Solution of Differential Equations

9.2 Euler's Method

Let $[a, b]$ be the interval over which we want to find the solution to the well-posed I.V.P. $y' = f(t, y)$ with $y(a) = y_0$.

First we choose the abscissas for the points. For convenience we subdivide the interval $[a, b]$ into M equal subintervals and select the mesh points

$$(1) \qquad t_k = a + kh \quad \text{for } k = 0, 1, \ldots, M \text{ where } h = \frac{b-a}{M}.$$

The value h is called the ***step size***. We now proceed to solve approximately

$$(2) \qquad y' = f(t, y) \quad \text{over} \quad [t_0, t_M] \quad \text{with} \quad y(t_0) = y_0.$$

Assume that $y(t)$, $y'(t)$, and $y''(t)$ are continuous and use Taylor's theorem to expand $y(t)$ about $t = t_0$. For each value t there exists a value c_1 that lies between t_0 and t so that

$$(3) \qquad y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(c_1)(t - t_0)^2}{2}.$$

When $y'(t_0) = f(t_0, y(t_0))$ and $h = t_1 - t_0$ are substituted in equation (3), the result is an expression for $y(t_1)$:

$$y(t_1) = y(t_0) + hf(t_0, y(t_0)) + y''(c_1)\frac{h^2}{2}. \tag{4}$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving h^2) and get

$$y_1 = y_0 + hf(t_0, y_0), \tag{5}$$

which is ***Euler's approximation***.

The process is repeated and generates a sequence of points that approximates the solution curve $y = y(t)$. The general step for Euler's method is

$$t_{k+1} = t_k + h, \qquad y_{k+1} = y_k + hf(t_k, y_k) \qquad \text{for } k = 0, 1, \ldots, M-1. \tag{6}$$

Geometric Description

If you start at the point (t_0, y_0) and compute the value of the slope $m_0 = f(t_0, y_0)$ and move horizontally the amount h and vertically $hf(t_0, y_0)$, then you are moving along the tangent line to $y(t)$ and will end up at the point (t_1, y_1) (see Figure 9.5). Notice that (t_1, y_1) is not on the desired solution curve! But this is the approximation that we are generating. Hence we must use (t_1, y_1) as though it were correct and proceed by computing the slope $m_1 = f(t_1, y_1)$ and using it to obtain the next vertical displacement $hf(t_1, y_1)$ to locate (t_2, y_2), and so on.

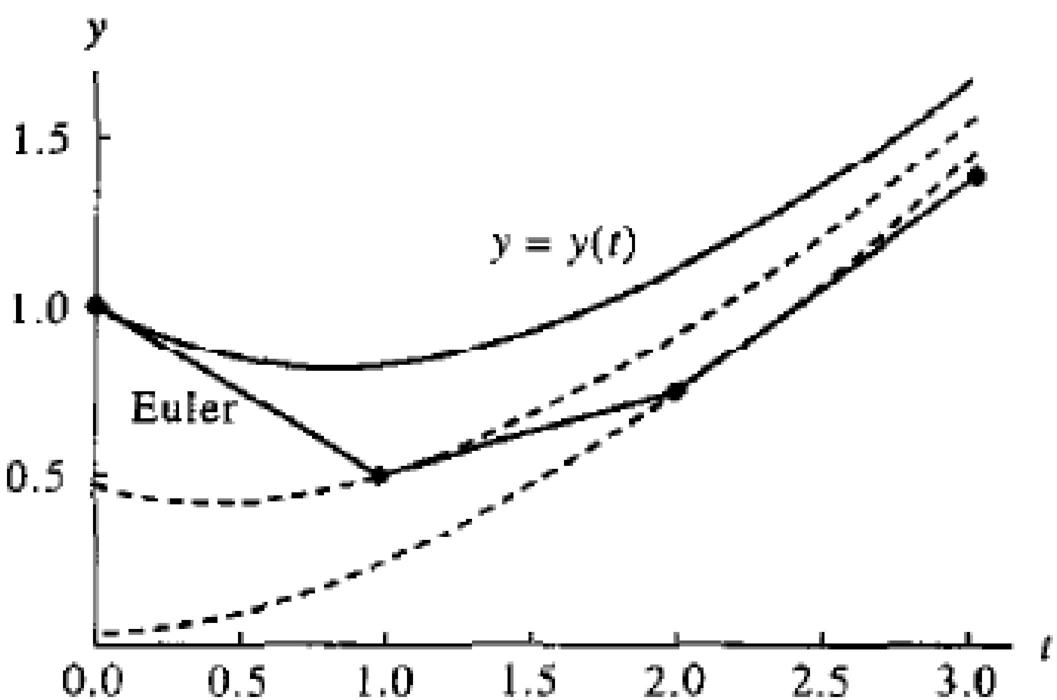

Figure 9.5 Euler's approximations $y_{k-1} = y_k + hf(t_k, y_k)$.

Example 9.2. Use Euler's method to solve approximately the initial value problem:

$$y' = Ry \quad \text{over } [0, 1] \text{ with } y(0) = y_0 \text{ and } R \text{ constant.} \tag{7}$$

The step size must be chosen, and then the second formula in (6) can be determined for computing the ordinates. This formula is sometimes called a difference equation and in this case it is

$$y_{k+1} = y_k(1 + hR) \quad \text{for } k = 0,\ 1,\ \ldots,\ M-1. \tag{8}$$

If we trace the solution values recursively, we see that

$$\begin{aligned} y_1 &= y_0(1 + hR) \\ y_2 &= y_1(1 + hR) = y_0(1 + hR)^2 \\ &\ \vdots \\ y_M &= y_{M-1}(1 + hR) = y_0(1 + hR)^M. \end{aligned} \tag{9}$$

For most problems there is no explicit formula for determining the solution points, and each new point must be computed successively from the previous point. However, for the initial value problem (7) we are fortunate; Euler's method has the explicit solution

$$t_k = kh \qquad y_k = y_0(1 - hR)^k \qquad \text{for } k = 0,\ 1,\ \ldots,\ M. \tag{10}$$

Formula (10) can be viewed as the "compound interest" formula, and the Euler approximation gives the future value of a deposit. ■

Example 9.3. Suppose that \$1000 is deposited and earns 10% interest compounded continuously over 5 years. What is the value at the end of 5 years?

We choose to use Euler approximations with $h = 1$, $\frac{1}{12}$, and $\frac{1}{360}$ to approximate $y(5)$ for the I.V.P.:

$$y' = 0.1y \qquad \text{over } [0, 5] \text{ with } y(0) = 1000.$$

Formula (10) with $R = 0.1$ produces Table 9.1. ■

Table 9.1 Compound Interest in Example 9.3

Step size, h	Number of iterations, M	Approximation to $y(5)$, y_M
1	5	$1000\left(1+\frac{0.1}{1}\right)^{5} = 1610.51$
$\frac{1}{12}$	60	$1000\left(1+\frac{0.1}{12}\right)^{60} = 1645.31$
$\frac{1}{360}$	1800	$1000\left(1+\frac{0.1}{360}\right)^{1800} = 1648.61$

Step Size versus Error

The methods we introduce for approximating the solution of an initial value problem are called ***difference methods*** or ***discrete variable methods***. The solution is approximated at a set of discrete points called a grid (or mesh) of points. An elementary single-step method has the form $y_{k+1} = y_k + h\Phi(t_k, y_k)$ for some function Φ called an ***increment function***.

When using any discrete variable method to approximately solve an initial value problem, there are two sources of error: discretization and round off.

Definition 9.3 (Discretization Error). Assume that $\{(t_k, y_k)\}_{k=0}^{M}$ is the set of discrete approximations and that $y = y(t)$ is the unique solution to the initial value problem.

The ***global discretization error*** e_k is defined by

$$e_k = y(t_k) - y_k \quad \text{for } k = 0, 1, \ldots, M. \tag{11}$$

It is the difference between the unique solution and the solution obtained by the discrete variable method.

The ***local discretization error*** ϵ_{k+1} is defined by

$$\epsilon_{k+1} = y(t_{k+1}) - y_k - h\Phi(t_k, y_k) \quad \text{for } k = 0, 1, \ldots, M-1. \tag{12}$$

It is the error committed in the single step from t_k to t_{k+1}.

When we obtained equation (6) for Euler's method, the neglected term for each step was $y^{(2)}(c_k)(h^2/2)$. If this was the only error at each step, then at the end of the interval $[a, b]$, after M steps have been made, the accumulated error would be

$$\sum_{k=1}^{M} y^{(2)}(c_k)\frac{h^2}{2} \approx My^{(2)}(c)\frac{h^2}{2} = \frac{hM}{2}y^{(2)}(c)h = \frac{(b-a)y^{(2)}(c)}{2}h = O(h^1).$$

Theorem 9.3 (Precision of Euler's Method). Assume that $y(t)$ is the solution to the I.V.P. given in (2). If $y(t) \in C^2[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Euler's method, then

$$(13)\qquad \begin{aligned} |e_k| &= |y(t_k) - y_k| = O(h), \\ |\epsilon_{k+1}| &= |y(t_{k+1}) - y_k - hf(t_k, y_k)| = O(h^2). \end{aligned}$$

The error at the end of the interval is called the ***final global error (F.G.E.)***:

$$(14)\qquad E(y(b), h) = |y(b) - y_M| = O(h).$$

Remark. The final global error $E(y(b), h)$ is used to study the behavior of the error for various step sizes. It can be used to give us an idea of how much computing effort must be done to obtain an accurate approximation.

Example 9.4. Use Euler's method to solve the I.V.P.

$$y' = \frac{t - y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

Figure 9.6 shows graphs of the four Euler solutions and the exact solution curve $y(t) = 3e^{-t/2} - 2 + t$. Table 9.2 gives the values for the four solutions at selected abscissas. For the step size $h = 0.25$, the calculations are

$$y_1 = 1.0 + 0.25\left(\frac{0.0 - 1.0}{2}\right) = 0.875,$$

$$y_2 = 0.875 + 0.25\left(\frac{0.25 - 0.875}{2}\right) = 0.796875, \qquad \text{etc.}$$

This iteration continues until we arrive at the last step:

$$y(3) \approx y_{12} = 1.440573 + 0.25\left(\frac{2.75 - 1.440573}{2}\right) = 1.604252. \qquad \blacksquare$$

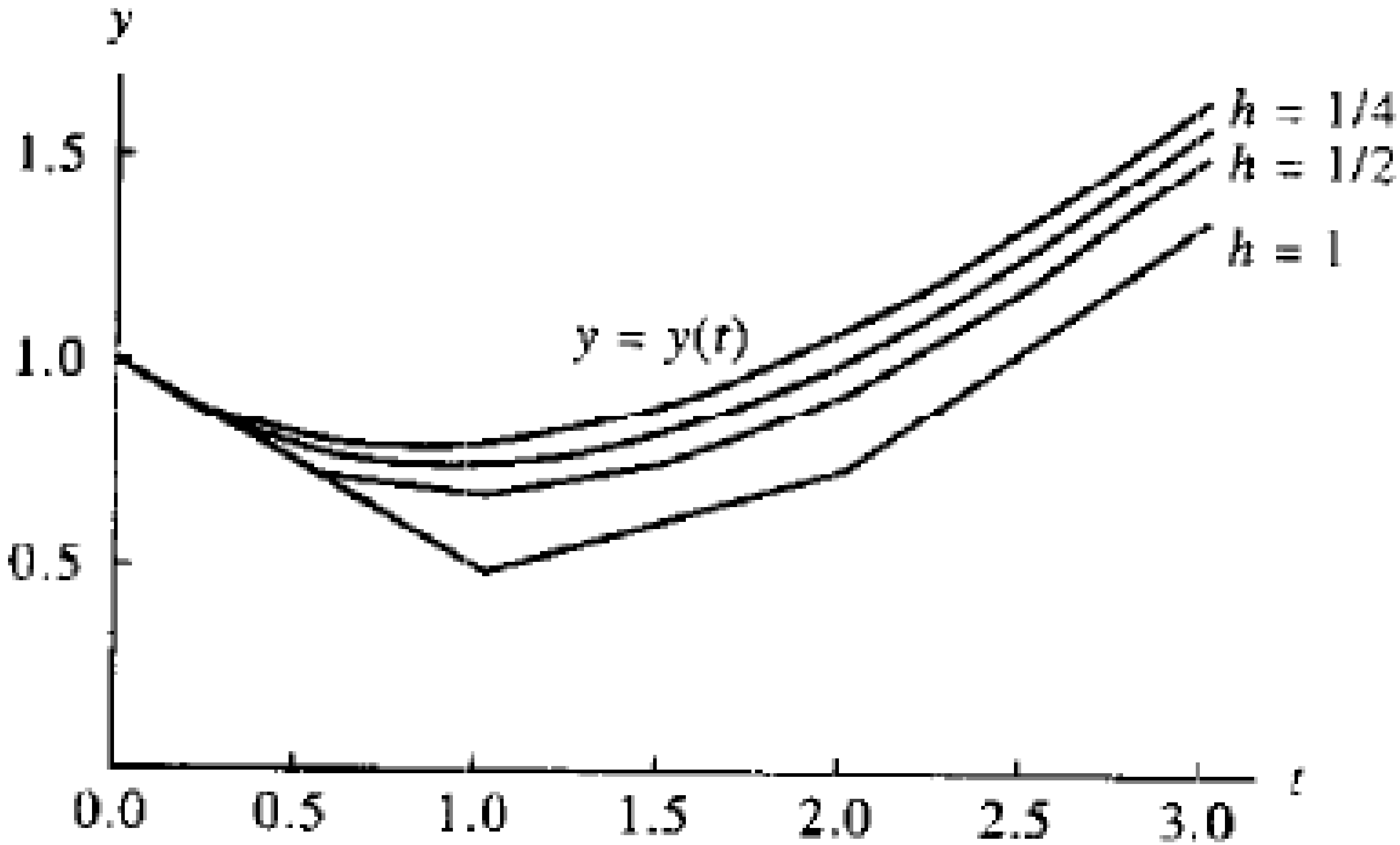

Figure 9.6 Comparison of Euler solutions with different step sizes for $y' = (t - y)/2$ over $[0, 3]$ with the initial condition $y(0) = 1$.

Table 9.2 Comparison of Euler Solutions with Different Step Sizes for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9375	0.943239
0.25			0.875	0.886719	0.897491
0.375				0.846924	0.862087
0.50		0.75	0.796875	0.817429	0.836402
0.75			0.759766	0.786802	0.811868
1.00	0.5	0.6875	0.758545	0.790158	0.819592
1.50		0.765625	0.846386	0.882855	0.917100
2.00	0.75	0.949219	1.030827	1.068222	1.103638
2.50		1.211914	1.289227	1.325176	1.359514
3.00	1.375	1.533936	1.604252	1.637429	1.669390

Example 9.5. Compare the F.G.E. when Euler's method is used to solve the I.V.P.

$$y' = \frac{t - y}{2} \quad \text{over } [0, 3] \text{ with } y(0) = 1,$$

using step sizes $1, \frac{1}{2}, \ldots, \frac{1}{64}$.

Table 9.3 gives the F.G.E. for several step sizes and shows that the error in the approximation to $y(3)$ decreases by about $\frac{1}{2}$ when the step size is reduced by a factor of $\frac{1}{2}$. For the smaller step sizes the conclusion of Theorem 9.3 is easy to see:

$$E(y(3), h) = y(3) - y_M = O(h^1) \approx Ch, \qquad \text{where} \qquad C = 0.256.$$

■

Table 9.3 Relation between Step Size and F.G.E. for Euler Solutions to $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$O(h) \approx Ch$ where $C = 0.256$
1	3	1.375	0.294390	0.256
$\frac{1}{2}$	6	1.533936	0.135454	0.128
$\frac{1}{4}$	12	1.604252	0.065138	0.064
$\frac{1}{8}$	24	1.637429	0.031961	0.032
$\frac{1}{16}$	48	1.653557	0.015833	0.016
$\frac{1}{32}$	96	1.661510	0.007880	0.008
$\frac{1}{64}$	192	1.665459	0.003931	0.004

Exercises for Euler's Method

In Exercises 1 through 5 solve the differential equations by the Euler method.

(a) Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

(b) Compare the exact solution $y(0.4)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

2. $y' = 3y + 3t$ with $y(0) = 1$, $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$

3. $y' = -ty$ with $y(0) = 1$, $y(t) = e^{-t^2/2}$

4. $y' = e^{-2t} - 2y$ with $y(0) = \frac{1}{10}$, $y(t) = \frac{1}{10}e^{-2t} + te^{-2t}$

5. $y' = 2ty^2$ with $y(0) = 1$, $y(t) = 1/(1 - t^2)$

7. Show that when Euler's method is used to solve the I.V.P.

$$y' = f(t) \quad \text{over } [a, b] \text{ with } y(a) = y_0 = 0$$

the result is

$$y(b) \approx \sum_{k=0}^{M-1} f(t_k)h,$$

which is a Riemann sum that approximates the definite integral of $f(t)$ taken over the interval $[a, b]$.

8. Show that Euler's method fails to approximate the solution $y(t) = t^{3/2}$ of the I.V.P.

$$y' = f(t, y) = 1.5y^{1/3} \quad \text{with} \quad y(0) = 0.$$

Justify your answer. What difficulties were encountered?

9. Can Euler's method be used to solve the I.V.P.

$$y' = 1 + y^2 \quad \text{over } [0, 3] \text{ with } y(0) = 0?$$

Hint. The exact solution curve is $y(t) = \tan(t)$.