PROBLEM SET 1.6

1–3 FAMILIES OF CURVES

Represent the given family of curves in the form $G(x, y; c) = 0$ and sketch some of the curves.

1. All ellipses with foci -3 and 3 on the x-axis.

2. All circles with centers on the cubic parabola $y = x^3$ and passing through the origin $(0, 0)$.

3. The catenaries obtained by translating the catenary $y = \cosh x$ in the direction of the straight line $y = x$.

4–10 ORTHOGONAL TRAJECTORIES (OTs)

Sketch or graph some of the given curves. Guess what their OTs may look like. Find these OTs.

4. $y = x^2 + c$ **5.** $y = cx$

6. $xy = c$ **7.** $y = c/x^2$

8. $y = \sqrt{x + c}$ **9.** $y = ce^{-x^2}$

10. $x^2 + (y - c)^2 = c^2$

11–16 APPLICATIONS, EXTENSIONS

11. Electric field. Let the electric **equipotential lines** (curves of constant potential) between two concentric cylinders with the z-axis in space be given by $u(x, y) = x^2 + y^2 = c$ (these are circular cylinders in the xyz-space). Using the method in the text, find their orthogonal trajectories (the curves of electric force).

12. Electric field. The lines of electric force of two opposite charges of the same strength at $(-1, 0)$ and $(1, 0)$ are the circles through $(-1, 0)$ and $(1, 0)$. Show that these circles are given by $x^2 + (y - c)^2 = 1 + c^2$. Show that the **equipotential lines** (which are orthogonal trajectories of those circles) are the circles given by $(x + c^*)^2 + \tilde{y}^2 = c^{*2} - 1$ (dashed in Fig. 25).

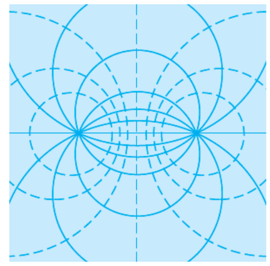

Fig. 25. Electric field in Problem 12

13. Temperature field. Let the **isotherms** (curves of constant temperature) in a body in the upper half-plane $y > 0$ be given by $4x^2 + 9y^2 = c$. Find the orthogonal trajectories (the curves along which heat will flow in regions filled with heat-conducting material and free of heat sources or heat sinks).

14. Conic sections. Find the conditions under which the orthogonal trajectories of families of ellipses $x^2/a^2 + y^2/b^2 = c$ are again conic sections. Illustrate your result graphically by sketches or by using your CAS. What happens if $a \to 0$? If $b \to 0$?

15. Cauchy–Riemann equations. Show that for a family $u(x, y) = c = \text{const}$ the orthogonal trajectories $v(x, y) = c^* = \text{const}$ can be obtained from the following *Cauchy–Riemann equations* (which are basic in complex analysis in Chap. 13) and use them to find the orthogonal trajectories of $e^x \sin y = \text{const}$. (Here, subscripts denote partial derivatives.)

$$u_x = v_y, \qquad u_y = -v_x$$

16. Congruent OTs. If $y' = f(x)$ with f independent of y, show that the curves of the corresponding family are congruent, and so are their OTs.

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1. $x^2/(c^2 + 9) + y^2/c^2 - 1 = 0$ **3.** $y - \cosh(x - c) - c = 0$

5. $y/x = c,\ y'/x = y/x^2,\ y' = y/x,\ \tilde{y}' = -x/\tilde{y},\ \tilde{y}^2 + x^2 = \tilde{c}$, circles

7. $2\tilde{y}^2 - x^2 = \tilde{c}$ **9.** $y' = -2xy,\ \tilde{y}' = 1/(2x\tilde{y}),\ x = \tilde{c}e^{\tilde{y}^2}$

11. $\tilde{y} = \tilde{c}x$

13. $y' = -4x/9y$. Trajectories $\tilde{y}' = 9\tilde{y}/4x,\ \tilde{y} = \tilde{c}x^{9/4}$ $(\tilde{c} > 0)$. Sketch or graph these curves.

15. $u = c,\ u_x\,dx + u_y\,dy = 0,\quad y' = -u_x/u_y$. Trajectories $\tilde{y}' = u_{\tilde{y}}/u_x$. Now $v = \tilde{c},\ v_x\,dx + v_y\,dy = 0,\quad y' = -v_x/v_y$. This agrees with the trajectory ODE in u if $u_x = v_y$ (equal denominators) and $u_y = -v_x$ (equal numerators). But these are just the Cauchy–Riemann equations.