

Week 11, 2024

Formulation of the problem:

The 2D Burger's equation:

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{1}{Re} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \quad (1)$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = \frac{1}{Re} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \quad (2)$$

Re – Reynold's number

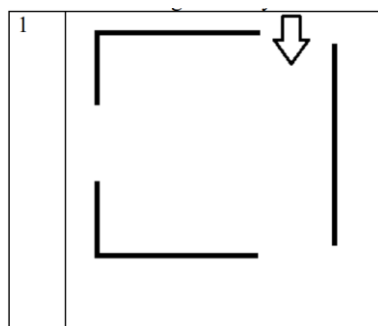
$$x \in (0, 1), \quad y \in (0, 1)$$

Initial condition:

$$U(t = 0, x, y) = 0$$

$$V(t = 0, x, y) = 0$$

And with the following boundary conditions:



Inlet: $U = 0, V = -1$

Outlet: $\frac{\partial U}{\partial n} = 0, \frac{\partial V}{\partial n} = 0$

Wall: $U = 0, V = 0$

Numerical method: Fractional step method

The same algorithm that we did in the last 2 weeks. But here's a little change. In Burger's equation we have convection term (the first derivatives by space) so linear differential operators Λ_1, Λ_2 will a little bit change. And everything else is the same as we did before.

P.S. I hope you know how to solve the Thomas algorithm and how to find coefficients alpha and beta.

a) Implicit method FSM (Fractional step method)

First step:

$$\frac{U_{ij}^{n+1/2} - U_{ij}^n}{\Delta t} = \frac{1}{2} (\Lambda_1 U^{n+1/2} + \Lambda_1 U^n) + \Lambda_2 U^n$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \frac{1}{2} (\Lambda_2 U^{n+1} - \Lambda_2 U^n)$$

Here is the Λ_1, Λ_2 operators have the following form:

$$\Lambda_1 = \frac{1}{Re} \frac{\partial^2}{\partial x^2} - U \frac{\partial}{\partial x}, \quad \Lambda_2 = \frac{1}{Re} \frac{\partial^2}{\partial y^2} - V \frac{\partial}{\partial y}$$

First step:

$$\begin{aligned} \frac{U_{ij}^{n+1/2} - U_{ij}^n}{\Delta t} = & \\ = \frac{1}{2} \left(\frac{1}{Re} \frac{U_{i+1j}^{n+1/2} - 2U_{ij}^{n+1/2} + U_{i-1j}^{n+1/2}}{\Delta x^2} - U_{ij}^n \frac{U_{i+1j}^{n+1/2} - U_{ij}^{n+1/2}}{\Delta x} + \frac{1}{Re} \frac{U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n}{\Delta x^2} - U_{ij}^n \frac{U_{i+1j}^n - U_{ij}^n}{\Delta x} \right) \\ & + \frac{1}{Re} \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2} - V_{ij}^n \frac{U_{ij+1}^n - U_{ij}^n}{\Delta y} \end{aligned}$$

*It also will work if you take backward or central scheme for the first derivative. *

We are going to perform in this form to solve the Thomas algorithm.

$$A_{ij} U_{i+1j}^{n+1/2} + B_{ij} U_{ij}^{n+1/2} + C_{ij} U_{i-1j}^{n+1/2} = D_{ij}, \quad i = \overline{1, N-1}, j \text{ is fixed}$$

$$\begin{aligned} A_{ij} &= -\frac{1}{2Re\Delta x^2} + \frac{U_{ij}^n}{2\Delta x}, \quad B_{ij} = \frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} - \frac{U_{ij}^n}{2\Delta x}, \quad C_{ij} = -\frac{1}{2Re\Delta x^2} \\ D_{ij} &= \frac{U_{ij}^n}{\Delta t} + \frac{1}{2} \left(\frac{1}{Re} \frac{U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n}{\Delta x^2} - U_{ij}^n \frac{U_{i+1j}^n - U_{ij}^n}{\Delta x} \right) + \frac{1}{Re} \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2} - V_{ij}^n \frac{U_{ij+1}^n - U_{ij}^n}{\Delta y} \end{aligned}$$

Second step:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n+1/2}}{\Delta t} = \frac{1}{2} \left(\frac{1}{Re} \frac{U_{ij+1}^{n+1} - 2U_{ij}^{n+1} + U_{ij-1}^{n+1}}{\Delta y^2} - V_{ij}^n \frac{U_{ij+1}^{n+1} - U_{ij}^{n+1}}{\Delta y} - \frac{1}{Re} \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2} + V_{ij}^n \frac{U_{ij+1}^n - U_{ij}^n}{\Delta y} \right)$$

Thomas algorithm for y:

$$A_{ij} U_{ij+1}^{n+1} + B_{ij} U_{ij}^{n+1} + C_{ij} U_{ij-1}^{n+1} = D_{ij}, \quad j = \overline{1, M-1}, i \text{ is fixed}$$

$$A_{ij} = -\frac{1}{2Re\Delta y^2} + \frac{V_{ij}^n}{2\Delta y}, \quad B_{ij} = \frac{1}{\Delta t} + \frac{1}{Re\Delta y^2} - \frac{V_{ij}^n}{2\Delta y}, \quad C_{ij} = -\frac{1}{2Re\Delta y^2}$$

$$D_{ij} = \frac{U_{ij}^{n+1/2}}{\Delta t} - \frac{1}{2} \left(\frac{1}{Re} \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2} - V_{ij}^n \frac{U_{ij+1}^n - U_{ij}^n}{\Delta y} \right)$$

Do the same for V.

Derivations of alpha and betta:

$$U_{ij} = \alpha_{i+1j}U_{i+1j} + \beta_{i+1j}, \quad i = \overline{N-1, 1} \quad (3)$$

$$A_{ij}U_{i+1j} + B_{ij}U_{ij} + C_{ij}U_{i-1j} = D_{ij}, \quad i = \overline{1, N-1} \quad (4)$$

$$U_{i-1j} = \alpha_{ij}U_{ij} + \beta_{ij}, \quad i = \overline{1, N-1} \quad (5)$$

Substitute equation (5) into (4)

$$A_{ij}U_{i+1j} + B_{ij}U_{ij} + C_{ij}(\alpha_{ij}U_{ij} + \beta_{ij}) = D_{ij}$$

$$A_{ij}U_{i+1j} + (B_{ij} + C_{ij}\alpha_{ij})U_{ij} + C_{ij}\beta_{ij} = D_{ij}$$

$$U_{ij} = -\frac{A_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}U_{i+1j} + \frac{D_{ij} - C_{ij}\beta_{ij}}{B_{ij} + C_{ij}\alpha_{ij}} \quad (6)$$

Equation (6) looks like (3) so we can find $\alpha_{i+1j}, \beta_{i+1j}$

$$\alpha_{i+1j} = -\frac{A_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}, \quad \beta_{i+1j} = \frac{D_{ij} - C_{ij}\beta_{ij}}{B_{ij} + C_{ij}\alpha_{ij}}, \quad (7)$$

$$i = \overline{1, N-1}, j \text{ is fixed}$$