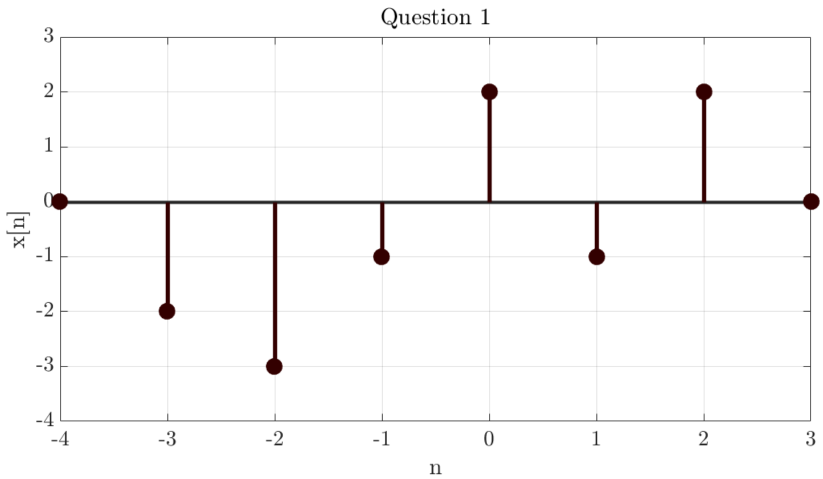
Signal Processing (Örgün Öğretim)
Midterm Exam Solutions

Istanbul University - Computer Engineering Department - FALL 2017

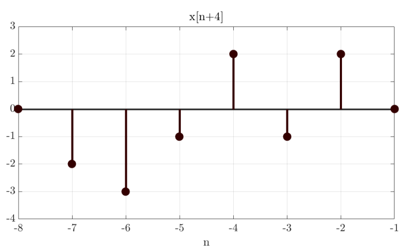
November 2$^{\text{nd}}$, 2017

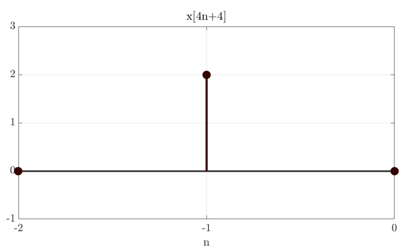
Q1: `(20 pts)` Consider the following DISCRETE TIME signal. Please carefully sketch $x[4n+4] + x[3n-3]$. Show your steps to receive credit.



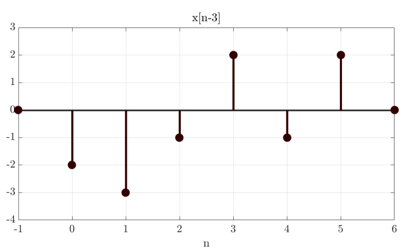
Solution 1:

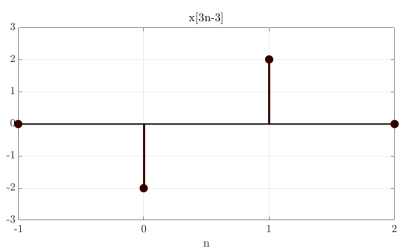
Let's sketch $x[n+4]$ and $x[4n+4]$.



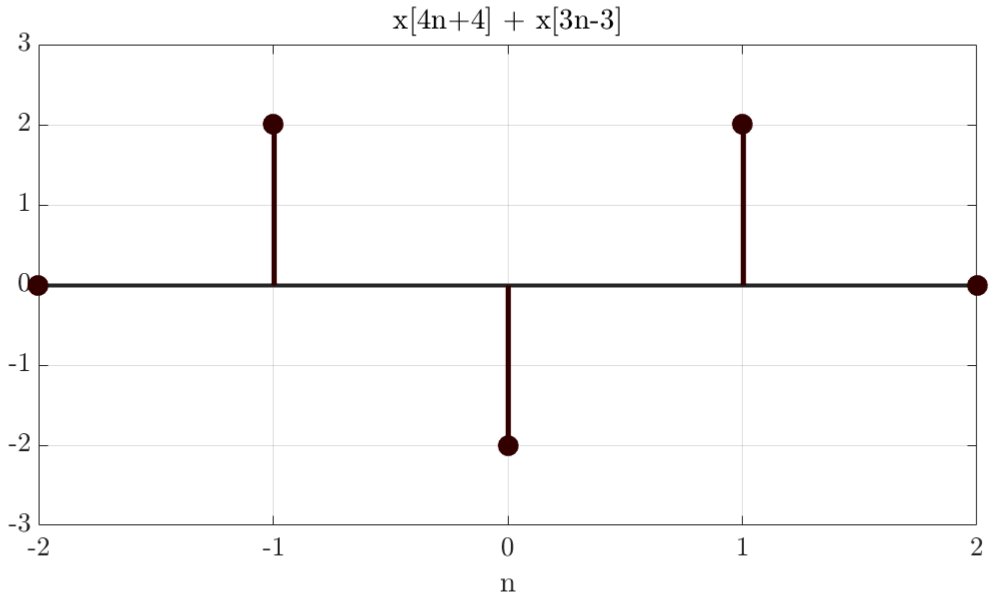


Let's sketch $x[n-3]$ and $x[3n-3]$.





Then, sketch $x[4n + 4] + x[3n - 3]$.



Q2: `(20 pts)` Show that the product of an even and an odd signal is an odd signal.

Solution 2:

Let's say $x(t) = x_e(t) \cdot x_o(t)$ where $x_e(t)$ is an even signal and $x_o(t)$ is an odd signal. From the definitions, we can see that:

$$
\begin{aligned}
x_e(t) &= x_e(-t) \\
x_o(t) &= -x_o(-t) \\
x(-t) &= x_e(-t) \cdot x_o(-t) \\
x(-t) &= x_e(t) \cdot \{-x_o(t)\} \\
x(-t) &= -\{x_e(t) \cdot x_o(t)\} \\
x(-t) &= -x(t)
\end{aligned}
$$

Which is the definition of an odd signal. ■

Q3: `(20 pts)` Consider the following DISCRETE TIME signal. Is $x[n]$ periodic? If so, calculate its fundamental period.

$$x[n] = 2\ \sin\left(\frac{3\pi}{5}n + \frac{\pi}{2}\right) - 4\ \cos\left(\frac{7\pi}{11}n + \frac{4\pi}{9}\right)$$

Solution 3:

Let's say

$$x[n] = \underbrace{2\ \sin\left(\frac{3\pi}{5}n + \frac{\pi}{2}\right)}_{x_1[n]} \underbrace{-\ 4\ \cos\left(\frac{7\pi}{11}n + \frac{4\pi}{9}\right)}_{x_2[n]}$$

$$x[n] = x_1[n] + x_2[n]$$

For $x_1[n]$:

$$\Omega_1 = \frac{3\pi}{5} = 2\pi\,\frac{m_1}{N_1}$$

where m_1 and N_1 are integers. The smallest $(m_1,\ N_1)$ pair is (3, 10) so, the period of $x_1[n]$ is $N_1 = 10$ cycles. Similiarly, for $x_2[n]$:

$$\Omega_2 = \frac{7\pi}{11} = 2\pi\,\frac{m_2}{N_2}$$

where m_2 and N_2 are integers. The smallest $(m_2,\ N_2)$ pair is (7, 22) so, the period of $x_2[n]$ is $N_1 = 22$ cycles. The period of $x[n]$ can be found using:

$$N = \text{LCM}(N_1, N_2) = \text{LCM}(10, 22) = 110 \quad \blacksquare$$

Q4: `(40 pts)` The systems below show the input as $x(t)$ or $x[n]$ and the output as $y(t)$ or $y[n]$. For each system, determine whether it is **(i)** `(2 pts each)` memoryless, **(ii)** `(2 pts each)` causal, **(iii)** `(4 pts each)` stable (show your work), **(iv)** `(6 pts each)` linear (show your work), and **(v)** `(6 pts each)` time-invariant (show your work).

$$\text{(a)}\quad y[n] = 2\,x[n-1]\,(\,u[n]\ -\ u[n-4]\,)$$

$$\text{(b)}\quad y(t) = \frac{\cos[x(t+1)]}{\sin[x(t+1)]}$$

Solution 4a:

$$(a)\qquad y[n] = 2\,x[n-1]\,(\,u[n]\ -\ u[n-4]\,)$$

(i) NOT-MEMORYLESS $\blacksquare$

(ii) CAUSAL ■

(iii) Assuming $|x[n]| \leq M_x < \infty$, for $\forall n \in \mathbb{N}$

$$|y[n]| = |2x[n-1] \cdot (\, u[n] \;-\; u[n-4]\,)|$$
$$|y[n]| \leq 2\,|x[n-1]| \;\cdot\; |(u[n] \;-\; u[n-4])|$$
$$|(u[n] \;-\; u[n-4])| \leq 1 \qquad \forall n \in \mathbb{N}$$
$$|y[n]| \leq 2M_x$$
$$|y[n]| \leq M_y < \infty \qquad \text{for } \forall n \in \mathbb{N}$$

Therefore, the system is BIBO-STABLE. ■

(iv) Homogenity:

$$\mathcal{H}\{\alpha\, x[n]\} = 2\,(\alpha\, x[n-1])\,(\, u[n] \;-\; u[n-4]\,)$$
$$\alpha\, y[n] = \alpha\, \{\underbrace{2\;\, x[n-1]\,(\, u[n] \;-\; u[n-4]\,)}_{y[n]}\}$$

$$\mathcal{H}\{\alpha\, x[n]\} = \alpha\, y[n]$$

Homogenity is satisfied.

Superposition:

Given the signals $x_1[n]$ and $x_2[n]$ and:

$$\mathcal{H}\{x_1[n]\} = y_1[n]$$
$$\mathcal{H}\{x_2[n]\} = y_2[n]$$

So,

$$\mathcal{H}\{x_1[n] + x_2[n]\} = 2(x_1[n-1] + x_2[n-1])\,(\, u[n] \;-\; u[n-4]\,)]$$
$$= \underbrace{2x_1[n-1]\,(\, u[n] \;-\; u[n-4]\,)}_{y_1[n]} + \underbrace{2x_2[n-1]\,(\, u[n] \;-\; u[n-4]\,)}_{y_2[n]}$$

$$= y_1[n] + y_2[n]$$

Superposition is satisfied. Therefore, $\mathcal{H}$ is LINEAR. ■

We could also check for both homogenity and superposition in a single step. We can also make our life a little bit easier by letting

$$b[n] \triangleq 2 \times (u[n] - u[n-4])$$

$$\text{So,} \qquad y_1[n] = x_1[n-1] \times b[n]$$

$$\text{And,} \qquad y_2[n] = x_2[n-1] \times b[n]$$

Then:

$$\begin{aligned}
\mathcal{H}\{\alpha x_1[n] + \beta x_2[n]\} &= (\alpha x_1[n] + \beta x_2[n]) \times b[n] \\
&= \alpha x_1[n-1] \times b[n] + \beta x_2[n-1] \times b[n] \\
&= \alpha \underbrace{x_1[n-1] \times b[n]}_{y_1[n]} + \beta \underbrace{x_2[n-1] \times b[n]}_{y_2[n]} \\
&= \alpha y_1[n] + \beta y_2[n]
\end{aligned}$$

Thus, linearity is satisfied ■

(v)

Let's say $y_1[n] = y[n-n_0]$ and $y_2[n] = \mathcal{H}\{x[n-n_0]\}$. We'll check if they are equal.

$$\begin{aligned}
y_2[n] &= 2x[n-n_0-1]\,(\,u[n] \;-\; u[n-4]\,) \\
y_1[n] &= 2x[n-n_0-1]\,(\,u[n-n_0] \;-\; u[n-n_0-4]\,) \\
y_1[n] &\neq y_2[n]
\end{aligned}$$

Therefore $\mathcal{H}$ is NOT TIME INVARIANT. ■

Solution 4b:

$$(b) \qquad y(t) = \frac{\cos[x(t+1)]}{\sin[x(t+1)]}$$

(i) NOT MEMORYLESS ■

(ii) NOT CAUSAL ■

(iii) Assuming $|x(t)| \leq M_z < \infty$

$$|y(t)| = \left| \frac{\cos[x(t+1)]}{\sin[x(t+1)]} \right|$$

Since $y(t) \to \infty$ when $x(t) = 0, \pi, 2\pi...$, the system is NOT STABLE. ■

(iv) Checking for homogenity

$$y_1(t) = \mathcal{H}\{\alpha\ x(t)\} = \frac{\cos[\alpha\, x(t+1)]}{\sin[\alpha\, x(t+1)]}$$

$$y_2(t) = \alpha\ y(t) = \alpha\ \frac{\cos[x(t+1)]}{\sin[x(t+1)]}$$

$$y_1(t) \neq y_2(t)$$

Therefore, $\mathcal{H}$ is NOT LINEAR. (No need to check for superposition but we'll check it anyway)
Let's say $\mathcal{H}\{x_1(t)\} = y_1(t)$ and $\mathcal{H}\{x_2(t)\} = y_2(t)$ and $y(t) = y_1(t) + y_2(t)$

$$\mathcal{H}\{x_1(t) + x_2(t)\} = \frac{\cos[x_1(t+1)\ +\ x_2(t+1)]}{\sin[x_1(t+1)\ +\ x_2(t+1)]}$$

$$y_1(t) + y_2(t) = \frac{\cos[x_1(t+1)]}{\sin[x_1(t+1)]} + \frac{\cos[x_2(t+1)]}{\sin[x_2(t+1)]}$$

$$\mathcal{H}\{x_1(t) + x_2(t)\} \neq y_1(t) + y_2(t)$$

It does not satisfy the superposition principle either. ■

(v) Let's say $y_1(t) = y(t - t_0)$ and $y_(t) = \mathcal{H}\{x(t - t_0)\}$. We'll check to see if they are equal.

$$y_2(t) = \frac{\cos[x(t - t_0 + 1)]}{\sin[x(t - t_0 + 1)]}$$

$$y_1(t) = \frac{\cos[x(t - t_0 + 1)]}{\sin[x(t - t_0 + 1)]}$$

$$y_1(t) = y_2(t)$$

Therefore, $\mathcal{H}$ is TIME-INVARIANT. ■