

Formulation of the problem:

The 2D Navier-Stokes equation for incompressible flow:

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \quad (1)$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \quad (2)$$

$$\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = 0 \quad (3)$$

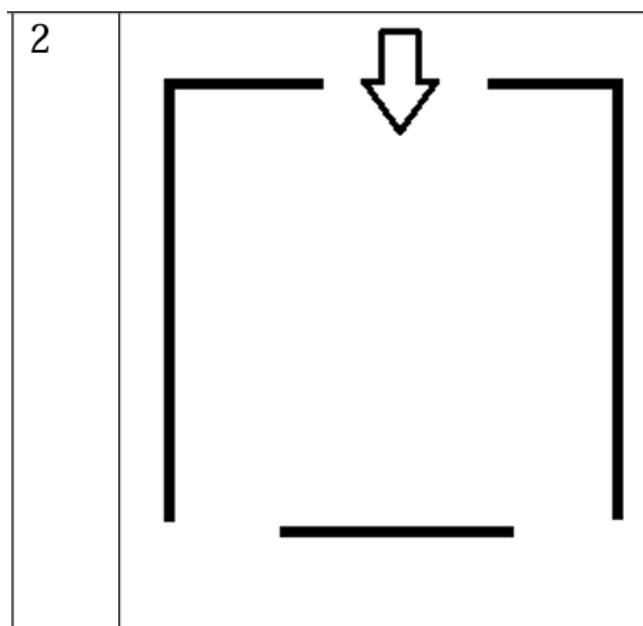
$U$  – velocity compnent of  $x$ ,  $V$  – velocity component of  $y$

$P$  – pressure,  $Re$  – Reynold's number

(1) - (2) Momentum equations

(3) Continuity equation

The following conditions:



$$x \in [0, 1], y \in [0, 1]$$

Initial condition:

$$U = 0, \quad V = 0, \quad P = 0$$

Inlet:

$$U = 0, \quad V = -1, \quad P = 1$$

Outlet:

$$\frac{\partial U}{\partial n} = 0, \quad \frac{\partial V}{\partial n} = 0, \quad P = 0$$

Walls:

$$U = 0, \quad V = 0, \quad \frac{\partial P}{\partial n} = 0$$

**Numerical method:** Projection method (Метод расщепления по физическим параметрам)

Projection method:

1)

$$\frac{U^{n+1} - U^n}{\Delta t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + L_1$$

$$\frac{V^{n+1} - V^n}{\Delta t} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + L_2$$

$$L_1 = -U \frac{\partial U}{\partial x} - V \frac{\partial U}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)$$

$$L_2 = -U \frac{\partial V}{\partial x} - V \frac{\partial V}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right)$$

$$\frac{U^{n+1} - U^n + U^* - U^*}{\Delta t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + L_1$$

$$\frac{V^{n+1} - V^n + V^* - V^*}{\Delta t} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + L_2$$

$$\frac{U^* - U^n}{\Delta t} = L_1, \text{Burger's equation}$$

$$\frac{V^* - V^n}{\Delta t} = L_2, \text{Burger's equation}$$

$$\frac{U^{n+1} - U^*}{\Delta t} = -\frac{1}{\rho} \frac{\partial P^{n+1}}{\partial x}$$

$$\frac{V^{n+1} - V^*}{\Delta t} = -\frac{1}{\rho} \frac{\partial P^{n+1}}{\partial y}$$

$$U^{n+1} = U^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial x}$$

$$V^{n+1} = V^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial y}$$

2)

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$\frac{\partial U^{n+1}}{\partial x} + \frac{\partial V^{n+1}}{\partial y} = 0$$

$$\frac{\partial}{\partial x} \left( U^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial x} \right) + \frac{\partial}{\partial y} \left( V^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial y} \right) = 0$$

$$\frac{\partial U^*}{\partial x} - \frac{\Delta t}{\rho} \frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial V^*}{\partial y} - \frac{\Delta t}{\rho} \frac{\partial^2 P^{n+1}}{\partial y^2} = 0$$

$$\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} = \frac{\rho}{\Delta t} \left( \frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y} \right), \quad \text{Poisson equation}$$

3)

$$U^{n+1} = U^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial x}$$

$$V^{n+1} = V^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial y}$$

Algorithm:

1) Solve Burger's equation to find  $U^*, V^*$ .

$$\frac{U^* - U^n}{\Delta t} = L_1$$
$$\frac{V^* - V^n}{\Delta t} = L_2$$

You can solve it with different numerical methods (Explicit, FSM, ADM, Tridiagonal matrix method).

2) Solve Poisson equation to find P.

$$\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} = \frac{\rho}{\Delta t} \left( \frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y} \right)$$

You can solve it with different numerical methods (Jacobi, Gauss-Seidel, Relaxation methods, Tridiagonal matrix method).

3) Correct velocity components  $U^{n+1}, V^{n+1}$ :

$$U^{n+1} = U^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial x}$$
$$V^{n+1} = V^* - \frac{\Delta t}{\rho} \frac{\partial P^{n+1}}{\partial y}$$

Numerical solution:

1) Burger's equation

$$\begin{aligned}\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \\ \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} &= \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right)\end{aligned}$$

Explicit method (Euler's method)

$$\begin{aligned}\frac{U^* - U^n}{\Delta t} &= -U \frac{\partial U}{\partial x} - V \frac{\partial U}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \\ U^* &= U^n + \Delta t \left( -U \frac{\partial U}{\partial x} - V \frac{\partial U}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \right) \\ \frac{V^* - V^n}{\Delta t} &= -U \frac{\partial V}{\partial x} - V \frac{\partial V}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \\ V^* &= V^n + \Delta t \left( -U \frac{\partial V}{\partial x} - V \frac{\partial V}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \right)\end{aligned}$$

Discretization:

$$\begin{aligned}U_{ij}^* &= U_{ij}^n + \Delta t \left( -U_{ij}^n \frac{U_{i+1j}^n - U_{i-1j}^n}{2\Delta x} - V_{ij}^n \frac{U_{ij+1}^n - U_{ij-1}^n}{2\Delta y} + \frac{1}{Re} \left( \frac{U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n}{\Delta x^2} + \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2} \right) \right) \\ V_{ij}^* &= V_{ij}^n + \Delta t \left( -U_{ij}^n \frac{V_{i+1j}^n - V_{i-1j}^n}{2\Delta x} - V_{ij}^n \frac{V_{ij+1}^n - V_{ij-1}^n}{2\Delta y} + \frac{1}{Re} \left( \frac{V_{i+1j}^n - 2V_{ij}^n + V_{i-1j}^n}{\Delta x^2} + \frac{V_{ij+1}^n - 2V_{ij}^n + V_{ij-1}^n}{\Delta y^2} \right) \right)\end{aligned}$$

2) Poisson equation

$$\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} = \frac{\rho}{\Delta t} \left( \frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y} \right)$$

Explicit method. Jacobi.

Discretization:

$$\begin{aligned}\frac{P_{i+1j}^n - 2P_{ij}^{n+1} + P_{i-1j}^n}{\Delta x^2} + \frac{P_{ij+1}^n - 2P_{ij}^{n+1} + P_{ij-1}^n}{\Delta y^2} &= \frac{\rho}{\Delta t} \left( \frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y} \right) \\ P_{ij}^{n+1} \left( \frac{2}{\Delta x^2} + \frac{2}{\Delta y^2} \right) &= \frac{P_{i+1j}^n + P_{i-1j}^n}{\Delta x^2} + \frac{P_{ij+1}^n + P_{ij-1}^n}{\Delta y^2} - \frac{\rho}{\Delta t} \left( \frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y} \right) \\ P_{ij}^{n+1} &= \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \left( \frac{P_{i+1j}^n + P_{i-1j}^n}{\Delta x^2} + \frac{P_{ij+1}^n + P_{ij-1}^n}{\Delta y^2} - \frac{\rho}{\Delta t} \left( \frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y} \right) \right)\end{aligned}$$

Final iterative formula:

$$P_{ij}^{n+1} = \frac{\Delta y^2 (P_{i+1j}^n + P_{i-1j}^n) + \Delta x^2 (P_{ij+1}^n + P_{ij-1}^n)}{2(\Delta x^2 + \Delta y^2)} - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \frac{\rho}{\Delta t} \left( \frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y} \right)$$

Do it until steady condition is reached:

$$\max(|P^{n+1} - P^n|) < \varepsilon$$

$\varepsilon$  – small number,  $10^{-6}$

3) Correction

$$U^{n+1} = U^* - \frac{\Delta t}{\rho} \frac{\partial P}{\partial x}$$

$$V^{n+1} = V^* - \frac{\Delta t}{\rho} \frac{\partial P}{\partial y}$$

Discretization:

$$U_{ij}^{n+1} = U_{ij}^* - \frac{\Delta t}{\rho} \frac{P_{i+1j}^{n+1} - P_{i-1j}^{n+1}}{2\Delta x}$$

$$V_{ij}^{n+1} = V_{ij}^* - \frac{\Delta t}{\rho} \frac{P_{ij+1}^{n+1} - P_{ij-1}^{n+1}}{2\Delta y}$$

Do it until steady condition is reached:

$$\max(|U^{n+1} - U^n|, |V^{n+1} - V^n|) < \varepsilon$$

$\varepsilon$  – small number,  $10^{-6}$