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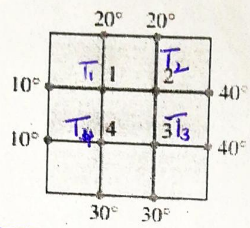
National University of Computer and Emerging Sciences

School of Computing Fall 2021 Islamabad Campus

Question # 1[50 marks]

(a) Assume that the plate shown in the figure given below represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1, T_2, T_3 and T_4 denote the temperatures at the four interior nodes of the mesh in the figure given below. Note that the temperature at a node is approximately equal to the average of the four nearest nodes-- to the left, above, to the right, and below. Write (**DON'T SOLVE**) the system of equations whose solution gives estimates for the temperatures T_1, T_2, T_3, T_4 at the four interior nodes: (4)



$$T_1 = \frac{10+20+T_2+T_4}{4}$$

$$T_2 = \frac{20+40+T_1+T_3}{4}$$

$$T_3 = \frac{30+40+T_4+T_2}{4}$$

$$T_4 = \frac{10+30+T_1+T_3}{4}$$

(b) Let A be a real 7×3 matrix such that its null space is spanned by the vectors

$$\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

Find the rank of the matrix A. (2)

7×3

$\dim(\text{Nul}(A)) = 3$

$\dim(\text{Col}(A)) = 4$

$\text{Rank}(A) = \dim(\text{Col}(A)) = 4.$

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(c) Which of the following statements are true? Explain your answer. (10)

- Matrices A and $B = RREF(A)$ always have the same column space.
  (RREF stands for row reduced echelon form)

No, Column space of both matrices would be different

- Matrices A and $B = RREF(A)$ always have the same row space .

Yes, Row space of both matrices would be same

- If A is an $m \times n$ matrix with linearly independent columns, then $m \geq n$.

Yes, true since "n" ~~column~~ vectors in $\mathbb{R}^m$ could be l.i only if $m \geq n$.
e.g 3 vectors in $\mathbb{R}^n$ can NOT l.i

- If A is an $m \times n$ matrix of $rank = m$, then the left nullspace $N(A^T)$ contains only the zero vector 0.

$A^T_{n \times m} \Rightarrow$ all Columns are linearly ind.

$\Rightarrow Nul(A^T) = 0$.

$\Rightarrow$ True .

- It is given that A and B are 3×3 matrices that satisfy $det(AB) = 20$ and $det(A^{-1}) = -4$. A solid S of volume $5\ cm^3$ is transformed by B to produce an image S'. Find the volume of S'

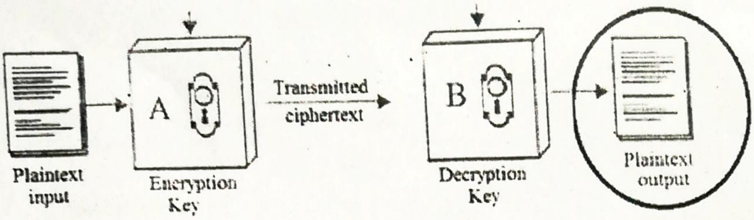
$det(AB) = det(A)det(B) = 20$

$\Rightarrow det(B) = \frac{20}{det(A)} = 20\,det(A^{-1})$
$= 20 \times -4 = -80$

Volume $(T(S)) = det(B) \times$ Volume of S.
$= -80 \times 5\ cm^3 =$

(d) Write the plaintext for encrypted message (LBSYMUUIO) with the following encrypted Key:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 3 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$



$L \to 12$
$B \to 2$
$S \to 19$

$Y \to 25$
$M \to 13$
$U \to 21$

$U \to 21$
$I \to 9$
$O \to 15$

Modular Inverses for Mod 26:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

(8)

$$A^{-1} = \begin{bmatrix} 7/31 & -6/31 & 4/31 \\ 1/31 & 8/31 & 5/31 \\ 11/31 & -5/31 & -7/31 \end{bmatrix} \approx \begin{bmatrix} 17 & 4 & 6 \\ 21 & 12 & 1 \\ 23 & 25 & 9 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 12 \\ 2 \\ 19 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 3 \end{bmatrix} \begin{matrix} \to N \\ \to I \\ \to C \end{matrix}$$

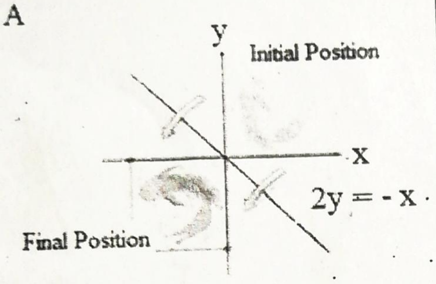
$$A^{-1}\begin{bmatrix} 25 \\ 13 \\ 21 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 23 \end{bmatrix} \begin{matrix} \to E \\ \to Z \\ \to W \end{matrix}$$

$$A^{-1}\begin{bmatrix} 21 \\ 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 18 \\ 11 \end{bmatrix} \begin{matrix} \to O \\ \to R \\ \to K \end{matrix}$$

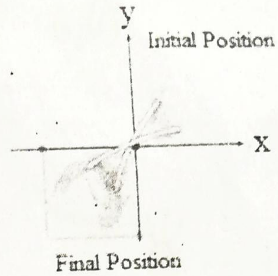
Plain text Nice Z Work.

(e) Find the matrices A and B that transform the picture of fish from initial to final position

A



B



(4)

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1/2\end{pmatrix} \qquad T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\0\end{pmatrix} \qquad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}0 & -2\\-1/2 & 0\end{pmatrix} \qquad B = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$$

(f) Due to earthquake the roof of a building collapses onto the ground. What type of transformation is this (assuming the whole roof lays flat on the ground) and what is the transformation matrix? If the coordinates of the roof were $(-3, 2, 4)$, $(4, -3, 2)$, $(-4, 8, 7)$, $(5, 3, -1)$, determine the new coordinates.

(6)

Transformation is projection on xy-plane

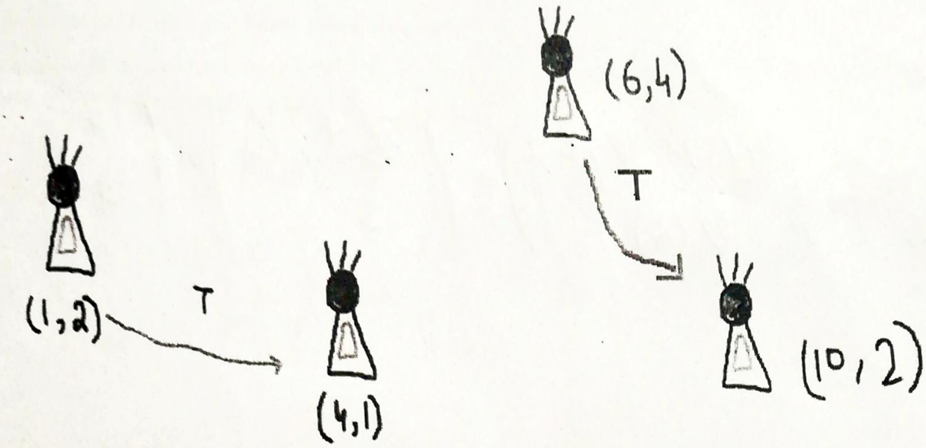
$$\Rightarrow T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$

New coordinates are

$(-3\ \ 2\ \ 0), (+4, -3, 0)\ (-4, 8, 0), (5, 3, 0)$

(g) Suppose you are developing a video game where the character moves as given in the following figure. Construct a transformation that will determine the next position of the character. If the character is at position $(5, 6)$, what is its next position? (8)



$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} \qquad T\begin{pmatrix}6\\4\end{pmatrix} = \begin{pmatrix}10\\2\end{pmatrix}$$

Since $A = \begin{pmatrix}T(e_1) & T(e_2)\end{pmatrix}$

we need to find $T(e_1), T(e_2)$

$$\because\ T\begin{pmatrix}1\\2\end{pmatrix} = T\left(1\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix}\right) = 1T(e_1) + 2T(e_2) = \begin{pmatrix}4\\1\end{pmatrix} \rightarrow \mathrm{I}$$

$$T\begin{pmatrix}6\\4\end{pmatrix} = T\left(6\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix}\right) = 6T(e_1) + 4T(e_2) = \begin{pmatrix}10\\2\end{pmatrix} \rightarrow \mathrm{II}$$

multiply eqn I by 2: $2T(e_1) + \cancel{4T(e_2)} = \begin{pmatrix}8\\2\end{pmatrix}$

$$\underline{-\,6T(e_1) - \cancel{4T(e_2)} = -\begin{pmatrix}10\\2\end{pmatrix}}$$

$$-4T(e_1) = \begin{pmatrix}-2\\0\end{pmatrix} \Rightarrow T(e_1) = \begin{pmatrix}1/2\\0\end{pmatrix}$$

from II

$$6\begin{pmatrix}1/2\\0\end{pmatrix} + 4T(e_2) = \begin{pmatrix}10\\2\end{pmatrix}$$

$$\begin{pmatrix}3\\0\end{pmatrix} + 4T(e_2) = \begin{pmatrix}10\\2\end{pmatrix} \Rightarrow 4T(e_2) = \begin{pmatrix}7\\2\end{pmatrix} \Rightarrow T(e_2) = \begin{pmatrix}7/4\\1/2\end{pmatrix}$$

Transformation Matrix $= \begin{pmatrix}1/2 & 7/4\\0 & 1/2\end{pmatrix}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 7/4 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{7}{4}y \\ \frac{1}{2}y \end{pmatrix}$$

Hence

$$T\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(5) + \frac{7}{4}(6) \\ \frac{1}{2}(6) \end{pmatrix} = \begin{pmatrix} \frac{5}{2} + \frac{21}{2} \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}$$

(h) Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

be a transformation matrix of linear transformation T. (8)

- Describe the kernel and range of T. 6
- Give the rank and nullity of T. 2

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$\ker(T) = Nul(A)$

i.e $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow Nul(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$

For range, consider arbitrary element $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ of $\mathbb{R}^3$ s.t $T(x) = Ax = b$

$\left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 1 & 1 & b_2 \\ 0 & 1 & b_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 2 & b_2 - b_1 \\ 0 & 1 & b_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & b \\ 0 & 2 & b_2 - \\ 0 & 0 & b_3 - \frac{}{2} \end{array}\right)$

which is consistent if

$b_3 - \frac{1}{2}(b_2 - b_1) = 0 \Rightarrow 2b_3 = b_2 - b_1$

So

$Range(T) = \{ \vec{b} \in \mathbb{R}^3 \mid 2b_3 = b_2 - b_1 \}$

Rank $= 2$

Nulity $= 0$

Question # 2 [16 marks]

(a) Let $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$.

- Show that B and B' are both bases for $\mathbb{R}^2$.
- Find the change-of-basis matrix $P_{B' \leftarrow B}$.
- Find the coordinate vector of $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with respect to B' by using change of basis matrix.

(2+4+4)

$B = \left\{ \underset{u_1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}} \right\}$

Since u_1 and u_2 are l.i
and two linearly independent vectors
will span whole of $\mathbb{R}^2$
$\Rightarrow$ B is basis.

$B' = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} -3 \\ 7 \end{pmatrix}} \right\}$

Since v_1 and v_2 are linearly independent (not scalar multiple) and two l.i vectors in $\mathbb{R}^2$ spans whole of $\mathbb{R}^2$
$\Rightarrow B'$ is also basis of $\mathbb{R}^2$

we want $P_{B' \leftarrow B}$ (taking element of B to B')

Let $u_1 = c_1 v_1 + c_2 v_2$ and $u_2 = d_1 v_1 + d_2 v_2$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 7 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 7 & 1 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 13 & 1-4 \end{array} \right)$

$\Rightarrow \boxed{c_2 = -3/13}$ $\quad c_1 = 2 + 3\left(\frac{-3}{13}\right) = 17/13$

$\boxed{c_1 = 17/13 \quad c_2 = -3/13}$

$u_2 = d_1 v_1 + d_2 v_2$

$$\begin{pmatrix}5\\2\end{pmatrix} = d_1\begin{pmatrix}1\\2\end{pmatrix} + d_2\begin{pmatrix}-3\\7\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1 & -3 & 5\\2 & 7 & 6\end{array}\right) \sim \left(\begin{array}{cc|c}1 & -3 & 5\\0 & 13 & -4\end{array}\right)$$

$c_2 = -4/13 \qquad c_1 = 5 + 3\left(-4/13\right) = \frac{53}{13}$

Hence

$$\underset{B' \leftarrow B}{P} = \begin{pmatrix}17/13 & 53/13\\ -3/13 & -4/13\end{pmatrix}$$

$$[x]_{B'} = \underset{B' \leftarrow B}{P}[x]_B = \begin{pmatrix}17/13 & 53/13\\ -3/13 & -4/13\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}17/13\\ -3/13\end{pmatrix}$$

$\because x = 1u_1 + 0u_2 \Rightarrow [x]_B = \begin{bmatrix}1\\0\end{bmatrix}$

(b) Let $Q = \{x^2+2x+1, 2x^2+3x+1, 2x^2, 2x^2+x+1\}$ be a subset of $\mathbb{P}_2$. Find a basis of the $span\{Q\}$. (6)

The equivalent set in $\mathbb{R}^3$ is

$$Q_{\mathbb{R}^3} = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right\}$$

To find basis we need to pick l.i vectors

$$\begin{pmatrix}1 & 2 & 2 & 2\\ 2 & 3 & 0 & 1\\ 1 & 1 & 0 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 2 & 2\\ 0 & -1 & -2 & -1\\ 0 & -1 & -2 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 2 & 2\\ 0 & -1 & -2 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Hence basis of $Span\{Q_{\mathbb{R}^3}\} = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix} \right\}$

asis of $Span\{Q\} = \{x^2+2x+1, 2x^2+3x+1\}$

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Question # 3 [30 marks]

(a) The eigenvalues of a 3×3 matrix are given by $3, 4/5$, and $3/5$, with corresponding eigenvectors

$$\left\{ \underset{V_1}{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}}, \underset{V_2}{\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}}, \underset{V_3}{\begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}} \right\}$$

respectively.

Let $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Find the solution of the system $\mathbf{x}_{n+1} = A\mathbf{x}_n$, and describe what happens when $n \to \infty$. (8)

First write $x_0 = C_1 V_1 + C_2 V_2 + C_3 V_3$

$$\begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + C_3 \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ 0 & 1 & -2 & -4 \end{array}\right] \quad R_3 + 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 - R_2$$

$\Rightarrow C_3 = 1 \qquad C_2 = -5 + 3(1) = -2 \qquad C_1 = -2 - 2C_2 + 3C_3$
$= -2 - 2(-2) + 3$
$= 5$

The solution of $x_{n+1} = A x_n$ is

$$x_n = C_1 \lambda_1^n V_1 + C_2 \lambda_2^n V_2 + C_3 \lambda_3^n V_3$$

$$= 5(3)^n \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} - 2\left(\tfrac{4}{5}\right)^n \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} - 1\left(\tfrac{3}{5}\right)^n \begin{pmatrix} -3 \\ -3 \\ 7 \end{pmatrix}$$

as $n \to \infty$ $\left(\tfrac{4}{5}\right)^n$ and $\left(\tfrac{3}{5}\right)^n \to 0$.

Hence solution will converge to 3-eigenspace i.e line generated by $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

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(b) The Lucas number are like the Fibonacci numbers except that they start with $L_1 = 1$ and $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$, the next Lucas numbers are 4, 7, 11, 18. Show that the general expression for Lucas number can be written as $L_k = \lambda_1^k + \lambda_2^k$. (8)

Given equation (linear recurrence relation)

$$L_{k+2} = L_{k+1} + L_k$$

also consider $L_{k+1} = L_{k+1}$

$\Rightarrow \begin{pmatrix} L_{k+2} \\ L_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_{k+1} \\ L_k \end{pmatrix}$ $\quad X_1 = \begin{pmatrix} L_2 \\ L_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\Rightarrow X_{n+1} = A X_n \; ; n = 1, \ldots$

its solution is $x_n = c_1 \lambda_1 v_1^n + c_2 \lambda_2^n v_2^n$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \lambda^2 - \lambda - 1 = 0 \quad \lambda = \frac{1 \pm \sqrt{1+4}}{2}$

eigenvalues of A are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ $\quad \lambda_2 = \frac{1-\sqrt{5}}{2}$

λ_1-Eigenspace $= Nul(A - \lambda_1 I)$

$\Rightarrow \left(\begin{array}{cc|c} 1-\lambda_1 & 1 & 0 \\ 1 & -\lambda_1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\lambda_1 & 0 \\ 1-\lambda_1 & 1 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cc|c} 1 & -\lambda_1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad R_2 - (1-\lambda_1) R_1$

$1 - (1-\lambda_1)(-\lambda_1)$

$1 + \left(1 - \frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)$

$\Rightarrow x_2 = x_2$

$x_1 = \lambda_1 x_2$ $\quad 1 + \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)$

λ_1-Eigenspan $= Span\left\{\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}\right\}$ $\quad 1 - 1 = 0$

λ_2- Eigenspace $= Nul(A-\lambda_2 I)$

$$\left(\begin{array}{cc|c} 1-\lambda_2 & 1 & 0 \\ 1 & -\lambda_2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\lambda_2 & 0 \\ 1-\lambda_2 & 1 & 0 \end{array}\right)$$

λ_2- Eigenspace $=$ Span $\left\{ \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \right\}$

$$X_n = c_1 \lambda_1^n \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + c_2 \lambda_2^n \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} L_{n+1} \\ L_n \end{pmatrix} = c_1 \lambda_1^n \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + c_2 \lambda_2^n \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$\Rightarrow L_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

where c_1 and c_2 is solution of the System

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = c_1 v_1 + c_2 v_2 \Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$\Rightarrow c_1 = 1 = c_2 .$$

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(c) Find the eigenvalues and eigenvectors and the k^{th} power of the following matrix A. For the following adjacency matrix the i, j entry of A^k counts the k-step paths form i to j.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(6)

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)(+\lambda^2) - 1(-\lambda - 0) + 1(\lambda) = 0$$

$$= (1-\lambda)\lambda^2 + \lambda + \lambda = 0$$

$$\lambda(\lambda - \lambda^2 + 1+1)$$

$$\Rightarrow \lambda = 0, \quad -\lambda^2 + \lambda + 2 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda^2 - 2\lambda + \lambda - 2 = 0$$

$$\lambda = +2 \quad \lambda = -1$$

$$\lambda_1 = 0 \quad \lambda_2 = +2 \quad \lambda_3 = -1$$

0- Eigenspace:

$$Nul(A) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_3 = x_3, \quad x_2 = -x_3, \quad x_1 = 0$$

$$0\text{-Eigenspace} = Span\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$$

-2-Eigenspace: $Nul(A - 2I)$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & +1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = x_3$

$x_2 = x_3 \quad \Rightarrow -2\text{-Eigenspace} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}$

$x_1 = -2x_3$

$-1\text{-Eigenspace} = \text{Nul}(A+I)$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3 = x_3$

$x_2 = x_3 \quad \Rightarrow -1\text{-Eigenspace} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$x_1 = -x_3$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & \cancel{2}1 \end{bmatrix}$$

$$A^k = P D^k P^{-1}$$

$$= P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} P^{-1}$$

(d) The circuit shown in figure can be described by the differential equation

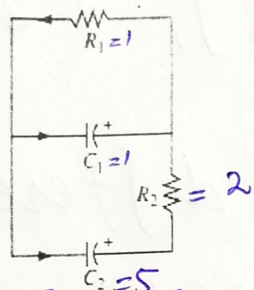
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -(1/R_1 + 1/R_2)C_1 & 1/(R_2C_1) \\ 1/(R_2C_2) & -1/(R_2C_2) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $x_1(t)$ and $x_2(t)$ are the voltages across the two capacitors at time t. Suppose resistor R_1 is 1 ohm, R_2 is 2 ohms, capacitor C_1 is 1 farad, and C_2 is 5 farad, and suppose there is an initial charge of 5 volts on capacitor C_1 and 4 volts on capacitor C_2. Find formulas for $x_1(t)$ and $x_2(t)$ that describes how the voltages change over time.

Initial Cdn

$x_1(0) = 5V$

$x_2(0) = 4V$



(8)

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 \\ 1/10 & -1/10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

A → 2

$|A - \lambda I| = 0$

$$\begin{vmatrix} -3/2 - \lambda & 1/2 \\ 1/10 & -1/10 - \lambda \end{vmatrix} = 0 \Rightarrow (-3/2 - \lambda)(-1/10 - \lambda) - \frac{1}{20} = 0$$

Chareteristic equation: is

$$\lambda^2 + \frac{8}{5}\lambda + \frac{1}{10} = 0$$

$$\lambda = \frac{-8/5 \pm \sqrt{64/25 - 2/5}}{2} = \frac{-8/5 \pm \sqrt{54/25}}{2}$$

$$= \frac{-8 \pm \sqrt{54}}{10} \Rightarrow \quad \lambda_1 = \frac{-8 + \sqrt{54}}{10}, \; \lambda_2 = \frac{-8 - \sqrt{54}}{2}.$$

D → 2

$\lambda = \lambda_1$ $\quad (A - \lambda_1 I)X = 0$

$$\begin{pmatrix} -\frac{3}{2} - \lambda_1 & \frac{1}{2} & 0 \\ \frac{1}{10} & -\frac{1}{10} - \lambda_1 & 0 \end{pmatrix}$$

$(-\frac{3}{2} - \lambda_1)x + \frac{1}{2}y = 0$

$(-3 - 2\lambda_1)x + y = 0.$

$y = (3 + 2\lambda_1)x$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (3+2\lambda_1)x \end{pmatrix}.$$

Eigenvectors

$$\begin{pmatrix} 1 \\ 3+2\lambda_1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3+2\lambda_2 \end{pmatrix}.$$

Solution of differential eq is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 3+2\lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 3+2\lambda_2 \end{pmatrix}.$$

Question # 4[26 marks]

(a) Let W be a subspace of $\mathbb{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$, $v = (2, 4, 7, 2, -1)$. Find an orthogonal basis of orthogonal complement of W. Moreover, write $b = (1, 1, 1, 1, 1)^T$ as sum of two vectors one from W other from orthogonal complement of W. (6+4)

Let $\vec{x} \in W^{\perp}$.

$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ and $\vec{x} \cdot u = 0$

$\Rightarrow x_1 + 2x_2 + 3x_3 - x_4 + 2x_5 = 0$.

$\vec{x} \cdot v = 0$

$2x_1 + 4x_2 + 7x_3 + 2x_4 - x_5 = 0$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 3 & -1 & 2 & 0 \\ 2 & 4 & 7 & 2 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 & -5 & 0 \end{array}\right) \quad R_2 - 2R_1$$

x_2, x_4, x_5 are free variable.

$x_3 = -4x_4 + 5x_5$

$x_1 = -2x_2 - 3x_3 + x_4 - 2x_5$

$= -2x_2 - 3(-4x_4 + 5x_5) + x_4 - 2x_5$

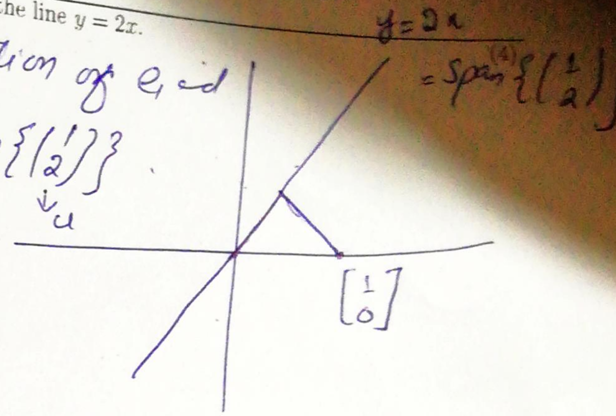
$= -2x_2 + 13x_4 - 17x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u_1} + x_4 \underbrace{\begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}}_{u_2} + x_5 \underbrace{\begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{u_3}.$$

Since we want orthogonal basis
so we will use Gram Schmidth method

(b) Find the matrix that projects each vector onto the line $y = 2x$. (4)

we want to find projection of e_1 and e_2 on line $y = 2x = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$.



$$\text{Proj}_u e_1 = \frac{u \cdot e_1}{u \cdot u} \cdot \vec{u}$$

$$= \frac{1}{5}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1/5\\2/5\end{pmatrix}$$

$$\text{Proj}_u e_2 = \frac{u \cdot e_2}{u \cdot u} \cdot \vec{u} = \frac{2}{5}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2/5\\4/5\end{pmatrix}$$

$$M = \begin{pmatrix}1/5 & 2/5\\2/5 & 4/5\end{pmatrix}$$

(c) Verify that the set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}3\\-1\\-1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\-4\\7\end{bmatrix}.$$

is an orthogonal set. Can B be a basis for the vector space $\mathbb{R}^3$? Justify your answer. (4)

$$V_1 \cdot V_2 = 3 - 2 - 1 = 0$$

$$V_2 \cdot V_3 = 3 + 4 - 7 = 0$$

$$V_1 \cdot V_3 = 1 - 8 + 7 = 0$$

orthogonal vectors are linearly independent and 3 l.i vector in $\mathbb{R}^3$ will form basis of $\mathbb{R}^3$.

(d) Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fit the data points $(2,1)$, $(5,2)$, $(7,3)$ and $(8,3)$. (8)

$$\beta_0 + 2\beta_1 = 1$$

$$\beta_0 + 5\beta_1 = 2$$

$$\beta_0 + 7\beta_1 = 3$$

$$\beta_0 + 8\beta_1 = 3$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} \hat{\beta_0} \\ \hat{\beta_1} \end{bmatrix}$$

we will solve $A^T A \hat{x} = A^T b$

$$A^T A = \underset{2\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix}} \underset{4\times 2}{\begin{pmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{pmatrix}} \begin{bmatrix} \hat{\beta_0} \\ \hat{\beta_1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 22 \\ 22 & 442 \end{bmatrix} \begin{bmatrix} \hat{\beta_0} \\ \hat{\beta_1} \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 4 & 22 & 9 \\ 22 & 442 & 57 \end{array}\right] \sim$$

Question # 5 [32 marks]

(a) For the following quadratic form $9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$, the matrix A has the eigenvalues 3, 9 and 15. Find an orthogonal matrix P required to transform given quadratic form into a quadratic form with no cross product terms. Write the new quadratic form. (8)

$$A = \begin{pmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{pmatrix}$$

$\lambda = 3, 9, 15$

$$V_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} -1/4 \\ 1/4 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

$3y_1^2 + 9y_2^2 + 15y_3^2$

$\lambda = 3$

$(A - 3I)V = 0$

$$\begin{pmatrix} 6 & -4 & 4 & 0 \\ -4 & 4 & 0 & 0 \\ 4 & 0 & 8 & 0 \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} 6 & -4 & 4 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & 4 & 8 & 0 \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} \cancel{3/2} & \cancel{-1} & \cancel{1} & \cancel{0} \\ \cancel{-4} & \cancel{4} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{4} & \cancel{8} & \cancel{0} \end{pmatrix}$$

$-\frac{8}{3} + 4$

$$\overset{R}{\sim} \begin{pmatrix} \cancel{1} & \cancel{-4/3} & \cancel{4/3} & \cancel{0} \\ \cancel{0} & \cancel{-7/3} & \cancel{16/3} & \cancel{0} \\ \cancel{0} & \cancel{3} & \cancel{8} & \cancel{0} \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} \cancel{3} & \cancel{-4} & \cancel{4} & \cancel{0} \\ \cancel{0} & \cancel{-7} & \cancel{16} & \cancel{0} \\ \cancel{0} & \cancel{3} & \cancel{8} & \cancel{0} \end{pmatrix}$$

$-\frac{16}{7}$

$$\overset{R}{\sim} \begin{pmatrix} \cancel{3} & \cancel{-4} & \cancel{4} & \cancel{0} \\ \cancel{0} & \cancel{-7} & \cancel{16} & \cancel{0} \\ \cancel{0} & & & \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} 1 & -2/3 & 2/3 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & 4 & 8 & 0 \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} 1 & -2/3 & 2/3 & 0 \\ 0 & 4/3 & 8/3 & 0 \\ 0 & 4 & 8 & 0 \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} 1 & -2/3 & 2/3 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R}{\sim} \begin{pmatrix} 3 & -2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$y + 2z = 0$

$\boxed{y = -2z}$

$3x - 2(-2z) + 2z = 0$

$3x + 6z = 0$

$\boxed{x = -2z}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

0050

(b) If A is any symmetric positive definite matrix, show that A^2 and A^{-1} are also positive definite. (4)

If A is +ve definite matrix
then $\lambda > 0$, $\forall \lambda$.

Eigenvalues of A^2 are $\lambda^2 > 0$
" " A^{-1} " $\frac{1}{\lambda} > 0$

(c) For which numbers b and c, the following matrix is positive definite?

$$A = \begin{bmatrix} c & b \\ b & c \end{bmatrix}.$$

(4)

M-I:

$\lambda^2 - 2c\lambda + (c^2 - b^2) = 0$

$\lambda = \frac{2c \pm \sqrt{4c^2 - 4c^2 + 4b^2}}{2}$

$\lambda = \frac{2c \pm 2b}{2}$

$\lambda = c \pm b.$

$\lambda_1 = c + b$, $\lambda_2 = c - b$.

$c + b > 0$, $c - b > 0$
$c > -b$ $c > b$

M-II:

$c > 0$

$c^2 - b^2 > 0$

$c^2 > b^2$

$|c| > |b|$.

0050

(d) Find the maximum value of $Q(\mathbf{x}) = 7x^2 + 3y^2 - 2xy$ subject to the constraint $x^2 + y^2 = 1$. (4)

$$A = \begin{pmatrix} 7 & -1 \\ -1 & 3 \end{pmatrix}.$$

Charcteristic Equation

$$\lambda^2 - 10\lambda + 20 = 0$$

$$\lambda = \frac{10 \pm \sqrt{100-80}}{2}$$

$$= \frac{10 \pm \sqrt{20}}{2}$$

$$\lambda = 5 \pm \sqrt{5}.$$

Maximum value: $5 + \sqrt{5}$

(e) For the given matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$, the eigenvalues of A^TA are 18 and 0 with corresponding eigenvectors given by $\mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ respectively. Find singular value decomposition of the matrix A. (8)

SVD: $U \Sigma V^T$

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Singular values are $\sigma_1 = \sqrt{18}$, $\sigma_2 = 0$

$$D = 3\sqrt{2}$$

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

For U.

$$Av_1 = \begin{pmatrix} \sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{pmatrix} \quad Av_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\|Av_1\| = 3\sqrt{2}, \quad \|Av_2\| = 0$$

$$u_1 = \frac{Av_1}{\|Av_1\|} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

We want to find x s.t $u_1^T x = 0 \Rightarrow x_1 - 2x_2 + 2x_3 = 0$

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{pmatrix}$$