Lineer Algebra

$f(x)$ x_1 x_2

$R \to$ sets of real number

$x \in R$

$\alpha_1 \in R$

$a_2 \in R$

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

$\alpha_1 = 1$ $\alpha_2 = 1$

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

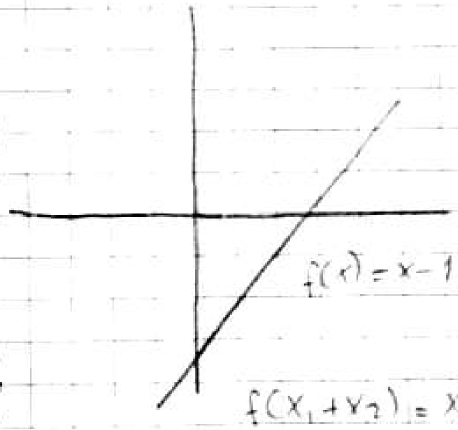
$$x_1 + x_2 - 1 = x_1 - 1 + x_2 - 1$$

$$x_1 + x_2 - 1 \neq x_1 + x_2 - 2$$

$f(x) = x$

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$x_1 + x_2 = x_1 + x_2$$



$$f(x_1 + x_2) = x_1 + x_2 - 1$$

$$f(x_1) = x_1 - 1$$

$$f(x_2) = x_2 - 1$$

1. Addition $+$
2. Multiplication $\cdot$
3. Division $\div$
4. Substraction $-$

$f(x) = ax = b$

$a, b \in R$ $\qquad ax = b$

$a, b \rightarrow$ known parameter
$x \rightarrow$ unknown parameter

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots a_{nn}x_n &= b_n \end{aligned}\right\}$$

1. $m < n$
2. $m > n$
3. $m = n$

$m \rightarrow$ equations

$n \rightarrow$ unknown parameter

d) rectangular Array
(dikdörtgen dizi)

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \rightarrow \text{rows (satır)}$$

Columns (sütun)

matrix $\qquad\qquad$ vector

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b} \Rightarrow Ax = b$$

$a_{ij} \qquad i \to 1, 2, \ldots m$

$\qquad\quad j \to 1, 2, \ldots n$

$$A = (ij)_{m \times n}$$

$Ax = b \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \qquad T \to$ transpoze

$y^T = [y_1 \; y_2 \ldots y_n] \qquad z^T = [z_1 \; z_2 \ldots z_n]$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$y^T z = [y_1 \; y_2 \ldots y_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = y_1 z_1 + y_2 z_2 + \ldots + y_n z_n$$

$$= \sum_{r=1}^{n} y_r z_r$$

GIPTA

$$a_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} \quad a_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix} \quad a_m = \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$a_1^T . x = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$

$$a_1^T x = b_1$$

$$a_2^T x = b_2$$

$$\vdots$$

$$a_m^T x = b_m$$

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Elementary Row Operations

$$k\,(a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n) = b_1 k$$

$$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m$$

GIPTA

$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$

$\int (\alpha_1 f(x_1) + \alpha_2 f(x_2))\, dx$

$\alpha_1 \int f(x_1) dx + \alpha_2 \int f(x_2) dx$

$\frac{d}{dx}(\alpha_1 x_1 + \alpha_2 x_2)$

$\alpha_1 \frac{dx_1}{dx} + \alpha_2 \frac{dx_2}{dx}$

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots a_{2n} x_n = b_2$

$\vdots$

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots a_{mn} x_n = b_m$

$Ax = b$ $\quad$ $A \rightarrow$ Augmented Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$\hat{A} = [A, b]$ $\quad$ $\hat{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & & a_{1n} & b_1 \\ a_{21} & a_{22} & & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{array}\right]$

1. Unique solution
2. Multiple solution
3. No solution

GIPTA

$$\left[\begin{array}{cccc|c} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \hat{b}_1 \\ 0 & \alpha_{22} & & \alpha_{2n} & \hat{b}_2 \\ \vdots & 0 & & & \vdots \\ 0 & 0 & \cdots & \alpha_{mn} & \hat{b}_m \end{array}\right]$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{2n} \\ \vdots & 0 & \\ 0 & 0 & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_n \end{bmatrix}$$

EX:

$$x_1 + x_2 + x_3 = 6$$
$$2x_1 - x_2 - 2x_3 = -6$$
$$x_1 + 3x_2 - x_3 = 4$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 3 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}$$

$$\hat{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -2 & -6 \\ 1 & 3 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -4 & -18 \\ 1 & 3 & -1 & 4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -4 & -18 \\ 0 & 2 & 2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -4 & -18 \\ 0 & 0 & -14/3 & -14 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 1 & 0 & -14/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -18 \\ -14 \end{bmatrix}$$

$$-\frac{14}{3} x_3 = -14 \qquad \underline{x_3 = 3}$$

$$-3x_2 - 4x_3 = -18 \qquad \underline{x_2 = 2}$$

$$\underline{x_1 = 1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -4/3 & -6 \\ 0 & 0 & -14/3 & -14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Gauss Elemination

$m = n \longrightarrow$ solution unque

$$\hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \Rightarrow \left[\begin{array}{ccccc|c} \alpha_{11} & \alpha_{12} & \cdots & \cdots & \alpha_{1n} & \beta_1 \\ 0 & \alpha_{22} & \cdots & \cdots & \alpha_{2n} & \beta_2 \\ 0 & 0 & \alpha_{33} & \cdots & \alpha_{3n} & \beta_3 \\ & & & & & \\ & & & & \alpha_{mn} & \beta_n \end{array}\right]$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & & & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & & \alpha_{3n} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \\ \beta_m \end{bmatrix} \qquad \alpha_{mn} x_n = \beta_m$$

EX:

$$2x_1 + 3x_2 - 2x_3 + 4x_4 = 18$$

$$3x_1 - 3x_2 + 4x_3 - 2x_4 = 3$$

$$3x_1 - 2x_2 - x_3 + 2x_4 = 4$$

$$-2x_1 + 4x_2 - 3x_3 - x_4 = 11$$

$$A = \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 3 & -3 & 4 & -2 & 3 \\ 3 & -2 & -1 & 2 & 4 \\ -2 & 4 & -3 & -1 & 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 3 & -2 & -1 & 2 & 4 \\ -2 & 4 & 3 & -1 & -11 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & -13/2 & 2 & -4 & -23 \\ -2 & 4 & 3 & -1 & -11 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & -13/2 & 2 & -4 & -23 \\ 0 & 7 & 1 & 3 & 29 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -21/2 & 9 & -12 & -42 \\ 0 & 0 & -25/7 & 24/7 & 3 \\ 0 & 7 & 1 & 3 & 29 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & 11/2 & 9 & -12 & -42 \\ 0 & 0 & -25/7 & 24/7 & 3 \\ 0 & 0 & 7 & -5 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 2 & 3 & -2 & 4 & 18 \\ 0 & -11/2 & 9 & -12 & -42 \\ 0 & 0 & -\frac{25}{7} & \frac{24}{7} & 3 \\ 0 & 0 & 0 & -3/25 & 172/25 \end{bmatrix}$$

$\frac{43}{25} x_4 = \frac{172}{25} \qquad x_4 = 4$

$-\frac{25}{7} x_3 + \frac{24}{7} x_3 = 3 \qquad x_3 = 3$

$\frac{21}{2} x_2 + 27 - 48 = -42 \qquad x_2 = 2$

$x_1 = 1$

EX:

$m = n$

$x_1 + 2x_2 + 2x_3 - x_4 = 4$

$-x_1 + x_2 + 4x_3 - 2x_4 = 8$

$2x_1 - 2x_2 - x_3 + x_4 = 8$

$-4x_1 - 12x_2 - 2x_3 + 2x_4 = 16$

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 & 4 \\ -1 & 1 & 4 & -2 & 8 \\ 2 & -2 & -1 & 1 & 8 \\ -4 & -12 & -2 & 2 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & -1 & 4 \\ 0 & 3 & 6 & -3 & 12 \\ 0 & 0 & 7 & -3 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 24 \\ 0 \end{bmatrix}$$

$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$

$x_4 = \alpha \qquad \alpha \in R$

$7x_3 - 3x_4 = 24$

$x_3 = \frac{24 + 3x_4}{7} = \frac{24 + 3\alpha}{7}$

$3x_2 + 6x_3 - \underset{\alpha}{3x_4} = 12 \qquad x_4 \text{ — constant variable}$

EX: $x_1 + x_2 + x_3 - x_4 = 1$

$2x_1 - 2x_2 + 2x_3 - 2x_4 = -2$

$2x_1 + 2x_2 + 2x_3 - 2x_4 = 2$

$-x_1 + x_2 - x_3 + x_4 = 1$

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 2 & -2 & 2 & -2 & -2 \\ 2 & 2 & 2 & -2 & 2 \\ -1 & 1 & -1 & 1 & 1 \end{array}\right]$$

(1) 1. satırı -2 ile çarp 2. satıra ekle
(2) 1. satırı -2 ile çarp 3'e ekle
(3) 1. satırı 4'e ekle
(4) 2. satırı 1/2 ile çarp 4'e ekle

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & -1 & 1 \\ 0 & -4 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \end{bmatrix}$$

$x_4 = \beta \quad x_3 = \alpha \quad x_2 = 1$

$x_1 + 1 + x_3 - x_4 = 1 \longrightarrow x_1 = -x_3 + x_4$

$x_1 = \beta - \alpha$

$m \neq n$

Number of constant of values is equal to the number of unknown parameters - the number of the rows that has all non zero elements

Ex! $\quad m=3 \quad n=2$

$$\left.\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \text{consistent değil}$$

$Ax = b$

$A = \{a_{ij}\}_{m \times n}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$n \rightarrow$ number of unknown variables

$m \rightarrow$ number of equations

1) $m = n$ $\qquad$ There are no nonzero rows

$$\begin{array}{r} k \\ 0 = n-k \\ \\ m = n \end{array}\left[\begin{array}{cccc} & & & \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & & \end{array}\right] \} \text{nonzero}$$

$n - k \longrightarrow$ constant variables

$k = n \qquad n - k = 0$

2) $n > m$ — unknown — equations

$n=2$
$m=1$

$x_1 + x_2 = 1$ → bilinmi constant olarak seç

$x_2 = \alpha$

$x_1 + x_2 + x_3 = 3$	$x_1 = 1-\alpha$	$x_1 + x_2 + x_3 = 3$
$x_1 - x_2 - x_3 = -1$	$2x_1 = 2$	$x_2 = \alpha$ $\quad$ $x_3 = \beta$
$n=3$ $\quad$ $m=2$	$x_1 = 1$	

3) $m > n$

$$\left.\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 4 \\ -x_1 + x_2 + 4x_3 - 2x_4 &= 8 \\ 2x_1 - 2x_2 - x_3 + x_4 &= 12 \\ -4x_1 - 12x_2 - 2x_3 + 2x_4 &= 16 \end{aligned}\right\}$$

1. denklemi -2 ile çarp 2'ye ekle

3. denklemi -2 ile çarp 4'e ekle

1. yol

$-3x_1 - 3x_2 = 0$

$x_1 + x_2 = 0$

$-8x_1 - 8x_2 = -8$

$x_1 + x_2 = 1$

2. yol

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ -1 & 1 & 4 & -2 & 8 \\ 2 & -2 & -1 & 1 & 12 \\ -4 & -12 & -2 & 2 & 16 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 3 & 6 & -3 & 12 \\ 0 & 0 & 7 & -3 & 18 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right]$$

$0x_1 + 0x_2 + 0x_3 + 0x_4 = -8$ → 0 olmalıydı

– consistant değil –

Vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow \text{column vector}$$

$x^T = [\, x_1 \;\; x_2 \; \ldots \; x_n \,] \longrightarrow$ row vector (x' in transpose)

$$x = 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \text{zero vector}$$

$x \neq 0 \longrightarrow$ nonzero vector $\quad x \in \mathbb{R}^n \quad x \in \mathbb{C}^n$

$x \in \mathbb{R}^n, \; y \in \mathbb{R}^n, \; z \in \mathbb{R}^n, \; \alpha \in \mathbb{R}, \; \beta \in \mathbb{R}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

1) $$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

2) $x + y = y + x$

3) $x + (y + z) = (x + y) + z$

4) $$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

5) $(\alpha + \beta) x = \alpha x + \beta x$

6) $\alpha (x + y) = \alpha x + \alpha y$

Inner Product

$x, y \quad x^T y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 \ldots x_n y_n$

i) $f(x,y) = f(y,x)$

ii) $f(x+y,z) = f(x,z) + f(y,z)$

iii) $f(\alpha x, y) = \alpha f(x,y)$

Dot product $\longrightarrow f(x,y) = x^T y$

Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad A = \{a_{ij}\}_{m \times n} \qquad A \in R^{m \times n}$$

Square Matrix $\longrightarrow m = n$

Diagonal Elements $\quad a_{ii} \, , \, i = 1, 2 \ldots n$

off-diagonal elements

$A = \{a_{ij}\}_{m \times n} \qquad B = \{b_{ij}\}_{m \times n}$

$C = A + B = \{a_{ij} + b_{ij}\}_{m \times n}$

$C = \{c_{ij}\}_{m \times n}$

$c_{ij} = a_{ij} + b_{ij} \qquad i, j = 1, 2 \ldots n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & & a_{2p} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{np} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ & & & \vdots \\ b_{p1} & b_{p2} & & b_{pn} \end{bmatrix} = B$$

$$C = A \cdot B$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ & & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \qquad c_{11} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{bmatrix}$$

$$a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{ip} \end{bmatrix} \qquad a_i^T = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{ip} \end{bmatrix}$$

$$b_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} \qquad i, j = 1, 2 \dots n$$

$$c_{ij} = a_i^T \cdot b_j = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$c_{13} = \sum_{k=1}^{3} a_{1k} \cdot b_{k3} = a_{11} b_{13} + a_{12} b_{23} + a_{13} b_{33}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$c_{32} = \sum_{k=1}^{3} a_{3k} b_{k2}$$

$n=4 \quad m=2$

$$\left.\begin{array}{l} x_1+x_2+x_3+x_4=1 \\ 2x_1+2x_2+2x_3+2x_4=2 \\ \begin{array}{c} k \\ m-k \end{array}\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad n-k \end{array}\right\} \quad \begin{array}{l} n-m+m-k \\ = n-k \end{array}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad A^T=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix}$$

$A=\{a_{ij}\}_{n\times n} \qquad A^T=\{a_{ji}\}_{n\times n}$

$$A=\begin{bmatrix} 1+j & j \\ -j & 1-j \end{bmatrix} \qquad j^2=-1$$

Symetric matrices $\rightarrow A^T=A \qquad a_{ij}=a_{ji} \qquad i,j=1,2\ldots n$

Diagonal matrices $\rightarrow D=\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \qquad \begin{array}{l} d_{ii}>0, \quad i=1,2\ldots n \\ d_{ij}\geq 0, \quad i\neq 1,2\ldots n \end{array}$

$$I=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & 1 \end{bmatrix} \qquad d_{ii}=1$$

Inverse of the matrix $A \qquad A^{-1} \qquad \boxed{A^{-1}\cdot A=I}$

If A^{-1} exists A is nonsingular otherwise A is singular

→ $[A : I]$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$

$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow A^{-1} . A = I$

→ $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\qquad AB \neq BA$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$ Tersi yok → No inverse

→ $\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & & & \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix}$

→ $(AB)^{-1} = B^{-1} . A^{-1}$

$(AB)^T = B^T A^T$

$(\alpha A)^{-1} = \alpha^{-1} A^{-1}$

$(A+B)^T = A^T + B^T$

$(\alpha A)^T = \alpha A^T$

$(A^T)^{-1} = (A^{-1})^T$

GIPTA

$$A^k = \underbrace{A\,A \ldots\, A}_{k \text{ times}}$$

$$(\alpha A)^k = \alpha^k A^k$$

$$(A+B)(A+B) = A^2 + AB + BA + B^2$$

Determinants

Upper Triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Lower matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = \Delta(A)$$

$$= a_{11}\, a_{22} \cdot a_{33}$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix} \qquad \det(A) = 1$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \dfrac{a_{12}\, a_{21}}{a_{11}} \end{bmatrix} \rightarrow a_{11} a_{22} - a_{12} a_{21} = \det(A)$$

Minors and Cofactors

$A = \{a_{ij}\}_{n\times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

minors $\longrightarrow M_{ij}$

cofactors $\longrightarrow C_{ij}$

a_{ij}

$(n-1)\times(n-1)$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$C_{ij} = (-1)^{i+j}\, M_{ij} \qquad i,j = 1,2 \ldots n$

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{23}.a_{32} \qquad C_{11} = (-1)^2 . M_{11} = M_{11}$

$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}.a_{33} - a_{31}.a_{23} \qquad C_{12} = (-1)^3 M_{12} = -M_{12}$

$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{31}\,a_{22} \qquad C_{13} = (-1)^4 M_{13} = M_{13}$

GIPTA

$c_{ij} = (-1)^{i+j} \quad m_{ij} \quad i,j = 1,2 \ldots n$

$a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$

$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{31}a_{23} - a_{21}a_{33}) +$

$a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{12}a_{31}a_{23} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32}$

$- a_{13}a_{31}a_{22}$

$\det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$

$= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$

$= a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32}$

j. column $\quad j = 1,2 \ldots n$

$\det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + \ldots \quad a_{nj}c_{nj}$

i. row

$\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \ldots \quad a_{in}c_{in}$

$a_{21}c_{11} + a_{22}c_{12} + a_{23}c_{13}$

$= a_{21}a_{22}a_{33} - a_{21}a_{12}a_{33} - a_{21}a_{32}a_{23} + a_{23}a_{21}a_{32} + a_{22}a_{31}a_{23} -$

$a_{22}a_{31}a_{23} = 0$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \quad C^T = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & & & \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

$$a_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} \quad c_j = \begin{bmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{bmatrix} \quad a_i^T = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

$$a_i^T c_j = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{bmatrix}$$

$a_i^T c_j = \det(A)$, if $j = i$

$a_i^T c_j = 0$ if $i \neq j$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \quad C^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

$$AC^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

$$AC^T = \begin{bmatrix} a_1^T c_1 & a_1^T c_2 & \cdots & a_1^T c_n \\ a_2^T c_1 & a_2^T c_2 & & a_2^T c_n \\ \vdots & & & \vdots \\ a_n^T c_1 & a_n^T c_2 & \cdots & a_n^T c_n \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 & \cdots & 0 \\ 0 & \det(A) & & & \\ \vdots & & & & \\ 0 & 0 & & \cdots & \det(A) \end{bmatrix}$$

GIPTA

$$A^{-1}\det(A) = A^{-1}AC^T = \det(A)I = AC^T$$

$$A^{-1} = \frac{1}{\det(A)} C^T$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad AX = b$$

$$A^{-1} \longrightarrow \text{exists}$$

$$x = A^{-1}b = \frac{1}{\det(A)} \operatorname{adj}(A)\, b = \frac{1}{\det(A)} C^T b = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \dots b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \dots b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \dots b_nC_{nn} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} \sum_{i=1}^{n} b_iC_{i1} \\ \sum_{i=1}^{n} b_iC_{i2} \\ \vdots \\ \sum_{i=1}^{n} b_iC_{in} \end{bmatrix}$$

$$x_j = \frac{1}{\det(A)} \sum_{i=1}^{n} b_iC_{ij}$$

$$x_1 = \frac{1}{\det(A)} \sum_{i=1}^{n} b_iC_{i1} \qquad x_2 = \frac{1}{\det(A)} \sum_{i=1}^{n} b_iC_{i2}$$

$$x_1 = \frac{1}{\det(A)} \left[b_1C_{11} + b_2C_{21} + \dots b_nC_{n1} \right]$$

leading principal minors

$\rightarrow \quad A = \{a_{ij}\}_{n\times n} \quad B = \{b_{ij}\}_{n\times n} \quad \alpha \in \mathbb{R}$

$\det(AB) = \det(A) \cdot \det(B)$

$\det(A^{-1}) = \dfrac{1}{\det(A)} \qquad \det(A^T) = \det(A)$

$\det(A^{-1}BA) = \det(B)$

$\det(A^{-1}A) = \det(I) = 1$

* $\det(\alpha A) = \alpha^n \det(A)$

* $\det(-A) = (-1)^n \det(A)$

$\det(A^k) = [\det(A)]^k$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad \Delta_{1234} = \det(A)$$

$$\Delta_{134} = \begin{bmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}$$

Lineer Combinations

$x_1, x_2, x_3 \dots x_n$ $\quad x_i \in \mathbb{R}, \forall i$

$\alpha_1, \alpha_2 \dots \alpha_n$ $\quad \alpha_i \in \mathbb{R}, \forall i$

$y = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \sum_{i=1}^{n} \alpha_i x_i$

$S = \{x_1, x_2 \dots x_n\}$ → set of vectors

span $\quad Y = span(S)$

Linearly independent vector sets → Lineer bağımsız

$x_1, x_2 \dots x_n$ $\quad \alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n = 0$

$\alpha_1 \alpha_2 \dots \alpha_n$ $\quad \alpha_1 = \alpha_2 \dots = \alpha_n = 0$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \alpha_1 \alpha_2$

$\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\alpha_1 + 2\alpha_2 = 0 \longrightarrow \alpha_1 = -2\alpha_2$

$\alpha_1 + 2\alpha_2 = 0$

$\alpha_1 = 2 \quad \alpha_2 = -1$

$\alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$S = \{a_1, a_2, \ldots, a_n\}$

$a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \quad i = 1, 2, \ldots, n$

$\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n = 0$

$$\alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \ldots + \alpha_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A\alpha = 0$$

$A = \{a_{ij}\}_{m \times n} \qquad \alpha = \begin{bmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha_n \end{bmatrix}$

n: unknown parameters m: equations

∼ ∼ ∼

* $a_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \det(A) \neq 0$

* $n > m \quad a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad n - m$

$\alpha_3 = 1$ olsun

$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\alpha_1 + 2\alpha_2 + \alpha_3 = 0$
$2\alpha_1 + \alpha_2 + \alpha_3 = 0$

GIPTA

* $m > n$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 4 \end{bmatrix} \quad a_3 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Vector space

$V \mapsto$ vector set

"addition" , "multiplication"

$x \in V$, $y \in V$, $z \in V$, $\alpha \in R$, $\beta \in R$

1. $x + y \in V$
2. $x + y = y + x$
3. $x + (y + z) = (x + y) + z$
4. $\alpha (x + y) = \alpha x + \alpha y$
5. $(\alpha + \beta) x = \alpha x + \beta x$
6. $\alpha (\beta x) = (\alpha \beta) x$
7. $\alpha x \in V$
8. $-x \in V$ $\quad (x + (-x) = 0$
9. $1 \cdot x \in V$
10. $0 + x \in V$

GIPTA

Column space

$C = [C_1 \ C_2 \ldots C_n]$

$C_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$ $\quad i = 1, 2, \ldots n$

$C(A) = \text{span}(C)$

$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ $\quad m = 2$, $n = 3$

$r_1 = [1 \ 2 \ 2] \quad r_2 [2 \ 1 \ 2]$

$\alpha_1 r_1 + \alpha_2 r_2 = 0$

$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \left.\begin{array}{l} \alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \\ 2\alpha_2 + 2\alpha_2 = 0 \end{array}\right\} \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \end{array}$

$C_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad C_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad C_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3 = 0$

$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$
$2\alpha_1 + \alpha_2 + 2\alpha_3 = 0$

$\alpha_3 = \alpha$

$\alpha_1 + 2\alpha_2 = -2\alpha$ $\quad n > m$
$2\alpha_1 + \alpha_2 = -2\alpha$ $\quad n - m > 0$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \alpha$

$\alpha_2 = -1/3 \quad \alpha_1 = -1/6 \qquad 2\alpha_1 - 2/3 = -1 \qquad 2\alpha_1 = -1 + 2/3 = -1/6$

GIPTA

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$n > m$
$n - m$: kadar göre seçebileceğin değişken sayısı

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$$
$$2\alpha_1 + \alpha_2 + 2\alpha_3 = 0$$

$\alpha_3 = 1/2$ dersek $\longrightarrow$

$$\alpha_1 + 2\alpha_2 = -1 \quad / -$$
$$2\alpha_1 + \alpha_2 = -1$$
$$\overline{\alpha_1 - \alpha_2 = 0} \longrightarrow \alpha_1 = \alpha_2$$

$$\left.\begin{matrix} 3\alpha_2 = -1 \\ 3\alpha_1 = -1 \end{matrix}\right\} \alpha_1 = \alpha_2 = -1/3$$

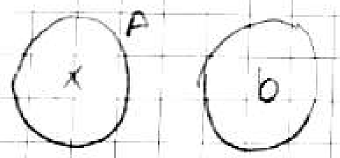
Rank:

$$\operatorname{Rank}(A) = \min\{\dim(C(A)), \dim(R(A))\}$$
$$\leq \min(m, n)$$

dim : dimension

Range Space

$$Ax = b$$



Bütün x'lerin çözümü Range space olur

$Ax = 0 \longrightarrow$ Null space

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 1$$
$$- \; x_1 + x_2 = 1$$

$x_3 = -1$; $x_1 = 0$ $x_2 = 1$

$$\begin{bmatrix} 0 \\ \alpha \\ -\alpha \end{bmatrix} \longrightarrow \alpha \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

GIPTA

$A = \{a_{ij}\}_{n \times n}$

$\det(\alpha A) = \alpha^n \det(A)$

$$\alpha A = (\alpha I)A = \begin{bmatrix} \alpha & 0 & & \\ 0 & \alpha & & \\ 0 & & \alpha & \\ \vdots & & & \\ 0 & & & \alpha \end{bmatrix} \cdot A$$

$\det(A \cdot B) = \det(A) \cdot \det(B)$

$\sim\sim$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 4/7 & -2/7 \\ 0 & 0 & -50/7 & -150/7 \end{array}\right]$$

$-\frac{50}{7} x_3 = -\frac{150}{7} \rightarrow \underline{\underline{x_3 = 3}}$

$x_2 + \frac{12}{7} = \frac{-2}{7} \rightarrow \underline{\underline{x_2 = -12}}$

$x_1 + (-4) + 9 = 6 \rightarrow \underline{\underline{x_1 = 1}}$

$\sim\sim$

$$\begin{aligned} x_1 + (k^2-5)x_2 + x_3 &= k \\ x_1 - x_2 + x_3 &= 2 \\ x_1 + x_2 + 2x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & k^2-5 & 1 & k \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right]$$

2. yi -1 ile çarp → 1. yi -1 ile çarp 3'e ekle →

2.'yi -2'ye böl → 2. yi $6-k^2$ ile çarp 3'e ekle →

$m < n$

$V = \mathrm{span}(\hat{S})$

$$v_1 = a_{11}\hat{v}_1 + a_{12}\hat{v}_2 + \dots a_{1m}\hat{v}_m$$

$$v_n = a_{n1}\hat{v}_1 + a_{n2}\hat{v}_2 + \dots a_{nm}\hat{v}_m$$

$$v = A\hat{v}$$

$$\alpha^T v = 0 \qquad \alpha^T v = \alpha^T A \hat{v} = 0$$

$$\hat{v}^T A^T \alpha = 0 \qquad A^T \alpha = 0$$

Row space

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$r_1 = [a_{11}, a_{12}, \dots a_{1n}]^T$$

$$r_2 \; [a_{21}, a_{22} \dots a_{2n}]^T$$

$$r_m \; [a_{m1}, a_{m2} \dots a_{mn}]^T$$

$$r_i = [a_{i1}\, a_{i2} \dots a_{in}]^T \quad i = 1, 2, \dots m$$

$$R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \qquad R(A) = \mathrm{span}(R)$$

GIPTA

$V\ \{v_1\, v_2 \ldots v_n\}$

Dimension

$\dim(V) = n$

$S = \{v_1, v_2 \ldots v_n\}$

$\hat{S} = \{\hat{v}_1, \hat{v}_2 \ldots \hat{v}_m\}$

$\hat{v} = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_n \end{bmatrix}$ $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$\left.\begin{aligned} \hat{v}_1 &= a_{11}v_1 + a_{12}v_2 + \ldots a_{1n}v_n \\ \hat{v}_2 &= a_{21}v_1 + a_{22}v_2 + \ldots a_{2n}v_n \\ &\vdots \\ \hat{v}_n &= a_{m1}v_1 + a_{m2}v_2 + \ldots a_{mn}v_n \end{aligned}\right\} \hat{v} = Av$$

$m > n \rightarrow$ Dependent

$\beta^T \hat{v} = \beta^T A v$

$\beta_1\hat{v}_1 + \beta_2\hat{v}_2 + \ldots \beta_m\hat{v}_m = 0$

$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \neq 0 \qquad \beta^T\hat{v} = 0$

$\beta^T A v = 0 \qquad v^T A^T \beta = 0 \qquad A^T\beta = 0$

GIPTA

If the above conditions are satisfied, then V defines a vector space over the field R
R → field

V → vector space

$S = \{ v_1, v_2, \ldots v_n \}$

Basis

1. S is a linearly independent set
2. V = Span(S)

$v_1 \; v_2 \; \ldots \; v_n$

$v_0 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

$\alpha = [\alpha_1 \; \alpha_2 \ldots \ldots \alpha_n]^T$

$v = [v_1 \; v_2 \ldots \ldots v_n]^T$

* α is the representation of v_0 with respect to the basis v

$v_0 \quad \alpha_1 \; \alpha_2 \ldots \ldots \alpha_n$
$\quad\;\; \hat{\alpha}_1 \; \hat{\alpha}_2 \ldots \ldots \hat{\alpha}_n$

$v_0 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i$

$v_0 = \hat{\alpha}_1 v_1 + \hat{\alpha}_2 v_2 + \ldots \hat{\alpha}_n v_n = \sum_{i=1}^{n} \hat{\alpha}_i v_i$

$\sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \hat{\alpha}_i v_i \qquad \sum_{i=1}^{n} (\alpha_i - \hat{\alpha}_i) v_i = 0$

$(\alpha_1 - \hat{\alpha}_1) v_1 + (\alpha_2 - \hat{\alpha}_2) v_2 + \ldots (\alpha_n - \hat{\alpha}_n) v_n = 0$

$\alpha_1 = \hat{\alpha}_1 \qquad \alpha_2 = \hat{\alpha}_2 \qquad \alpha_n = \hat{\alpha}_n$

GIPTA