The Bisection Method of Bolzano

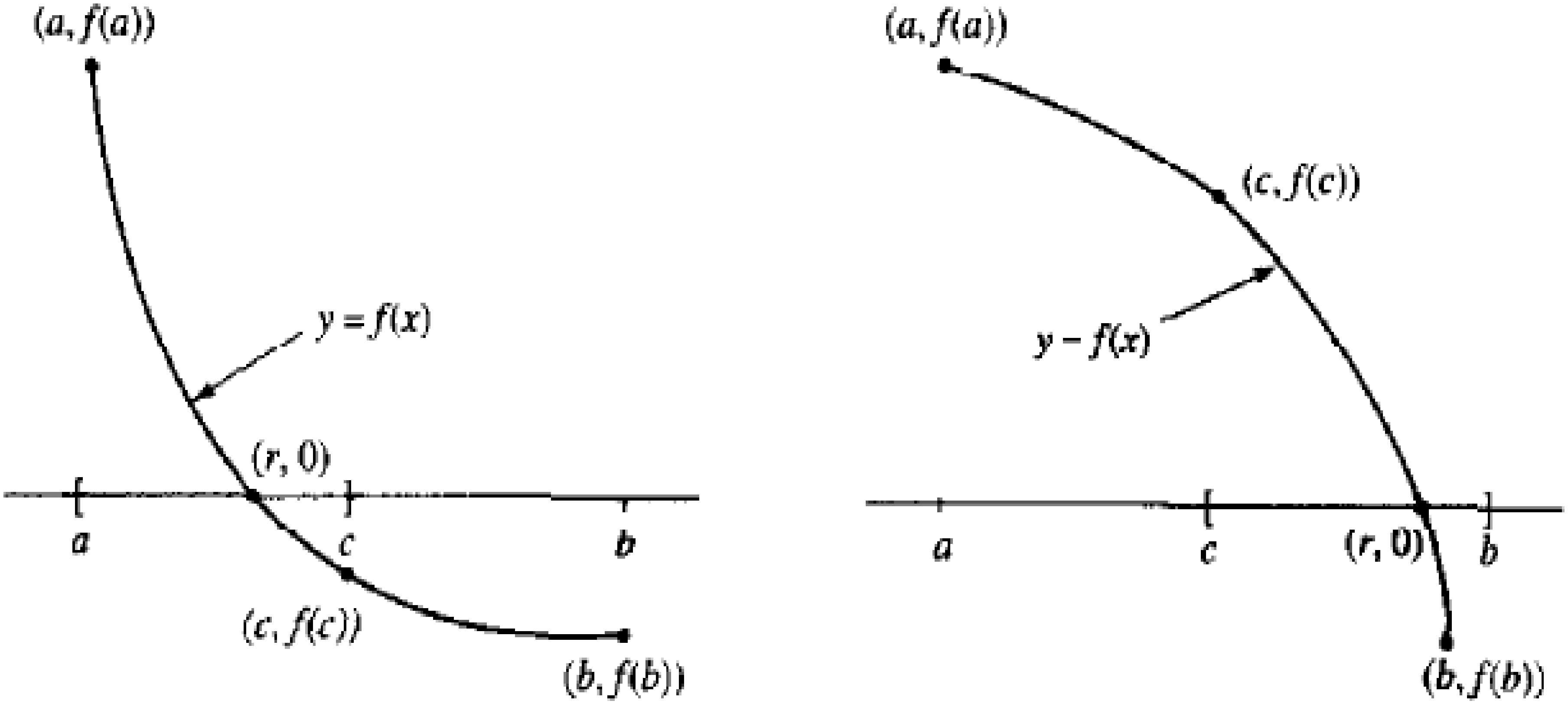
In this section we develop our first bracketing method for finding a zero of a continuous function. We must start with an initial interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs. Since the graph $y = f(x)$ of a continuous function is unbroken, it will cross the x-axis at a zero $x = r$ that lies somewhere in the interval (see Figure 2.6). The bisection method systematically moves the end points of the interval closer and closer together until we obtain an interval of arbitrarily small width that brackets the zero. The decision step for this process of interval halving is first to choose the midpoint

$c = (a + b)/2$ and then to analyze the three possibilities that might arise:

(4) If $f(a)$ and $f(c)$ have opposite signs, a zero lies in $[a, c]$.

(5) If $f(c)$ and $f(b)$ have opposite signs, a zero lies in $[c, b]$.

(6) If $f(c) = 0$, then the zero is c.



(a) If $f(a)$ and $f(c)$ have opposite signs then squeeze from the right.

(b) If $f(c)$ and $f(b)$ have opposite signs then squeeze from the left.

Figure 2.6 The decision process for the bisection process.

If either case (4) or (5) occurs, we have found an interval half as wide as the original interval that contains the root, and we are "squeezing down on it" (see Figure 2.6). To continue the process, relabel the new smaller interval $[a, b]$ and repeat the process until the interval is as small as desired. Since the bisection process involves sequences of nested intervals and their midpoints, we will use the following notation to keep track of the details in the process:

$[a_0, b_0]$ is the starting interval and $c_0 = \frac{a_0+b_0}{2}$ is the midpoint.

(7) $[a_1, b_1]$ is the second interval, which brackets the zero r, and c_1 is its midpoint; the interval $[a_1, b_1]$ is half as wide as $[a_0, b_0]$.

After arriving at the nth interval $[a_n, b_n]$, which brackets r and has midpoint c_n, the interval $[a_{n+1}, b_{n+1}]$ is constructed, which also brackets r and is half as wide as $[a_n, b_n]$.

It is left as an exercise for the reader to show that the sequence of left end points is increasing and the sequence of right end points is decreasing; that is,

$$(8) \qquad a_0 \le a_1 \le \cdots \le a_n \le \cdots \le r \le \cdots \le b_n \le \cdots \le b_1 \le b_0,$$

where $c_n = \frac{a_n + b_n}{2}$, and if $f(a_{n+1})f(b_{n+1}) < 0$, then

$$(9) \qquad [a_{n+1}, b_{n+1}] = [a_n, c_n] \quad \text{or} \quad [a_{n+1}, b_{n+1}] = [c_n, b_n] \quad \text{for all } n.$$

Theorem 2.4 (Bisection Theorem). Assume that $f \in C[a, b]$ and that there exists a number $r \in [a, b]$ such that $f(r) = 0$. If $f(a)$ and $f(b)$ have opposite signs, and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection process of (8) and (9), then

$$(10) \qquad |r - c_n| \le \frac{b - a}{2^{n+1}} \quad \text{for } n = 0, 1, \ldots,$$

and therefore the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the zero $x = r$; that is,

$$(11) \qquad \lim_{n \to \infty} c_n = r.$$

Proof. Since both the zero r and the midpoint c_n lie in the interval $[a_n, b_n]$, the distance between c_n and r cannot be greater than half the width of this interval (see Figure 2.7). Thus

$$|r - c_n| \le \frac{b_n - a_n}{2} \quad \text{for all } n. \tag{12}$$

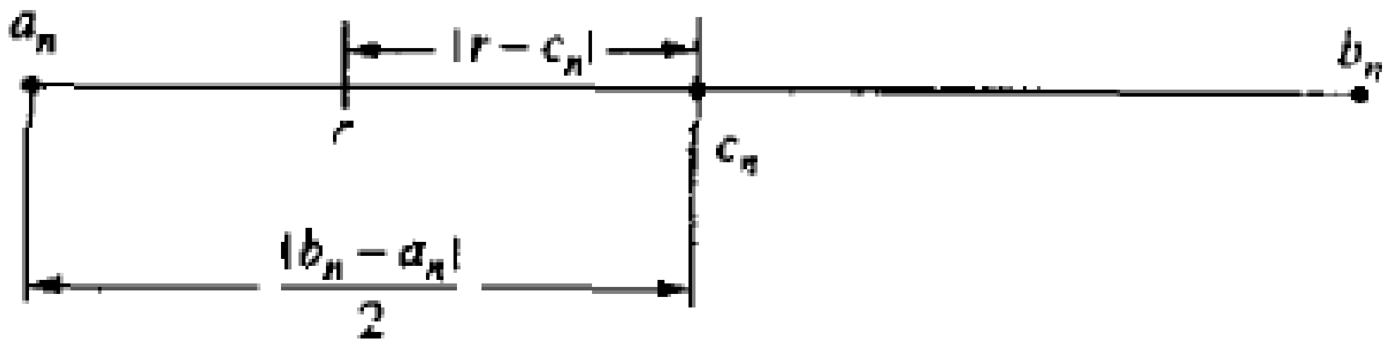


Figure 2.7 The root r and midpoint c_n of $[a_n, b_n]$ for the bisection method.

Observe that the successive interval widths form the pattern

$$b_1 - a_1 = \frac{b_0 - a_0}{2^1},$$
$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}.$$

It is left as an exercise for the reader to use mathematical induction and show that

$$b_n - a_n = \frac{b_0 - a_0}{2^n}. \tag{13}$$

Combining (12) and (13) results in

$$|r - c_n| \le \frac{b_0 - a_0}{2^{n+1}} \quad \text{for all } n. \tag{14}$$

Now an argument similar to the one given in Theorem 2.3 can be used to show that (14) implies that the sequence $\{c_n\}_{n=0}^{\infty}$ converges to r and the proof of the theorem is complete. •

Example 2.7. The function $h(x) = x \sin(x)$ occurs in the study of undamped forced oscillations. Find the value of x that lies in the interval $[0, 2]$, where the function takes on the value $h(x) = 1$ (the function $\sin(x)$ is evaluated in radians).

We use the bisection method to find a zero of the function $f(x) = x\sin(x) - 1$. Starting with $a_0 = 0$ and $b_0 = 2$, we compute

$$f(0) = -1.000000 \quad \text{and} \quad f(2) = 0.818595,$$

so a root of $f(x) = 0$ lies in the interval $[0, 2]$. At the midpoint $c_0 = 1$, we find that $f(1) = -0.158529$. Hence the function changes sign on $[c_0, b_0] = [1, 2]$.

To continue, we squeeze from the left and set $a_1 = c_0$ and $b_1 = b_0$. The midpoint is $c_1 = 1.5$ and $f(c_1) = 0.496242$. Now, $f(1) = -0.158529$ and $f(1.5) = 0.496242$ imply that the root lies in the interval $[a_1, c_1] = [1.0, 1.5]$. The next decision is to squeeze from the right and set $a_2 = a_1$ and $b_2 = c_1$. In this manner we obtain a sequence $\{c_k\}$ that converges to $r \approx 1.114157141$. A sample calculation is given in Table 2.1. ∎

Table 2.1 Bisection Method Solution of $x \sin(x) - 1 = 0$

k	Left end point, a_k	Midpoint, c_k	Right end point, b_k	Function value, $f(c_k)$
0	0	1.	2.	−0.158529
1	1.0	1.5	2.0	0.496242
2	1.00	1.25	1.50	0.186231
3	1.000	1.125	1.250	0.015051
4	1.0000	1.0625	1.1250	−0.071827
5	1.06250	1.09375	1.12500	−0.028362
6	1.093750	1.109375	1.125000	−0.006643
7	1.1093750	1.1171875	1.1250000	0.004208
8	1.10937500	1.11328125	1.11718750	−0.001216
⋮	⋮	⋮	⋮	⋮

3. For each function, find an interval $[a, b]$ so that $f(a)$ and $f(b)$ have opposite signs.

(a) $f(x) = e^x - 2 - x$

(b) $f(x) = \cos(x) + 1 - x$

(c) $f(x) = \ln(x) - 5 + x$

(d) $f(x) = x^2 - 10x + 23$

In Exercises 4 through 7 start with $[a_0, b_0]$ and use the false position method to compute c_0, c_1, c_2, and c_3.

4. $e^x - 2 - x = 0$, $[a_0, b_0] = [-2.4, -1.6]$

5. $\cos(x) + 1 - x = 0$, $[a_0, b_0] = [0.8, 1.6]$

6. $\ln(x) - 5 + x = 0$, $[a_0, b_0] = [3.2, 4.0]$

7. $x^2 - 10x + 23 = 0$, $[a_0, b_0] = [6.0, 6.8]$