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Formulation of the problem:

The 2D Navier-Stokes equation for incompressible flow:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \quad (1)$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \quad (2)$$

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (3)$$

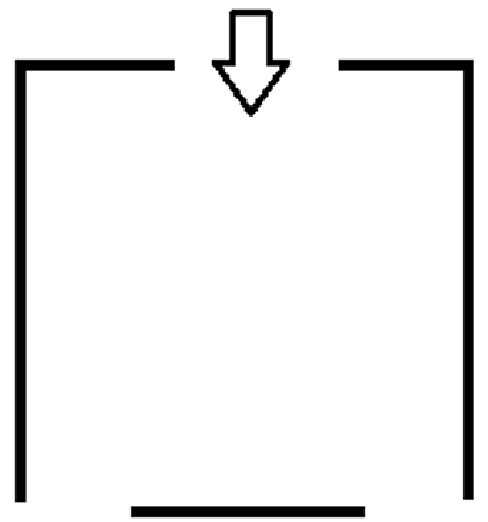
$$U - velocity\ compent\ of\ x, V - velocity\ component\ of\ y$$

$$P - pressure, Re - Reynold's\ number$$

(1) - (2) Momentum equations

(3) Continuity equation

The following conditions:

2	

$x \epsilon [0, 1], y \epsilon [0, 1]$

Initial condition:

$U = 0, \quad V = 0, \quad P = 0$

Inlet:

$U = 0, \quad V = -1, \quad P = 1$

Outlet:

$$\frac{\partial U}{\partial n} = 0, \quad \frac{\partial V}{\partial n} = 0, \quad P = 0$$

Walls:

$$U = 0, \quad V = 0, \quad \frac{\partial P}{\partial n} = 0$$

Numerical method: Projection method (Метод расщепления по физическим параметрам)

Projection method:

1)

$$\frac{U^{n+1}-U^{n}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+L_1$$

$$L_1=-U\frac{\partial U}{\partial x}-V\frac{\partial U}{\partial y}+\frac{1}{Re}(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{dy^2})$$

$$\frac{U^{n+1}-U^{n}+U^{*}-U^{*}}{\Delta \mathrm{t}}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+L_1$$

$$\frac{U^{*}-U^{n}}{\Delta t}=L_1, Burger's\ equation$$

$$\frac{U^{n+1}-U^{*}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P^{n+1}}{\partial x}$$

$$U^{n+1}=U^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}$$

$$\frac{V^{n+1}-V^{n}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+L_2$$

$$L_2=-U\frac{\partial V}{\partial x}-V\frac{\partial V}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{dy^2}\right)$$

$$\frac{V^{n+1}-V^{n}+V^{*}-V^{*}}{\Delta \mathrm{t}}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+L_2$$

$$\frac{V^{*}-V^{n}}{\Delta t}=L_2, Burger's\ equation$$

$$\frac{V^{n+1}-V^{*}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

$$V^{n+1}=V^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

2)

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0$$

$$\frac{\partial U^{n+1}}{\partial x}+\frac{\partial V^{n+1}}{\partial y}=0$$

$$\frac{\partial}{\partial x}\left(U^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}\right)+\frac{\partial}{\partial y}\left(V^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}\right)=0$$

$$\frac{\partial U^{*}}{\partial x}-\frac{\Delta t}{\rho}\frac{\partial^2 P^{n+1}}{\partial x^2}+\frac{\partial V^{*}}{\partial y}-\frac{\Delta t}{\rho}\frac{\partial^2 P^{n+1}}{\partial y^2}=0$$

$$\frac{\partial^2 P^{n+1}}{\partial x^2}+\frac{\partial^2 P^{n+1}}{\partial y^2}=\frac{\rho}{\Delta t}\left(\frac{\partial U^{*}}{\partial x}+\frac{\partial V^{*}}{\partial y}\right), \qquad Poisson\ equation$$

3)

$$U^{n+1}=U^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}$$

$$V^{n+1}=V^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

Algorithm:

1) Solve Burger's equation to find U^*, V^*.

$$\frac{U^* - U^n}{\Delta t} = L_1$$
$$\frac{V^* - V^n}{\Delta t} = L_2$$

You can solve it with different numerical methods (Explicit, FSM, ADM, Tridiagonal matrix method).

2) Solve Poisson equation to find P.

$$\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} = \frac{\rho}{\Delta t}\left(\frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y}\right)$$

You can solve it with different numerical methods (Jacobi, Gauss-Seidel, Relaxation methods, Tridiagonal matrix method).

3) Correct velocity components U^{n+1}, V^{n+1}:

$$U^{n+1} = U^* - \frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}$$
$$V^{n+1} = V^* - \frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

Numerical solution:

1) Burger's equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$
$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

a) Explicit method (Euler's method)

$$\frac{U^* - U^n}{\Delta t} = -U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$
$$U^* = U^n + \Delta t\left(-U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)\right)$$
$$\frac{V^* - V^n}{\Delta t} = -U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$
$$V^* = V^n + \Delta t\left(-U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)\right)$$

Discretization:

$$U_{ij}^* = U_{ij}^n + \Delta t\left(-U_{ij}^n\frac{U_{i+1j}^n - U_{i-1j}^n}{2\Delta x} - V_{ij}^n\frac{U_{ij+1}^n - U_{ij-1}^n}{2\Delta y} + \frac{1}{Re}\left(\frac{U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n}{\Delta x^2} + \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2}\right)\right)$$
$$V_{ij}^* = V_{ij}^n + \Delta t\left(-U_{ij}^n\frac{V_{i+1j}^n - V_{i-1j}^n}{2\Delta x} - V_{ij}^n\frac{V_{ij+1}^n - V_{ij-1}^n}{2\Delta y} + \frac{1}{Re}\left(\frac{V_{i+1j}^n - 2V_{ij}^n + V_{i-1j}^n}{\Delta x^2} + \frac{V_{ij+1}^n - 2V_{ij}^n + V_{ij-1}^n}{\Delta y^2}\right)\right)$$

b) Implicit method FSM (Fractional step method)

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^n}{\Delta t} = \frac{1}{2}\left(\Lambda_1 U^{n+1/2} + \Lambda_1 U^n\right) + \Lambda_2 U^n$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \frac{1}{2}\left(\Lambda_2 U^{n+1} - \Lambda_2 U^n\right)$$

Here is the Λ_1, Λ_2 operators have the following form:

$$\Lambda_1 = \frac{1}{Re}\frac{\partial^2}{\partial x^2} - U\frac{\partial}{\partial x}, \qquad \Lambda_2 = \frac{1}{Re}\frac{\partial^2}{\partial y^2} - V\frac{\partial}{\partial y}$$

First step:

$$\frac{U_{ij}^{n+1/2}-U_{ij}^{n}}{\Delta t}=$$

$$=\frac{1}{2}\left(\frac{1}{Re}\frac{U_{i+1j}^{n+1/2}-2U_{ij}^{n+1/2}+U_{i-1j}^{n+1/2}}{\Delta x^2}-U_{ij}^{n}\frac{U_{i+1j}^{n+1/2}-U_{ij}^{n+1/2}}{\Delta x}+\frac{1}{Re}\frac{U_{i+1j}^{n}-2U_{ij}^{n}+U_{i-1j}^{n}}{\Delta x^2}-U_{ij}^{n}\frac{U_{i+1j}^{n}-U_{ij}^{n}}{\Delta x}\right)$$
$$+\frac{1}{Re}\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}-V_{ij}^{n}\frac{U_{ij+1}^{n}-U_{ij}^{n}}{\Delta y}$$

*It also will work if you take a backward or central scheme for the first derivative. *

We are going to perform in this form to solve the Thomas algorithm.

$$A_{ij}U_{i+1j}^{n+1/2}+B_{ij}U_{ij}^{n+1/2}+C_{ij}U_{i-1j}^{n+1/2}=D_{ij},\qquad i=\overline{1,N-1}, j\ is\ fixed$$

$$A_{ij}=-\frac{1}{2Re\Delta x^2}+\frac{U_{ij}^{n}}{2\Delta x},\qquad B_{ij}=\frac{1}{\Delta t}+\frac{1}{Re\Delta x^2}-\frac{U_{ij}^{n}}{2\Delta x},\qquad C_{ij}=-\frac{1}{2Re\Delta x^2}$$

$$D_{ij}=\frac{U_{ij}^{n}}{\Delta t}+\frac{1}{2}\left(\frac{1}{Re}\frac{U_{i+1j}^{n}-2U_{ij}^{n}+U_{i-1j}^{n}}{\Delta x^2}-U_{ij}^{n}\frac{U_{i+1j}^{n}-U_{ij}^{n}}{\Delta x}\right)+\frac{1}{Re}\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}-V_{ij}^{n}\frac{U_{ij+1}^{n}-U_{ij}^{n}}{\Delta y}$$

Second step:

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\Delta t}=\frac{1}{2}\left(\frac{1}{Re}\frac{U_{ij+1}^{n+1}-2U_{ij}^{n+1}+U_{ij-1}^{n+1}}{\Delta y^2}-V_{ij}^{n}\frac{U_{ij+1}^{n+1}-U_{ij}^{n+1}}{\Delta y}-\frac{1}{Re}\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}+V_{ij}^{n}\frac{U_{ij+1}^{n}-U_{ij}^{n}}{\Delta y}\right)$$

Thomas algorithm for y:

$$A_{ij}U_{ij+1}^{n+1}+B_{ij}U_{ij}^{n+1}+C_{ij}U_{ij-1}^{n+1}=D_{ij},\qquad j=\overline{1,M-1}, i\ is\ fixed$$

$$A_{ij}=-\frac{1}{2Re\Delta y^2}+\frac{V_{ij}^{n}}{2\Delta y},\qquad B_{ij}=\frac{1}{\Delta t}+\frac{1}{Re\Delta y^2}-\frac{V_{ij}^{n}}{2\Delta y},\qquad C_{ij}=-\frac{1}{2Re\Delta y^2}$$

$$D_{ij}=\frac{U_{ij}^{n+1/2}}{\Delta t}-\frac{1}{2}\left(\frac{1}{Re}\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}-V_{ij}^{n}\frac{U_{ij+1}^{n}-U_{ij}^{n}}{\Delta y}\right)$$

Do the same for V.

Derivations of alpha and betta:

$$U_{ij}=\alpha_{i+1j}U_{i+1j}+\beta_{i+1j},\qquad i=\overline{N-1,1}\tag{3}$$

$$A_{ij}U_{i+1j}+B_{ij}U_{ij}+C_{ij}U_{i-1j}=D_{ij},\qquad i=\overline{1,N-1}\tag{4}$$

$$U_{i-1j}=\alpha_{ij}U_{ij}+\beta_{ij},\qquad i=\overline{1,N-1}\tag{5}$$

Substitute equation (5) into (4)

$$A_{ij}U_{i+1j}+B_{ij}U_{ij}+C_{ij}\left(\alpha_{ij}U_{ij}+\beta_{ij}\right)=D_{ij}$$

$$A_{ij}U_{i+1j} + \left(B_{ij} + C_{ij}\alpha_{ij}\right)U_{ij} + C_{ij}\beta_{ij} = D_{ij}$$

$$U_{ij} = -\frac{A_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}U_{i+1j} + \frac{D_{ij} - C_{ij}\beta_{ij}}{B_{ij} + C_{ij}\alpha_{ij}} \tag{6}$$

Equation (6) looks like (3) so we can find $\alpha_{i+1j}, \beta_{i+1j}$

$$\alpha_{i+1j} = -\frac{A_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}, \qquad \beta_{i+1j} = \frac{D_{ij} - C_{ij}\beta_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}, \tag{7}$$

$$i = \overline{1, N-1}, j\ is\ fixed$$

c) Implicit method ADM (Alternating direction method)

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^{n}}{\Delta t} = \Lambda_1 U^{n+1/2} + \Lambda_2 U^{n}$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \Lambda_1 U^{n+1/2} + \Lambda_2 U^{n+1}$$

Here is the Λ_1 and Λ_2 operators have the following form:

$$\Lambda_1 = \frac{1}{Re}\frac{\partial^2}{\partial x^2} - U\frac{\partial}{\partial x}, \qquad \Lambda_2 = \frac{1}{Re}\frac{\partial^2}{\partial y^2} - V\frac{\partial}{\partial y}$$

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^{n}}{\Delta t} = \frac{1}{Re}\frac{U_{i+1j}^{n+1/2} - 2U_{ij}^{n+1/2} + U_{i-1j}^{n+1/2}}{\Delta x^2} - U_{ij}^{n}\frac{U_{i+1j}^{n+1/2} - U_{ij}^{n+1/2}}{\Delta x} + \frac{1}{Re}\frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2} - V_{ij}^{n}\frac{U_{ij+1}^{n} - U_{ij}^{n}}{\Delta y}$$

We are going to perform in this form to solve the Thomas algorithm.

$$A_{ij}U_{i+1j}^{n+1/2} + B_{ij}U_{ij}^{n+1/2} + C_{ij}U_{i-1j}^{n+1/2} = D_{ij}, \qquad i = \overline{1, N-1}, j\ is\ fixed$$

$$A_{ij} = -\frac{1}{Re\Delta x^2} + \frac{U_{ij}^{n}}{\Delta x}, \qquad B_{ij} = \frac{1}{\Delta t} + \frac{2}{Re\Delta x^2} - \frac{U_{ij}^{n}}{\Delta x}, \qquad C_{ij} = -\frac{1}{Re\Delta x^2}$$

$$D_{ij} = \frac{U_{ij}^{n}}{\Delta t} + \frac{1}{Re}\frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2} - V_{ij}^{n}\frac{U_{ij+1}^{n} - U_{ij}^{n}}{\Delta y}$$

Second step:

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\Delta t}=$$

$$=\frac{1}{Re}\frac{U_{i+1j}^{n+1/2}-2U_{ij}^{n+1/2}+U_{i-1j}^{n+1/2}}{\Delta x^2}-U_{ij}^{n}\frac{U_{i+1j}^{n+1/2}-U_{ij}^{n+1/2}}{\Delta x}+\frac{1}{Re}\frac{U_{ij+1}^{n+1}-2U_{ij}^{n+1}+U_{ij-1}^{n+1}}{\Delta y^2}-V_{ij}^{n}\frac{U_{ij+1}^{n+1}-U_{ij}^{n+1}}{\Delta y}$$

Thomas algorithm for y:

$$A_{ij}U_{ij+1}^{n+1}+B_{ij}U_{ij}^{n+1}+C_{ij}U_{ij-1}^{n+1}=D_{ij},\qquad j=\overline{1,M-1}, i\ is\ fixed$$

$$A_{ij}=-\frac{1}{Re\Delta y^2}+\frac{V_{ij}^{n}}{\Delta y},\qquad B_{ij}=\frac{1}{\Delta t}+\frac{2}{Re\Delta y^2}-\frac{V_{ij}^{n}}{\Delta y},\qquad C_{ij}=-\frac{1}{Re\Delta y^2}$$

$$D_{ij}=\frac{U_{ij}^{n+1/2}}{\Delta t}+\frac{1}{Re}\frac{U_{i+1j}^{n+1/2}-2U_{ij}^{n+1/2}+U_{i-1j}^{n+1/2}}{\Delta x^2}-U_{ij}^{n}\frac{U_{i+1j}^{n+1/2}-U_{ij}^{n+1/2}}{\Delta x}$$

Do the same for V.

2) Poisson equation

$$\frac{\partial^2 P^{n+1}}{\partial x^2}+\frac{\partial^2 P^{n+1}}{\partial y^2}=\frac{\rho}{\Delta t}\left(\frac{\partial U^*}{\partial x}+\frac{\partial V^*}{\partial y}\right)$$

We need to solve this equation until steady condition is reached:

$$\max(|P^{n+1}-P^{n}|)<\varepsilon$$
$$\varepsilon - small\ number, 10^{-6}$$

a) Explicit method. Jacobi.

Discretization:

$$\frac{P_{i+1j}^{n}-2P_{ij}^{n+1}+P_{i-1j}^{n}}{\Delta x^2}+\frac{P_{ij+1}^{n}-2P_{ij}^{n+1}+P_{ij-1}^{n}}{\Delta y^2}=\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*}-U_{i-1j}^{*}}{2\Delta x}+\frac{V_{ij+1}^{*}-V_{ij-1}^{*}}{2\Delta y}\right)$$

$$P_{ij}^{n+1}\left(\frac{2}{\Delta x^2}+\frac{2}{\Delta y^2}\right)=\frac{P_{i+1j}^{n}+P_{i-1j}^{n}}{\Delta x^2}+\frac{P_{ij+1}^{n}+P_{ij-1}^{n}}{\Delta y^2}-\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*}-U_{i-1j}^{*}}{2\Delta x}+\frac{V_{ij+1}^{*}-V_{ij-1}^{*}}{2\Delta y}\right)$$

$$P_{ij}^{n+1}=\frac{\Delta x^2\Delta y^2}{2(\Delta x^2+\Delta y^2)}\left(\frac{P_{i+1j}^{n}+P_{i-1j}^{n}}{\Delta x^2}+\frac{P_{ij+1}^{n}+P_{ij-1}^{n}}{\Delta y^2}-\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*}-U_{i-1j}^{*}}{2\Delta x}+\frac{V_{ij+1}^{*}-V_{ij-1}^{*}}{2\Delta y}\right)\right)$$

Final iterative formula:

$$P_{ij}^{n+1}=\frac{\Delta y^2\left(P_{i+1j}^{n}+P_{i-1j}^{n}\right)+\Delta x^2(P_{ij+1}^{n}+P_{ij-1}^{n})}{2(\Delta x^2+\Delta y^2)}-\frac{\Delta x^2\Delta y^2}{2(\Delta x^2+\Delta y^2)}\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*}-U_{i-1j}^{*}}{2\Delta x}+\frac{V_{ij+1}^{*}-V_{ij-1}^{*}}{2\Delta y}\right)$$

b) Explicit method. Gauss-Seidel.

Discretization:

$$\frac{P_{i+1j}^{n} - 2P_{ij}^{n+1} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n} - 2P_{ij}^{n+1} + P_{ij-1}^{n+1}}{\Delta y^2} = \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{i-1j}^{*}}{2\Delta y}\right)$$

$$P_{ij}^{n+1}\left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right) = \frac{P_{i+1j}^{n} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n} + P_{ij-1}^{n+1}}{\Delta y^2} - \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

$$P_{ij}^{n+1} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\left(\frac{P_{i+1j}^{n} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n} + P_{ij-1}^{n+1}}{\Delta y^2} - \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)\right)$$

Final iterative formula.

$$P_{ij}^{n+1} = \frac{\Delta y^2\left(P_{i+1j}^{n} + P_{i-1j}^{n+1}\right) + \Delta x^2(P_{ij+1}^{n} + P_{ij-1}^{n+1})}{2(\Delta x^2 + \Delta y^2)} - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

c) Explicit method. Relaxation methods.

Discretization:

$$\frac{P_{i+1j}^{n} - 2P_{ij}^{n+1} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n} - 2P_{ij}^{n+1} + P_{ij-1}^{n+1}}{\Delta y^2} = \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

$$P_{ij}^{n+1}\left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right) = \frac{P_{i+1j}^{n} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n} + P_{ij-1}^{n+1}}{\Delta y^2} - \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

$$P_{ij}^{n+1} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\left(\frac{P_{i+1j}^{n} + P_{i-1j}^{n+1}}{\Delta x^2} + \frac{P_{ij+1}^{n} + P_{ij-1}^{n+1}}{\Delta y^2} - \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)\right)$$

$$P_{ij}^{n+1} = \frac{\Delta y^2\left(P_{i+1j}^{n} + P_{i-1j}^{n+1}\right) + \Delta x^2\left(P_{ij+1}^{n} + P_{ij-1}^{n+1}\right)}{2(\Delta x^2 + \Delta y^2)} - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

$$\frac{P_{ij}^{n+1}}{w} + \left(1 - \frac{1}{w}\right)P_{ij}^{n} = \frac{\Delta y^2\left(P_{i+1j}^{n} + P_{i-1j}^{n+1}\right) + \Delta x^2\left(P_{ij+1}^{n} + P_{ij-1}^{n+1}\right)}{2(\Delta x^2 + \Delta y^2)} - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

Final iterative formula.

$$P_{ij}^{n+1} =$$

$$= (1 - w)P_{ij}^{n} + w\frac{\Delta y^2\left(P_{i+1j}^{n} + P_{i-1j}^{n+1}\right) + \Delta x^2\left(P_{ij+1}^{n} + P_{ij-1}^{n+1}\right)}{2(\Delta x^2 + \Delta y^2)} - w\frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^{*} - U_{i-1j}^{*}}{2\Delta x} + \frac{V_{ij+1}^{*} - V_{ij-1}^{*}}{2\Delta y}\right)$$

$$w - relaxation\ parameter$$

$$w \in (0, 1)\ Under\ relaxation$$

$$w \in (1, 2)\ Over\ relaxation$$

3) Correction

$$U^{n+1} = U^* - \frac{\Delta t}{\rho}\frac{\partial P}{\partial x}$$

$$V^{n+1} = V^* - \frac{\Delta t}{\rho}\frac{\partial P}{\partial y}$$

Discretization:

$$U_{ij}^{n+1} = U_{ij}^* - \frac{\Delta t}{\rho}\frac{P_{i+1j}^{n+1} - P_{i-1j}^{n+1}}{2\Delta x}$$

$$V_{ij}^{n+1} = V_{ij}^* - \frac{\Delta t}{\rho}\frac{P_{ij+1}^{n+1} - P_{ij-1}^{n+1}}{2\Delta y}$$

Do it this algorithm until steady condition is reached:

$$\max(|U^{n+1} - U^n|, |V^{n+1} - V^n|) < \varepsilon$$

$$\varepsilon - small\ number, 10^{-6}$$

Flow chart of the Projection method