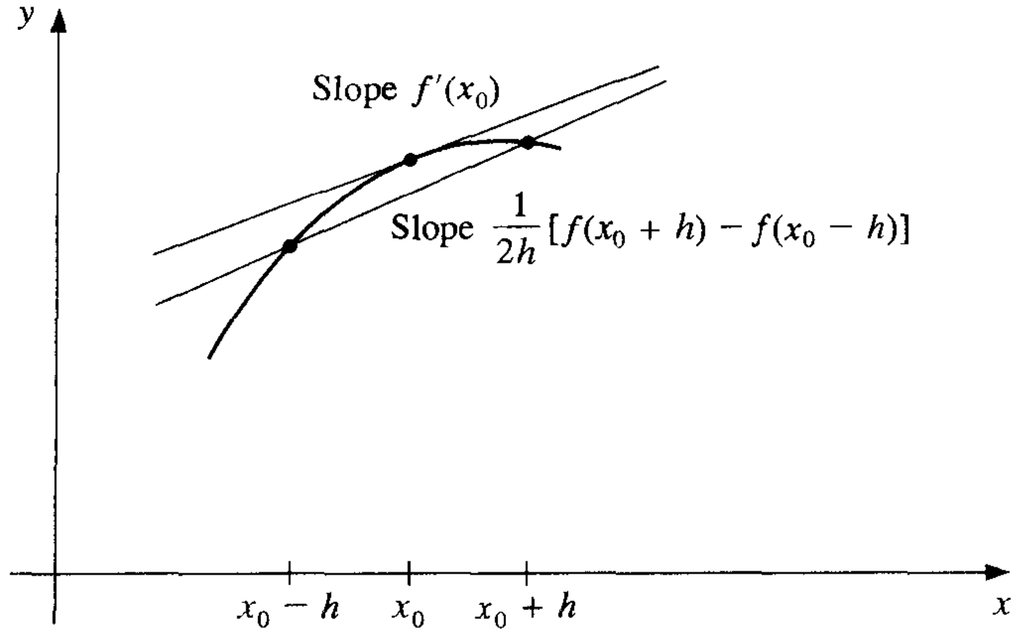
$$f'(x_0+h) = \frac{1}{h}\left[-\frac{1}{2}f(x_0) + \frac{1}{2}f(x_0+2h)\right] - \frac{h^2}{6}f^{(3)}(\xi_1), \quad \text{and}$$

$$f'(x_0+2h) = \frac{1}{h}\left[\frac{1}{2}f(x_0) - 2f(x_0+h) + \frac{3}{2}f(x_0+2h)\right] + \frac{h^2}{3}f^{(3)}(\xi_2).$$

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi_0),$$

$$f'(x_0) = \frac{1}{2h}[-f(x_0-h) + f(x_0+h)] - \frac{h^2}{6}f^{(3)}(\xi_1), \quad \text{and}$$

$$f'(x_0) = \frac{1}{2h}[f(x_0-2h) - 4f(x_0-h) + 3f(x_0)] + \frac{h^2}{3}f^{(3)}(\xi_2).$$



y
Slope $f'(x_0)$
Slope $\frac{1}{2h}[f(x_0 + h) - f(x_0 - h)]$
$x_0 - h$
x_0
$x_0 + h$
x

$$f(x) = xe^x$$

$$f'(x) = (x+1)e^x,$$

$$f'(2.0) = 22.167168$$

x	$f(x)$
1.8	10.889365
1.9	12.703199
2.0	14.778112
2.1	17.148957
2.2	19.855030

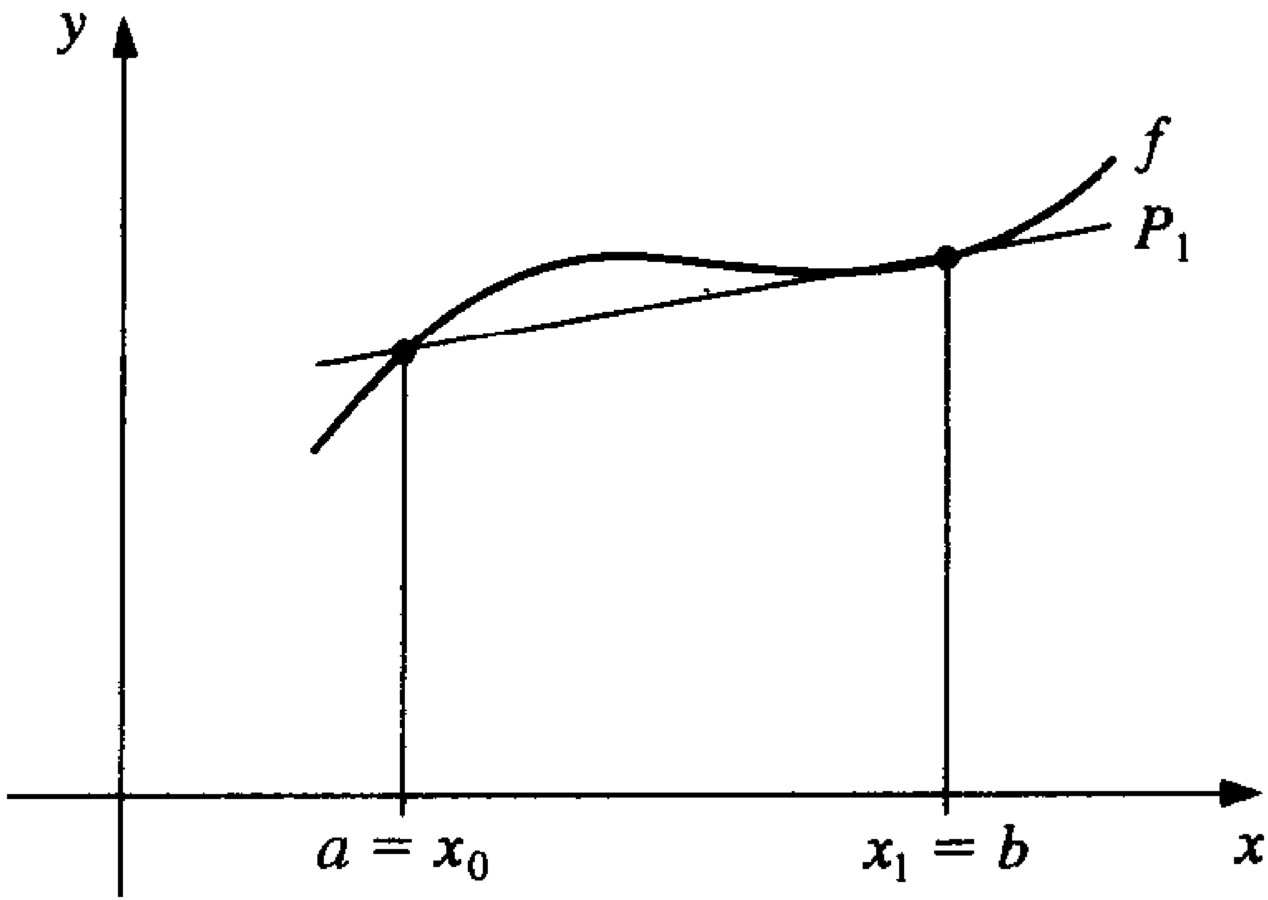
$$h = 0.1 : \tfrac{1}{0.2}[-3f(2.0) + 4f(2.1) - f(2.2)] = 22.032310,$$

$$h = -0.1 : \tfrac{1}{-0.2}[-3f(2.0) + 4f(1.9) - f(1.8)] = 22.054525,$$

$$h = 0.1 : \tfrac{1}{0.2}[f(2.1) - f(1.9)] = 22.228790,$$

$$h = 0.2 : \tfrac{1}{0.4}[f(2.2) - f(1.8)] = 22.414163.$$

Sayısal İntegral



$n = 1$: Trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \text{where} \quad x_0 < \xi < x_1.$$

$n = 2$: Simpson's rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi), \quad \text{where} \quad x_0 < \xi < x_2.$$

$n = 3$: Simpson's Three-Eighths rule

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi),$$

$$\text{where} \quad x_0 < \xi < x_3.$$

$n = 4$:

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8h^7}{945} f^{(6)}(\xi),$$

$$\text{where} \quad x_0 < \xi < x_4.$$

$$\int_0^4 e^x\,dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

$$\begin{aligned}\int_0^4 e^x\,dx &= \int_0^2 e^x\,dx + \int_2^4 e^x\,dx \\ &\approx \frac{1}{3}[e^0 + 4e + e^2] + \frac{1}{3}[e^2 + 4e^3 + e^4] \\ &= \frac{1}{3}[e^0 + 4e + 2e^2 + 4e^3 + e^4] \\ &= 53.86385.\end{aligned}$$

$$\begin{aligned}\int_0^4 e^x\,dx &= \int_0^1 e^x\,dx + \int_1^2 e^x\,dx + \int_2^3 e^x\,dx + \int_3^4 e^x\,dx \\ &\approx \frac{1}{6}[e_0 + 4e^{1/2} + e] + \frac{1}{6}[e + 4e^{3/2} + e^2] \\ &\quad + \frac{1}{6}[e^2 + 4e^{5/2} + e^3] + \frac{1}{6}[e^3 + 4e^{7/2} + e^4] \\ &= \frac{1}{6}[e^0 + 4e^{1/2} + 2e + 4e^{3/2} + 2e^2 + 4e^{5/2} + 2e^3 + 4e^{7/2} + e^4] \\ &= 53.61622.\end{aligned}$$

Composite Simpson's Rule

To approximate the integral $I = \int_a^b f(x)\,dx$:

INPUT endpoints a, b; even positive integer n.

OUTPUT approximation XI to I.

Step 1 Set $h = (b - a)/n$.

Step 2 Set $XI0 = f(a) + f(b)$;
$XI1 = 0$; *(Summation of $f(x_{2i-1})$.)*
$XI2 = 0$. *(Summation of $f(x_{2i})$.)*

Step 3 For $i = 1, \ldots, n - 1$ do Steps 4 and 5.

Step 4 Set $X = a + ih$.

Step 5 If i is even then set $XI2 = XI2 + f(X)$
else set $XI1 = XI1 + f(X)$.

Step 6 Set $XI = h(XI0 + 2 \cdot XI2 + 4 \cdot XI1)/3$.

Step 7 OUTPUT (XI);
STOP.

Adi Diferansiyel Denklemler

Euler Metodu :

$y'=f(t,y)$, $[t_0,t_M]$, $y(t_0)=y_0$

$t_k=a+kh$, $k=0,1,2,\ldots,M$, $h=(b-a)/M$

$y(t)$ fonksiyonu $t=t_0$ noktasında Taylor serisine açıldığında :

$$y(t) = y(t_0) + y'(t_0)(t-t_0) + \frac{y''(c_1)(t-t_0)^2}{2}$$

$y'(t_0)=f(t_0,y(t_0))$, $h=t_1-t_0$, değerlerini yerine yazarak

$$y(t_1) = y(t_0) + hf(t_0, y(t_0)) + y''(c_1)\frac{h^2}{2}$$

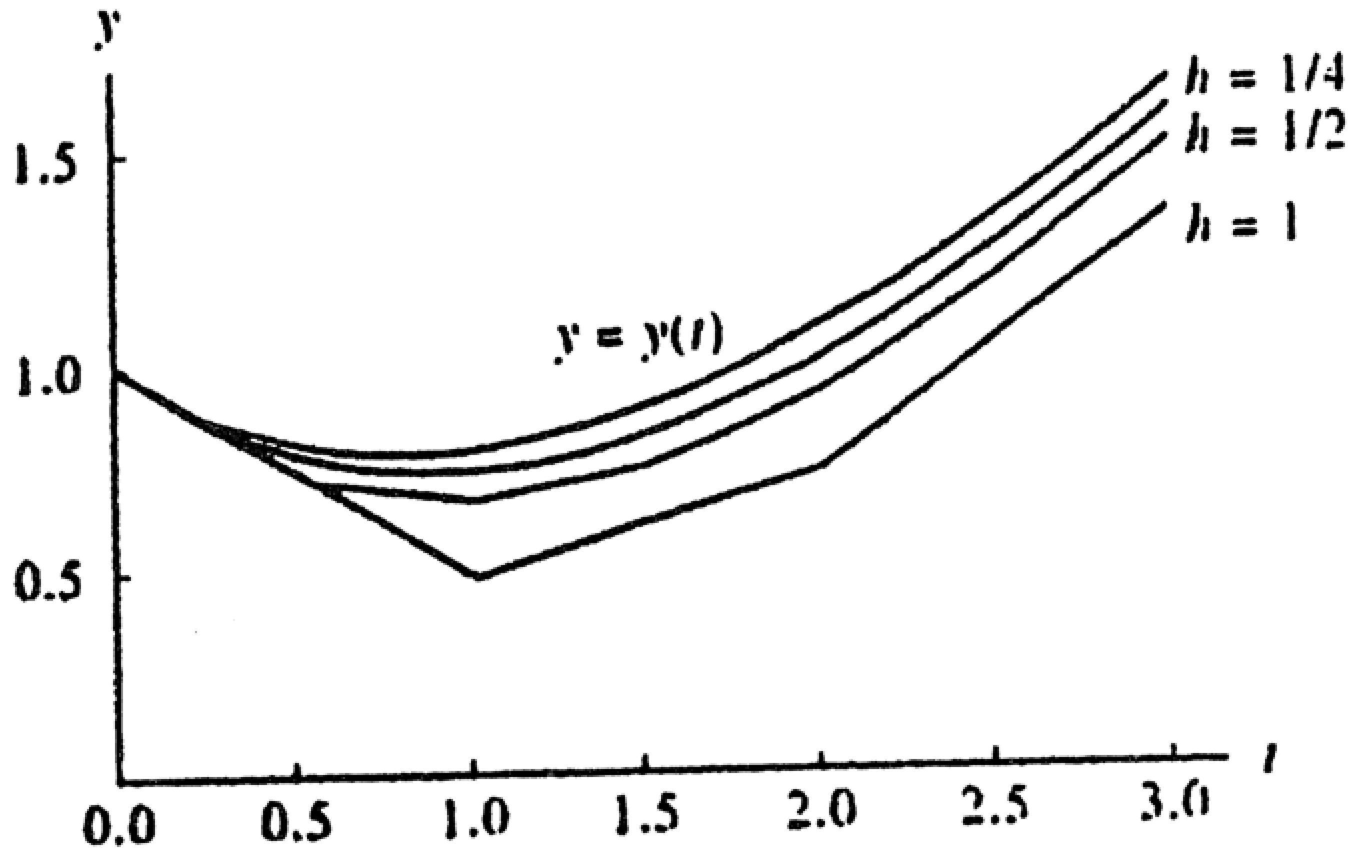
elde edilir.

h yeterince küçük seçilirse ikinci türeve sahip terim ihmal edilebilir.
Bu durumda :

$$y_1 = y_0 + hf(t_0, y_0).$$

$$y' = \frac{t - y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

$h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.



y
1.5
1.0
0.5
h = 1/4
h = 1/2
h = 1
y = y(t)
t
0.0
0.5
1.0
1.5
2.0
2.5
3.0

$$y_1 = 1.0 + 0.25\left(\frac{0.0 - 1.0}{2}\right) = 0.875.$$

$$y_2 = 0.875 + 0.25\left(\frac{0.25 - 0.875}{2}\right) = 0.796875. \qquad \text{etc.}$$

$$y(3) \approx y_{12} = 1.440573 + 0.25\left(\frac{2.75 - 1.440573}{2}\right) = 1.604252.$$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9375	0.943239
0.25			0.875	0.886719	0.897491
0.375				0.846924	0.862087
0.50		0.75	0.796875	0.817429	0.836402
0.75			0.759766	0.786802	0.811868
1.00	0.5	0.6875	0.758545	0.790158	0.819592
1.50		0.765625	0.846386	0.882855	0.917100
2.00	0.75	0.949219	1.030827	1.068222	1.103638
2.50		1.211914	1.289227	1.325176	1.359514
3.00	1.375	1.533936	1.604252	1.637429	1.669390

Heun Metodu :

$$y'(t) = f(t, y(t)) \quad \text{over} \quad [a, b] \quad \text{with} \quad y(t_0) = y_0.$$

$$\int_{t_0}^{t_1} f(t, y(t))\, dt = \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0).$$

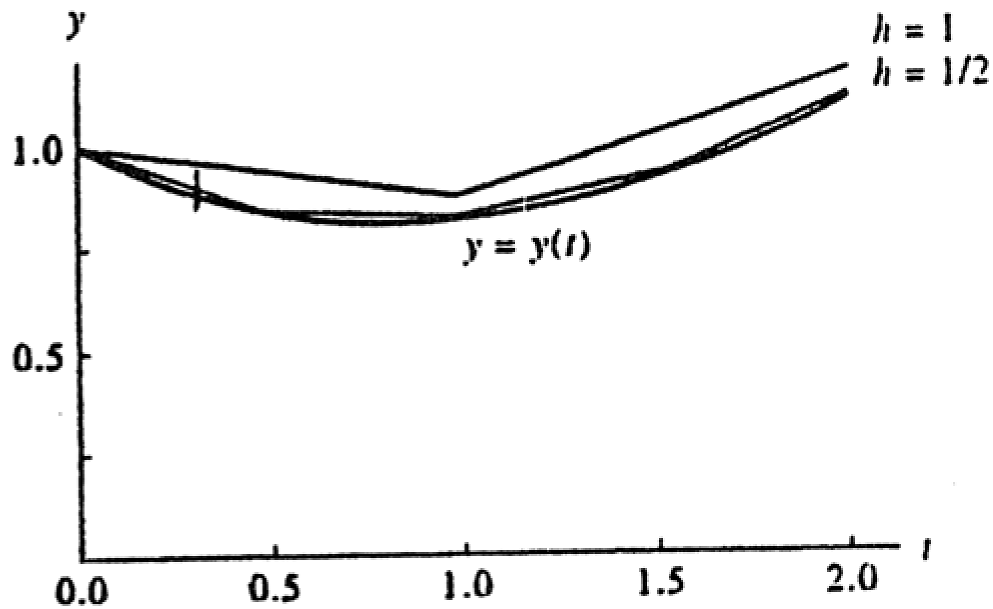
$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(t, y(t))\, dt.$$

$$y(t_1) \approx y(t_0) + \frac{h}{2}(f(t_0, y(t_0)) + f(t_1, y(t_1))).$$

$$y_1 = y(t_0) + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))).$$

$$p_{k+1} = y_k + hf(t_k, y_k). \qquad t_{k+1} = t_k + h.$$

$$y_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, p_{k+1})).$$



y
h = 1
h = 1/2
1.0
y = y(t)
0.5
0.0
0.5
1.0
1.5
2.0
t

$$y' = \frac{t - y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

$$f(t_0, y_0) = \frac{0 - 1}{2} = -0.5$$

$$p_1 = 1.0 + 0.25(-0.5) = 0.875.$$

$$f(t_1, p_1) = \frac{0.25 - 0.875}{2} = -0.3125.$$

$$y_1 = 1.0 + 0.125(-0.5 - 0.3125) = 0.8984375.$$

$$y(3) \approx y_{12} = 1.511508 + 0.125(0.619246 + 0.666840) = 1.672269.$$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.943359	0.943239
0.25			0.898438	0.897717	0.897491
0.375				0.862406	0.862087
0.50		0.84375	0.838074	0.836801	0.836402
0.75			0.814081	0.812395	0.811868
1.00	0.875	0.831055	0.822196	0.820213	0.819592
1.50		0.930511	0.920143	0.917825	0.917100
2.00	1.171875	1.117587	1.106800	1.104392	1.103638
2.50		1.373115	1.362593	1.360248	1.359514
3.00	1.732422	1.682121	1.672269	1.670076	1.669390

Taylor Metodu :

$$P = \left(\frac{\partial}{\partial t} + f \frac{\partial}{\partial y} \right)$$

$$y_{k+1} = y_k + d_1 h + \frac{d_2 h^2}{2!} + \frac{d_3 h^3}{3!} + \cdots + \frac{d_N h^N}{N!},$$

$$y' = \frac{t - y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

$$y'(t) = \frac{t - y}{2}.$$

$$y^{(2)}(t) = \frac{d}{dt}\left(\frac{t - y}{2}\right) = \frac{1 - y'}{2} = \frac{1 - (t - y)/2}{2} = \frac{2 - t + y}{4}.$$

$$y^{(3)}(t) = \frac{d}{dt}\left(\frac{2 - t + y}{4}\right) = \frac{0 - 1 + y'}{4} = \frac{-1 + (t - y)/2}{4} = \frac{-2 + t - y}{8},$$

$$y^{(4)}(t) = \frac{d}{dt}\left(\frac{-2 + t - y}{8}\right) = \frac{-0 + 1 - y'}{8} = \frac{1 - (t - y)/2}{8} = \frac{2 - t + y}{16}.$$

$$d_1 = y'(0) = \frac{0.0 - 1.0}{2} = -0.5.$$

$$d_2 = y^{(2)}(0) = \frac{2.0 - 0.0 + 1.0}{4} = 0.75.$$

$$d_3 = y^{(3)}(0) = \frac{-2.0 + 0.0 - 1.0}{8} = -0.375.$$

$$d_4 = y^{(4)}(0) = \frac{2.0 - 0.0 + 1.0}{16} = 0.1875.$$

$$y_1 = 1.0 + 0.25\left(-0.5 + 0.25\left(\frac{0.75}{2} + 0.25\left(\frac{-0.375}{6} + 0.25\left(\frac{0.1875}{24}\right)\right)\right)\right)$$
$$= 0.8974915.$$

$$d_1 = y'(0.25) = \frac{0.25 - 0.8974915}{2} = -0.3237458.$$

$$d_2 = y^{(2)}(0.25) = \frac{2.0 - 0.25 + 0.8974915}{4} = 0.6618729.$$

$$d_3 = y^{(3)}(0.25) = \frac{-2.0 + 0.25 - 0.8974915}{8} = -0.3309364.$$

$$d_4 = y^{(4)}(0.25) = \frac{2.0 - 0.25 + 0.8974915}{16} = 0.1654682.$$

$$y_2 = 0.8974915 + 0.25\Bigg(-0.3237458 + 0.25\left(\frac{0.6618729}{2} + 0.25\left(\frac{-0.3309364}{6} + 0.25\left(\frac{0.1654682}{24}\right)\right)\right)\Bigg)$$
$$= 0.8364037.$$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

Runge-Kutta Metodu :

$$y_{k+1} = y_k + \frac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}.$$

$$f_1 = f(t_k, y_k),$$

$$f_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right),$$

$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right).$$

$$f_4 = f(t_k + h, y_k + h f_3).$$

$$y' = \frac{t - y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

$$f_1 = \frac{0.0 - 1.0}{2} = -0.5,$$

$$f_2 = \frac{0.125 - (1 + 0.25(0.5)(-0.5))}{2} = -0.40625,$$

$$f_3 = \frac{0.125 - (1 + 0.25(0.5)(-0.40625))}{2} = -0.4121094,$$

$$f_4 = \frac{0.25 - (1 + 0.25(-0.4121094))}{2} = -0.3234863,$$

$$y_1 = 1.0 + 0.25\left(\frac{-0.5 + 2(-0.40625) + 2(-0.4121094) - 0.3234863}{6}\right)$$

$$= 0.8974915.$$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

Matrix Factorization

$$\begin{aligned}
x_1 + x_2 \phantom{{}-x_3} + 3x_4 &= 8 \\
2x_1 + x_2 - x_3 + x_4 &= 7 \\
3x_1 - x_2 - x_3 + 2x_4 &= 14 \\
-x_1 + 2x_2 + 3x_3 - x_4 &= \text{-}7
\end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.$$

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix},$$

$$LU\mathbf{x} = L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

$$\begin{aligned} y_1 &= 8; \\ 2y_1 + y_2 &= 7, \quad \text{so } y_2 = 7 - 2y_1 = -9; \\ 3y_1 + 4y_2 + y_3 &= 14, \quad \text{so } y_3 = 14 - 3y_1 - 4y_3 = 26; \\ -y_1 - 3y_2 + y_4 &= -7, \quad \text{so } y_4 = -7 + y_1 + 3y_2 = -26. \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}.$$

$$x_4 = 2,\ x_3 = 0,\ x_2 = -1,\ x_1 = 3.$$

Örnek :

$$\begin{aligned} x_1 + \ 2x_2 + \ 4x_3 + \ x_4 &= 21 \\ 2x_1 + \ 8x_2 + \ 6x_3 + 4x_4 &= 52 \\ 3x_1 + 10x_2 + \ 8x_3 + 8x_4 &= 79 \\ 4x_1 + 12x_2 + 10x_3 + 6x_4 &= 82. \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 8 & 6 & 4 \\ 3 & 10 & 8 & 8 \\ 4 & 12 & 10 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix} = LU.$$

$$\begin{aligned} y_1 \qquad\qquad\qquad &= 21 \\ 2y_1 + y_2 \qquad\qquad &= 52 \\ 3y_1 + y_2 + \ y_3 \qquad &= 79 \\ 4y_1 + y_2 + 2y_3 + y_4 &= 82. \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + \ x_4 &= \ 21 \\ 4x_2 - 2x_3 + 2x_4 &= \ 10 \\ -2x_3 + 3x_4 &= \ 6 \\ -6x_4 &= -24. \end{aligned}$$

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}'.$$