9.3 Heun's Method

The next approach, Heun's method, introduces a new idea for constructing an algorithm to solve the I.V.P.

(1) $$y'(t) = f(t, y(t)) \quad \text{over} \quad [a, b] \quad \text{with} \quad y(t_0) = y_0.$$

To obtain the solution point (t_1, y_1), we can use the fundamental theorem of calculus and integrate $y'(t)$ over $[t_0, t_1]$ to get

(2) $$\int_{t_0}^{t_1} f(t, y(t))\, dt = \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0),$$

where the antiderivative of $y'(t)$ is the desired function $y(t)$. When equation (2) is solved for $y(t_1)$, the result is

(3) $$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(t, y(t))\, dt.$$

Now a numerical integration method can be used to approximate the definite integral in (3). If the trapezoidal rule is used with the step size $h = t_1 - t_0$, then the result is

$$y(t_1) \approx y(t_0) + \frac{h}{2}(f(t_0, y(t_0)) + f(t_1, y(t_1))). \tag{4}$$

Notice that the formula on the right-hand side of (4) involves the yet to be determined value $y(t_1)$. To proceed, we use an estimate for $y(t_1)$. Euler's solution will suffice for this purpose. After it is substituted into (4), the resulting formula for finding (t_1, y_1) is called ***Heun's method***:

$$y_1 = y(t_0) + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))). \tag{5}$$

The process is repeated and generates a sequence of points that approximates the solution curve $y = y(t)$. At each step, Euler's method is used as a prediction, and then the trapezoidal rule is used to make a correction to obtain the final value. The general step for Heun's method is

$$\begin{aligned} p_{k+1} &= y_k + hf(t_k, y_k), \qquad t_{k+1} = t_k + h, \\ y_{k+1} &= y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, p_{k+1})). \end{aligned} \tag{6}$$

Step Size versus Error

The error term for the trapezoidal rule used to approximate the integral in (3) is

$$-y^{(2)}(c_k)\frac{h^3}{12}. \tag{7}$$

If the only error at each step is that given in (7), after M steps the accumulated error for Heun's method would be

$$-\sum_{k=1}^{M} y^{(2)}(c_k)\frac{h^3}{12} \approx \frac{b-a}{12} y^{(2)}(c)h^2 = O(h^2). \tag{8}$$

Theorem 9.4 (Precision of Heun's Method). Assume that $y(t)$ is the solution to the I.V.P. (1). If $y(t) \in C^3[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{M}$ is the sequence of approximations generated by Heun's method, then

$$\begin{aligned} |e_k| &= |y(t_k) - y_k| = O(h^2), \\ |\epsilon_{k+1}| &= |y(t_{k+1}) - y_k - h\Phi(t_k, y_k)| = O(h^3), \end{aligned} \tag{9}$$

where $\Phi(t_k, y_k) = y_k + (h/2)(f(t_k, y_k) + f(t_{k+1}, y_k + hf(t_k, y_k)))$.

In particular, the final global error (F.G.E.) at the end of the interval will satisfy

$$E(y(b), h) = |y(b) - y_M| = O(h^2). \tag{10}$$

Example 9.6. Use Heun's method to solve the I.V.P.

$$y' = \frac{t - y}{2} \quad \text{on } [0, 3] \text{ with } y(0) = 1.$$

Compare solutions for $h = 1, \frac{1}{2}, \frac{1}{4}$, and $\frac{1}{8}$.

Figure 9.8 shows the graphs of the first two Heun solutions and the exact solution curve $y(t) = 3e^{-t/2} - 2 + t$. Table 9.4 gives the values for the four solutions at selected abscissas. For the step size $h = 0.25$, a sample calculation is

$$f(t_0, y_0) = \frac{0 - 1}{2} = -0.5$$

$$p_1 = 1.0 + 0.25(-0.5) = 0.875,$$

$$f(t_1, p_1) = \frac{0.25 - 0.875}{2} = -0.3125,$$

$$y_1 = 1.0 + 0.125(-0.5 - 0.3125) = 0.8984375.$$

This iteration continues until we arrive at the last step:

$$y(3) \approx y_{12} = 1.511508 + 0.125(0.619246 + 0.666840) = 1.672269.$$

■

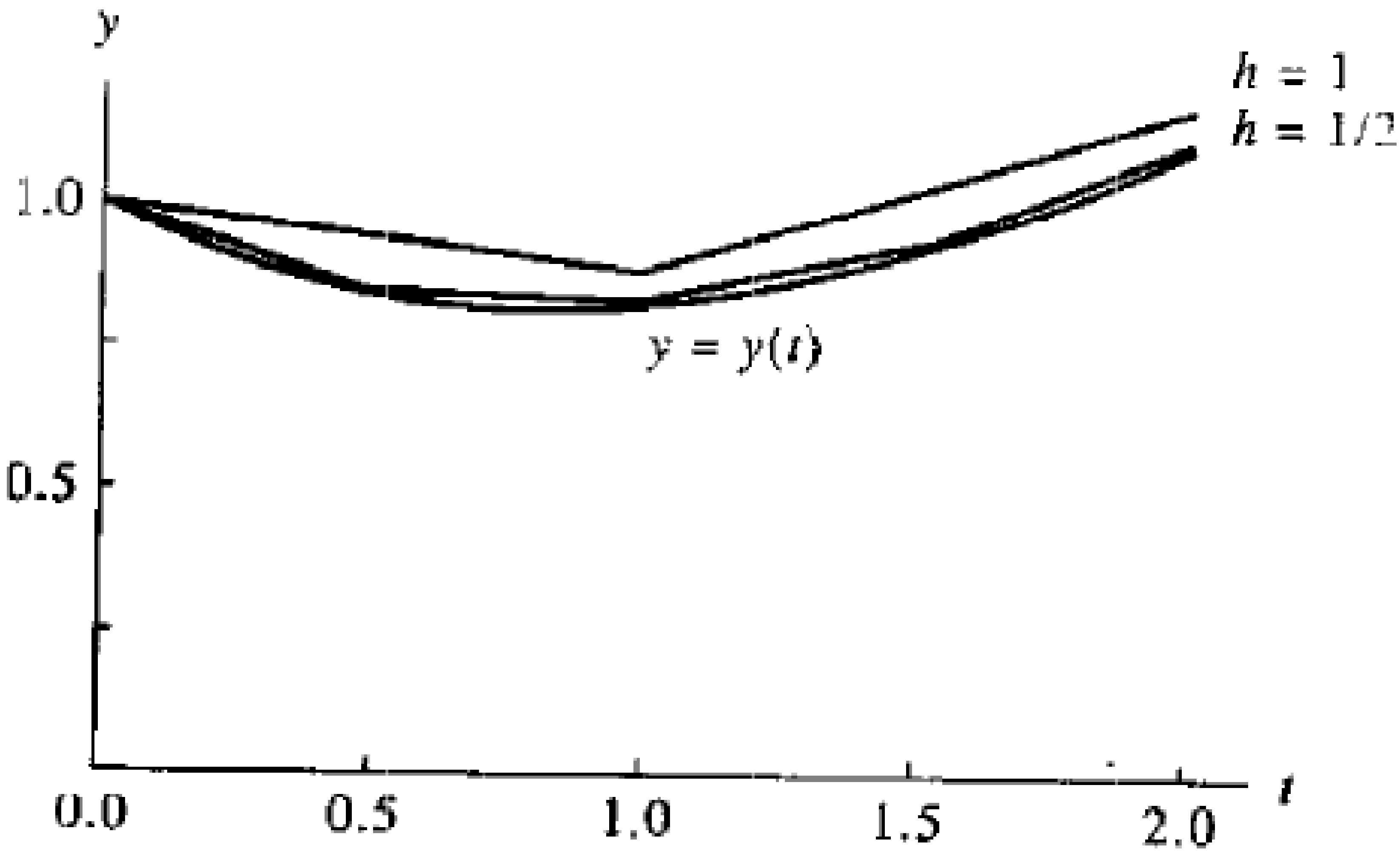

Figure 9.8 Comparison of Heun solutions with different step sizes for $y' = (t - y)/2$ over $[0, 2]$ with the initial condition $y(0) = 1$.

Table 9.4 Comparison of Heun Solutions with Different Step Sizes for $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

t_k	y_k				$y(t_k)$ Exact
	$h = 1$	$h = \frac{1}{2}$	$h = \frac{1}{4}$	$h = \frac{1}{8}$	
0	1.0	1.0	1.0	1.0	1.0
0.125				0.943359	0.943239
0.25			0.898438	0.897717	0.897491
0.375				0.862406	0.862087
0.50		0.84375	0.838074	0.836801	0.836402
0.75			0.814081	0.812395	0.811868
1.00	0.875	0.831055	0.822196	0.820213	0.819592
1.50		0.930511	0.920143	0.917825	0.917100
2.00	1.171875	1.117587	1.106800	1.104392	1.103638
2.50		1.373115	1.362593	1.360248	1.359514
3.00	1.732422	1.682121	1.672269	1.670076	1.669390

Example 9.7. Compare the F.G.E. when Heun's method is used to solve

$$y' = \frac{t - y}{2} \quad \text{over } [0, 3] \text{ with } y(0) = 1,$$

using step sizes $1, \frac{1}{2}, \ldots, \frac{1}{64}$.

Table 9.5 gives the F.G.E. and shows that the error in the approximation to $y(3)$ decreases by about $\frac{1}{4}$ when the step size is reduced by a factor of $\frac{1}{2}$:

$$E(y(3), h) = y(3) - y_M = O(h^2) \approx Ch^2, \quad \text{where } C = -0.0432. \quad \blacksquare$$

Table 9.5 Relation between Step Size and F.G.E. for Heun Solutions to $y' = (t - y)/2$ over $[0, 3]$ with $y(0) = 1$

Step size, h	Number of steps, M	Approximation to $y(3)$, y_M	F.G.E. Error at $t = 3$, $y(3) - y_M$	$O(h^2) \approx Ch^2$ where $C = -0.0432$
1	3	1.732422	−0.063032	−0.043200
$\frac{1}{2}$	6	1.682121	−0.012731	−0.010800
$\frac{1}{4}$	12	1.672269	−0.002879	−0.002700
$\frac{1}{8}$	24	1.670076	−0.000686	−0.000675
$\frac{1}{16}$	48	1.669558	−0.000168	−0.000169
$\frac{1}{32}$	96	1.669432	−0.000042	−0.000042
$\frac{1}{64}$	192	1.669401	−0.000011	−0.000011

In Exercises 1 through 5 solve the differential equations by Heun's method.

(a) Let $h = 0.2$ and do two steps by hand calculation. Then let $h = 0.1$ and do four steps by hand calculation.

(b) Compare the exact solution $y(0.4)$ with the two approximations in part (a).

(c) Does the F.G.E. in part (a) behave as expected when h is halved?

1. $y' = t^2 - y$ with $y(0) = 1$, $y(t) = -e^{-t} + t^2 - 2t + 2$

2. $y' = 3y + 3t$ with $y(0) = 1$, $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$

3. $y' = -ty$ with $y(0) = 1$, $y(t) = e^{-t^2/2}$

4. $y' = e^{-2t} - 2y$ with $y(0) = \frac{1}{10}$, $y(t) = \frac{1}{10}e^{-2t} + te^{-2t}$

5. $y' = 2ty^2$ with $y(0) = 1$, $y(t) = 1/(1 - t^2)$
Notice that Heun's method will generate an approximation to $y(1)$ even though the solution curve is not defined at $t = 1$.

6. Show that when Heun's method is used to solve the I.V.P. $y' = f(t)$ over $[a, b]$ with $y(a) = y_0 = 0$ the result is

$$y(b) = \frac{h}{2} \sum_{k=0}^{M-1} (f(t_k) + f(t_{k+1})),$$

which is the trapezoidal rule approximation for the definite integral of $f(t)$ taken over the interval $[a, b]$.