Distance between Points on the Earth's Surface

Abstract

During a casual conversation with one of my students, he asked me how one could go about computing the distance between two points on the surface of the Earth, in terms of their respective latitudes and longitudes. This is an interesting exercise in spherical coordinates, and relates to the so-called **haversine**.

The calculation of the distance between two points on the surface of the Earth proceeds in two stages: (1) to compute the "straight-line" Euclidean distance these two points (obtained by burrowing through the Earth), and (2) to convert this distance to one measured along the surface of the Earth. Figure 1 depicts the **spherical coordinates** we shall use.[1] We orient this coordinate system so that

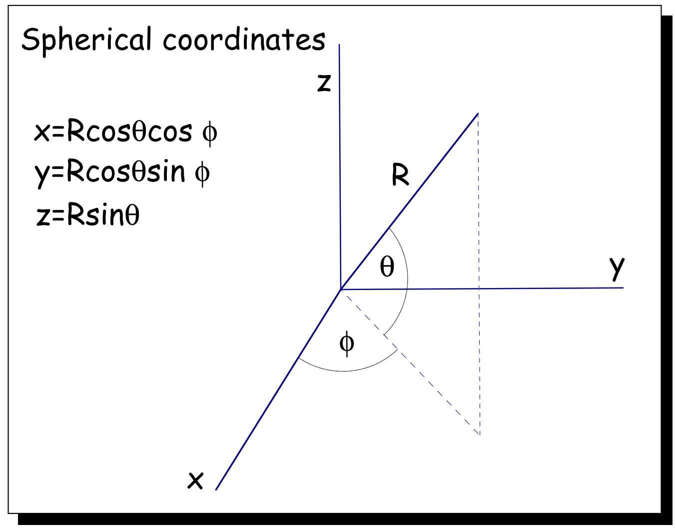

Figure 1: Spherical Coordinates

(i) The origin is at the Earth's center;

(ii) The x-axis passes through the Prime Meridian (0° longitude);

(iii) The xy-plane contains the Earth's equator (and so the positive z-axis will pass through the North Pole)

Note that the angle θ is the measurement of lattitude, and the angle ϕ is the measurement of longitude, where $0 \leq \phi < 360^\circ$, and $-90^\circ \leq \theta \leq 90^\circ$. Negative values of θ correspond to points in the Southern Hemisphere, and positive values of θ correspond to points in the Northern Hemisphere.

When one uses spherical coordinates it is typical for the radial distance R to vary; however, in our discussion we may fix it to be the average radius of the Earth:

$$R \approx 6,378 \text{ km}.$$

[1]What is depicted are not the usual spherical coordinates, as the angle θ is usually measure from the "zenith", or z-axis.

Thus, we assume that we are given two points P_1 and P_2 determined by their respective lattitude-longitude pairs: $P_1(\theta_1, \phi_1),\ P_2(\theta_2, \phi_2)$. In cartesian coordinates we have $P_1 = P_1(x_1, y_1, z_1)$ and $P_2 = P_2(x_2, y_2, z_2)$, where $x,\ y,$ and z are determined by the spherical coordinates through the familiar equations:

$$\begin{aligned} x &= R\cos\theta\cos\phi \\ y &= R\cos\theta\sin\phi \\ z &= R\sin\theta \end{aligned}$$

The Euclidean distance d between P_1 and P_2 is given by the three-dimensional Pythagorean theorem:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

The bulk of our work will be in computing this distance in terms of the spherical coordinates. Converting the cartesian coordinates to spherical coordinates, we get

$$\begin{aligned} d^2/R^2 &= (\cos\theta_1\cos\phi_1 - \cos\theta_2\cos\phi_2)^2 \\ &\quad +(\cos\theta_1\sin\phi_1 - \cos\theta_2\sin\phi_2)^2 \\ &\quad +(\sin\theta_1 - \sin\theta_2)^2 \\ &= \cos^2\theta_1\cos^2\phi_1 - 2\cos\theta_1\cos\phi_1\cos\theta_2\cos\phi_2 + \cos^2\theta_2\cos^2\phi_2 \\ &\quad + \cos^2\theta_1\sin^2\phi_1 - 2\cos\theta_1\sin\phi_1\cos\theta_2\sin\phi_2 + \cos^2\theta_2\sin^2\phi_2 \\ &\quad + \sin^2\theta_1 - 2\sin\theta_1\sin\theta_2 + \sin^2\theta_2 \\ &= 2 - 2\cos\theta_1\cos\theta_2\cos(\phi_1 - \phi_2) - 2\sin\theta_1\sin\theta_2 \end{aligned}$$

Next in order to compute the distance D along the surface of the Earth, we need only analyze Figure 2 in detail. Notice that D is the arc length along the indicated sector. From

$$\sin(\alpha/2) = \frac{d}{2R},$$

we get

$$\begin{aligned}
\sin\alpha &= 2\sin(\alpha/2)\cos(\alpha/2) \\
&= \frac{d}{R} \times \sqrt{1 - \left(\frac{d}{2R}\right)^2} \\
&= \frac{d}{2R^2}\sqrt{4R^2 - d^2}.
\end{aligned}$$

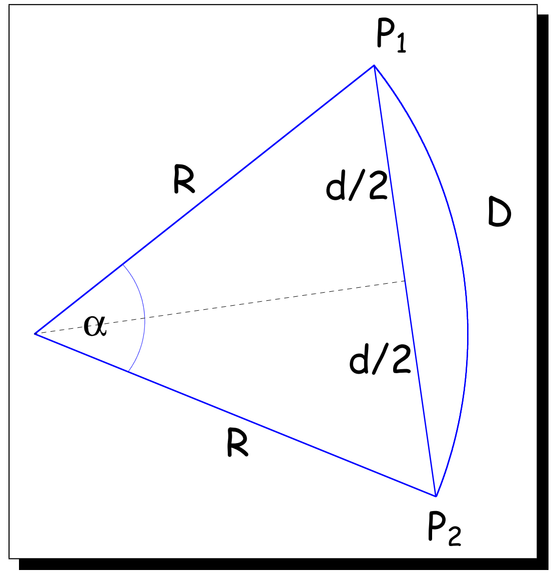

Figure 2: Arc Length

Therefore, in terms of d and R, the distance D is given by

$$D = R\alpha = R\sin^{-1}\left(\frac{d}{2R^2}\sqrt{4R^2 - d^2}\right),$$

which, in principle, concludes this narrative.

The distance is often represented in terms of the so-called **haversine** function, defined by

$$\text{haversin A} = \sin^2\left(\frac{\text{A}}{2}\right) = \frac{1 - \cos \text{A}}{2}.$$

This says that $\cos A = 1 - 2\,\text{haversin A}$.

Returning to the formula for d above, we continue, feeding in the new notation:

$$\begin{aligned}
d^2/R^2 &= 2 - 2\cos\theta_1\cos\theta_2\cos(\phi_1 - \phi_2) - 2\sin\theta_1\sin\theta_2 \\
&= 2 - 2\cos\theta_1\cos\theta_2[1 - 2\,\text{haversin}\ (\phi_1 - \phi_2)] - 2\sin\theta_1\sin\theta_2 \\
&= 2 - 2\cos(\theta_1 - \theta_2) + 4\cos\theta_1\cos\theta_2\,\text{haversin}\ (\phi_1 - \phi_2) \\
&= 4\,\text{haversin}\ (\theta_1 - \theta_2) + 4\cos\theta_1\cos\theta_2\,\text{haversin}\ (\phi_1 - \phi_2)
\end{aligned}$$

Which says that

$$\left(\frac{d}{2R}\right)^2 = \text{haversin}\ (\theta_1 - \theta_2) + \cos\theta_1\cos\theta_2\,\text{haversin}\ (\phi_1 - \phi_2),$$

and so

$$\boxed{\text{haversin } \alpha = \text{haversin } (\theta_1 - \theta_2) + \cos\theta_1 \cos\theta_2 \, \text{haversin } (\phi_1 - \phi_2),}$$

Finally, we have (refer to Figure 2)

$$\left(\frac{d}{2R}\right)^2 = \text{haversin } (\theta_1 - \theta_2) + \cos\theta_1 \cos\theta_2 \, \text{haversin } (\phi_1 - \phi_2), \sin^2(\alpha/2) = \text{haversin } \alpha,$$

meaning that

$$\boxed{D = R\alpha = 2R\sin^{-1}\left(\sqrt{\text{haversin } \alpha}\right).}$$