Week 8, 2024

Fractional step and alternating direction methods

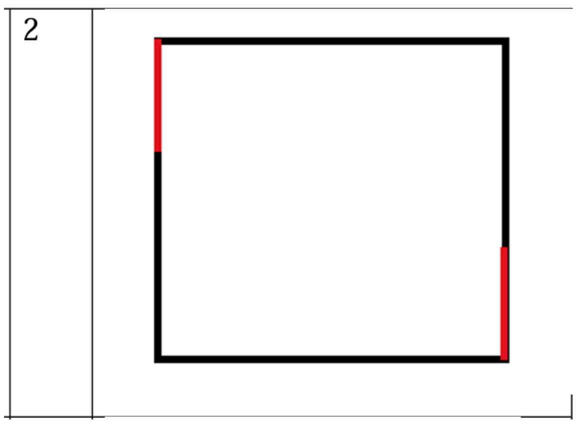
Formulation of the problem:

The 2D Heat equation:

$$\frac{\partial U}{\partial t} = \alpha^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \qquad x \in (0,1), \qquad y \in (0,1)$$

With the following boundary conditions:

$$U(x = 0, 0 < y < 0.7) = 0$$
$$U(x = 0, 0.7 < y < 1) = 1$$
$$U(x = 1, 0 < y < 0.3) = 1$$
$$U(x = 1, 0.3 < y < 1) = 0$$
$$U(x, y = 0) = 0$$
$$U(x, y = 1) = 0$$



Initial condition: $U(t = 0, x, y) = 0$

Numerical method: Fractional step and alternating direction methods

These two implicit methods are very similar to each other because the main idea of this methods based on fractioning time internal on 2 or 3 parts for each own dimension so problem is converted into one-dimensional problem, and we can solve it easily as we solved in first 3 week doing simple Thomas algorithm. The approximation error of methods $O(\Delta t^2, \Delta x^2, \Delta y^2)$.

P.S. I hope you know how to solve the Thomas algorithm and how to find coefficients alpha and betta.

a) Implicit method FSM (Fractional step method)

First step:

$$\frac{U_{ij}^{n+1/2}-U_{ij}^{n}}{\Delta t}=\frac{1}{2}\left(\Lambda_1 U^{n+1/2}+\Lambda_1 U^{n}\right)+\Lambda_2 U^{n}$$

Second step:

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\Delta t}=\frac{1}{2}(\Lambda_2 U^{n+1}-\Lambda_2 U^{n})$$

Here is the Λ_1 and Λ_2 operators have the following form:

$$\Lambda_1=\alpha^2\frac{\partial^2}{\partial x^2},\qquad \Lambda_2=\alpha^2\frac{\partial^2}{\partial y^2}$$

First step:

$$\frac{U_{ij}^{n+1/2}-U_{ij}^{n}}{\Delta t}=\frac{1}{2}\left(\alpha^2\frac{U_{i+1j}^{n+1/2}-2U_{ij}^{n+1/2}+U_{i-1j}^{n+1/2}}{\Delta x^2}+\alpha^2\frac{U_{i+1j}^{n}-2U_{ij}^{n}+U_{i-1j}^{n}}{\Delta x^2}\right)+\alpha^2\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}$$

The equation is converted in 1D, and we can solve it by the Thomas algorithm.

$$A_{ij}U_{i+1j}^{n+1/2}+B_{ij}U_{ij}^{n+1/2}+C_{ij}U_{i-1j}^{n+1/2}=D_{ij},\qquad i=\overline{1,N-1}, j\ is\ fixed$$

$$A_{ij}=-\frac{\alpha^2}{2\Delta x^2},\qquad B_{ij}=\frac{1}{\Delta t}+\frac{\alpha^2}{\Delta x^2},\qquad C_{ij}=-\frac{\alpha^2}{2\Delta x^2}$$

$$D_{ij}=\frac{U_{ij}^{n}}{\Delta t}+\frac{1}{2}\left(\alpha^2\frac{U_{i+1j}^{n}-2U_{ij}^{n}+U_{i-1j}^{n}}{\Delta x^2}\right)+\alpha^2\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}$$

Second step:

$$\frac{U_{ij}^{n+1}-U_{ij}^{n+1/2}}{\Delta t}=\frac{1}{2}\left(\alpha^2\frac{U_{ij+1}^{n+1}-2U_{ij}^{n+1}+U_{ij-1}^{n+1}}{\Delta y^2}-\alpha^2\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}\right)$$

Thomas algorithm for y:

$$A_{ij}U_{ij+1}^{n+1}+B_{ij}U_{ij}^{n+1}+C_{ij}U_{ij-1}^{n+1}=D_{ij},\qquad j=\overline{1,M-1}, i\ is\ fixed$$

$$A_{ij}=-\frac{\alpha^2}{2\Delta y^2},\qquad B_{ij}=\frac{1}{\Delta t}+\frac{\alpha^2}{\Delta y^2},\qquad C_{ij}=-\frac{\alpha^2}{2\Delta y^2}$$

$$D_{ij}=\frac{U_{ij}^{n+1/2}}{\Delta t}-\frac{1}{2}\left(\alpha^2\frac{U_{ij+1}^{n}-2U_{ij}^{n}+U_{ij-1}^{n}}{\Delta y^2}\right)$$

a) Implicit method ADM (Alternating direction method)

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^{n}}{\Delta t} = \Lambda_1 U^{n+1/2} + \Lambda_2 U^{n}$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \Lambda_1 U^{n+1/2} + \Lambda_2 U^{n+1}$$

Here is the Λ_1 and Λ_2 operators have the following form:

$$\Lambda_1 = \alpha^2 \frac{\partial^2}{\partial x^2}, \qquad \Lambda_2 = \alpha^2 \frac{\partial^2}{\partial y^2}$$

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^{n}}{\Delta t} = \alpha^2 \frac{U_{i+1j}^{n+1/2} - 2U_{ij}^{n+1/2} + U_{i-1j}^{n+1/2}}{\Delta x^2} + \alpha^2 \frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2}$$

The equation is converted in 1D, and we can solve it by the Thomas algorithm.

$$A_{ij} U_{i+1j}^{n+1/2} + B_{ij} U_{ij}^{n+1/2} + C_{ij} U_{i-1j}^{n+1/2} = D_{ij}, \qquad i = \overline{1, N-1}, j\ is\ fixed$$

$$A_{ij} = -\frac{\alpha^2}{\Delta x^2}, \qquad B_{ij} = \frac{1}{\Delta t} + \frac{2\alpha^2}{\Delta x^2}, \qquad C_{ij} = -\frac{\alpha^2}{\Delta x^2}$$

$$D_{ij} = \frac{U_{ij}^{n}}{\Delta t} + \alpha^2 \frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2}$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \alpha^2 \frac{U_{i+1j}^{n+1/2} - 2U_{ij}^{n+1/2} + U_{i-1j}^{n+1/2}}{\Delta x^2} + \alpha^2 \frac{U_{ij+1}^{n+1} - 2U_{ij}^{n+1} + U_{ij-1}^{n+1}}{\Delta y^2}$$

Thomas algorithm for y:

$$A_{ij} U_{ij+1}^{n+1} + B_{ij} U_{ij}^{n+1} + C_{ij} U_{ij-1}^{n+1} = D_{ij}, \qquad j = \overline{1, M-1}, i\ is\ fixed$$

$$A_{ij} = -\frac{\alpha^2}{\Delta y^2}, \qquad B_{ij} = \frac{1}{\Delta t} + \frac{2\alpha^2}{\Delta y^2}, \qquad C_{ij} = -\frac{\alpha^2}{\Delta y^2}$$

$$D_{ij} = \frac{U_{ij}^{n+1/2}}{\Delta t} + \alpha^2 \frac{U_{i+1j}^{n+1/2} - 2U_{ij}^{n+1/2} + U_{i-1j}^{n+1/2}}{\Delta x^2}$$