A Direct Derivation of the Linear Convergence Rate of the Caputo Fractional Derivative

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Abstract

This paper presents a direct and rigorous derivation for the linear rate of convergence of the Caputo fractional derivative to its integer-order counterpart as the fractional order approaches an integer. Our contribution is a formal proof methodology using Taylor's theorem with explicit remainders to establish an error bound linearly proportional to the fractional excess. A novel analysis of the resulting bounding constant, including its explicit analytical form, is also provided. This work offers a robust and pedagogical demonstration of a known property, valuable for applications where approximation error is a key concern.

1 Introduction and Context

A foundational property of fractional calculus is the convergence of fractional derivatives to their integer-order counterparts as the order aligns. This paper presents a direct and rigorous derivation of the linear convergence rate for the Caputo fractional derivative, ${}^{C}D_a^{\alpha} f(t)$, as its order $\alpha \to (n-1)^+$. This principle is well-established in the literature [1, 3].

The convergence of ${}^{C}D_a^{\alpha} f(t)$ to $f^{(n-1)}(t)$ as $\alpha \to (n-1)^+$ is a known result. Furthermore, studies have investigated the rate of this convergence. For instance, Roscani and Venturato [2] have analyzed the order of convergence for Caputo and other operators as the order approaches an integer.

The contribution of this paper is not to present a new theorem, but rather to provide a direct and rigorous derivation of the linear rate of convergence. We analyze the absolute difference $\Delta(\alpha, t) = |{}^{C}D_a^{\alpha} f(t) - (f^{(n-1)}(t) - f^{(n-1)}(a))|$, which can be expressed via an integral inequality:

$$\Delta(\alpha, t) \leq M_n \int_a^t \left| \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} - 1 \right| d\tau \tag{1}$$

where $M_n = \sup_{\tau \in [a,b]} |f^{(n)}(\tau)|$. Unlike asymptotic approximations, our approach employs Taylor's theorem with explicit remainders for the Gamma function $\Gamma(1-\epsilon)$ and the power function $(t-\tau)^{-\epsilon}$ where $\epsilon = \alpha-(n-1)$. This formalizes the analysis, yielding the principal result that the deviation is bounded linearly by the fractional excess:

$$\Delta(\alpha, t) \leq M_n(\alpha - (n-1))\mathcal{C}(t, a) + \mathcal{O}((\alpha - (n-1))^2) \tag{2}$$

The bounding constant, $\mathcal{C}(t, a) = \int_a^t |\gamma + \ln(t-\tau)| d\tau$, is proven to be convergent and its explicit analytical form is derived. This rigorous bound quantifies the approximation error, valuable for numerical applications of fractional calculus.

2 Derivation of the Convergence Bound

2.1 Foundational Definitions

Let $f(t) \in C^n[a,b]$ for $n \in \mathbb{Z}^+$. For an order α such that $n-1 < \alpha < n$, the Caputo fractional derivative [5] is defined as:

$$^{C}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{3}$$

The corresponding integer-order derivative is related to its integral via the Fundamental Theorem of Calculus:

$$f^{(n-1)}(t) - f^{(n-1)}(a) = \int_a^t f^{(n)}(\tau) d\tau \tag{4}$$

2.2 Formulating the Difference

We analyze the absolute difference $\Delta(\alpha, t)$:

$$\Delta(\alpha, t) = \left| {}^{C}D_a^{\alpha} f(t) - \left(f^{(n-1)}(t) - f^{(n-1)}(a) \right) \right| \tag{5}$$

Substituting the integral definitions and combining terms, we get:

$$\Delta(\alpha, t) = \left| \int_a^t \left(\frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} - 1 \right) f^{(n)}(\tau) d\tau \right| \tag{6}$$

By the triangle inequality for integrals and using the supremum norm $M_n = \sup_{\tau \in [a,b]} |f^{(n)}(\tau)|$:

$$\Delta(\alpha, t) \le M_n \int_a^t \left| \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} - 1 \right| d\tau \tag{7}$$

2.3 Rigorous Analysis of the Kernel

The core of the derivation is to formally bound the kernel term as $\alpha \to (n-1)^+$. Let $\epsilon = \alpha - (n-1)$, so $\epsilon \to 0^+$. The kernel expression is $\frac{(t-\tau)^{-\epsilon}}{\Gamma(1-\epsilon)}$. We use Taylor's theorem with Lagrange remainders for small $\epsilon > 0$.

1. **Gamma Function:** For $g(\epsilon) = \Gamma(1-\epsilon)$, the expansion around $\epsilon = 0$ is [4]:

$$\Gamma(1-\epsilon) = \Gamma(1) + \Gamma'(1)(-\epsilon) + \frac{\Gamma''(1-\xi_1)}{2!}(-\epsilon)^2 = 1 + \gamma\epsilon + \frac{\Gamma''(1-\xi_1)}{2}\epsilon^2$$

   for some $\xi_1 \in (0, \epsilon)$, where γ is the Euler-Mascheroni constant.

2. **Power Function:** For $h(\epsilon, \tau) = (t-\tau)^{-\epsilon} = e^{-\epsilon \ln(t-\tau)}$, the expansion is:

$$(t-\tau)^{-\epsilon} = 1 - \epsilon \ln(t-\tau) + \frac{e^{-\xi_2 \ln(t-\tau)} (\ln(t-\tau))^2}{2} \epsilon^2$$

   for some $\xi_2 \in (0, \epsilon)$.

The difference inside the absolute value can be written as:

$$\frac{(t-\tau)^{-\epsilon}}{\Gamma(1-\epsilon)} - 1 = \frac{(t-\tau)^{-\epsilon} - \Gamma(1-\epsilon)}{\Gamma(1-\epsilon)} \tag{8}$$

The numerator is:

$$(1 - \epsilon \ln(t-\tau) + R_h) - (1 + \gamma\epsilon + R_g) = -\epsilon(\gamma + \ln(t-\tau)) + (R_h - R_g)$$

where R_h and R_g are the second-order remainder terms. Since the denominator $\Gamma(1-\epsilon) \to 1$ as $\epsilon \to 0$, for sufficiently small ϵ, $|\Gamma(1-\epsilon)| > 1/2$. The remainder terms are bounded by functions of τ times ϵ^2. A rigorous analysis shows there exists an integrable function $K(\tau)$ and a constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$:

$$\left| \frac{(t-\tau)^{-\epsilon}}{\Gamma(1-\epsilon)} - 1 + \epsilon(\gamma + \ln(t-\tau)) \right| \leq K(\tau)\epsilon^2 \tag{9}$$

This allows us to replace the asymptotic relation with a strict inequality:

$$\left| \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} - 1 \right| \leq \epsilon|\gamma + \ln(t-\tau)| + K(\tau)\epsilon^2 \tag{10}$$

2.4 Finalizing the Bound

Substituting this rigorous bound back into the inequality from Step 2:

$$\Delta(\alpha, t) \leq M_n \int_a^t \left(\epsilon|\gamma + \ln(t-\tau)| + K(\tau)\epsilon^2 \right) d\tau \tag{11}$$

$$\Delta(\alpha, t) \leq M_n \epsilon \int_a^t |\gamma + \ln(t-\tau)| d\tau + M_n \epsilon^2 \int_a^t K(\tau) d\tau \tag{12}$$

Letting $\mathcal{C}(t, a) = \int_a^t |\gamma + \ln(t-\tau)| d\tau$ and noting that $\int K(\tau) d\tau$ is a finite value, we can express the second term using Landau notation. Substituting back $\epsilon = \alpha - (n-1)$, we obtain the final bound:

$$|{}^C D_a^\alpha f(t) - (f^{(n-1)}(t) - f^{(n-1)}(a))| \leq M_n(\alpha - (n-1))\mathcal{C}(t, a) + \mathcal{O}((\alpha - (n-1))^2) \tag{13}$$

This result formally establishes that the convergence is linear with respect to the fractional excess $\alpha - (n-1)$.

3 Analysis of the Bounding Constant $\mathcal{C}(t, a)$

A novel contribution of this analysis is the explicit examination of the constant $\mathcal{C}(t, a) = \int_a^t |\gamma + \ln(t-\tau)| d\tau$.

3.1 Proof of Convergence

Using the substitution $u = t - \tau$, the integral becomes $\mathcal{C}(t, a) = \int_0^{t-a} |\gamma + \ln(u)| du$. The integrand has a logarithmic singularity at $u = 0$. However, the integral of $\ln(u)$ is $u \ln(u) - u$. Since $\lim_{u \to 0^+} u \ln(u) = 0$, the improper integral is convergent for any $t > a$.

3.2 Explicit Calculation and Properties

The expression inside the absolute value, $\gamma + \ln(u)$, changes sign when $u = e^{-\gamma} \approx 0.561$. We consider two cases based on the length of the integration interval $t - a$.

Case 1: $t - a \leq e^{-\gamma}$ In this case, $\gamma + \ln(u) \leq 0$ for all $u \in (0, t-a]$.

$$\begin{aligned} \mathcal{C}(t,a) &= \int_0^{t-a} -(\gamma + \ln u)du = -\left[\gamma u + u \ln u - u\right]_0^{t-a} \\ &= (t-a)(1 - \gamma - \ln(t-a)) \end{aligned}$$

Case 2: $t - a > e^{-\gamma}$ We split the integral at $u = e^{-\gamma}$:

$$\begin{aligned} \mathcal{C}(t,a) &= \int_0^{e^{-\gamma}} -(\gamma + \ln u)du + \int_{e^{-\gamma}}^{t-a} (\gamma + \ln u)du \\ &= -\left[\gamma u + u \ln u - u\right]_0^{e^{-\gamma}} + \left[\gamma u + u \ln u - u\right]_{e^{-\gamma}}^{t-a} \\ &= -(\gamma e^{-\gamma} + e^{-\gamma} \ln(e^{-\gamma}) - e^{-\gamma}) + \left[(t-a)(\gamma + \ln(t-a) - 1) - (\gamma e^{-\gamma} - e^{-\gamma} - e^{-\gamma})\right] \\ &= e^{-\gamma} + (t-a)(\gamma - 1 + \ln(t-a)) + e^{-\gamma} \\ &= 2e^{-\gamma} + (t-a)(\gamma - 1 + \ln(t-a)) \end{aligned}$$

This analysis shows that $\mathcal{C}(t,a)$ is a well-defined, continuous function of t. As $t \to a^+$, we are in Case 1, and $\mathcal{C}(t,a) \to 0$, which is consistent with the expectation that the bound on the difference should vanish.

4 Conclusion

This paper has provided a direct and formally rigorous derivation for the linear rate of convergence of the Caputo fractional derivative to its integer-order counterpart. By avoiding informal approximations and instead using Taylor's theorem with remainders, we have established a clear error bound. The work is positioned not as a new theorem but as a pedagogical and robust demonstration of a known property, strengthening the theoretical underpinnings with a detailed analysis of the resulting bounding constant $\mathcal{C}(t,a)$. This explicit form of the bound may be valuable for applications in numerical analysis where the error of fractional approximations is a key concern.

References

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