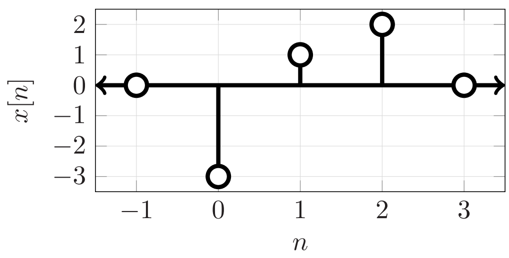
Signal Processing (Örgün Öğretim)
Final Exam Solutions

Istanbul University - Computer Engineering Department - FALL 2017

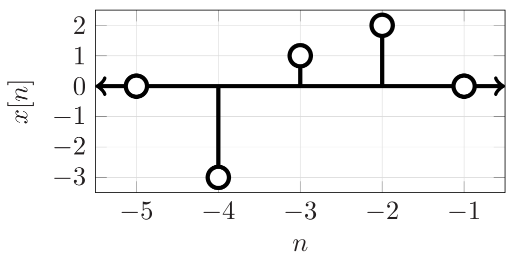
December 28$^{\text{th}}$, 2017

Q1: `(15 pts)` Consider the following DISCRETE TIME signal. Please carefully sketch $x[4n+4]$. Show your steps to receive credit.

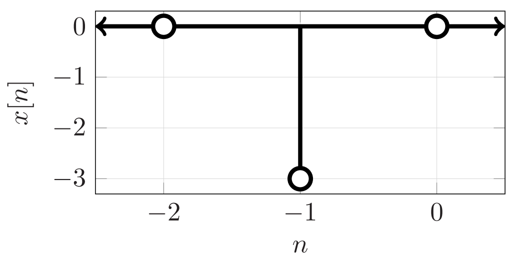


Solution 1:

First, let's shift $x[n]$ to find $x[n+4]$.



First, let's scale the signal to find $x[4n+4]$.



Q2: Consider the following discrete time system, $\mathcal{H}_1$. Answer the following questions.

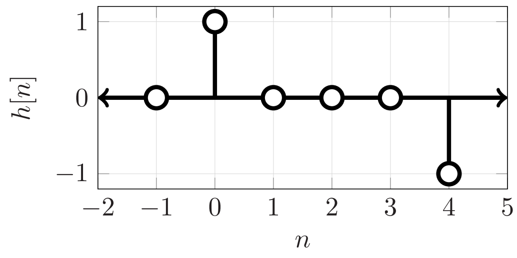
$$y[n] = \mathcal{H}_1\{x[n]\} = x[n] - x[n-4]$$

(a) `(10 pts)` Find and SKETCH the impulse response of $\mathcal{H}_1$.

Solution 2a:

The *Impulse Response* is the output of a system when the impulse function is applied to the input of the system. SO,

$$h[n] = \mathcal{H}_1\{\delta[n]\}$$
$$h[n] = \delta[n] - \delta[n-4]$$

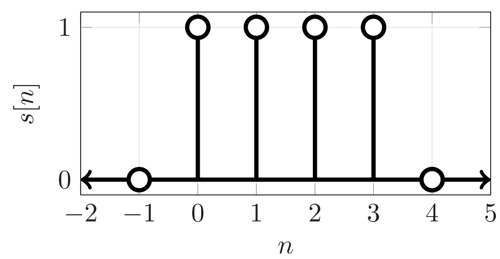


(b) `(15 pts)` Find and SKETCH the step response of $\mathcal{H}_1$.

Solution 2b:

The *Step Response* is the output of a system when the *step* function is applied to the input of the system. So,

$$s[n] = \mathcal{H}_1\{u[n]\}$$
$$s[n] = u[n] - u[n-4]$$



Q3: For a discrete time system, $\mathcal{H}_2$, the impulse response is given below as $h[n]$.

$$h[n] = \mathrm{e}^{-2n}\, u[n-3]$$

(a) `(10 pts)` Please state whether or not $\mathcal{H}_2$ is memoryless and/or causal. (No explanation necessary)

Solution 3a:

Not memoryless & Causal ■

(b) `(15 pts)` Find the output of $\mathcal{H}_2$ when the input is given as the following:

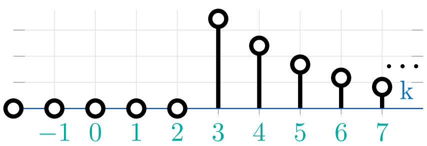
$$x[n] = u[1-n]$$

Solution 3b:

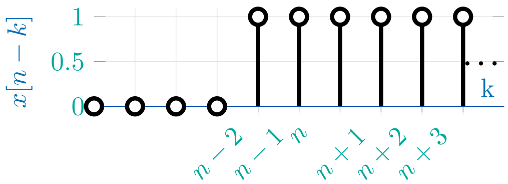
Since we know the impulse response, we can use

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \end{aligned}$$

$$h[k] = \mathrm{e}^{-2k}\, u[k-3]$$



$$x[n-k] = u[k-(n-1)]$$



For $n-1 < 3$, therefore for $n < 4$

$$\begin{aligned} y[n] &= \sum_{k=3}^{\infty} \mathrm{e}^{-2k} \\ &= \sum_{k=3}^{\infty} (\mathrm{e}^{-2})^k \\ &= \frac{(\mathrm{e}^{-2})^3}{1-\mathrm{e}^{-2}} \\ &= \frac{\mathrm{e}^{-6}}{1-\mathrm{e}^{-2}} \\ &= 0.002867 \end{aligned}$$

For $n \geq 4$

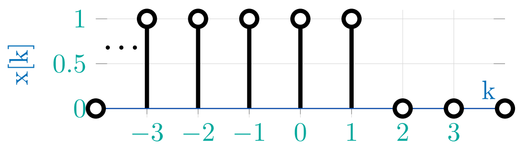
$$\begin{aligned} y[n] &= \sum_{k=n-1}^{\infty} \mathrm{e}^{-2k} \\ &= \sum_{k=n-1}^{\infty} (\mathrm{e}^{-2})^k \\ &= \frac{(\mathrm{e}^{-2})^{n-1}}{1-\mathrm{e}^{-2}} \\ &= \frac{\mathrm{e}^{2-2n}}{1-\mathrm{e}^{-2}} \end{aligned}$$

So,

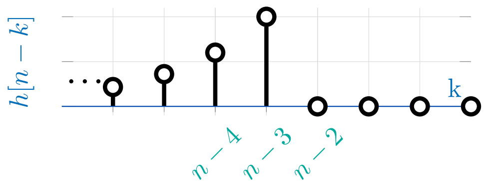
$$y[n] = \begin{cases} \dfrac{\mathrm{e}^{-6}}{1-\mathrm{e}^{-2}} & , \quad n < 4 \\ \dfrac{\mathrm{e}^{2-2n}}{1-\mathrm{e}^{-2}} & , \quad n \geq 4 \end{cases}$$ ■

We could also flip & shift $h[n]$ instead of $x[n]$.

$$x[k] = u[1-k]$$



$$h[n-k] = \mathrm{e}^{-2(n-k)}\, u[n-k-3]$$



For $n-3<4$, therefore for $n<4$

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{n-3} \mathrm{e}^{-2n-2k} \\
&= \mathrm{e}^{-2n} \sum_{k=-\infty}^{n-3} (\mathrm{e}^{2})^{k} \\
&= \mathrm{e}^{-2n} (\mathrm{e}^{2})^{n-3} \frac{e^{2}}{\mathrm{e}^{2}-1} \\
&= \mathrm{e}^{-6} \frac{\mathrm{e}^{2}}{1-\mathrm{e}^{2}} \\
&= 0.002867
\end{aligned}
$$

For $n \geq 4$

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{1} \mathrm{e}^{-2n-2k} \\
&= \cdots \\
&= \frac{\mathrm{e}^{2-2n}}{1-\mathrm{e}^{-2}}
\end{aligned}
$$

Same result will be obtained.

$$
y[n] = \begin{cases} \dfrac{\mathrm{e}^{-6}}{1-\mathrm{e}^{-2}} &, \quad n<4 \\ \dfrac{\mathrm{e}^{2-2n}}{1-\mathrm{e}^{-2}} &, \quad n \geq 4 \end{cases} \qquad \blacksquare
$$

Q4: `(10 pts)` The step response of a continuous time system, $\mathcal{H}_3$, is given as the following. Determine the impulse response of $\mathcal{H}_3$.

$$
s(t) = (1-\mathrm{e}^{-2t})\, u(t)
$$

Solution (4):

$$
\begin{aligned}
h(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\, s(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\{(1-\mathrm{e}^{-2t})\, u(t)\} \\
&= \{\frac{\mathrm{d}}{\mathrm{d}t}\, (1-\mathrm{e}^{-2t})\}\, u(t) \; + (1-\mathrm{e}^{-2t})\, \{\frac{\mathrm{d}}{\mathrm{d}t}\, u(t)\} \\
&= 2\, \mathrm{e}^{-2t}\, u(t) + (1-\mathrm{e}^{-2t})\delta(t)
\end{aligned}
$$

Since $x(t)\ \delta(t) = x(0)\ \delta(t)$,

$$
\begin{aligned}
(1-\mathrm{e}^{-2t})\ \delta(t) &= (1-\mathrm{e}^{-2\cdot 0})\ \delta(t) \\
&= (1-1)\ \delta(t) \\
&= 0 \\
h(t) &= 2\ \mathrm{e}^{-2t}\ u(t) \qquad \blacksquare
\end{aligned}
$$

Q5: `(15 pts)` For a continuous time system, $\mathcal{H}_4$, the impulse response is given below as $h(t)$. Find the output when the input signal is $x(t)$.

$$
h(t) = u(t+1) - u(t-1)
$$

$$
x(t) = \mathrm{e}^{-t}\, u(t-1)
$$

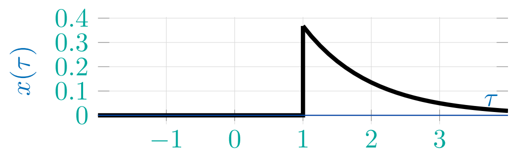
Solution (5):

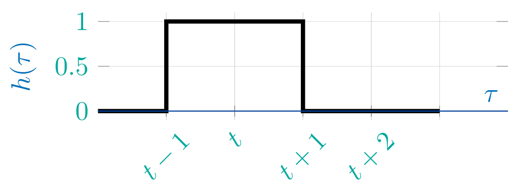
$$
y(t) = x(t) * h(t)
$$

$$
= h(t) * x(t)
$$

$$
x(t) = \int_{-\infty}^{\infty} h(\tau)\ x(t-\tau)\ \mathrm{d}\tau
$$

$$
x(\tau) = \mathrm{e}^{-\tau}\ u(\tau-1)
$$





⊙ For $t+1<1 \implies t<0 \quad y(t)=0$.

⊙ For $t+1 \geq 1 \implies t \geq 0$ and, for $t-1<1 \implies t \leq 2$, which means $0 \leq t \leq 2$

$$
\begin{aligned}
y(t) &= \int\limits_{1}^{t+1} \mathrm{e}^{-\tau}\ \mathrm{d}\tau \\
&= \mathrm{e}^{-1} - \mathrm{e}^{-t-1} \\
&= \mathrm{e}^{-1}\ (1 - \mathrm{e}^{-t})
\end{aligned}
$$

⊙ For $t - 1 > 1 \implies t > 2$

$$
\begin{aligned}
y(t) &= \int\limits_{t-1}^{t+1} \mathrm{e}^{-\tau}\ \mathrm{d}\tau \\
&= \mathrm{e}^{-t+1} - \mathrm{e}^{-t-1} \\
&= \mathrm{e}^{-t}\ (\mathrm{e} - \mathrm{e}^{-1})
\end{aligned}
$$

Therefore:

$$
y(t) = \begin{cases} 0 & , \quad t < 0 \\ \mathrm{e}^{-1}\ (1 - \mathrm{e}^{-t}) & , \quad 0 \le t \le 2 \\ \mathrm{e}^{-t}\ (\mathrm{e} - \mathrm{e}^{-1}) & , \quad t > 2 \end{cases} \qquad \blacksquare
$$

Q6: `(10 pts)` Consider the following discrete time system, $\mathcal{H}_5$. Determine whether it is linear.

$$
y[n] = \mathcal{H}_5\{x[n]\} = 2\, x[n] + 3
$$

Solution 6:

Checking for homogenity:

$$
\begin{aligned}
y_1[n] &= \mathcal{H}_5\{\alpha\ x[n]\} = 2\ \alpha\ x[n] + 3 \\
y_2[n] &= \alpha\ y[n] = 2\ \alpha\ x[n] + 3\ \alpha \\
y_1[n] &\neq y_2[n]
\end{aligned}
$$

Therefore $\mathcal{H}_5$ is not LINEAR. (No need to check for superposition) $\blacksquare$

Q7: `(10 pts)` (BONUS QUESTION) Is the signal $x(t)$ given in Q5 an energy signal, power signal, or neither? Calculate its average power and energy.

Solution 7:

Let's check its average power first.

$$
\begin{aligned}
P_{avg} &= \lim_{T\to\infty} \frac{1}{T} \int\limits_{0}^{T} x^2(t)\ \mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{T} \int\limits_{1}^{T} (\mathrm{e}^{-t})^2\ \mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{2T} (\mathrm{e}^{-2T} - \mathrm{e}^{-2}) \\
&= 0 \quad \blacksquare
\end{aligned}
$$

Therefore it is not a power signal. Let's check its energy.

$$
\begin{aligned}
E &= \int\limits_{-\infty}^{\infty} x^2(t)\ \mathrm{d}t \\
&= \int\limits_{1}^{\infty} (\mathrm{e}^{-t})^2\ \mathrm{d}t \\
&= \frac{1}{2}\mathrm{e}^{-2} \\
&< \infty
\end{aligned}
$$

Therefore $x(t)$ is an energy signal. $\blacksquare$