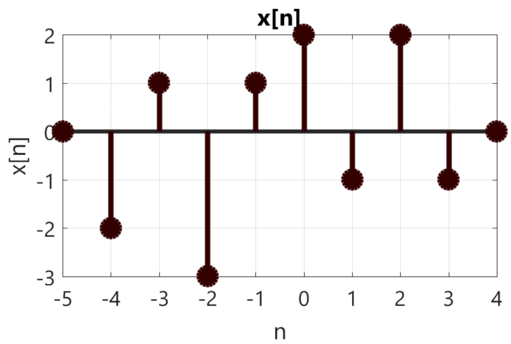
Signal Processing (Örgün)- Midterm Exam

Istanbul University - Computer Engineering Department - FALL 2016

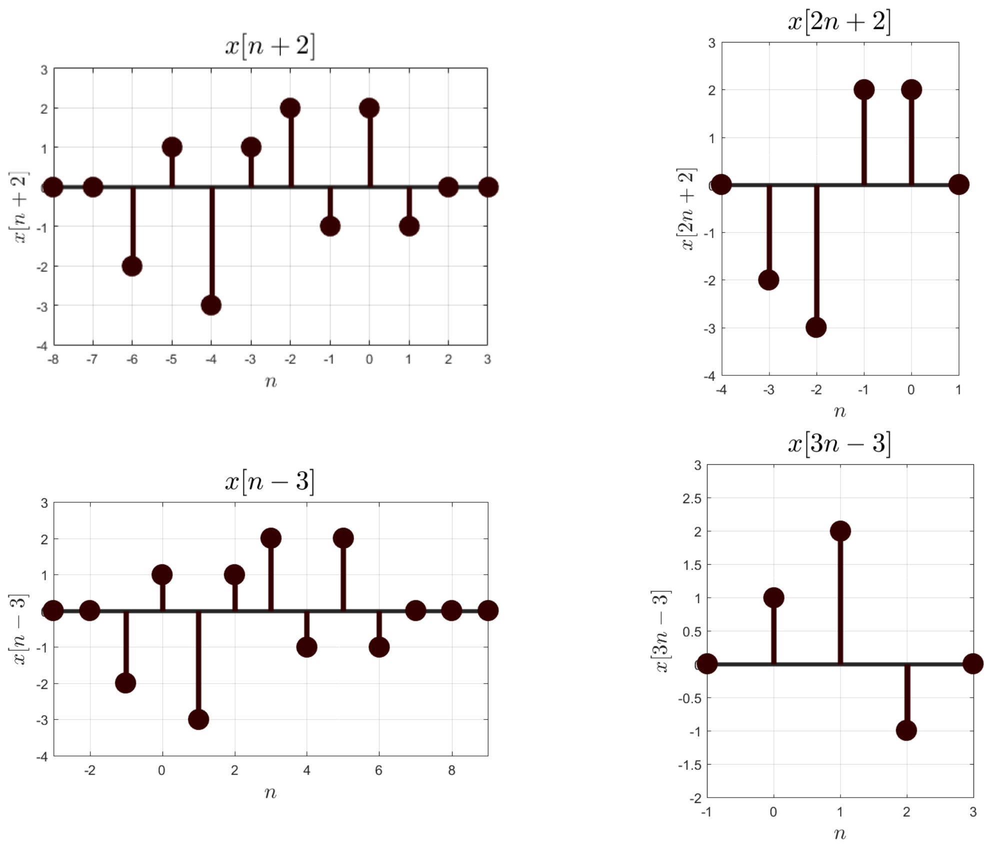
November 10$^{\text{th}}$, 2016

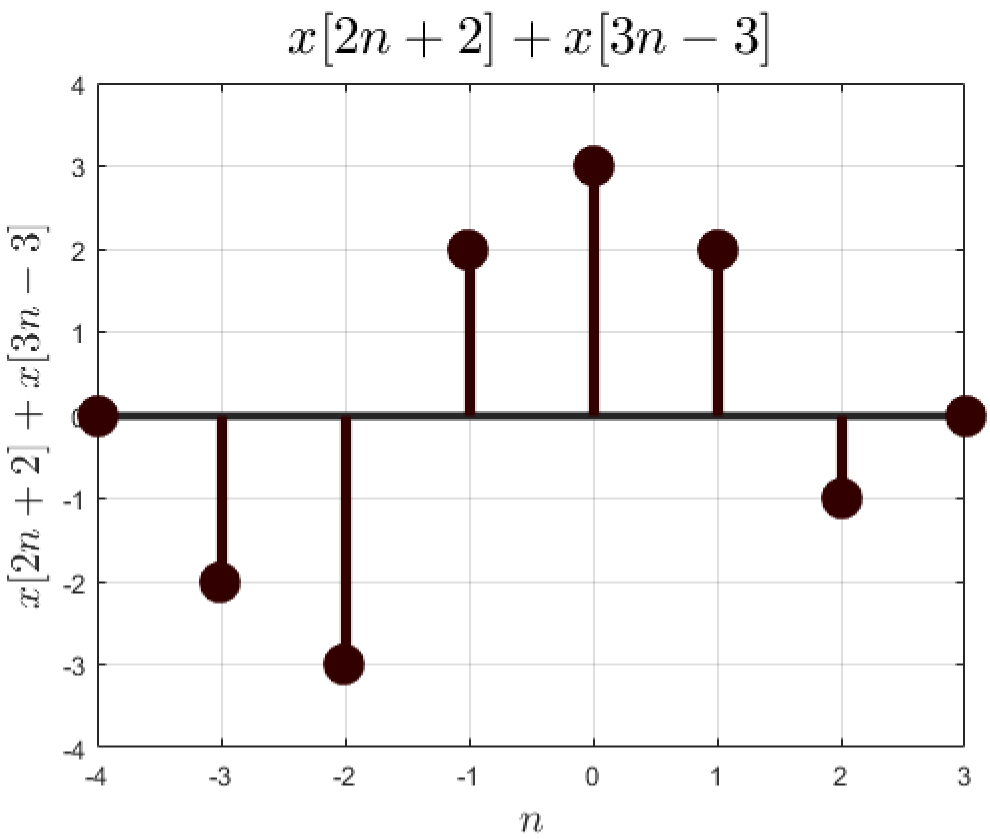
Q1: Consider the following DISCRETE TIME signals and answer the following questions.



(a) `(20 pts)` Please carefully sketch $x[2n+2]+x[3n-3]$.

Solution 1a:





(b) `(10 pts)` Please determine whether $x[n]$ is an energy or power signal. Calculate its power or energy, whichever applies.

Solution 1b:

Let's first calculate its energy.

$$
\begin{aligned}
E &= \sum_{-\infty}^{\infty} x^2[n] \\
&= (-2)^2 + 1^2 + (-3)^2 + 1^2 + 2^2 + (-1)^2 + 2^2 + (-1)^2 \\
&= 25
\end{aligned}
$$

Since $0 < 25 < \infty$, it's an energy signal and its energy is 25. ■

Q2: `(30 pts)` The systems that follow have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$. For each system, determine whether it is (i) memoryless, (ii) causal , (iii) stable , (iv) linear, and (v) time-invariant.

$$
\textbf{(a)} \qquad y[n] = \begin{cases} x[2n] & , \quad n < 0 \\ \dfrac{n}{n+1} & , \quad n \geq 0 \end{cases}
$$

$$
\textbf{(b)} \qquad y(t) = (t+1)^2 x(t)
$$

Solution 2a:

i) Not memoryless

ii) We know that for $n < 0$, $x[2n]$ shows the past values of the input. But as n becomes positive, input has no bearing. So, the output never depends on the future. Therefore, the SYSTEM is CAUSAL.

iii) Let's look at the system. Assuming $x[n]$ is bounded, that is $|x[n]| \leq M_x > \infty$, let's check the output.

$$\begin{aligned}
\text{when} \quad n < 0 \quad |y[n]| &= |x[2n]| \leq M_x \\
\text{when} \quad n \geq 0 \quad |y[n]| &= |\frac{n}{n+1}| < \frac{n+1}{n+1} = 1 \\
|y[n]| &< 1
\end{aligned}$$

So, we can see that for a bounded input, output is bounded for all values of n. Therefore, this SYSTEM is BIBO-stable.

iv) Let's write the system this way.

$$y[n] = x[2n]u[1-n] + \frac{n}{n+1}u[n]$$

So, let's check homogenity.

$$\begin{aligned}
y_1[n] = \mathcal{H}\{ax[n]\} &= ax[2n]u[1-n] + \frac{n}{n+1}u[n] \\
ay[n] &= a\{x[2n]u[1-n] + \frac{n}{n+1}u[n]\} \\
y_1[n] &\neq y[n]
\end{aligned}$$

Therefore, this SYSTEM is not LINEAR.

v)

$$\begin{aligned}
y_1[n] = \mathcal{H}\{x[n-n_0]\} &= x[2(n-n_0)]u[1-n] + \frac{n}{n+1}u[n] \\
y[n-n_0] &= x[2(n-n_0)]u[1-n+n_0] + \frac{n-n_0}{n-n_0+1}u[n-n_0] \\
y_1[n] &\neq y[n-n_0]
\end{aligned}$$

Therefore, this SYSTEM is not TIME-INVARIANT.

Solution 2b:

i) Memoryless.

ii) Causal.

iii) Assuming $x(t)$ is bounded, that is $|x(t)| \leq M_x < \infty$,

$$\begin{aligned} |y(t)| &= |(t+1)^2||x(t)| \\ &\leq |(t+1)^2|M_x \end{aligned}$$

Even if the input is bounded, $(t+1)^2$ is not, that is, when t goes to infinity, $(t+1)^2$ goes to infinity. Therefore, this system is not BIBO-stable.

iv) Let's check for homogenity and superposition together.

$$\begin{aligned} y_1(t) &= \mathcal{H}\{x_1(t)\} = (t+1)^2 x_1(t) \\ y_2(t) &= \mathcal{H}\{x_2(t)\} = (t+1)^2 x_2(t) \\ y(t) &= \mathcal{H}\{ax_1(t) + bx_2(t)\} = (t+1)^2[ax_1(t) + bx_2(t)] \\ &= a(t+1)^2 x_1(t) + b(t+1)^2 x_2(t) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, $\mathcal{H}$ is linear.

v)

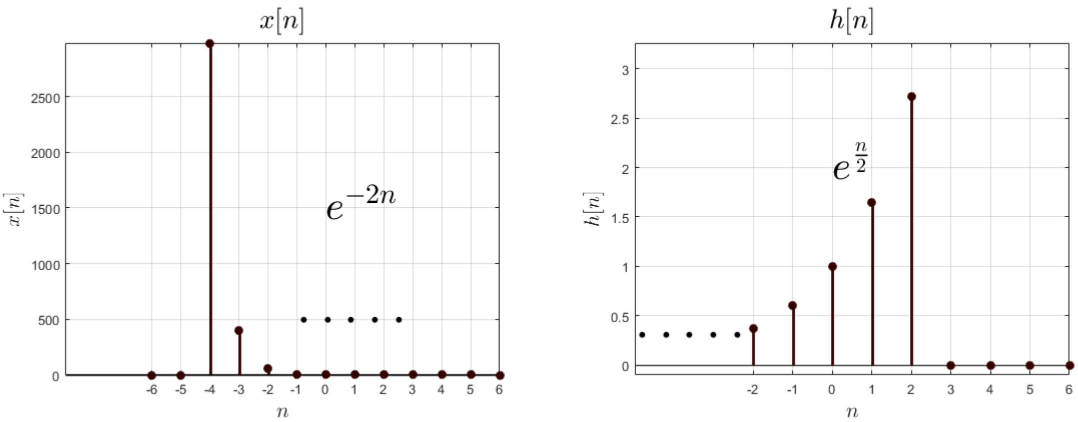
$$\begin{aligned} y_1(t) &= \mathcal{H}\{x[t-t_0]\} = (t+1)^2 x(t-t_0) \\ y(t-t_0) &= (t-t_0+1)^2 x(t-t_0) \\ y_1(t) &\neq y(t-t_0) \end{aligned}$$

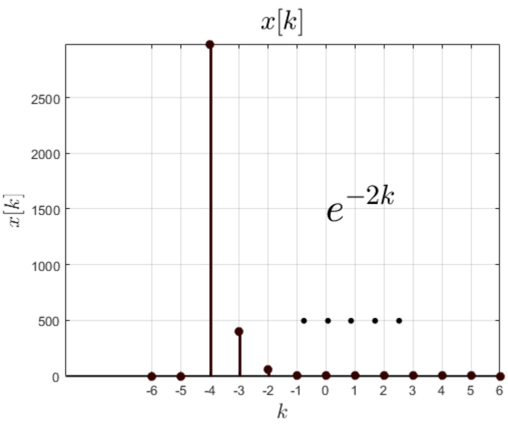
Therefore, $\mathcal{H}$ is not time-invariant.

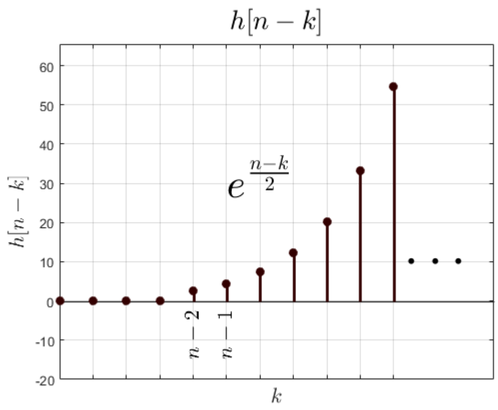
Q3: `(30 pts)` Find the DISCRETE TIME convolution sum of the following two signals.

$$\begin{aligned} x[n] &= e^{-2n} \times u[n+4] \\ h[n] &= e^{n/2} \times u[2-n] \end{aligned}$$

Solution 1a:







Solution 3:

For $n - 2 < -4$, which is $n < -2$:

$$
\begin{aligned}
y[n] &= \sum_{k=-4}^{\infty} e^{-2k} \times e^{(n-k)/2} \\
&= e^{n/2} \sum_{k=-4}^{\infty} e^{-2k} \times e^{-k/2} \\
&= e^{n/2} \sum_{k=-4}^{\infty} e^{-5k/2} \\
&= e^{n/2} \frac{e^{-5\times -4/2}}{1 - e^{-5/2}} \\
&= e^{n/2} \frac{e^{-10}}{1 - e^{-5/2}}
\end{aligned}
$$

For $n - 2 \geq -4$, which is $n \geq -2$:

$$
\begin{aligned}
y[n] &= \sum_{k=n-2}^{\infty} e^{-2k} \times e^{(n-k)/2} \\
&= e^{n/2} \sum_{k=n-2}^{\infty} e^{-2k} \times e^{-k/2} \\
&= e^{n/2} \sum_{k=n-2}^{\infty} e^{-2k-k/2} \\
&= e^{n/2} \sum_{k=n-2}^{\infty} e^{-5k/2} \\
&= e^{n/2} \, \frac{e^{-5(n-2)/2}}{1-e^{-5/2}} \\
&= \frac{e^{-2n+5}}{1-e^{-5/2}} \\
&= \frac{e^{5}}{1-e^{-5/2}} \, e^{-2n}
\end{aligned}
$$

Therefore,

$$
y[n] = \begin{cases} \dfrac{e^{5}}{1-e^{-5/2}} \, e^{-2n} & , \quad n < -2 \\ e^{n/2} \, \dfrac{e^{-10}}{1-e^{-5/2}} & , \quad n \geq -2 \end{cases}
$$

Q4: `(10 pts)` Consider the following signal. Determine if this signal is periodic and if so, find the period, frequency and angular frequency.

$$
x(t) = \cos(3\pi t) + \sin(2t)
$$

Solution 4:

Let's write this as a superposition of two signal

$$
x(t) = \underbrace{\cos(3\pi t)}_{x_1(t)} + \underbrace{\sin(2t)}_{x_2(t)}
$$

For $x_1(t)$, $\omega_1 = 3\ \pi$, for $x_2(t)$, $\omega_2 = 2$. Both signals are periodic. However, in order to the superposition of these two periodic signals to be periodic $\frac{\omega_1}{\omega_2}$ must be a rational number. In this case,

$$
\frac{\omega_1}{\omega_2} = \frac{3\pi}{2}
$$

is not a rational number. Therefore, $x(t)$ is not periodic. ■