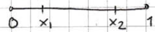
Q1 - 20 pts



Let's say $r = 0.2277$ and define (p) as the answer to the question.

The points x_1 & x_2 satisfy the following

$$\frac{x_1}{1-x_1} < r \quad \text{and} \quad \frac{1-x_2}{x_2} < r$$

- The maximum value of x_1 can be found as such:

$$\frac{x_1}{1-x_1} = r \rightarrow x_1 = r - r x_1$$

$$x_1 = \frac{r}{1+r}$$

- The minimum value of x_2

$$\frac{1-x_2}{x_2} = r \rightarrow 1 - x_2 = r x_2 \quad 1 = r x_2$$

$$x_2 = \frac{1}{1+r}$$

Since $r < 1$ $x_2 < x_1$

$$p = P[(X < x_1) \cup (X > x_2)]$$

($x_1 < x_2$ ∴ these two events are mutually exclusive)

$$= P(X < x_1) + P(X > x_2)$$

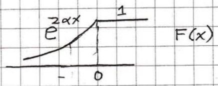
$$= x_1 + (1 - x_2)$$

X: uniformly distributed R.V.

$$p = \frac{r}{1+r} + \left(1 - \frac{1}{1+r}\right)$$

$$p = \frac{2r}{1+r} = \frac{2 \times 0.2277}{1 + 0.2277} = \boxed{0.3709} \ /\!/$$

② $\boxed{a-15p}$



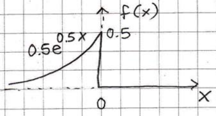
$$P(x>-1) = 1- P(x\leqslant -1)$$
$$= 1 - F(-1) = 0.393369$$
$$= 1 - e^{-2\alpha} = 0.39369$$
$$\therefore \quad e^{-2\alpha} = 1-0.39369$$
$$-2\alpha = \ln(0.60631)$$
$$\alpha = \frac{-0.50036}{-2} = +0.2502 \cong \boxed{+0.25}$$

/*Note: The reason for the small inaccuracy (0.2×10^{-3}) is because I actually meant to say 0.393469. No matter, since I told you the value during the exam*/

$\boxed{b-15}$ Lets make things easier : $2\alpha \triangleq \beta = 0.5$

$$f(x) = \frac{d}{dx} F(x)$$

$$\therefore \quad f(x) = \begin{cases} \beta e^{\beta x}, & x<0 \\ 0, & x\geqslant 0 \end{cases}$$



c-15 pts

$$\mu = E(X) = \int_{-\infty}^{0} x \cdot \beta \cdot e^{\beta x}\, dx$$

Integration by parts $\int u\,dv = uv - \int v\,du$

$u = x \qquad dv = \beta e^{\beta x}\,dx$

$du = dx \qquad v = \int \beta e^{\beta x}\,dx = e^{\beta x}$

Indefine integral would be

$$\int x \cdot \beta \cdot e^{\beta x}\,dx = x \cdot e^{\beta x} - \int e^{\beta x}\,dx$$

$$= \left(x - \frac{1}{\beta}\right) e^{\beta x}$$

So: $\mu = E(X) = \left(x - \frac{1}{\beta}\right) e^{\beta x} \Big]_{-\infty}^{0}$

$$= -\frac{1}{\beta} = -\frac{1}{0.5} = \boxed{-2} \;\text{//} \; = \mu$$

(Note: If α were 0.5, $E(X)$ would be -1)

d-15 pts

$E(h(X))$

$= E(X^2) - 2\mu + 3$

$= 8 - 2(-2) + 3$

$= 15$ //

$\sigma^2 = V(X) = E(X^2) - \mu^2$

$E(X^2) = \int_{-\infty}^{0} x^2 \beta e^{\beta x}\,dx$

$u = x^2$

$du = 2x\,dx$

$dv = \beta \cdot e^{\beta x}\,dx$

$v = e^{\beta x}$

$$\int x^2 f(x)\,dx = x^2 e^{\beta x} - \underbrace{2\int x e^{\beta x}\,dx}_{\text{similar to above}}$$

$$= x^2 e^{\beta x} - \frac{2}{\beta}\left(x - \frac{1}{\beta}\right) e^{\beta x}$$

$$= \left(x^2 - \frac{2x}{\beta} + \frac{2}{\beta^2}\right) e^{\beta x}$$

$E(X^2) = \left[x^2 - 2x/\beta + 2/\beta^2\right] e^{\beta x} \Big]_{-\infty}^{0} = \frac{2}{\beta^2} = 8$

$\sigma^2 = V(X) = 8 - (-2)^2 = 4$ //

$\boxed{Q3}$ 20 pts

⊙ We should look at $x=0$ first

$$\underbrace{f(0-1)}_{0}f(1) = 0 \qquad f(2)>0$$

$\therefore$ For $x=0$ the property is satisfied.

⊙ For $x \geqslant 1$

$$f(x-1) = e^{\lambda} \cdot \frac{\lambda^{x-1}}{(x-1)!} = e^{\lambda} \cdot \frac{x}{\lambda} \cdot \frac{\lambda^{x}}{x!}$$

$$= f(x) \cdot \frac{x}{\lambda}$$

$$f(x+1) = e^{\lambda} \frac{\lambda^{x+1}}{(x+1)!} = e^{\lambda} \frac{\lambda}{x+1} \cdot \frac{\lambda^{x}}{x!}$$

$$= f(x) \frac{\lambda}{x+1}$$

$$\therefore \quad f(x-1)\, f(x+1) = f^2(x) \frac{x}{\lambda} \frac{\lambda}{x+1} = f^2(x) \frac{x}{x+1}$$

Since $\frac{x}{x+1} < 1$ for $\forall x \in \mathbb{Z}^{+}$

$$f^2(x) \frac{x}{x+1} < f^2(x)$$

▲ Q.E.D.

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