Theme Examples (1.10)

DT Moving Average Systems (1.10.4 @ p 75)

This DT-system is used to identify the underlying "trend" within fluctuating data.

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$$

N: degree of smoothing.

Generalized version

$$y[n] = \sum_{k=0}^{N-1} \alpha_k \cdot x[n-k]$$

α_k → weights.

Recursive DT Computation (1.10.6 @ p 79)

In this form of computation the current value of the system's output depend on

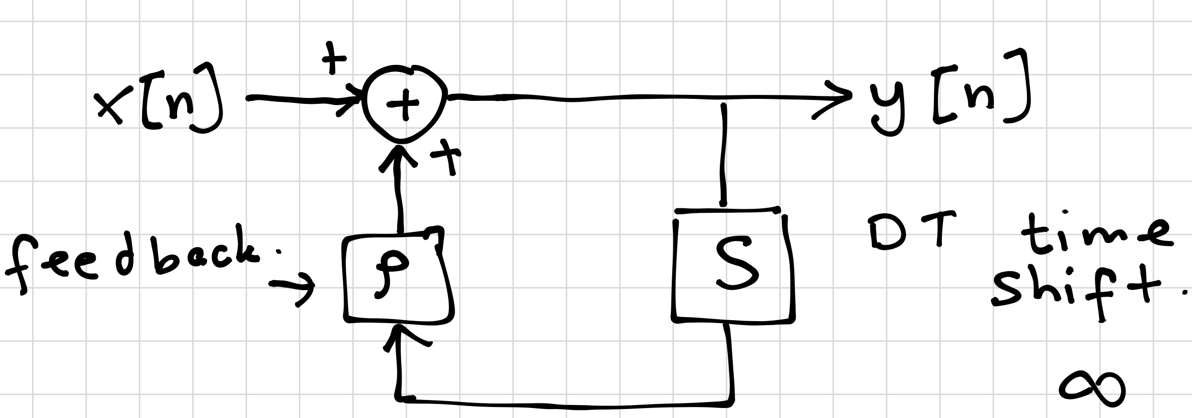
(1) The current and/or past values of the input signal.

(2) The past values of the output.

- Example: 1st order recursive DT filter.

$$y[n] = x[n] - \rho \cdot y[n-1]$$

ρ → feedback coefficient.



The solution : $y[n] = \sum_{k=0}^{\infty} \rho^k \, x[n-k]$

Proof :

$$y[n] = \rho^0 \cdot x[n] + \sum_{k=1}^{\infty} \rho^k \cdot x[n-k]$$

Let $k = m+1$

$$y[n] = x[n] + \sum_{m=0}^{\infty} \rho^{m+1} \cdot x[n-1-m]$$

$$= x[n] + \rho \underbrace{\sum_{m=0}^{\infty} \rho^m \, x[n-1-m]}_{y[n-1]}$$

$$y[n] = x[n] + \rho \, y[n-1] \qquad \ominus$$

Depending on ρ :

1. $\rho = 1 \Rightarrow y[n] = \sum_{k=0}^{\infty} x[n-k]$

   This is called an accumulator, (DT integrator).

2. $|\rho| < 1 \Rightarrow$ leaky accumulator.

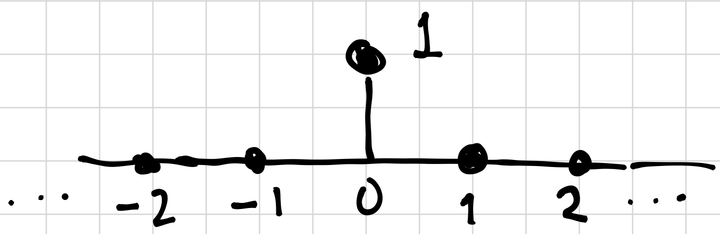
3. $|\rho| > 1 \Rightarrow y[n]$ is amplified.

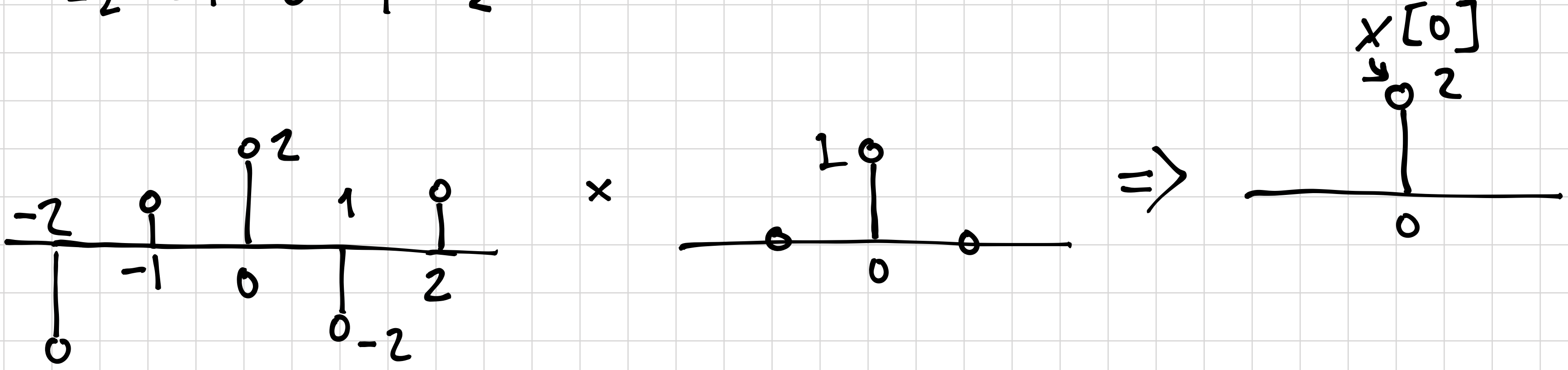
Linear Time-Invariant Systems in Time Domain

The Convolution Sum.

Impulse Signal

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



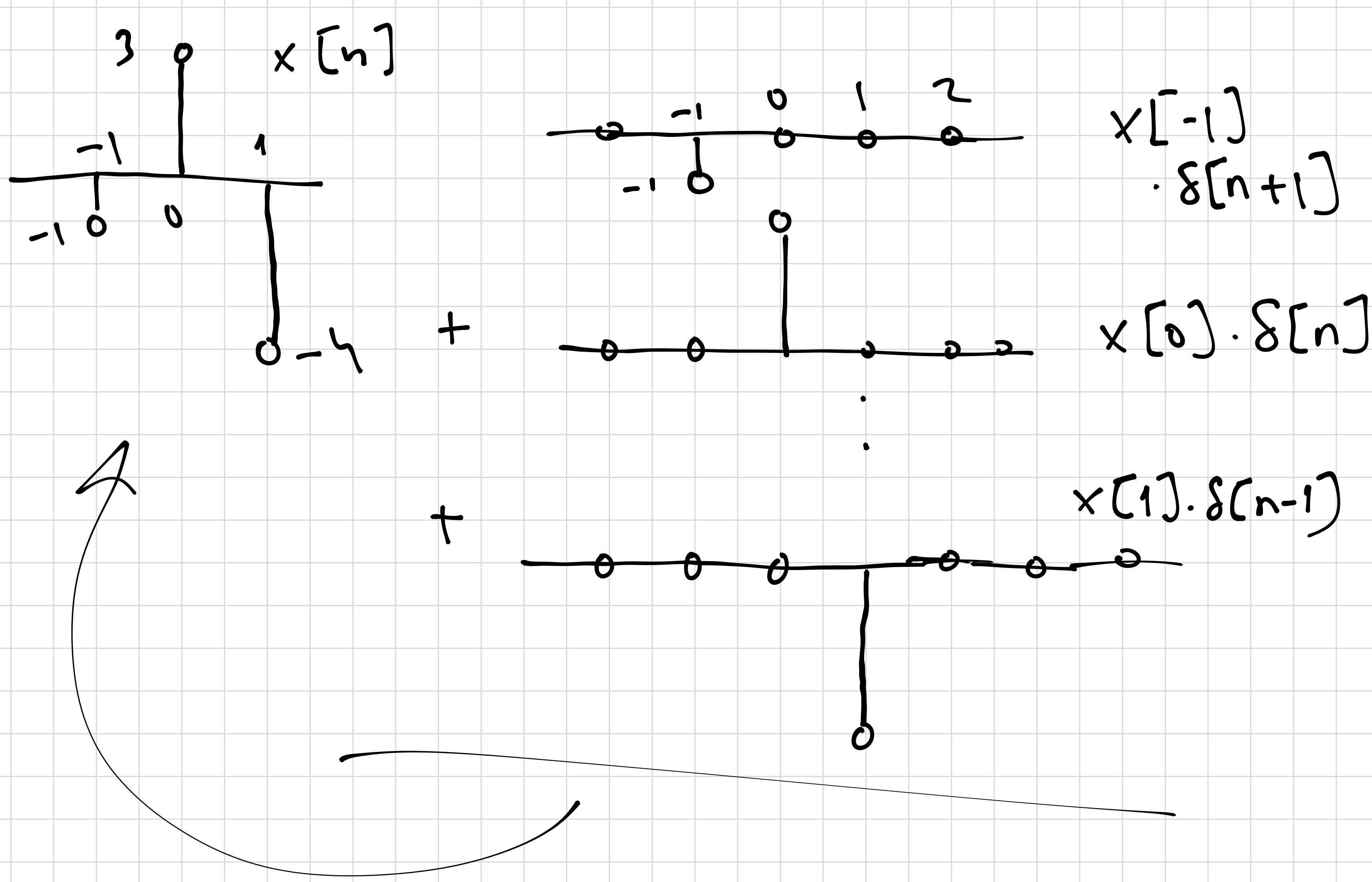


$$x[n] \cdot \delta[n] = x[0]\,\delta[n]$$

Similarly

$$x[n]\,\delta[n-k] = x[k] \cdot \delta[n-k]$$

- n is time index, thus $x[n]$ denotes the entire signal, while $x[k]$ denotes the specific value of $x[n]$ at time k



$$x[n] = \dots + x[-2]\cdot\delta[n+2] + x[-1]\cdot\delta[n+1] + x[0]\cdot\delta[n] + x[1]\cdot\delta[n-1] + \dots$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\cdot\delta[n-k]$$

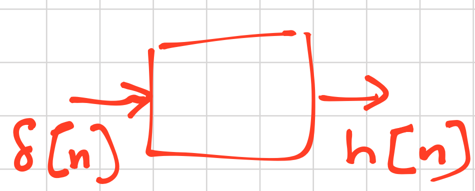
For the system $\mathcal{H}$,

$$y[n] = \mathcal{H}\{x[n]\} = \mathcal{H}\left\{\sum_{k=-\infty}^{+\infty} x[k]\cdot\delta[n-k]\right\}$$

If H is linear,

$$y[n] = \sum_{k=-\infty}^{+\infty} \mathcal{H}\{x[k]\cdot\delta[n-k]\}$$ (Superposition)

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\cdot\mathcal{H}\{\delta[n-k]\}$$



Let's say $h[n-k] \triangleq \mathcal{H}\{\delta[n-k]\}$

$\Rightarrow h[n] = \mathcal{H}\{\delta[n]\}$ → Impulse response

Thus:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k]$$

$$y[n] = x[n] * \underline{h[n]}$$

Ex

$$y[n] = x[n] + \frac{1}{2} x[n-1]$$

a) Impulse response?

set $x[n] = \delta[n]$

$$h[n] = \delta[n] + \frac{1}{2}\delta[n-1] = \begin{cases} 1, & n=0 \\ 1/2, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

b) $$x[n] = \begin{cases} 2, & n=0 \\ 4, & n=1 \\ -2, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

$$= 2 \cdot \delta[n] + 4\,\delta[n-1] - 2\,\delta[n-2]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$= x[0] \cdot h[n-0] + x[1]\, h[n-1] + x[2] \cdot h[n-2]$$

$$= 2\,\underline{h[n]} + 4\,h[n-1] - 2\,h[n-2]$$

	0	1	2	3	
	2	1			$2h[n]$
		4	2		$4h[n-1]$
			-2	-1	$-2h[n-2]$
+ ...	2	5	0	-1	

$$y[n] = \begin{cases} 0, & n < 0 \\ 2, & n = 2 \\ 5, & n = 1 \\ 0, & n = 2 \\ -1, & n = 3 \\ 0, & n > 3 \end{cases}$$

Convolution Sum Evaluation Procedure.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]\cdot h[n-k]}$$

we define an intermediate signal

$$w_n[k] = x[k]\cdot h[n-k]$$

$\hookrightarrow$ k is the independent variable of the intermediate signal.

$$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$$

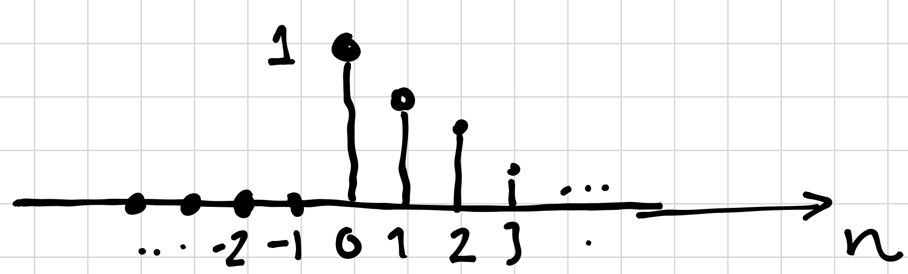
Ex For an LTI system, $\mathcal{H}$, the impulse response is given as :

$$h[n] = \left(\frac{3}{4}\right)^n \cdot u[n],$$

The input signal is :

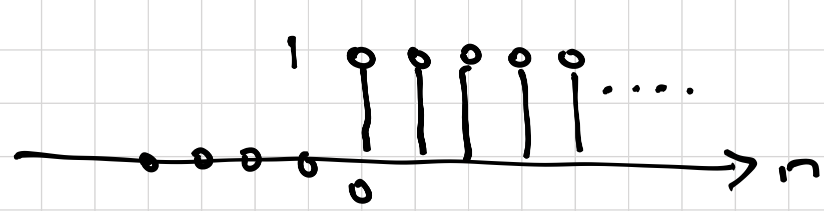
$$x[n] = u[n]$$

Determine $y[-5]$, $y[5]$ and $y[10]$



h[n]

n ↓ h[n] → h[n−k] ← k



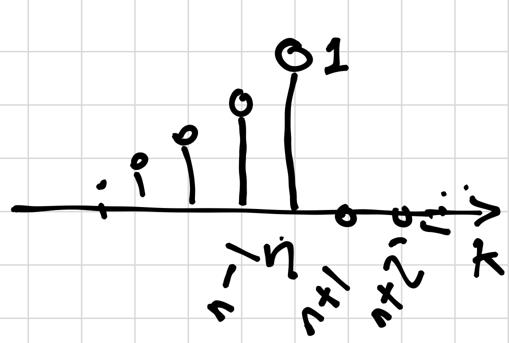
x[n]

$$- \; \underline{\underline{w_n[k]}} = x[k]\cdot h[n-k]$$

$$h[n-\underline{k}] = \left(\frac{3}{4}\right)^{n-k} \cdot u[n-k]$$

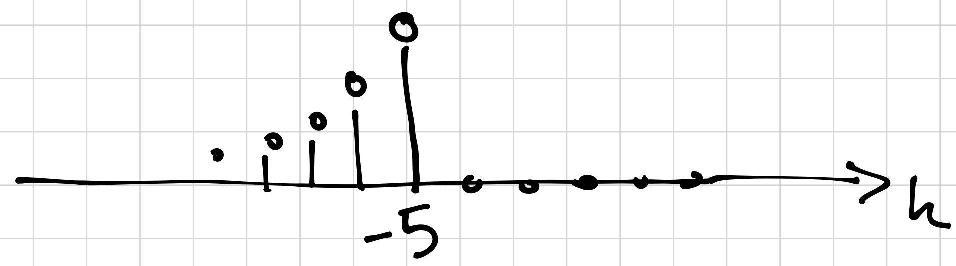
$$= \begin{cases} (3/4)^{n-k}, & k \le n \\ 0, & k > n \end{cases}$$

h[n−k]

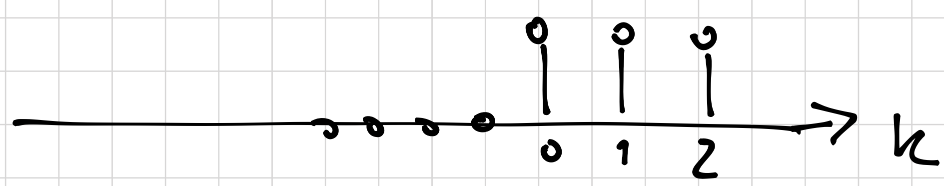


$$y[-5] = \sum_{k=-\infty}^{+\infty} w_{-5}[k]$$

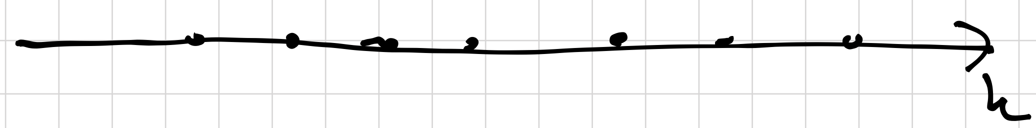
$h[-5-k]$



$x[k]$



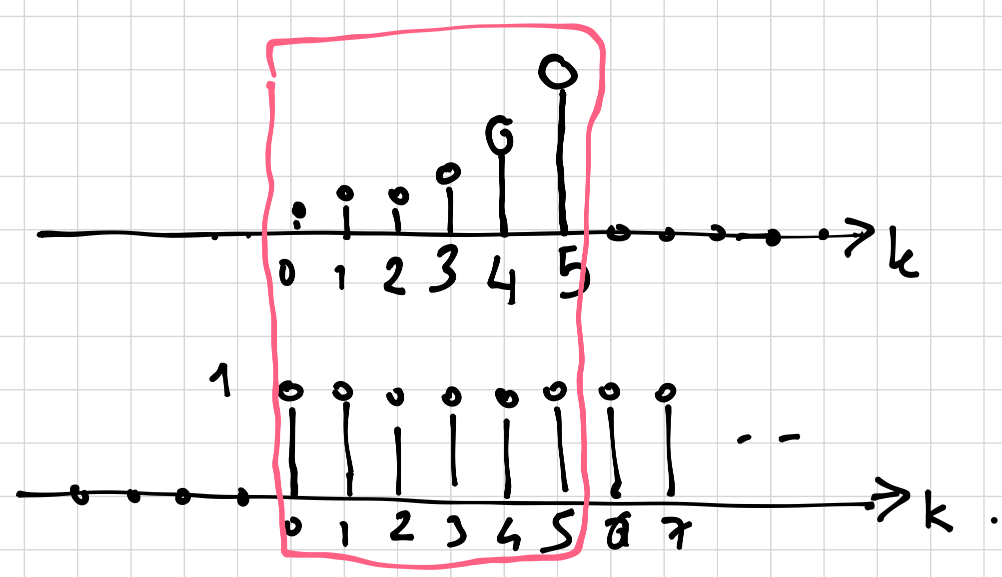
$\times$



$$y[-5] = \sum_{k=-\infty}^{+\infty} 0 = 0$$

$h[5-k]$

$x[k]$



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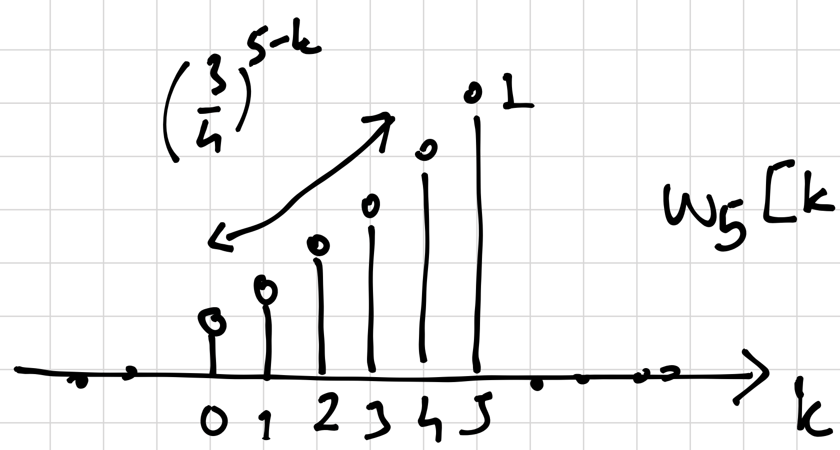
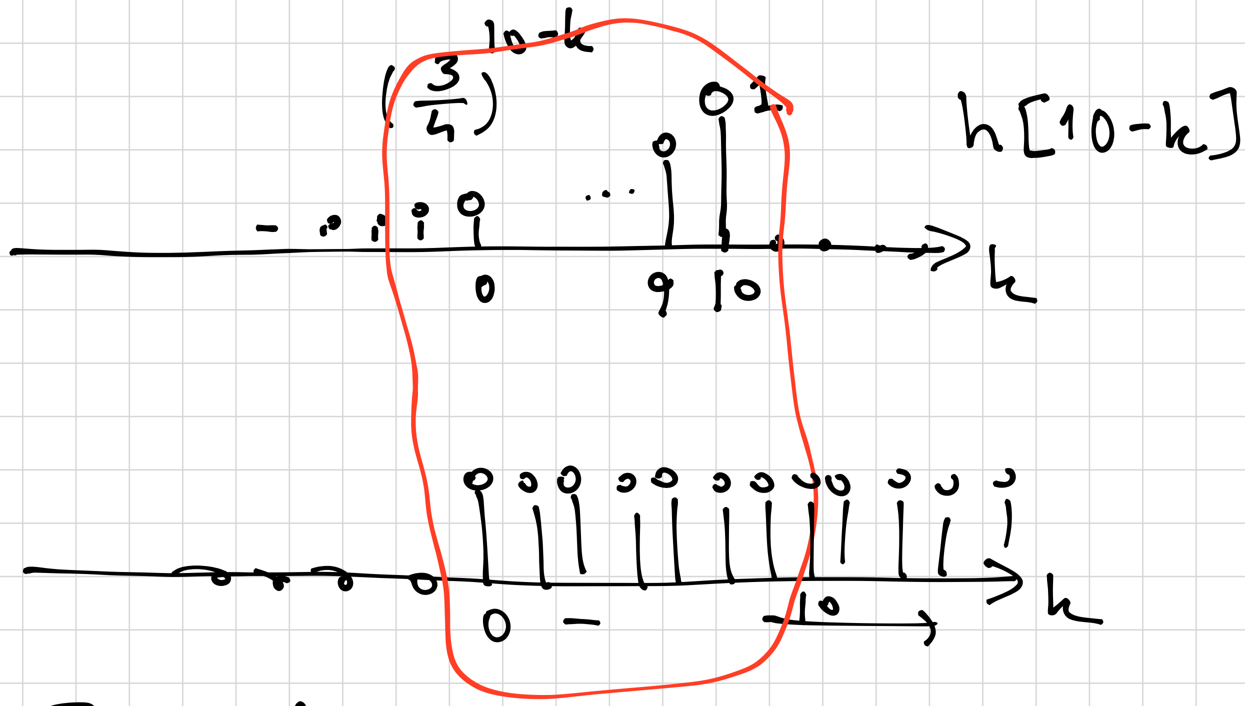


$w_5[k]$

$$y[5] = \sum_{k=0}^{5} \left(\frac{3}{4}\right)^{5-k} = \dots = \underline{3.288}$$

$$y[10] \rightarrow \quad w_{10}[k] = h[10-k]\,x[k]$$



$$w_{10}[k] = \begin{cases} (3/4)^{10-k}, & 0 \le k \le 10 \\ 0, & \text{otherwise} \end{cases}$$

$$y[10] = \sum_{k=0}^{10} \left(\frac{3}{4}\right)^{10-k} = \underline{3.831}$$