Parabolic Equations

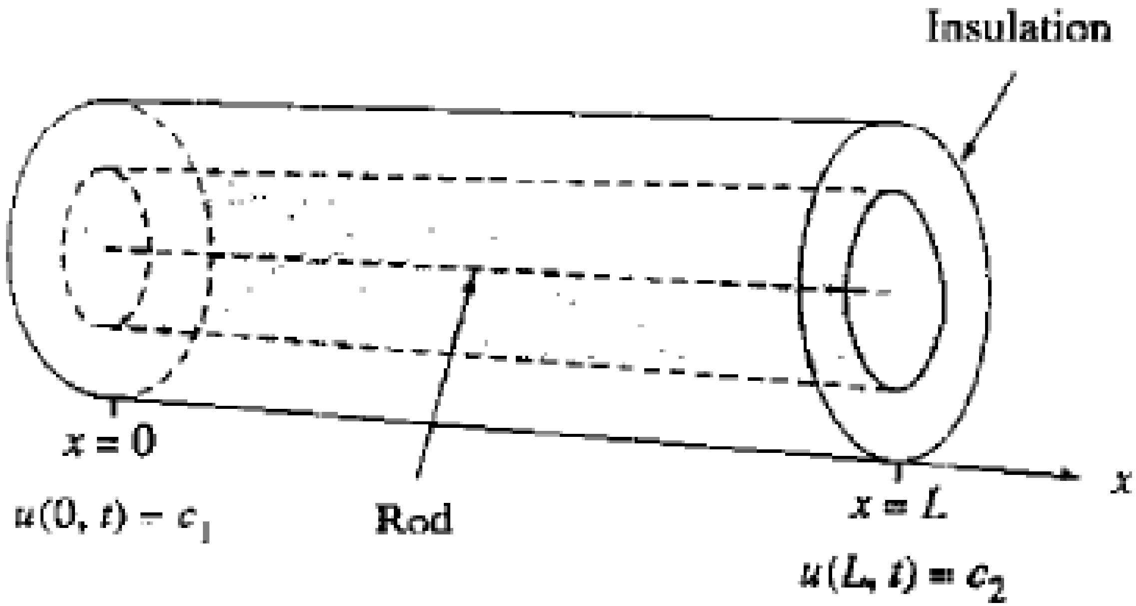

Figure 10.2 The heat equation models the temperature in an insulated rod.

As an example of a parabolic equation, we consider the one-dimensional model for heat flow in an insulated rod of length L (see Figure 10.2). The heat equation, which involves the temperature $u(x, t)$ in the rod at the position x and time t, is

$$(8) \qquad \kappa u_{xx}(x, t) = \sigma \rho u_t(x, t) \quad \text{for } 0 < x < L \text{ and } 0 < t < \infty.$$

the initial temperature distribution at $t = 0$ is

$$u(x, 0) = f(x) \quad \text{for } t = 0 \text{ and } 0 \le x \le L, \tag{9}$$

and the boundary values at the ends of the rod are

$$\begin{aligned} u(0, t) &= c_1 \quad \text{for } x = 0 \text{ and } 0 \le t < \infty, \\ u(L, t) &= c_2 \quad \text{for } x = L \text{ and } 0 \le t < \infty. \end{aligned} \tag{10}$$

The constant κ is the coefficient of thermal conductivity, σ is the specific heat, and ρ is the density of the material in the rod.

Heat Equation

As an example of parabolic differential equations, we consider the one-dimensional heat equation

$$(1) \qquad u_t(x,t) = c^2 u_{xx}(x,t) \qquad \text{for } 0 \le x < a \text{ and } 0 < t < b,$$

with the initial condition

$$(2) \qquad u(x,0) = f(x) \qquad \text{for } t = 0 \text{ and } 0 \le x \le a,$$

and the boundary conditions

$$(3) \qquad \begin{aligned} u(0,t) &= g_1(t) \equiv c_1 \qquad \text{for } x = 0 \text{ and } 0 \le t \le b, \\ u(a,t) &= g_2(t) \equiv c_2 \qquad \text{for } x = a \text{ and } 0 \le t \le b. \end{aligned}$$

Derivation of the Difference Equation

Assume that the rectangle $R = \{(x, t) : 0 \le x \le a, 0 \le t \le b\}$ is subdivided into $n - 1$ by $m - 1$ rectangles with sides $\Delta x = h$ and $\Delta t = k$, as shown in Figure 10.8. Start at the bottom row, where $t = t_1 = 0$, and the solution is $u(x_i, t_1) = f(x_i)$. A method for computing the approximations to $u(x, t)$ at grid points in successive rows $\{u(x_i, t_j) : i = 1, 2, \ldots, n\}$, for $j = 2, 3, \ldots, m$, will be developed.

The difference formulas used for $u_t(x, t)$ and $u_{xx}(x, t)$ are

$$u_t(x, t) = \frac{u(x, t + k) - u(x, t)}{k} + O(k) \tag{4}$$

and

$$u_{xx}(x, t) = \frac{u(x - h, t) - 2u(x, t) + u(x + h, t)}{h^2} + O(h^2). \tag{5}$$

The grid spacing is uniform in every row: $x_{i+1} = x_i + h$ (and $x_{i-1} = x_i - h$), and it is uniform in every column: $t_{j+1} = t_j + k$. Next, we drop the terms $O(k)$ and $O(h^2)$ and use the approximation $u_{i,j}$ for $u(x_i, t_j)$ in equations (4) and (5), which are in turn substituted into equation (1) to obtain

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \tag{6}$$

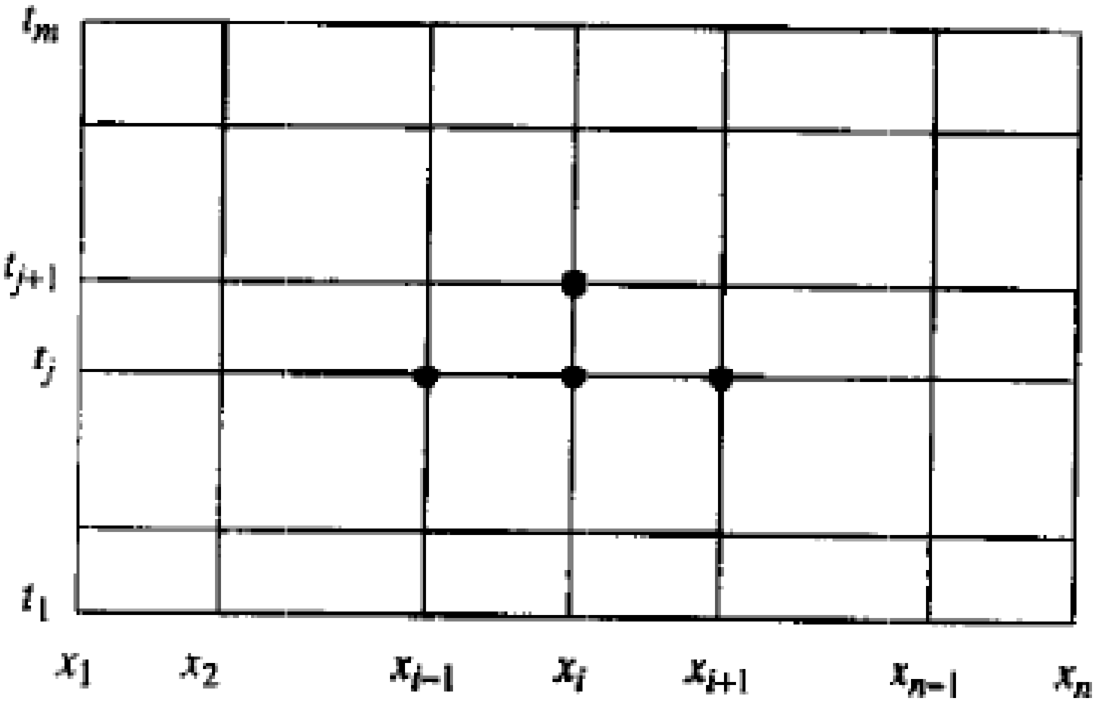

Figure 10.8 The grid for solving $u_t(x, t) = c^2 u_{xx}(x, t)$ over R.

which approximates the solution to (1). For convenience, the substitution $r = c^2k/h^2$ is introduced in (6), and the result is the explicit forward-difference equation

$$u_{i,j+1} = (1-2r)u_{i,j} + r(u_{i-1,j} + u_{i+1,j}). \tag{7}$$

Equation (7) is employed to create the $(j+1)$th row across the grid, assuming that approximations in the jth row are known. Notice that this formula explicitly gives the value $u_{i,j+1}$ in terms of $u_{i-1,j}$, $u_{i,j}$, and $u_{i+1,j}$. The computational stencil representing the situation in formula (7) is given in Figure 10.9.

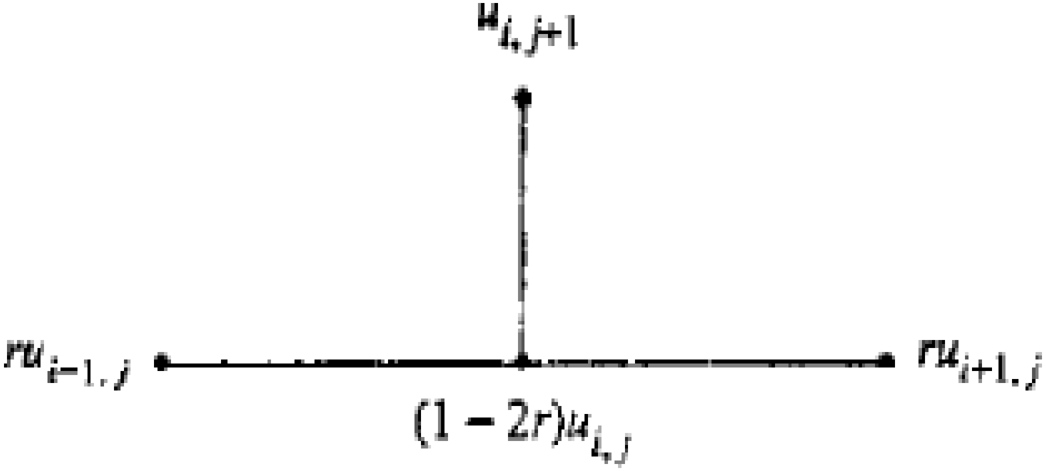

Figure 10.9 The forward difference stencil.

The simplicity of formula (7) makes it appealing to use. However, it is important to use numerical techniques that are stable. If any error made at one stage of the calculations is eventually dampened out, the method is called stable. The explicit forward-difference equation (7) is stable if and only if r is restricted to the interval $0 \le r \le \frac{1}{2}$. This means that the step size k must satisfy $k \le h^2/(2c^2)$. If this condition is not fulfilled, errors committed in one line $\{u_{i,j}\}$ might be magnified in subsequent lines $\{u_{i,p}\}$ for some $p > j$. The next example illustrates this point.

Example 10.3. Use the forward-difference method to solve the heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.20, \tag{8}$$

with the initial condition

$$u(x,0) = f(x) = 4x - 4x^2 \quad \text{for } t = 0 \text{ and } 0 \le x \le 1, \tag{9}$$

and the boundary conditions

$$\begin{aligned} u(0,t) &= g_1(t) \equiv 0 \quad \text{for } x = 0 \text{ and } 0 \le t \le 0.20, \\ u(1,t) &= g_2(t) \equiv 0 \quad \text{for } x = 1 \text{ and } 0 \le t \le 0.20. \end{aligned} \tag{10}$$

For the first illustration, we use the step sizes $\Delta x = h = 0.2$ and $\Delta t = k = 0.02$ and $c = 1$, so the ratio is $r = 0.5$. The grid will be $n = 6$ columns wide by $m = 11$ rows high. In this case, formula (7) becomes

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}. \tag{11}$$

Formula (11) is stable for $r = 0.5$ and can be used successfully to generate reasonably accurate approximations to $u(x, t)$. Successive rows in the grid are given in Table 10.3. A three-dimensional presentation of the data in Table 10.3 is given in Figure 10.10.

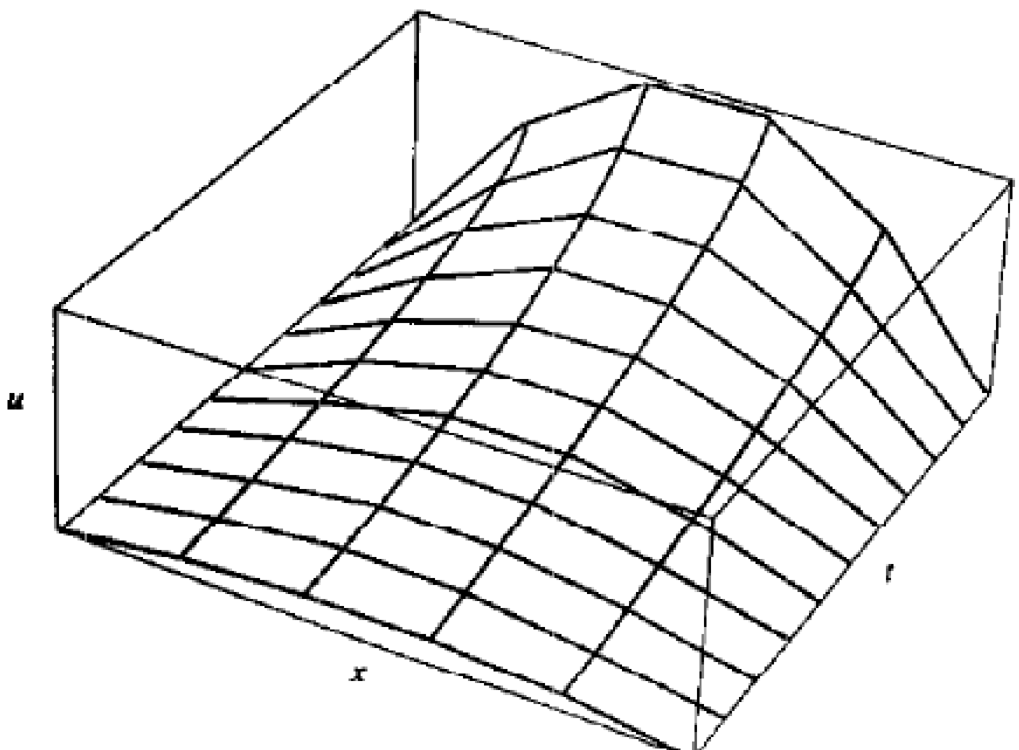

Table 10.3 Using the Forward-difference Method with $r = 0.5$

	$x_1 = 0.00$	$x_2 = 0.20$	$x_3 = 0.40$	$x_4 = 0.60$	$x_5 = 0.80$	$x_6 = 1.00$
$t_1 = 0.00$	0.000000	0.640000	0.960000	0.960000	0.640000	0.000000
$t_2 = 0.02$	0.000000	0.480000	0.800000	0.800000	0.480000	0.000000
$t_3 = 0.04$	0.000000	0.400000	0.640000	0.640000	0.400000	0.000000
$t_4 = 0.06$	0.000000	0.320000	0.520000	0.520000	0.320000	0.000000
$t_5 = 0.08$	0.000000	0.260000	0.420000	0.420000	0.260000	0.000000
$t_6 = 0.10$	0.000000	0.210000	0.340000	0.340000	0.210000	0.000000
$t_7 = 0.12$	0.000000	0.170000	0.275000	0.275000	0.170000	0.000000
$t_8 = 0.14$	0.000000	0.137500	0.222500	0.222500	0.137500	0.000000
$t_9 = 0.16$	0.000000	0.111250	0.180000	0.180000	0.111250	0.000000
$t_{10} = 0.18$	0.000000	0.090000	0.145625	0.145625	0.090000	0.000000
$t_{11} = 0.20$	0.000000	0.072812	0.117813	0.117813	0.072812	0.000000

For our second illustration, we use the step sizes $\Delta x = h = 0.2$ and $\Delta t = k = \frac{1}{30} \approx 0.033333$, so that the ratio is $r = 0.833333$. In this case, formula (7) becomes

$$u_{i,j+1} = -0.666665u_{i,j} + 0.833333(u_{i-1,j} + u_{i+1,j}). \tag{12}$$

Formula (12) is unstable in this case, because $r > \frac{1}{2}$, and errors committed at one row will be magnified in successive rows. Numerical values that turn out to be imprecise approximations to $u(x, t)$, for $0 \le t \le 0.33333$, are given in Table 10.4. A three-dimensional presentation of the data in Table 10.4 is given in Figure 10.11.

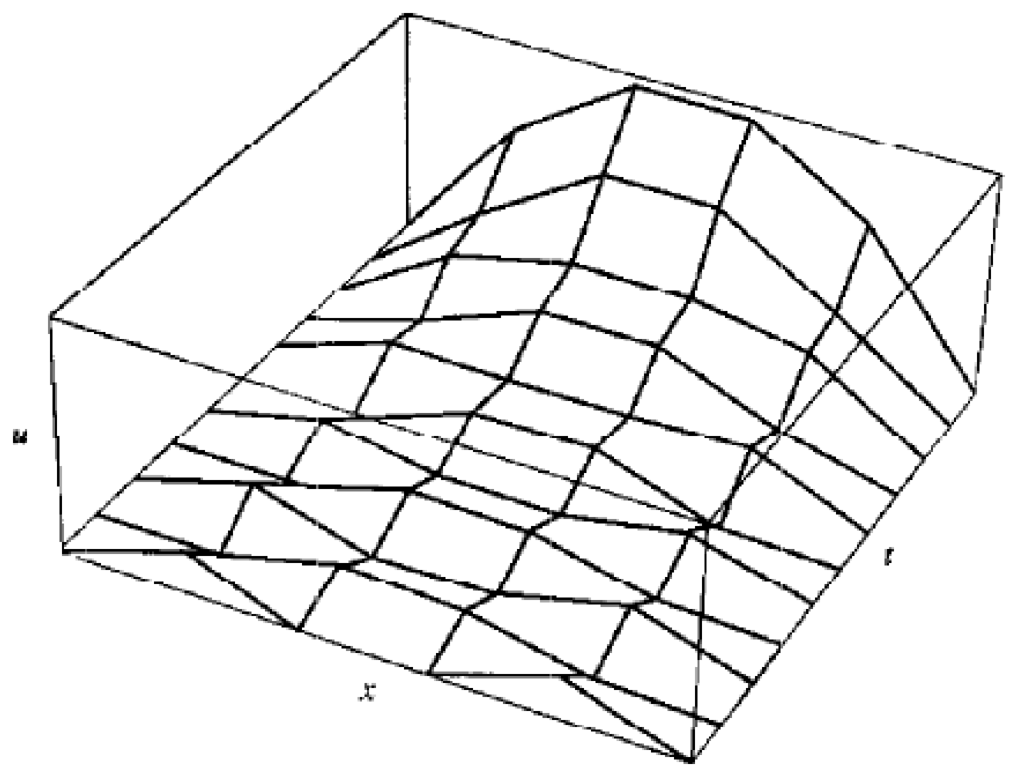

Table 10.4 Using the Forward-difference Method with $r = 0.833333$

	$x_1 = 0.00$	$x_2 = 0.20$	$x_3 = 0.40$	$x_4 = 0.60$	$x_5 = 0.80$	$x_6 = 1.00$
$t_1 = 0.000000$	0.000000	0.640000	0.960000	0.960000	0.640000	0.000000
$t_2 = 0.033333$	0.000000	0.373333	0.693333	0.693333	0.373333	0.000000
$t_3 = 0.066667$	0.000000	0.328889	0.426667	0.426667	0.328889	0.000000
$t_4 = 0.100000$	0.000000	0.136296	0.345185	0.345185	0.136296	0.000000
$t_5 = 0.133333$	0.000000	0.196790	0.171111	0.171111	0.196790	0.000000
$t_6 = 0.166667$	0.000000	0.011399	0.192510	0.192510	0.011399	0.000000
$t_7 = 0.200000$	0.000000	0.152826	0.041584	0.041584	0.152826	0.000000
$t_8 = 0.233333$	0.000000	−0.067230	0.134286	0.134286	−0.067230	0.000000
$t_9 = 0.266667$	0.000000	0.156725	−0.033644	−0.033644	0.156725	0.000000
$t_{10} = 0.300000$	0.000000	−0.132520	0.124997	0.124997	−0.132520	0.000000
$t_{11} = 0.333333$	0.000000	0.192511	−0.089601	−0.089601	0.192511	0.000000

The difference equation (7) has accuracy of the order $O(k) + O(h^2)$. Because the term $O(k)$ decreases linearly as k tends to zero, it is not surprising that it must be made small to produce good approximations. However, the stability requirement introduces further considerations. Suppose that the solutions over the grid are not sufficiently accurate and that both the increments $\Delta x = h_0$ and $\Delta t = k_0$ must be reduced. For simplicity, suppose that the new x increment is $\Delta x = h_1 = h_0/2$. If the same ratio r is used, k_1 must satisfy

$$k_1 = \frac{r(h_1)^2}{c^2} = \frac{r(h_0)^2}{4c^2} = \frac{k_0}{4}.$$

This results in a doubling and quadrupling of the number of grid points along the x-axis and t-axis, respectively. Consequently, there must be an eightfold increase in the total computational effort when reducing the grid size in this manner. This extra effort is usually prohibitive and demands that we explore a more efficient method that does not have stability restrictions. The method proposed will be implicit rather than explicit. The apparent rise in the level of complexity will have the immediate payoff of being unconditionally stable.

1. (a) Verify by direct substitution that $u(x,t) = \sin(n\pi x)e^{-4n^2\pi^2 t}$ is a solution to the heat equation $u_t(x,t) = 4u_{xx}(x,t)$ for each positive integer $n = 1, 2, \ldots$.

   (b) Verify by direct substitution that $u(x,t) = \sin(n\pi x)e^{-(cn\pi)^2 t}$ is a solution to the heat equation $u_t(x,t) = c^2 u_{xx}(x,t)$ for each positive integer $n = 1, 2, \ldots$.

2. What difficulty might occur if $\Delta t = k = h^2/c^2$ is used with formula (7)?

In Exercises 3 and 4, use the forward-difference method to calculate the first three rows of the approximate solution for the given heat equation. Carry out your calculations by hand (calculator).

3. $u_t(x,t) = u_{xx}(x,t)$, for $0 < x < 1$ and $0 \le t \le 0.1$, with the initial condition $u(x,0) = f(x) = \sin(\pi x)$, for $t = 0$ and $0 \le x \le 1$, and the boundary conditions

$$\begin{aligned} u(0,t) &= c_1 = 0 \quad \text{for } x = 0 \text{ and } 0 \le t \le 0.1, \\ u(1,t) &= c_2 = 0 \quad \text{for } x = 1 \text{ and } 0 \le t \le 0.1. \end{aligned}$$

   Let $h = 0.2$, $k = 0.02$, and $r = 0.5$.

4. $u_t(x,t) = u_{xx}(x,t)$, for $0 < x < 1$ and $0 \le t \le 0.1$, with the initial condition $u(x,0) = f(x) = 1 - |2x - 1|$, for $t = 0$ and $0 \le x \le 1$, and the boundary conditions

$$\begin{aligned} u(0,t) &= c_1 = 0 \quad \text{for } x = 0 \text{ and } 0 \le t \le 0.1, \\ u(1,t) &= c_2 = 0 \quad \text{for } x = 1 \text{ and } 0 \le t \le 0.1. \end{aligned}$$

6. Show that $u(x,t) = \sum_{j=1}^{N} a_j e^{-(j\pi)^2 t} \sin(j\pi x)$ is a solution to $u_t(x,t) = u_{xx}(x,t)$, for $0 \le x \le 1$ and $0 < t$, and has the boundary values $u(0,t) = 0$, $u(1,t) = 0$, and $u(x,0) = \sum_{j=1}^{N} a_j \sin(j\pi x)$.

7. Consider the analytic solution $u(x,t) = \sin(\pi x)e^{-\pi^2 t} + \sin(3\pi x)e^{-(3\pi)^2 t}$ that was discussed in Example 10.4.
   (a) Hold x fixed and determine $\lim_{t\to\infty} u(x,t)$.
   (b) What does this mean physically?

8. Suppose that we wish to solve the parabolic equation $u_t(x,t) - u_{xx}(x,t) = h(x)$.
   (a) Derive the explicit forward-difference equation for this situation.
   (b) Derive the implicit difference formula for this situation.

9. Suppose that equation (11) is used and that $f(x) \ge 0$, $g_1(t) = 0$, and $g_2(t) = 0$.
   (a) Show that the maximum value of $u(x_i, t_{j+1})$ in row $j+1$ is less than or equal to the maximum of $u(x_i, t_j)$ in row j.
   (b) Make a conjecture concerning the maximum of $u(x_i, t_n)$ in row n as n tends to infinity.