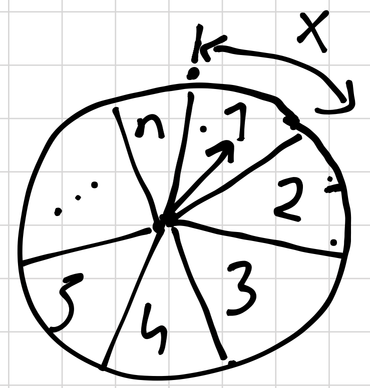
Ex



Y = the number of the arc in which the pointer stops

The PMF of Y | The circumference of the wheel is 1 meter

$$P_Y(y) = \begin{cases} 1/n, & y = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

X : the distance from the marked point. in meter

$$P[X = x] \leq P[Y = \lceil nx \rceil] = \frac{1}{n}$$

$$P[X = x] \leq \lim_{n \to \infty} P[Y = \lceil nx \rceil] = \lim_{n \to \infty} \frac{1}{n} = 0$$

$P[X = x] \leq 0$

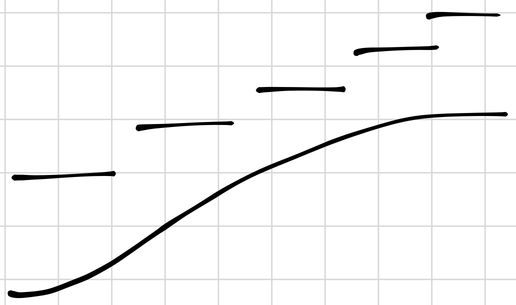
1st axiom of probability states that
$P[X = x] \geq 0$

$$\Rightarrow P[X = x] = 0$$

Cumulative Distribution Function (CDF)

Given a Continuous R.V., X, the CDF of X is

$$F_X(x) = P[X \leq x]$$



Theorem

For a R.V., X

(a) $F_X(-\infty) = 0$

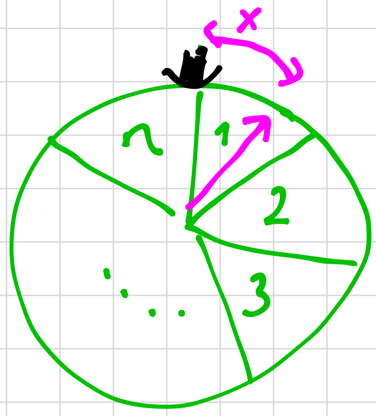
(b) $F_X(\infty) = 1$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

Definition

X is continuous R.V. if the CDF, $F_X(x)$ is a continuous function.

Example



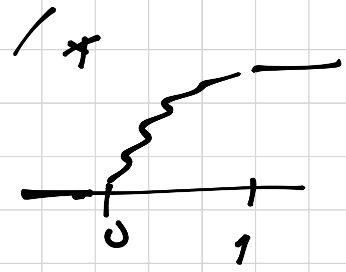
X is the distance from the marker.
$0 \leq X \leq 1$

$x \in S_X = \underline{[0, 1)} \Rightarrow F_X(x) = 0$ for $x < 0$

$F_X(x) = 1$ for $x \geq 1$

Between 0, 1 we need to ~~find~~ consider the event $\underline{\{X \leq x\}}$ with x growing from 0 to 1

$F_X(x) = P[X \leq x]$



Y: the number of the arc in which the pointer stops

$$\{Y \leq \lceil nx \rceil - 1\} \subset \{X \leq x\} \subset \{Y \leq \lceil nx \rceil\}$$

$$F_Y(\lceil nx \rceil - 1) \leq F_X(x) \leq F_Y(\lceil nx \rceil)$$

CDF of Y (Y is a discrete uniform R.V)

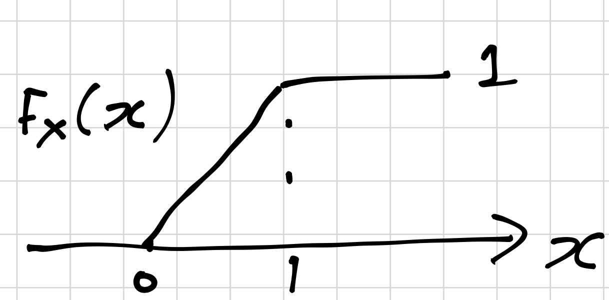
$$F_Y(y) = \begin{cases} 0, & y < 0 \\ k/n, & (k-1)/n \leq y < k/n, \quad k = 1, 2, 3, \dots n \\ 1, & y > 1 \end{cases}$$

For $x \in [0, 1)$ and for all n

$$\frac{\lceil nx \rceil - 1}{n} \leq F_X(x) \leq \frac{\lceil nx \rceil}{n}$$

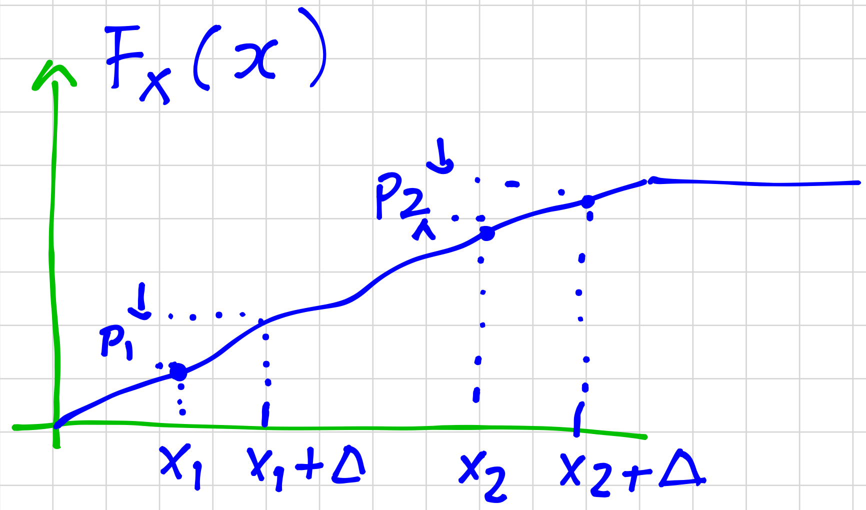
$$\lim_{n \to \infty} \frac{\lceil nx \rceil}{n} = x \qquad \lim_{n \to \infty} \frac{\lceil nx \rceil - 1}{n} = x$$

$F_X(x) = x \qquad 0 \leq x < 1$



$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Probability Density Function



CDF of an arbitrary X

Definition

The Probability Density Function (PDF) of a continuous R.V. X is

$$f_X(x) = \frac{d}{dx} F_X(x)$$

From above figure

$$p_1 = P[x_1 < X \leq x_1 + \Delta] = F_X(x_1+\Delta) - F_X(x_1)$$

$$p_2 = P[x_2 < X \leq x_2 + \Delta] = F_X(x_2+\Delta) - F_X(x_2)$$

$$P[x_1 < X \leq x_1 + \Delta] = \frac{F_X[x_1+\Delta] - F_X(x_1)}{\Delta} \cdot \Delta$$

$\Delta \to 0 \quad \longrightarrow \quad$ derivative of $F_X(x)$

Theorem

X is a C. R.V with P.D.F $f_X(x)$

(a) $f_X(x) \geq 0$ for all x

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$

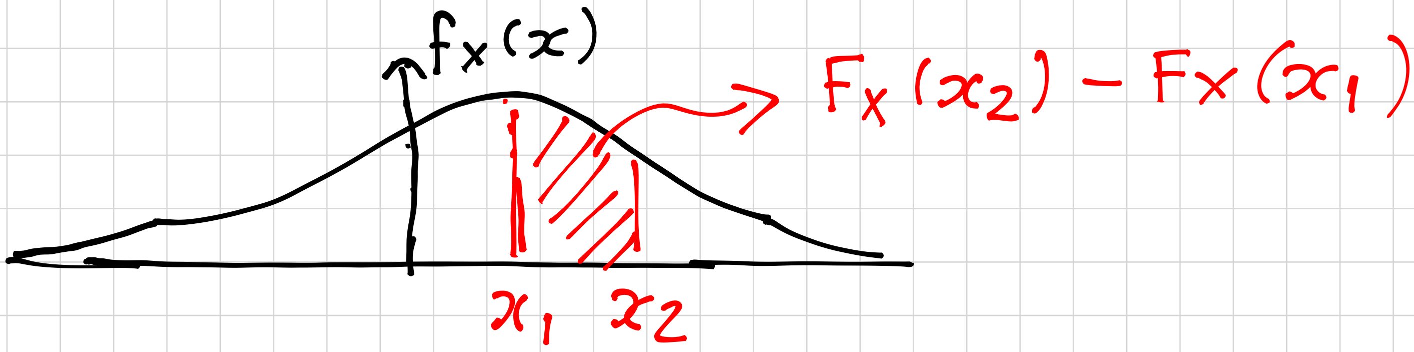
(c) $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

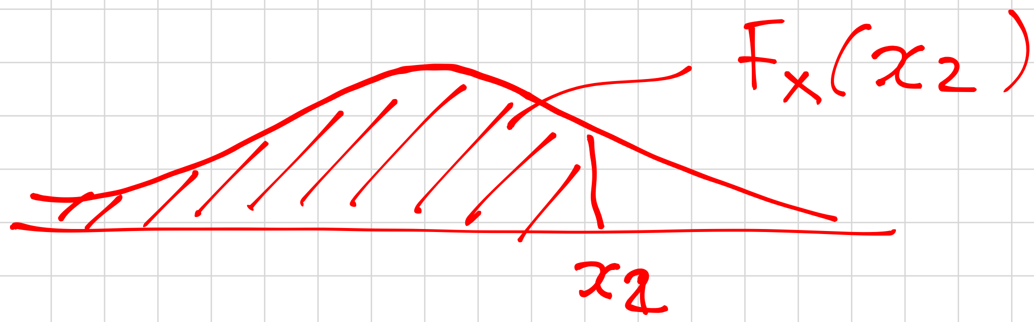
Theorem

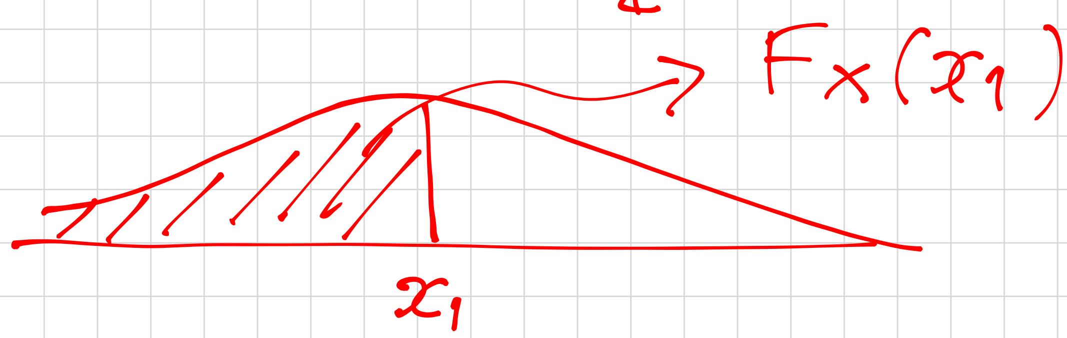
$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$$

$$P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$$

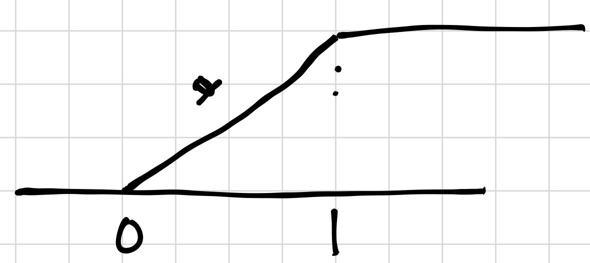
$$= \int_{-\infty}^{x_2} f_X(x)\, dx - \int_{-\infty}^{x_1} f_X(x)\, dx$$





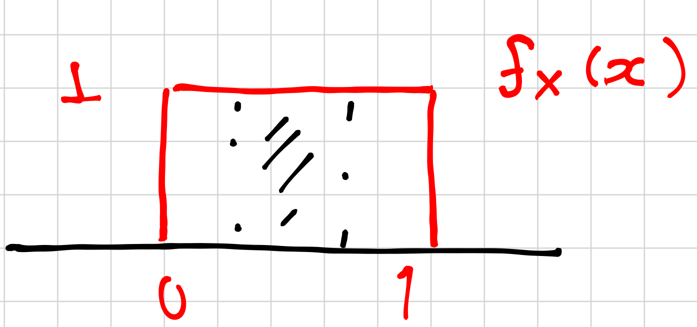


Example



$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



$$P[1/4 < X \leq \tfrac{3}{4}] = \int_{1/4}^{3/4} 1 \cdot dx = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$= F_X(\tfrac{3}{4}) - F_X(\tfrac{1}{4}) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

Example Y, its CDF is given as

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y^3, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

PDF ?

$$f_Y(y) = \begin{cases} 3y^2, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P[1/4 < Y < \tfrac{3}{4}] = F_Y(\tfrac{3}{4}) - F_Y(\tfrac{1}{4}) = (\tfrac{3}{4})^3 - (\tfrac{1}{4})^3$$

$$= \int_{1/4}^{3/4} 3y^2 \, dy = \;\longrightarrow\; = 13/32$$

Ex X is C.R.V. with PDF :

$$f_X(x) = \begin{cases} c \cdot x \cdot e^{-x/2} & , \ x \geq 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

a) $c = ?$ $\quad \int_0^\infty c \cdot x \cdot e^{-x/2} \cdot dx = 1 \quad (c = 1/4)$

b) CDF = ?

$x < 0 \Rightarrow F_X(x) = 0$

$x \geq 0 \quad F_X(x) = \int_0^x f_X(u)\, du$

$$= \int_0^x \frac{u}{4} \cdot e^{-u/2} \cdot du$$

$$= -\frac{u}{2} e^{-u/2} \Big|_0^x + \int_0^x \frac{1}{2} e^{-u/2} du$$

$$= 1 - \frac{x}{2} e^{-x/2} - e^{-x/2}$$

$$F_X(x) = \begin{cases} 1 - \left(\frac{x}{2} + 1\right) e^{-x/2} & , \ x > 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

Expected Value

The expected value of a C.R.V., X is

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$$

Ex. $\quad f_X(x) = \begin{cases} 1 , & 0 \leq x < 1 \\ 0 , & \text{otherwise} \end{cases}$

$$E[X] = \int_0^1 x \cdot 1 \cdot dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

Ex

Y is C.R.V.

$$f_Y(y) = \begin{cases} 3y^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = \int_0^1 y \cdot (3y^2) \cdot dy = 3 \cdot \frac{y^4}{4} \Big|_0^1 = \frac{3}{4}$$

Example

X is a CRV

$$f_X(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$W = g(X) = 0$ if $X \leq \frac{1}{2}$

$W = g(X) = 1$ if $X > 1/2$

Then W is a discrete R.V. with range

$S_W = \{0, 1\}$

Expectation of a function of a C.R.V.

Theorem

The expected value of a function, $g(X)$ of C.R.V. X is

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x)\, dx$$

Theorem

For any random variable X

a) $E[X - \mu_X] = 0$

$$E[X-\mu_X] = \int_{-\infty}^{+\infty} (x-\mu_X) \cdot f_X(x) \cdot dx$$

$$= \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx - \mu_X \underbrace{\int_{-\infty}^{+\infty} f_X(x)\, dx}_{1}$$

$$= \mu_X - \mu_X = 0$$

(b) $E[aX+b] = a \cdot E[X] + b$

Variance $\quad Var(X) = E[(X-\mu_X)^2]$

$$Var(X) = E[X^2 - 2\mu_X \cdot X + \mu_X^2]$$

$$= E[X^2] - 2\mu_X \cdot \mu_X + \mu_X^2$$

$$= E[X^2] - \mu_X^2$$

=

- $Var(aX+b) = a^2 \, Var(X)$

$$= Var[(aX+b-\mu_X)^2] =$$

Standard Deviation $\quad \sigma = \sqrt{Var(X)}$

Ex

X is a C.R.V

$Var(X) = ?$

$\sigma_X = ?$

$$f_X(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X^2] = \int_0^1 x^2 \cdot 3 \cdot x^2 \, dx = \frac{3}{5}$$

$$E[X] = \frac{3}{4}$$

$$Var(X) = \left(\frac{3}{5}\right) - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$$

$$\sigma_X = \sqrt{Var(X)} = \underline{\underline{0.194}}$$

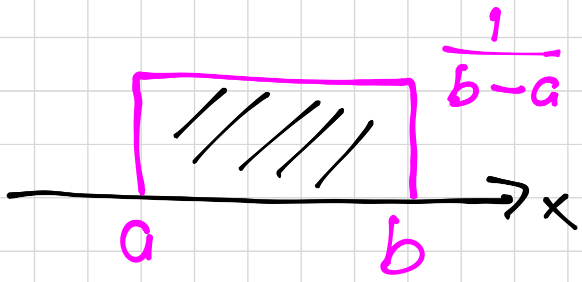
Families of C.R.V.s

Uniform Random Variable

$\underline{\underline{b > a}}$

If X is a Uniform(a, b) random variable if its PDF is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & \text{otherwise} \end{cases}$$



X is uniform random variable
X has a uniform distribution
X is uniformly distributed

} Synonyms (eşanlamlı)

If X is uniformly distributed, its CDF

$x < a \quad F_X(x) = 0$

$x \geq b \quad F_X(x) = 1$

$a \leq x < b \quad \displaystyle\int_a^x \frac{1}{b-a} \cdot dx = \frac{x-a}{b-a}$

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

Expected value

$$E[X] = \int_a^b x \cdot \frac{1}{b-a} \, dx = \frac{x^2}{2} \cdot \frac{1}{b-a} \Big|_a^b$$

$$= \frac{1}{2(b-a)} \cdot (b-a)^2 = \frac{b+a}{2}$$

Variance

$$Var(X) = \frac{(b-a)^2}{12}$$