Motion in 2-Dimension

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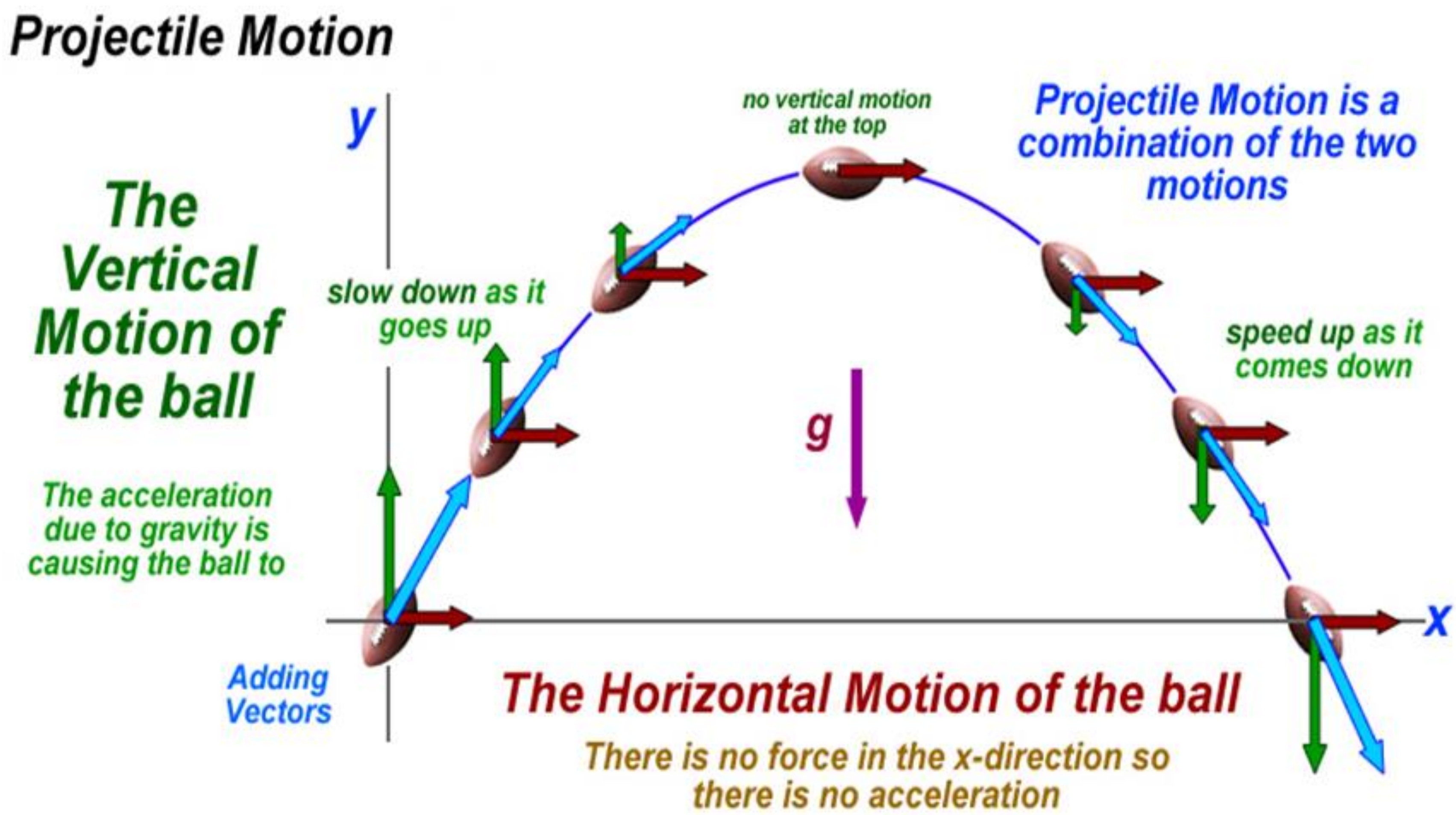
Projectile & Projectile Motion

- A particle moves in a vertical plane with some initial velocity but its acceleration is always the freefall acceleration , which is downward.
- Such a particle is called a **projectile (meaning** that it is projected or launched), and its motion is called **projectile motion.**
- In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

Pictorial View

A stroboscopic photograph of a yellow tennis ball bouncing off a hard surface. Between impacts, the ball has projectile motion.

Vector Perspective



What may be the Projectiles?

Projectile might be ;

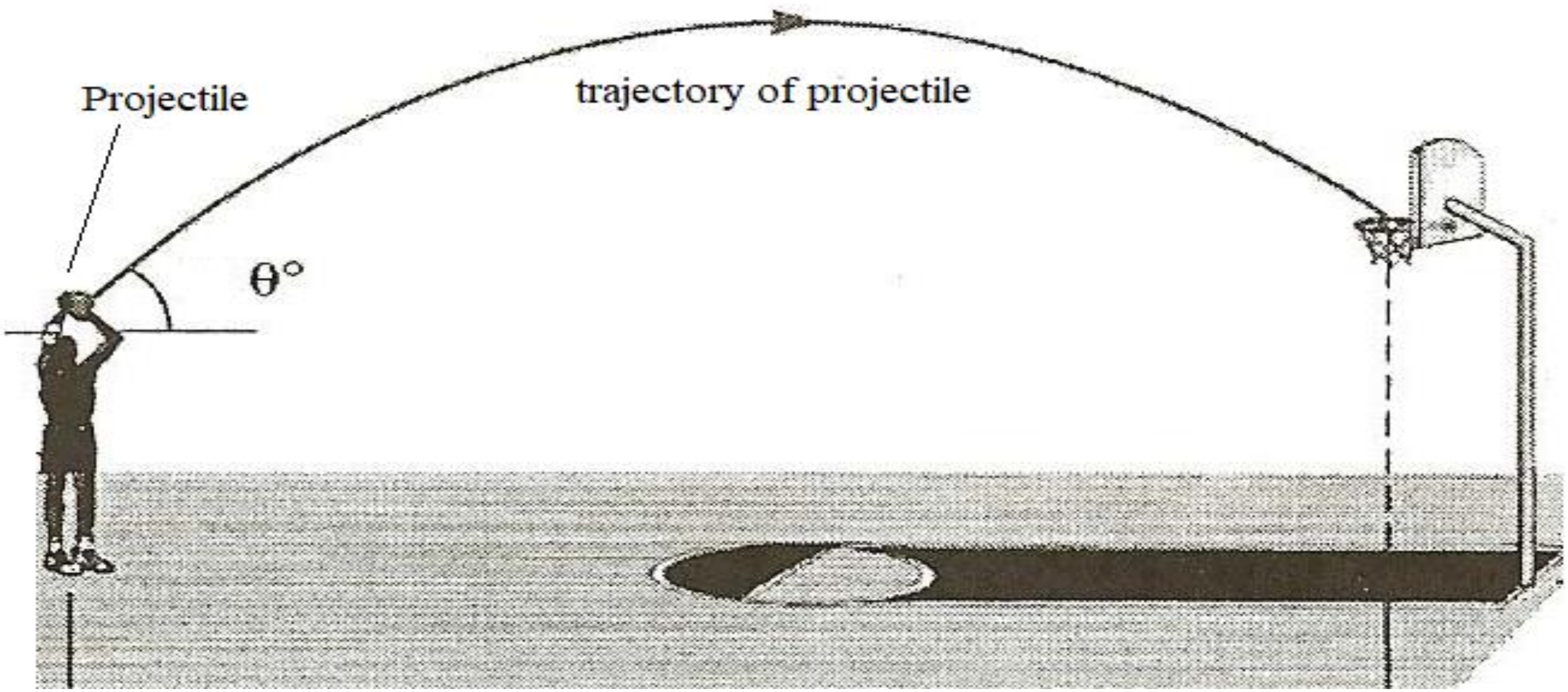
- a tennis ball
- a baseball in flight
- Cannon ball shot from a cannon
- A basket ball thrown in a basket
- A kicked football

Many sports (from golf and football to lacrosse and Racquetball)involve the projectile motion of a ball, and much effort is spent in trying to control that motion for an advantage.

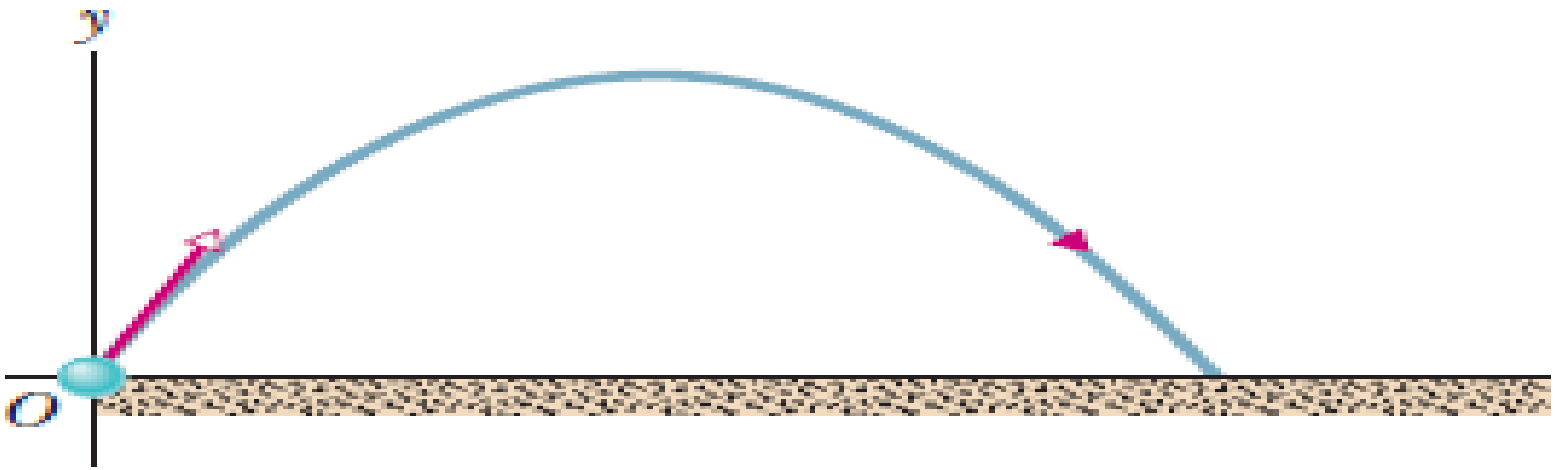
Note: Projectile is not an airplane or a duck in flight

Trajectory of Projectile

- Path followed by the projectile is called as **trajectory of that projectile.**



Pictorial Representation



The *projectile motion* of an object launched into the air at the origin of a coordinate system and with launch velocity $\vec{v}_0$ at angle θ_0. The motion is a combination of vertical motion (constant acceleration) and horizontal motion (constant velocity), as shown by the velocity components.

Pictorial Representation

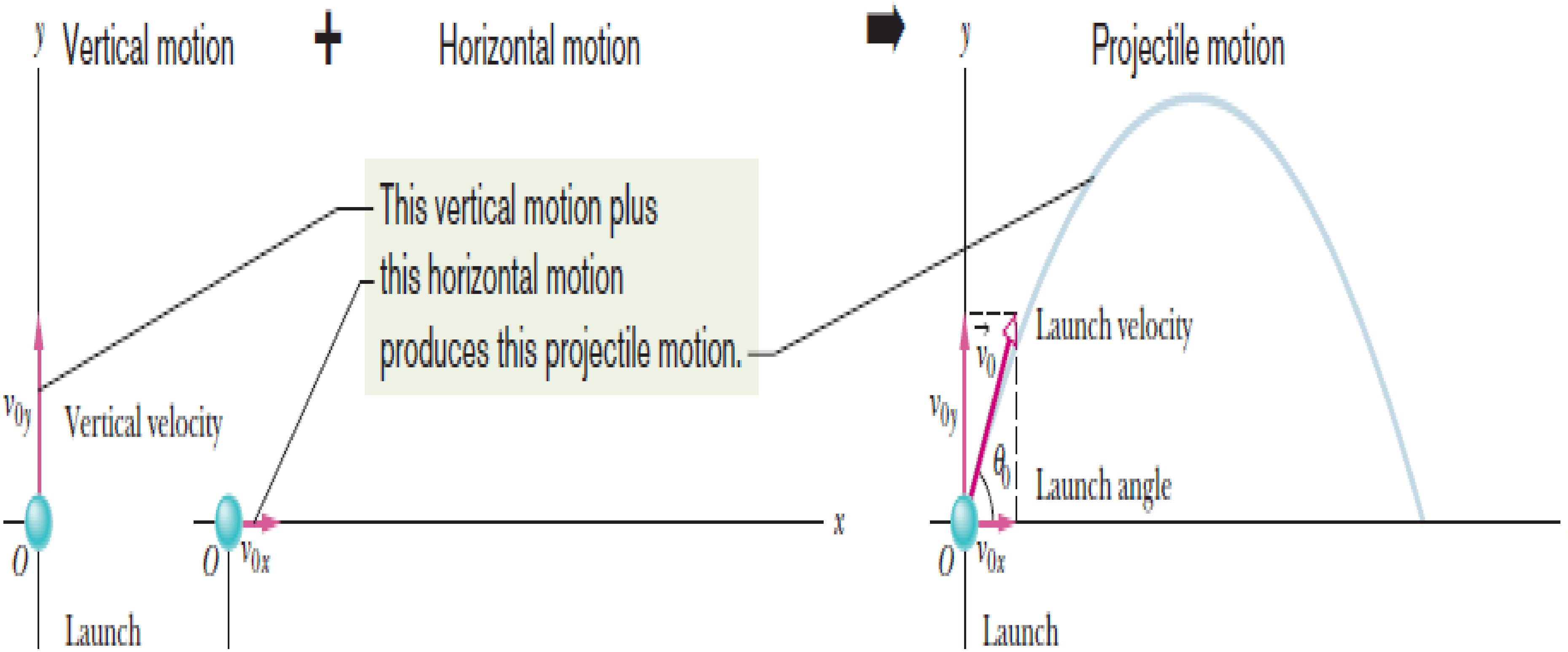


Fig 1(a)

Pictorial Representation(cont'd)

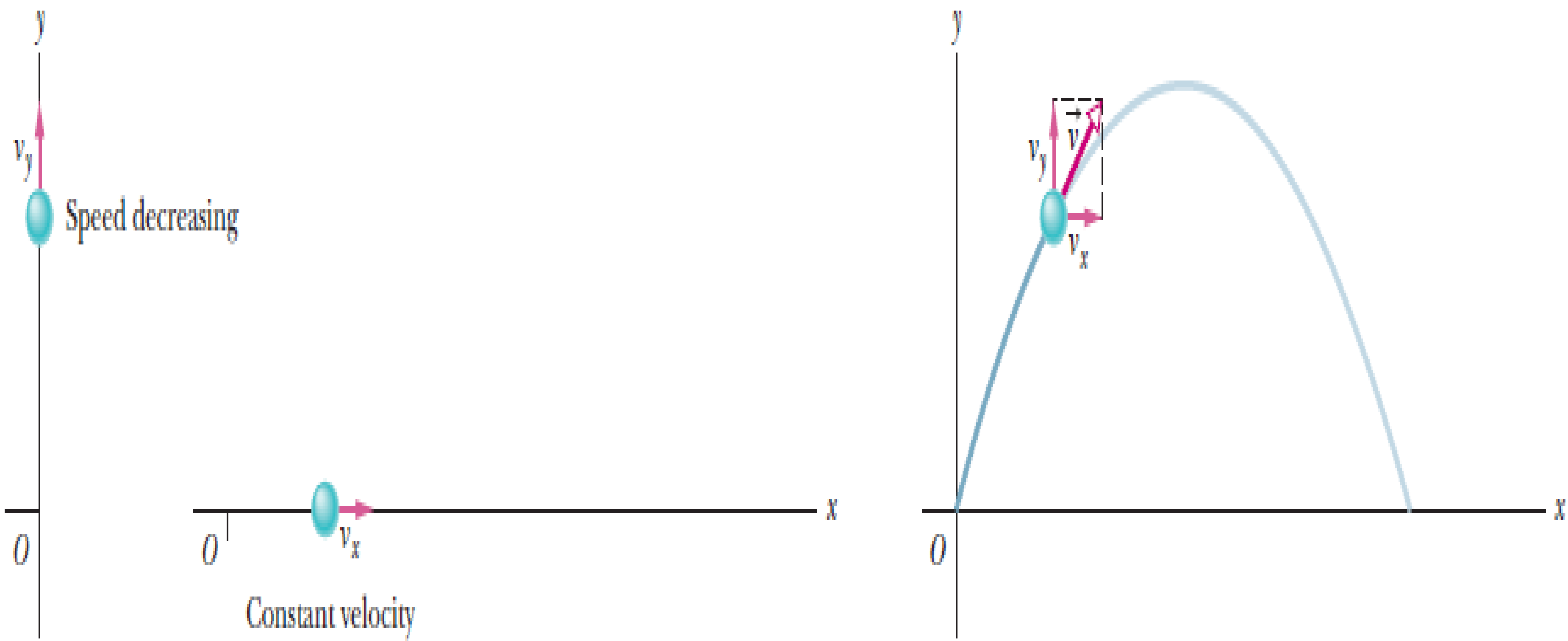


Fig 1(b)

Pictorial Representation(cont'd)

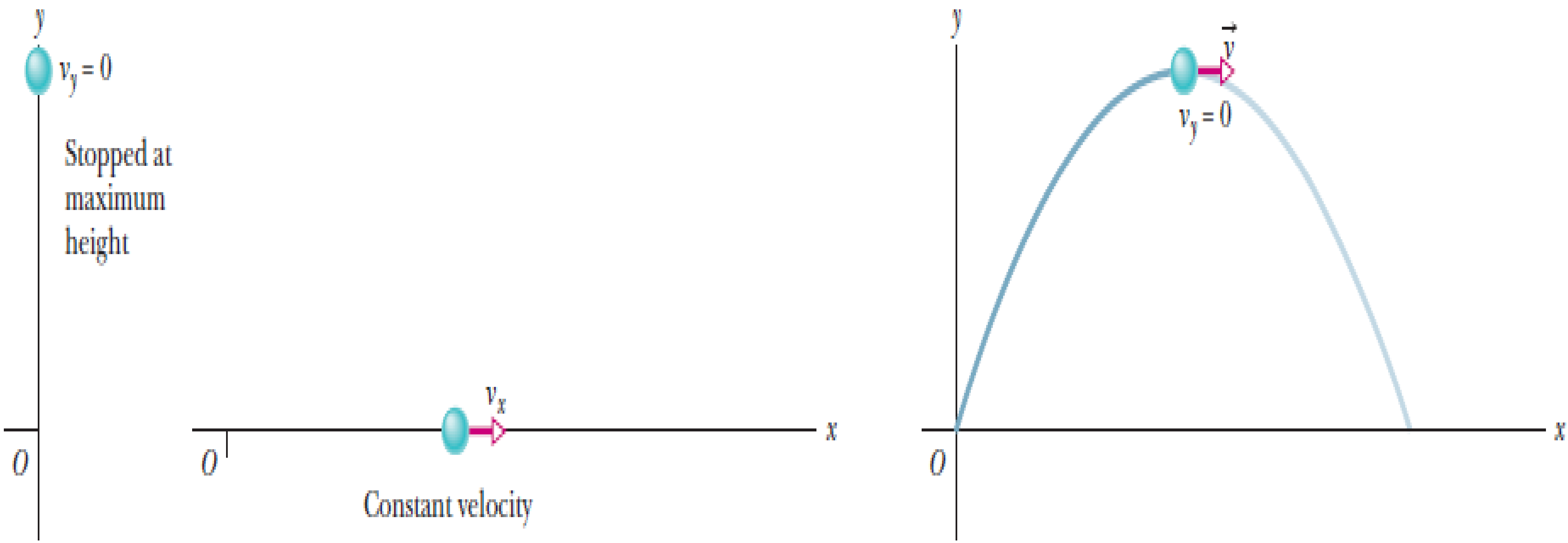


Fig 1(c)

Pictorial Representation(cont'd)

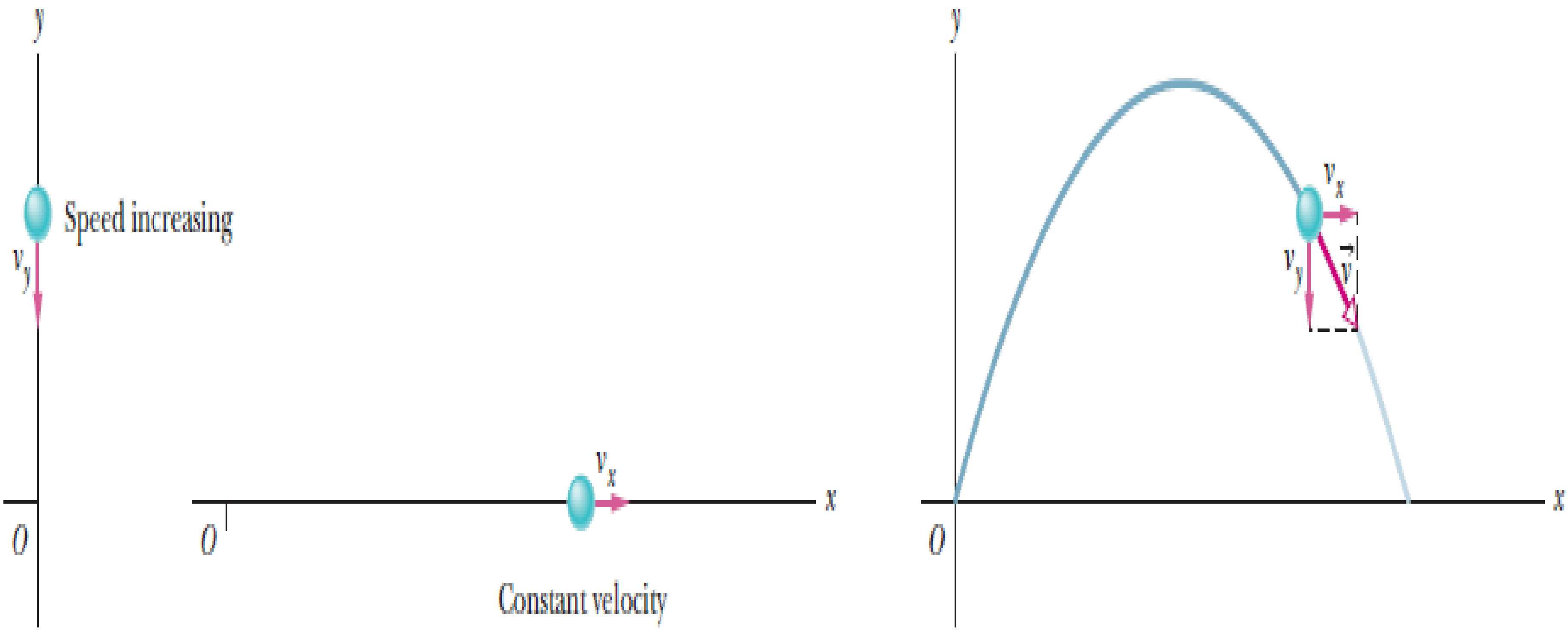


Fig 1(d)

Pictorial Representation(cont'd)

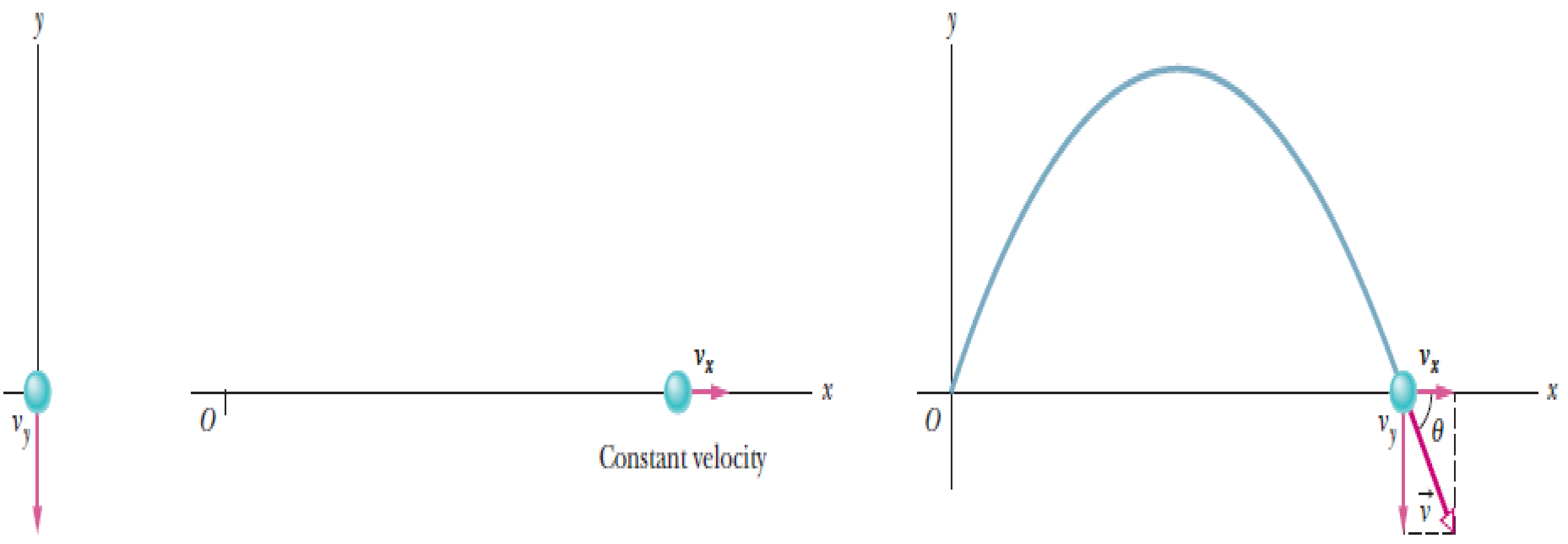


Fig 1(e)

Brainstorming

- Question:
- At a certain instant, a fly ball has velocity $\vec{v} = 25\hat{\imath} - 4.9\,\hat{\jmath}$ (the x- axis is horizontal, the y- axis is upward, and $\vec{v}$ is in meters per second).Has the ball passed its highest point?
- Answer:
- The answer is NO the ball did not pass the highest point because we have already seen that velocity at highest point should be zero and the y-component is negative which shows that the speed of fly ball is low.

Analysis of Projectile Motion

The Horizontal Motion

The Vertical Motion

The Horizontal Motion

Because there is *no acceleration* in the horizontal direction, the horizontal component v_x of the projectile's velocity remains unchanged from its initial value v_{0x} throughout the motion, as demonstrated in Fig. 1(a) to 1(e)

At any time t, the projectile's horizontal displacement $x - x_0$ from an initial position x_0 is given by

$$x - x_0 = v_0 t + \tfrac{1}{2} a t^2.$$

where a=0

Therefore the above equation will become,

$$x - x_0 = v_{0x} t.$$

Because $v_{0x} = v_0 \cos \theta_0$, this becomes

$$\boxed{x - x_0 = (v_0 \cos \theta_0) t.}$$

The Vertical Motion

- The vertical motion is a motion of a particle in a free fall.
- The acceleration in free fall remains constant
- We substitute "negative acceleration" because the direction of motion of the body is against the gravity.
- As the motion is vertical or in y-axis we switch to the y-component notation, therefore

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2$$

$$y - y_0 = (v_0 \sin \theta_0)t - \frac{1}{2}gt^2 \longrightarrow \textbf{Eq (i)}$$

The Vertical Motion(cont'd)

where the initial vertical velocity component v_{0y} is replaced with the equivalent $v_0 \sin\theta_0$.

By using first equation of motion in y-direction

$$v_y = v_{oy} + (-gt)$$

$$v_y = v_{oy} - gt$$

$$v_y = v_o \sin\theta_o - gt \longrightarrow \textbf{Eq (ii)}$$

The Vertical Motion(cont’d)

$$v^2{}_y = (v_o \sin\theta_o - gt)^2$$

$$v^2{}_y = (v_o \sin\theta_o)^2 - 2(v_o \sin\theta_o)(gt) + (gt)^2$$

$$v^2{}_y = (v_o \sin\theta_o)^2 - 2g(v_o \sin\theta_o\, t + \frac{1}{2} g t^2)$$

$$\boxed{v^2{}_y = (v_o \sin\theta_o)^2 - 2g(y - y_o)}$$

The Vertical Motion(cont'd)

$$\boxed{v_y = v_0 \sin\theta_0 - gt}$$

$$\boxed{v^2{}_y = (v_o \sin\theta_o)^2 - 2g(y - y_o)}$$

Conclusion

the vertical velocity component behaves just as for a ball thrown vertically upward. It is directed upward initially, and its magnitude steadily decreases to zero, *which marks the maximum height of the path.* The vertical velocity component then reverses direction, and its magnitude becomes larger with time.

Equation of trajectory of Projectile

We can find the equation of the projectile's path (its **trajectory**) by eliminating time t

$$x - x_0 = (v_0 \cos \theta_0)t. \qquad \text{........(Eq iii)}$$

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2$$

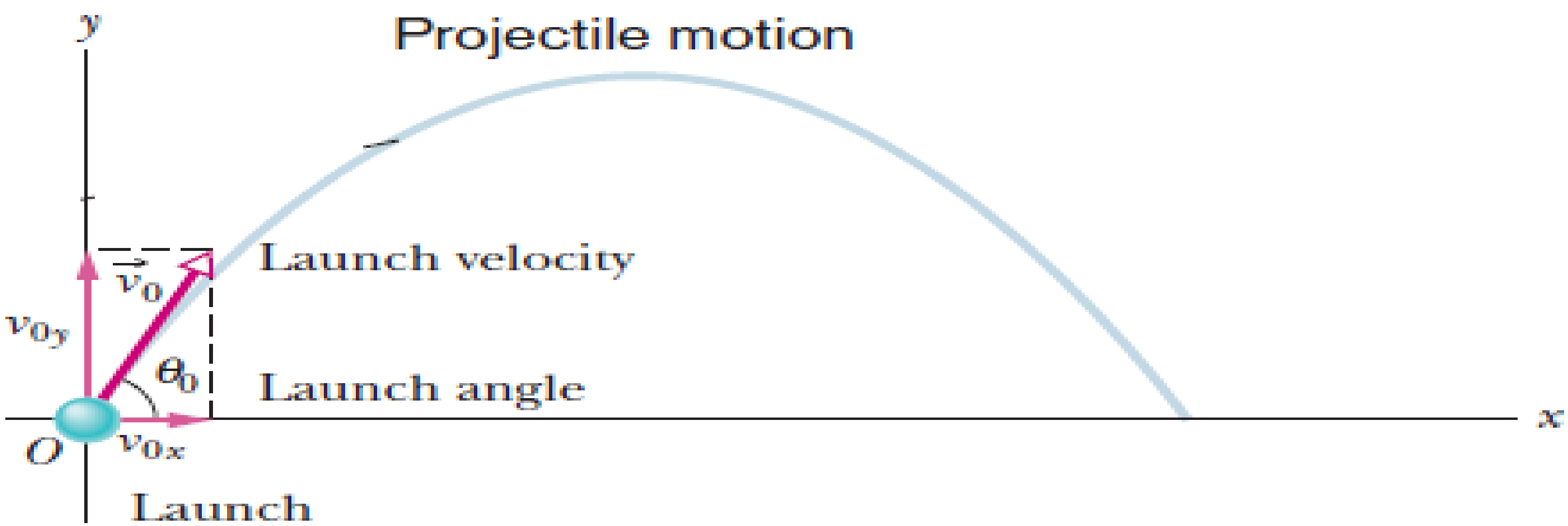
$$= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2 \text{ ....(Eq iv)}$$

Solving Eq. iii for t and substituting into

Eq. iv we obtain, after a little rearrangement.

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2} \qquad \text{(trajectory)} \longrightarrow \text{(Eq v)}$$

Equation of trajectory of Projectile



This is the equation of the path shown in Figure In deriving it, for simplicity we let $x_0 = 0$ and $y_0 = 0$ in Eqs. iii and iv respectively. Because g, θ_0, and v_0 are constants, Eq. v is of the form $y = ax + bx^2$, in which a and b are constants. This is the equation of a parabola, so the path is *parabolic*.

The Horizontal Range

The *horizontal range* R of the projectile is the *horizontal* distance the projectile has traveled when it returns to its initial height (the height at which it is launched). To find range R, let us put

$$x - x_0 = R$$

in Eq. $x - x_0 = (v_0 \cos \theta_0)t$.

$$R = (v_0 \cos \theta_0)t \quad \text{......(Eq vi)}$$

and $y - y_0 = 0$ in Eq. $y - y_0 = (v_0 \sin \theta_0)t - \frac{1}{2}gt^2$,

obtaining $$0 = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2. \quad \text{........(Eq vii)}$$

Eliminating t between these two equations yields

$$R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0.$$

Using the identity $\sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0$

we obtain

$$R = \frac{v_0^2}{g} \sin 2\theta_0. \quad \text{......(Eq viii)}$$

The Horizontal Range(Cont'd)

Caution: This equation does *not* give the horizontal distance traveled by a projectile when the final height is not the launch height.

Note that R in Eq viii has its maximum value when $\sin 2\theta_0 = 1$, which corresponds to $2\theta_0 = 90°$ or $\theta_0 = 45°$.

The horizontal range R is maximum for a launch angle of 45°.

However, when the launch and landing heights differ, as in shot put, hammer throw, and basketball, a launch angle of 45° does not yield the maximum horizontal distance.

Effects of Air on a Projectile

- We have assumed that the air through which the projectile moves has no effect its motion.
- However, in many situations, the disagreement between our calculations and the actual motion of the projectile can be large because the air resists the motion.
- As an example figure below shows two paths for a fly ball that leaves the bat at an angle of 60° with the horizontal and an initial speed of 44.7 m/s.

Graphical View

- Path I (the baseball player's fly ball) is a calculated path that
- approximates normal conditions of play, in air. Path II (the physics professor's fly ball) is the path the ball would follow in a vacuum.

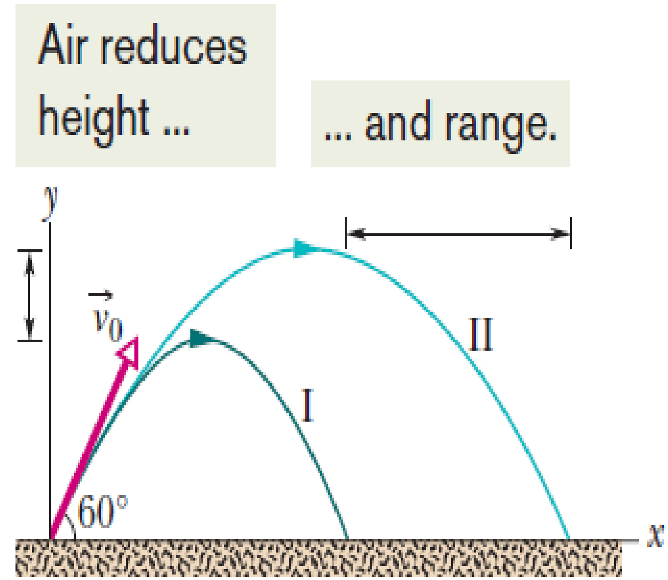


Figure (I) The path of a fly ball calculated by taking air resistance into account. (II) The path the ball would follow in a vacuum, calculated by the methods of this chapter.

Corresponding data of the fly balls

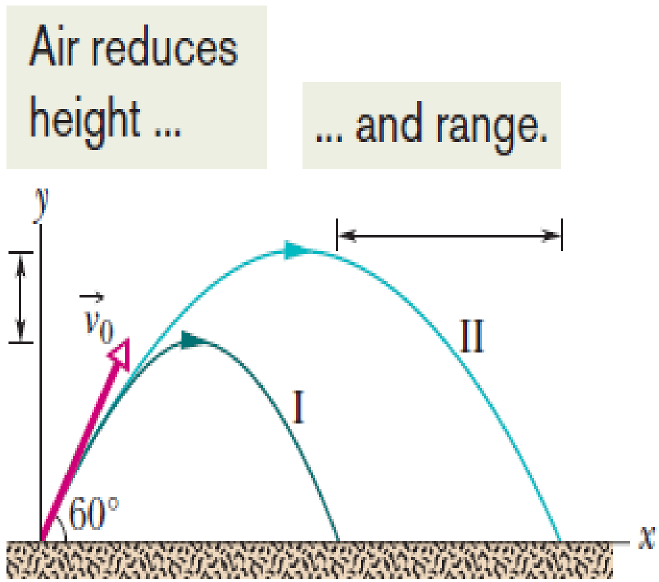


Figure (I) The path of a fly ball calculated by taking air resistance into account. (II) The path the ball would follow in a vacuum, calculated by the methods of this chapter.

Table 1 Two Fly Balls[a]

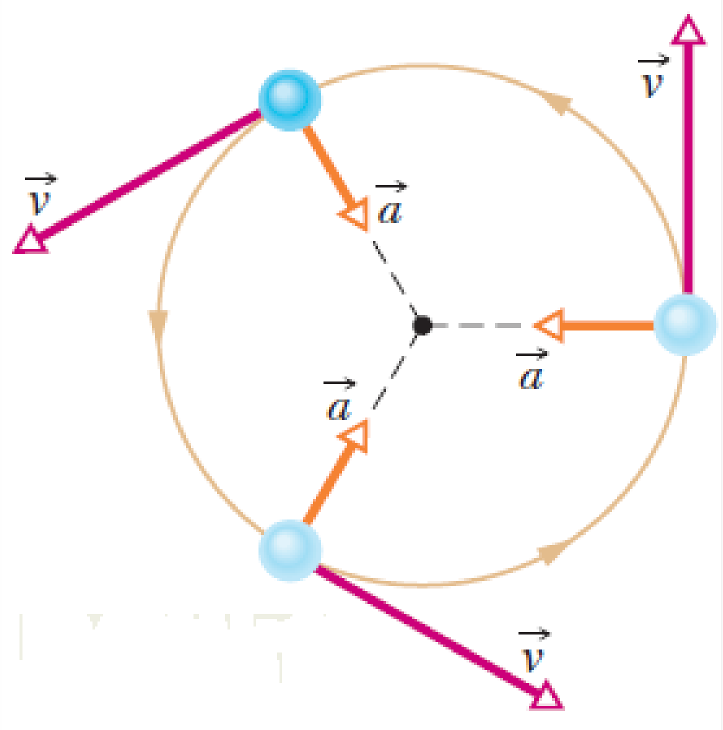
	Path I (Air)	Path II (Vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s

The launch angle is 60° and the launch speed is 44.7 m/s.

Uniform Circular Motion

$$a = \frac{v^2}{r} \quad \text{(centripetal acceleration)},$$

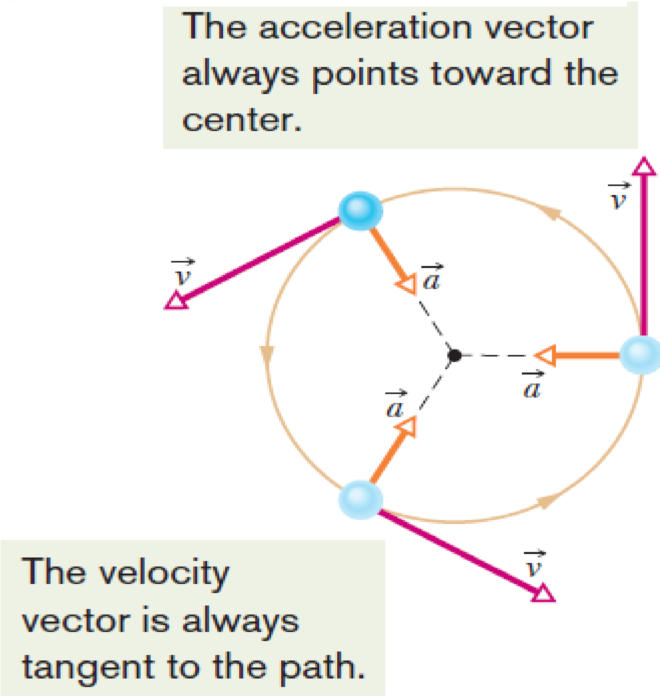
$$T = \frac{2\pi r}{v} \quad \text{(period)}.$$



Uniform Circular Motion

A particle is in **uniform circular motion** if it travels around a circle or a circular arc at constant (*uniform*) speed. Although the speed does not vary, *the particle is accelerating* because the velocity changes in direction.

Figure shows the relationship between the velocity and acceleration vectors at various stages during uniform circular motion.



Velocity and acceleration vectors for uniform circular motion.

Uniform Circular Motion(Cont'd)

- Both "velocity & acceleration vectors" have constant magnitude, but their directions change continuously.
- The velocity is always directed **tangent to the circle** in the direction of motion.
- The acceleration is always directed **radially inward.** Because of this, the acceleration associated with uniform circular motion is called a **centripetal acceleration.**
- Centripetal means centre seeking.
- Mathematically, the magnitude of acceleration is,

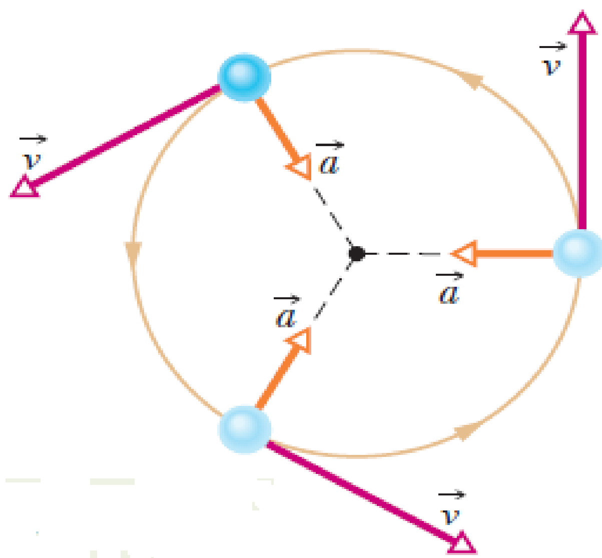
$$a_c = \frac{v^2}{r}$$

Uniform Circular Motion(Cont'd)

- During this acceleration at constant speed, the particle travels the circumference of the circle (a distance of $2\pi r$) time t.

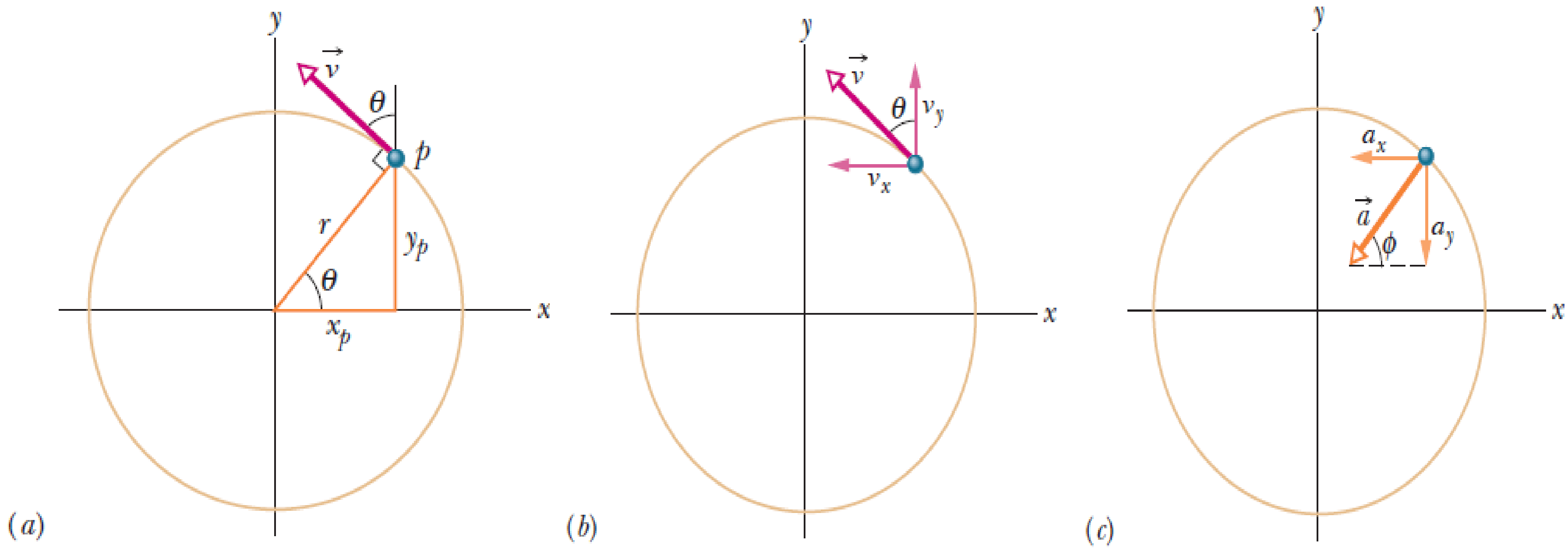
$$T = \frac{2\pi r}{v} \text{ (period)}$$

(the time for a particle to go around a closed path exactly once)



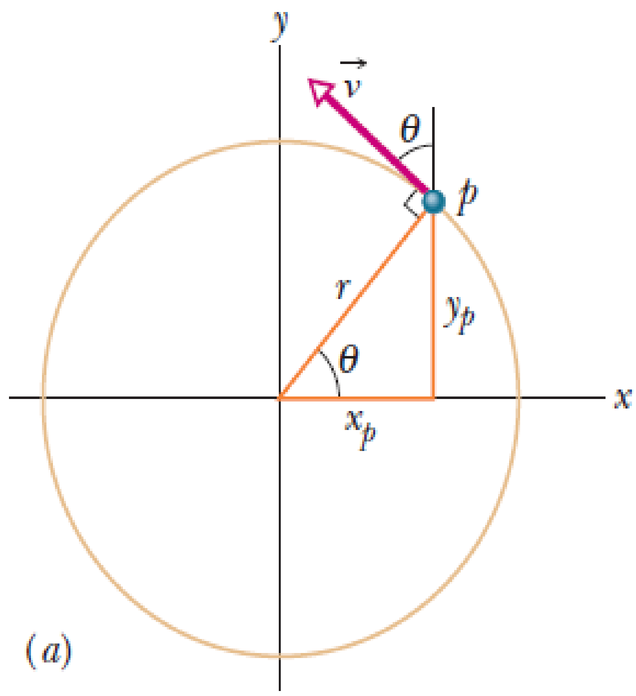
Proof : $a_c = \frac{v^2}{r}$

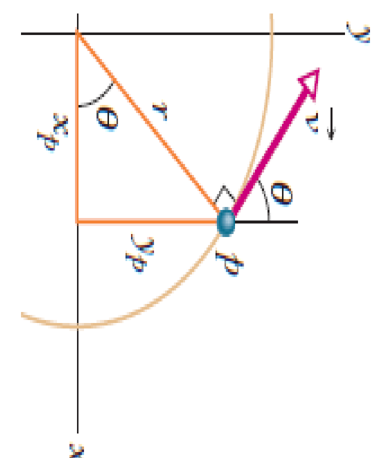
- To find the magnitude and direction of the acceleration for uniform circular motion, we consider the following figures;



Proof (Cont'd)

- In Fig. a, particle ***P*** moves at constant speed v around a circle of radius r.
- At the instant shown, ***P*** has x_p *and* y_p coordinates
- v is tangent to the path hence perpendicular to a radius r drawn to the particle's position. Then the angle (theta) that v makes with a vertical at P equals the angle (theta) that radius r makes with the x axis.

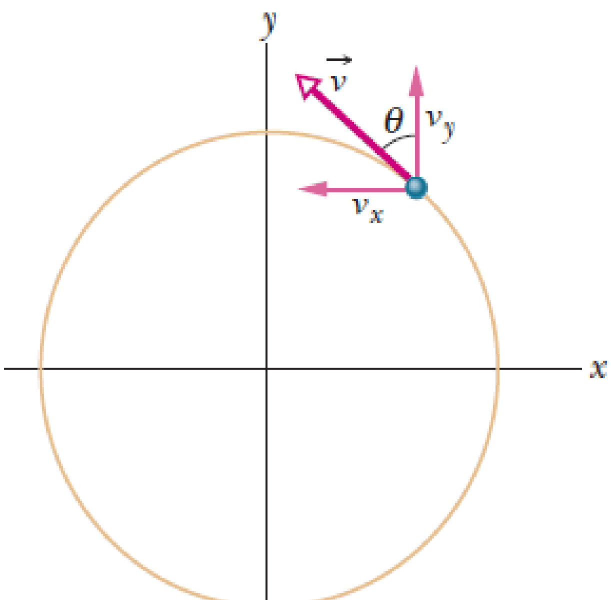




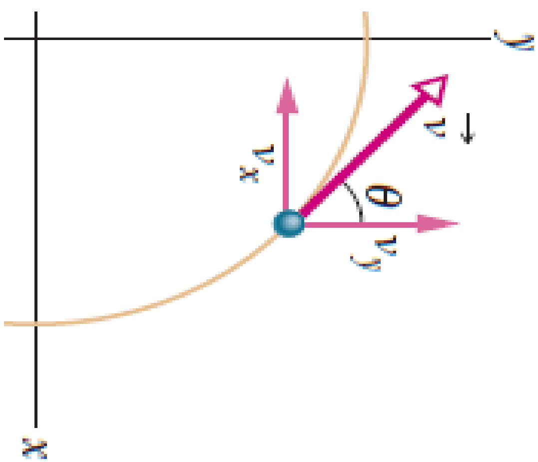
Proof (Cont'd)

- The scalar components of v are shown in Fig b. With them, we can write the velocity "v" as

$$\vec{v} = v_x\hat{\text{i}} + v_y\hat{\text{j}}$$
$$\vec{v} = (-v\sin\theta)\hat{\text{i}} + (v\cos\theta)\hat{\text{j}}$$



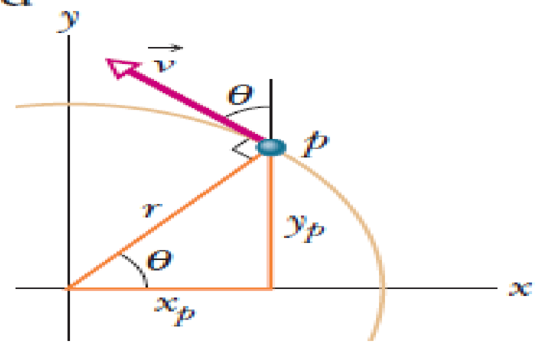
(b)



Proof (Cont'd)

Now, using the right triangle in Fig. a, we can replace $\sin\theta$ with y_p/r and $\cos\theta$ with x_p/r to write

$$\vec{v} = \left(-\frac{v y_p}{r}\right)\hat{\mathrm{i}} + \left(\frac{v x_p}{r}\right)\hat{\mathrm{j}}$$



To find the acceleration $\vec{a}$ of particle p, we must take the time derivative of this equation. Noting that speed v and radius r do not change with time, we obtain

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d}{dt}\left(-\frac{v y_p}{r}\right)\hat{\imath} + \frac{d}{dt}\left(\frac{v x_p}{r}\right)\hat{\jmath}$$

$$= \left(-\frac{v}{r}\frac{dy_p}{dt}\right)\hat{\mathrm{i}} + \left(\frac{v}{r}\frac{dx_p}{dt}\right)\hat{\mathrm{j}}. \longrightarrow \textbf{Eq (i)}$$

Proof (Cont'd)

Now note that the rate dy_p/dt at which y_p changes is equal to the velocity component v_y. Similarly, $dx_p/dt = v_x$, and, from figure (b) we have the further substitutions

Making these substitutions in Eq (i) i.e.,

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_p}{dt}\right)\hat{\mathrm{i}} + \left(\frac{v}{r}\frac{dx_p}{dt}\right)\hat{\mathrm{j}}$$

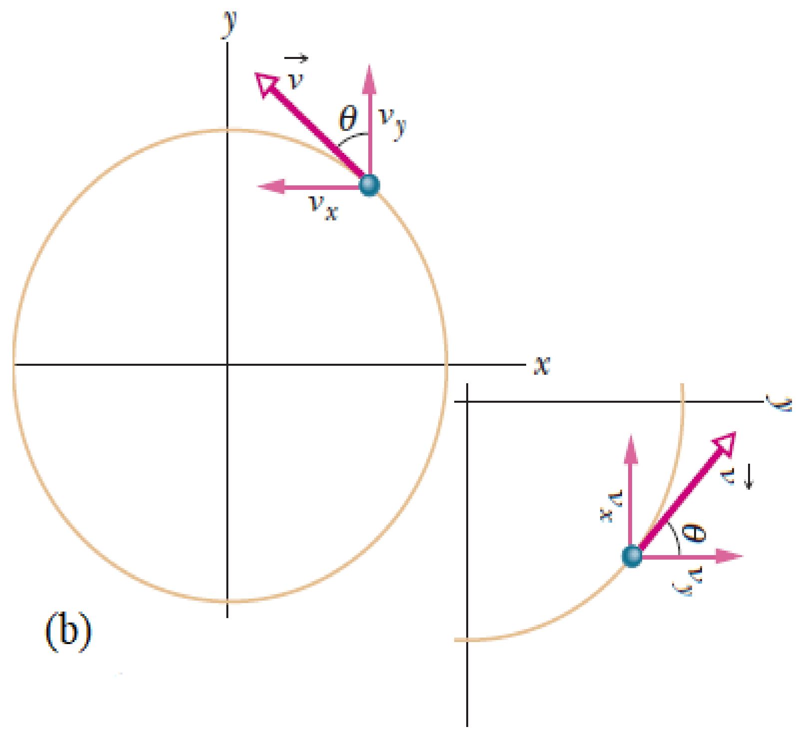
$$\vec{a} = \frac{d}{dt}\left(-\frac{vv_y}{r}\right)\hat{\imath} + \frac{d}{dt}\left(\frac{vv_x}{r}\right)\hat{\jmath}$$

$$v_x = -v\sin\theta$$

and

$$v_y = v\cos\theta$$

$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{\mathrm{i}} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{\mathrm{j}}$$



(b)

This vector and its components are shown in Fig. *c*.

Proof (Cont'd)

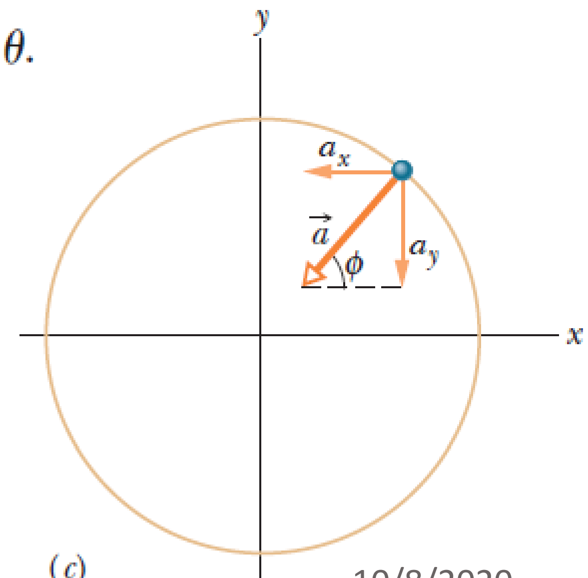
$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{\mathrm{i}} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{\mathrm{j}}.$$

$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \frac{v^2}{r}\sqrt{1} = \frac{v^2}{r},$$

as we wanted to prove. To orient $\vec{a}$, we find the angle ϕ shown in Fig. c

$$\tan\phi = \frac{a_y}{a_x} = \frac{-(v^2/r)\sin\theta}{-(v^2/r)\cos\theta} = \tan\theta.$$

Thus, $\phi = \theta$, which means that $\vec{a}$ is directed along the radius r of Fig. a, toward the circle's center, as we wanted to prove.



(c)