Runge-Kutta Methods

(1) $$y'(t) = f(t, y(t)) \quad \text{over} \quad [a, b] \quad \text{with} \quad y(t_0) = y_0.$$

the standard Runge-Kutta method of order $N = 4$, which is stated as follows. Start with the initial point (t_0, y_0) and generate the sequence of approximations using

$$y_{k+1} = y_k + \frac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6},$$

$$f_1 = f(t_k, y_k),$$

$$f_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right),$$

$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right),$$

$$f_4 = f(t_k + h, y_k + h f_3).$$

To obtain the solution point (t_1, y_1), we can use the fundamental theorem of calculus and integrate $y'(t)$ over $[t_0, t_1]$ to get

$$\int_{t_0}^{t_1} f(t, y(t))\,dt = \int_{t_0}^{t_1} y'(t)\,dt = y(t_1) - y(t_0), \tag{2}$$

where the antiderivative of $y'(t)$ is the desired function $y(t)$. When equation (2) is solved for $y(t_1)$, the result is

$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(t, y(t))\,dt. \tag{3}$$

If Simpson's rule is applied with step size $h/2$, the approximation to the integral in (8) is

$$\int_{t_0}^{t_1} f(t, y(t))\,dt \approx \frac{h}{6}\left(f(t_0, y(t_0)) + 4f(t_{1/2}, y(t_{1/2})) + f(t_1, y(t_1))\right), \tag{9}$$

where $t_{1/2}$ is the midpoint of the interval. Three function values are needed; hence we make the obvious choice $f(t_0, y(t_0)) = f_1$ and $f(t_1, y(t_1)) \approx f_4$. For the value in the middle we chose the average of f_2 and f_3:

$$f(t_{1/2}, y(t_{1/2})) \approx \frac{f_2 + f_3}{2}.$$

These values are substituted into (9), which is used in equation (8) to get y_1:

(10) $$y_1 = y_0 + \frac{h}{6}\left(f_1 + \frac{4(f_2 + f_3)}{2} + f_4\right).$$

Step Size versus Error

The error term for Simpson's rule with step size $h/2$ is

(11) $$-y^{(4)}(c_1)\frac{h^5}{2880}.$$

If the only error at each step is that given in (11), after M steps the accumulated error for the RK4 method would be

(12) $$-\sum_{k=1}^{M} y^{(4)}(c_k)\frac{h^5}{2880} \approx \frac{b-a}{5760}y^{(4)}(c)h^4 \approx O(h^4).$$

Theorem 9.7 (Precision of the Runge-Kutta Method). Assume that $y(t)$ is the solution to the I.V.P. If $y(t) \in C^5[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by the Runge-Kutta method of order 4, then

$$
\begin{aligned}
|e_k| &= |y(t_k) - y_k| = O(h^4), \\
|\epsilon_{k+1}| &= |y(t_{k+1}) - y_k - hT_N(t_k, y_k)| = O(h^5).
\end{aligned} \tag{13}
$$

In particular, the F.G.E. at the end of the interval will satisfy

$$
E(y(b), h) = |y(b) - y_M| = O(h^4). \tag{14}
$$

Examples 9.10 and 9.11 illustrate Theorem 9.7. If approximations are computed using the step sizes h and $h/2$, we should have

$$
E(y(b), h) \approx Ch^4 \tag{15}
$$

for the larger step size, and

$$
E\left(y(b), \frac{h}{2}\right) \approx C\frac{h^4}{16} = \frac{1}{16}Ch^4 \approx \frac{1}{16}E(y(b), h). \tag{16}
$$

Example 9.10. Use the RK4 method to solve the I.V.P. $y' = (t - y)/2$ on $[0, 3]$ with $y(0) = 1$. Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

Table 9.8 gives the solution values at selected abscissas. For the step size $h = 0.25$, a sample calculation is

$$f_1 = \frac{0.0 - 1.0}{2} = -0.5,$$

$$f_2 = \frac{0.125 - (1 + 0.25(0.5)(-0.5))}{2} = -0.40625,$$

$$f_3 = \frac{0.125 - (1 + 0.25(0.5)(-0.40625))}{2} = -0.4121094,$$

$$f_4 = \frac{0.25 - (1 + 0.25(-0.4121094))}{2} = -0.3234863,$$

$$y_1 = 1.0 + 0.25\left(\frac{-0.5 + 2(-0.40625) + 2(-0.4121094) - 0.3234863}{6}\right)$$

$$= 0.8974915.$$

■

Table 9.8 Comparison of the RK4 Solutions with Different Step Sizes for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.9432392	0.9432392
0.25			0.8974915	0.8974908	0.8974917
0.375				0.8620874	0.8620874
0.50		0.8364258	0.8364037	0.8364024	0.8364023
0.75			0.8118696	0.8118679	0.8118678
1.00	0.8203125	0.8196285	0.8195940	0.8195921	0.8195920
1.50		0.9171423	0.9171021	0.9170998	0.9170997
2.00	1.1045125	1.1036826	1.1036408	1.1036385	1.1036383
2.50		1.3595575	1.3595168	1.3595145	1.3595144
3.00	1.6701860	1.6694308	1.6693928	1.6693906	1.6693905

Table 9.9 Relation between Step Size and F.G.E. for the RK4 Solutions to $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$O(h^4) \approx Ch^4$ where $C = -0.000614$
1	3	1.6701860	−0.0007955	−0.0006140
$\frac{1}{2}$	6	1.6694308	−0.0000403	−0.0000384
$\frac{1}{4}$	12	1.6693928	−0.0000023	−0.0000024
$\frac{1}{8}$	24	1.6693906	−0.0000001	−0.0000001

Exercises for Runge-Kutta Methods

In Exercises 1 through 5, solve the differential equations by the Runge-Kutta method of order $N = 4$.

(a) Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

(b) Compare the exact solution $y(0.4)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

2. $y' = 3y + 3t$ with $y(0) = 1$, $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$

3. $y' = -ty$ with $y(0) = 1$, $y(t) = e^{-t^2/2}$

4. $y' = e^{-2t} - 2y$ with $y(0) = \frac{1}{10}$, $y(t) = \frac{1}{10}e^{-2t} + te^{-2t}$

5. $y' = 2ty^2$ with $y(0) = 1$, $y(t) = 1/(1 - t^2)$