Signal Processing
Final Exam Solutions

Istanbul University - Cerrahpaşa
Computer Engineering Department - FALL 2019

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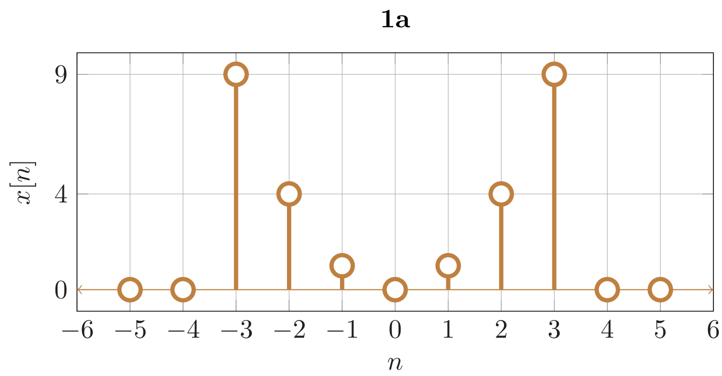
$\boxed{\textbf{S1:}}$ Consider the following DISCRETE TIME signal. Answer the following questions.

$$x[n] = \sum_{k=-3}^{3} k^2 \ \delta[n-k]$$

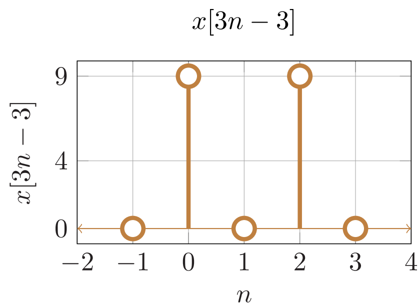
(a) `(5 pts)` Carefully sketch $x[n]$.

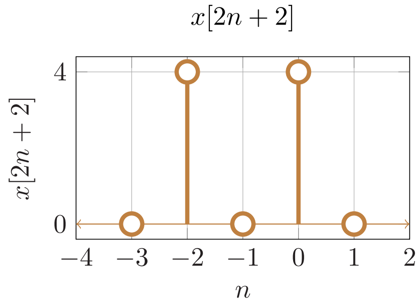
Solution 1a:



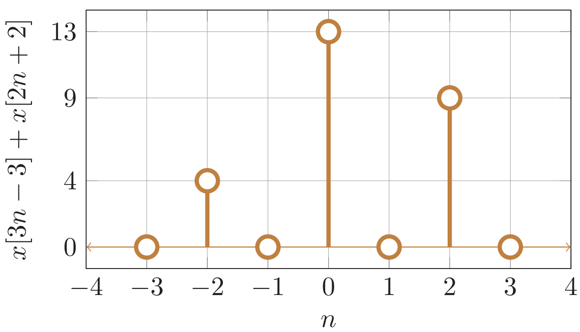
(b) `(10 pts)` Carefully sketch $x[3n-3] + x[2n+2]$.

Solution 1b:





1b



(c) `(5 pts)` Is $x[n]$ an even signal, odd signal or neither? Explain.

Solution 1c:

$x[n]$ is an even signal because $x[n] = x[-n]$ for $\forall\ n\ \in \mathbb{Z}$. We know that $\delta[n] = \delta[-n]$, therefore $\delta[n-k] = \delta[k-n]$. By setting $l = -k$ in the following:

$$
\begin{aligned}
x[n] &= \sum_{k=-3}^{3} k^2\ \delta[n-k] \\
&= \sum_{l=3}^{-3} (-l)^2\ \delta[n-(-l)] \\
&= \sum_{l=-3}^{3} l^2\ \delta[n+l] \\
&= \sum_{l=-3}^{3} l^2\ \delta[-(n+l)] \\
&= \sum_{l=-3}^{3} l^2\ \delta[-n-l] \quad \text{let } k = l \\
&= \sum_{k=-3}^{3} k^2\ \delta[-n-k] \\
&= x[-n] \qquad \blacksquare
\end{aligned}
$$

(d) `(10 pts)` Is $x[n]$ a power signal, energy signal or neither? Calculate its average power and total energy.

Solution 1d:

Let's check the total energy:

$$
\begin{aligned}
E &= \sum_{n=-\infty}^{\infty} x^2[n] \\
&= 9^2 + 4^2 + 1^2 + 1^2 + 4^2 + 9^2 \\
&= 196
\end{aligned}
$$

Since it has finite energy, it is an energy signal. Thus its average power would be zero. ■

S2: For the CT LTI system $\mathcal{H}_1$, the step response is given as:

$$
s(t) = \begin{cases} 0 \ , & t < 0 \\ 1 - \mathrm{e}^{-t} \ , & 0 \leq t \end{cases}
$$

(a) `(5 pts)` What is the impulse response of this system?

Solution 2a:

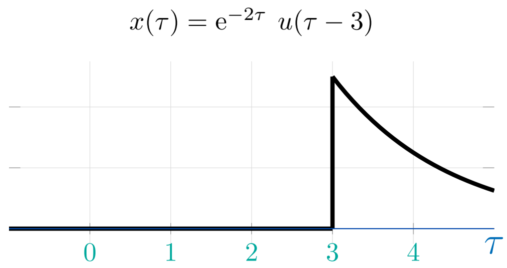
$$
\begin{aligned}
s(t) &= \left(1 - \mathrm{e}^{-t}\right)\, u(t) \\
h(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\, s(t) \\
&= \underbrace{\left(1 - \mathrm{e}^{-t}\right)\, \delta(t)}_{=0} + \mathrm{e}^{-t}\, u(t) \\
&= \mathrm{e}^{-t}\, u(t)
\end{aligned}
$$

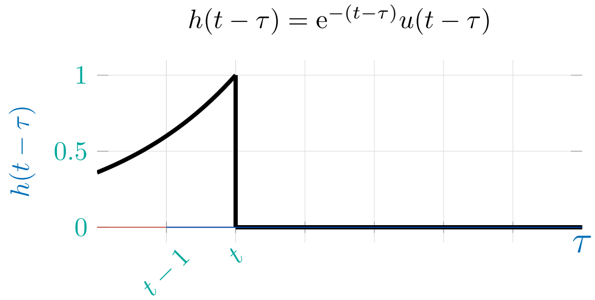
(b) `(10 pts)` Find the output of this system when the input is the following:

$$
x(t) = \mathrm{e}^{-2t}\, u(t-3)
$$

Solution 2b:

Let's first sketch the graphs:





For $t < 3$, $y(t) = 0$

For $t > 3$:

$$\begin{aligned} y(t) &= \int_3^t \mathrm{e}^{-2\tau}\ \mathrm{e}^{\tau - t}\ \mathrm{d}\tau \\ &= \mathrm{e}^{-t} \int_3^t \mathrm{e}^{-2\tau}\ \mathrm{e}^{\tau}\ \mathrm{d}\tau \\ &= \mathrm{e}^{-t} \left[-\ \mathrm{e}^{-\tau} \right]_3^t \\ &= \mathrm{e}^{-3-t}\ -\ \mathrm{e}^{-2t} \end{aligned}$$

So,

$$y(t) = \begin{cases} 0 \ , & t < 3 \\ \mathrm{e}^{-3-t}\ -\ \mathrm{e}^{-2t} \ , & t \geq 3 \end{cases} \qquad \blacksquare$$

S3: Consider the following DISCRETE TIME system. Answer the following questions.

$$y[n] = \mathcal{H}_2\{x[n]\} = \sum_{m=-2}^{2} x[n-m]$$

(a) `(5 pts)` Is $\mathcal{H}_2$ stable? Show your work.

Solution 3a:

Assume that $x[n] \leq M_x < \infty$ for all n.

$$
\begin{aligned}
y[n] = \mathcal{H}_2\{x[n]\} &= \sum_{m=-2}^{2} x[n-m] \\
&\leq \sum_{m=-2}^{2} M_x = 5M_x < \infty
\end{aligned}
$$

Therefore $\mathcal{H}_2$ is BIBO stable. ■

(b) `(10 pts)` Is $\mathcal{H}_2$ linear? Show your work.

Solution 3b:

Let $y_1[n] = \mathcal{H}_2\{x_1[n]\}$ and $y_2[n] = \mathcal{H}_2\{x_2[n]\}$. We can check both for superposition and homogeneity at the same time:

$$
\begin{aligned}
\mathcal{H}_2\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} &= \sum_{m=-2}^{2} \Big(\alpha_1 x_1[n-m] + \alpha_2 x_2[n-m]\Big) \\
&= \sum_{m=-2}^{2} \alpha_1 x_1[n-m] + \sum_{m=-2}^{2} \alpha_2 x_2[n-m] \\
&= \alpha_1 \sum_{m=-2}^{2} x_1[n-m] + \alpha_2 \sum_{m=-2}^{2} x_2[n-m] \\
&= \alpha_1 y_1[n] + \alpha_2 y_2[n]
\end{aligned}
$$

So, $\mathcal{H}_2$ is linear. ■

(c) `(10 pts)` Is $\mathcal{H}_2$ time invariant? Show your work.

Solution 3c:

For some $n_0 \in \mathbb{Z}$

$$
\begin{aligned}
\mathcal{H}_2\{x[n-n_0]]\} &= \sum_{m=-2}^{2} x[n-n_0-m] \\
&= y[n-n_0]
\end{aligned}
$$

So, $\mathcal{H}_2$ is time-invariant. ■

(d) `(10 pts)` Find and sketch the impulse response of $\mathcal{H}_2$.

Solution 3d:

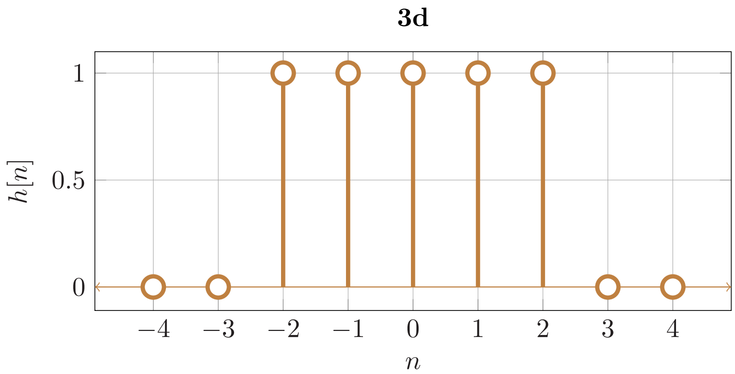
We can find the impulse respone by setting the input to $\delta[n]$,

$$
\begin{aligned}
h[n] &= \mathcal{H}_2\{\delta[n]\} \\
&= \sum_{m=-2}^{2} \delta[n-m]
\end{aligned}
$$

So,

$$
h[n] = \begin{cases} 1 \ , & n = -2, -1, 0, 1, 2 \\ 0 \ , & \text{otherwise} \end{cases} \qquad \blacksquare
$$

Plotting this:



(e) `(10 pts)` Find the frequency response of $\mathcal{H}_2$.

Solution 3e:

$$
\begin{aligned}
H(e^{j\Omega}) &= \sum_{k=-\infty}^{\infty} h[k] \ e^{-j\Omega k} \\
&= \sum_{k=-2}^{2} 1 \ e^{-j\Omega k} \\
&= e^{j\Omega 2} + e^{j\Omega 1} + e^{j\Omega 0} + e^{-j\Omega 1} + e^{-j\Omega 2} \\
&= 1 + 2 \ \cos(\Omega) + 2 \ \cos(2\Omega) \qquad \blacksquare
\end{aligned}
$$

(f) `(10 pts)` Find the output of this system when the input signal is $x[n] = \delta[n-1]$.

Solution 3f:

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \delta[n-1] * h[n] \\
&= h[n-1]
\end{aligned}
$$

So,

$$h[n] = \begin{cases} 1 \ , & n = -1, 0, 1, 2, 3 \\ 0 \ , & \text{otherwise} \end{cases} \qquad \blacksquare$$

S4: `(10 pts)` Determine whether the following CT signal is periodic. If it is periodic then calculate its period.

$$x(t) = \cos\left(\frac{3}{4}t + \frac{4}{7}\right) + \sin\left(\frac{5}{16}t + \frac{3}{8}\right)$$

Solution 4:

Let's write $x(t) = x_1(t) + x_2(t)$ where:

$$x_1(t) = \cos\left(\frac{3}{4}t + \frac{4}{7}\right)$$
$$x_2(t) = \sin\left(\frac{5}{16}t + \frac{3}{8}\right)$$

Let's say the fundamental frequency of $x_1(t)$ and $x_2(t)$ are ω_1 and ω_2, respectively. Let T, T_1 and T_2 be the fundamental period of $x(t)$, $x_1(t)$ and $x_2(t)$.

$$\omega_1 = \frac{3}{4}, \quad T_1 = \frac{2\pi}{\omega_1} = \frac{8\pi}{3}$$
$$\omega_2 = \frac{5}{16}, \quad T_2 = \frac{2\pi}{\omega_2} = \frac{32\pi}{5}$$

Now, we know that T must be an integer multiple of both T_1 and T_2 such that:

$$\begin{aligned} T = kT_1 &= mT_2 \qquad \text{where } k, m \in \mathbb{Z}^+ \\ k\frac{8\pi}{3} &= m\frac{32\pi}{5} \\ k &= m\frac{12}{5} \end{aligned}$$

For $k = 12$ and $m = 5$ the equation will be satisfied. Therefore:

$$\begin{aligned} T &= kT_1 \\ &= 12\ \frac{8\pi}{3} \\ T &= 32\pi \qquad \blacksquare \end{aligned}$$