AMS-131: Introduction to Probability Theory (Spring 2009)
Midterm Exam Solutions

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1. (20 points). A box contains 5 red balls, 10 blue balls, and 5 yellow balls. Consider the following experiment. Four balls will be selected at random, without replacement, from the box (the balls will be selected one at a time). The color of each selected ball will be observed. Moreover, if the selected ball is red or yellow, 3 new balls of the same color will be put into the box. If the selected ball is blue, no balls will be added to the box.
• Obtain the probability that the first two balls will be red, and the third ball will be blue, and the fourth ball will be yellow.

Solution: For $i = 1, ..., 4$, let R_i be the event that a red ball is drawn on the i-th draw, B_i be the event that a blue ball is drawn on the i-th draw, and Y_i be the event that a yellow ball is drawn on the i-th draw. Then, the probability that the first two balls will be red, and the third ball will be blue, and the fourth ball will be yellow can be expressed as $\Pr(R_1 \cap R_2 \cap B_3 \cap Y_4)$.

Using the multiplication rule for conditional probabilities, and the problem assumption regarding red or yellow balls added to the box, we obtain

$$\begin{aligned}
\Pr(R_1 \cap R_2 \cap B_3 \cap Y_4) &= \Pr(R_1)\Pr(R_2|R_1)\Pr(B_3|R_1 \cap R_2)\Pr(Y_4|R_1 \cap R_2 \cap B_3) \\
&= \frac{5}{20} \times \frac{7}{22} \times \frac{10}{24} \times \frac{5}{23} = 0.0072.
\end{aligned}$$

2. (25 points). A system consists of two components A and B. The probability that component B functions during its design life is 0.9; the probability that neither of the components functions is 0.04; and the probability that exactly one of the two components functions is 0.21.

• Compute the conditional probability that component B functions during its design life given that component A functions during its design life.

Solution: Denote by A the event that component A functions during its design life; by B the event that component B functions during its design life; by C the event that exactly one of the two components functions; and by D the event that neither of the components functions. Note that $C = (A \cap B^c) \cup (B \cap A^c)$, and $D = A^c \cap B^c = (A \cup B)^c$.

Based on the problem assumptions, we have that $\Pr(B) = 0.9$, $\Pr(C) = 0.21$, and $\Pr(D) = 0.04$. We need to compute $\Pr(B|A)$.

We have $A \cup B = C \cup (A \cap B)$, with the events C and $(A \cap B)$ disjoint. Therefore, $\Pr(A \cup B) = \Pr(C) + \Pr(A \cap B)$, and so $\Pr(A \cap B) = \Pr(A \cup B) - \Pr(C) = 1 - \Pr((A \cup B)^c) - \Pr(C) = 1 - \Pr(D) - \Pr(C) = 0.75$.

Moreover, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, and, therefore, $\Pr(A) = 1 - \Pr((A \cup B)^c) - \Pr(B) + \Pr(A \cap B) = 1 - \Pr(D) - \Pr(B) + \Pr(A \cap B) = 0.81$.

Finally,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.75}{0.81} = 0.9259.$$

3. (25 points). Each one of three contestants on a quiz show is asked to specify one of six possible categories of questions. Assume that Pr(contestant requests category i) $= 1/6$, $i = 1,\ldots,6$, the same for all three contestants. Suppose that the contestants choose their categories independently of one another. Note that it is possible for two (or all three) contestants to choose the same category. Hence, the sample space consists of all possible triplets (x_1, x_2, x_3), where $x_j = 1, \ldots, 6$ is the choice of the j-th contestant (for $j = 1, 2, 3$).

(a) (10 points). Are all simple outcomes of the sample space equally likely? Justify your answer.

Solution: Based on the multiplication rule, the sample space consists of 6^3 simple outcomes. Moreover, for a generic simple outcome, we have

$$\Pr(\{(i,j,k)\}) = \Pr(x_1 = i, x_2 = j, x_3 = k) = \Pr(x_1 = i)\Pr(x_2 = j)\Pr(x_3 = k) = \frac{1}{6^3}$$

using the assumption of independence. Therefore, all outcomes are equally likely.

(b) (15 points). Obtain the conditional probability that exactly one contestant selects category 1, given that all three contestants select different categories.

Solution: Let A be the event that exactly one contestant selects category 1, and B be the event that all three contestants select different categories.

We have

$$\Pr(B) = \frac{6 \times 5 \times 4}{6^3}$$

and

$$\Pr(A \cap B) = \frac{3 \times (1 \times 5 \times 4)}{6^3}$$

(note that $\Pr(A \cap B)$ arises through the union of three disjoint events, one for each contestant).

Finally, $\Pr(A|B) = \Pr(A \cap B)/\Pr(B) = 0.5$.

4. (30 points). Consider a machine that produces items that can be classified as being of either high quality or medium quality. Assume that, given that the machine is adjusted properly, 50 percent of the items produced by it are of high quality and the other 50 percent are of medium quality. Moreover, assume that, given that the machine is improperly adjusted, only 25 percent of the items produced by it are of high quality and the other 75 percent are of medium quality. Finally, suppose that the probability that the machine is improperly adjusted is given by 0.1. Denote by B_1 the event that the machine is adjusted properly and by B_2 the event that the machine is improperly adjusted.

Six items produced by the machine at a certain time are selected at random and inspected. (Assume that the outcomes for these six items are conditionally independent given B_1, and also that they are conditionally independent given B_2.) Suppose that it is observed that four of the items are of high quality and two items are of medium quality.

• Given the observation from these six items, compute the posterior probability that the machine was adjusted properly at that time.

Solution: Denote by D the event that an item produced by the machine is of high quality, and by A the event that 4 of the 6 selected items are of high quality. It is given that $\Pr(B_1) = 0.9$; $\Pr(B_2) = 0.1$; $\Pr(D|B_1) = 0.5$; and $\Pr(D|B_2) = 0.25$. We need to compute $\Pr(B_1|A)$.

Conditionally on B_1, we have a Binomial experiment with probability of success given by $\Pr(D|B_1)$, and, therefore,

$$\Pr(A|B_1) = \binom{6}{4}(0.5)^4(0.5)^2.$$

Similarly, conditionally on B_2, we have a Binomial experiment, now with probability of success given by $\Pr(D|B_2)$, and, hence,

$$\Pr(A|B_2) = \binom{6}{4}(0.25)^4(0.75)^2.$$

Therefore, using Bayes theorem,

$$\Pr(B_1|A) = \frac{\Pr(B_1)\Pr(A|B_1)}{\Pr(B_1)\Pr(A|B_1) + \Pr(B_2)\Pr(A|B_2)} = 0.98461.$$