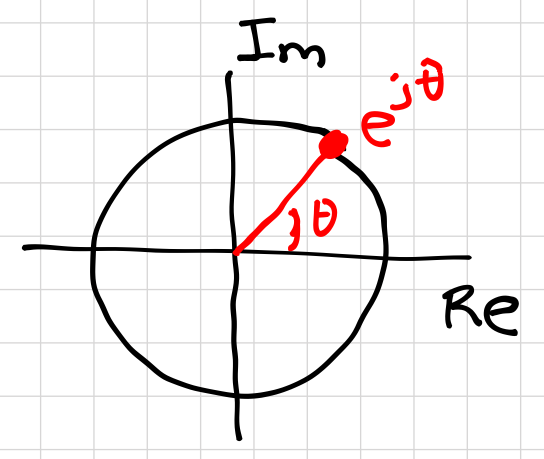
Fourier Representation of Signals and LTI systems

— Read Introduction @ p 195

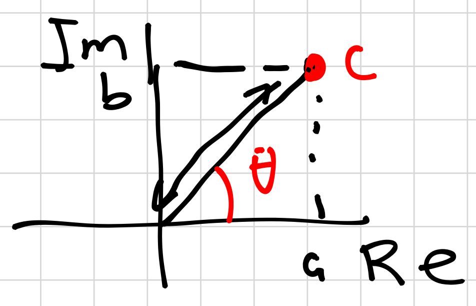
* Representing signals as weighted superposition of complex sinusoids.

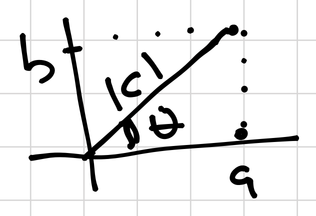
* Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$



* Polar Form





$$\left\{ \begin{array}{ll} c = a + jb & \\ |c| = \sqrt{a^2+b^2} & : \text{Magnitude} \\ \arg\{c\} = \theta = \arctan\left(\frac{b}{a}\right) & : \text{Phase} \\ \rightarrow c = |c|\, e^{j\arg\{c\}} & \text{Polar form} \end{array} \right.$$

Complex Sinusoids and Frequency Response of LTI Systems.

DT

$$x[n] \rightarrow \boxed{\mathcal{H}} \rightarrow y[n] = \mathcal{H}\{x[n]\}$$

$h[n]$ is the impulse response
$h[n] = \mathcal{H}\{\delta[n]\}$

For a given input, $x[n]$

$$y[n] = \mathcal{H}\{x[n]\} = x[n] * h[n] = h[n] * x[n]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$

* Let $x[n] = e^{j\Omega n}$ $\quad \Omega$: frequency.

Then the output

$$y[n] = \mathcal{H}\{e^{j\Omega n}\} = \sum_{k=-\infty}^{+\infty} h[k]\, e^{j\Omega(n-k)}$$

$$= e^{j\Omega n} \sum_{k=-\infty}^{+\infty} h[k] \cdot e^{-j\Omega k}$$

Let's define $\underbrace{H(e^{j\Omega})}_{\text{Frequency Response}} \triangleq \left\{ \sum_{k=-\infty}^{+\infty} h[k] \cdot e^{-j\Omega k} \right\}$

— Not a function of time, but FREQUENCY.

$$y[n] = \underline{H(e^{j\Omega})} \cdot \underline{e^{j\Omega n}} = \mathcal{H}\{e^{j\Omega n}\}$$

$$e^{j\Omega n} \rightarrow \boxed{\mathcal{H}} \rightarrow \underbrace{H(e^{j\Omega})}_{\substack{\Downarrow \\ \text{Frequency Response}}} \cdot e^{j\Omega n}$$

CT

$$x(t) \rightarrow \boxed{\mathcal{H}} \rightarrow y(t) = \mathcal{H}\{x(t)\}$$

$\llcorner h(t)$: impulse response

Let $x(t) = e^{j\omega t}$, then the output

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

Frequency Response : $H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$

$$y(t) = \mathcal{H}\{e^{j\omega t}\} = e^{j\omega t} \cdot H(j\omega)$$

$$e^{j\omega t} \rightarrow \boxed{\mathcal{H}} \rightarrow e^{j\omega t} \cdot \underline{H(j\omega)}$$

Polar form $\qquad \boxed{\exp(\cdot) = e^{\cdot}}$

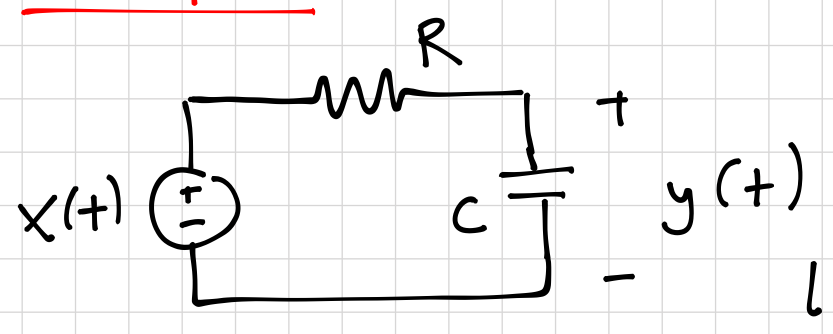
$$H(j\omega) = |H(j\omega)| \cdot \exp\left[j \cdot \arg\{H(j\omega)\}\right]$$

$$\mathcal{H}\{e^{j\omega t}\} = H(j\omega) \cdot e^{j\omega t}$$

$$= |H(j\omega)| \cdot \exp\left[j\,(\omega t + \arg\{H(j\omega)\})\right]$$

$$\left[\begin{array}{ll} |H(j\omega)| : & \text{Magnitude Response} \\ \arg\{H(j\omega)\} : & \text{Phase Response} \end{array} \right]$$

Example



$x(t)$: input voltage
$y(t)$: output voltage
let $\alpha \triangleq RC$

Impulse Response

$$h(t) = \frac{1}{\alpha} \cdot e^{-t/\alpha} \, u(t)$$

$$\int e^{-at} dt = -\frac{1}{a} e^{-at}$$

Frequency Response

$$H(jw) = \int_{-\infty}^{+\infty} h(\tau) \, e^{-jw\tau} \, d\tau$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\alpha} \, e^{-\tau/\alpha} u(t) \, e^{-jw\tau} \, d\tau$$

$$= \frac{1}{\alpha} \int_{0}^{\infty} \exp\left\{-\left(jw + \frac{1}{\alpha}\right)\tau\right\} d\tau$$

$$= \frac{1}{\alpha} \cdot \frac{-1}{jw + 1/\alpha} \cdot \left[\exp\left\{-\left(jw + \frac{1}{\alpha}\right)\tau\right\}\right]_0^{\infty}$$

$$= \frac{1}{\alpha} \cdot \frac{-1}{jw + 1/\alpha} \quad [0 - 1]$$

$$\boxed{H(jw) = \frac{1/\alpha}{jw + 1/\alpha}}$$

Magnitude Response $|H(jw)|$

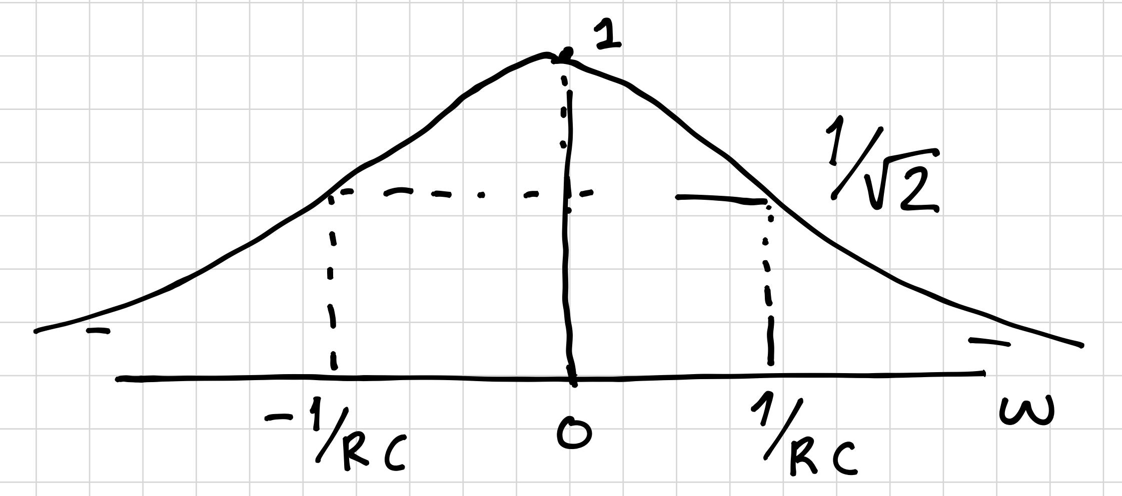
$$H(jw) = \frac{1/\alpha}{1/\alpha + jw} \cdot \left[\frac{1/\alpha - jw}{1/\alpha - jw}\right] = \frac{1}{\alpha} \cdot \frac{1/\alpha - jw}{\frac{1}{\alpha^2} - (-1 \cdot w^2)}$$

$$= \frac{1}{\alpha} \cdot \frac{1/\alpha - jw}{\left(\frac{1}{\alpha}\right)^2 + w^2}$$

$$= \frac{1/\alpha}{(1/\alpha)^2 + w^2} \left(\frac{1}{\alpha} - jw\right)$$

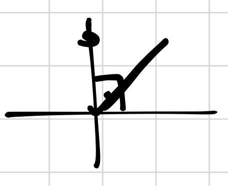
$$|H(jw)| = \frac{1/\alpha}{(1/\alpha)^2 + w^2} \sqrt{\left(\frac{1}{\alpha}\right)^2 + w^2}$$

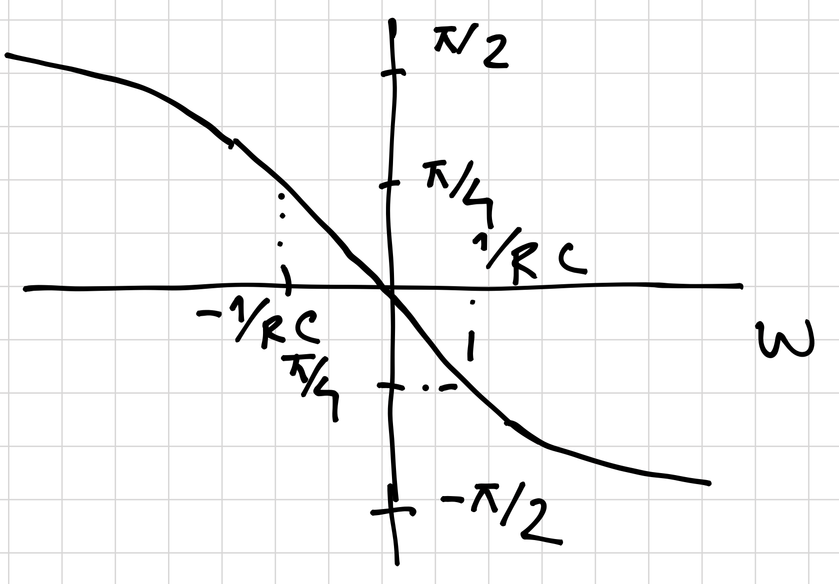
$$= \frac{1/\alpha}{\sqrt{\frac{1}{\alpha^2} + w^2}} = \frac{1/RC}{\sqrt{\frac{1}{(RC)^2} + w^2}}$$



Phase Response

$$\arg\{H(jw)\} = \arctan\left(\frac{-w}{\alpha}\right) = -\arctan(wRC)$$





— Eigenfunction

$\psi(t) = e^{jwt}$ is an $\boxed{\text{eigenfunction}}$ of the LTI system associated with the eigenvalue $\lambda = H(jw)$ because ψ satisfies the eigenvalue problem described by

$$\mathcal{H}\{\psi(t)\} = \lambda \cdot \psi(t)$$

$$\psi(t) \longrightarrow \boxed{\quad} \longrightarrow H(jw) \cdot \psi(t)$$
$$\psi[n] \qquad\qquad\qquad H(e^{j\Omega}) \cdot \psi[n]$$

/* If e_k is an eigenvector of a matrix A with eigenvalue λ_k then $A e_k = \lambda_k e_k$ */

→ By representing arbitrary signals as weighted superposition of eigenfunction we transform convolution operation to multiplication ←

→ For example if $x(t)$ is a weighted sum of M complex sinusoids

$$x(t) = \sum_{k=1}^{M} a_k \cdot \exp\{j w_k t\}$$

If $e^{jw_k t}$ is eigenfunction of the system with eigenvalue $H(jw)$ then each term in $x(t)$, $a_k \cdot e^{jw_k t}$ produces an output $a_k H(jw_k) e^{jw_k t}$

Output $y(t) = \sum_{k=1}^{M} a_k H(jw_k) e^{jw_k t}$

– ex $\quad y(t) = \underline{x(t-3)}$

If the input to this system is $x(t) = e^{j2t}$

then the output

$$y(t) = e^{j2(t-3)} = e^{j2t} \cdot e^{-j6}$$

e^{j2t} is the eigenfunction associated with the eigenvalue $\underline{H(j2)} = e^{-j6}$ //

$y(t) = H(jw) \cdot e^{j2t}$

– we can show this by

Impulse Response $\quad h(t) = \delta(t-3)$

$$\underline{H(jw)} = \int_{-\infty}^{+\infty} h(\tau) \cdot e^{-jw\tau} \, d\tau$$

Frequency response

$$= \int_{-\infty}^{+\infty} \delta(t-3) \, e^{-jw\tau} \, d\tau$$

$$= e^{-3jw}$$

$$\underline{H(j2)} = e^{-3j2} = e^{-6j} \text{ //}$$