The Solution of Nonlinear Equations $f(x) = 0$

2.1 Iteration for Solving $x = g(x)$

A fundamental principle in computer science is *iteration*. As the name sugge
process is repeated until an answer is achieved. Iterative techniques are used t
roots of equations, solutions of linear and nonlinear systems of equations, and sol
of differential equations. In this section we study the process of iteration using rep
substitution.

A rule or function $g(x)$ for computing successive terms is needed, together starting value p_0. Then a sequence of values $\{p_k\}$ is obtained using the iterativ

$p_{k+1} = g(p_k)$. The sequence has the pattern

$$
\begin{aligned}
&p_0 && \text{(starting value)} \\
&p_1 = g(p_0) \\
&p_2 = g(p_1) \\
&\vdots \\
&p_k = g(p_{k-1}) \\
p_{k-1} &= g(p_k) \\
&\vdots
\end{aligned}
\tag{1}
$$

Finding Fixed Points

Definition 2.1 (Fixed Point). A ***fixed point*** of a function $g(x)$ is a real number P such that $P = g(P)$. ▲

Geometrically, the fixed points of a function $y = g(x)$ are the points of intersection of $y = g(x)$ and $y = x$.

Definition 2.2 (Fixed-point Iteration). The iteration $p_{n+1} = g(p_n)$ for $n = 0, 1, \ldots$ is called ***fixed-point iteration***. ▲

Theorem 2.1. Assume that g is a continuous function and that $\{p_n\}_{n=0}^{\infty}$ is a sequence generated by fixed-point iteration. If $\lim_{n\to\infty} p_n = P$, then P is a fixed point of $g(x)$.

Proof. If $\lim_{n\to\infty} p_n = P$, then $\lim_{n\to\infty} p_{n+1} = P$. It follows from this result, the continuity of g, and the relation $p_{n+1} = g(p_n)$ that

$$g(P) = g\left(\lim_{n\to\infty} p_n\right) = \lim_{n\to\infty} g(p_n) = \lim_{n\to\infty} p_{n+1} = P. \tag{2}$$

Therefore, P is a fixed point of $g(x)$. ∎

Theorem 2.2. Assume that $g \in C[a, b]$.

(3) If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.

(4) Furthermore, suppose that $g'(x)$ is defined over (a, b) and that a positive constant $K < 1$ exists with $|g'(x)| \leq K < 1$ for all $x \in (a, b)$, then g has a unique fixed point P in $[a, b]$.

Example 2.3. Apply Theorem 2.2 to rigorously show that $g(x) = \cos(x)$ has a unique fixed point in $[0, 1]$.

Clearly, $g \in C[0, 1]$. Secondly, $g(x) = \cos(x)$ is a decreasing function on $[0, 1]$, thus its range on $[0, 1]$ is $[\cos(1), 1] \subseteq [0, 1]$. Thus condition (3) of Theorem 2.2 is satisfied and g has a fixed point in $[0, 1]$. Finally, if $x \in (0, 1)$, then $|g'(x)| = |-\sin(x)| = \sin(x) \leq \sin(1) < 0.8415 < 1$. Thus $K = \sin(1) < 1$, condition (4) of Theorem 2.2 is satisfied, and g has a unique fixed point in $[0, 1]$. ∎

Theorem 2.3 (Fixed-point Theorem). Assume that (i) $g, g' \in C[a, b]$, (ii) K is a positive constant, (iii) $p_0 \in (a, b)$, and (iv) $g(x) \in [a, b]$ for all $x \in [a, b]$.

(6) If $|g'(x)| \le K < 1$ for all $x \in [a, b]$, then the iteration $p_n = g(p_{n-1})$ will converge to the unique fixed point $P \in [a, b]$. In this case, P is said to be an attractive fixed point.

(7) If $|g'(x)| > 1$ for all $x \in [a, b]$, then the iteration $p_n = g(p_{n-1})$ will not converge to P. In this case, P is said to be a repelling fixed point and the iteration exhibits local divergence.

Graphical Interpretation of Fixed-point Iteration

Since we seek a fixed point P to $g(x)$, it is necessary that the graph of the curve $y = g(x)$ and the line $y = x$ intersect at the point (P, P). Two simple types of convergent iteration, monotone and oscillating, are illustrated in Figure 2.4(a) and (b), respectively.

To visualize the process, start at p_0 on the x-axis and move vertically to the point $(p_0, p_1) = (p_0, g(p_0))$ on the curve $y = g(x)$. Then move horizontally from (p_0, p_1) to the point (p_1, p_1) on the line $y = x$. Finally, move vertically downward to p_1 on the x-axis. The recursion $p_{n+1} = g(p_n)$ is used to construct the point (p_n, p_{n+1}) on the graph, then a horizontal motion locates (p_{n+1}, p_{n+1}) on the line $y = x$, and then a vertical movement ends up at p_{n+1} on the x-axis. The situation is shown in Figure 2.4.

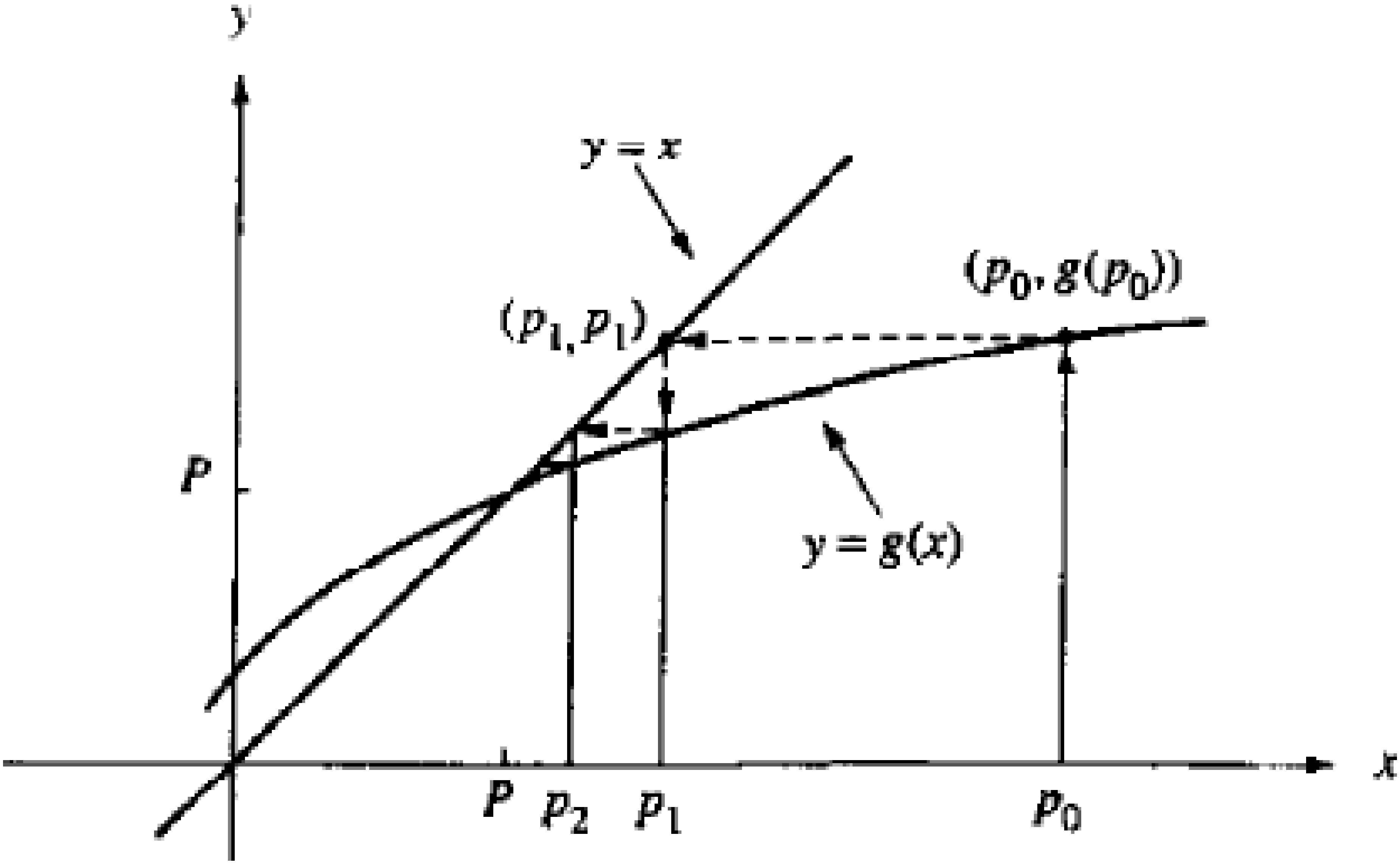

Figure 2.4 (a) Monotone convergence when $0 < g'(P) < 1$.

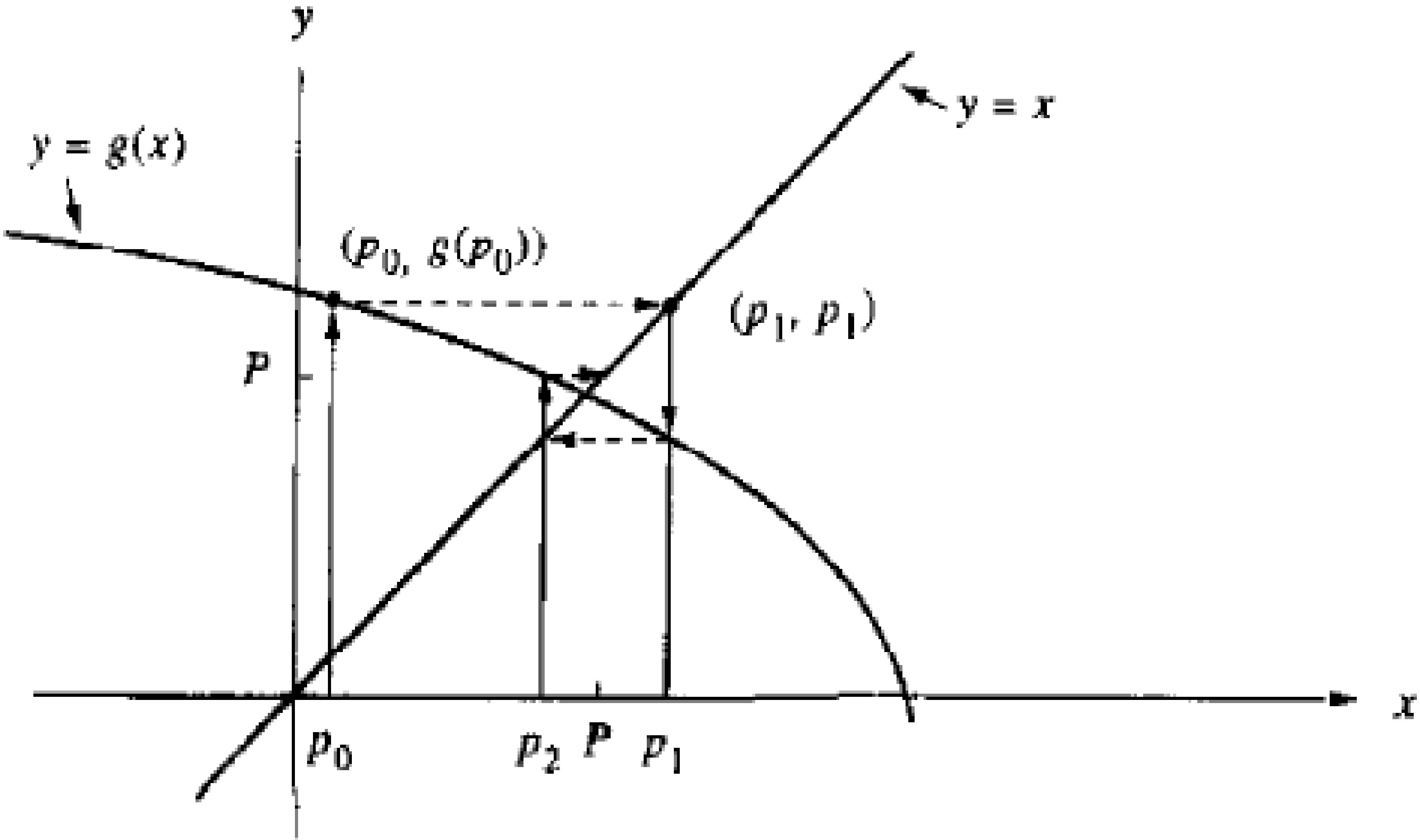

Figure 2.4 (b) Oscillating convergence when $-1 < g'(P) < 0$.

If $|g'(P)| > 1$, then the iteration $p_{n+1} = g(p_n)$ produces a sequence that diverges away from P. The two simple types of divergent iteration, monotone and oscillating, are illustrated in Figure 2.5(a) and (b), respectively.

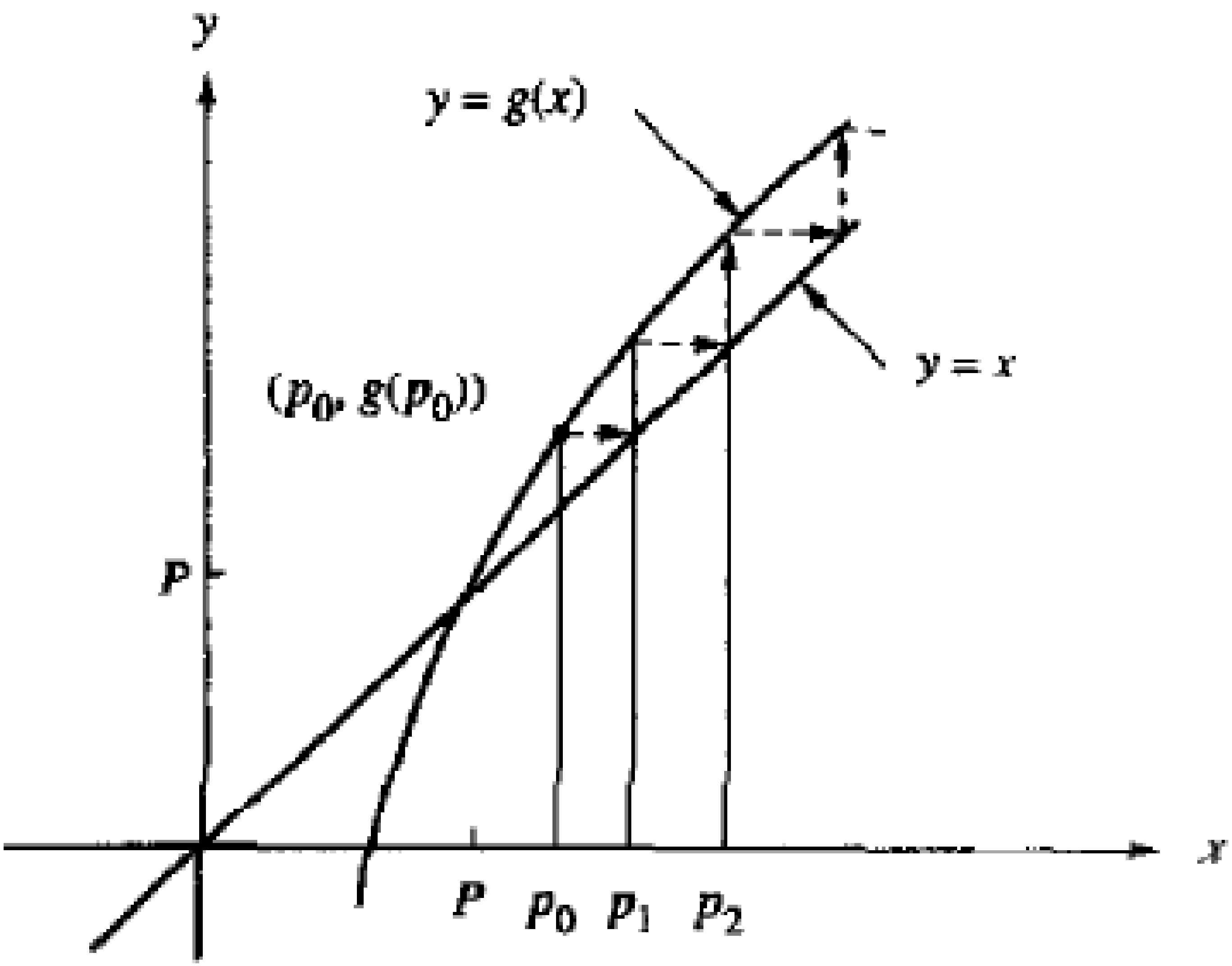

Figure 2.5 (a) Monotone divergence when $1 < g'(P)$.

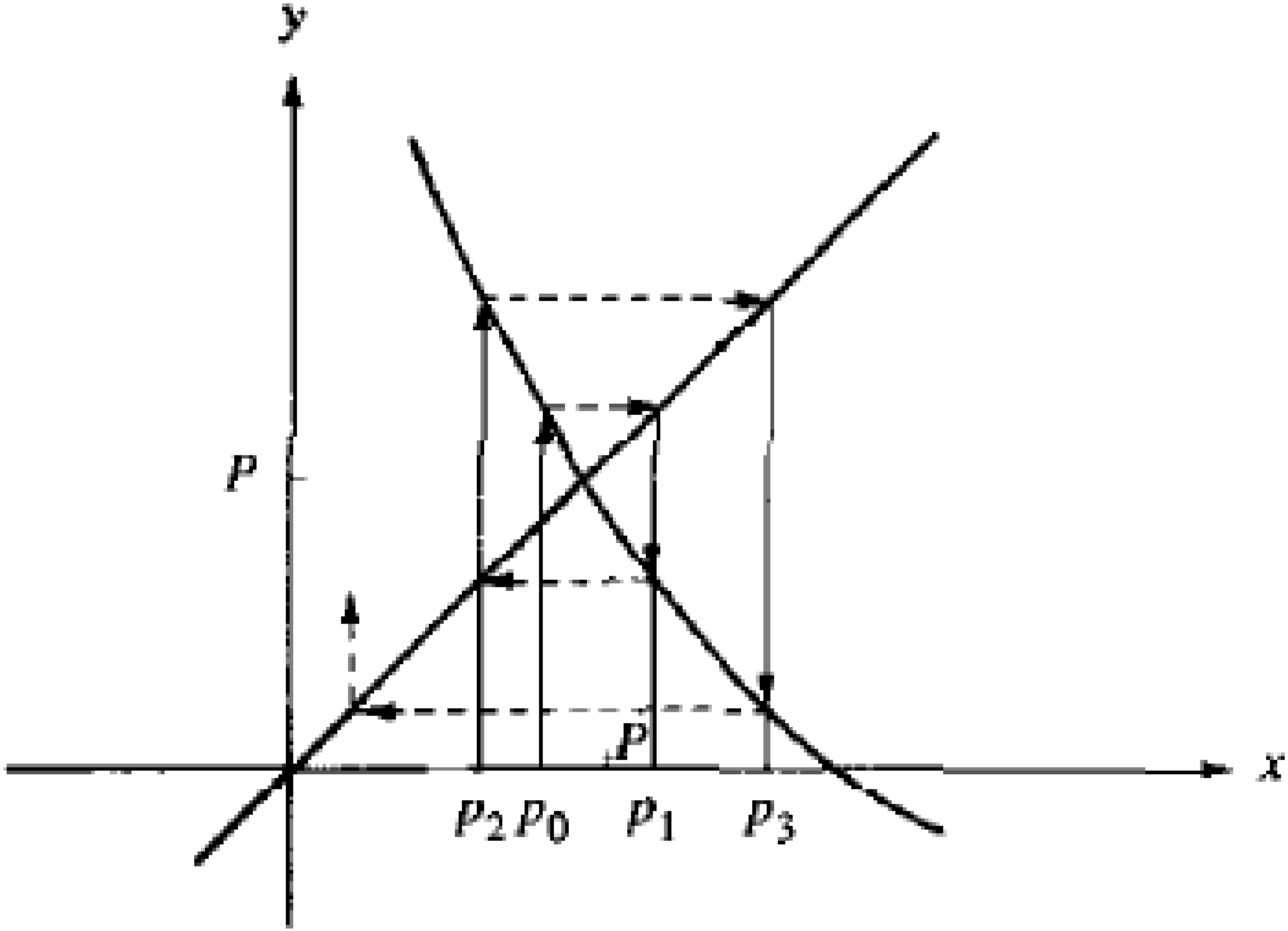

Figure 2.5 (b) Divergent oscillation when $g'(P) < -1$.

Example 2.4. Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 1+x-x^2/4$ is used. The fixed points can be found by solving the equation $x = g(x)$. The two solutions (fixed points of g) are $x = -2$ and $x = 2$. The derivative of the function is $g'(x) = 1-x/2$, and there are only two cases to consider.

Case (i): $P = -2$

Start with $p_0 = -2.05$

then get

$$p_1 = -2.100625$$
$$p_2 = -2.20378135$$
$$p_3 = -2.41794441$$
$$\vdots$$
$$\lim_{n\to\infty} p_n = -\infty.$$

Since $|g'(x)| > \frac{3}{2}$ on $[-3, -1]$, by Theorem 2.3, the sequence will not converge to $P = -2$.

Case (ii): $P = 2$

Start with $p_0 = 1.6$

then get

$$p_1 = 1.96$$
$$p_2 = 1.9996$$
$$p_3 = 1.99999996$$
$$\vdots$$
$$\lim_{n\to\infty} p_n = 2.$$

Since $|g'(x)| < \frac{1}{2}$ on $[1, 3]$, by Theorem 2.3, the sequence will converge to $P = 2$.

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Exercises for Iteration for Solving $x = g(x)$

1. Determine rigorously if each function has a unique fixed point on the given interval (follow Example 2.3).
   (a) $g(x) = 1 - x^2/4$ on $[0, 1]$
   (b) $g(x) = 2^{-x}$ on $[0, 1]$
   (c) $g(x) = 1/x$ on $[0.5, 5.2]$
2. Investigate the nature of the fixed-point iteration when

$$g(x) = -4 + 4x - \frac{1}{2}x^2.$$

   (a) Solve $g(x) = x$ and show that $P = 2$ and $P = 4$ are fixed points.
   (b) Use the starting value $p_0 = 1.9$ and compute p_1, p_2, and p_3.
   (c) Use the starting value $p_0 = 3.8$ and compute p_1, p_2, and p_3.
   (d) Find the errors E_k and relative errors R_k for the values p_k in parts (b) and (c).
   (e) What conclusions can be drawn from Theorem 2.3?
3. Graph $g(x)$, the line $y = x$, and the given fixed point P on the same coordinate system. Using the given starting value p_0, compute p_1 and p_2. Construct figures similar to Figures 2.4 and 2.5. Based on your graph, determine geometrically if fixed-point iteration converges.
   (a) $g(x) = (6 + x)^{1/2}$, $P = 3$, and $p_0 = 7$
   (b) $g(x) = 1 + 2/x$, $P = 2$, and $p_0 = 4$
   (c) $g(x) = x^2/3$, $P = 3$, and $p_0 = 3.5$
   (d) $g(x) = -x^2 + 2x + 2$, $P = 2$, and $p_0 = 2.5$

4. Let $g(x) = x^2 + x - 4$. Can fixed-point iteration be used to find the solution(s) to the equation $x = g(x)$? Why?

5. Let $g(x) = x\cos(x)$. Solve $x = g(x)$ and find all the fixed points of g (there are infinitely many). Can fixed-point iteration be used to find the solution(s) to the equation $x = g(x)$? Why?

6. Suppose that $g(x)$ and $g'(x)$ are defined and continuous on (a, b); $p_0, p_1, p_2 \in (a, b)$; and $p_1 = g(p_0)$ and $p_2 = g(p_1)$. Also, assume that there exists a constant K such that $|g'(x)| < K$. Show that $|p_2 - p_1| < K|p_1 - p_0|$. *Hint.* Use the Mean Value Theorem.

7. Suppose that $g(x)$ and $g'(x)$ are continuous on (a, b) and that $|g'(x)| > 1$ on this interval. If the fixed point P and the initial approximations p_0 and p_1 lie in the interval (a, b), then show that $p_1 = g(p_0)$ implies that $|E_1| = |P - p_1| > |P - p_0| = |E_0|$. Hence statement (7) of Theorem 2.3 is established (local divergence).

8. Let $g(x) = -0.0001x^2 + x$ and $p_0 = 1$, and consider fixed-point iteration.

(a) Show that $p_0 > p_1 > \cdots > p_n > p_{n+1} > \cdots$.

(b) Show that $p_n > 0$ for all n.

(c) Since the sequence $\{p_n\}$ is decreasing and bounded below, it has a limit. What is the limit?

9. Let $g(x) = 0.5x + 1.5$ and $p_0 = 4$, and consider fixed-point iteration.

(a) Show that the fixed point is $P = 3$.

(b) Show that $|P - p_n| = |P - p_{n-1}|/2$ for $n = 1, 2, 3, \ldots$

(c) Show that $|P - p_n| = |P - p_0|/2^n$ for $n = 1, 2, 3, \ldots$.

10. Let $g(x) = x/2$, and consider fixed-point iteration.

(a) Find the quantity $|p_{k+1} - p_k|/|p_{k+1}|$.

(b) Discuss what will happen if only the relative error stopping criterion were used in Program 2.1.

11. For fixed-point iteration, discuss why it is an advantage to have $g'(P) \approx 0$.

Theorem 2.3 does not state what will happen when $g'(P) = 1$. The next example has been specially constructed so that the sequence $\{p_n\}$ converges whenever $p_0 > P$ and it diverges if we choose $p_0 < P$.

Example 2.5. Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 2(x-1)^{1/2}$ for $x \geq 1$ is used. Only one fixed point $P = 2$ exists. The derivative is $g'(x) = 1/(x-1)^{1/2}$ and $g'(2) = 1$, so Theorem 2.3 does not apply. There are two cases to consider when the starting value lies to the left or right of $P = 2$.

Case (i): Start with $p_0 = 1.5$, then get

$$\begin{aligned} p_1 &= 1.41421356 \\ p_2 &= 1.28718851 \\ p_3 &= 1.07179943 \\ p_4 &= 0.53590832 \\ &\vdots \\ p_5 &= 2(-0.46409168)^{1/2}. \end{aligned}$$

Since p_4 lies outside the domain of $g(x)$, the term p_5 cannot be computed.

Case (ii): Start with $p_0 = 2.5$, then get

$$\begin{aligned} p_1 &= 2.44948974 \\ p_2 &= 2.40789513 \\ p_3 &= 2.37309514 \\ p_4 &= 2.34358284 \\ &\vdots \\ \lim_{n\to\infty} p_n &= 2. \end{aligned}$$

This sequence is converging too slowly to the value $P = 2$; indeed, $P_{1000} = 2.00398714$.

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