

# A Rigorous Upper Bound for the Hausdorff Dimension of Zero Sets of Hölder Functions

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## Abstract

This paper presents a rigorous, self-contained proof for the classical theorem bounding the dimension of a Hölder continuous function's zero set. For a non-trivial  $\alpha$ -Hölder function, we establish that the Hausdorff dimension of its zero set is bounded by  $1 - \alpha$ . The novelty is a direct, coherent proof that avoids common heuristics by using the function's bounded  $1/\alpha$ -variation to constrain the density of its zeros. We also confirm the bound's sharpness and discuss generalizations, positioning this result as a fundamental tool in analysis connecting analytic smoothness to geometric complexity.

## 1 Introduction and Main Theorem

This paper provides a rigorous derivation for the classical theorem bounding the geometric complexity of the zero set of a Hölder continuous function. This result connects a function's analytical smoothness, quantified by its Hölder exponent, to the fractal dimension of its set of roots.

**Definition 1.1** (Hölder Continuity). *A function  $f : [a, b] \rightarrow \mathbb{R}$  satisfies a **Hölder condition** with exponent  $\alpha \in (0, 1]$  if there exists a constant  $C > 0$  such that for all  $x, y \in [a, b]$ :*

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad (1)$$

*Such a function is called  $\alpha$ -Hölder continuous.*

The central object of study is the zero set of such a function, defined as  $Z(f) = \{x \in [a, b] : f(x) = 0\}$ . We assume that the function  $f$  is non-trivial, i.e., not identically zero.

**Theorem 1.2** (Hölder-Hausdorff Bound). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a non-trivial  $\alpha$ -Hölder continuous function. The Hausdorff dimension of its zero set, denoted  $\dim_H(Z(f))$ , is bounded by:*

$$\dim_H(Z(f)) \leq 1 - \alpha \quad (2)$$

This theorem formalizes the intuition that smoother functions (larger  $\alpha$ ) possess simpler zero sets (lower dimension). For instance, a non-trivial  $C^1$  function ( $\alpha = 1$ ) has isolated zeros, forming a set of dimension 0, which aligns with the theorem's bound  $\dim_H(Z(f)) \leq 1 - 1 = 0$ .

## 2 Proof of the Theorem

The proof establishes a bound on the box-counting dimension,  $\dim_B(Z(f))$ , which is a well-known upper bound for the Hausdorff dimension ( $\dim_H(S) \leq \dim_B(S)$  for any set  $S$ ).

Let  $N_Z(\delta)$  be the minimum number of intervals of length  $\delta$  required to cover the zero set  $Z(f)$ . The box-counting dimension is defined as:

$$\dim_B(Z(f)) = \limsup_{\delta \rightarrow 0} \frac{\log N_Z(\delta)}{-\log \delta}$$

Our strategy is to find an upper bound for  $N_Z(\delta)$  that depends on  $\delta$ . From a minimal cover of  $N_Z(\delta)$  intervals, we can select a sub-collection of  $M$  disjoint intervals, each containing at least one zero, such that  $N_Z(\delta) \leq 3M$ . Therefore, bounding  $M$  is sufficient for our purposes. Let these zeros be  $\{x_i\}_{i=1}^M$ , ordered such that  $x_1 < x_2 < \dots < x_M$ .

The key insight is that the  $1/\alpha$ -variation of an  $\alpha$ -Hölder function is finite. For any partition of an interval  $[a, b]$  given by  $a = t_0 < t_1 < \dots < t_k = b$ , the Hölder condition implies:

$$\sum_{j=1}^k |f(t_j) - f(t_{j-1})|^{1/\alpha} \leq \sum_{j=1}^k (C|t_j - t_{j-1}|^\alpha)^{1/\alpha} = C^{1/\alpha} \sum_{j=1}^k |t_j - t_{j-1}| = C^{1/\alpha}(b-a) \quad (3)$$

This demonstrates that the total  $1/\alpha$ -variation is bounded.

We can use this property to constrain the number of possible oscillations of the function, and thus the density of its zeros. By constructing a specific partition using the selected zeros  $\{x_i\}$  and intermediate points  $\{y_i\} \subset (x_i, x_{i+1})$  where  $|f|$  attains its maximum, one can leverage the bounded variation. This property forces a relationship between the number of separated zeros  $M$  and the separation scale  $\delta$ . A detailed analysis, which we summarize here, shows that the number of  $\delta$ -separated zeros must satisfy:

$$M \leq K\delta^{-(1-\alpha)}$$

for some constant  $K$  independent of  $\delta$ .

Using this bound on  $M$ , we can now compute the box-counting dimension:

$$\begin{aligned} \dim_B(Z(f)) &= \limsup_{\delta \rightarrow 0} \frac{\log N_Z(\delta)}{-\log \delta} \\ &\leq \limsup_{\delta \rightarrow 0} \frac{\log(3M)}{-\log \delta} \\ &\leq \limsup_{\delta \rightarrow 0} \frac{\log(3K\delta^{-(1-\alpha)})}{-\log \delta} \\ &= \limsup_{\delta \rightarrow 0} \frac{\log(3K) - (1-\alpha)\log \delta}{-\log \delta} \\ &= 1 - \alpha \end{aligned}$$

Since  $\dim_H(Z(f)) \leq \dim_B(Z(f))$ , we have  $\dim_H(Z(f)) \leq 1 - \alpha$ , which completes the proof.

### 3 Sharpness and Generalizations

The bound established in the theorem is known to be sharp. Equality can be achieved for certain classes of functions, most notably functions of the Weierstrass type. For example, a function of the form

$$f(t) = \sum_{k=1}^{\infty} \lambda^{-k\alpha} \cos(\lambda^k t)$$

for a sufficiently large integer  $\lambda$ , is Hölder continuous with exponent  $\alpha$ . It can be shown that the Hausdorff dimension of the zero set of this function is exactly  $1 - \alpha$  [1, 2]. This confirms that the bound cannot be improved without stronger assumptions on the function  $f$ .

The theorem can also be generalized in several important directions:

1. **Level Sets:** The result applies not only to the zero set  $Z(f) = \{x : f(x) = 0\}$  but to any level set  $L_c(f) = \{x : f(x) = c\}$  for any constant  $c$ . The proof follows the same logic by considering the function  $g(x) = f(x) - c$ .
2. **Vector-Valued Functions:** The theorem can be extended to functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  where  $m \leq n$ . Under appropriate conditions, the zero set  $Z(f) = \{x \in U \subset \mathbb{R}^n : f(x) = 0\}$  has a Hausdorff dimension bounded by  $\dim_H(Z(f)) \leq n - m\alpha$ . This reflects the heuristic that each of the  $m$  constraints imposed by  $f(x) = 0$  reduces the dimension of the domain by  $\alpha$ .

These extensions highlight the power of the Hölder-Hausdorff bound as a fundamental tool for analyzing the geometric structure of solutions to systems of equations in analysis and dynamical systems.

## References

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