Assignment #3.

Roll no = 20K-0409. BS(CS)

Ques 1 - 4. Find Laplace.

1) $F(t) = \begin{cases} e^t & t \leq 2 \\ 3 & t > 2 \end{cases}$

Sol: $\int_0^2 e^{-st}(e^t)\,dt + \int_2^\infty e^{-st}(3)\,dt.$

$$\int_0^2 e^{t-st}\,dt + 3\int_2^\infty e^{-st}\,dt.$$

$$= \int_0^2 e^{t(1-s)}\,dt + 3\int_2^\infty e^{-st}\,dt$$

$$= \lim_{b\to 2} \left.\frac{e^{t(1-s)}}{1-s}\right|_0^b + 3\lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_2^b$$

$$= \left(\frac{e^{2(1-s)}}{1-s} - \frac{1}{1-s}\right) - 3\left(\frac{e^{-\infty s}}{s} + \frac{e^{-2s}}{s}\right).$$

$$= \frac{e^{2(1-s)}}{1-s} - \frac{1}{1-s} - \frac{3e^{-2s}}{s}.$$

$$= \boxed{\frac{e^{2(1-s)} - 1}{1-s} - \frac{3e^{-2s}}{s}} \text{ // Ans.}$$

2. $F(t) = 3 + 2t^2$. $\quad \because \mathcal{L}(1) = 1/s$.

$\mathcal{L}\{F(t)\} = \mathcal{L}(3) + 2\mathcal{L}(t^2)$ $\quad \because \mathcal{L}(t^2) = \frac{2!}{s^3}$

$= \frac{3}{s} + \frac{2 \cdot 2!}{s^3}$.

$\boxed{= \frac{3}{s} + \frac{4}{s^3}}$ Ans.

3. $5\sin 3t - 17e^{-2t}$.

$= 5\mathcal{L}(\sin 3t) - 17\mathcal{L}(e^{-2t})$

$= 5\left(\frac{9}{s^2+9}\right) - 17 \cdot \frac{1}{s+2}$.

$= \boxed{\frac{45}{s^2+9} - \frac{17}{s+2}}$ Ans.

4. te^{4t}.

$= \int_0^\infty te^{4t} \cdot e^{-st}\, dt$.

$= \int_0^\infty te^{t(4-s)}\, dt$.

$= uv - \int v\, du$. $\quad \because u = t$, $du = dt$.

$$= \int_0^\infty \frac{e^{(4-s)t}}{4-s}\,dt - \int_0^\infty v\,du \qquad \because du = e^{t(4-s)}dt$$

$$v = \frac{e^{t(4-s)}}{(4-s)}$$

$$= \frac{e^{(4-s)t}}{4-s}\Big|_0^\infty - \frac{e^{(4-s)t}}{4-s}\,dt\Big|_0^\infty$$

$$= -0 + 0 \mp 0 + \frac{1}{(s-4)^2}$$

$$= \frac{1}{(s-4)^2} \quad \text{Ans.}$$

Ques 1-4. Find inverse (laplace).

1. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+5)}\right\}$

Sol:- partial fraction. 1st step.

$$\frac{1}{"} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}.$$

$$A = \frac{1}{5}, \quad B = \frac{-s}{5}, \quad C = \frac{-2}{5}.$$

$$\frac{1}{s^2+2s+5} \therefore \frac{1}{5s} + \frac{s-2}{5(s^2+2s+5)}$$

$$= \frac{1}{5s} - \frac{1}{5}\frac{s+1}{[(s+1)^2+4]} - \frac{1}{5}\frac{1}{(s+1)^2+4}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{5s}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}$$

$$- \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= \frac{1}{5} - \frac{1}{5}e^{-t}\cos(2t) - \frac{1}{5}e^{-t}\frac{\sin(2t)}{2}.$$

$$= \frac{1}{5} - \frac{1}{5}e^{-t}\cos(2t) - \frac{1}{10}e^{-t}\sin(2t).$$

Ans.

Ques 2.

$$\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s+3)}\right\}.$$

1st step partial fraction-

$$7s-1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2).$$

$$7s-1 = A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2)$$

$$A = -4, \quad B = 15, \quad C = -11$$

$$-\mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{15}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{11}{s+3}\right\}$$

$$= 4e^{-t} + 15e^{-2t} - 11e^{-3t}$$

Ans.

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3. $\mathcal{L}^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

1st step partial fraction.

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$s^2+9s+2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2.$$

$$'' = A(s^2+2s-3) + B(s+3) + C(s^2+1-2s)$$

s^2 :- $A+C = 1.$

s :- $2A + B - 2C = 9. \rightarrow (2$

s^0 :- $-3A + 3B + C = 2 - (3$

Solving simultaneously equation we get,

$A = 2,\ B = 3,\ C = -1.$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \underline{2e^{t} + 3te^{t} - e^{-3t}}$$ Ans.

Q4. $\mathcal{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$

$$\frac{2s^2+10s}{(s+1)(s^2-2s+5)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s+1}$$

$$2s^2+10s = A(s^2+s)+B(s+1)+C(s^2-2s+5)$$

s^2:- $A+C=2$.

s:- $A+B-2C=10$.

s^0:- $B+5C=0$.

$A=3, \quad B=5, \quad C=-1.$

$$\mathcal{L}^{-1}\left\{\frac{3s+5}{s^2-2s+5} - \frac{1}{s+1}\right\}.$$

$$\frac{3s+5}{s^2-2s+5} : \frac{3s}{s^2-2s+5} + \frac{5}{s^2-2s+5} \quad \frac{1}{s^2-2s+5} = \frac{1}{(s+1)^2+4}.$$

$$= 3\left[\frac{(s-1)+1}{(s-1)^2+4}\right] + 5\cdot\frac{1}{(s-1)^2+4}$$

$$= 3\left[\frac{(s-1)}{(s-1)^2+4}\right] + \frac{8}{(s-1)^2+4}.$$

$$= 3\mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+4}\right] + 8\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+4}\right]$$

$$- \mathcal{L}^{-1}\left[\frac{1}{s+1}\right].$$

$$= 3e^t \cos(2t) + \frac{8e^t \sin(2t)}{2} - e^{-t}$$

$$= \underline{3e^t \cos(2t) + 4e^t \sin(2t) - e^{-t}}$$

Ans.

Ques 1-4

Solve by Laplace, confirm by analytical method

1. $y' - 5y = e^{5x}$; $y(0) = 0$

$\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{e^{5x}\}$.

$sY - y(0) - 5Y = \frac{1}{s-5}$

$Y(s-5) - 0 = 1/s-5$

$Y = 1/(s-5)^2$

<u>Analytical</u>

$y' - 5y = e^{5x}$

$\because P(x) = -5$, $x = 0$

$I.F = e^{\int -5dx} \rightarrow I.F = e^{-5x}$

$\Rightarrow e^{-5x} \cdot y = \int e^{-5x} \cdot e^{5x} + c.$

$y e^{-5x} = \int e^{0} + c.$

$y e^{-5x} = x + c. \quad \because x = 0.$

$\boxed{c = 0} \quad \boxed{y = x e^{5x}}$

proved.

Q: 2.

$y' + y = \sin x \; ; \; y(0) = 1$

$sY - y(0) + Y = \frac{1}{s^2+1}$

$sY - 1 + Y = \frac{1}{s^2+1}$

$Y(s+1) = \frac{1}{s^2+1} + 1$

$(s+1)Y = \frac{1+s^2+1}{s^2+1} \rightarrow Y = \frac{s^2+2}{(s+1)(s^2+1)}$

partial fraction

$s^2+1 = A(s^2+1) + (Bs+C)(s+1)$

$\because s = 0.$

$\because s = -1. \quad A = 3/2, \; B = -1/2$

$\because s = 1, \quad C = 1/2$

$$\mathcal{L}\{y\} = \frac{s^2+2}{(s+1)(s^2+1)} = \frac{3}{2(s+1)} - \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s^2+2}{(s+1)(s^2+1)}\right\}$$

$$= \frac{3}{2}e^{-x} - \frac{\cos x}{2} + \frac{\sin x}{2}$$

Analytical:-

$y' + y = \sin x$, $P(x) = 1$,

$I.F = e^{\int 1 dx}$

$I.F = e^{x}$

$$ye^{x} = \int e^{x}.\sin x + C$$

$$= \sin x . e^{x} - \int \cos x . e^{x}$$

$$= \sin x e^{x} - [\cos x \cdot e^{x} - \int - \sin x e^{x}]$$

$$= e^{x}\sin x - e^{x}\cos x - \int e^{x}\sin x.$$

$$\int e^{x}\sin x + \int e^{x}\sin x = e^{x}\sin x - e^{x}\cos x$$

$$2\left(\int e^{x}\sin x\right) = e^{x}\sin x - e^{x}\cos x$$

$$\int e^{x}\sin x = \frac{e^{x}\sin x - e^{x}\cos x}{2} + C$$

$e^{x}y = $ ''

$$e^{0}(1) = \frac{e^{0}\sin 0 - e^{0}\cos 0}{2} + C.$$

$c = 3/2$

$$e^{x} y = \frac{e^{x} \sin x}{2} - \frac{e^{x} \cos x}{2} + \frac{3}{2}.$$

$$\boxed{y = \frac{\sin x}{2} - \frac{\cos x}{2} + \frac{3}{2} e^{-x}}$$

Proved

Ques: 3-

$y'' - y' = 2x \; ; \; y(0) = 1, \; y'(0) = -2.$

$$s^{2} Y - s\, y(0) - y'(0) - sY + y(0) = \frac{2}{s^{2}}$$

$$(s^{2} - s) L(Y) - s + 2 + 1 = \frac{2}{s^{2}}$$

$$(s^{2} - s) L(Y) = \frac{s^{3} - 3s^{2} + 2}{s^{2}}$$

$$L(Y) = \frac{s^{3} - 3s^{2} + 2}{(s^{2} - s)(s^{2})}$$

$$= \frac{A}{s} + \frac{B}{s^{2}} + \frac{C}{s^{3}} + \frac{D}{s-1}.$$

$$s^{3} - 3s^{2} + 2 = As^{2}(s-1) + B(s^{2} - s) + C(s-1) + Ds^{3}$$

$\because s = 0 \rightarrow \boxed{C = -2.}$

$\because s = 1 \rightarrow \boxed{D = 0.}$

s^{3}:- $A + D = 1.$, $\boxed{A = 1}$

s^{2}:- $-B + C = 0.$ $\boxed{B = -2}$

$$\mathcal{L}\{y\} = \frac{1}{s} - \frac{2}{s^2} \quad \frac{-2}{s^3} + 0$$

$$\boxed{y = 1 - 2x - x^2}$$

Ans.

Analytical

$$y'' - y' = 2x$$

$$m^2 - m = 0, \quad m = 0, m = 1$$

$$y_c = c_1 e^{0x} + c_2 e^{x}$$

$$y_c = c_1 + c_2 e^{x} \quad , y_1 = 1, \; y_2 = e^{x}$$

$$\therefore y_p = x^r (Ax + B). \quad \boxed{r = 1}$$

$$y_p = Ax^2 + Bx$$

$$y_p' = 2Ax + B \quad , y_p'' = 2A.$$

$$\therefore 2A - 2Ax - B = 2x.$$

$$-2A = 2 \quad , \quad 2A - B = 0$$

$$\boxed{A = -1} \quad \boxed{B = -2}$$

$$\boxed{y_p = -x^2 - 2x}$$

$$y = c_1 + c_2 e^{x} - x^2 - 2x$$

$$1 = c_1 + c_2 e^{0} - (0)^2 - 0.$$

$$\boxed{c_1 + c_2 = 1} \quad , \quad y' = c_2 e^{x} - 2x - 2.$$

$$-2 = c_2 e^{0} - 0 - 2.$$

$$\boxed{c_1 = 1} \quad \boxed{c_2 = 0}$$

$$\boxed{y = 1 - x^2 - 2x} \quad \text{Proved}$$

24. $y'' - 2y' + 5y = -8e^{7-x}$

$y(7) = 2, \quad y'(7) = 12.$

Sol:- $x_0 = 7$, and $x = x_1 + 7$.

$y'' - 2y' + 5y = -8e^{7-(x_1+7)}$

$y'' - 2y' + 5y = -8e^{-x_1}$

$s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0) - 2s\,\mathcal{L}\{y\}$

$+ 2y(0) + 5\,\mathcal{L}\{y\} = \frac{-8}{s+1}$

$(s^2 - 2s + 5)\,\mathcal{L}\{y\} - 2s - 12 + 4 = \frac{-8}{s+1}$

$\{s^2 - 2s + 5\}\,\mathcal{L}\{y\} = \frac{-8}{s+1} + 2s + 8$

$\mathcal{L}\{y\} = \frac{\cancel{-8} + 2s^2 + 2s + 8s + \cancel{8}}{(s+1)(s^2 - 2s + 5)}$

$= \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 5}$

$\therefore s = -1, \; A = -1, \; A + B = 2, \; B = 3, \; SA + C = 0, \; \boxed{C = 5}$

$\mathcal{L}\{y\} = \frac{-1}{s+1} + \frac{3s+5}{s^2 - 2s + 5}.$

$$\frac{3s+5}{s^2-2s+5} = \frac{3s+5}{(s-1)^2+4}$$

$$= 3\left[\frac{s}{(s-1)^2+4}\right] + \frac{5}{(s-1)^2+4}$$

$$= 3\left[\frac{s-1}{(s-1)^2+4}\right] + 8 \cdot \frac{1}{(s-1)^2+4}$$

$$= 3e^{x'}\cos 2x' + 4e^{x'}\sin 2x'$$

$$\therefore \mathcal{L}\{y\} = \frac{-1}{s+1} + \frac{3s+5}{s^2-2s+5}$$

$$y = -e^{-x'} + 3e^{x'}\cos 2x + 4e^{x'}\sin 2x,$$

$$x' = x-7$$

$$y = -e^{7-x} + 3e^{x-7}\cos 2x + 4e^{x-7}\sin 2(x-7).$$

Analytical. $y'' - 2y' + 5y = -8^{7-x}$

$$m^2 - 2m + 5 = 0$$

$$m = 1+2i, \quad m = 1-2i$$

$$y_c = e^{x}(c_1\cos 2x + c_2\sin 2x).$$

Now, y_p $\quad y_p = x^r(Ae^{-x})$

$r=0$, $\quad y_p = Ae^{-x}$

$$y_p' = -Ae^{-x}$$

$$y_p'' = Ae^{-x}$$

$$Ae^{-x} - 2(-Ae^{-x}) + 5Ae^{-x} = -8e^{7}\cdot e^{-x}$$

$$8A = -8e^{7}$$

$$\boxed{A = -e^{7}}$$

$$y_P = -e^{7}\cdot e^{-x} \rightarrow y_P = -e^{7-x}$$

$$y = c_1 e^{x}\cos 2x + c_2 e^{x}\sin 2x - e^{7-x}$$

$$2 = c_1 e^{7}\cos 2(7) + c_2 e^{7}\sin 2(7) - e^{7-7}$$

assume $\cos 14 = 1$, $\sin 14 = 0$.

$$\boxed{3 = c_1 e^{7}}$$

$$y' = c_1(e^{x} - 2\sin 2x + e^{x}\cos 2x)$$
$$+ c_2(e^{x}\cdot 2\cos 2x + e^{x}\sin 2x)$$
$$+ e^{7-x}$$

$$12 = -2c_1 e^{7}\sin 14 + c_1 e^{7}\cos 14$$
$$+ 2c_2 e^{7}\cos 14 + c_2 e^{7}\sin 14 + e^{7-7}$$

$$11 = c_1 e^{7} + 2c_2 e^{7}$$

$$11 = 3 + 2c_2 e^{7}$$

$$4 = c_2 e^{7}$$

$$\boxed{y = 3e^{x-7}\cos 2(x-7) + 4e^{x-7}\sin 2(x-7) - e^{x-7}}$$

Proved