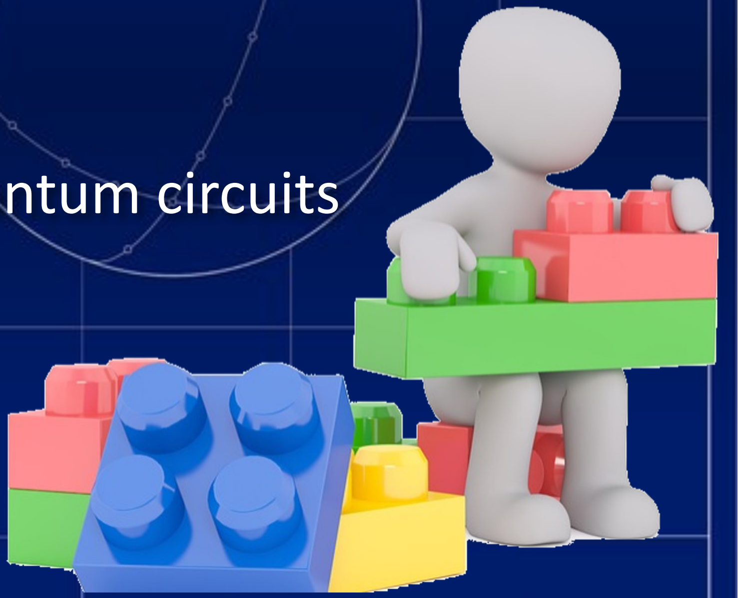
Qiskit | Global Summer School 2022

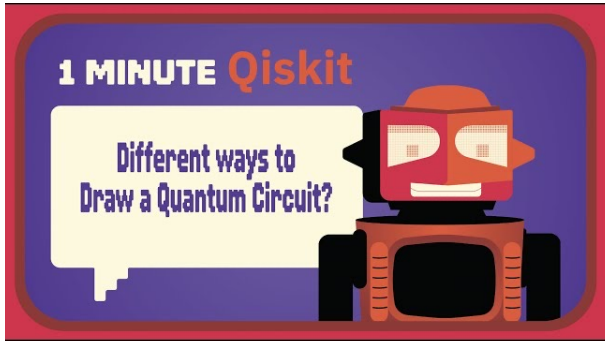
Lecture 2

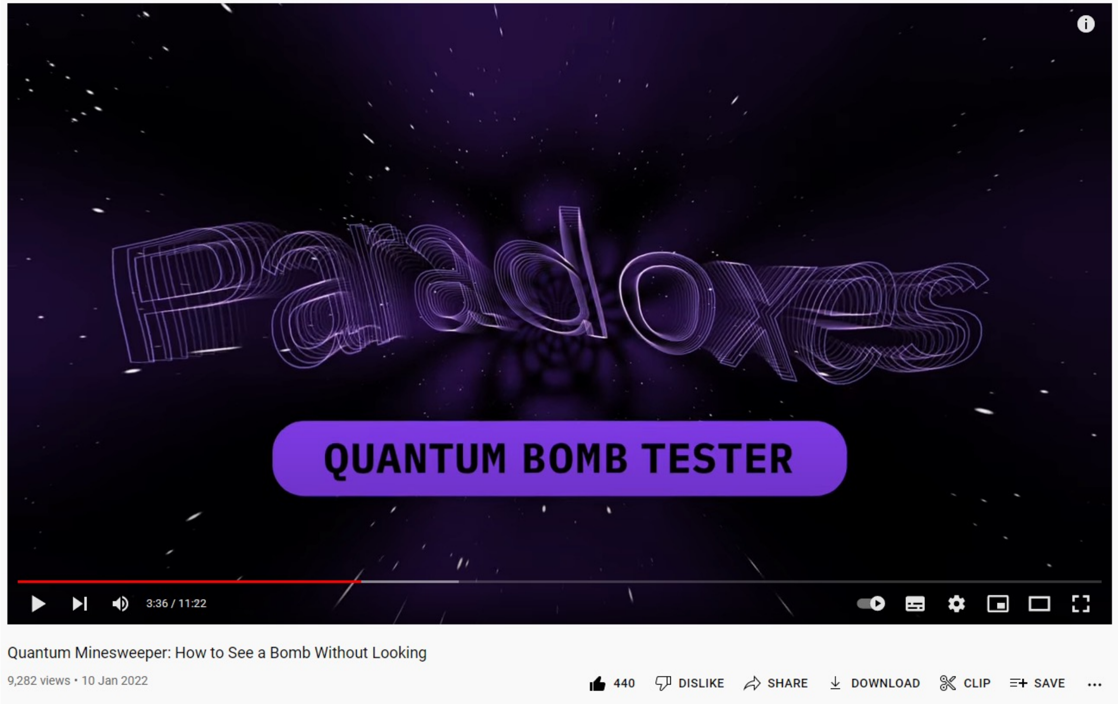
Building blocks of quantum: from linear algebra to quantum circuits



Quantum Simulation

Hello!


1 MINUTE Qiskit
Different ways to Draw a Quantum Circuit?


Paradoxes
QUANTUM BOMB TESTER
3:36 / 11:22
Quantum Minesweeper: How to See a Bomb Without Looking
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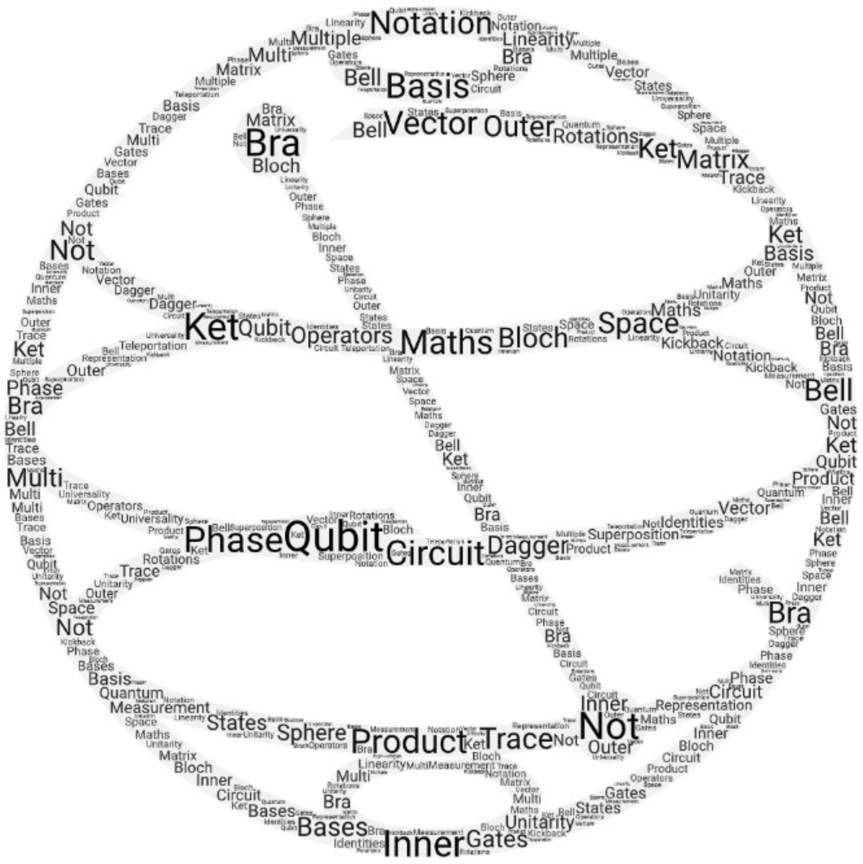
Building blocks of quantum

- Linear algebra
  - Vector space, basis, bra-ket notation, inner product, dagger, operators, outer product
- Single qubits
  - Bloch sphere, superposition, NOT, matrix representation, linearity, rotations, Y, Z, H, P, S, T, I, U, unitarity, measurement, bases, global and relative phases
- Multiple qubits
  - Multiple qubits, Bell states, multi-qubit gates, C-NOT, pure and mixed states, density matrices, universality, quantum teleportation



Vector space

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

1. $V_1 + V_2 = V_3$

2. $\alpha V_1 = V_2$

real real

$$\boxed{\alpha \begin{pmatrix} a \\ b \end{pmatrix} + \beta \begin{pmatrix} c \\ d \end{pmatrix}} = \begin{pmatrix} \overbrace{\alpha a + \beta c}^{\text{real}} \\ \underbrace{\alpha b + \beta d}_{\text{complex}} \end{pmatrix}$$

complex complex

linear combination.

Dimensions

$$v_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

complex.

$$\alpha v_1 + \beta v_2 = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

2-dimensional

$$u_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad u_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\alpha u_1 + \beta u_2 + \gamma u_3 = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

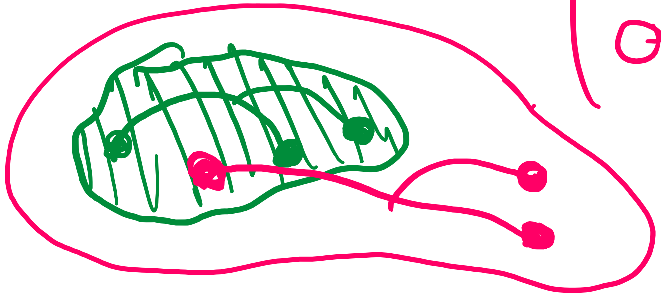
3-dimensional.

subspace:

$$u_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\alpha u_1 + \beta u_2 = \begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}$$

2-dimensional



Basis

$\alpha = \beta = \frac{1}{\sqrt{2}}$ $\quad \alpha = -\beta = \frac{1}{\sqrt{2}}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \text{basis}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \longrightarrow \text{basis}$$

$$\alpha \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

1 $\quad$ 0

$+\beta$

Inner product $= 0 \rightarrow$ orthogonal.

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}^{*T} = \begin{pmatrix} a \\ b \end{pmatrix}^{\dagger} = \begin{pmatrix} a^* & b^* \end{pmatrix}$$

$$\begin{pmatrix} a_1^* & \cdots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = a^* c + b^* d$$

$$= {}^* b_2 + \cdots + a_n^* b_n.$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & & 1 \end{pmatrix}$$

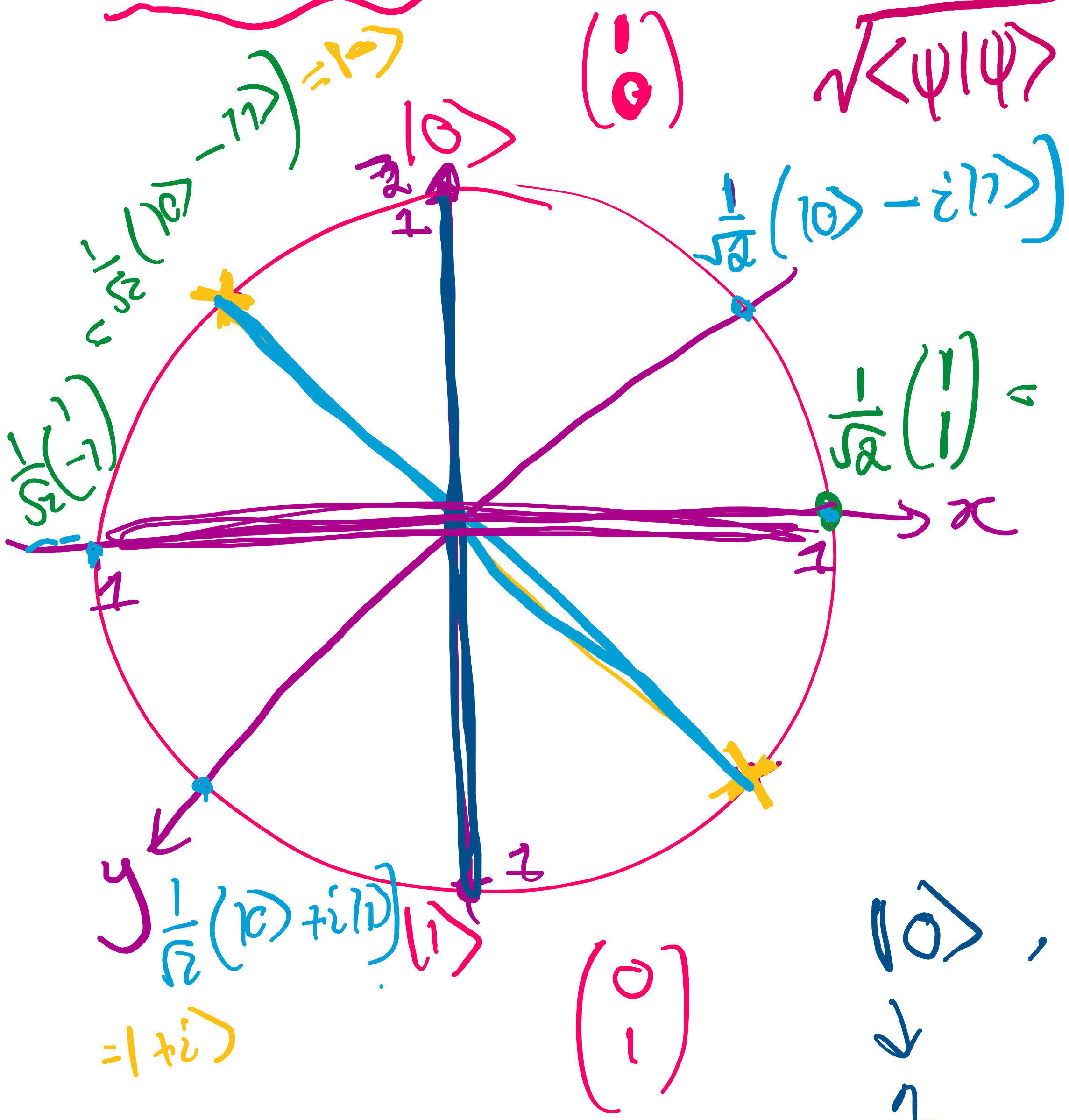
Bra-ket notation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \qquad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = |a\rangle$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{+} = \begin{pmatrix} 1 & 0 \end{pmatrix} = \langle 0| \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+} = \begin{pmatrix} 0 & 1 \end{pmatrix} = \langle 1| \qquad \begin{pmatrix} a_1^* & a_2^* & \cdots & a_n^* \end{pmatrix} = \langle a|$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad \begin{pmatrix} a_1^* & \cdots & a_n^* \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \langle a| \, |b\rangle = \boxed{\langle a|b\rangle}$$

Bloch sphere



$\alpha|0\rangle + \beta|1\rangle$

$\sqrt{\langle\psi|\psi\rangle} \rightarrow$ length

$\langle 0|0\rangle = (1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$

$\langle 1|1\rangle = 1$

$\langle\psi|\psi\rangle = 1$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

$\langle +|+\rangle = \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle + |1\rangle)$

$= \frac{1}{2}(1+1) = \frac{2}{2} = 1$

$|0\rangle, |1\rangle, |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\downarrow$ 1 $\quad \downarrow$ 1 $\quad \frac{1}{2}, \frac{1}{2}$

Measurement

$\underline{\alpha}|0\rangle + \underline{\beta}|1\rangle$

prob. $|0\rangle$: $|\alpha|^2$ prob. $|1\rangle$: $|\beta|^2$

$\underbrace{|\alpha|^2 + |\beta|^2 = 1}$

$\alpha|0\rangle + \beta|1\rangle$

$= \alpha\left(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\right) + \beta\left(\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right)$

$= \underbrace{\frac{1}{\sqrt{2}}(\alpha+\beta)}|+\rangle + \frac{1}{\sqrt{2}}(\alpha-\beta)|-\rangle$

prob. $|+\rangle$: $\frac{1}{2}|\alpha+\beta|^2$ prob. $|-\rangle$: $\frac{1}{2}|\alpha-\beta|^2$

$\langle +|(\alpha|0\rangle + \beta|1\rangle)$

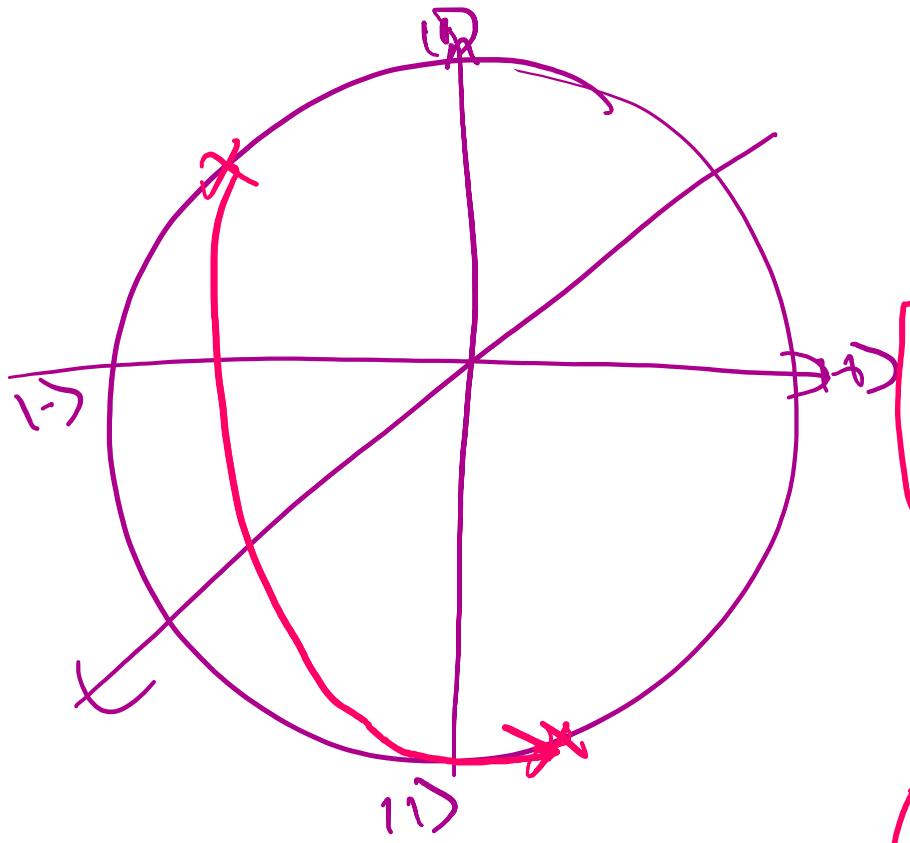
$= \alpha\langle +|0\rangle + \beta\langle +|1\rangle$

$= \alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{\sqrt{2}} = \overbrace{\frac{1}{\sqrt{2}}(\alpha+\beta)}$

$|\langle a|b\rangle|^2$

Quantum gates

Linear Operators



$$|\psi\rangle \longrightarrow |\psi'\rangle$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} a' \\ b' \end{pmatrix}$$

$$\boxed{M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}}$$

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m_1 a + m_2 b \\ m_3 a + m_4 b \end{pmatrix}$$

$$M \begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} a' \\ b' \end{pmatrix}$$

$$M|\psi\rangle = |\psi'\rangle$$

$$M\left(\frac{|\psi\rangle + |\phi\rangle}{\sqrt{2}}\right) = \frac{M|\psi\rangle + M|\phi\rangle}{\sqrt{2}}$$

Unitarity

$$U^\dagger U = \mathbb{1}$$

$$U(\theta,\phi,\lambda) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\phi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}$$

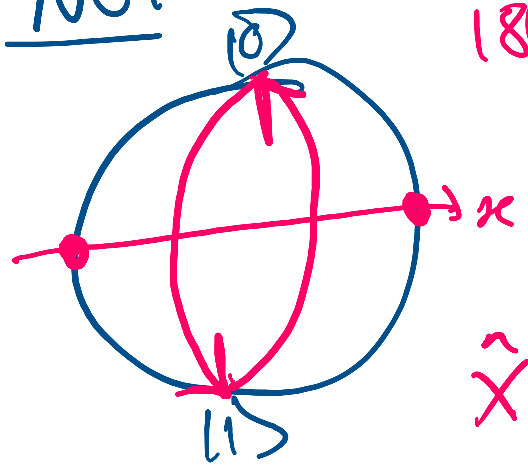
check!

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\langle\psi'|\psi'\rangle = \langle\psi|\underbrace{U^\dagger U}_{U^\dagger U = \mathbb{1}}|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \langle\psi|\psi\rangle = 1$$

Single qubit gates

NOT



180° about x

$$\hat{X}|0\rangle = |1\rangle$$

$$\hat{X}|1\rangle = |0\rangle$$

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ eigenvectors + eigenvalues

Y: 180° y $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ± 1

Z: 180° z $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ± 1

$$\hat{X}|+\rangle = \frac{1}{\sqrt{2}}X(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2}}(X|0\rangle + X|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$$

$$= |+\rangle \qquad +1$$

$$\hat{X}|-\rangle = \frac{X}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = -|-\rangle \qquad -1$$

global phase:
→ doesn't affect measurement

relative phase.

$$\hat{X}\Big(\underbrace{\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)}_{|0\rangle}\Big) = \underbrace{\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)}_{|1\rangle}$$

$e^{i\varphi}$

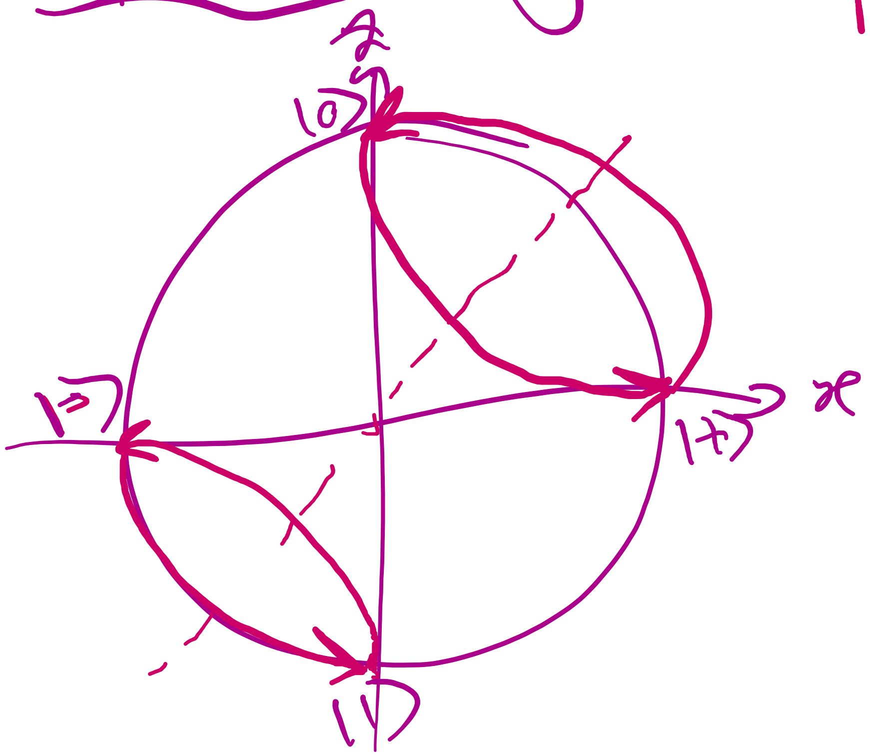
$$U|\psi\rangle = a|\psi\rangle$$

$$(U|\psi\rangle)^\dagger = \langle\psi|U^\dagger = \langle\psi|a^*$$

$$\langle\psi|\underbrace{U^\dagger U}_{\mathbb{1}}|\psi\rangle = aa^*\langle\psi|\psi\rangle = |a|^2 = 1$$

$$a = e^{i\varphi}$$

Hadamard gate



180° x-z axis

$\hat{H}|0\rangle = |+\rangle$

$\hat{H}|+\rangle = |0\rangle$

$\hat{H}|-\rangle = |1\rangle$

$\hat{H}|1\rangle = |-\rangle$

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$H = H^{-1}$

$HH|\psi\rangle = \mathbb{1}|\psi\rangle = |\psi\rangle$

Phase gate

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

Phase gate family tree!

ϕ: $-\pi$, $-\frac{\pi}{2}$, $-\frac{\pi}{4}$, 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, π

$-\pi$: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

$-\frac{\pi}{2}$: $\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = S^\dagger$

$-\frac{\pi}{4}$: $\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix} = T^\dagger$

0: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

$\frac{\pi}{4}$: $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = T$

$\frac{\pi}{2}$: $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = S$

π: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

Bra-ket for gates

Outer product: $|a\rangle\langle b| = |a\rangle\langle b|$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$X|0\rangle = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle$$

$$= \cancel{|0\rangle\underbrace{\langle 1|0\rangle}_{0}} + |1\rangle\underbrace{\langle 0|0\rangle}_{1}$$

$$= |1\rangle$$

$$\begin{cases} |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ |0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ |1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

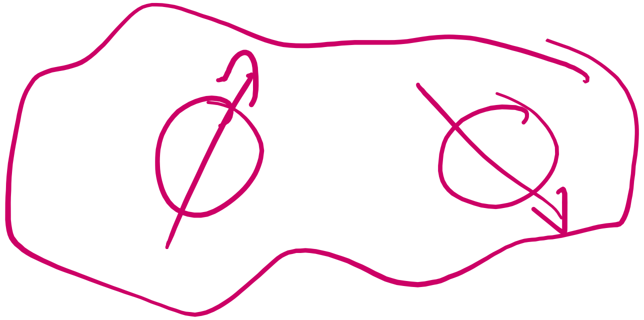
$$\alpha|0\rangle\langle 0| + \beta|0\rangle\langle 1| + \gamma|1\rangle\langle 0| + \delta|1\rangle\langle 1|$$

$$= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\alpha'|+\rangle\langle +| + \beta|+\rangle\langle -| + \gamma'|-\rangle\langle +| + \delta'|-\rangle\langle -|$$

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

Multiple qubits!



$$(\hat{X}\otimes\hat{Z})|01\rangle = |11\rangle$$

$$(\hat{X}\otimes\hat{Z})|0+\rangle = |1-\rangle$$

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad |b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$|a\rangle$, $|b\rangle$

Tensor product! $\otimes$

$$|a\rangle \otimes |b\rangle = |ab\rangle$$

$$|a\rangle\otimes|b\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

$$(A\otimes B)(|a\rangle\otimes|b\rangle) = A|a\rangle \otimes B|b\rangle$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} A_1\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} & A_2\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \\ A_3\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} & A_4\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \end{pmatrix}$$

2-qubit gates

C-NOT gate

controlled-NOT

CNOT , CX

$CX|00\rangle = |00\rangle$

$CX|01\rangle = |01\rangle$

$CX|10\rangle = |11\rangle$

$CX|11\rangle = |10\rangle$

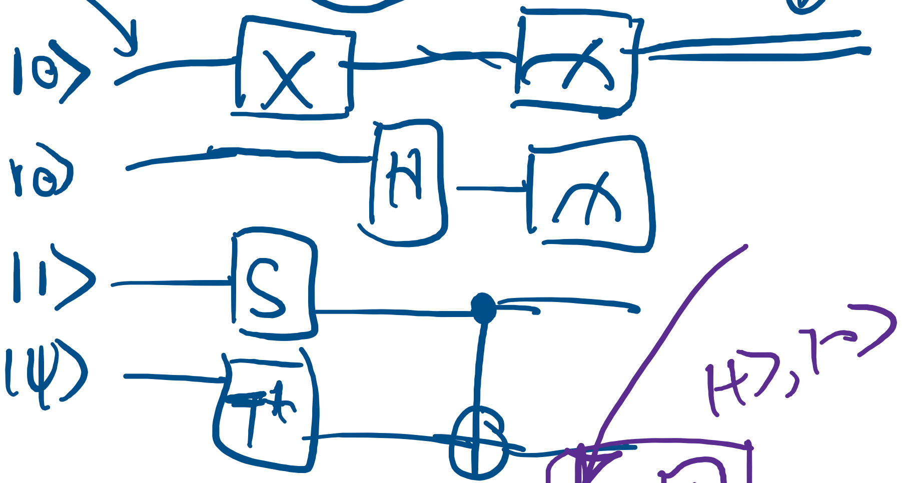
$|0\rangle, |1\rangle$

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

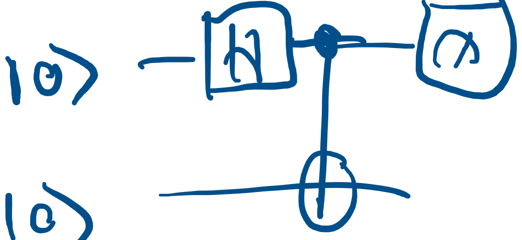
Quantum circuits

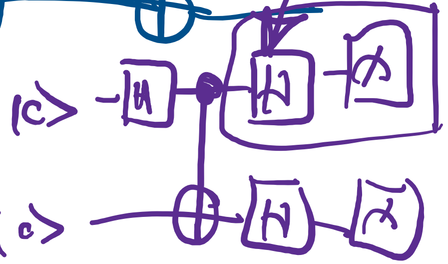
wires

classical



$|+\rangle, |-\rangle$





Maximally entangled!

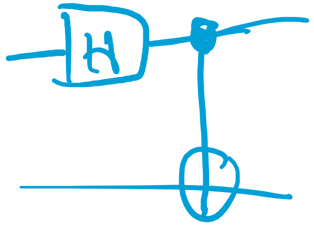
$$|00\rangle \xrightarrow{H} \underbrace{(|0\rangle+|1\rangle)|0\rangle}_{|00\rangle+|10\rangle} \xrightarrow{CNOT} |00\rangle + |11\rangle$$

$$|++\rangle + |--\rangle$$

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

Bell states

$|0\rangle$



$|0\rangle$

$|\phi^+\rangle = |00\rangle + |11\rangle$

$$\langle \phi_i | \phi_j \rangle = 0$$

$|0\rangle$

$|1\rangle$

$|\phi^-\rangle = |00\rangle - |11\rangle$

$|1\rangle$

$|0\rangle$

$|\psi^+\rangle = |01\rangle + |10\rangle$

$|1\rangle$

$|1\rangle$

$|\psi^-\rangle = |01\rangle - |10\rangle$

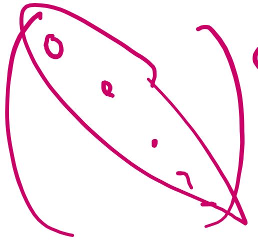
state vector

Pure and mixed states

$$\boxed{|\psi\rangle = |00\rangle + |11\rangle}$$

$$\rho = |\psi\rangle\langle\psi| = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|$$

density matrix



$$\sum_j \langle\psi|_j \, \rho_{12} \, |\psi\rangle_j$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

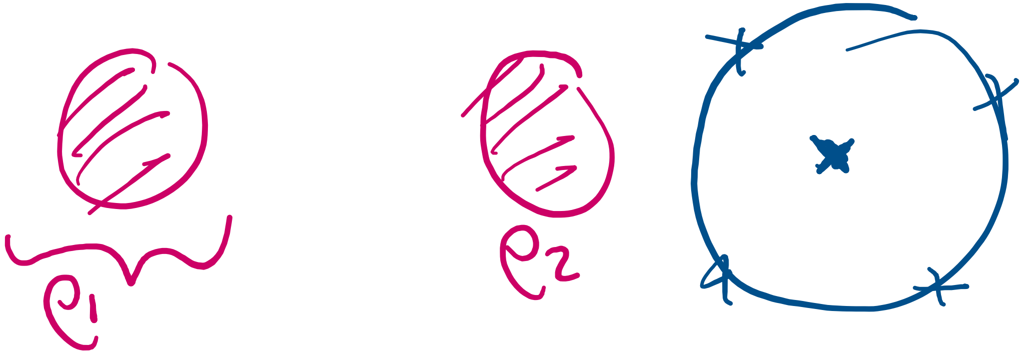
$$\underbrace{|0\rangle\langle 0|}\otimes\underbrace{|0\rangle\langle 0|}_{1}$$
$$+\, |0\rangle\langle 1| \otimes |0\rangle\langle 1| \leftarrow 0$$
$$+\, |1\rangle\langle 0| \otimes |1\rangle\langle 0| \leftarrow 0$$
$$+\, |1\rangle\langle 1| \otimes |1\rangle\langle 1| \leftarrow 1$$

mixed

$$\rho_1 = \mathrm{tr}_2\, \rho_{12} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \quad \boxed{\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

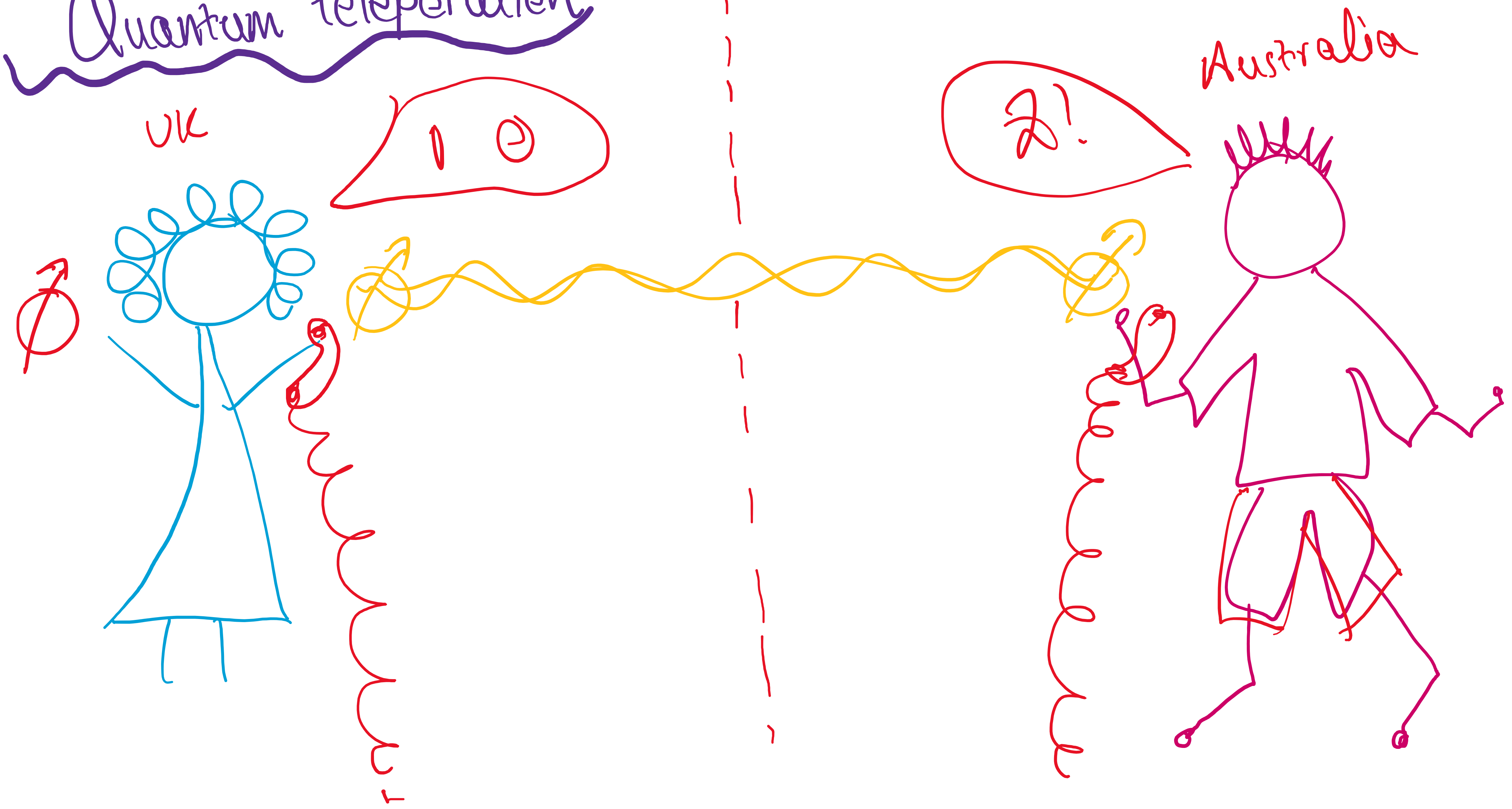
$$\rho_2 = \mathrm{tr}_1\, \rho_{12} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \quad \boxed{\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

mixed.

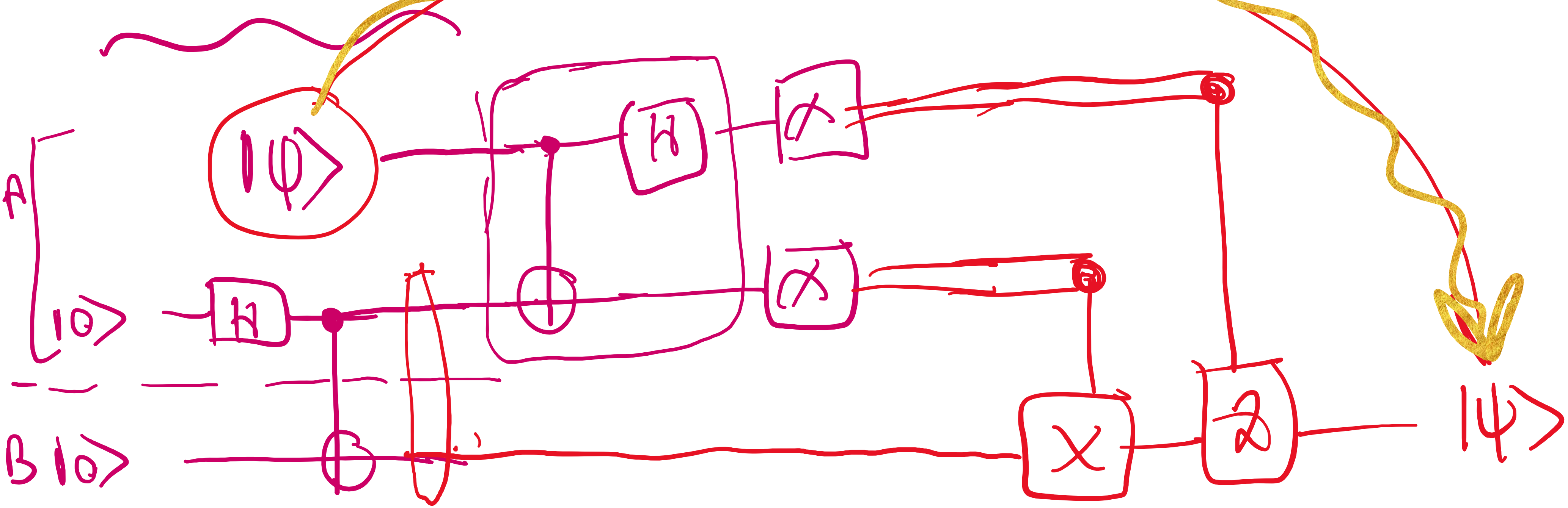


$|0\rangle + |1\rangle$

Quantum teleportation



Q Circuit



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle|00\rangle \rightarrow (\alpha|0\rangle + \beta|1\rangle)(|00\rangle + |11\rangle) = \alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle$

$\xrightarrow{\text{CNOT}} \alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle$

$\xrightarrow{H} \alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle$

$= |00\rangle(\alpha|0\rangle + \beta|1\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |11\rangle(\alpha \ldots$