3.6 Iterative Methods for Linear Systems

The goal of this chapter is to extend some of the iterative methods introduced in Chapter 2 to higher dimensions. We consider an extension of fixed-point iteration that applies to systems of linear equations.

Jacobi Iteration

Example 3.26. Consider the system of equations

$$
\begin{aligned}
4x - y + z &= 7 \\
4x - 8y + z &= -21 \\
-2x + y + 5z &= 15.
\end{aligned}
$$

(1)

These equations can be written in the form

$$
\begin{aligned}
x &= \frac{7 + y - z}{4} \\
y &= \frac{21 + 4x + z}{8} \\
z &= \frac{15 + 2x - y}{5}.
\end{aligned}
$$

(2)

This suggests the following Jacobi iterative process:

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$

(3)
$$y_{k+1} = \frac{21 + 4x_k + z_k}{8}$$

$$z_{k+1} = \frac{15 + 2x_k - y_k}{5}.$$

Let us show that if we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then the iteration in (3) appears to converge to the solution $(2, 4, 3)$.

Substitute $x_0 = 1$, $y_0 = 2$, and $z_0 = 2$ into the right-hand side of each equation in (3) to obtain the new values

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75$$

$$y_1 = \frac{21 + 4 + 2}{8} = 3.375$$

$$z_1 = \frac{15 + 2 - 2}{5} = 3.00.$$

The new point $P_1 = (1.75, 3.375, 3.00)$ is closer to $(2, 4, 3)$ then P_0. Iteration using (3) generates a sequence of points $\{P_k\}$ that converges to the solution $(2, 4, 3)$ (see Table 3.2). ∎

Table 3.2 Convergent Jacobi Iteration for the Linear System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

This process is called *Jacobi iteration* and can be used to solve certain types of linear systems. After 19 steps, the iteration has converged to the nine-digit machine approximation (2.00000000, 4.00000000, 3.00000000).

Sometimes the Jacobi method does not work. Let us experiment and see that a rearrangement of the original linear system can result in a system of iteration equations that will produce a divergent sequence of points.

Example 3.27. Let the linear system (1) be rearranged as follows:

$$
\begin{aligned}
-2x + y + 5z &= 15 \\
4x - 8y + z &= -21 \\
4x - y + z &= 7.
\end{aligned}
$$

(4)

These equations can be written in the form

$$
x = \frac{-15 + y + 5z}{3}
$$

(5)

$$
y = \frac{21 + 4x + z}{8}
$$

$$
z = 7 - 4x + y.
$$

This suggests the following Jacobi iterative process:

$$x_{k+1} = \frac{-15 + y_k + 5z_k}{3}$$

(6)

$$y_{k+1} = \frac{21 + 4x_k + z_k}{8}$$

$$z_{k+1} = 7 - 4x_k + y_k.$$

See that if we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$ then the iteration using (6) will diverge away from the solution $(2, 4, 3)$.

Substitute $x_0 = 1$, $y_0 = 2$, and $z_0 = 2$ into the right-hand side of each equation in (6) to obtain the new values x_1, y_1, and z_1:

$$x_1 = \frac{-15 + 2 + 10}{2} = -1.5$$

$$y_1 = \frac{21 + 4 + 2}{8} = 3.375$$

$$z_1 = 7 - 4 + 2 = 5.00.$$

The new point $P_1 = (-1.5, 3.375, 5.00)$ is farther away from the solution $(2, 4, 3)$ than P_0. Iteration using the equations in (6) produces a divergent sequence (see Table 3.3).

Table 3.3 Divergent Jacobi Iteration for the Linear System (4)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	−1.5	3.375	5.0
2	6.6875	2.5	16.375
3	34.6875	8.015625	−17.25
4	−46.617188	17.8125	−123.73438
5	−307.929688	−36.150391	211.28125
6	502.62793	−124.929688	1202.56836
⋮	⋮	⋮	⋮

In view of Examples 3.26 and 3.27, it is necessary to have some criterion to determine whether the Jacobi iteration will converge. Hence we make the following definition.

Definition 3.6. A matrix A of dimension $N \times N$ is said to be *strictly diagonally dominant* provided that

$$(8) \qquad |a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{N} |a_{kj}| \qquad \text{for } k = 1, 2, \ldots, N. \qquad \blacktriangle$$

This means that in each row of the matrix the magnitude of the element on the main diagonal must exceed the sum of the magnitudes of all other elements in the row The coefficient matrix of the linear system (1) in Example 3.26 is strictly diagonally dominant because

$$
\begin{aligned}
\text{In row 1:} &\qquad |4| > |-1| + |1| \\
\text{In row 2:} &\qquad |-8| > |4| + |1| \\
\text{In row 3:} &\qquad |5| > |-2| + |1|.
\end{aligned}
$$

All the rows satisfy relation (8) in Definition 3.6; therefore, the coefficient matrix A for the linear system (1) is strictly diagonally dominant.

The coefficient matrix A of the linear system (4) in Example 3.27 is not strictly diagonally dominant because

$$\text{In row 1:} \quad |-2| < |1| + |5|$$
$$\text{In row 2:} \quad |-8| > |4| + |1|$$
$$\text{In row 3:} \quad |1| < |4| + |-1|.$$

Rows 1 and 3 do not satisfy relation (8) in Definition 3.6; therefore, the coefficient matrix A for the linear system (4) is not strictly diagonally dominant.

We now generalize the Jacobi and Gauss-Seidel iteration processes. Suppose that the given linear system is

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2N}x_N &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jj}x_j + \cdots + a_{jN}x_N &= b_j \\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\
a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{Nj}x_j + \cdots + a_{NN}x_N &= b_N.
\end{aligned}
\tag{9}
$$

Let the kth point be $P_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_j^{(k)}, \ldots, x_N^{(k)})$; then the next point is $P_{k+1} = (x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_j^{(k+1)}, \ldots, x_N^{(k+1)})$. The superscript (k) on the coordinates of P_k enables us to identify the coordinates that belong to this point. The iteration formulas use row j of (9) to solve for $x_j^{(k+1)}$ in terms of a linear combination of the previous values $x_1^{(k)}, x_2^{(k)}, \ldots, x_j^{(k)}, \ldots, x_N^{(k)}$:

Jacobi iteration:

$$(10) \quad x_j^{(k+1)} = \frac{b_j - a_{j1}x_1^{(k)} - \cdots - a_{jj-1}x_{j-1}^{(k)} - a_{jj+1}x_{j+1}^{(k)} - \cdots - a_{jN}x_N^{(k)}}{a_{jj}}$$

for $j = 1, 2, \ldots, N$.

Theorem 3.15 (Jacobi Iteration). Suppose that A is a strictly diagonally dominant matrix. Then $AX = B$ has a unique solution $X = P$. Iteration using formula (10) will produce a sequence of vectors $\{P_k\}$ that will converge to P for any choice of the starting vector P_0.

Proof. The proof can be found in advanced texts on numerical analysis. ●

In Exercises 1 through 8:

(a) Start with $P_0 = 0$ and use Jacobi iteration to find P_k for $k = 1, 2, 3$. Will Jacobi iteration converge to the solution?

(b) Start with $P_0 = 0$ and use Gauss-Seidel iteration to find P_k for $k = 1, 2, 3$. Will Gauss-Seidel iteration converge to the solution?

1.
$$4x - y = 15$$
$$x + 5y = 9$$

2.
$$8x - 3y = 10$$
$$-x + 4y = 6$$

3.
$$-x + 3y = 1$$
$$6x - 2y = 2$$

4.
$$2x + 3y = 1$$
$$7x - 2y = 1$$

5.
$$5x - y + z = 10$$
$$2x + 8y - z = 11$$
$$-x + y + 4z = 3$$

6.
$$2x + 8y - z = 11$$
$$5x - y + z = 10$$
$$-x + y + 4z = 3$$

7.
$$x - 5y - z = -8$$
$$4x + y - z = 13$$
$$2x - y - 6z = -2$$

8.
$$4x + y - z = 13$$
$$x - 5y - z = -8$$
$$2x - y - 6z = -2$$

Gauss-Seidel Iteration

Sometimes the convergence can be speeded up. Observe that the Jacobi iterative process (3) yields three sequences $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ that converge to 2, 4, and 3, respectively (see Table 3.2). It seems reasonable that x_{k+1} could be used in place of x_k in the computation of y_{k+1}. Similarly, x_{k+1} and y_{k+1} might be used in the computation of z_{k+1}. The next example shows what happens when this is applied to the equations in Example 3.26.

Example 3.28. Consider the system of equations given in (1) and the Gauss-Seidel iterative process suggested by (2):

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$

(7)
$$y_{k+1} = \frac{21 + 4x_{k+1} + z_k}{8}$$

$$z_{k+1} = \frac{15 + 2x_{k+1} - y_{k+1}}{5}.$$

See that if we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then iteration using (7) will converge to the solution $(2, 4, 3)$.

Substitute $y_0 = 2$ and $z_0 = 2$ into the first equation of (7) and obtain

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75.$$

Then substitute $x_1 = 1.75$ and $z_0 = 2$ into the second equation and get

$$y_1 = \frac{21 + 4(1.75) + 2}{8} = 3.75.$$

Finally, substitute $x_1 = 1.75$ and $y_1 = 3.75$ into the third equation to get

$$z_1 = \frac{15 + 2(1.75) - 3.75}{5} = 2.95.$$

The new point $P_1 = (1.75, 3.75, 2.95)$ is closer to $(2, 4, 3)$ than P_0 and is better than the value given in Example 3.26. Iteration using (7) generates a sequence $\{P_k\}$ that converges to $(2, 4, 3)$ (see Table 3.4). ∎

Table 3.4 Convergent Gauss-Seidel Iteration for the System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.75	2.95
2	1.95	3.96875	2.98625
3	1.99625	3.9609375	2.9903125
⋮	⋮	⋮	⋮
8	1.9999983	3.9999988	2.9999996
9	1.9999998	3.9999999	3.0000000
10	2.0000000	4.0000000	3.0000000

Jacobi iteration uses all old coordinates to generate all new coordinates, whereas Gauss-Seidel iteration uses the new coordinates as they become available:

Gauss-Seidel Iteration:

$$(11) \quad x_j^{(k+1)} = \frac{b_j - a_{j1}x_1^{(k+1)} - \cdots - a_{jj-1}x_{j-1}^{(k+1)} - a_{jj+1}x_{j+1}^{(k)} - \cdots - a_{jN}x_N^{(k)}}{a_{jj}}$$

for $j = 1, 2, \ldots, N$.

It can be proved that the Gauss-Seidel method will also converge when the matrix A is strictly diagonally dominant. In many cases the Gauss-Seidel method will converge faster than the Jacobi method; hence it is usually preferred (compare Examples 3.26 and 3.28). It is important to understand the slight modification of formula (10) that has been made to obtain formula (11). In some cases the Jacobi method will converge even though the Gauss-Seidel method will not.