Week 8, 2024

Fractional step and alternating direction methods

Formulation of the problem:

The 3D Heat equation:

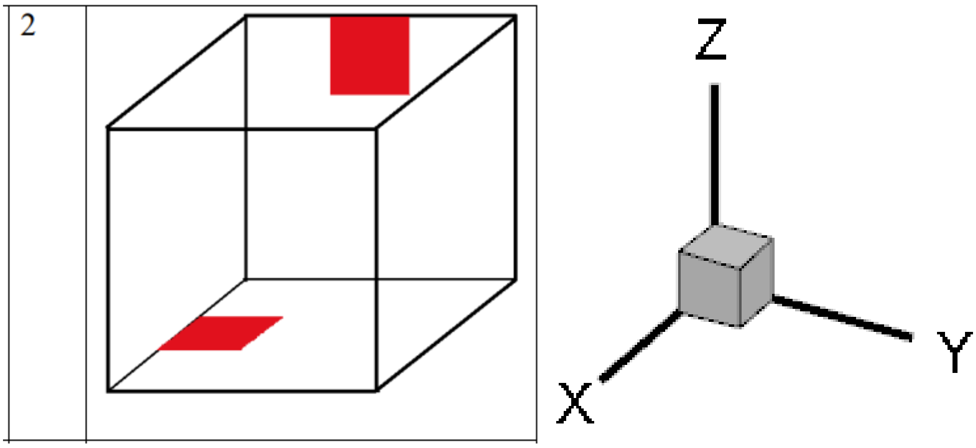
$$\frac{\partial U}{\partial t} = \alpha^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right), \qquad x \in (0,1), \qquad y \in (0,1), \qquad z \in (0,1)$$

Initial condition: $U(t=0,x,y,z) = 0$

And with the following boundary conditions:

$$U(t, x=0, 0.3 < y < 0.6, 0.7 < z < 1) = 1$$
$$U(t, 0.4 < x < 0.7, 0 < y < 0.3, z = 0) = 1$$

Otherwise, is 0



Numerical method: Fractional step and alternating direction methods

These two implicit methods are very similar to each other because the main idea of this methods based on fractioning time internal on 3 parts for each own dimension so problem is converted into one-dimensional problem, and we can solve it easily as we solved in first 3 week doing simple Thomas algorithm. The approximation error of methods $O(\Delta t^2, \Delta x^2, \Delta y^2, \Delta z^2)$.

P.S. I hope you know how to solve the Thomas algorithm and how to find coefficients alpha and betta.

a) Implicit method FSM (Fractional step method)

First step:

$$\frac{U_{ijk}^{n+1/3} - U_{ijk}^{n}}{\Delta t} = \frac{1}{2}\left(\Lambda_1 U^{n+1/3} + \Lambda_1 U^n\right) + \Lambda_2 U^n + \Lambda_3 U^n$$

Second step:

$$\frac{U_{ijk}^{n+2/3} - U_{ijk}^{n+1/3}}{\Delta t} = \frac{1}{2}(\Lambda_2 U^{n+2/3} - \Lambda_2 U^n)$$

Third step:

$$\frac{U_{ijk}^{n+1} - U_{ijk}^{n+2/3}}{\Delta t} = \frac{1}{2}(\Lambda_3 U^{n+1} - \Lambda_3 U^n)$$

Here is the Λ_1, Λ_2 and $\Lambda_3$operators have the following form:

$$\Lambda_1 = \alpha^2 \frac{\partial^2}{\partial x^2}, \qquad \Lambda_2 = \alpha^2 \frac{\partial^2}{\partial y^2}, \qquad \Lambda_3 = \alpha^2 \frac{\partial^2}{\partial z^2}$$

First step:

$$\frac{U_{ijk}^{n+1/3} - U_{ijk}^{n}}{\Delta t} =$$

$$= \frac{1}{2}\left(\alpha^2 \frac{U_{i+1jk}^{n+1/3} - 2U_{ijk}^{n+1/3} + U_{i-1jk}^{n+1/3}}{\Delta x^2} + \alpha^2 \frac{U_{i+1jk}^{n} - 2U_{ijk}^{n} + U_{i-1jk}^{n}}{\Delta x^2}\right)$$
$$+ \alpha^2 \frac{U_{ij+1k}^{n} - 2U_{ijk}^{n} + U_{ij-1k}^{n}}{\Delta y^2} + \alpha^2 \frac{U_{ijk+1}^{n} - 2U_{ijk}^{n} + U_{ijk-1}^{n}}{\Delta z^2}$$

The equation is converted in 1D, and we can solve it by the Thomas algorithm.

$$A_{ijk} U_{i+1jk}^{n+1/3} + B_{ijk} U_{ijk}^{n+1/3} + C_{ijk} U_{i-1jk}^{n+1/3} = D_{ijk}, \qquad i = \overline{1, N-1}, j, k \ is \ fixed$$

$$A_{ijk} = -\frac{\alpha^2}{2\Delta x^2}, \qquad B_{ijk} = \frac{1}{\Delta t} + \frac{\alpha^2}{\Delta x^2}, \qquad C_{ijk} = -\frac{\alpha^2}{2\Delta x^2}$$
$$D_{ijk} = \frac{U_{ijk}^{n}}{\Delta t} + \frac{1}{2}\left(\alpha^2 \frac{U_{i+1jk}^{n} - 2U_{ijk}^{n} + U_{i-1jk}^{n}}{\Delta x^2}\right) + \alpha^2 \frac{U_{ij+1k}^{n} - 2U_{ijk}^{n} + U_{ij-1k}^{n}}{\Delta y^2}$$
$$+ \alpha^2 \frac{U_{ijk+1}^{n} - 2U_{ijk}^{n} + U_{ijk-1}^{n}}{\Delta z^2}$$

Second step:

$$\frac{U_{ijk}^{n+2/3}-U_{ijk}^{n+1/3}}{\Delta t}=\frac{1}{2}\left(\alpha^2\frac{U_{ij+1k}^{n+2/3}-2U_{ijk}^{n+2/3}+U_{ij-1k}^{n+2/3}}{\Delta y^2}-\alpha^2\frac{U_{ij+1k}^{n}-2U_{ijk}^{n}+U_{ij-1k}^{n}}{\Delta y^2}\right)$$

Thomas algorithm for y:

$$A_{ijk}U_{ij+1k}^{n+2/3}+B_{ijk}U_{ijk}^{n+2/3}+C_{ijk}U_{ij-1k}^{n+2/3}=D_{ijk},\qquad j=\overline{1,M-1}, i,k\ is\ fixed$$

$$A_{ijk}=-\frac{\alpha^2}{2\Delta y^2},\qquad B_{ijk}=\frac{1}{\Delta t}+\frac{\alpha^2}{\Delta y^2},\qquad C_{ijk}=-\frac{\alpha^2}{2\Delta y^2}$$

$$D_{ijk}=\frac{U_{ijk}^{n+1/3}}{\Delta t}-\frac{1}{2}\left(\alpha^2\frac{U_{ij+1k}^{n}-2U_{ijk}^{n}+U_{ij-1k}^{n}}{\Delta y^2}\right)$$

Third step:

$$\frac{U_{ijk}^{n+1}-U_{ijk}^{n+2/3}}{\Delta t}=\frac{1}{2}\left(\alpha^2\frac{U_{ijk+1}^{n+1}-2U_{ijk}^{n+1}+U_{ijk-1}^{n+1}}{\Delta z^2}-\alpha^2\frac{U_{ijk+1}^{n}-2U_{ijk}^{n}+U_{ijk-1}^{n}}{\Delta z^2}\right)$$

Thomas algorithm for z:

$$A_{ijk}U_{ijk+1}^{n+1}+B_{ijk}U_{ijk}^{n+1}+C_{ijk}U_{ijk-1}^{n+1}=D_{ijk},\qquad k=\overline{1,P-1}, i,j\ is\ fixed$$

$$A_{ijk}=-\frac{\alpha^2}{2\Delta z^2},\qquad B_{ijk}=\frac{1}{\Delta t}+\frac{\alpha^2}{\Delta z^2},\qquad C_{ijk}=-\frac{\alpha^2}{2\Delta z^2}$$

$$D_{ijk}=\frac{U_{ijk}^{n+2/3}}{\Delta t}-\frac{1}{2}\left(\alpha^2\frac{U_{ijk+1}^{n}-2U_{ijk}^{n}+U_{ijk-1}^{n}}{\Delta z^2}\right)$$

a) Implicit method ADM (Alternating direction method)

First step:

$$\frac{U_{ijk}^{n+1/3} - U_{ijk}^{n}}{\Delta t} = \Lambda_1 U^{n+1/3} + \Lambda_2 U^{n} + \Lambda_3 U^{n}$$

Second step:

$$\frac{U_{ijk}^{n+2/3} - U_{ijk}^{n+1/3}}{\Delta t} = \Lambda_1 U^{n+1/3} + \Lambda_2 U^{n+2/3} + \Lambda_3 U^{n+1/3}$$

Third step:

$$\frac{U_{ijk}^{n+1} - U_{ijk}^{n+2/3}}{\Delta t} = \Lambda_1 U^{n+2/3} + \Lambda_2 U^{n+2/3} + \Lambda_3 U^{n+1}$$

Here is the Λ_1, Λ_2 and $\Lambda_3$operators have the following form:

$$\Lambda_1 = \alpha^2 \frac{\partial^2}{\partial x^2}, \qquad \Lambda_2 = \alpha^2 \frac{\partial^2}{\partial y^2}, \qquad \Lambda_3 = \alpha^2 \frac{\partial^2}{\partial z^2}$$

First step:

$$\frac{U_{ijk}^{n+1/3} - U_{ijk}^{n}}{\Delta t} =$$

$$= \alpha^2 \frac{U_{i+1jk}^{n+1/3} - 2U_{ijk}^{n+1/3} + U_{i-1jk}^{n+1/3}}{\Delta x^2} + \alpha^2 \frac{U_{ij+1k}^{n} - 2U_{ijk}^{n} + U_{ij-1k}^{n}}{\Delta y^2} + \alpha^2 \frac{U_{ijk+1}^{n} - 2U_{ijk}^{n} + U_{ijk-1}^{n}}{\Delta z^2}$$

The equation is converted in 1D, and we can solve it by the Thomas algorithm.

$$A_{ijk} U_{i+1jk}^{n+1/3} + B_{ijk} U_{ijk}^{n+1/3} + C_{ijk} U_{i-1jk}^{n+1/3} = D_{ijk}, \qquad i = \overline{1, N-1}, j, k \ is \ fixed$$

$$A_{ijk} = -\frac{\alpha^2}{\Delta x^2}, \qquad B_{ijk} = \frac{1}{\Delta t} + \frac{2\alpha^2}{\Delta x^2}, \qquad C_{ijk} = -\frac{\alpha^2}{\Delta x^2}$$

$$D_{ijk} = \frac{U_{ijk}^{n}}{\Delta t} + \alpha^2 \frac{U_{ij+1k}^{n} - 2U_{ijk}^{n} + U_{ij-1k}^{n}}{\Delta y^2} + \alpha^2 \frac{U_{ijk+1}^{n} - 2U_{ijk}^{n} + U_{ijk-1}^{n}}{\Delta z^2}$$

Second step:

$$\frac{U_{ijk}^{n+2/3}-U_{ijk}^{n+1/3}}{\Delta t}=\alpha^2\frac{U_{i+1jk}^{n+1/3}-2U_{ijk}^{n+1/3}+U_{i-1jk}^{n+1/3}}{\Delta x^2}+\alpha^2\frac{U_{ij+1k}^{n+2/3}-2U_{ijk}^{n+2/3}+U_{ij-1k}^{n+2/3}}{\Delta y^2}$$
$$+\alpha^2\frac{U_{ijk+1}^{n+1/3}-2U_{ijk}^{n+1/3}+U_{ijk-1}^{n+1/3}}{\Delta z^2}$$

Thomas algorithm for y:

$$A_{ijk}U_{ij+1k}^{n+2/3}+B_{ijk}U_{ijk}^{n+2/3}+C_{ijk}U_{ij-1k}^{n+2/3}=D_{ijk},\qquad j=\overline{1,M-1}, i,k\ is\ fixed$$
$$A_{ijk}=-\frac{\alpha^2}{\Delta y^2},\qquad B_{ijk}=\frac{1}{\Delta t}+\frac{2\alpha^2}{\Delta y^2},\qquad C_{ijk}=-\frac{\alpha^2}{\Delta y^2}$$
$$D_{ijk}=\frac{U_{ijk}^{n+1/3}}{\Delta t}+\alpha^2\frac{U_{i+1jk}^{n+1/3}-2U_{ijk}^{n+1/3}+U_{i-1jk}^{n+1/3}}{\Delta x^2}+\alpha^2\frac{U_{ijk+1}^{n+1/3}-2U_{ijk}^{n+1/3}+U_{ijk-1}^{n+1/3}}{\Delta z^2}$$

Third step:

$$\frac{U_{ijk}^{n+1}-U_{ijk}^{n+\frac{2}{3}}}{\Delta t}=$$
$$=\alpha^2\frac{U_{i+1jk}^{n+2/3}-2U_{ijk}^{n+2/3}+U_{i-1jk}^{n+2/3}}{\Delta x^2}+\alpha^2\frac{U_{ij+1k}^{n+2/3}-2U_{ijk}^{n+2/3}+U_{ij-1k}^{n+2/3}}{\Delta y^2}$$
$$+\alpha^2\frac{U_{ijk+1}^{n+1}-2U_{ijk}^{n+1}+U_{ijk-1}^{n+1}}{\Delta z^2}$$

Thomas algorithm for z:

$$A_{ijk}U_{ijk+1}^{n+1}+B_{ijk}U_{ijk}^{n+1}+C_{ijk}U_{ijk-1}^{n+1}=D_{ijk},\qquad k=\overline{1,P-1}, i,j\ is\ fixed$$
$$A_{ijk}=-\frac{\alpha^2}{\Delta z^2},\qquad B_{ijk}=\frac{1}{\Delta t}+\frac{2\alpha^2}{\Delta z^2},\qquad C_{ijk}=-\frac{\alpha^2}{\Delta z^2}$$
$$D_{ijk}=\frac{U_{ijk}^{n+2/3}}{\Delta t}+\alpha^2\frac{U_{i+1jk}^{n+2/3}-2U_{ijk}^{n+2/3}+U_{i-1jk}^{n+2/3}}{\Delta x^2}+\alpha^2\frac{U_{ij+1k}^{n+2/3}-2U_{ijk}^{n+2/3}+U_{ij-1k}^{n+2/3}}{\Delta y^2}$$