Week 12, 2024

Formulation of the problem:

The 2D Navier-Stokes equation for incompressible flow:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{dy^2}\right) \tag{1}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{dy^2}\right) \tag{2}$$

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{3}$$

$$U - velocity\ compent\ of\ x, V - velocity\ component\ of\ y$$

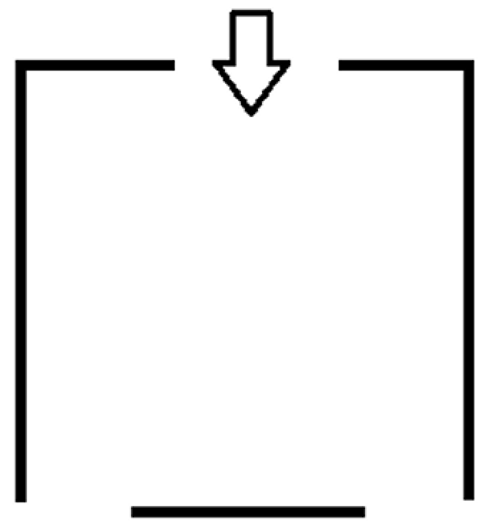
$$P - pressure, Re - Reynold's\ number$$

(1) - (2) Momentum equations

(3) Continuity equation

The following conditions:

2



$$x \epsilon [0,1], y \epsilon [0,1]$$

Initial condition:

$$U = 0, \qquad V = 0, \qquad P = 0$$

Inlet:

$$U = 0, \qquad V = -1, \qquad P = 1$$

Outlet:

$$\frac{\partial U}{\partial n} = 0, \qquad \frac{\partial V}{\partial n} = 0, \qquad P = 0$$

Walls:

$$U = 0, \qquad V = 0, \qquad \frac{\partial P}{\partial n} = 0$$

Numerical method: Projection method (Метод расщепления по физическим параметрам)

Projection method:

1)

$$\frac{U^{n+1}-U^{n}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P}{dx}+L_1$$

$$L_1=-U\frac{\partial U}{\partial x}-V\frac{\partial U}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{dy^2}\right)$$

$$\frac{U^{n+1}-U^{n}+U^{*}-U^{*}}{\Delta \mathrm{t}}=-\frac{1}{\rho}\frac{\partial P}{dx}+L_1$$

$$\frac{U^{*}-U^{n}}{\Delta t}=L_1, Burger's\ equation$$

$$\frac{U^{n+1}-U^{*}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P^{n+1}}{\partial x}$$

$$U^{n+1}=U^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}$$

$$\frac{V^{n+1}-V^{n}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P}{dy}+L_2$$

$$L_2=-U\frac{\partial V}{\partial x}-V\frac{\partial V}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{dy^2}\right)$$

$$\frac{V^{n+1}-V^{n}+V^{*}-V^{*}}{\Delta \mathrm{t}}=-\frac{1}{\rho}\frac{\partial P}{dy}+L_2$$

$$\frac{V^{*}-V^{n}}{\Delta t}=L_2, Burger's\ equation$$

$$\frac{V^{n+1}-V^{*}}{\Delta t}=-\frac{1}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

$$V^{n+1}=V^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

2)

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0$$

$$\frac{\partial U^{n+1}}{\partial x}+\frac{\partial V^{n+1}}{\partial y}=0$$

$$\frac{\partial}{\partial x}\left(U^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}\right)+\frac{\partial}{\partial y}\left(V^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}\right)=0$$

$$\frac{\partial U^{*}}{\partial x}-\frac{\Delta t}{\rho}\frac{\partial^2 P^{n+1}}{\partial x^2}+\frac{\partial V^{*}}{\partial y}-\frac{\Delta t}{\rho}\frac{\partial^2 P^{n+1}}{\partial y^2}=0$$

$$\frac{\partial^2 P^{n+1}}{\partial x^2}+\frac{\partial^2 P^{n+1}}{\partial y^2}=\frac{\rho}{\Delta t}\left(\frac{\partial U^{*}}{\partial x}+\frac{\partial V^{*}}{\partial y}\right),\qquad Poisson\ equation$$

3)

$$U^{n+1}=U^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}$$

$$V^{n+1}=V^{*}-\frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

Algorithm:

1) Solve Burger's equation to find U^*, V^*.

$$\frac{U^* - U^n}{\Delta t} = L_1$$
$$\frac{V^* - V^n}{\Delta t} = L_2$$

You can solve it with different numerical methods (Explicit, FSM, ADM, Tridiagonal matrix method).

2) Solve Poisson equation to find P.

$$\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} = \frac{\rho}{\Delta t}\left(\frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y}\right)$$

You can solve it with different numerical methods (Jacobi, Gauss-Seidel, Relaxation methods, Tridiagonal matrix method).

3) Correct velocity components U^{n+1}, V^{n+1}:

$$U^{n+1} = U^* - \frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial x}$$
$$V^{n+1} = V^* - \frac{\Delta t}{\rho}\frac{\partial P^{n+1}}{\partial y}$$

Numerical solution:

1) Burger's equation

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{dy^2}\right)$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{dy^2}\right)$$

Explicit method (Euler's method)

$$\frac{U^* - U^n}{\Delta t} = -U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{dy^2}\right)$$

$$U^* = U^n + \Delta t\left(-U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{dy^2}\right)\right)$$

$$\frac{V^* - V^n}{\Delta t} = -U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{dy^2}\right)$$

$$V^* = V^n + \Delta t\left(-U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{dy^2}\right)\right)$$

Discretization:

$$U_{ij}^* = U_{ij}^n + \Delta t\left(-U_{ij}^n\frac{U_{i+1j}^n - U_{i-1j}^n}{2\Delta x} - V_{ij}^n\frac{U_{ij+1}^n - U_{ij-1}^n}{2\Delta y} + \frac{1}{Re}\left(\frac{U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n}{\Delta x^2} + \frac{U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n}{\Delta y^2}\right)\right)$$

$$V_{ij}^* = V_{ij}^n + \Delta t\left(-U_{ij}^n\frac{V_{i+1j}^n - V_{i-1j}^n}{2\Delta x} - V_{ij}^n\frac{V_{ij+1}^n - V_{ij-1}^n}{2\Delta y} + \frac{1}{Re}\left(\frac{V_{i+1j}^n - 2V_{ij}^n + V_{i-1j}^n}{\Delta x^2} + \frac{V_{ij+1}^n - 2V_{ij}^n + V_{ij-1}^n}{\Delta y^2}\right)\right)$$

2) Poisson equation

$$\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} = \frac{\rho}{\Delta t}\left(\frac{\partial U^*}{\partial x} + \frac{\partial V^*}{\partial y}\right)$$

Explicit method. Jacobi.

Discretization:

$$\frac{P_{i+1j}^n - 2P_{ij}^{n+1} + P_{i-1j}^n}{\Delta x^2} + \frac{P_{ij+1}^n - 2P_{ij}^{n+1} + P_{ij-1}^n}{\Delta y^2} = \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y}\right)$$

$$P_{ij}^{n+1}\left(\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}\right) = \frac{P_{i+1j}^n + P_{i-1j}^n}{\Delta x^2} + \frac{P_{ij+1}^n + P_{ij-1}^n}{\Delta y^2} - \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y}\right)$$

$$P_{ij}^{n+1} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\left(\frac{P_{i+1j}^n + P_{i-1j}^n}{\Delta x^2} + \frac{P_{ij+1}^n + P_{ij-1}^n}{\Delta y^2} - \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y}\right)\right)$$

Final iterative formula:

$$P_{ij}^{n+1} = \frac{\Delta y^2\left(P_{i+1j}^n + P_{i-1j}^n\right) + \Delta x^2(P_{ij+1}^n + P_{ij-1}^n)}{2(\Delta x^2 + \Delta y^2)} - \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \frac{\rho}{\Delta t}\left(\frac{U_{i+1j}^* - U_{i-1j}^*}{2\Delta x} + \frac{V_{ij+1}^* - V_{ij-1}^*}{2\Delta y}\right)$$

Do it until steady condition is reached:

$$\max(|P^{n+1} - P^n|) < \varepsilon$$

$$\varepsilon - small\ number, 10^{-6}$$

3) Correction

$$U^{n+1} = U^* - \frac{\Delta t}{\rho}\frac{\partial P}{\partial x}$$

$$V^{n+1} = V^* - \frac{\Delta t}{\rho}\frac{\partial P}{\partial y}$$

Discretization:

$$U_{ij}^{n+1} = U_{ij}^* - \frac{\Delta t}{\rho}\frac{P_{i+1j}^{n+1} - P_{i-1j}^{n+1}}{2\Delta x}$$

$$V_{ij}^{n+1} = V_{ij}^* - \frac{\Delta t}{\rho}\frac{P_{ij+1}^{n+1} - P_{ij-1}^{n+1}}{2\Delta y}$$

Do it until steady condition is reached:

$$\max(|U^{n+1} - U^n|, |V^{n+1} - V^n|) < \varepsilon$$

$$\varepsilon - small\ number, 10^{-6}$$