Gate-Level Minimization

Bölüm 3

3-1 Introduction

- Gate-level minimization refers to the design task of finding an optimal gate-level implementation of Boolean functions describing a digital circuit.

3-2 The Map Method

- The complexity of the digital logic gates
  - The complexity of the algebraic expression
- Logic minimization
  - Algebraic approaches: lack specific rules
  - The Karnaugh map
    - A simple straight forward procedure
    - A pictorial form of a truth table
    - Applicable if the # of variables < 7
- A diagram made up of squares
  - Each square represents one minterm

Review of Boolean Function

- Boolean function
  - Sum of minterms
  - Sum of products (or product of sum) in the simplest form
  - A minimum number of terms
  - A minimum number of literals
  - The simplified expression may not be unique

Two-Variable Map

- A two-variable map
  - Four minterms
  - *x'* = row 0; *x* = row 1
  - *y'* = column 0; *y* = column 1
  - A truth table in square diagram
  - Fig. 3.2(a): $xy = m_3$
  - Fig. 3.2(b): $x+y = x'y+xy'+xy = m_1+m_2+m_3$

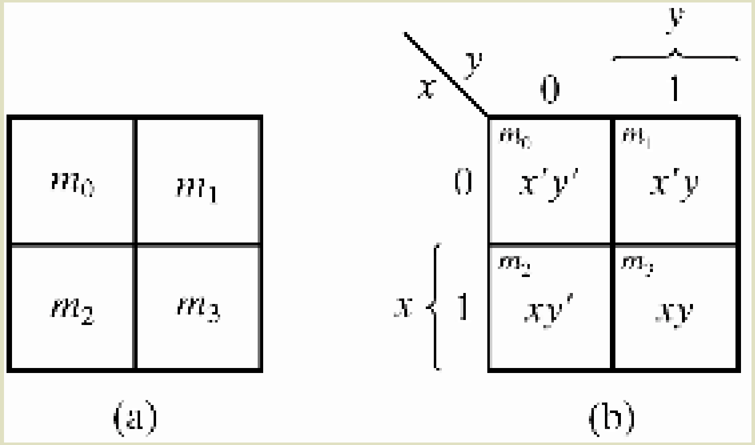


Figure 3.1 Two-variable Map

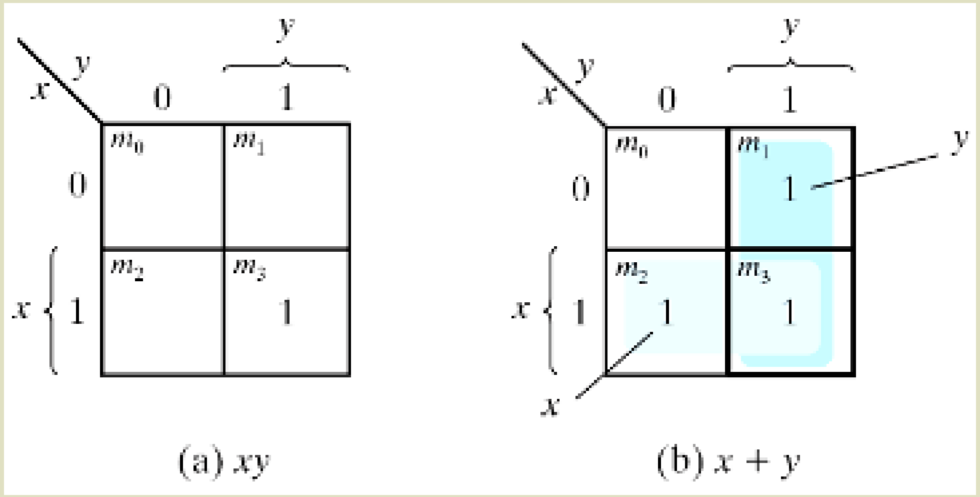


Figure 3.2 Representation of functions in the map

A Three-variable Map

- A three-variable map
  - Eight minterms
  - The Gray code sequence
  - Any two adjacent squares in the map differ by only on variable
    - Primed in one square and unprimed in the other
    - e.g., m_5 and m_7 can be simplified
    - $m_5 + m_7 = xy'z + xyz = xz\,(y'+y) = xz$

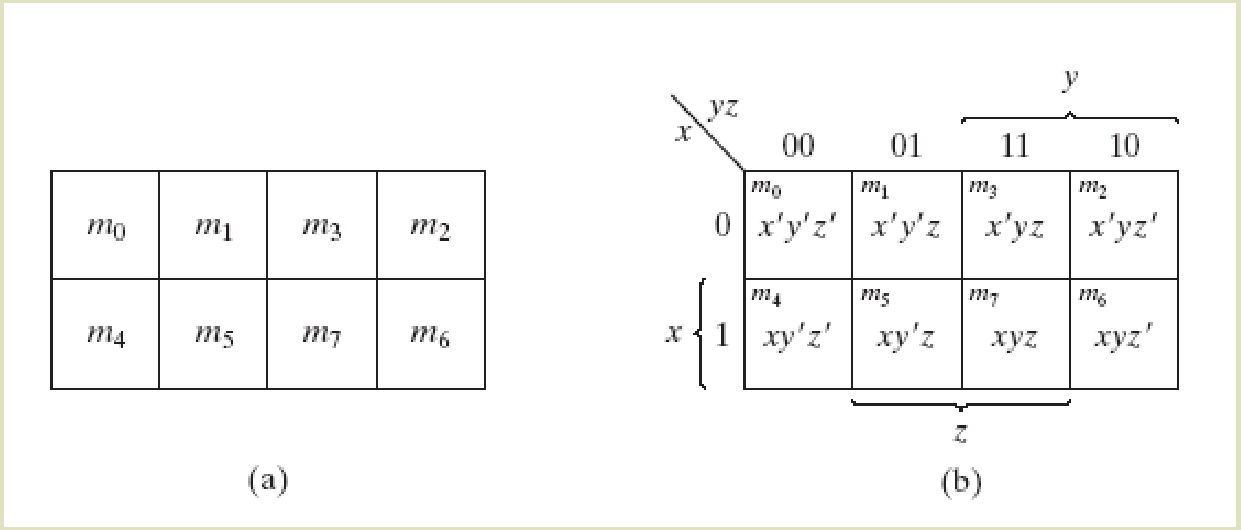


Figure 3.3 Three-variable Map

A Three-variable Map

- m_0 and m_2 (m_4 and m_6) are adjacent
- $m_0 + m_2 = x'y'z' + x'yz' = x'z'(y'+y) = x'z'$
- $m_4 + m_6 = xy'z' + xyz' = xz'(y'+y) = xz'$

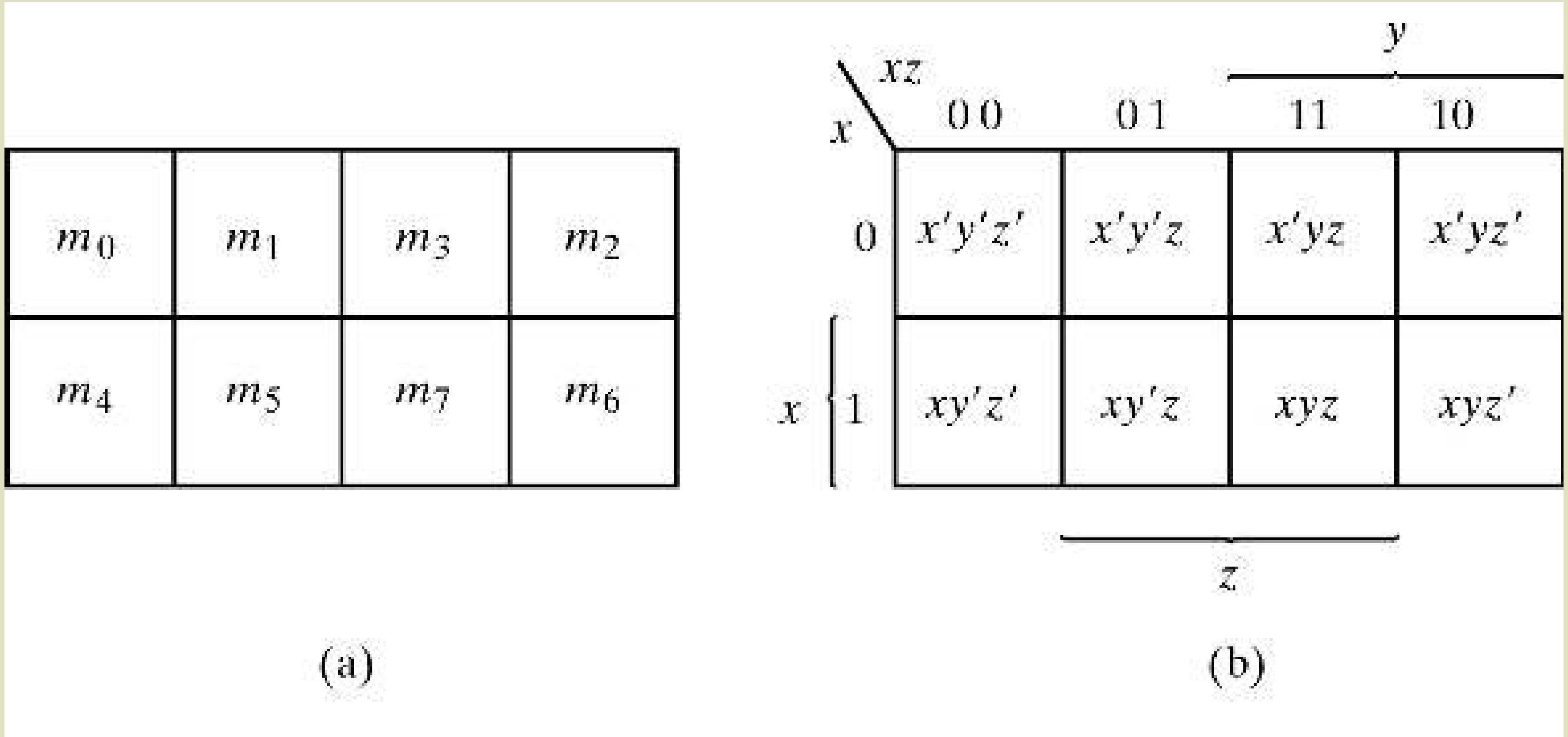


Fig. 3-3 Three-variable Map

Example 3.1

- Example 3.1: simplify the Boolean function F(x, y, z) = $\Sigma(2, 3, 4, 5)$
  - $F(x, y, z) = \Sigma(2, 3, 4, 5) = x'y + xy'$

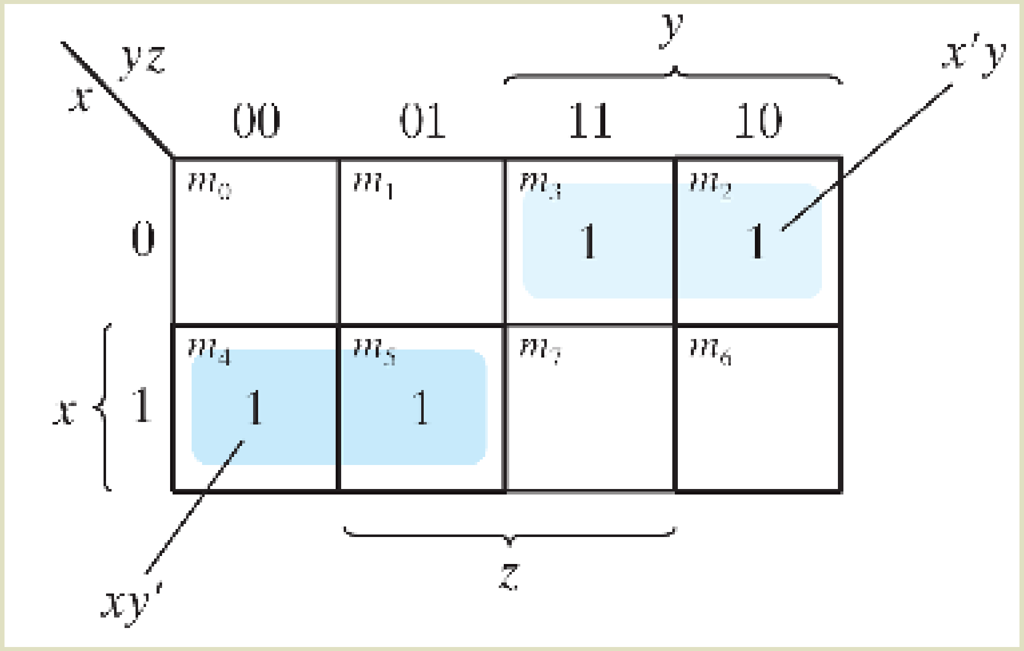


Figure 3.4 Map for Example 3.1, $F(x, y, z) = \Sigma(2, 3, 4, 5) = x'y + xy'$

Example 3.2

- Example 3.2: simplify $F(x, y, z) = \Sigma(3, 4, 6, 7)$
  - $F(x, y, z) = \Sigma(3, 4, 6, 7) = yz + xz'$

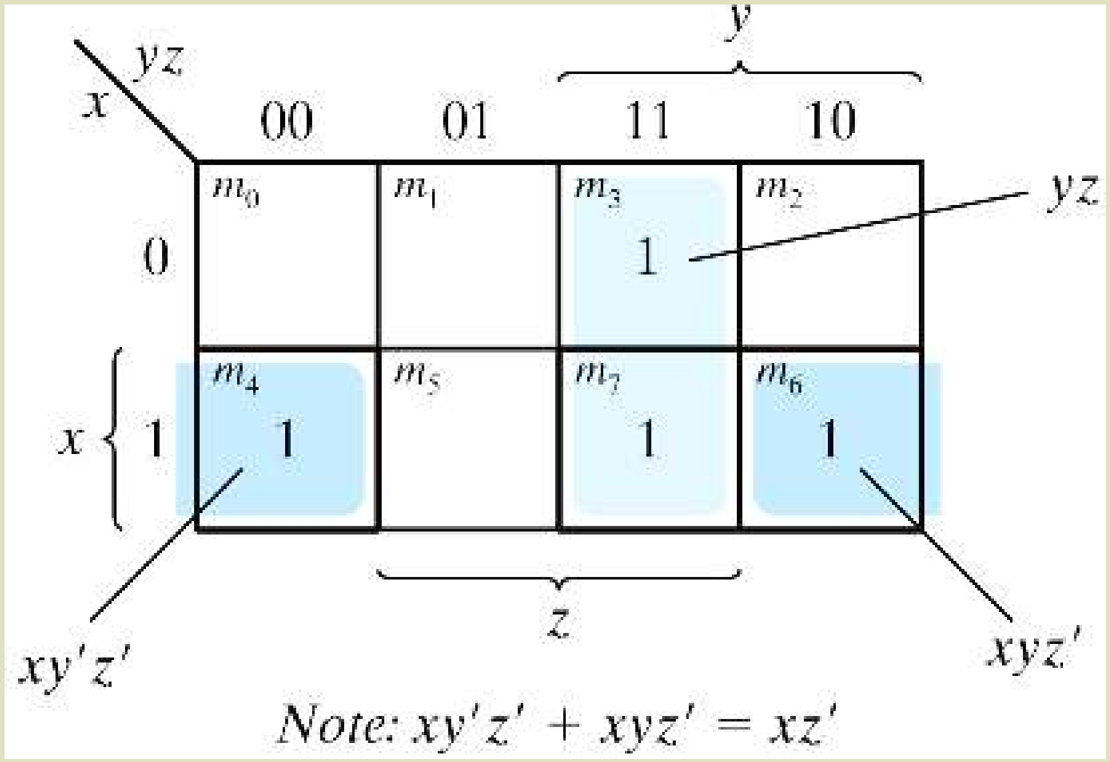


Figure 3.5 Map for Example 3-2; $F(x, y, z) = \Sigma(3, 4, 6, 7) = yz + xz'$

Four adjacent Squares

- Consider four adjacent squares
  - 2, 4, and 8 squares
  - $m_0+m_2+m_4+m_6 = x'y'z'+x'yz'+xy'z'+xyz' = x'z'(y'+y) +xz'(y'+y) = x'z' + xz' = z'$
  - $m_1+m_3+m_5+m_7 = x'y'z+x'yz+xy'z+xyz =x'z(y'+y) + xz(y'+y) =x'z + xz = z$

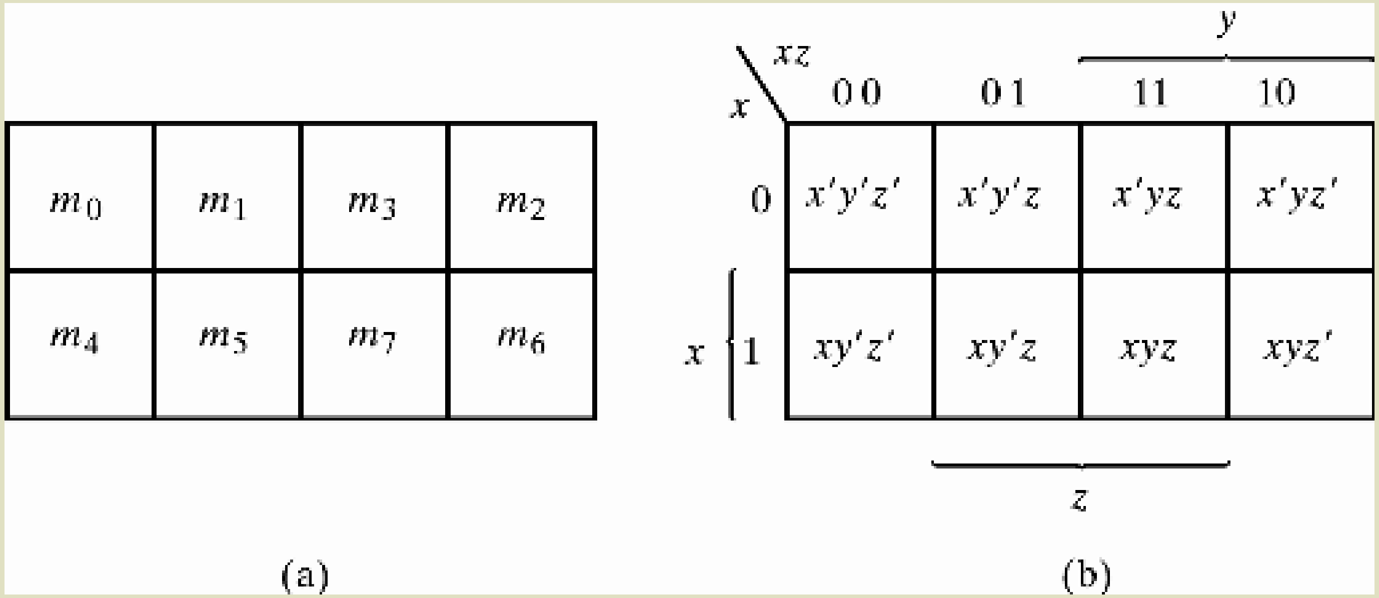


Figure 3.3 Three-variable Map

Example 3.3

- Example 3.3: simplify *F(x, y, z)* = Σ(0, 2, 4, 5, 6)
  - *F*(*x*, *y*, *z*) = Σ(0, 2, 4, 5, 6) = *z'*+ *xy'*

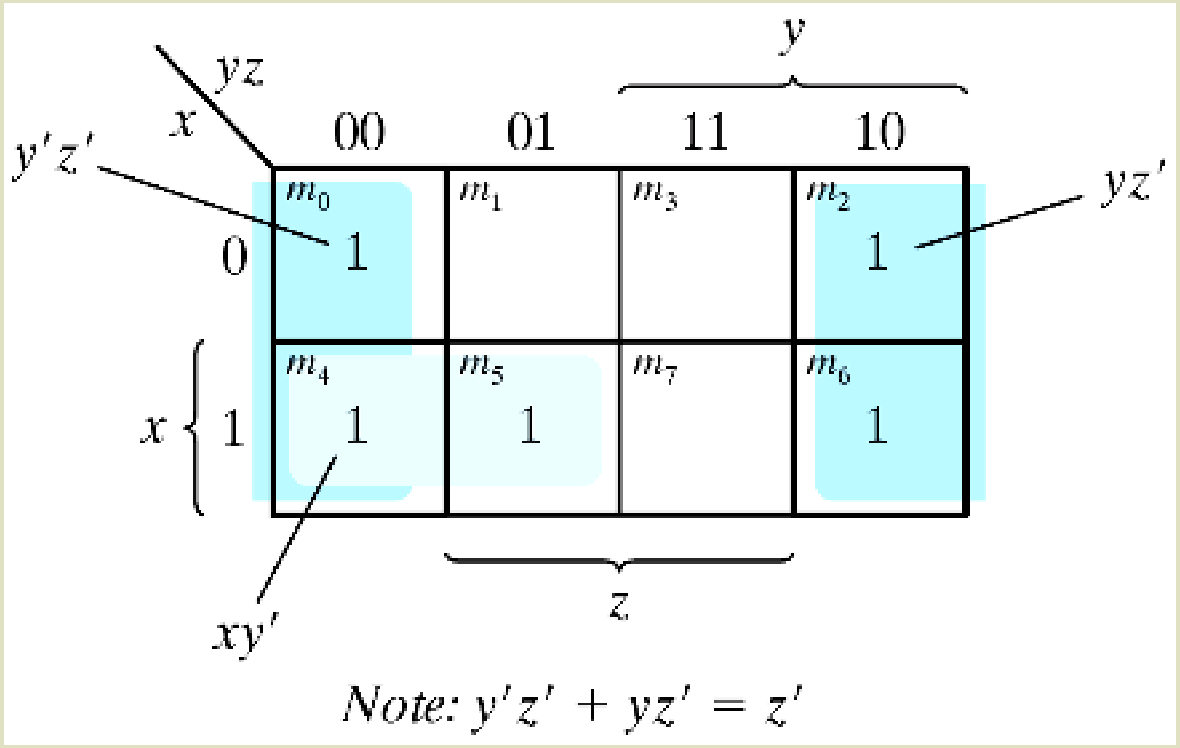


Figure 3.6 Map for Example 3-3, *F*(*x*, *y*, *z*) = Σ(0, 2, 4, 5, 6) = *z'* +*xy'*

Example 3.4

- Example 3.4: let $F = A'C + A'B + AB'C + BC$
  a) Express it in sum of minterms.
  b) Find the minimal sum of products expression.

  Ans:

  $F(A, B, C) = \Sigma(1, 2, 3, 5, 7) = C + A'B$

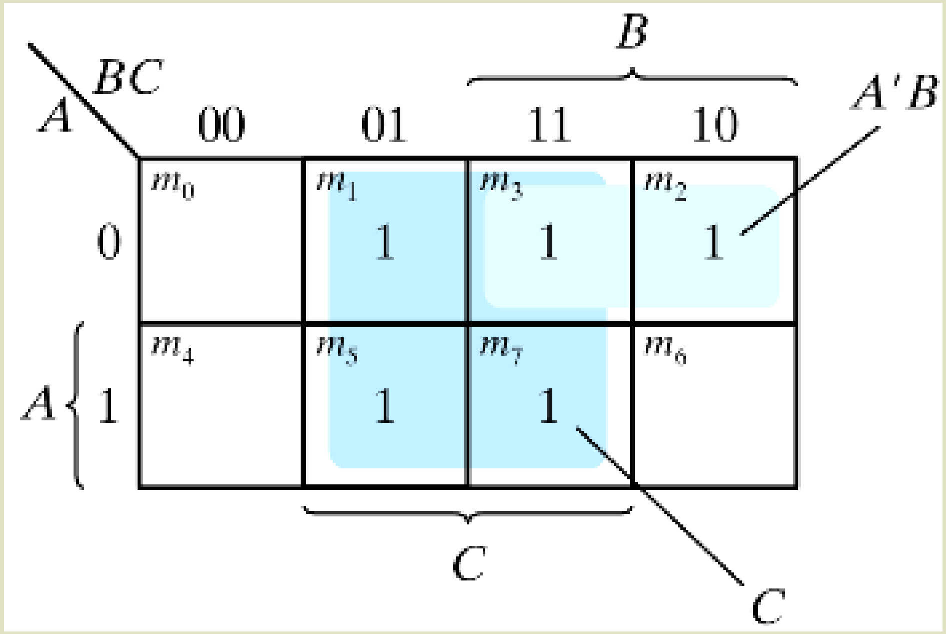
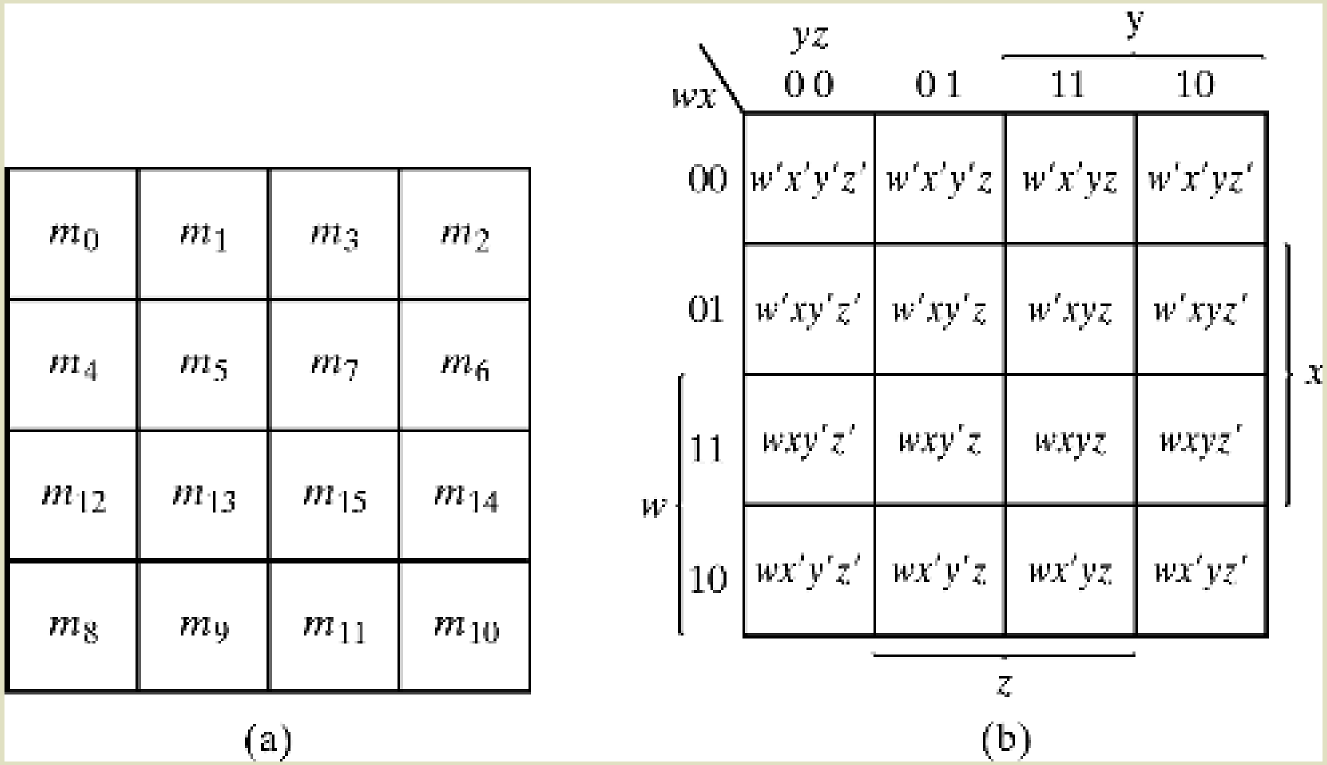


Figure 3.7 Map for Example 3.4, $A'C + A'B + AB'C + BC = C + A'B$

3.3 Four-Variable Map

- The map
  - 16 minterms
  - Combinations of 2, 4, 8, and 16 adjacent squares

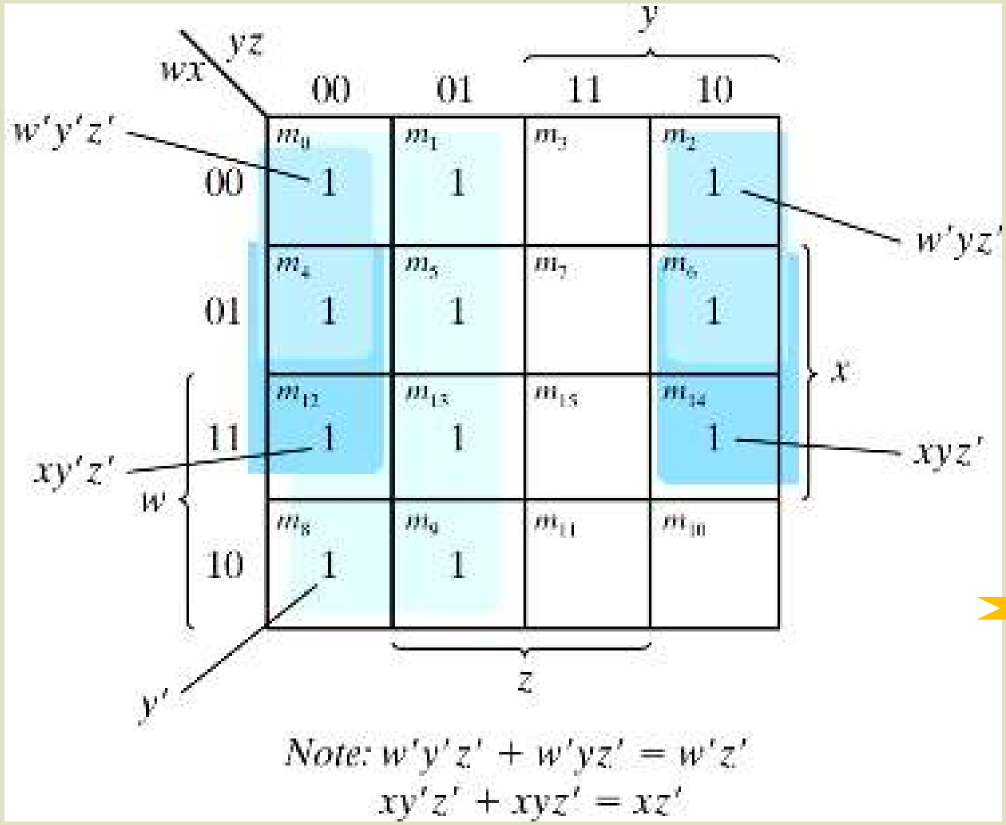


m_0 m_1 m_3 m_2
m_4 m_5 m_7 m_6
m_{12} m_{13} m_{15} m_{14}
m_8 m_9 m_{11} m_{10}
(a)
yz
wx
00 01 11 10
y
x
w
z
$w'x'y'z'$ $w'x'y'z$ $w'x'yz$ $w'x'yz'$
$w'xy'z'$ $w'xy'z$ $w'xyz$ $w'xyz'$
$wxy'z'$ $wxy'z$ $wxyz$ $wxyz'$
$wx'y'z'$ $wx'y'z$ $wx'yz$ $wx'yz'$
(b)

Figure 3.8 Four-variable Map

Example 3.5

- Example 3.5: simplify $F(w, x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14)$



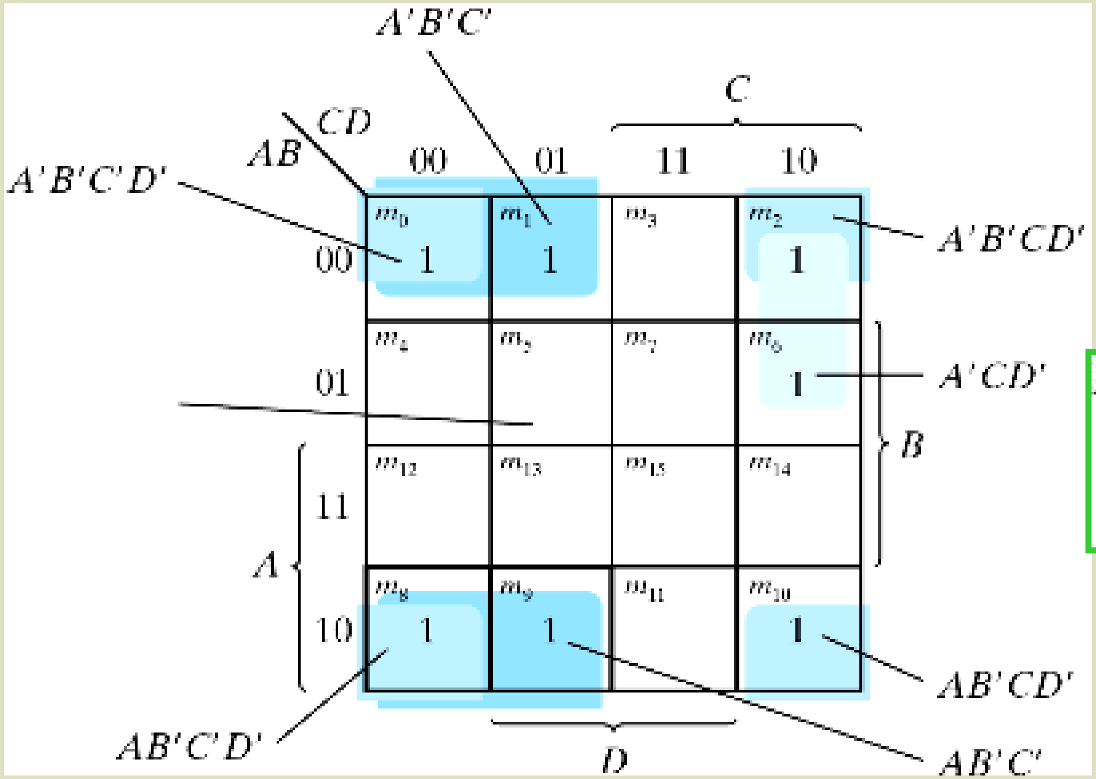
Note: $w'y'z' + w'yz' = w'z'$
$xy'z' + xyz' = xz'$

$$\rightarrow \quad F = y' + w'z' + xz'$$

Figure 3.9 Map for Example 3-5; $F(w, x, y, z) = \Sigma(0, 1, 2, 4, 5, 6, 8, 9, 12, 13, 14) = y' + w'z' + xz'$

Example 3.6

- Example 3-6: simplify $F = A'B'C' + B'CD' + A'B'C'D' + AB'C'$



Note: $A'B'C'D' + A'B'CD' = A'B'D$
$AB'C'D' + AB'CD' = AB'D'$
$A'B'D' + AB'D' = B'D'$
$A'B'C' + AB'C' = B'C'$

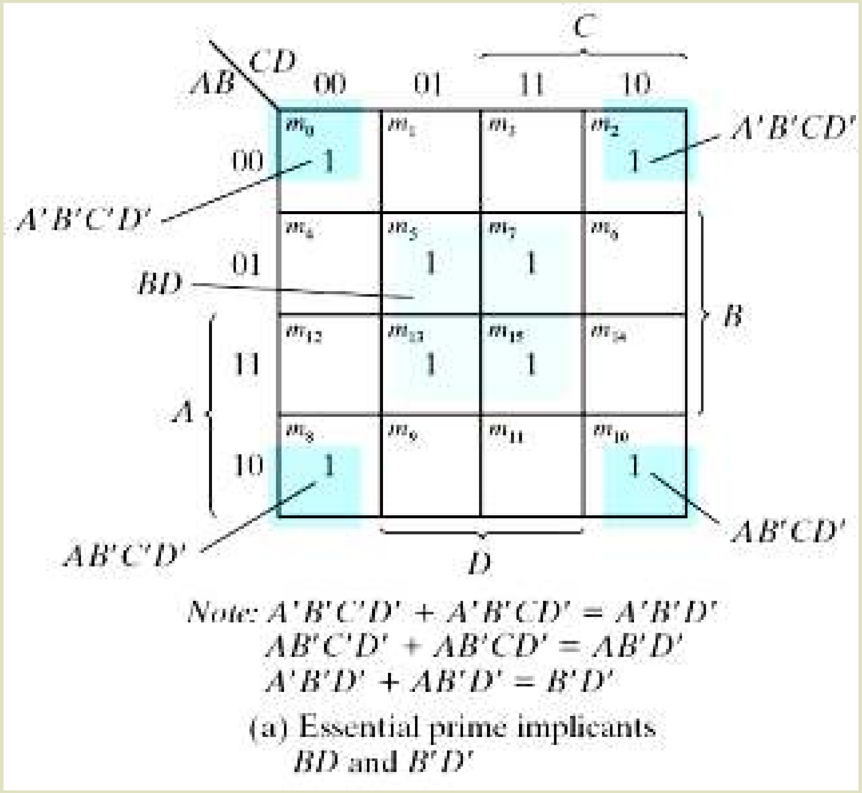
Figure 3.9 Map for Example 3-6; $A'B'C' + B'CD' + A'B'C'D' + AB'C' = B'D' + B'C' + A'CD'$

Prime Implicants

- Prime Implicants
  - All the minterms are covered.
  - Minimize the number of terms.
  - A prime implicant: a product term obtained by combining the maximum possible number of adjacent squares (combining all possible maximum numbers of squares).
  - Essential P.I.: a minterm is covered by only one prime implicant.
  - The essential P.I. must be included.

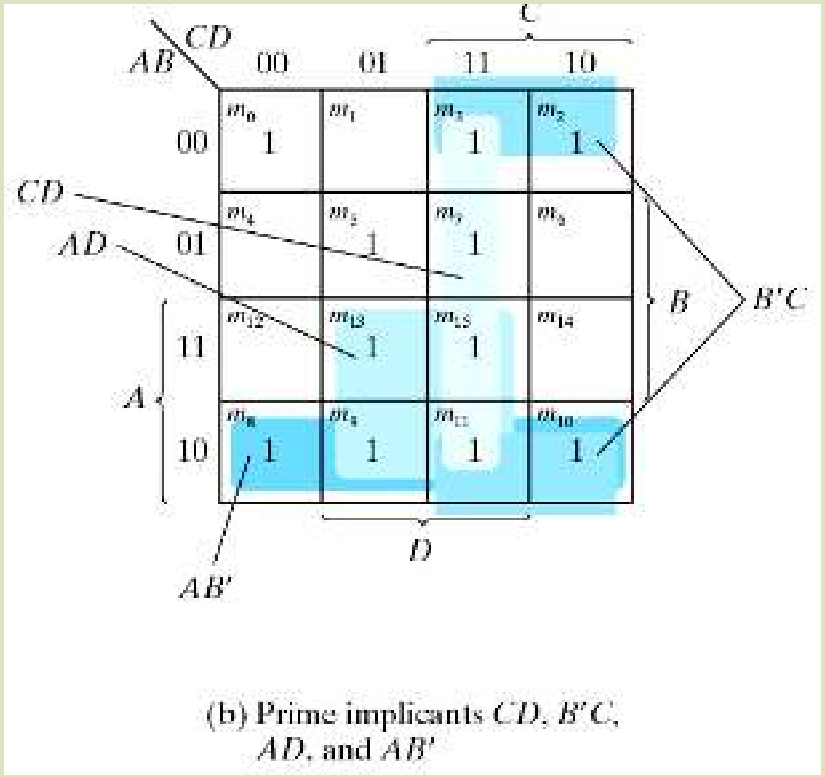
Prime Implicants

- Consider $F(A, B, C, D) = \Sigma(0, 2, 3, 5, 7, 8, 9, 10, 11, 13, 15)$
  - The simplified expression may not be unique
  - $F = BD+B'D'+CD+AD = BD+B'D'+CD+AB'$
    $= BD+B'D'+B'C+AD = BD+B'D'+B'C+AB'$



Note: $A'B'C'D' + A'B'CD' = A'B'D'$
$AB'C'D' + AB'CD' = AB'D'$
$A'B'D' + AB'D' = B'D'$

(a) Essential prime implicants BD and $B'D'$



(b) Prime implicants CD, $B'C$, AD, and AB'

Figure 3.11 Simplification Using Prime Implicants

3.4 Five-Variable Map

- Map for more than four variables becomes complicated
  - Five-variable map: two four-variable map (one on the top of the other).

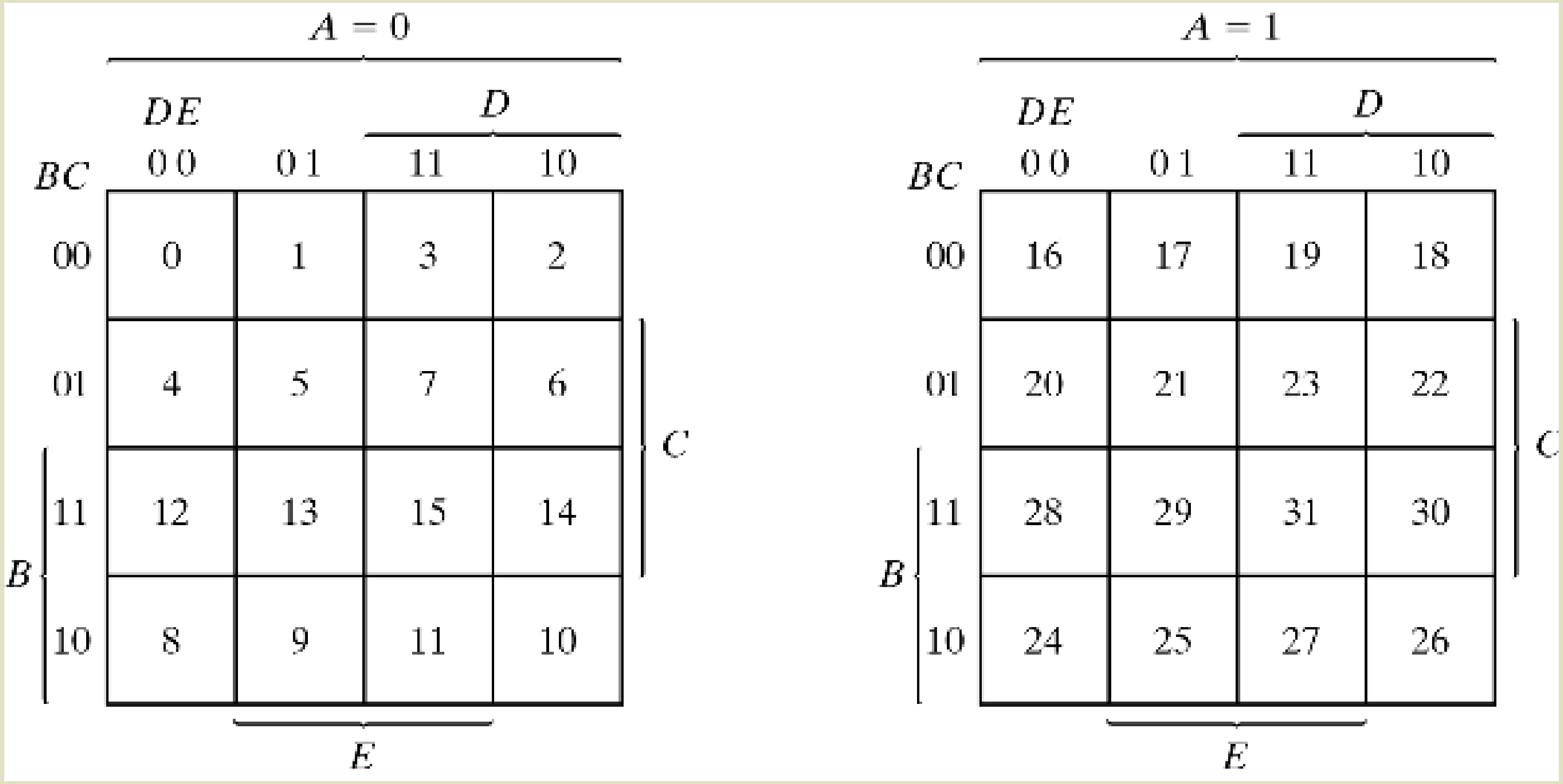


Figure 3.12 Five-variable Map

- Table 3.1 shows the relationship between the number of adjacent squares and the number of literals in the term.

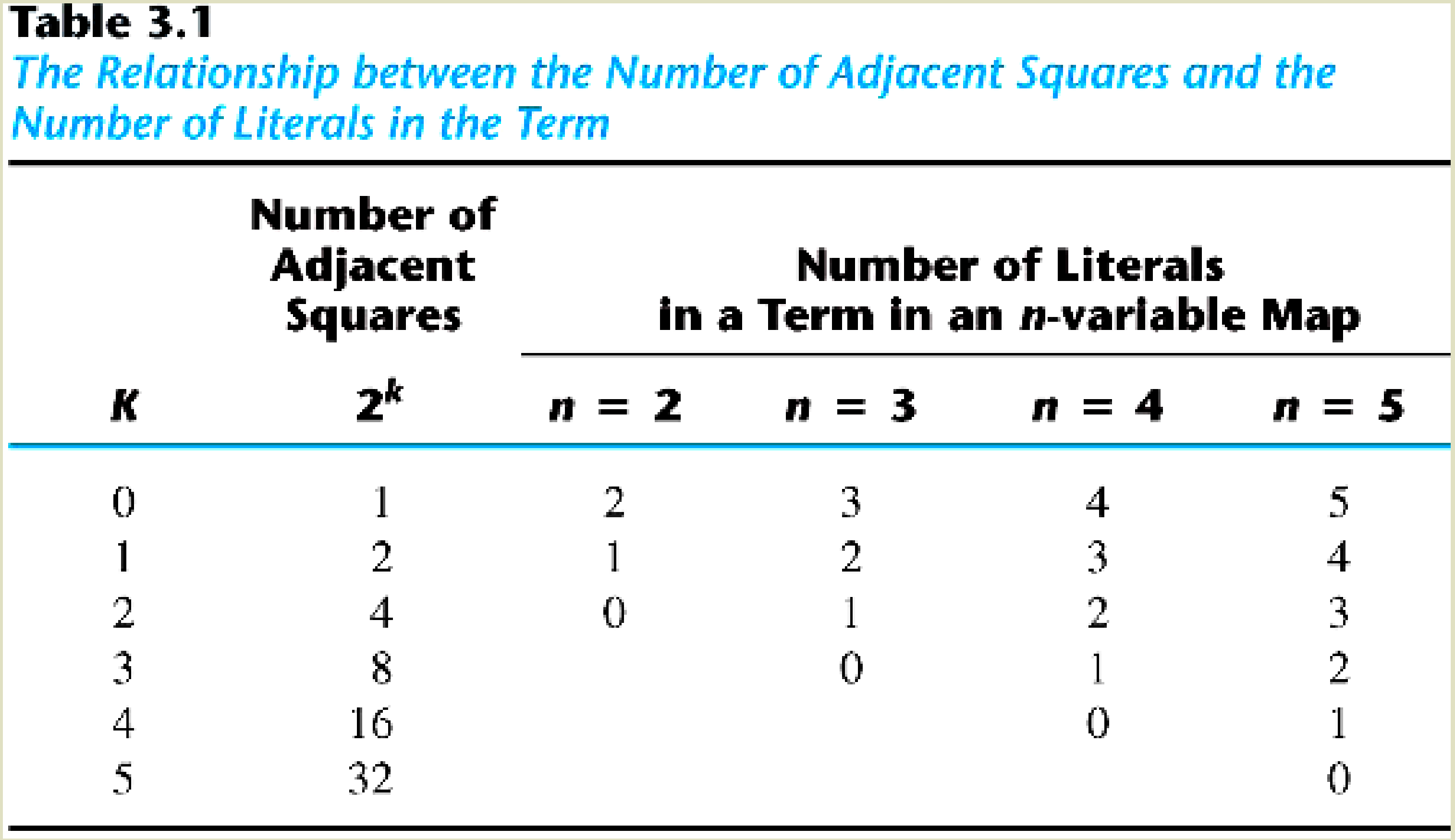


Table 3.1
The Relationship between the Number of Adjacent Squares and the Number of Literals in the Term

	Number of Adjacent Squares	Number of Literals in a Term in an *n*-variable Map			
K	2^k	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	1	2	3	4	5
1	2	1	2	3	4
2	4	0	1	2	3
3	8		0	1	2
4	16			0	1
5	32				0

Example 3.7

- Example 3.7: simplify $F = \Sigma(0, 2, 4, 6, 9, 13, 21, 23, 25, 29, 31)$

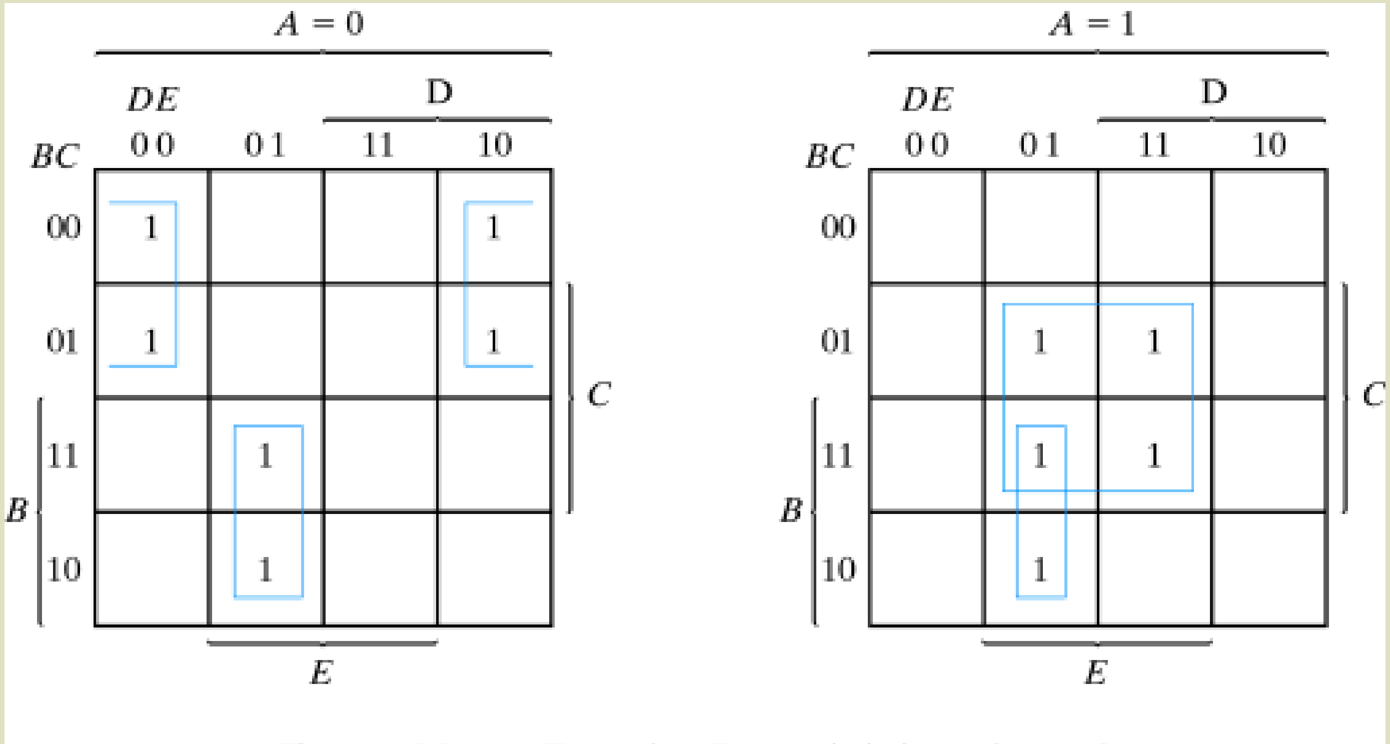


Fig. 3-13 Map for Example 3-7; $F = A'B'E' + BD'E + ACE$

$$F = A'B'E' + BD'E + ACE$$

3-5 Product of Sums Simplification

- Approach #1
  - Simplified F' in the form of sum of products
  - Apply DeMorgan's theorem $F = (F')'$
  - F': sum of products → F: product of sums
- Approach #2: duality
  - Combinations of maxterms (it was minterms)
  - $M_0M_1 = (A+B+C+D)(A+B+C+D') = (A+B+C)+(DD') = A+B+C$

AB \ CD	00	01	11	10
00	M_0	M_1	M_3	M_2
01	M_4	M_5	M_7	M_6
11	M_{12}	M_{13}	M_{15}	M_{14}
10	M_8	M_9	M_{11}	M_{10}

Example 3.8

- Example 3.8: simplify $F = \Sigma(0, 1, 2, 5, 8, 9, 10)$ into (a) sum-of-products form, and (b) product-of-sums form:

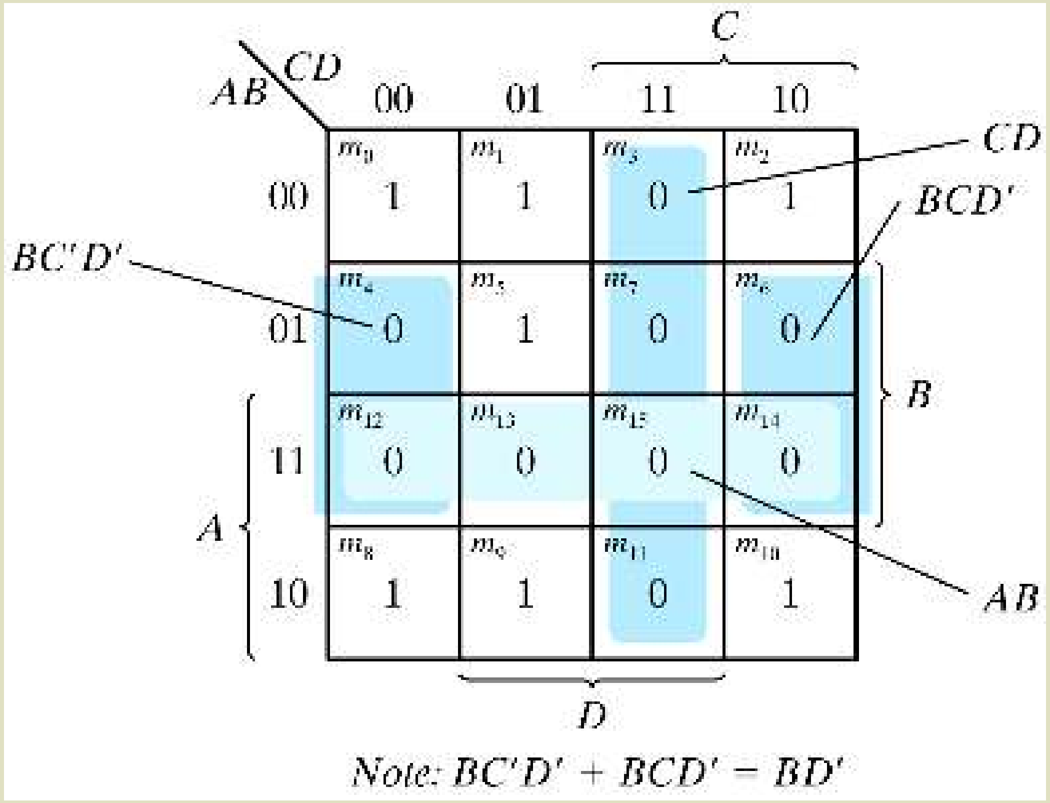


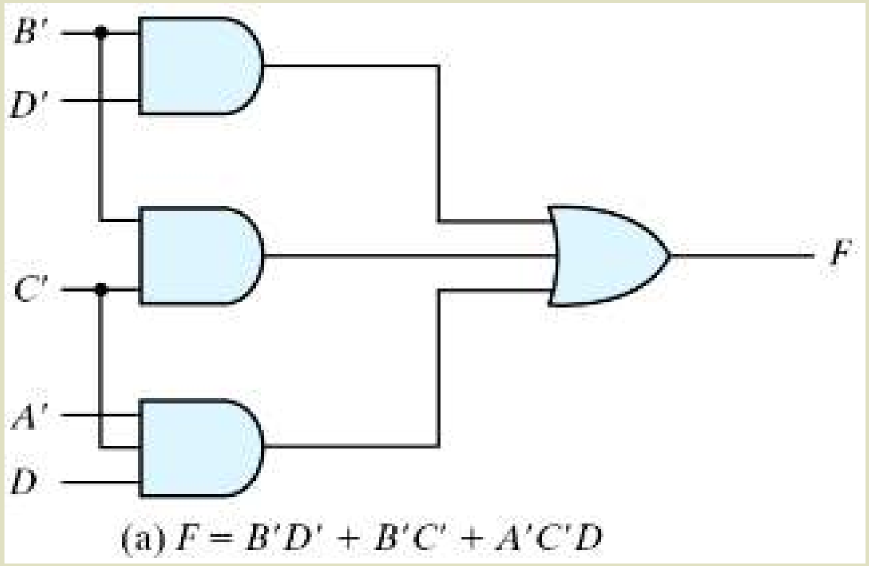
Figure 3.14 Map for Example 3.8, $F(A, B, C, D) = \Sigma(0, 1, 2, 5, 8, 9, 10) = B'D' + B'C' + A'C'D$

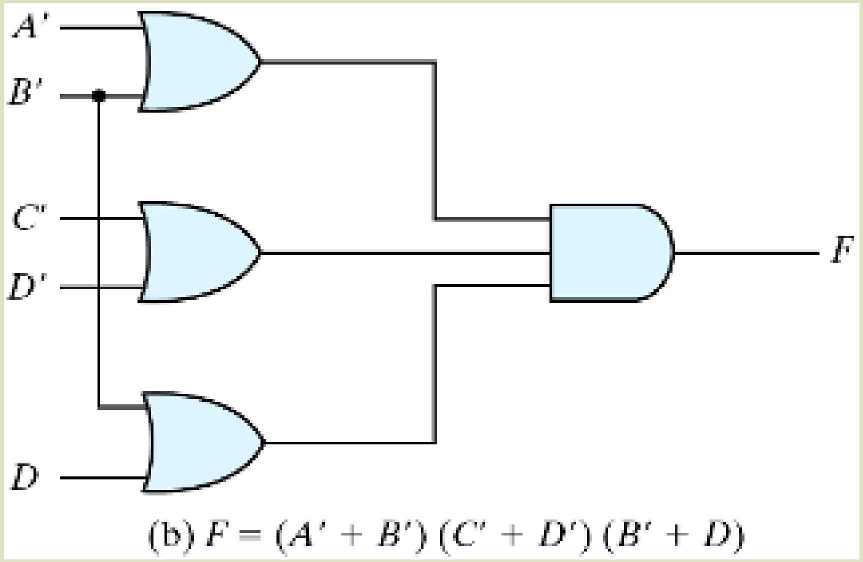
a) $F(A, B, C, D) = \Sigma(0, 1, 2, 5, 8, 9, 10) = B'D' + B'C' + A'C'D$

b) $F' = AB + CD + BD'$
   - » Apply DeMorgan's theorem; $F = (A' + B')(C' + D')(B' + D)$
   - » Or think in terms of maxterms

Example 3.8 (cont.)

- Gate implementation of the function of Example 3.8





Sum-of products form

Product-of sums form

Figure 3.15 Gate Implementation of the Function of Example 3.8

Sum-of-Minterm Procedure

- Consider the function defined in Table 3.2.
  - In sum-of-minterm:

$$\boxed{F(x,y,z)=\sum(1,3,4,6)}$$

  - F(x,y,z)=x'z + xz'
  - In sum-of-maxterm:

$$\boxed{F'(x,y,z)=\Pi(0,2,5,7)}$$

  - F'(x,y,z)=xz + x'z'
  - Taking the complement of F′

$$\boxed{F(x,y,z)=(x'+z')(x+z)}$$

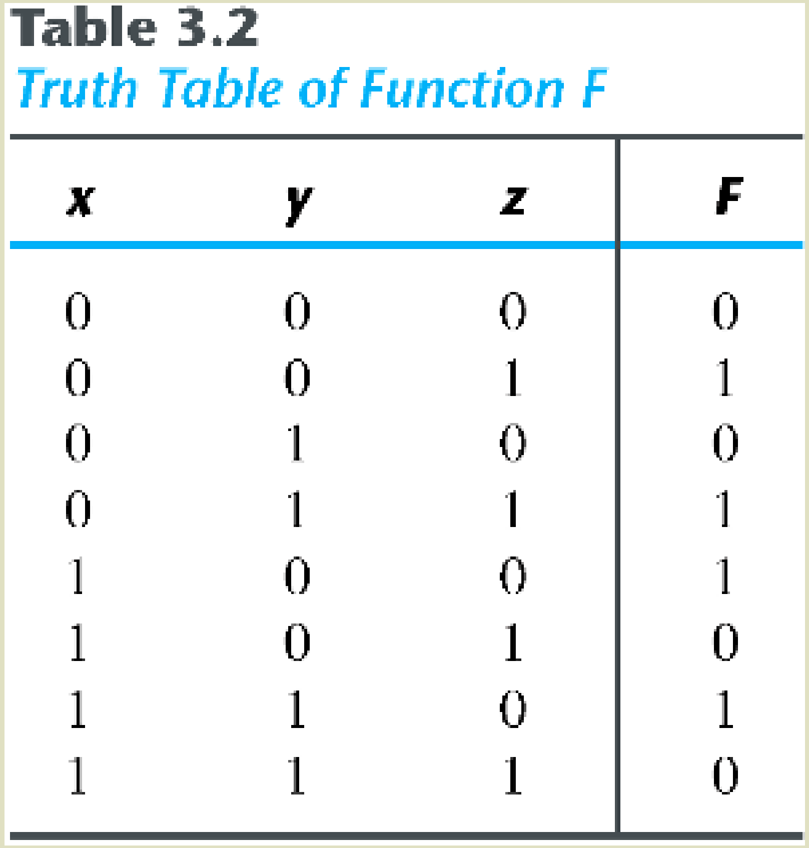


Table 3.2
Truth Table of Function F

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Sum-of-Minterm Procedure

- Consider the function defined in Table 3.2.
  - Combine the 1's:

$$\boxed{F(x,y,z) = x'z + xz'}$$

  - Combine the 0's :

$$\boxed{F'(x,y,z) = xz + x'z'}$$

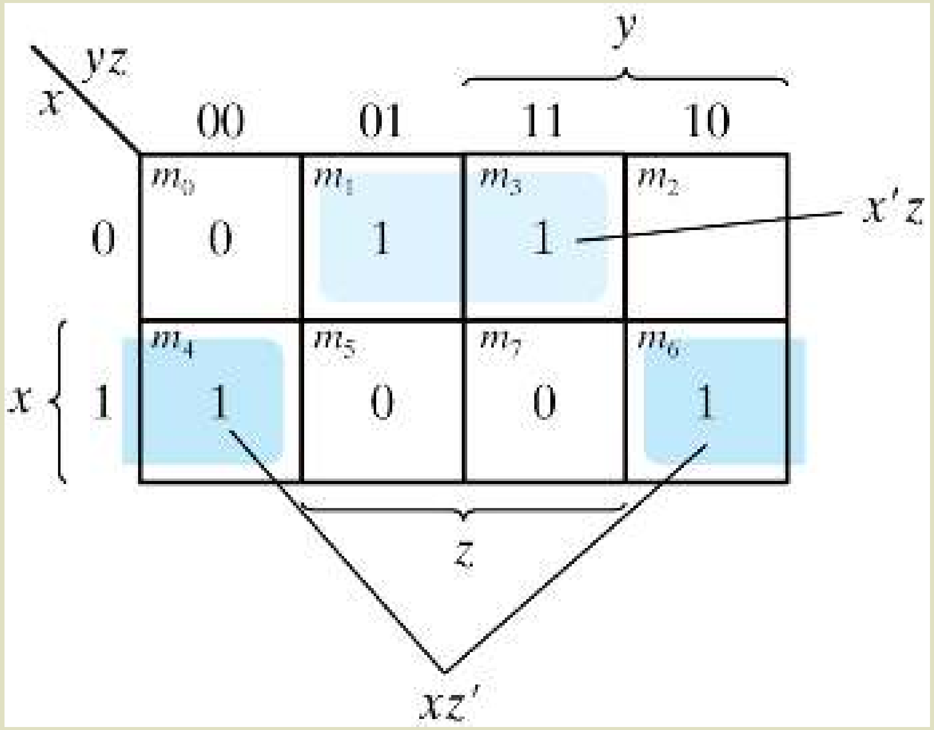


Figure 3.16 Map for the function of Table 3.2

3-6 Don't-Care Conditions

- The value of a function is not specified for certain combinations of variables
  - BCD; 1010-1111: don't care
- The don't-care conditions can be utilized in logic minimization
  - Can be implemented as 0 or 1
- Example 3.9: simplify $F(w, x, y, z) = \Sigma(1, 3, 7, 11, 15)$ which has the don't-care conditions $d(w, x, y, z) = \Sigma(0, 2, 5)$.

Example 3.9 (cont.)

- $F = yz + w'x'$; $F = yz + w'z$
- $F = \Sigma(0, 1, 2, 3, 7, 11, 15)$; $F = \Sigma(1, 3, 5, 7, 11, 15)$
- Either expression is acceptable

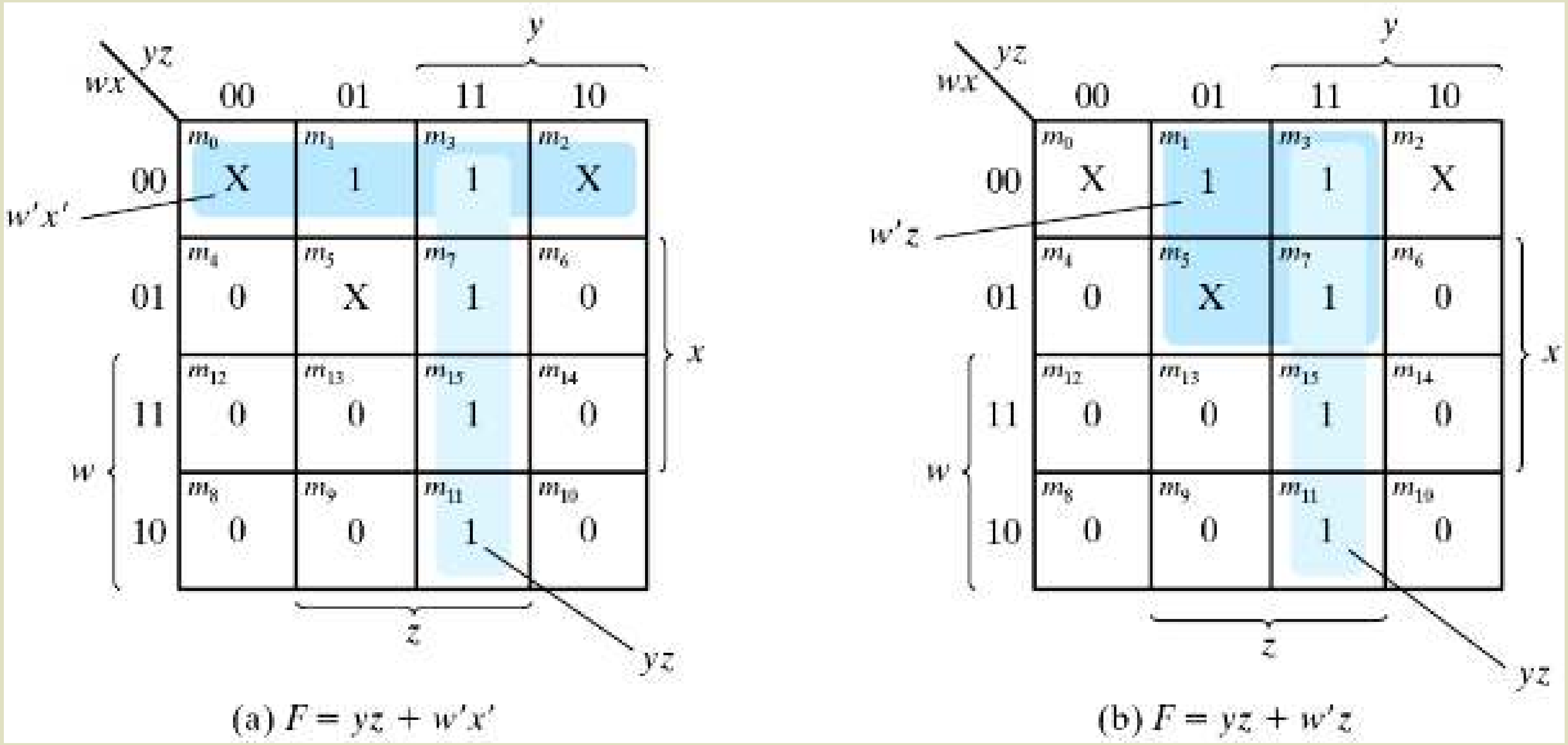


Figure 3.17 Example with don't-care Conditions

TABLE METHOD (QUINN MC-CUSKEY)

- The expression is represented in the canonical SOP form (minterm form) if not already in that form.
- The function is converted into numeric notation.
-
- The numbers are converted into binary form.
- The minterms are arranged in a column divided into groups.

Figure 3.17 Example with don't-care Conditions

MINIMIZATION PROCEDURE

Each minterm of one group is compared with each minterm in the group immediately below.

Each time a number is found in one group which is the same as a number in the group below except for one digit, the numbers pair is ticked and a new composite is created.

This composite number has the same number of digits as the numbers in the pair except the digit different which is replaced by an "x".

The above procedure is repeated on the second column to generate a third column.

The next step is to identify the essential prime implicants, which can be done using a prime implicant chart.

Where a prime implicant covers a minterm, the intersection of the corresponding row and column is marked with a cross.

Those columns with only one cross identify the essential prime implicants. -> These prime implicants must be in the final answer.

The single crosses on a column are circled and all the crosses on the same row are also circled, indicating that these crosses are covered by the prime implicants selected.

Once one cross on a column is circled, all the crosses on that column can be circled since the minterm is now covered.

If any non-essential prime implicant has all its crosses circled, the prime implicant is redundant and need not be considered further.

Next, a selection must be made from the remaining nonessential prime implicants, by considering how the non-circled crosses can be covered best.

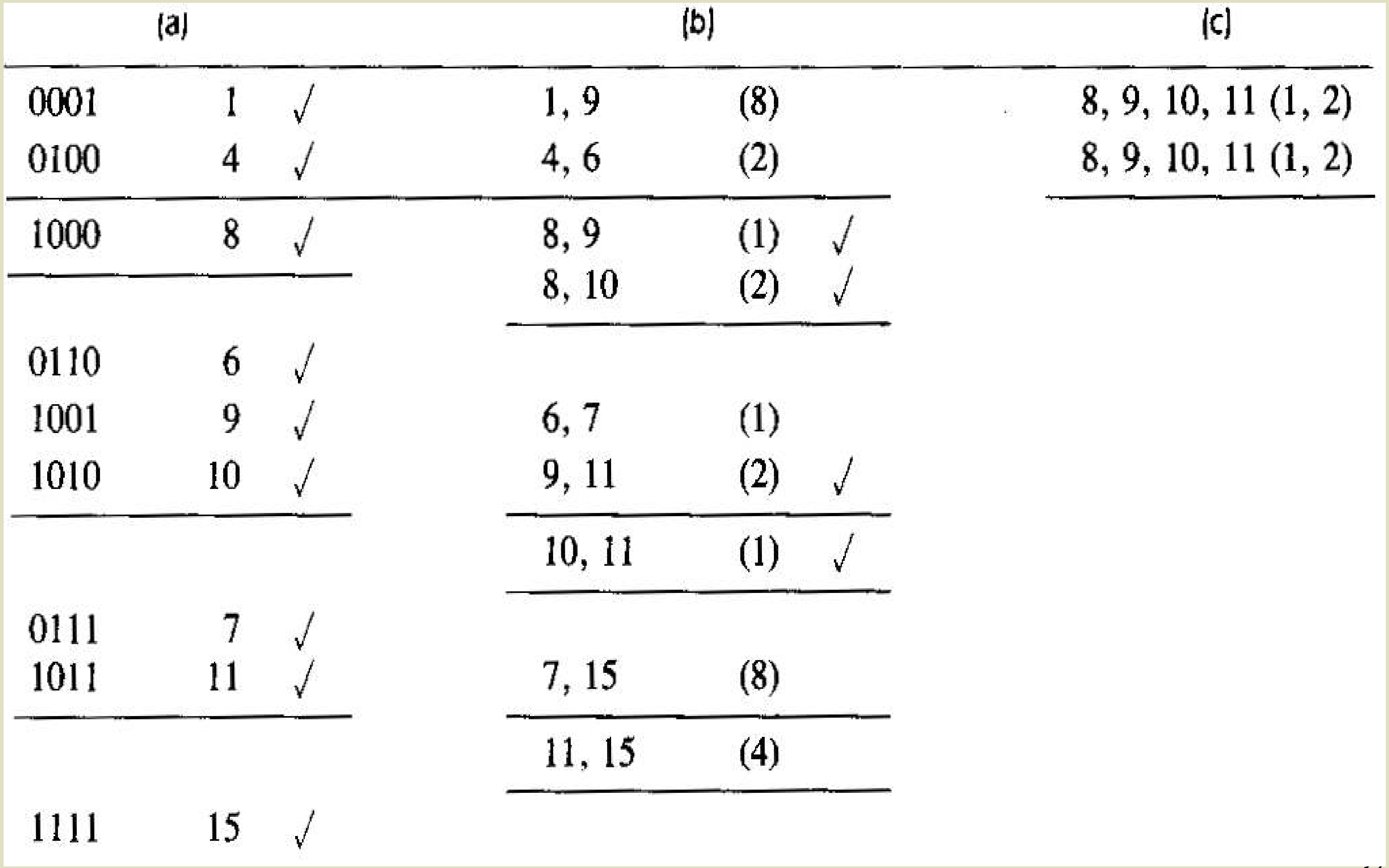
One generally would take those prime implicants which cover the greatest number of crosses on their row.

If all the crosses in one row also occur on another row which includes further crosses, then the latter is said to dominate the former and can be selected.

The dominated prime implicant can then be deleted.

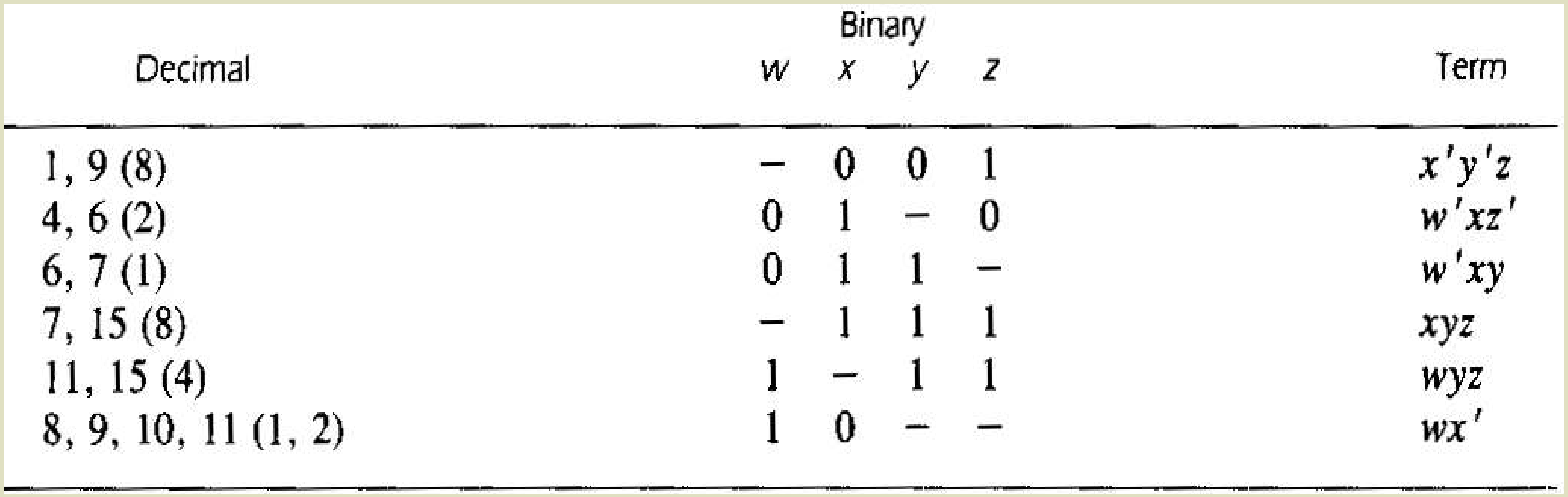
Example: F(w,x,y,z)=□(1,4,6,7,8,9,10,11,15)

- Grouping & Combining the minterms



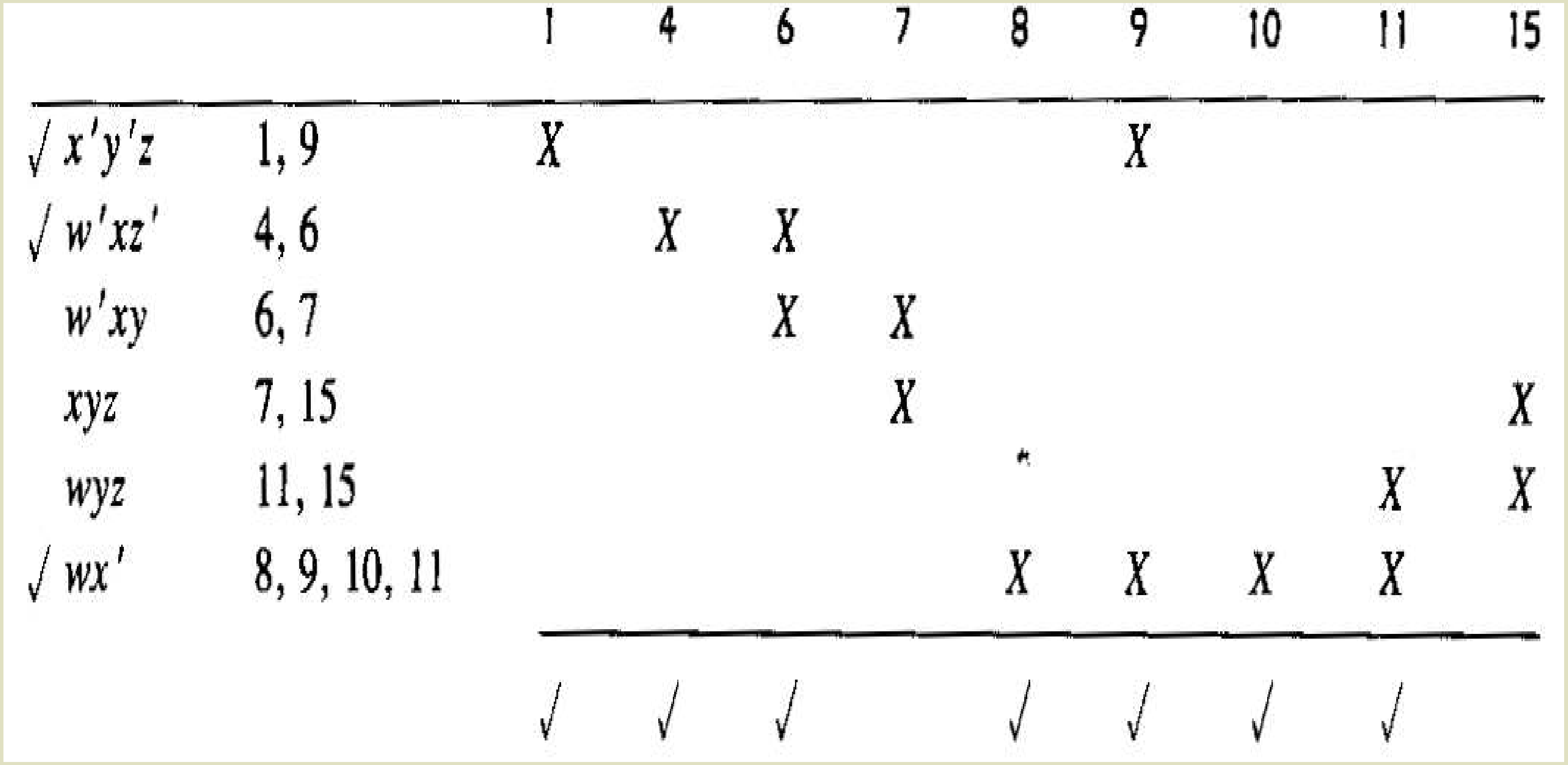
(a)			(b)			(c)
0001	1	✓	1, 9	(8)		8, 9, 10, 11 (1, 2)
0100	4	✓	4, 6	(2)		8, 9, 10, 11 (1, 2)
1000	8	✓	8, 9	(1)	✓	
			8, 10	(2)	✓	
0110	6	✓				
1001	9	✓	6, 7	(1)		
1010	10	✓	9, 11	(2)	✓	
			10, 11	(1)	✓	
0111	7	✓				
1011	11	✓	7, 15	(8)		
			11, 15	(4)		
1111	15	✓				

Prime implicants



Decimal	Binary				Term
	w	x	y	z	
1, 9 (8)	–	0	0	1	$x'y'z$
4, 6 (2)	0	1	–	0	$w'xz'$
6, 7 (1)	0	1	1	–	$w'xy$
7, 15 (8)	–	1	1	1	xyz
11, 15 (4)	1	–	1	1	wyz
8, 9, 10, 11 (1, 2)	1	0	–	–	wx'

Prime implicant chart



		1	4	6	7	8	9	10	11	15
√ $x'y'z$	1, 9	X					X			
√ $w'xz'$	4, 6		X	X						
$w'xy$	6, 7			X	X					
xyz	7, 15				X					X
wyz	11, 15								X	X
√ wx'	8, 9, 10, 11					X	X	X	X	
		√	√	√		√	√	√	√	

$$F = x'y'z + w'xz' + wx' + xyz$$

3-7 NAND and NOR Implementation

- NAND gate is a universal gate
  - Can implement any digital system

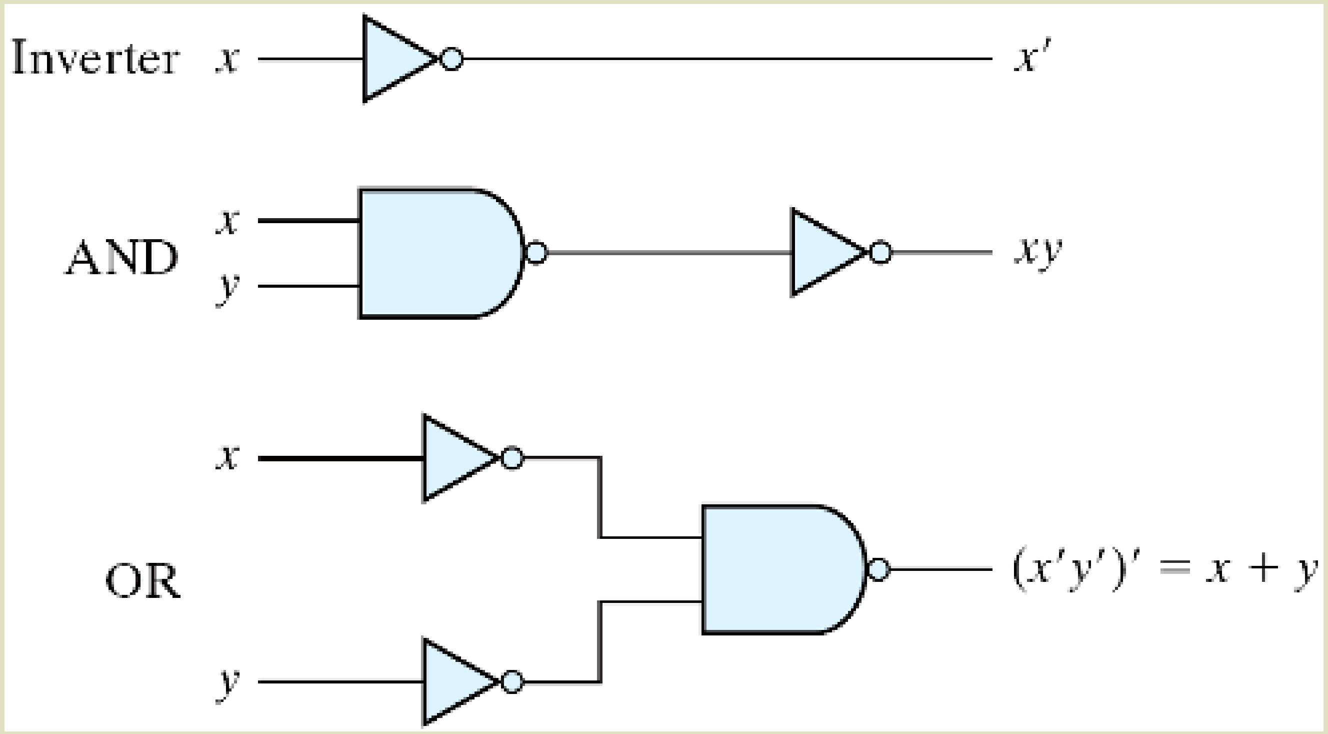


Figure 3.18 Logic Operations with NAND Gates

NAND Gate

- Two graphic symbols for a NAND gate

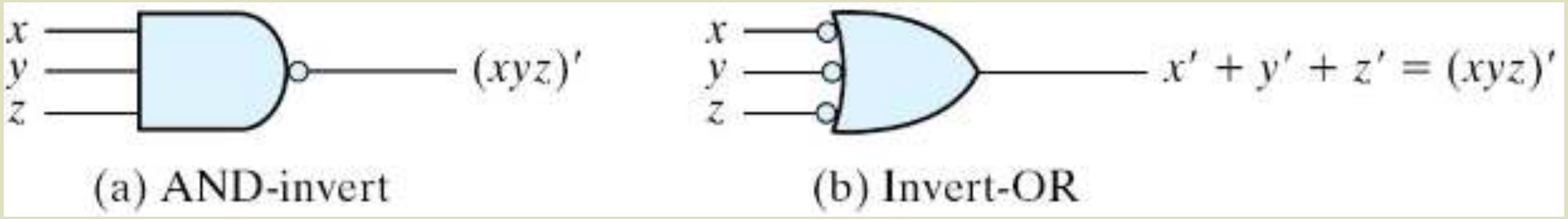


Figure 3.19 Two Graphic Symbols for NAND Gate

Two-level Implementation

- Two-level logic
  - NAND-NAND = sum of products
  - Example: $F = AB+CD$
  - $F = ((AB)' (CD)')' =AB+CD$

Figure 3.20 Three ways to implement $F = AB + CD$

Example 3.10

- Example 3-10: implement *F*(*x, y, z*) =

$$F(x,y,z)=\sum(1,2,3,4,5,7) \longrightarrow F(x,y,z)=xy'+x'y+z$$

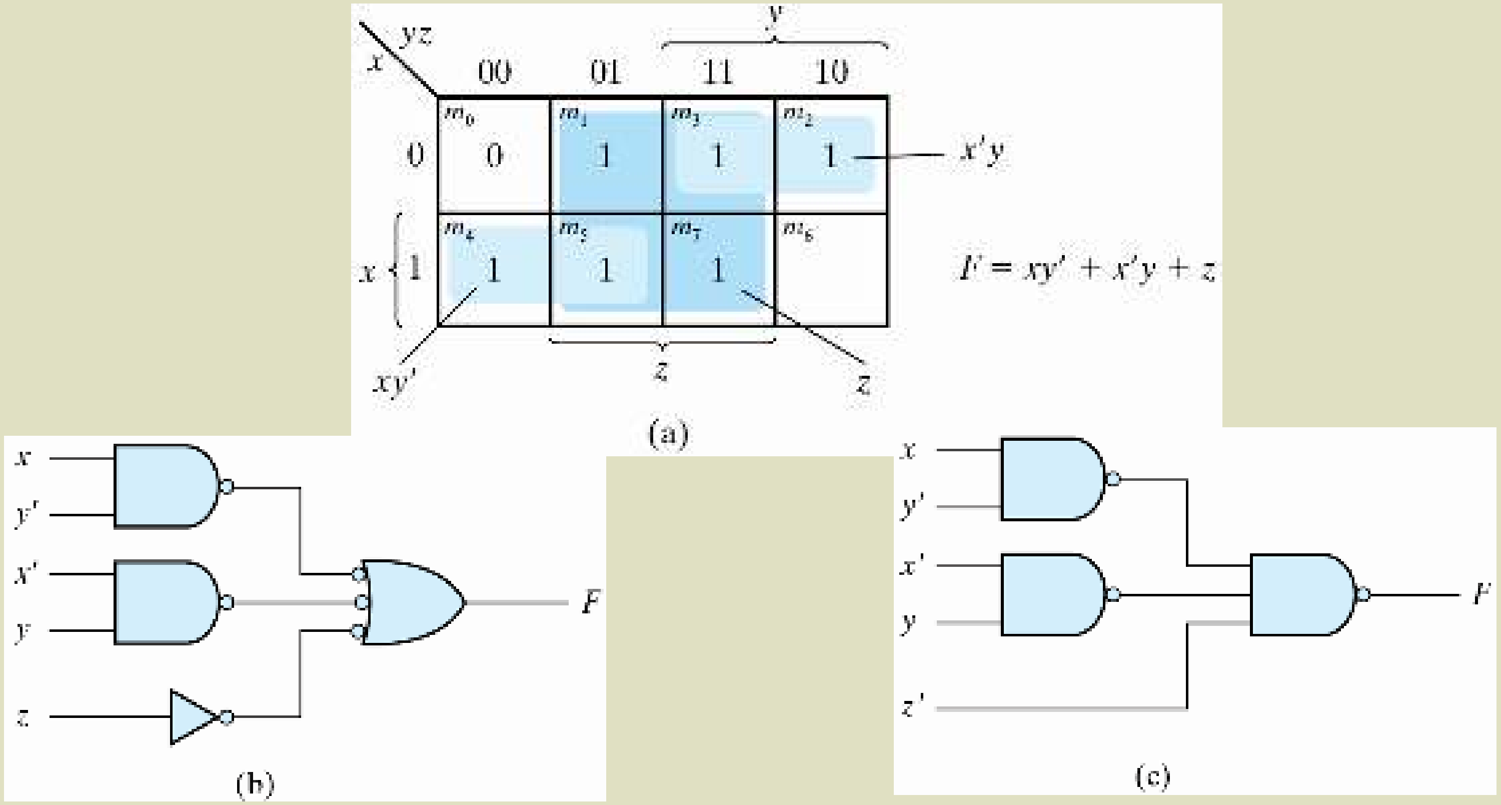


Figure 3.21 Solution to Example 3-10

Procedure with Two Levels NAND

- The procedure
  - Simplified in the form of sum of products;
  - A NAND gate for each product term; the inputs to each NAND gate are the literals of the term (the first level);
  - A single NAND gate for the second sum term (the second level);
  - A term with a single literal requires an inverter in the first level.

Multilevel NAND Circuits

- Boolean function implementation
  - AND-OR logic → NAND-NAND logic
    - AND → AND + inverter
    - OR: inverter + OR = NAND
    - For every bubble that is not compensated by another small circle along the same line, insert an inverter.

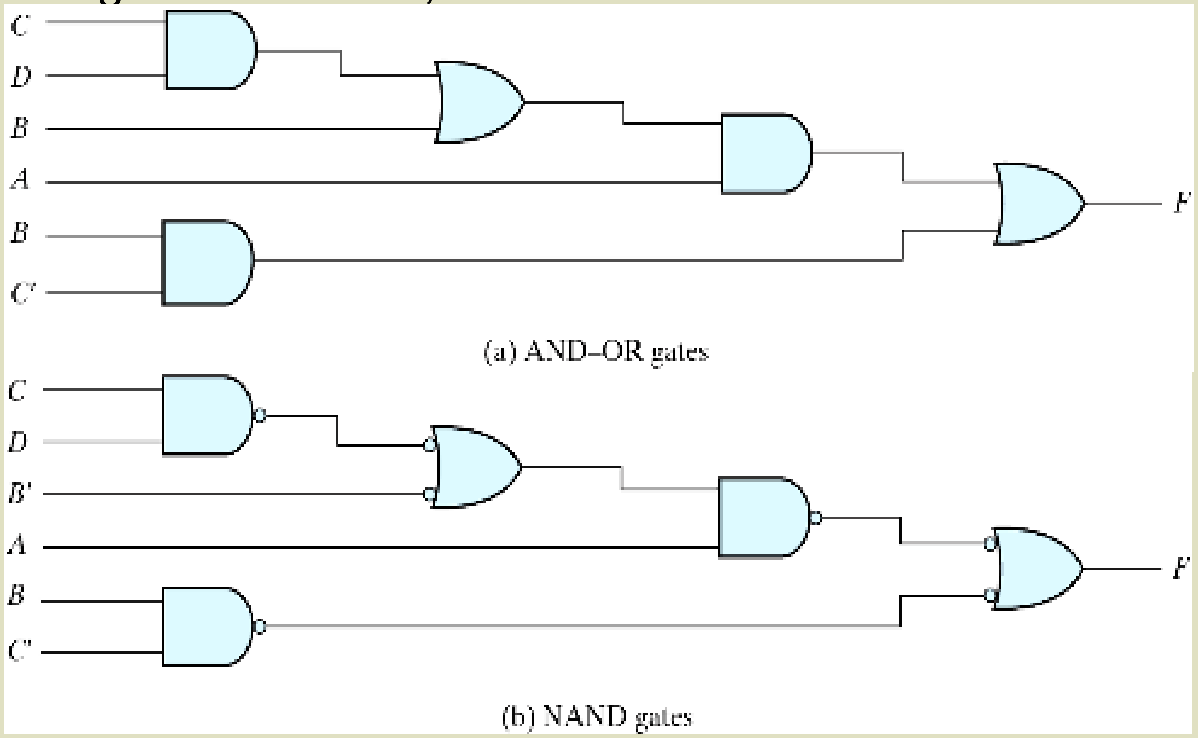
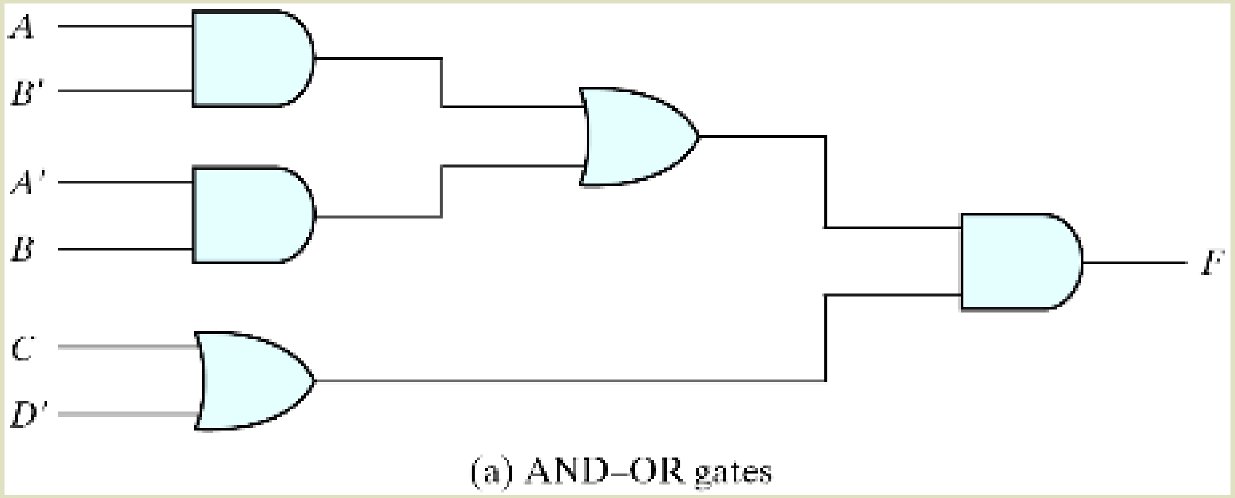


Figure 3.22 Implementing $F = A(CD + B) + BC'$

NAND Implementation



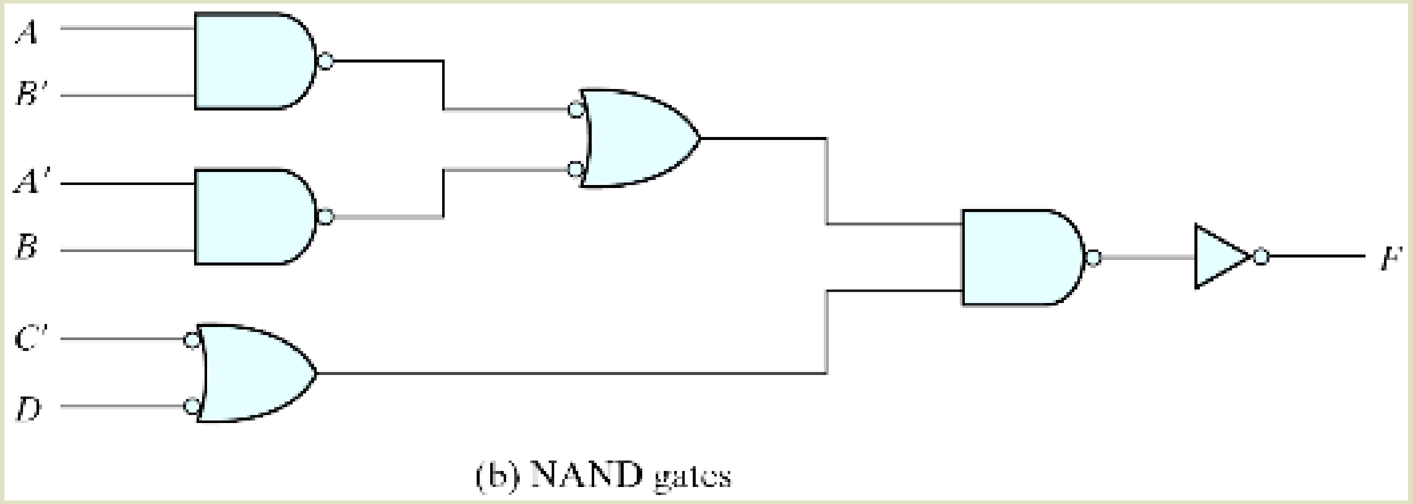


Figure 3.23 Implementing $F = (AB' + A'B)(C + D')$

NOR Implementation

- NOR function is the dual of NAND function.
- The NOR gate is also universal.

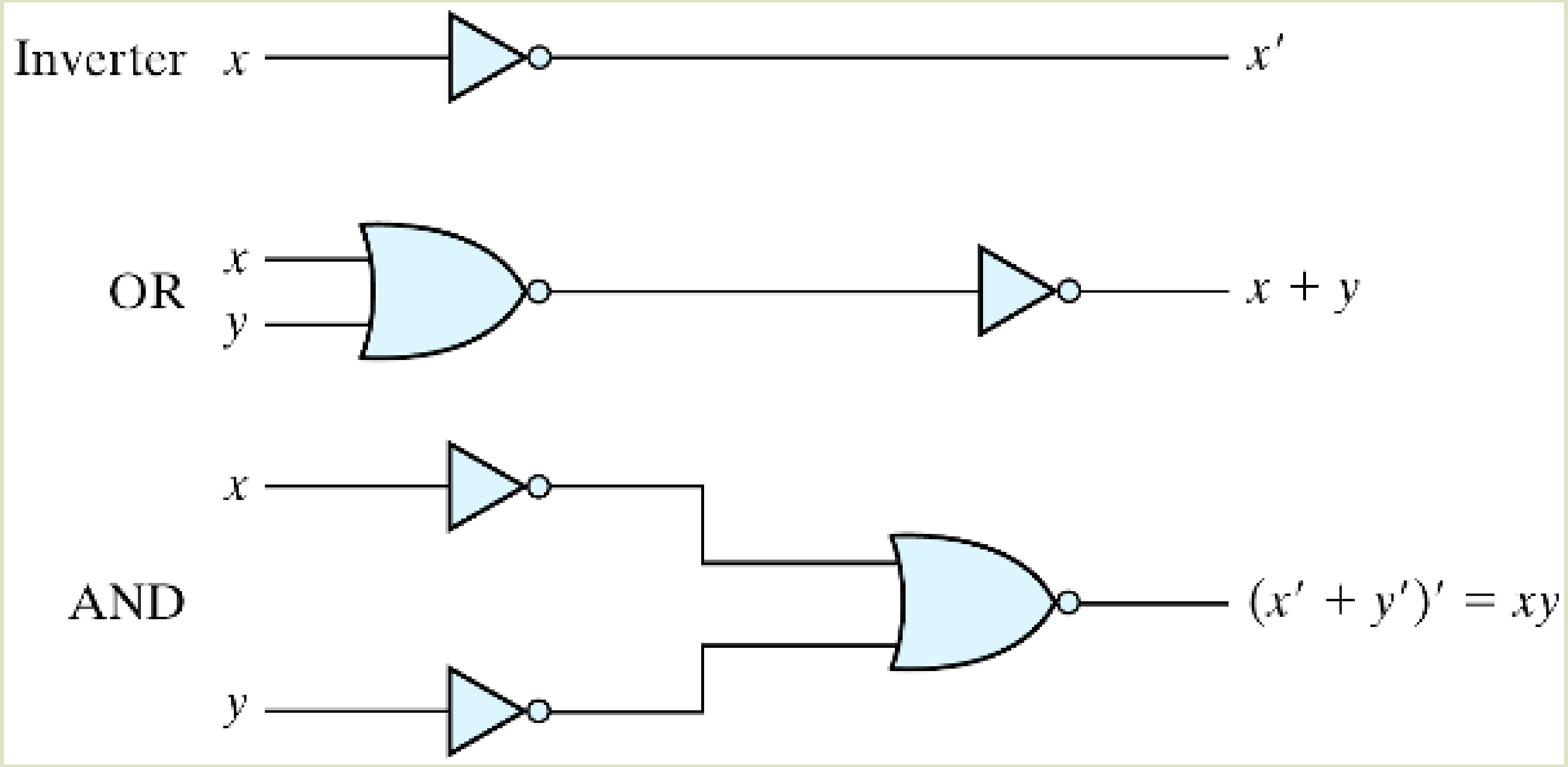


Figure 3.24 Logic Operation with NOR Gates

Two Graphic Symbols for a NOR Gate

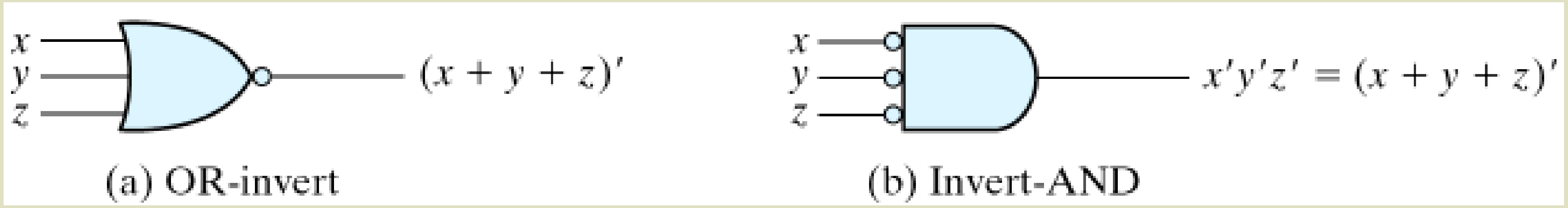


Figure 3.25 Two Graphic Symbols for NOR Gate

Example: $F = (A + B)(C + D)E$

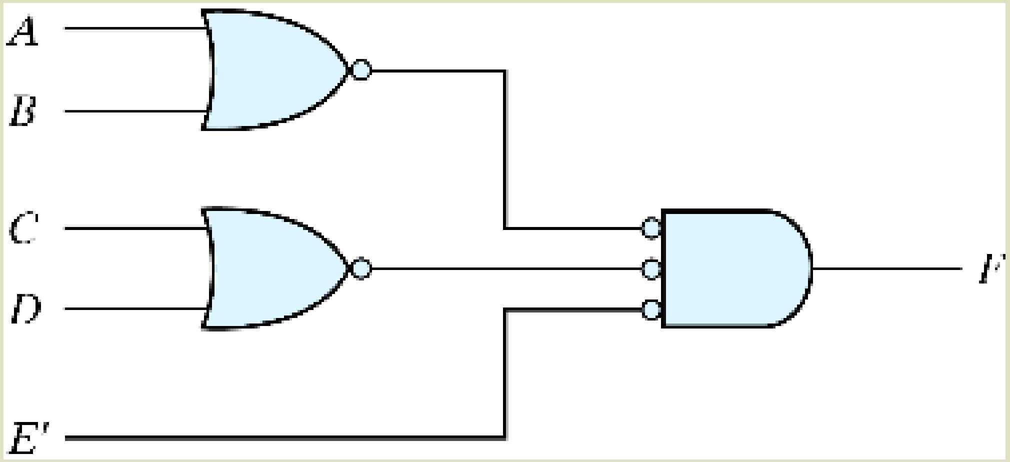


Figure 3.26 Implementing $F = (A + B)(C + D)E$

Example

Example: $F = (AB' + A'B)(C + D')$

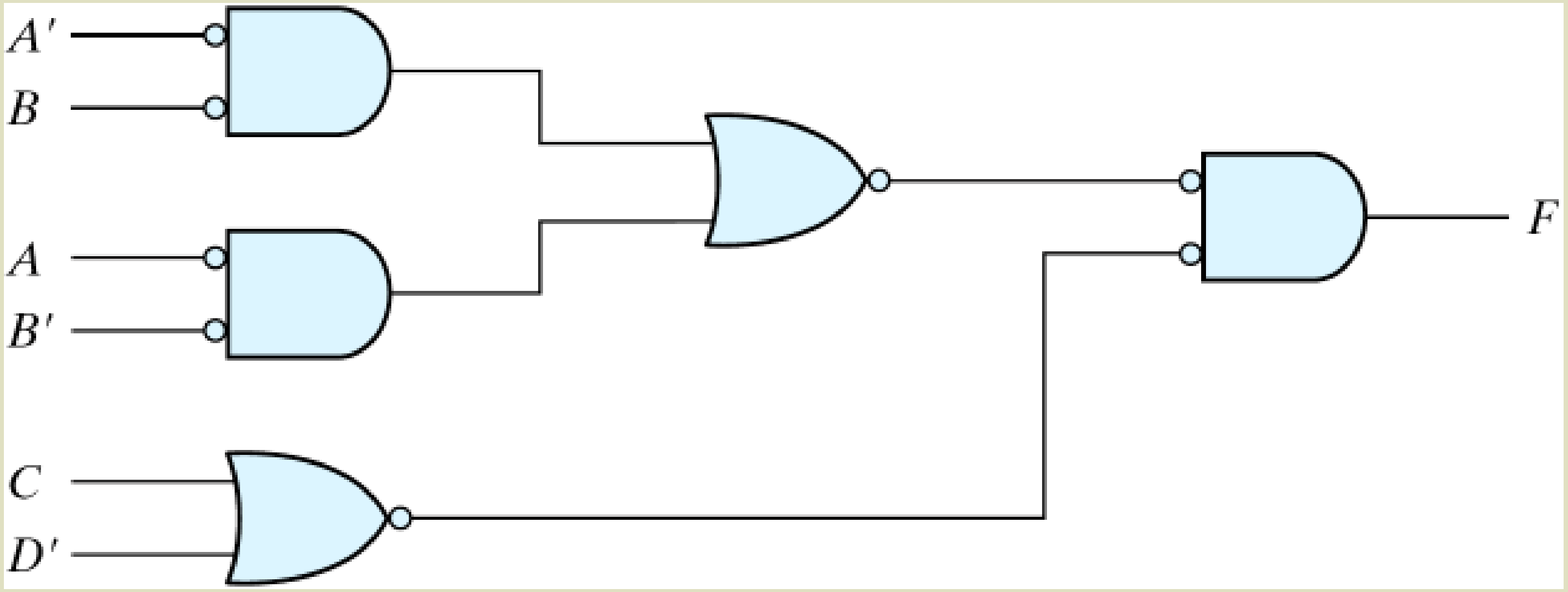


Figure 3.27 Implementing $F = (AB' + A'B)(C + D')$ with NOR gates

3-8 Other Two-level Implementations (

- Wired logic
  - A wire connection between the outputs of two gates
  - Open-collector TTL NAND gates: wired-AND logic
  - The NOR output of ECL gates: wired-OR logic

$$F = (AB)' \cdot (CD)' = (AB + CD)' = (A' + B')(C' + D')$$

AND-OR-INVERT function

$$F = (A + B)' + (C + D)' = [(A + B)(C + D)]'$$

OR-AND-INVERT function

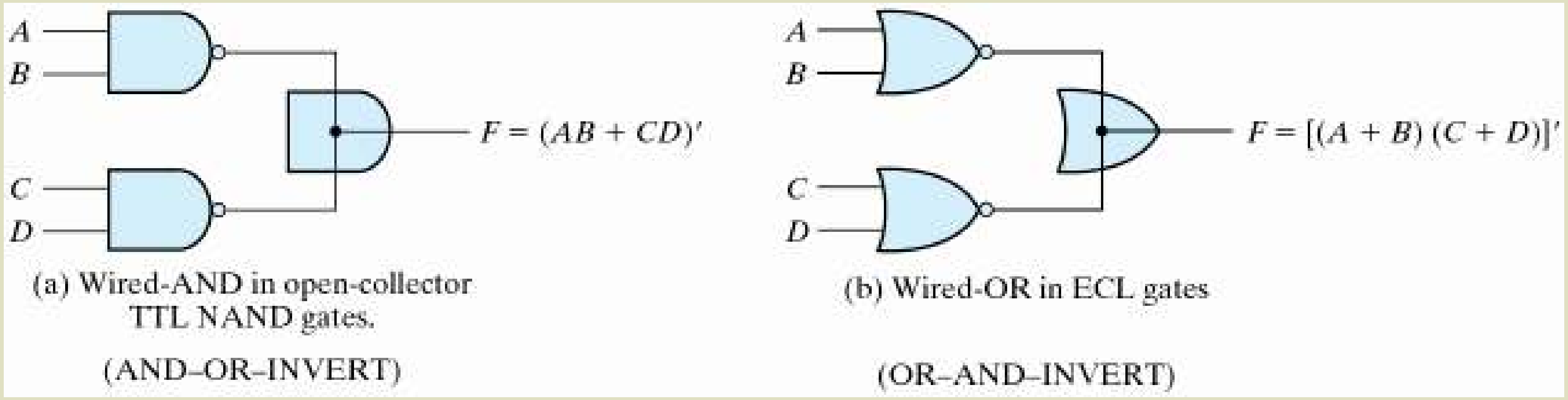


Figure 3.28 Wired Logic

Non-degenerate Forms

- 16 possible combinations of two-level forms
  - Eight of them: degenerate forms = a single operation
    - AND-AND, AND-NAND, OR-OR, OR-NOR, NAND-OR, NAND-NOR, NOR-AND, NOR-NAND.
  - The eight non-degenerate forms
    - AND-OR, OR-AND, NAND-NAND, NOR-NOR, NOR-OR, NAND-AND, OR-NAND, AND-NOR.
    - AND-OR and NAND-NAND = sum of products.
    - OR-NAND   NOR-OR
    - OR-AND and NOR-NOR = product of sums.
    - NAND-AND, AND-NOR

AND-OR-Invert Implementation

- AND-OR-INVERT (AOI) Implementation
  - NAND-AND = AND-NOR = AOI
  - $F = (AB+CD+E)'$
  - $F' = AB+CD+E$ (sum of products)

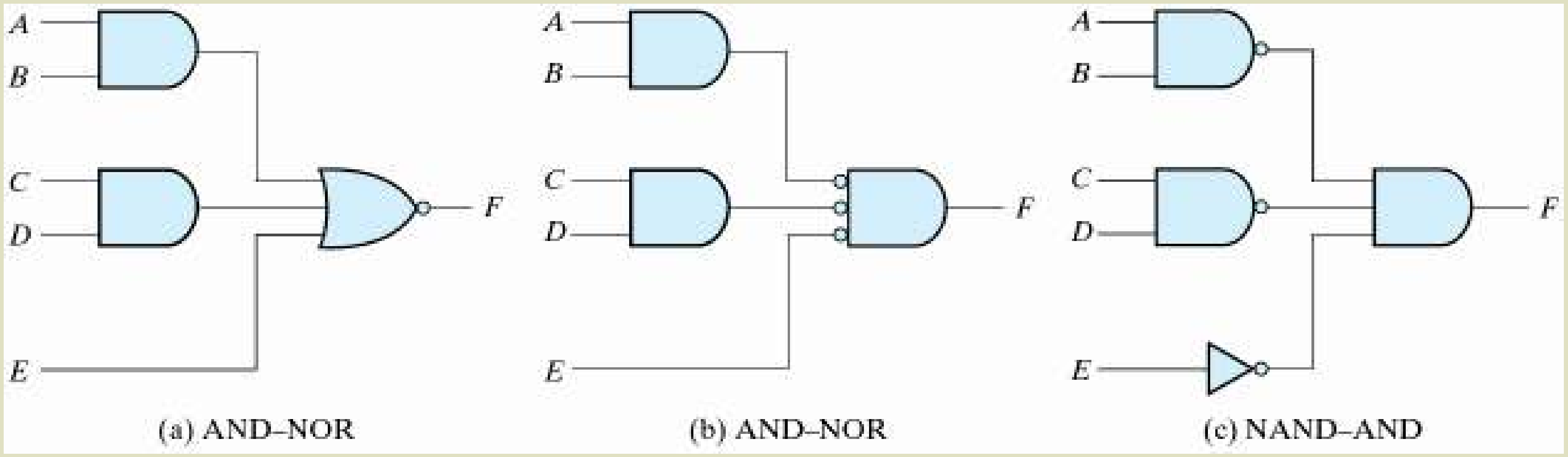


Figure 3.29 AND-OR-INVERT circuits, $F = (AB + CD + E)'$

OR-AND-Invert Implementation

- OR-AND-INVERT (OAI) Implementation
  - OR-NAND = NOR-OR = OAI
  - $F = ((A+B)(C+D)E)'$
  - $F' = (A+B)(C+D)E$ (product of sums)

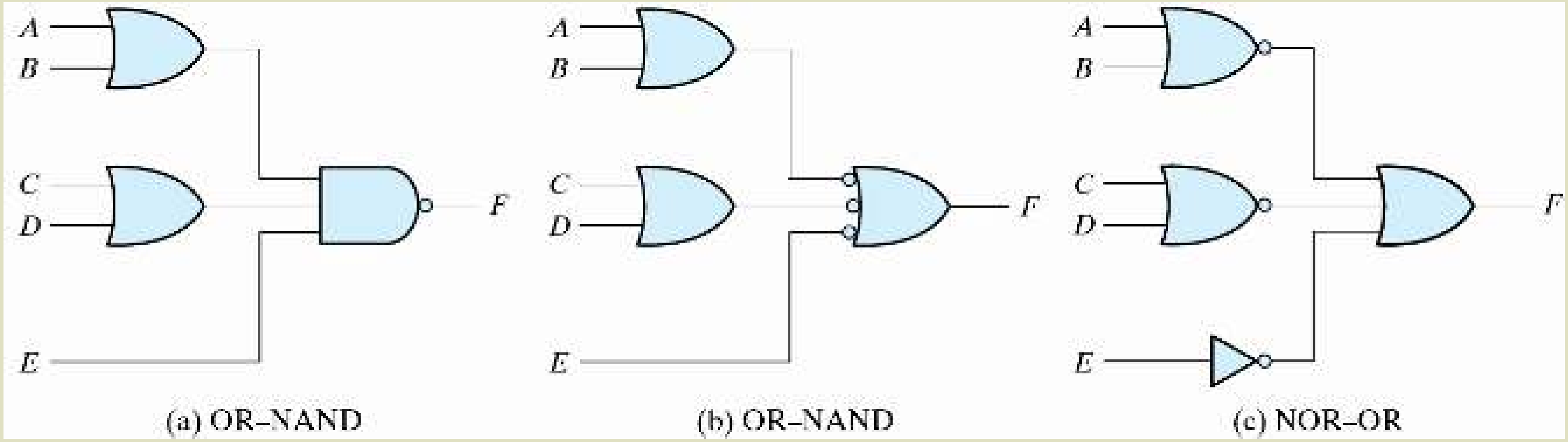


Figure 3.30 OR-AND-INVERT circuits, $F = ((A+B)(C+D)E)'$

Tabular Summary and Examples

- Example 3-11: $F = x'y'z'+xyz'$
  - $F' = x'y+xy'+z$ (*F'*: sum of products)
  - $F = (x'y+xy'+z)'$ (*F*: AOI implementation)
  - $F = x'y'z' + xyz'$ (*F*: sum of products)
  - $F' = (x+y+z)(x'+y'+z)$ (*F'*: product of sums)
  - $F = ((x+y+z)(x'+y'+z))'$ (F: OAI)

Tabular Summary and Examples

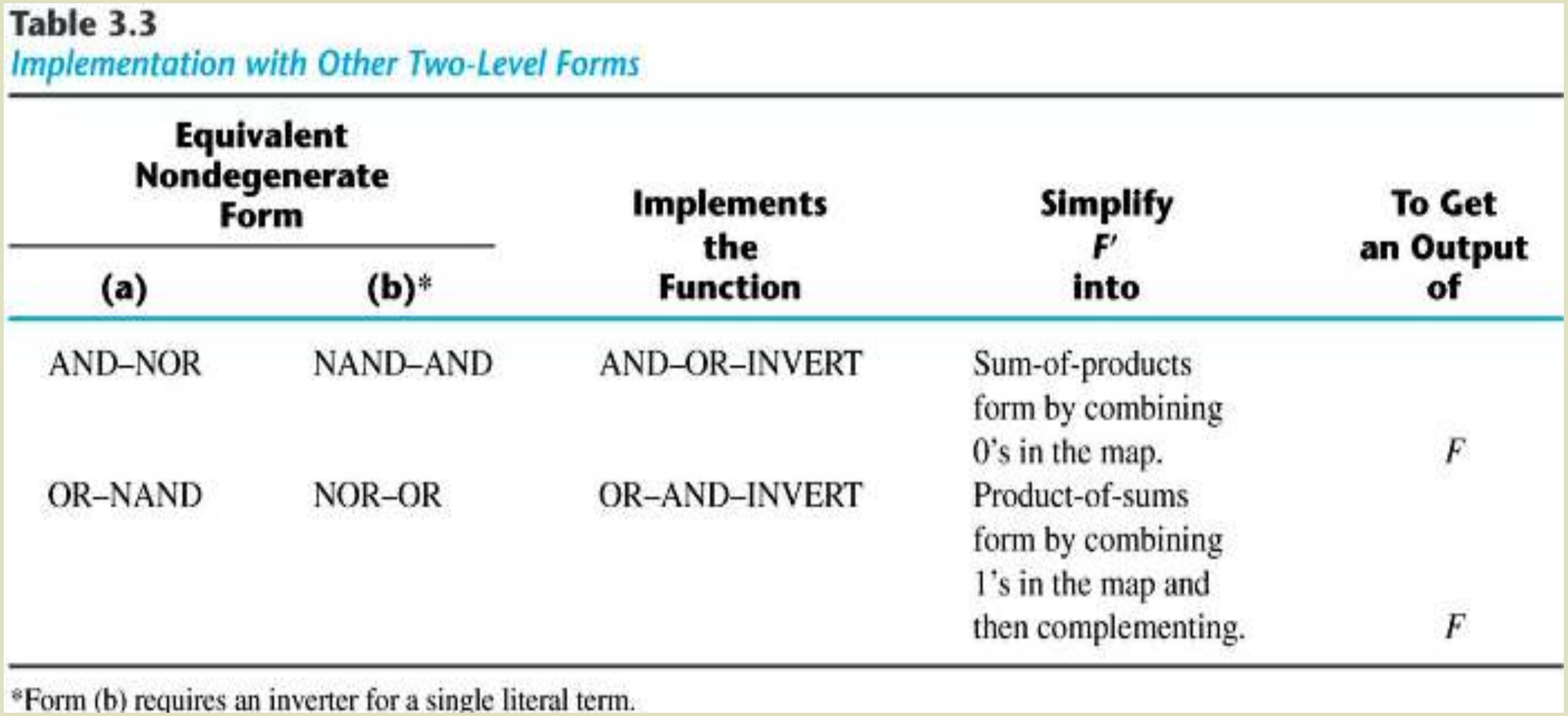
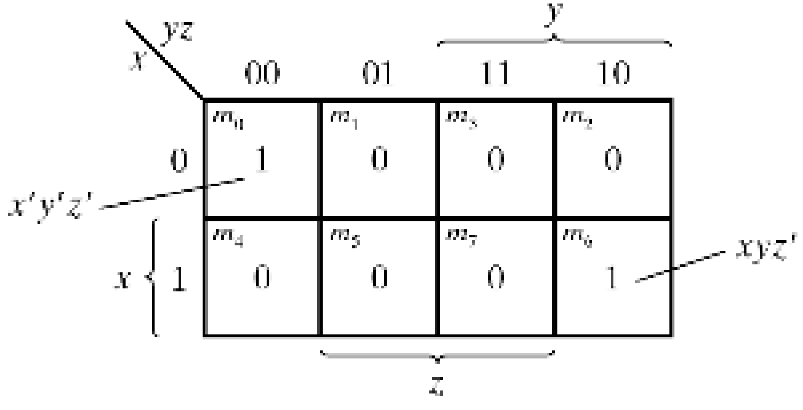


Table 3.3
Implementation with Other Two-Level Forms

Equivalent Nondegenerate Form		Implements the Function	Simplify F' into	To Get an Output of
(a)	(b)*			
AND–NOR	NAND–AND	AND–OR–INVERT	Sum-of-products form by combining 0's in the map.	F
OR–NAND	NOR–OR	OR–AND–INVERT	Product-of-sums form by combining 1's in the map and then complementing.	F

*Form (b) requires an inverter for a single literal term.

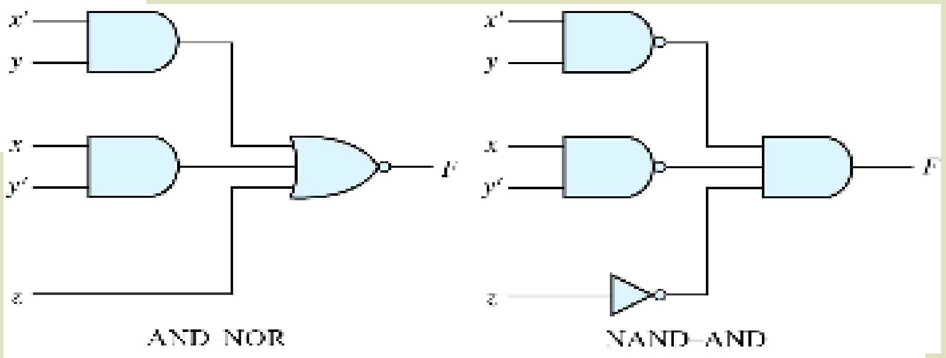
October 3, 2017



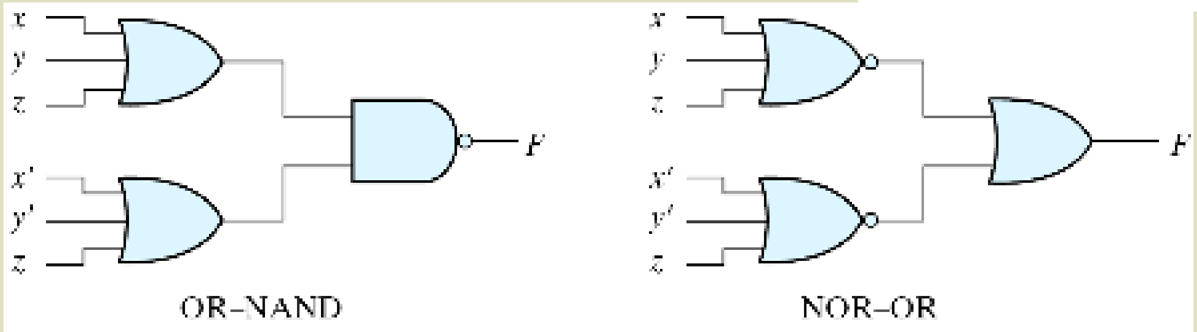
$F = x'y'z' + xyz'$

$F' = x'y + xy' + z$

(a) Map simplification in sum of products



(b) $F = (x'y + xy' + z)'$



(c) $F = [(x + y + z)(x' + y' + z)]'$

Figure 3.31 Other Two-level Implementations

3-9 Exclusive-OR Function

- Exclusive-OR (XOR)
  - $x \oplus y = xy' + x'y$
- Exclusive-NOR (XNOR)
  - $(x \oplus y)' = xy + x'y'$
- Some identities
  - $x \oplus 0 = x$
  - $x \oplus 1 = x'$
  - $x \oplus x = 0$
  - $x \oplus x' = 1$
  - $x \oplus y' = (x \oplus y)'$
  - $x' \oplus y = (x \oplus y)'$
- Commutative and associative
  - $A \oplus B = B \oplus A$
  - $(A \oplus B) \oplus C = A \oplus (B \oplus C) = A \oplus B \oplus C$

Exclusive-OR Implementations

- Implementations
  - $(x'+y')x + (x'+y')y = xy'+x'y = x \oplus y$

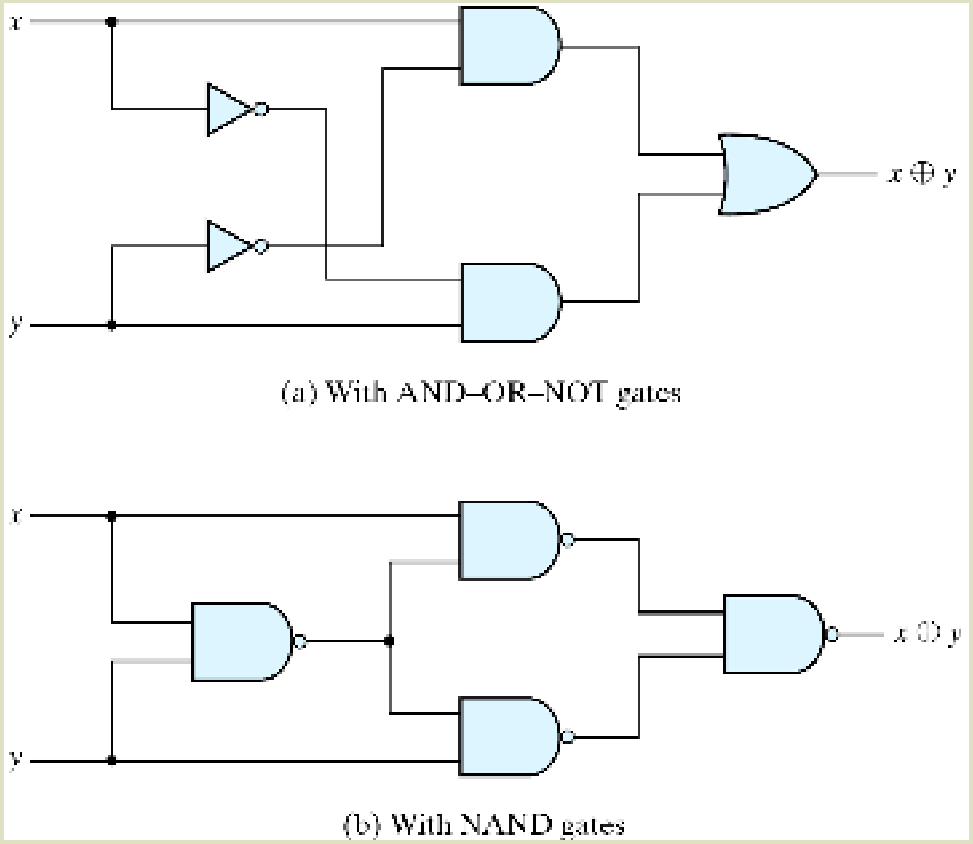


Figure 3.32 Exclusive-OR Implementations

Odd Function

- $A \oplus B \oplus C = (AB'+A'B)C' +(AB+A'B')C = AB'C'+A'BC'+ABC+A'B'C = \Sigma(1, 2, 4, 7)$
- XOR is a odd function → an odd number of 1's, then *F* = 1.
- XNOR is a even function → an even number of 1's, then *F* = 1.

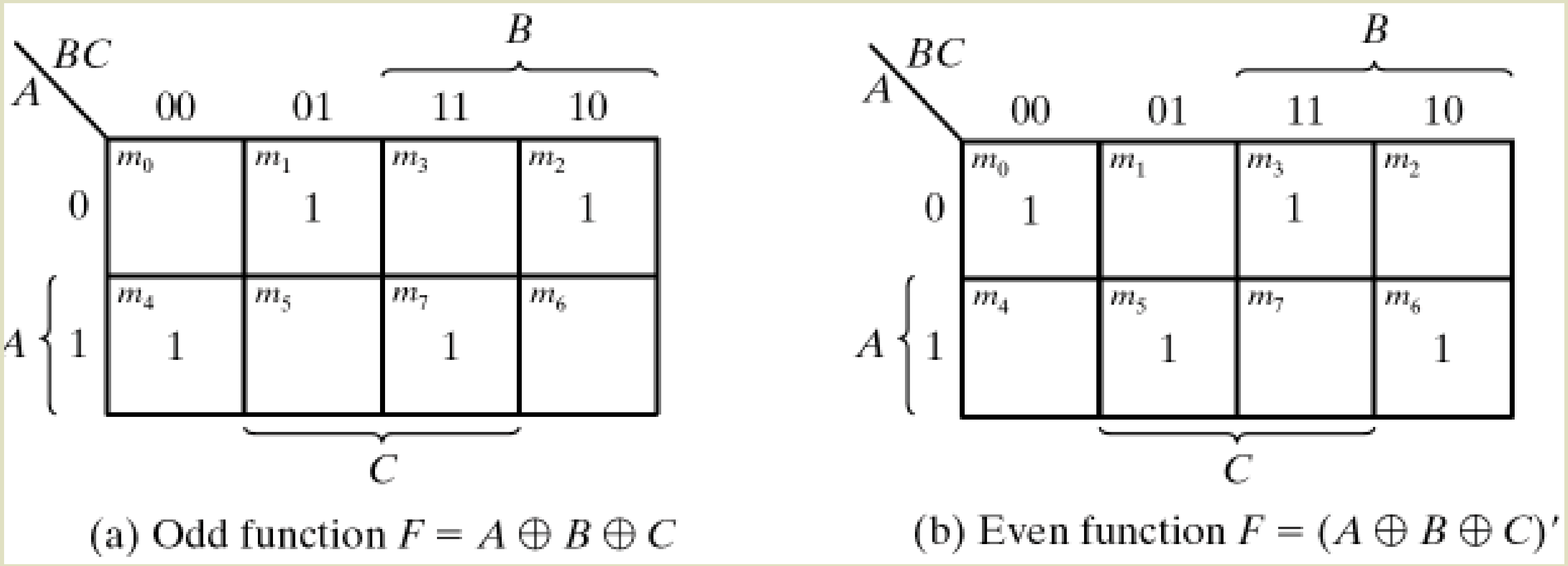


Figure 3.33 Map for a Three-variable Exclusive-OR Function

XOR and XNOR

- Logic diagram of odd and even functions

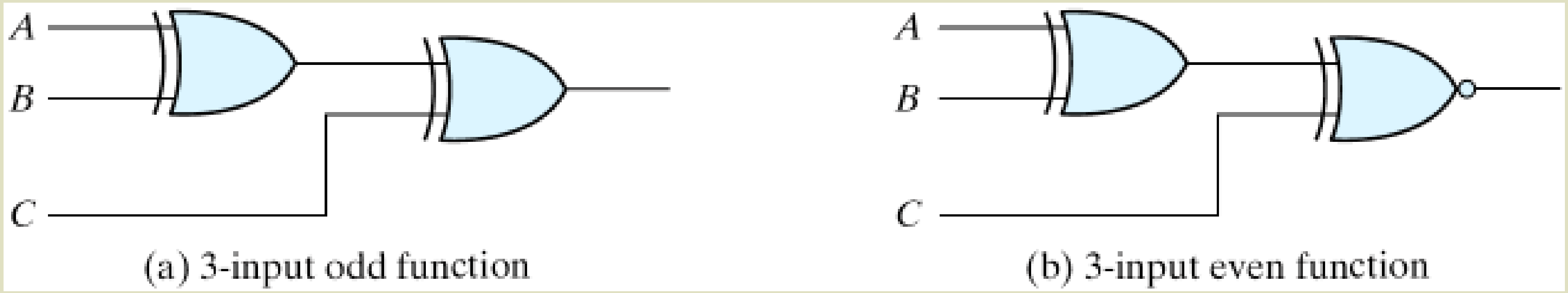
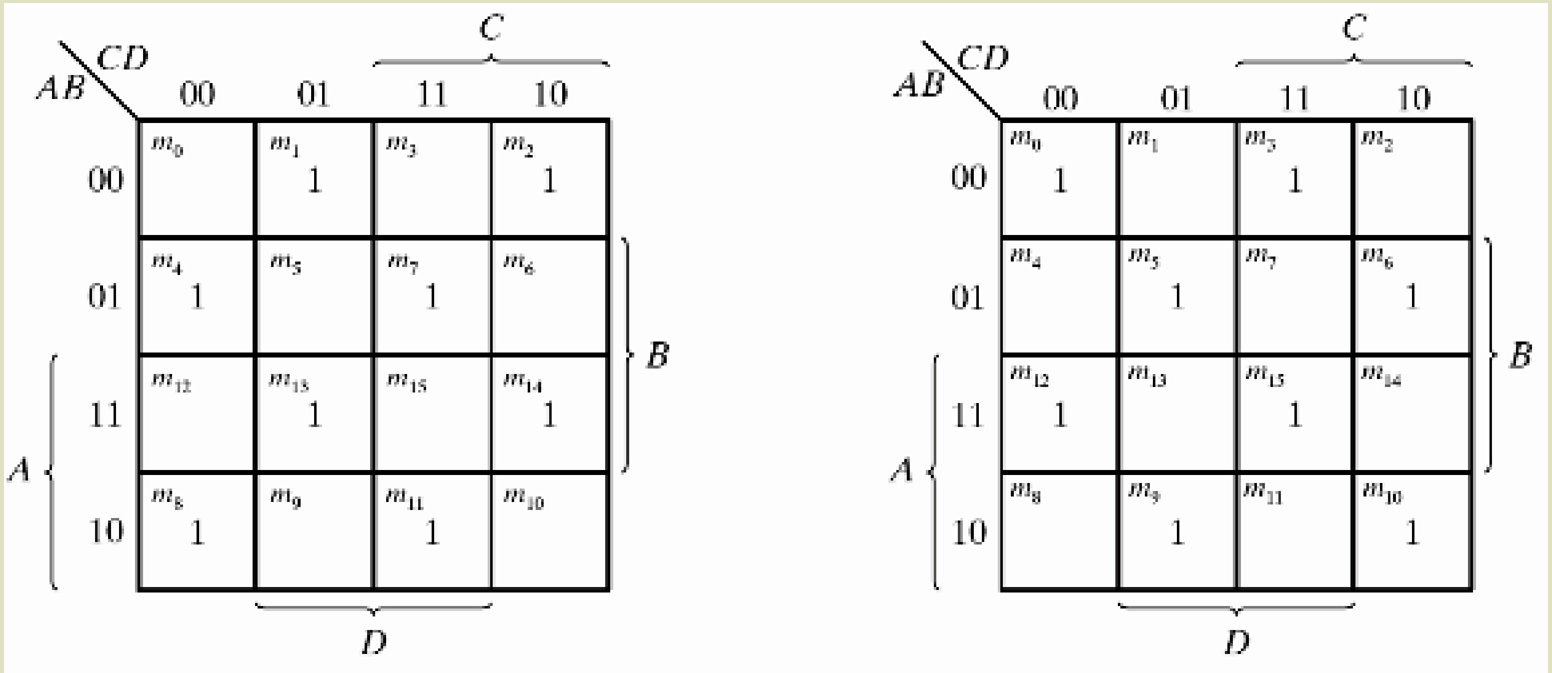


Figure 3.34 Logic Diagram of Odd and Even Functions

Four-variable Exclusive-OR function

- Four-variable Exclusive-OR function
  - $A \oplus B \oplus C \oplus D = (AB'+A'B) \oplus (CD'+C'D) = (AB'+A'B)(CD+C'D')+(AB+A'B')(CD'+C'D)$



(a) Odd function $F = A \oplus B \oplus C \oplus D$ (b) Even function $F = (A \oplus B \oplus C \oplus D)'$

Figure 3.35 Map for a Four-variable Exclusive-OR Function

Parity Generation and Checking

- Parity Generation and Checking
  - A parity bit: P = $x \oplus y \oplus z$
  - Parity check: C = $x \oplus y \oplus z \oplus P$
    - C=1: one bit error or an odd number of data bit error
    - C=0: correct or an even # of data bit error

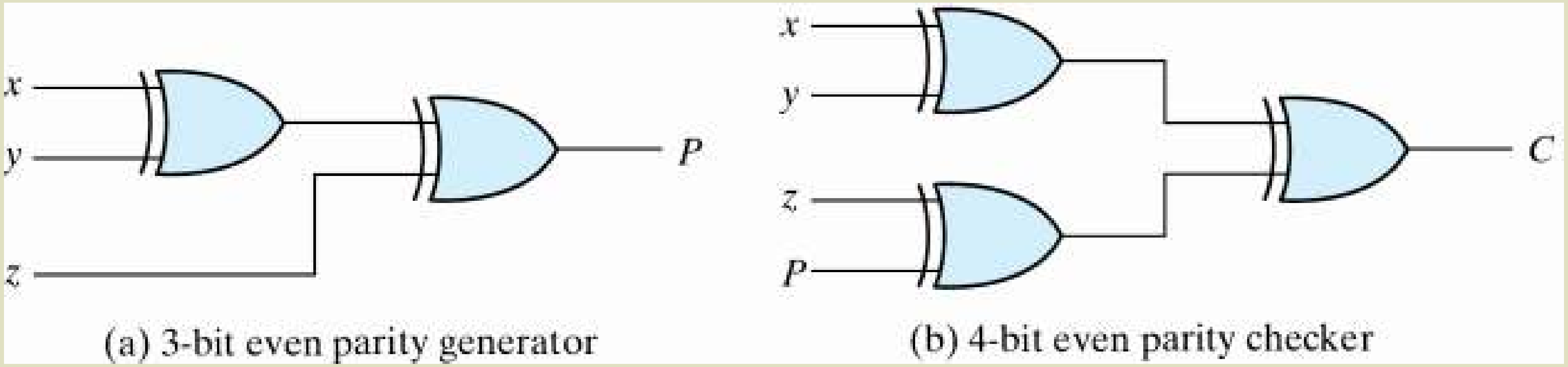

x
y
z
P
(a) 3-bit even parity generator
x
y
z
P
C
(b) 4-bit even parity checker

Figure 3.36 Logic Diagram of a Parity Generator and Checker

Parity Generation and Checking

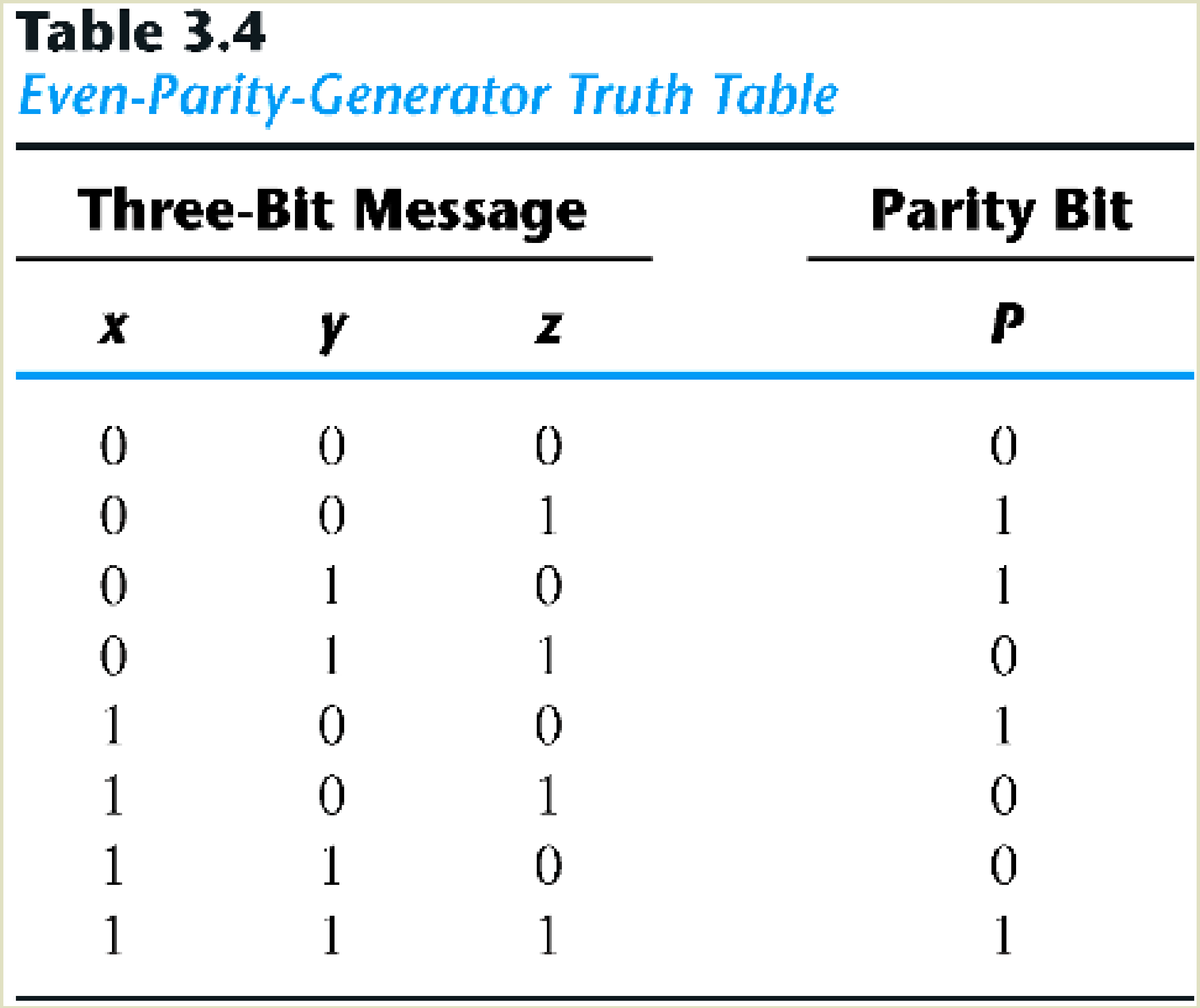


Table 3.4
Even-Parity-Generator Truth Table

Three-Bit Message			Parity Bit
x	*y*	*z*	*P*
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Parity Generation and Checking

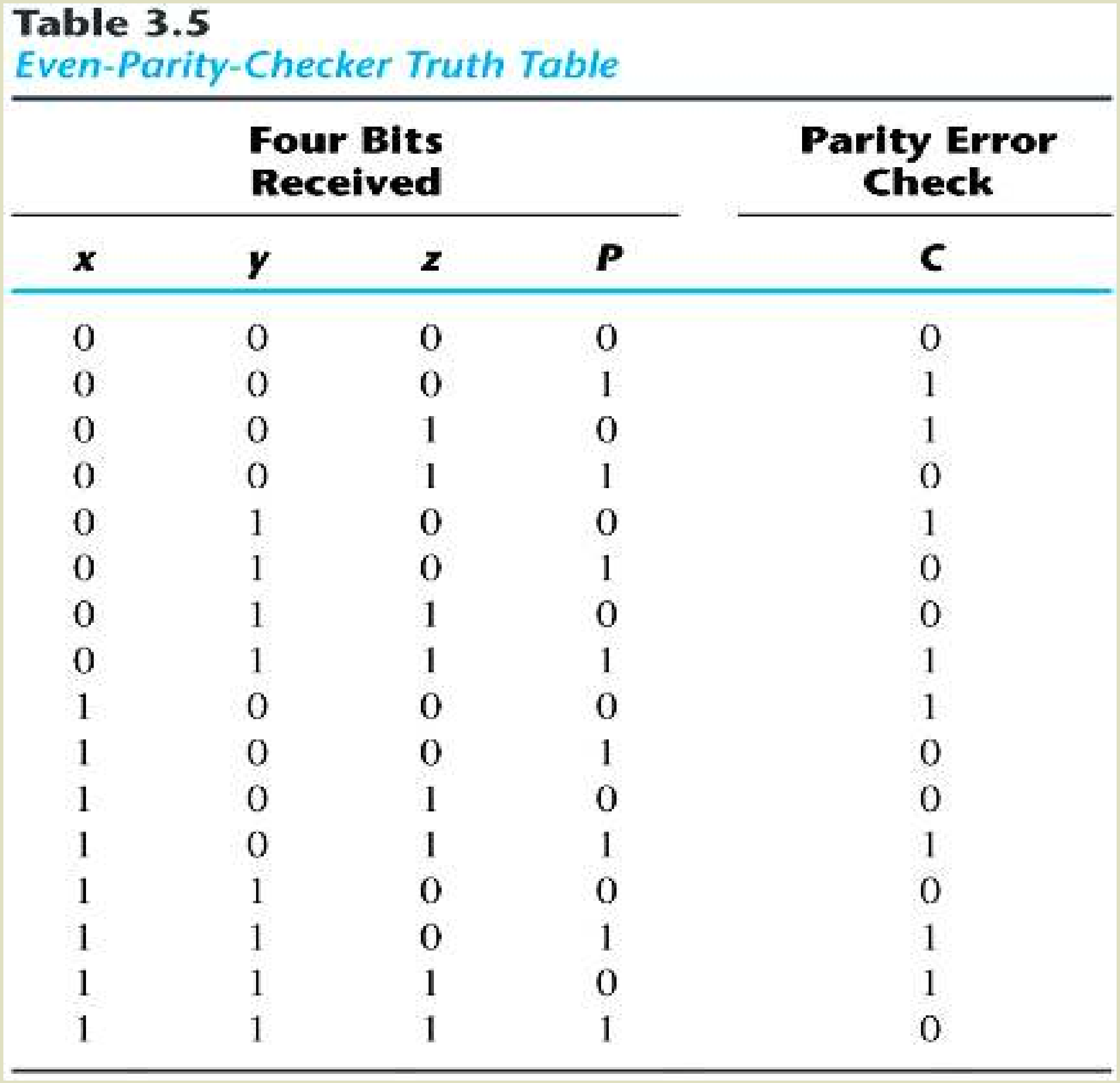


Table 3.5
Even-Parity-Checker Truth Table

Four Bits Received				Parity Error Check
x	***y***	***z***	***P***	***C***
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	0