General Equilibrium with Taxes: A Computational Procedure and an Existence Proof [1, 2]

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1. INTRODUCTION

The purpose of this paper is to describe a computational procedure for the determination of a competitive equilibrium for an economy with producer and consumer commodity [3] taxes. This procedure enables a proof of the existence of such an equilibrium to be obtained. While no properties of efficiency are claimed for such an equilibrium, the procedure is of great potential use in the consideration of the efficiency losses and distributional impacts from alternative tax schemes. Application [4] of such a technique may be found in such areas as the tax reform programmes being discussed in the USA, Canada, the UK and other countries.[5]

We consider a general Walrasian model with an arbitrary set of *ad valorem* commodity taxes. The method of computing an equilibrium is based on Scarf's Algorithm [7, 8]. The tax rates need not be uniform across production sectors or individuals. The revenue generated from the tax system is dispersed among the individual consumers, each of whom is assigned an arbitrary share of the total, or retained by the government for the purchase of goods and services. The only restriction on revenue distribution schemes is that the shares (those of individuals and the government) sum to unity and that each recipient's allocation be a continuous function of total tax revenue.

The major theorem of this paper is a substantial extension of the existence proof of a competitive equilibrium in the absence of taxes (see, for example, Arrow and Hahn [1], Scarf [7], McKenzie [6], or Debreu [2]) and is of particular importance if the general equilibrium framework is to be useful for the evaluation of economic policy regarding taxes and tariffs. Without such an existence proof, it is difficult to contemplate the computation of equilibria under various tax regimes. For convenience most of the exposition will be concerned only with an economy with differential producer taxes; the case of differential producer and consumer taxes is discussed in Section 4.

The economy considered here is characterized by (1) a set of market demand functions, (2) a description of the technological production possibilities through a listing of activity

1 *First version received June* 1972; *final version received December* 1972 (*Eds.*).
2 We are indebted to Herbert E. Scarf for valuable assistance, guidance, and comments. Portions of this work form part of our respective dissertations presented for the degree of Doctor of Philosophy in Yale University. This work was completed while the first author was supported by a Federal Deposit Insurance Corporation fellowship.

3 The term commodity refers here to both outputs and inputs as is the convention in general equilibrium analysis. This is in contrast with some of the taxation literature in which it refers only to outputs.

4 In addition, this procedure could be applied to the problem examined by the authors in a recent paper [9] where the efficiency loss from differential taxation of income from capital in the US economy was considered. While the method used in that paper was perfectly adequate for that particular problem, this procedure is more general in that it allows for taxes on all commodities which can be directly imposed.

5 Each of the authors is pursuing work on empirical evaluation of such tax proposals utilizing the methods outlined in this paper.

vectors, and (3) the economy's initial endowments of the n commodities, $(w_1, \ldots, w_n)$, $w_i \geqq 0$ for $i = 1, \ldots, n$. The market demand functions are simply the sum of individual demands, each of which may be derived from utility maximization subject to a budget constraint. Each person's income is given by the value of his initial holdings plus his portion of the tax proceeds. The government, which is treated as a consumer, may be given a claim to a fraction of the total revenue. The market demands are functions of the n commodity prices, $\pi_1, \ldots, \pi_n$, and the total tax revenue R and will be denoted by

$$\begin{matrix} \xi_1(\pi_1, \ldots, \pi_n, R) \\ \vdots \\ \xi_n(\pi_1, \ldots, \pi_n, R). \end{matrix} \qquad \ldots(1.1)$$

They are assumed to be non-negative, continuous, homogeneous of degree zero in all prices and revenue, and to satisfy Walras' Law as long as at least one price is positive. The homogeneity assumption allows us to consider only augmented price vectors

$$P = (\pi_1, \ldots, \pi_n, R)'$$

which are non-negative (i.e. $\pi_i \geqq 0 \; \forall_i$, $R \geqq 0$) and whose components sum to unity. These augmented vectors may be contrasted with price vectors conventionally used in economics whose components only refer to commodity prices. For an economy with only producer taxes the π_i's are the prices relevant to the consumer and may be thought of as sellers' prices for inputs (i.e. net of producer input tax prices) and buyers' prices for outputs (i.e. gross of producer output taxes). Walras' Law in this framework states that

$$\sum_{i=1}^{n} \pi_i \xi_i(\pi_1, \ldots, \pi_n, R) \equiv \sum_{i=1}^{n} \pi_i w_i + R \qquad \ldots(1.2)$$

for all π, R. There is some difficulty in the interpretation of Walras' Law as all commodity prices approach zero and revenue approaches unity. A neighbourhood around this point will be excluded later.

Production is described by an activity analysis matrix

$$A = \begin{bmatrix} -1 & 0 & \ldots & 0 & a_{1,n+1} & \ldots & a_{1,m} \\ 0 & -1 & \ldots & 0 & a_{2,n+1} & \ldots & a_{2,m} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -1 & a_{n,n+1} & \ldots & a_{n,m} \end{bmatrix} \qquad \ldots(1.3)$$

in which each column represents a feasible activity. Outputs are represented by positive and inputs by negative coefficients, and each activity may be operated at any non-negative level. The first n columns indicate the feasibility of free disposal of each commodity. It is assumed that A is such that the set of non-negative vectors x which satisfy $Ax+w \geqq 0$, where w is the vector of initial holdings, is bounded. This can be interpreted as implying that the production possibilities frontier is finite in all n dimensions.[1]

For each production activity j (i.e. for each column of A) there is an associated producer tax vector $T^j = (t_{1j}, \ldots, t_{nj})$, whose components are the *ad valorem* tax rates applying to inputs and outputs when the jth activity is utilized. Thus the tax system is characterized by m of these tax vectors T^j, $j = 1, \ldots, m$. We have adopted the convention that t_{ij}, the tax rate on the ith commodity for the jth production activity, has the same sign as a_{ij} for all i and j, and hence $t_{ij}a_{ij}$ is non-negative. The revenue produced by operating the jth activity at the unit level with prices π is thus given by

$$r_j = \sum_{i=1}^{n} \pi_i t_{ij} a_{ij}. \qquad \ldots(1.4)$$

[1] The technological production possibilities set given by $\{Ax \mid x \geqq 0, \; Ax+w \geqq 0\}$ is convex and unaffected by the presence of taxes. It is this set which is relevant for the analysis of this paper, and not the distorted production possibility set as used by H. G. Johnson in [3], which indeed may be non-con ex under some circumstances with factor input taxes.

In this notation a competitive equilibrium is defined by a vector $P^* = (\pi^*, R^*)$ and a vector of non-negative activity levels x^* such that

(1) Supply, $w_i + \sum_{j=1}^{m} a_{ij}x_j^*$, is equal to market demand for each commodity i.

$$\xi_i(\pi^*, R^*) - \sum_{j=1}^{m} a_{ij}x_j^* = w_i \quad \text{for } i = 1, \ldots, n \qquad \ldots(1.5)$$

and

(2) Profit is maximized at the prices π^*, or

$$\sum_{i=1}^{n} \pi_i^* a_{ij} - \sum_{i=1}^{n} \pi_i^* t_{ij} a_{ij} \leqq 0 \quad \text{for all } j = 1, \ldots, m \qquad \ldots(1.6)$$

with equality if $x_j^* > 0$.

Multiplying each of the n equations (1.5) by π_i^* and summing gives

$$\sum_{i=1}^{n} \pi_i^* \xi_i(\pi^*, R^*) - \sum_{i=1}^{n} \sum_{j=1}^{m} \pi_i^* a_{ij} x_j^* = \sum_{i=1}^{n} \pi_i^* w_i. \qquad \ldots(1.7)$$

By Walras' Law (equation (1.2)), this implies that

$$R^* - \sum_{i=1}^{n} \sum_{j=1}^{m} \pi_i^* a_{ij} x_j^* = 0. \qquad \ldots(1.8)$$

If for each j the corresponding relation (1.6) is multiplied by x_j^* and the resulting m equations are summed, one gets

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \pi_i^* a_{ij} x_j^* - \sum_{j=1}^{m} \sum_{i=1}^{n} \pi_i^* t_{ij} a_{ij} x_j^* = 0. \qquad \ldots(1.9)$$

Adding equation (1.9) to (1.8) gives

$$R^* = \sum_{j=1}^{m} \sum_{i=1}^{n} \pi_i^* t_{ij} a_{ij} x_j^*. \qquad \ldots(1.10)$$

Thus at a competitive equilibrium defined by (1.5) and (1.6) the amount of revenue distributed among consumers is the same as the amount generated on the production side of the economy.

2. FUNDAMENTAL CONCEPTS OF SCARF'S ALGORITHM

The arguments used in this paper hinge on the fundamental theorem which forms the basis for Scarf's Algorithm (Scarf [7, 8]). For this reason, an outline of the concepts of Scarf's computational algorithm, modified for the inclusion of taxes, is appropriate.

As stated previously, due to the homogeneity of degree zero of the demand functions, we may consider only augmented price vectors P which are on the unit simplex, i.e.

$$R + \sum_{i=1}^{n} \pi_i = 1; \; R \geqq 0, \pi_i \geqq 0 \quad \text{all } i = 1, \ldots, n. \qquad \ldots(2.1)$$

The algorithm is essentially a search procedure on this unit simplex for an approximate equilibrium $\hat{P} = (\hat{\pi}, \hat{R})$. The objective is to find a $\hat{P}$ and an associated $\hat{x}$ which approximately meet the two conditions (1.5) and (1.6). For this purpose a fine grid of vectors, $P^{n+2}, \ldots, P^k$, is created on the unit simplex, where k is a very large number.

The vectors $P^1, \ldots, P^n, P^{n+1}$, which represent the sides of the simplex $s_1, \ldots, s_n, s_R$, are also created. A two commodity example is shown in Figure 1.[1]

It is assumed that any two vectors in the list $P^1, \ldots, P^k$ differ in all coordinates. This non-degeneracy assumption allows us to unambiguously determine the vector with the

[1] The side s_i is defined as all vectors P on the simplex with $P_i = 0$.

smallest ith component for any subset of the vectors $P^1, \ldots, P^k$. The algorithm works with a subset of $n+1$ of the augmented price vectors in the list $P^1, \ldots, P^k$ which are referred to as a primitive set and which are " close to each other " in a particular sense. Let c_i be the smallest ith component among an arbitrary collection of $(n+1)$ vectors, $P^{j_1}, \ldots, P^{j_{n+1}}$, being considered. If there is no vector P^j among the entire list $P^1, \ldots, P^k$ for which

$$P_i^j \geqq c_i \quad \text{for all } i = 1, \ldots, n+1 \qquad \ldots(2.2)$$

then the vectors $P^{j_1}, \ldots, P^{j_{n+1}}$ form a primitive set. As defined above, a primitive set may be thought of as a square matrix of size $n+1$, each column of which refers to an augmented price vector $P = (\pi, R)$ and whose columns have the property (2.2). For the two-commodity example, such as that depicted in Figure 1, a primitive set may be thought of

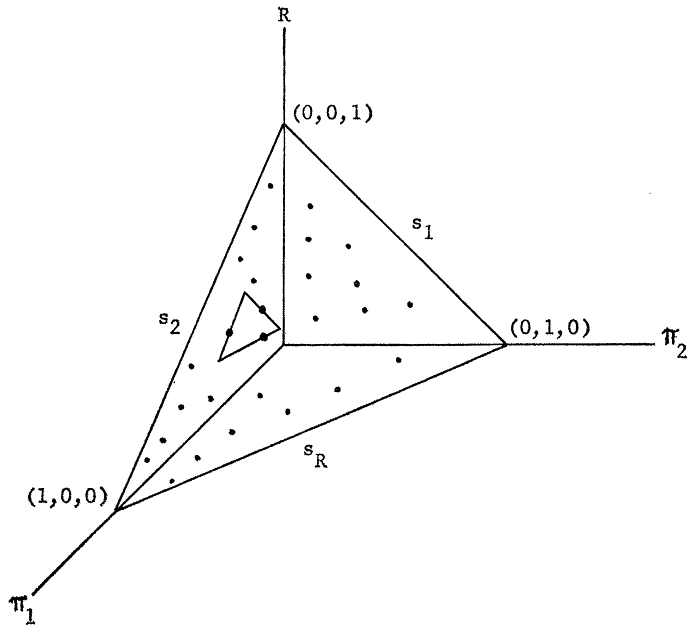

FIGURE 1

as a collection of three vectors in the list $P^1, \ldots, P^k$ with the property that a triangle with the same orientation as the full simplex may be drawn through these points. In addition, to form a primitive set one and only one vector must lie on each side of the triangle, and no other vector in the list $P^1, \ldots, P^k$ may be interior to the triangle. If it is desired to remove a column vector from a primitive set, there is a replacement rule which results in a unique replacement such that the new collection of vectors remains a primitive set.

Each vector $P^h = (\pi^h, R^h)$ in the list $P^1, \ldots, P^k$ is associated with a specific $n+1$ dimensional column vector b^h by the following rules:

(1) If any of the elements $P_1^h, \ldots, P_{n+1}^h$ are zero, the associated vector contains a 1 in place of the first zero entry and 0's elsewhere.

(2) The after-tax profitability of each activity in the technology matrix A is evaluated at the prices π^h. Let

$$(a_{1l}, \ldots, a_{nl})'$$

be the activity which yields the largest net profit if each activity is operated at unit intensity. Thus l is equal to j, the solution for

$$\max_j \left(\sum_{i=1}^{n} \pi_i^h a_{ij}(1-t_{ij}) \right)$$

(i) If the maximum per unit profit at prices π^h is positive, then b^h is defined to be

$$(-a_{1l}, \ldots, -a_{nl}, r_l)'$$

where r_l is given by expression (1.4) and is the per unit revenue generated by activity l with prices π^h.

(ii) If the maximum per unit profit is nonpositive at prices π^h then the column b^h corresponding to $P^h = (\pi^h, R^h)$ is an augmented column of market demands

$$(\xi_1(\pi^h, R^h)+\theta, \ldots, \xi_n(\pi^h, R^h)+\theta, -R^h+\Delta)'$$

where Δ, θ are finite positive constants.[1]

These rules result in a matrix B whose columns correspond to the augmented price vectors $P^1, \ldots, P^k$.

$$\begin{array}{cc} & \begin{array}{ccccccc} P^1 & P^2 & & P^{n+1} & P^{n+2} & & P^k \end{array} \\ B = & \begin{bmatrix} 1 & 0 & \ldots & 0 & b_{1,n+2} & \cdots & b_{1,k} \\ 0 & 1 & \ldots & 0 & b_{2,n+2} & \cdots & b_{2,k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & b_{n+1,n+2} & \cdots & b_{n+1,k} \end{bmatrix}. \end{array} \qquad \ldots(2.3)$$

Consider the vector $\tilde{w}$, which may be referred to as the augmented vector of the economy's endowment,

$$\tilde{w} = \begin{Bmatrix} w_1+\theta \\ w_2+\theta \\ \vdots \\ w_n+\theta \\ \Delta \end{Bmatrix}. \qquad \ldots(2.4)$$

Given this and the previous constructions, the theorem which is the basis of Scarf's Algorithm may be stated as the following:

Theorem. *There exists a primitive set $P^{j_1}, \ldots, P^{j_{n+1}}$ such that*

$$By = \tilde{w}$$

has a non-negative solution where $y_j = 0$ for $j \neq j_1, \ldots, j_{n+1}$.

For those familiar with the notion of a feasible basis from linear programming, this theorem can be restated even more briefly.

Theorem. *There exists a primitive set $P^{j_1}, \ldots, P^{j_{n+1}}$ such that the columns $j_1, \ldots, j_{n+1}$ form a feasible basis for $By = \tilde{w}$.*

The proof of this theorem, which is slightly modified by the inclusion of taxes, follows from Scarf [7, 8]. The algorithm starts in a " corner " of the simplex taking as the initial primitive set of vectors $(P^2, P^3, \ldots, P^{n+1}, P^{j*})$, where P^{j*} is that vector among the list $P^{n+2}, \ldots, P^k$ which has the largest first coordinate. The initial feasible basis for the equations $By = \tilde{w}$ is taken to be the first $n+1$ columns of the matrix B. There exists a perfect correspondence between the augmented price vectors of the initial primitive set and this feasible basis with two exceptions. The column corresponding to P^{j*} is not in the feasible basis, but instead the feasible basis contains the unmatched column $(1, 0, \ldots, 0)'$. This is a typical situation in that there is one unmatched column in both the primitive set and in the feasible basis. In general there are two steps which may be taken to attempt to bring about a complete correspondence. Either the unmatched primitive set vector

[1] For computational examples with specific grids of price vectors on the simplex explicit bounds on the magnitude of Δ and θ are needed. This matter is discussed in the first author's Yale Ph.D. thesis.

may be removed by the aforementioned replacement operation or the column in B which corresponds to this unmatched primitive set vector can be introduced into the feasible basis by a linear programming pivot operation. If the first of these options is followed and the replacement vector happens to correspond to the unmatched feasible basis vector (always $(1, 0, \ldots, 0)'$), a perfect correspondence has been achieved. If not, there are once again two unmatched vectors. If the second procedure is followed and the vector thrown out of the feasible basis by the pivot operation is the previously unmatched one, again a perfect correspondence will have been achieved. If not, one of the vectors corresponding to a primitive set vector will have been removed, and there will still be one unmatched vector in each of the two constructions.

The only exception to the above is the case of the initial primitive set, $(P^2, \ldots, P^{n+1}, P^{j*})$. The only replacement operation which is not permitted occurs when the primitive set is composed of n sides of the simplex plus one interior vector. The interior vector in this case may not be replaced. Therefore, in the initial position only one operation may be taken—namely bringing into the feasible basis the column b^{j*} which corresponds to P^{j*}.

At any intermediate stage one of the two possible steps has just been taken to arrive at the present state. The algorithm proceeds by taking the other possible continuation. That is, primitive set replacement operations and feasible basis pivot steps are performed alternately, until a perfect correspondence is achieved. The algorithm cannot cycle, since if the first state that it returns to is not the initial position, there must be three ways to exit from this position rather than two. If the first position which is repeated is the initial position, then there would be two ways to exit from this position rather than one. This argument that there cannot be cycling is due to Lemke and Howson [5]. Given that cycling has been ruled out and that there is a finite number of possible primitive sets and feasible bases, the algorithm must terminate. This can only happen when the vectors corresponding to the primitive set columns are a feasible basis for $By = \tilde{w}$.

3. EXISTENCE OF A COMPETITIVE EQUILIBRIUM

The only remaining argument necessary is to show how the primitive set whose corresponding columns form a feasible basis for $By = \tilde{w}$ defines an approximate competitive equilibrium, the degree of the approximation depending only on the grid size. We are concerned with non-negative solutions to $By = \tilde{w}$ where all $y_j = 0$ except for $j = j_1, \ldots, j_{n+1}$ corresponding to the final primitive set vectors. Writing out these equations explicitly, they become the $n+1$ equations

$$-\sum_j a_{il}x_j + \sum_j (\xi_i(\pi^j, R^j)+\theta)z_j = w_i+\theta, \quad i = 1, \ldots, n \qquad \ldots(3.1)$$

$$\sum_j r_l x_j + \sum_j (-R^j+\Delta)z_j + y = \Delta, \qquad \ldots(3.2)$$

where those positive y_j corresponding to augmented demand columns have been renamed z_j's and those corresponding to the negative of augmented activity columns have been replaced by x_j's. The y term in equation (3.2) is the weight associated with the $(n+1)$st column of B, namely $(0, 0, \ldots, 0, 1)'$. This weight can only be positive if a vector on the side s_R of the simplex is a member of the final primitive set. Recall that the subscript l in the first set of terms in equations (3.1) and (3.2) depends on j and refers to that activity which maximizes profit at prices π^j, should that profit net of tax be positive.

From the rules of construction of the matrix B, the following conditions must be satisfied.

(1) If $z_j > 0$, then

$$\sum_{i=1}^{n} \pi_i^j a_{il} - \sum_{i=1}^{n} \pi_i^j t_{il} a_{il} \leqq 0 \quad \text{for all } l = 1, \ldots, m. \qquad \ldots(3.3)$$

(2) If any l, other than a disposal activity (i.e. other than $l = 1, \ldots, n$), has a positive weight x_j, then

$$\sum_{i=1}^{n} \pi_i^j a_{il} - \sum_{i=1}^{n} \pi_i^j t_{il} a_{il} > 0. \qquad \ldots(3.4)$$

Imagine sequentially increasing the grid size. As the grid size approaches infinity and the augmented price vectors become everywhere dense on the simplex, the vectors of the final primitive set all approach the vector $\hat{P}$. Initially we will consider only those vectors which do not approach $\hat{R} = 1$ (for which $\sum_{i=1}^{n} \hat{\pi}_i = 0$), a situation which will be excluded later as a possible solution. In this case the demand functions converge to $\xi(\hat{P})$ and the weights $\{x_j\}$, $\{z_j\}$, and y approach $\{\hat{x}_j\}$, $\{\hat{z}_j\}$, and $\hat{y}$. In the limit equations (3.1) and (3.2) become

$$-\sum_j a_{il}\hat{x}_j + (\sum_j \hat{z}_j)(\xi_i(\hat{\pi}, \hat{R}) + \theta) = w_i + \theta, \quad i = 1, \ldots, n \qquad \ldots(3.5)$$

$$\sum_j \hat{r}_l \hat{x}_j + (\sum_j \hat{z}_j)(-\hat{R} + \Delta) + \hat{y} = \Delta \qquad \ldots(3.6)$$

where $\hat{r}_l = \sum_{i=1}^{n} \hat{\pi}_i t_{il} q_{il}$.

Moreover, conditions (3.3) and (3.4) become the following.

(1) If $\hat{z}_j > 0$, then

$$\sum_{i=1}^{n} \hat{\pi}_i a_{il} - \sum_{i=1}^{n} \hat{\pi}_i t_{il} a_{il} \leqq 0 \quad \text{for all } l = 1, \ldots, m. \qquad \ldots(3.7)$$

So, if at least one $\hat{z}_j$ is positive, no activity makes a positive profit.

(2) If any activity l, including disposal activities, has a positive weight $\hat{x}_j$, then

$$\sum_{i=1}^{n} \hat{\pi}_i a_{il} - \sum_{i=1}^{n} \hat{\pi}_i t_{il} a_{il} \geqq 0. \qquad \ldots(3.8)$$

Note that the disposal activities are included in (2) since commodity i can only be disposed of if $\hat{\pi}_i = 0$. If at least one $\hat{z}_j$ is positive, these conditions together imply that those activities which are operated make a zero profit. That is, the second of the equilibrium conditions (1.6) is satisfied.

With this dense grid covering the simplex, first consider the case where the limiting value of R is on the side of the simplex with $\hat{R} = 0$. For this situation $\hat{y}$ in equations (3.6) is greater than or equal to zero. Noting that $\hat{R} = 0$, equation (3.6) states,

$$(\sum_j \hat{z}_j)\Delta = \Delta - \hat{y} - \sum_j \hat{r}_l \hat{x}_j \qquad \ldots(3.9)$$

or

$$(\sum_j \hat{z}_j) = \frac{\Delta - \hat{y} - \sum_j \hat{r}_l \hat{x}_j}{\Delta}. \qquad \ldots(3.10)$$

Since $\sum_j \hat{r}_l \hat{x}_j \geqq 0$ and $\hat{y} \geqq 0$, (3.10) implies

$$\sum_j \hat{z}_j \leqq 1. \qquad \ldots(3.11)$$

Multiplying each of the n equations (3.5) by $\hat{\pi}_i$ and summing gives

$$-\sum_{i=1}^{n} \sum_j \hat{\pi}_i a_{ij} \hat{x}_j + (\sum_j \hat{z}_j) \sum_{i=1}^{n} \hat{\pi}_i(\xi_i(\hat{\pi}, \hat{R}) + \theta) = \sum_{i=1}^{n} \hat{\pi}_i w_i + \theta \sum_{i=1}^{n} \hat{\pi}_i. \qquad \ldots(3.12)$$

Solving this for $\sum_j \hat{z}_j$ results in

$$\sum_j \hat{z}_j = \frac{\sum_{i=1}^{n} \hat{\pi}_i w_i + \theta \sum_{i=1}^{n} \hat{\pi}_i + \sum_{i=1}^{n} \sum_j \hat{\pi}_i a_{ij} \hat{x}_j}{\sum_{i=1}^{n} \hat{\pi}_j \xi_i(\hat{\pi}, \hat{R}) + \theta \sum_{i=1}^{n} \hat{\pi}_i}. \quad \ldots(3.13)$$

Walras' Law for this case with $\hat{R} = 0$ states that

$$\sum_{i=1}^{n} \hat{\pi}_i \xi_i(\hat{\pi}, \hat{R}) = \sum_{i=1}^{n} \hat{\pi}_i w_i$$

and hence (3.13) may be written as

$$\sum_j \hat{z}_j = \frac{\sum_{i=1}^{n} \hat{\pi}_i w_i + \theta \sum_{i=1}^{n} \hat{\pi}_i + \sum_{i=1}^{n} \sum_j \hat{\pi}_i a_{ij} \hat{x}_j}{\sum_{i=1}^{n} \hat{\pi}_i w_i + \theta \sum_{i=1}^{n} \hat{\pi}_i}. \quad \ldots(3.14)$$

The final term in the numerator is the gross (before-tax) production profit and by (3.8) is non-negative. Hence (3.14) implies that

$$\sum_j \hat{z}_j \geqq 1. \quad \ldots(3.15)$$

Inequalities (3.11) and (3.15) together require that

$$\sum_j \hat{z}_j = 1. \quad \ldots(3.16)$$

Given the result (3.16) that the sum of the weights on the demand columns is unity for the case where $\hat{R} = 0$, equation (3.6) becomes

$$\sum_j \hat{r}_l \hat{x}_j + \Delta + \hat{y} = \Delta$$

or

$$\sum_j \hat{r}_l \hat{x}_j + \hat{y} = 0. \quad \ldots(3.17)$$

Since both left-hand side terms in (3.17) are non-negative, they both must be zero. That is, the weight on the vector $(0, \ldots, 0, 1)'$ must be zero as must the total revenue generated. With the result (3.16), equations (3.5) satisfy the first condition for a competitive equilibrium (1.5), namely that demand equal supply for all commodities. Since at least one of the $\hat{z}_j$ is positive, (3.7) and (3.8) imply the second condition for an equilibrium—that no production activity is profitable with those in use breaking even. Thus, for the case where the final primitive set lies on the side s_R of the simplex, it describes a competitive equilibrium.

To analyze the situation when the final primitive set is not on s_R, multiply each of the equations (3.5) by $\hat{\pi}_i$ and sum giving

$$-\sum_{i=1}^{n} \sum_j \hat{\pi}_i a_{il} \hat{x}_j + \left(\sum_j \hat{z}_j\right) \sum_{i=1}^{n} \hat{\pi}_i(\xi_i(\hat{\pi}, \hat{R}) + \theta) = \sum_{i=1}^{n} \hat{\pi}_i w_i + \theta \sum_{i=1}^{n} \hat{\pi}_i. \quad \ldots(3.18)$$

With $\hat{y} = 0$, we add (3.6) to this resulting in

$$-\left(\sum_{i=1}^{n} \sum_j \hat{\pi}_i a_{il} \hat{x}_j - \sum_j r_l \hat{x}_j\right) + \sum_j \hat{z}_j \left\{\sum_{i=1}^{n} \hat{\pi}_i \xi_i(\hat{\pi}, \hat{R}) + \theta \sum_{i=1}^{n} \hat{\pi}_i - \hat{R} + \Delta\right\}$$
$$= \sum_{i=1}^{n} \hat{\pi}_i w_i + \theta \sum_{i=1}^{n} \hat{\pi}_i + \Delta. \quad \ldots(3.19)$$

The first term in parentheses on the left-hand side is the aggregate production profit and is hence zero if one of the $\hat{z}_j$ is positive. Thus, in this case

$$\sum_j \hat{z}_j = \frac{\sum_{i=1}^n \hat{\pi}_i w_i + \theta \sum_{i=1}^n \hat{\pi}_i + \Delta}{\sum_{i=1}^n \hat{\pi}_i \xi_i(\hat{\pi}, \hat{R}) + \theta \sum_{i=1}^n \hat{\pi}_i - \hat{R} + \Delta}. \qquad \ldots(3.20)$$

The right-hand side of this expression is unity (by Walras' Law (1.2)) and hence if one $\hat{z}_j$ is positive, they sum to unity and, as before, this in turn implies that all the conditions of a competitive equilibrium are met.

It remains to show for this case that indeed one of the $\hat{z}_j$ must be positive. If all of the $\hat{z}_j$ are zero, (3.19) becomes

$$-\left(\sum_{i=1}^n \sum_j \hat{\pi}_i a_{il} \hat{x}_j - \sum_j \hat{r}_l \hat{x}_j\right) = \sum_{i=1}^n \hat{\pi}_i w_i + \theta \sum_{i=1}^n \hat{\pi}_i + \Delta. \qquad \ldots(3.21)$$

Since Δ is a positive constant, the right-hand side of this expression is strictly positive However, the condition for $\hat{x}_j$ to be positive, namely (3.8), requires that the left-hand side be non-positive, hence a contradiction is obtained. Therefore, at least one $\hat{z}_j$ must be positive and, from the above, $\hat{P} = (\hat{\pi}, \hat{R})$ meets the conditions of a competitive equilibrium.

It should be noted that the limiting solution $\hat{R} = 1$, $\sum_{i=1}^n \hat{\pi}_i = 0$ can be excluded by an upper bound on the size of the Δ term in equation (3.2). As the limit $\hat{R} = 1$, $\sum_{i=1}^n \hat{\pi}_i = 0$ is approached, equations (3.1) and (3.2) imply

$$\sum_j a_{il} x_j + w_i + \theta \geqq \theta \sum_j z_j, \quad i = 1, \ldots, n \qquad \ldots(3.22)$$

and

$$\sum_j r_l x_j + \sum_j (-R^j + \Delta) z_j = \Delta. \qquad \ldots(3.23)$$

As $\theta > 0$ and $\Sigma_j z_j \geqq 0$, it follows from the assumption that the set of x satisfying $Ax + w \geqq 0$ is bounded, that the x_j in (3.22) and hence (3.23) must be finite. Thus we know that

$$\sum_j r_e x_j = \sum_j \sum_{i=1}^n \pi_i t_{il} a_{il} x_j \qquad \ldots(3.24)$$

approaches zero in the limit and that (3.23) becomes

$$\sum_j z_j(-1 + \Delta) = \Delta \qquad \ldots(3.25)$$

which is excluded by restricting Δ such that $0 < \Delta < 1$.

4. COMPETITIVE EQUILIBRIUM WITH DIFFERENTIAL PRODUCER AND CONSUMER TAXES

This section seeks to illustrate how, with appropriate modification, the above procedure and proof can be extended to include the case of an economy with differential producer and consumer taxes. Such an extension is computationally extremely useful allowing consideration of a wide range of fiscal phenomena which are not capable of being directly included with only producer taxes. Thus such problems as the operation of statewide sales taxes in a federal economy, citywide sales taxes in a state economy, and the movement from a national sales tax levied on an origin principle to one levied on a destination principle are capable of more meaningful analysis.

No change need be made as regards assumptions on the supply side of the economy.

On the demand side of the economy, however, we assume that there are a finite number of individuals Q, each of whom has initial endowments

$$w_i^q \geqq 0 \; (q = 1, \ldots, Q); \; (i = 1, \ldots, n)$$

and each of whom faces a vector of finite *ad valorem* consumption tax rates

$$e^q = (e_1^q, \ldots, e_n^q) \geqq 0$$

payable on purchases of all goods. As before, each individual receives from the government R^q of the total revenue R from the tax system of the economy

$$R^q = \alpha^q R; \; \sum_{q=1}^{Q} \alpha^q = 1; \; \alpha^q \geqq 0. \qquad \ldots(4.1)$$

The possibility of the government being endowed with a share in the revenue which it then spends on goods and services is not excluded.

Each individual may be considered to face a vector of prices $\tilde{\pi}^q$ as consumer prices

$$\tilde{\pi}^q = (1 + e^q)\pi. \qquad \ldots(4.2)$$

Individuals pay no taxes, nor receive subsidies on the value of endowments they own. The payments by each individual may be denoted as E^q, where

$$E^q = \sum_{i=1}^{n} e_i^q \pi_i \xi_i^q(\pi, R). \qquad \ldots(4.3)$$

It follows immediately that $E^q \geqq 0$. As before, we assume there exists a set of market demand functions

$$\begin{matrix} \xi_1(\pi_1, \ldots, \pi_n, R) \geqq 0 \\ \vdots \\ \xi_n(\pi_1, \ldots, \pi_n, R) \geqq 0, \end{matrix} \qquad \ldots(4.4)$$

which are continuous, homogeneous of degree zero in all prices and revenue and satisfy Walras' Law provided at least one price is strictly positive. The demand for any commodity is assumed bounded from below by zero. Walras' Law for this case states

$$\sum_{i=1}^{n} \pi_i \xi_i(\pi, R) + E = \sum_{i=1}^{n} \pi_i w_i + R; \; E = \sum_{q=1}^{Q} E^q. \qquad \ldots(4.5)$$

It follows immediately that

$$E \geqq 0; \; \sum_{i=1}^{n} \pi_i w_i + R - E \geqq 0. \qquad \ldots(4.6)$$

A competitive equilibrium for such an economy may be characterized as before. It may easily be shown for such a state that

$$R - E^* = \sum_j r_j x_j^* \qquad \ldots(4.7)$$

where $E^* = \sum_{q=1}^{Q} \sum_{i=1}^{n} e_i^q \pi_i^* \xi_i^q(\pi^*, R^*)$.

An approximation to a competitive equilibrium may be obtained by a procedure similar to that described above. Only one change needs to be made to Section 2; in the event that the corresponding column be an augmented demand vector, the last element of that vector becomes $(-R^h + E^h + \Delta)$ in place of $(-R^h + \Delta)$ where

$$E^h = \sum_{q=1}^{Q} \sum_{i=1}^{n} e_i^q \pi_i^h \xi_i^q(\pi^h, R^h).[1]$$

[1] It may be noted that if demands for each individual are not single valued for any price vector, then different tax revenues may be associated with the same market demands by different aggregations over individuals. This problem is overcome by selecting the demand vector with the lexicographically largest first component for each individual.

With this one change, the theorem that there exists a primitive set $P^{j_1}, \ldots, P^{j_{n+1}}$ such that $By = \tilde{w}$ has a non-negative solution where $y_j = 0$ for $j \neq j_1, \ldots, j_{n+1}$ is unaltered. Excluding for the moment the case where the limiting vector $\hat{P}$ is such that $\hat{R} = 1$ the remainder of the proof that such a solution represents a competitive equilibrium goes through by simply making the replacement of $-R^j+E^j$ for $-R^j$ in equation (3.2) and $-\hat{R}+\hat{E}$ for $-\hat{R}$ in equations (3.6), (3.14), (3.19) and (3.20), and by noting property (4.6) together with the requirement that $\theta>0$. The term $\{\hat{E}\}$ represents the limit of the excise revenue collected as the grid size becomes dense on the simplex and follows from the convergence properties assumed for P and $\xi(P)$.

Returning now to the case where the limiting vector P approaches $\hat{R} = 1, \sum_{i=1}^{n} \hat{\pi}_i = 0$. For each of the sequence of increasingly fine grids

$$-\sum_j a_{il}x_j + \sum_j (\xi_i(\pi^j, R^j)+\theta)z_j = w_i+\theta \qquad \ldots(4.8)$$

$$\sum_j r_l x_j + \sum_j (-R^j+E^j+\Delta)z_j = \Delta \qquad \ldots(4.9)$$

must be satisfied. Given the assumption on A, the non-negativity of demands implies that the x_j in (4.8) and hence (4.9) are in a bounded set. Since all π_i approach zero $\Sigma_j r_l x_j$ in (4.9) approach zero. It also follows from equation (4.8) and from the positivity of the θ that $\Sigma_j z_j$ is bounded from above.

However for each individual q

$$\sum_{i=1}^{n} \pi_i^j \xi_i^q(\pi^j, R^j)(1+e_i^q) \leqq \sum_{i=1}^{n} \pi_i^j w_i^q + R^{qj}, \qquad \ldots(4.10)$$

where R^{qj} is individual q's revenue claim with an augmented price vector P^j.

Summing over the Q individuals,

$$E^j = \sum_{i=1}^{n} \sum_{q=1}^{Q} \pi_i^j \xi_i^q(\pi^j, R^j)e_i^q \leqq \sum_{i=1}^{n} \pi_i^j w_i + R^j. \qquad \ldots(4.11)$$

But as $\hat{R} = 1, \sum_{i=1}^{n} \hat{\pi}_i = 0$, there exists a finite constant $c>0$, such that

$$E^j \leqq R^j < c. \qquad \ldots(4.12)$$

Thus, as $\Delta>0$, from (4.9) there exists a $c'>0$, such that

$$\sum_j z_j \geqq c' > 0. \qquad \ldots(4.13)$$

Hence from (4.8) $\sum_j z_j \xi_i(\pi^j, R^j)$ approach a finite limit and as $\sum_{i=1}^{n} \hat{\pi}_i = 0, \sum_j z_j E^j$ approaches zero. Thus (4.9) approaches

$$\sum_j (-1+\Delta)z_j = \Delta \qquad \ldots(4.14)$$

which may be excluded by a value of Δ such that $0<\Delta<1$.

5. A COMPUTATIONAL EXAMPLE

In this section a simple computational example is presented involving an economy in two different tax situations. In this example[1], there are five commodities which may be interpreted as:

1. durable consumer goods
2. nondurable consumer goods
3. capital available at the beginning of the current period
4. labour.
5. capital available at the end of the current period.

[1] This example is the same as that presented in [7] except that a producer tax vector corresponding to each commodity has been introduced. The terms good and commodity are used interchangably throughout.

For expositional simplicity an economy with only differential producer taxation is considered. The economy can be thought of as consisting of three production sectors: (i) the durable goods sector; (ii) the nondurable goods sector; and (iii) a sector for the construction of new capital available at the end of the period.

There are three activities available in each of the durable and nondurable goods sectors, and a further two activities available for the construction of new capital.[1] These may be listed as follows.

	Activity							
Good	1	2	3	4	5	6	7	8
1	1	1	1	0	0	0	0	0
2	0	0	0	1	1	1	0	0
3	−2	−3	−6	−1	−3	−6	−2	−2
4	−3	−2	−1	−6	−3	−2	−3	−6
5	1·5	2·5	5·0	0·9	2·5	5·0	4·5	6·0

There are three people in the economy who are endowed with initial holdings of the commodites.

	Individual		
Good	1	2	3
1	2·0	0·0	1·0
2	3·0	1·0	1·0
3	1·0	3·0	8·0
4	3·0	8·0	10·0
5	0·0	0·0	0·0

Moreover, individuals receive shares in the distribution of revenue.

Individual	1	2	3
Revenue Share	0·34	0·33	0·33

The demand functions of each individual are assumed to be derived from a utility function such that the elasticity of substitution s_q between any two commodities is a constant,

$$U^q = \left[\sum_{i=1}^{5} a_{iq}^{1/s_q} X_i^{(s_q-1)/s_q} \right]^{s_q/(s_q-1)} \qquad q = (1, 2, 3).$$

For the case $s_q = 1$, the limit of the function is such that it follows from utility maximization that the terms a_{iq} represent the proportion of expenditure of the ith individual on the ith good. For convenience, values are chosen such that $\sum_{i=1}^{5} a_{iq} = 1$, these are

	Commodity					
Individual	1	2	3	4	5	s_q
1	0·3	0·3	0·0	0·3	0·1	1·5
2	0·4	0·2	0·0	0·2	0·2	0·8
3	0·2	0·4	0·0	0·1	0·3	1·0

Each individual gives no weight to capital available at the beginning of the period but has a positive weight on capital available at the end of the period. This has an interpretation as a preference for some savings. Corresponding to each activity, there are five *ad valorem* tax rates. Following the convention adopted earlier, if an input coefficient is negative, the corresponding component in the matrix of tax vectors is negative.

[1] The description of the production possibilities of the economy could be modified so as to permit the use of continuous production functions. This point is discussed in [9].

	Activity							
	1	2	3	4	5	6	7	8
Tax on Good 1	0·2	0·2	0·2	0·0	0·0	0·0	0·0	0·0
Tax on Good 2	0·0	0·0	0·0	0·2	0·2	0·2	0·0	0·0
Tax on Good 3	−0·6	−0·6	−0·6	−0·1	−0·1	−0·1	−0·05	−0·05
Tax on Good 4	−0·1	−0·1	−0·1	−0·1	−0·1	−0·1	−0·1	−0·1
Tax on Good 5	0·05	0·05	0·05	0·05	0·05	0·05	0·05	0·05

Thus there are taxes which are identical within any of the three sectors, however across sectors there are differences in the rate of taxation on capital available at the beginning of the current period.

When programmed and run on an IBM 360 a solution was obtained after 2327 iterations working with a grid whose components sum to 200. The solution was refined using a termination routine which involved the solution of a linear programming problem designed to minimize a set of bounds on the excess demands for price vectors in the neighbourhood of the final primitive set.

The following equilibrium set of prices and revenue was obtained.

Good 1	0·30
Good 2	0·19
Good 3	0·06
Good 4	0·06
Good 5	0·08
Revenue	0·31

The sense of the approximation may be gauged from market demands and supplies at these prices.

	Good 1	Good 2	Good 3	Good 4	Good 5	Revenue
Demand	3·68	6·52	0·00	15·32	11·21	0·31
Supply	3·68	6·52	0·00	15·32	11·22	0·31

At these prices activities 2, 6, and 7 were in use and profitabilities and activity levels were as follows.

Activity	Level	Profit
1	0·00	−0·051
2	0·68	0·004
3	0·00	−0·022
4	0·00	−0·268
5	0·00	−0·064
6	1·52	0·000
7	0·43	−0·000
8	0·00	−0·101

Corresponding to the solution the allocation of commodities between individuals was

	Good 1	Good 2	Good 3	Good 4	Good 5
Individual 1	1·03	1·94	0·00	10·13	2·51
Individual 2	1·50	1·06	0·00	2·55	2·18
Individual 3	1·15	3·52	0·00	2·64	6·52

A change in the tax scheme may be introduced where the tax rate on the use of capital available at the beginning of the period is uniform across activities at the rate −0·2, all other components of the tax scheme being unchanged. This has the interpretation of the removal by the government of the distortionary aspects of the taxation system of inputs in the economy. Working again with a grid size of 200 a solution was obtained after

2352 iterations, a refinement of the solution once more being accomplished by a use of a termination routine.

Under the new tax regime the equilibrium set of prices and revenue is

Good 1	0·23
Good 2	0·25
Good 3	0·07
Good 4	0·06
Good 5	0·08
Revenue	0·31

At these prices market demands and supplies are

	Good 1	Good 2	Good 3	Good 4	Good 5	Revenue
Demand	4·93	5·00	0·00	18·41	10·62	0·31
Supply	4·93	5·00	0·00	18·42	10·62	0·31

The shift in production to more of good 1 and less of good 2 reflects both the fact that before the change the tax rate on capital available at the beginning of the period was $-0{\cdot}6$ in sector 1 as against $-0{\cdot}1$ in sector 2 and that activities in sector 1 are more capital intensive than those in sector 2. At these prices, activities 3 and 7 were in use reflecting the tax incentive to shift to more capital intensive activities in sector 1; moreover, the tax change results in no production in sector 2.

Activity	Level	Profit
1	0·0	−0·055
2	0·0	−0·001
3	1·93	0·000
4	0·0	−0·185
5	0·0	−0·047
6	0·0	−0·046
7	0·22	−0·001
8	0·0	−0·066

The allocation of commodities between individuals was

	Good 1	Good 2	Good 3	Good 4	Good 5
Individual 1	1·49	1·32	0·00	12·35	2·27
Individual 2	1·94	0·91	0·00	2·99	2·17
Individual 3	1·50	2·77	0·00	3·07	6·18

CONCLUSION

This paper has described a procedure for the computation of a competitive equilibrium for an economy with differential producer and consumer taxes. For such an economy a proof of the existence of an equilibrium has also been presented.

This computational procedure is of great value for the analysis of the impact of various tax reform proposals presently under consideration in a number of countries. To illustrate the application of the method, a simple numerical example has been outlined.

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