Week 7, 2024

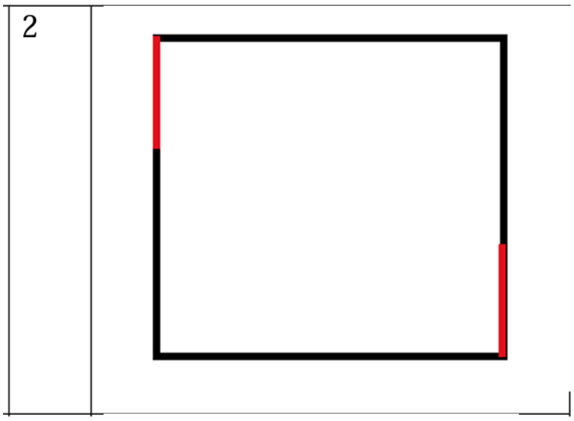
Formulation of the problem:

The 2D Laplace equation:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0, \qquad x \in (0,1), \qquad y \in (0,1)$$

With the following boundary conditions:

$$P(x = 0, 0 < y < 0.7) = 0$$
$$P(x = 0, 0.7 < y < 1) = 1$$
$$P(x = 1, 0 < y < 0.3) = 1$$
$$P(x = 1, 0.3 < y < 1) = 0$$
$$P(x, y = 0) = 0$$
$$P(x, y = 1) = 0$$



Numerical method: Tridiagonal matrix method (Метод матричной прогонки)

This is the implicit method used to solve 2D equations and based on linear algebra. This is an extension of Thomas algorithm which designed for 1D equations that we solved in first 5 weeks. The algorithm is the same as in Thomas algorithm but with a little bit difference: we will figure out with vector's not with point as we did in Thomas algorithm. And respectively the coefficients A, B, C, D will be in matrix form.

Discretization:

$$\frac{P_{i+1j}^{n+1} - 2P_{ij}^{n+1} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n+1} - 2P_{ij}^{n+1} + P_{ij-1}^{n+1}}{\Delta y^2} = 0 \qquad (1)$$

Let's derive the Tridiagonal matrix method:

$$A_i \vec{P}_{i+1} + B_i \vec{P}_i + C_i \vec{P}_{i-1} = \vec{D}_i, \qquad i = \overline{1, N-1} \qquad (2)$$

Here $\vec{P}_i = (P_{i1}, P_{i2}, \dots, P_{iN-2}, P_{iN-1})^T$ vector notation

$$\vec{P}_i = \alpha_{i+1}\vec{P}_{i+1} + \vec{\beta}_{i+1}, \qquad i = \overline{1, N-1} \tag{3}$$

$$\vec{P}_{i-1} = \alpha_i\vec{P}_i + \vec{\beta}_i, \qquad i = \overline{1, N-1} \tag{4}$$

Substitute equation (4) into (2)

$$A_i\vec{P}_{i+1} + B_i\vec{P}_i + C_i\left(\alpha_i\vec{P}_i + \vec{\beta}_i\right) = \vec{D}_i$$

$$A_i\vec{P}_{i+1} + (B_i + C_i\alpha_i)\vec{P}_i + C_i\vec{\beta}_i = \vec{D}_i$$

$$(B_i + C_i\alpha_i)\vec{P}_i = -A_i\vec{P}_{i+1} + \vec{D}_i - C_i\vec{\beta}_i$$

$$\vec{P}_i = -(B_i + C_i\alpha_i)^{-1}A_i\vec{P}_{i+1} + (B_i + C_i\alpha_i)^{-1}\left(\vec{D}_i - C_i\vec{\beta}_i\right)$$

$$\alpha_{i+1} = -(B_i + C_i\alpha_i)^{-1}A_i \tag{5}$$

$$\vec{\beta}_{i+1} = (B_i + C_i\alpha_i)^{-1}\left(\vec{D}_i - C_i\vec{\beta}_i\right) \tag{6}$$

$$A_i = \begin{pmatrix} \frac{1}{\Delta x^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\Delta x^2} & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\Delta x^2} \end{pmatrix}, \qquad C_i = \begin{pmatrix} \frac{1}{\Delta x^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\Delta x^2} & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\Delta x^2} \end{pmatrix}$$

$$B_i = \begin{pmatrix} -\frac{2}{\Delta x^2} - \frac{2}{\Delta x^2} & \frac{1}{\Delta y^2} & \cdots & 0 & 0 \\ \frac{1}{\Delta y^2} & -\frac{2}{\Delta x^2} - \frac{2}{\Delta x^2} & \frac{1}{\Delta y^2} & 0 & 0 \\ \vdots & \frac{1}{\Delta y^2} & \ddots & \frac{1}{\Delta y^2} & \vdots \\ 0 & 0 & \frac{1}{\Delta y^2} & -\frac{2}{\Delta x^2} - \frac{2}{\Delta x^2} & \frac{1}{\Delta y^2} \\ 0 & 0 & \cdots & \frac{1}{\Delta y^2} & -\frac{2}{\Delta x^2} - \frac{2}{\Delta x^2} \end{pmatrix}$$

$$\vec{D}_N = \{0, 0, \dots, 0, 0\}^T$$

We found the final formula for coefficients alpha and betta. Now let's find $\alpha_1, \vec{\beta}_1, \vec{U}_N$ from boundary conditions.

$$\alpha_1 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \qquad \vec{\beta}_1 = \{0, 0, \dots, 0, 1, 1\}^T, \qquad \vec{U}_N = \{1, 1, 0, \dots, 0, 0\}^T$$