We have a car PUF that uses 2 arbiter PUFs – a working PUF and a reference PUF, as well as a secret threshold value τ > 0.

Let Δ_w, Δ_r be the difference in timings experienced for the two PUFs on the same challenge. The response to this challenge is 0 if |Δ_w − Δ_r| ≤ τ and the response is 1 if |Δ_w − Δ_r| > τ where τ > 0 is the secret threshold value.

We have

$$\Delta_{31} = w_0 \cdot x_0 + w_1 \cdot x_1 + \cdots + w_{31} \cdot x_{31} + \beta_{31} = \mathbf{w}^\top \mathbf{x} + b$$

where

$$x_i = d_i \cdot d_{i+1} \cdot \ldots \cdot d_{31}$$
$$w_0 = \alpha_0$$
$$w_i = \alpha_i + \beta_{i-1} \text{ (for } i > 0)$$

Where $d_i = (1 - 2c_i)$; c_i is the i^{th} index of challenge vector **C.**

where $\alpha_i = \frac{(p_i - q_i + r_i - s_i)}{2}$ and $\beta_i = \frac{(p_i - q_i - r_i + s_i)}{2}$

If $\Delta_{31} < 0$, upper signal wins and the answer is 0.

If $\Delta_{31} > 0$, lower signal wins and the answer is 1.

Thus, answer is simply $\frac{\text{sign}(\mathbf{w}^\top \mathbf{x} + b) + 1}{2}$

Now we will implement same model for our both working and reference PUFs:

Let (**u**, p),(**v**, q) be the two linear models that can exactly predict the outputs of the two arbiter PUFs sitting inside the CAR-PUF.

For working PUF:

$$\Delta_w = w_0 \cdot x_0 + w_1 \cdot x_1 + \cdots + w_{31} \cdot x_{31} + \beta_{31} = \mathbf{u}^\top \mathbf{x} + p$$

For reference PUF:

$$\Delta_r = w_0 \cdot x_0 + w_1 \cdot x_1 + \cdots + w_{31} \cdot x_{31} + \beta_{31} = \mathbf{v}^\top \mathbf{x} + q$$

If |Δ_w − Δ_r| - τ ≤ 0 → 0

If |Δ_w − Δ_r| - τ > 0 → 1

And |Δ_w − Δ_r| - τ = $|(\mathbf{u} - \mathbf{v})^\top \mathbf{x} + p - q| - \tau$ {since we are using same challenge for the both PUFs}

For our problem response is:

Response = $\frac{1 + \text{sign}(\mathbf{W}^\top \phi(\text{c}) + b)}{2}$ =r →given ....eqn(1)

Response = $\frac{1 + \text{sign}(|\Delta_w - \Delta_r| - \tau)}{2} = \frac{1 + \text{sign}(|(\mathbf{u} - \mathbf{v})^\top \mathbf{x} + p - q| - \tau)}{2}$ → we have.........eqn(2)

Let $\Delta = \Delta_w - \Delta_r = (\mathbf{u} - \mathbf{v})^\top \mathbf{x} + p - q$

If we multiply $|\Delta| - \tau$ by a positive number, then the $\text{sign}(|\Delta_w - \Delta_r| - \tau)$ remains same. We will multiply it by $|\Delta| + \tau$ (which is always positive).

→ $(|\Delta| - \tau) * (|\Delta| + \tau) = \Delta^2 - \tau^2$

So, we can write the above response as:

Response $= \frac{1+\mathrm{sign}(|\Delta_w - \Delta_r| - \tau)}{2} = \frac{1+\mathrm{sign}(\Delta^2 - \tau^2)}{2}$ ....eqn(3)

Let $\mathbf{w} = (\mathbf{u} - \mathbf{v})^\top$ and k = p-q.

$$\Delta^2 = (w_0 \cdot x_0 + w_1 \cdot x_1 + \cdots + w_{31} \cdot x_{31} + k) * (w_0 \cdot x_0 + w_1 \cdot x_1 + \cdots + w_{31} \cdot x_{31} + k)$$

$$\Delta^2 = \sum_{i=0}^{31} w_i \cdot \mathrm{x}^2{}_i + \sum_{i=0}^{31} 2kw_i \cdot \mathrm{x}_i + \sum_{i=0\ \&\ i!=j}^{31} \sum_{j=0}^{31} 2w_i w_j \cdot \mathrm{x}_i \cdot \mathrm{x}_j + k^2$$

This equation has total 32+32+496+1=561 terms but $\mathrm{x}^2{}_i$ terms will merge into constant since $\mathrm{x}_i = \pm 1$ => its square will be 1 always. Let this whole constant is K which further will merge into τ^2 term and make new constant. In such way $\Delta^2 - \tau^2$ has 561-32-1=528 variables in the form 2nd and 3rd term in the above equation.

So,

$$\Delta^2 - \tau^2 = \sum_{i=0}^{31} 2kw_i \cdot \mathrm{x}_i + \sum_{i=0\ \&\ i!=j}^{31} \sum_{j=0}^{31} 2w_i w_j \cdot \mathrm{x}_i \cdot \mathrm{x}_j + b \quad \text{........eqn(4)}$$

Where $\mathrm{b} = \mathrm{K} - \tau^2$

As we are given our response in the form of above-mentioned equation.

From equations 1,2 and 3:

$$\mathbf{W}^\top \phi(\mathrm{c}) + b = \Delta^2 - \tau^2$$

After putting the values of right-hand side term from the equation (3),

$$\phi(\mathbf{c}) = (x_0, x_1 \ldots . x_{31}, x_0 x_1 \ldots \ldots . . x_{30} x_{31})$$

Thus, the dimensionality of the function $\phi(\mathbf{c})$ is 528.