Week 11, 2024

Formulation of the problem:

The 2D Burger's equation:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{dy^2}\right) \tag{1}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{dy^2}\right) \tag{2}$$

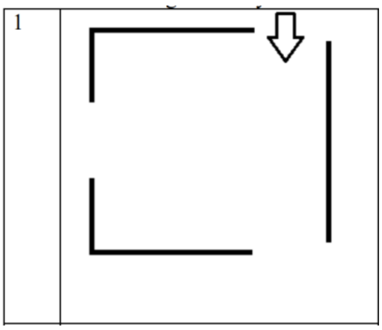
$$Re - Reynold's\ number$$

$$x \in (0,1), \qquad y \in (0,1)$$

Initial condition:

$$U(t=0,x,y)=0$$
$$V(t=0,x,y)=0$$

And with the following boundary conditions:



Inlet: $U=0, V=-1$

Outlet: $\frac{\partial U}{\partial n}=0, \frac{\partial V}{\partial n}=0$

Wall: $U=0, V=0$

Numerical method: Fractional step method

The same algorithm that we did in the last 2 weeks. But here's a little change. In Burger's equation we have convection term (the first derivatives by space) so linear differential operators Λ_1, Λ_2 will a little bit change. And everything else is the same as we did before.

P.S. I hope you know how to solve the Thomas algorithm and how to find coefficients alpha and betta.

a) Implicit method FSM (Fractional step method)

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^{n}}{\Delta t} = \frac{1}{2}\left(\Lambda_1 U^{n+1/2} + \Lambda_1 U^{n}\right) + \Lambda_2 U^{n}$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \frac{1}{2}\left(\Lambda_2 U^{n+1} - \Lambda_2 U^{n}\right)$$

Here is the Λ_1, Λ_2 operators have the following form:

$$\Lambda_1 = \frac{1}{Re}\frac{\partial^2}{\partial x^2} - U\frac{\partial}{\partial x}, \qquad \Lambda_2 = \frac{1}{Re}\frac{\partial^2}{\partial y^2} - V\frac{\partial}{\partial y}$$

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^{n}}{\Delta t} =$$

$$= \frac{1}{2}\left(\frac{1}{Re}\frac{U_{i+1j}^{n+1/2} - 2U_{ij}^{n+1/2} + U_{i-1j}^{n+1/2}}{\Delta x^2} - U_{ij}^{n}\frac{U_{i+1j}^{n+1/2} - U_{ij}^{n+1/2}}{\Delta x} + \frac{1}{Re}\frac{U_{i+1j}^{n} - 2U_{ij}^{n} + U_{i-1j}^{n}}{\Delta x^2} - U_{ij}^{n}\frac{U_{i+1j}^{n} - U_{ij}^{n}}{\Delta x}\right)$$

$$+ \frac{1}{Re}\frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2} - V_{ij}^{n}\frac{U_{ij+1}^{n} - U_{ij}^{n}}{\Delta y}$$

*It also will work if you take backward or central scheme for the first derivative. *

We are going to perform in this form to solve the Thomas algorithm.

$$A_{ij}U_{i+1j}^{n+1/2} + B_{ij}U_{ij}^{n+1/2} + C_{ij}U_{i-1j}^{n+1/2} = D_{ij}, \qquad i = \overline{1, N-1}, j\ is\ fixed$$

$$A_{ij} = -\frac{1}{2Re\Delta x^2} + \frac{U_{ij}^{n}}{2\Delta x}, \qquad B_{ij} = \frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} - \frac{U_{ij}^{n}}{2\Delta x}, \qquad C_{ij} = -\frac{1}{2Re\Delta x^2}$$

$$D_{ij} = \frac{U_{ij}^{n}}{\Delta t} + \frac{1}{2}\left(\frac{1}{Re}\frac{U_{i+1j}^{n} - 2U_{ij}^{n} + U_{i-1j}^{n}}{\Delta x^2} - U_{ij}^{n}\frac{U_{i+1j}^{n} - U_{ij}^{n}}{\Delta x}\right) + \frac{1}{Re}\frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2} - V_{ij}^{n}\frac{U_{ij+1}^{n} - U_{ij}^{n}}{\Delta y}$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \frac{1}{2}\left(\frac{1}{Re}\frac{U_{ij+1}^{n+1} - 2U_{ij}^{n+1} + U_{ij-1}^{n+1}}{\Delta y^2} - V_{ij}^{n}\frac{U_{ij+1}^{n+1} - U_{ij}^{n+1}}{\Delta y} - \frac{1}{Re}\frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2} + V_{ij}^{n}\frac{U_{ij+1}^{n} - U_{ij}^{n}}{\Delta y}\right)$$

Thomas algorithm for y:

$$A_{ij}U_{ij+1}^{n+1} + B_{ij}U_{ij}^{n+1} + C_{ij}U_{ij-1}^{n+1} = D_{ij}, \qquad j = \overline{1, M-1}, i\ is\ fixed$$

$$A_{ij} = -\frac{1}{2Re\Delta y^2} + \frac{V_{ij}^n}{2\Delta y}, \qquad B_{ij} = \frac{1}{\Delta t} + \frac{1}{Re\Delta y^2} - \frac{V_{ij}^n}{2\Delta y}, \qquad C_{ij} = -\frac{1}{2Re\Delta y^2}$$

$$D_{ij} = \frac{U_{ij}^{n+1/2}}{\Delta t} - \frac{1}{2}\left(\frac{1}{Re}\frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2} - V_{ij}^n \frac{U_{ij+1}^n - U_{ij}^n}{\Delta y}\right)$$

Do the same for V.

Derivations of alpha and betta:

$$U_{ij} = \alpha_{i+1j} U_{i+1j} + \beta_{i+1j}, \qquad i = \overline{N-1,1} \tag{3}$$

$$A_{ij} U_{i+1j} + B_{ij} U_{ij} + C_{ij} U_{i-1j} = D_{ij}, \qquad i = \overline{1,N-1} \tag{4}$$

$$U_{i-1j} = \alpha_{ij} U_{ij} + \beta_{ij}, \qquad i = \overline{1,N-1} \tag{5}$$

Substitute equation (5) into (4)

$$A_{ij} U_{i+1j} + B_{ij} U_{ij} + C_{ij}\left(\alpha_{ij} U_{ij} + \beta_{ij}\right) = D_{ij}$$

$$A_{ij} U_{i+1j} + \left(B_{ij} + C_{ij}\alpha_{ij}\right) U_{ij} + C_{ij}\beta_{ij} = D_{ij}$$

$$U_{ij} = -\frac{A_{ij}}{B_{ij} + C_{ij}\alpha_{ij}} U_{i+1j} + \frac{D_{ij} - C_{ij}\beta_{ij}}{B_{ij} + C_{ij}\alpha_{ij}} \tag{6}$$

Equation (6) looks like (3) so we can find $\alpha_{i+1j}, \beta_{i+1j}$

$$\alpha_{i+1j} = -\frac{A_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}, \qquad \beta_{i+1j} = \frac{D_{ij} - C_{ij}\beta_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}, \tag{7}$$

$$i = \overline{1,N-1}, j\ is\ fixed$$