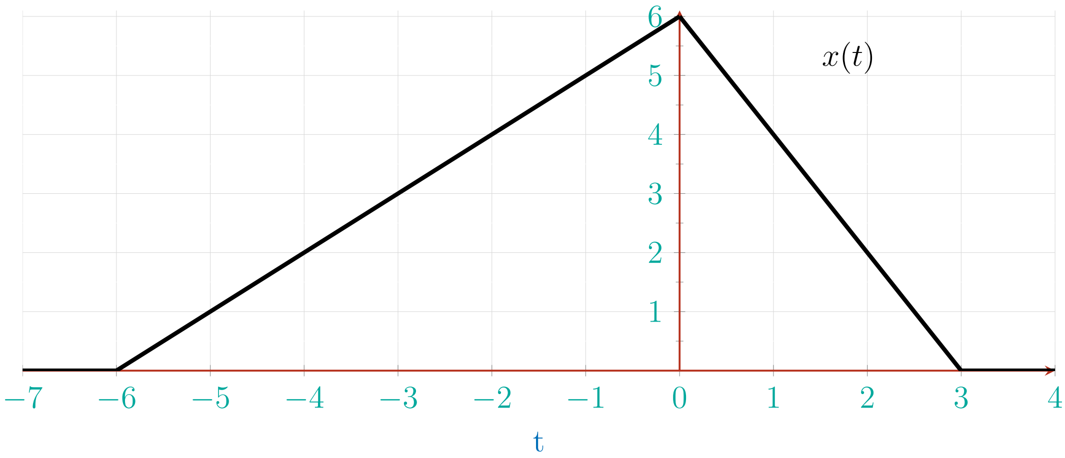
Signal Processing (İkinci)
Midterm Exam Solutions

Istanbul University - Computer Engineering Department - FALL 2017

November 2$^{\text{nd}}$, 2017

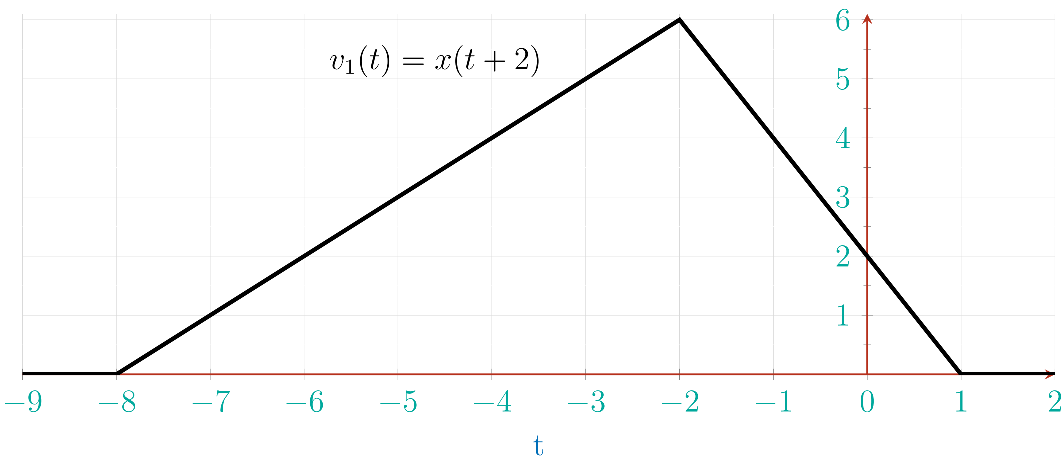
Q1: Consider the following CONTINUOUS TIME signal and answer the following questions.



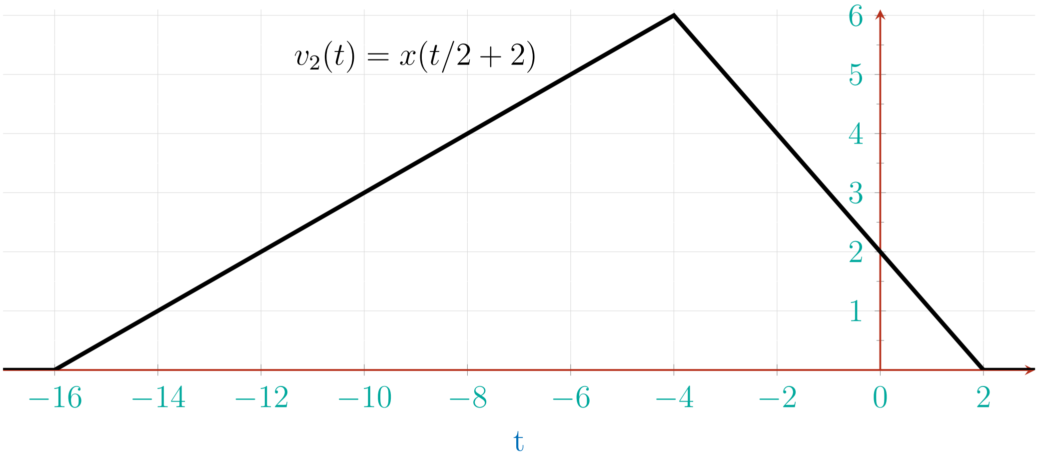
(a) `(20 pts)` Please carefully sketch $x(2 - \frac{t}{2})$. Show your steps to receive credit.

Solution 1a:

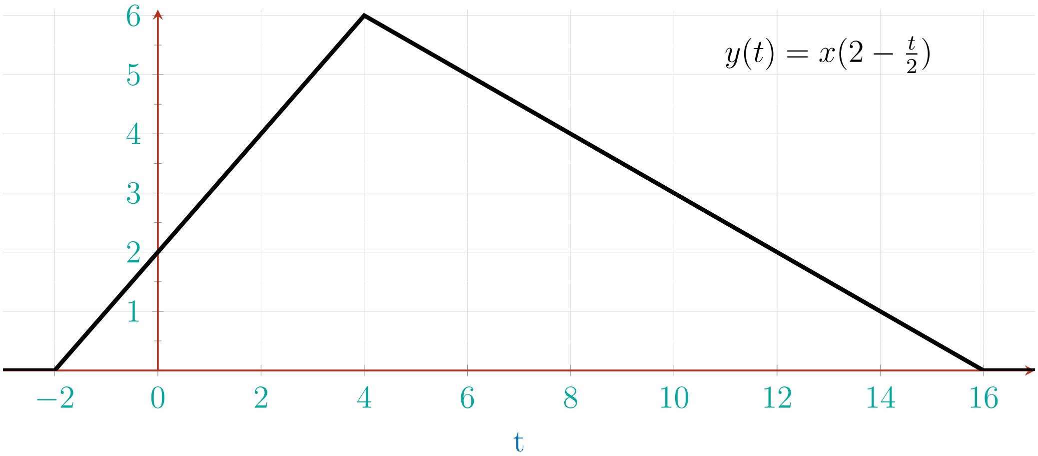
Let's say $v_1(t) = x(t + 2)$.



Let $v_2(t) = v_1(\frac{1}{2}t) = x(\frac{1}{2}t + 2)$



Finally, let $y(t) = v_2(-t) = x(\frac{-t}{2} + 2)$. This time we are *reflect*ing the signal around y axis.



You can easily verify your sketch by checking a couple of points:

$$\begin{aligned}
y(0) &= x(2 - \frac{0}{2}) = x(2) = 2 \\
y(4) &= x(2 - \frac{4}{2}) = x(0) = 6 \\
y(-2) &= x(2 - \frac{-2}{2}) = x(3) = 0 \\
y(8) &= x(2 - \frac{16}{2}) = x(6) = 0
\end{aligned}$$

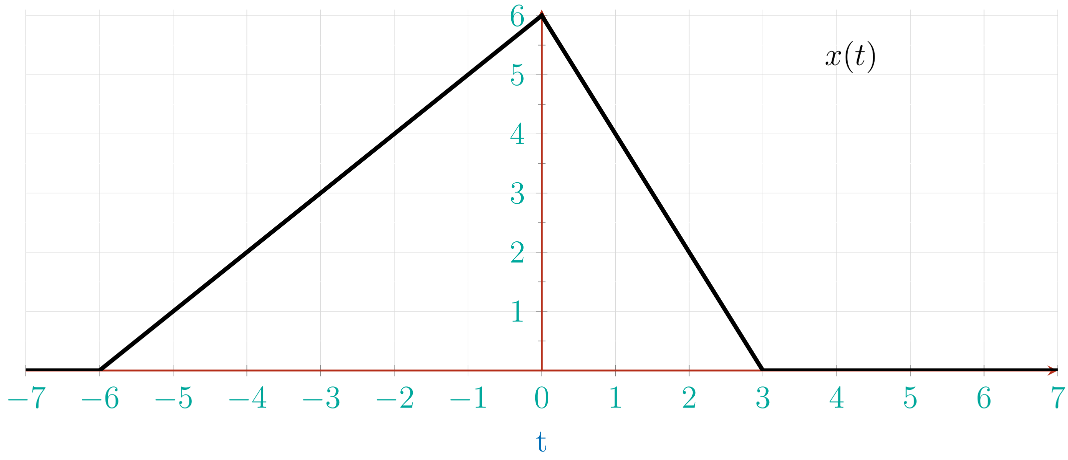
(b) `(20 pts)` Please sketch the even portion of $x(t)$.

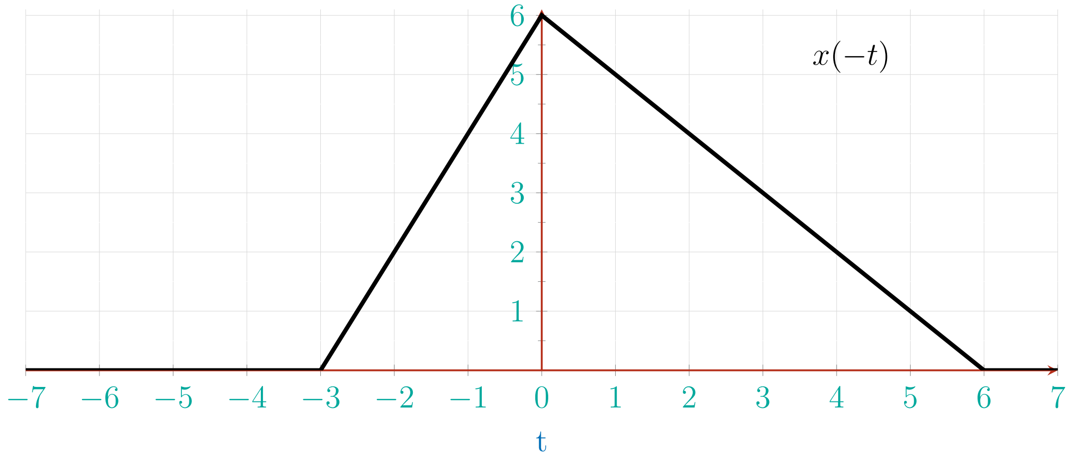
Solution 1b:

The even portion of a signal is found using

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

Let's put $x(t)$ and $x(-t)$ on top of each other.

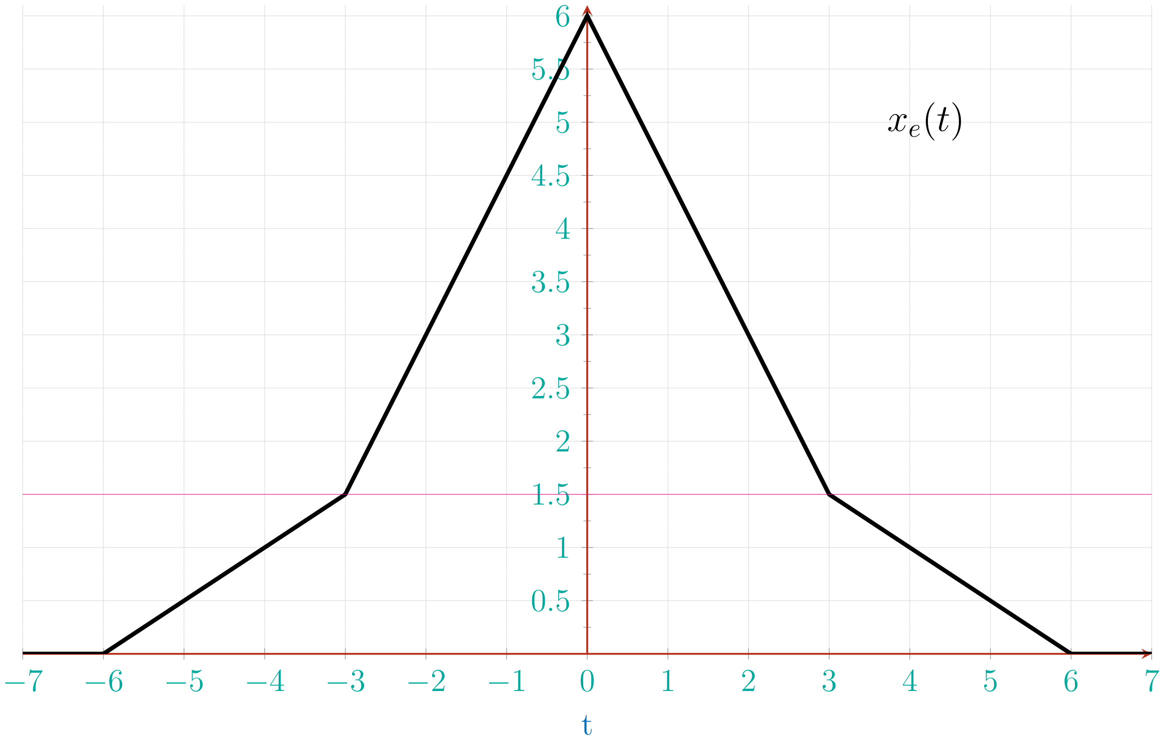




Now, we mark the points where either signal changes. These are $t = -6, -3, 0, 3, 6.$. Let's calculate the value of the even portion at these points.

$$
\begin{aligned}
x_e(-6) &= \frac{1}{2}[x(6) + x(-6)] = 0 \\
x_e(-3) &= \frac{1}{2}[x(3) + x(-3)] = \frac{3+0}{2} = 1.5 \\
x_e(0) &= x(0) = 6 \\
x_e(3) &= \frac{1}{2}[x(3) + x(-3)] = \frac{3+0}{2} = 1.5 \\
x_e(6) &= \frac{1}{2}[x(6) + x(-6)] = 0
\end{aligned}
$$

Using these points, we can sketch the even portion of the signal.



Q2: `(20 pts)` Consider the following DISCRETE TIME signal. Is $x[n]$ periodic? If so, calculate its fundamental period.

$$x[n] = \cos\left(\frac{\pi}{2}n\right) \cdot \cos\left(\frac{\pi}{4}n\right)$$

Solution 2:

Using trigonometric identities:

$$\begin{aligned}
x[n] &= \frac{1}{2}\left[\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}n\right) + \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}n\right)\right] \\
&= \frac{1}{2}\cos\left(\frac{\pi}{4}n\right) + \frac{1}{2}\cos\left(\frac{3\pi}{4}n\right) \\
&= \underbrace{\frac{1}{2}\cos\left(\frac{\pi}{4}n\right)}_{x_1[n]} + \underbrace{\frac{1}{2}\cos\left(\frac{3\pi}{4}n\right)}_{x_2[n]} \\
&= x_1[n] + x_2[n]
\end{aligned}$$

For $x_1[n]$:

$$\Omega_1 = \frac{\pi}{4} = 2\pi\,\frac{m_1}{N_1}$$

where m_1 and N_1 are integers. The smallest $(m_1,\, N_1)$ pair is $(1,\, 8)$ so, the period of $x_1[n]$ is $N_1 = 8$ cycles. Similiarly, for $x_2[n]$:

$$\Omega_2 = \frac{3\pi}{4} = 2\pi\,\frac{m_2}{N_2}$$

where m_2 and N_2 are integers. The smallest $(m_2,\, N_2)$ pair is $(3,\, 8)$ so, the period of $x_2[n]$ is $N_2 = 8$ cycles. The superposition of periodic sinusoidal discrete-time signals are also periodic and we can find the period by finding the *least common multiple* of their respective periods.

The period of $x[n]$ is therefore

$$N = \mathrm{LCM}(N_1, N_2) = \mathrm{LCM}(8, 8) = 8 \quad \blacksquare$$

Q3: `(40 pts)` The systems below show the input as $x(t)$ or $x[n]$ and the output as $y(t)$ or $y[n]$. For each system, determine whether it is **(i)** `(2 pts each)` memoryless, **(ii)** `(2 pts each)` causal, **(iii)** `(4 pts each)` stable (show your work), **(iv)** `(6 pts each)` linear (show your work), and **(v)** `(6 pts each)` time-invariant (show your work).

(a) $y[n] = 2\,x[1-n]\,(\,u[n] \;-\; u[n-4]\,)$

(b) $y(t) = \dfrac{x(t)}{x(t-1)}$

Solution 3a:

$$\boxed{\text{(a)}\quad y[n] = 2\,x[1-n]\,(\,u[n] \;-\; u[n-4]\,)}$$

(i) NOT-MEMORYLESS $\blacksquare$

(ii) NON-CAUSAL (For example: $y[0]$ depends on $x[1]$) $\blacksquare$

(iii) Assuming $|x[n]| \leq M_x < \infty$, for $\forall n \in \mathbb{N}$

$$
\begin{aligned}
|y[n]| &= |2x[1-n] \cdot (\, u[n] \;-\; u[n-4]\,)| \\
|y[n]| &\leq 2\,|x[1-n]| \;\cdot\; |(u[n] \;-\; u[n-4])| \\
|(u[n] \;-\; u[n-4])| &\leq 1 \qquad \forall n \in \mathbb{N} \\
|y[n]| &\leq 2M_x \\
|y[n]| &\leq M_y < \infty \qquad \text{for } \forall n \in \mathbb{N}
\end{aligned}
$$

Therefore, the system is BIBO-STABLE. ■

(iv) Homogenity:

$$
\begin{aligned}
\mathcal{H}\{\alpha\, x[n]\} &= 2\,(\alpha\, x[1-n])\,(\, u[n] \;-\; u[n-4]\,) \\
\alpha\, y[n] &= \alpha\,\{\underbrace{2\;\, x[1-n]\,(\, u[n] \;-\; u[n-4]\,)}_{y[n]}\}
\end{aligned}
$$

$$
\mathcal{H}\{\alpha\, x[n]\} = \alpha\, y[n]
$$

Homogenity is satisfied.

Superposition:

Given the signals $x_1[n]$ and $x_2[n]$ and:

$$
\begin{aligned}
\mathcal{H}\{x_1[n]\} &= y_1[n] \\
\mathcal{H}\{x_2[n]\} &= y_2[n]
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathcal{H}\{x_1[n] + x_2[n]\} &= 2(x_1[1-n] + x_2[1-n])\,(\, u[n] \;-\; u[n-4]\,)] \\
&= \underbrace{2x_1[n]\,(\, u[n] \;-\; u[n-4]\,)}_{y_1[n]} + \underbrace{2x_2[1-n]\,(\, u[n] \;-\; u[n-4]\,)}_{y_2[n]} \\
&= y_1[n] + y_2[n]
\end{aligned}
$$

Superposition is satisfied. Therefore, $\mathcal{H}$ is LINEAR. ■

(v)

Let's say $y_1[n] = y[n-n_0]$ and $y_2[n] = \mathcal{H}\{x[n-n_0]\}$. We'll check if they are equal.

$$
\begin{aligned}
y_2[n] &= 2x[1-n+n_0]\,(\,u[n] \;-\; u[n-4]\,) \\
y_1[n] &= 2x[1-n+n_0]\,(\,u[n-n_0] \;-\; u[n-n_0-4]\,) \\
y_1[n] &\neq y_2[n]
\end{aligned}
$$

Therefore $\mathcal{H}$ is NOT TIME INVARIANT. ■

Solution 3b:

$$\text{(b)}\quad y(t) = \frac{x(t)}{x(t-1)}$$

(i) NOT-MEMORYLESS ■

(ii) CAUSAL ■

(iii) Assuming $|x(t)| \leq M_x < \infty$, for $\forall t \in \mathbb{R}$, since $y(t) \to \infty$ when $x(t-1) = 0$, the system is **NOT** BIBO-STABLE. ■

(iv) Cheking for homogenity:

$$
\begin{aligned}
\mathcal{H}\{\alpha x(t)\} &= \frac{\alpha\ x(t)}{\alpha\ x(t-1)} = \frac{x(t)}{x(t-1)} \\
\alpha y(t) &= \alpha\ \frac{x(t)}{x(t-1)} \\
\alpha y(t) &\neq \mathcal{H}\{\alpha x(t)\}
\end{aligned}
$$

It does not satisfy the homogenity principle. The system $\mathcal{H}$ is NOT LINEAR. ■

(v)

$$
\begin{aligned}
y_1(t) &= \mathcal{H}\{x(t-t_0)\} = \frac{x(t-t_0)}{x(t-t_0-1)} \\
y_2(t) &= y(t-t_0) = \frac{x(t-t_0)}{x(t-t_0-1)} \\
y_1(t) &= y_2(t)
\end{aligned}
$$

The system $\mathcal{H}$ is TIME-INVARIANT. ■