Solution of Partial Differential Equations

Many problems in applied science, physics, and engineering are modeled mathematically with partial differential equations. A differential equation involving more than one independent variable is called a ***partial differential equation*** (PDE). It is not necessary to have taken a specialized course in PDEs to understand the rudimentary principles involved in obtaining computer solutions. In this chapter we will study finite-difference methods which are based on formulas for approximating the first and second derivatives of a function. We start by classifying the three types of equations under investigation and introduce a physical problem for each case. A partial differential equation of the form

$$A\Phi_{xx} + B\Phi_{xy} + C\Phi_{yy} = f(x, y, \Phi, \Phi_x, \Phi_y), \tag{1}$$

where A, B, and C are constants, is called ***quasilinear***. There are three types of quasilinear equations:

(2) If $B^2 - 4AC < 0$, the equation is called ***elliptic***.

(3) If $B^2 - 4AC = 0$, the equation is called ***parabolic***.

(4) If $B^2 - 4AC > 0$, the equation is called ***hyperbolic***.

As an example of a hyperbolic equation, we consider the one-dimensional model for a vibrating string. The displacement $u(x, t)$ is governed by the wave equation

$$\rho u_{tt}(x, y) = T u_{xx}(x, t) \quad \text{for } 0 < x < L \text{ and } 0 < t < \infty, \tag{5}$$

with the given initial position and velocity functions

$$\begin{aligned} u(x, 0) &= f(x) \quad \text{for } t = 0 \text{ and } 0 \le x \le L, \\ u_t(x, 0) &= g(x) \quad \text{for } t = 0 \text{ and } 0 < x < L, \end{aligned} \tag{6}$$

and the boundary values

$$\begin{aligned} u(0, t) &= 0 \quad \text{for } x = 0 \text{ and } 0 \le t < \infty, \\ u(L, t) &= 0 \quad \text{for } x = L \text{ and } 0 \le t < \infty. \end{aligned} \tag{7}$$

The constant ρ is the mass of the string per unit length and T is the tension in the string. A diagram of a string with fixed ends at the locations $(0, 0)$ and $(L, 0)$ is shown in Figure 10.1.

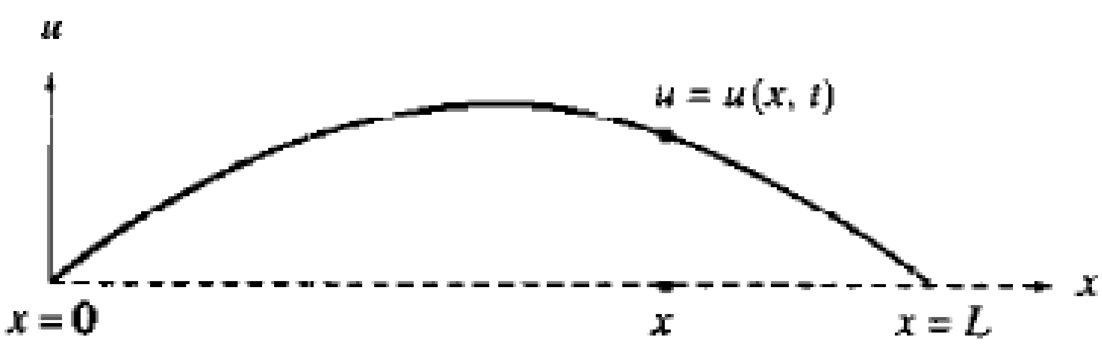


Figure 10.1 The wave equation models a vibrating string.

Hyperbolic Equations

Wave Equation

As an example of a hyperbolic partial differential equation, we consider the wave equation

$$(1) \qquad u_{tt}(x,t) = c^2 u_{xx}(x,t) \quad \text{for } 0 < x < a \text{ and } 0 < t < b,$$

with the boundary conditions

$$(2) \qquad \begin{aligned} &u(0,t) = 0 \quad \text{and} \quad u(a,t) = 0 && \text{for } 0 \le t \le b, \\ &u(x,0) = f(x) && \text{for } 0 \le x \le a, \\ &u_t(x,0) = g(x) && \text{for } 0 < x < a. \end{aligned}$$

The wave equation models the displacement u of a vibrating elastic string with fixed ends at $x = 0$ and $x = a$. Although analytic solutions to the wave equation can be obtained with Fourier series, we use the problem as a prototype of a hyperbolic equation.

Derivation of the Difference Equation

Partition the rectangle $R = \{(x, t) : 0 \le x \le a, 0 \le t \le b\}$ into a grid consisting of $n-1$ by $m-1$ rectangles with sides $\Delta x = h$ and $\Delta t = k$, as shown in Figure 10.4. Start at the bottom row, where $t = t_1 = 0$ and the solution is known to be $u(x_i, t_1) = f(x_i)$. We shall use a difference-equation method to compute approximations

$$\{u_{i,j} : i = 1, 2, \ldots, n\} \text{ in successive rows} \qquad \text{for } j = 2, 3, \ldots, m.$$

The true solution value at the grid points is $u(x_i, t_j)$.

The central-difference formulas for approximating $u_{tt}(x, t)$ and $u_{xx}(x, t)$ are

$$u_{tt}(x, t) = \frac{u(x, t+k) - 2u(x, t) + u(x, t-k)}{k^2} + O(k^2) \tag{3}$$

and

$$u_{xx}(x, t) = \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2} + O(h^2). \tag{4}$$

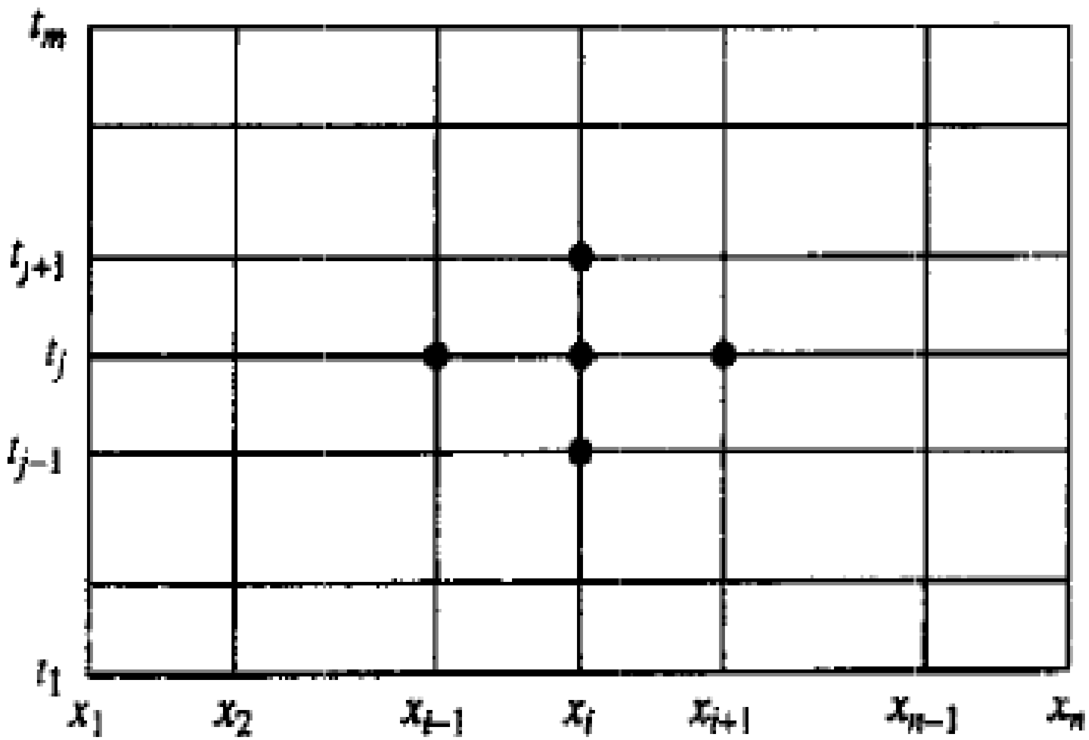


Figure 10.4 The grid for solving $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ over R.

The grid spacing is uniform in every row: $x_{i+1} = x_i + h$ (and $x_{i-1} = x_i - h$); and it is uniform in every column: $t_{j+1} = t_j + k$ (and $t_{j-1} = t_j - k$). Next, we drop the terms $O(k^2)$ and $O(h^2)$ and use the approximation $u_{i,j}$ for $u(x_i, t_j)$ in equations (3) and (4), which in turn are substituted into (1); this produces the difference equation

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \tag{5}$$

which approximates the solution to (1). For convenience, the substitution $r = ck/h$ is introduced in (5), and we obtain the relation

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = r^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}). \tag{6}$$

Equation (6) is employed to find row $j + 1$ across the grid, assuming that approximations in both rows j and $j - 1$ are known:

$$u_{i,j+1} = (2 - 2r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \tag{7}$$

for $i = 2, 3, \ldots, n - 1$. The four known values on the right side of equation (7), which are used to create the approximation $u_{i,j+1}$, are shown in Figure 10.5.

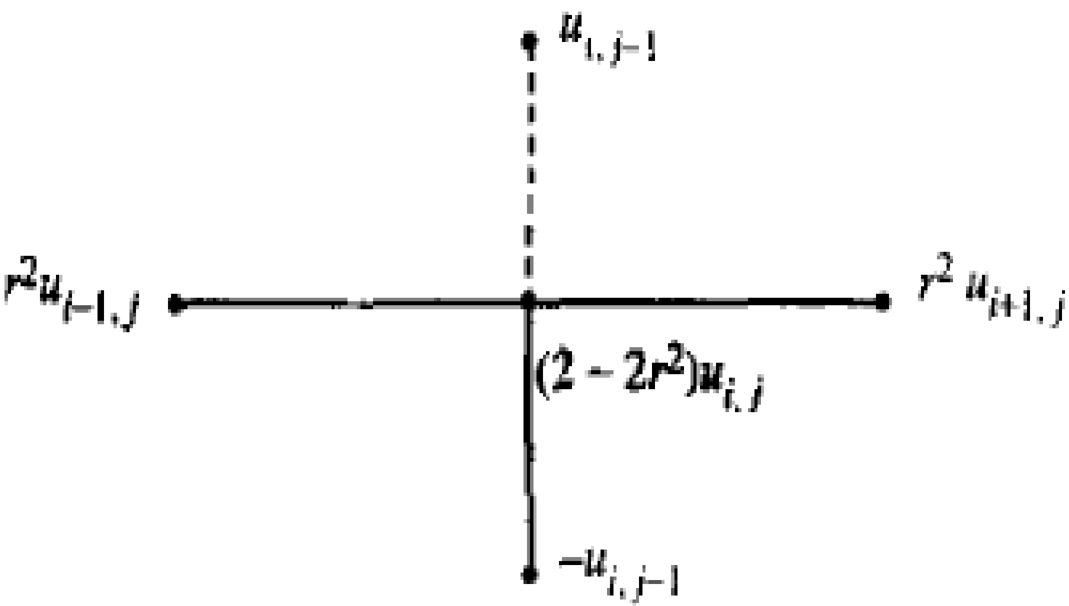


Figure 10.5 The wave equation stencil.

Caution must be taken when using formula (7). If the error made at one stage of the calculations is eventually dampened out, the method is called stable. To guarantee stability in formula (7), it is necessary that $r = ck/h \leq 1$. There are other schemes, called implicit methods, that are more complicated to implement, but do not have stability restrictions for r (see Reference [90]).

Starting Values

Two starting rows of values corresponding to $j = 1$ and $j = 2$ must be supplied in order to use formula (7) to compute the third row. Since the second row is not usually given, the boundary function $g(x)$ is used to help produce starting approximations in the second row. Fix $x = x_i$ at the boundary and apply Taylor's formula of order 1 for expanding $u(x, t)$ about $(x_i, 0)$. The value $u(x_i, k)$ satisfies

$$u(x_i, k) = u(x_i, 0) + u_t(x_i, 0)k + O(k^2). \tag{8}$$

Then use $u(x_i, 0) = f(x_i) = f_i$ and $u_t(x_i, 0) = g(x_i) = g_i$ in (8) to produce the formula for computing the numerical approximations in the second row:

$$u_{i,2} = f_i + kg_i \quad \text{for} \quad i = 2,\ 3,\ \ldots,\ n-1. \tag{9}$$

Usually, $u(x_i, t_2) \neq u_{i,2}$, and such errors introduced by formula (9) will propagate throughout the grid and will not be dampened out when the scheme in (7) is implemented. Hence it is prudent to use a very small step size for k so that the values for $u_{i,2}$ given in (9) do not contain a large amount of truncation error.

Often, the boundary function $f(x)$ has a second derivative $f''(x)$ over the interval. In this case we have $u_{xx}(x, 0) = f''(x)$, and it is beneficial to use the Taylor formula of order $n = 2$ to help construct the second row. To do this, we go back to the wave equation and use the relationship between the second-order partial derivatives to obtain

$$u_{tt}(x_i, 0) = c^2 u_{xx}(x_i, 0) = c^2 f''(x_i) = c^2 \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O(h^2). \tag{10}$$

Recall that Taylor's formula of order 2 is

$$u(x, k) = u(x, 0) + u_t(x, 0)k + \frac{u_{tt}(x, 0)k^2}{2} + O(k^3). \tag{11}$$

Applying formula (11) at $x = x_i$, together with (9) and (10), we get

$$u(x_i, k) = f_i + kg_i + \frac{c^2k^2}{2h^2}(f_{i+1} - 2f_i + f_{i-1}) + O(h^2)O(k^2) + O(k^3). \tag{12}$$

Using $r = ck/h$, formula (12) can be simplified to obtain a difference formula for the improved numerical approximations in the second row:

$$u_{i,2} = (1 - r^2)f_i + kg_i + \frac{r^2}{2}(f_{i+1} + f_{i-1}) \tag{13}$$

for $i = 2, 3, \ldots, n - 1$.

Example 10.1. Use the finite-difference method to solve the wave equation for a vibrating string:

(19) $$u_{tt}(x,t) = 4u_{xx}(x,t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.5,$$

with the boundary conditions

(20) $$\begin{aligned} u(0,t) = 0 \quad &\text{and} \quad u(1,t) = 0 && \text{for } 0 \le t \le 0.5, \\ u(x,0) = f(x) &= \sin(\pi x) + \sin(2\pi x) && \text{for } 0 \le x \le 1, \\ u_t(x,0) = g(x) &= 0 && \text{for } 0 \le x \le 1. \end{aligned}$$

For convenience we choose $h = 0.1$ and $k = 0.05$. Since $c = 2$, this yields $r = ck/h = 2(0.05)/0.1 = 1$. Since $g(x) = 0$ and $r = 1$, formula (13) for creating the second row is

(21) $$u_{i,2} = \frac{f_{i-1} + f_{i+1}}{2} \quad \text{for} \quad i = 2, 3, \ldots, 9.$$

Substituting $r = 1$ into equation (7) gives the simplified difference equation

(22) $$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}.$$

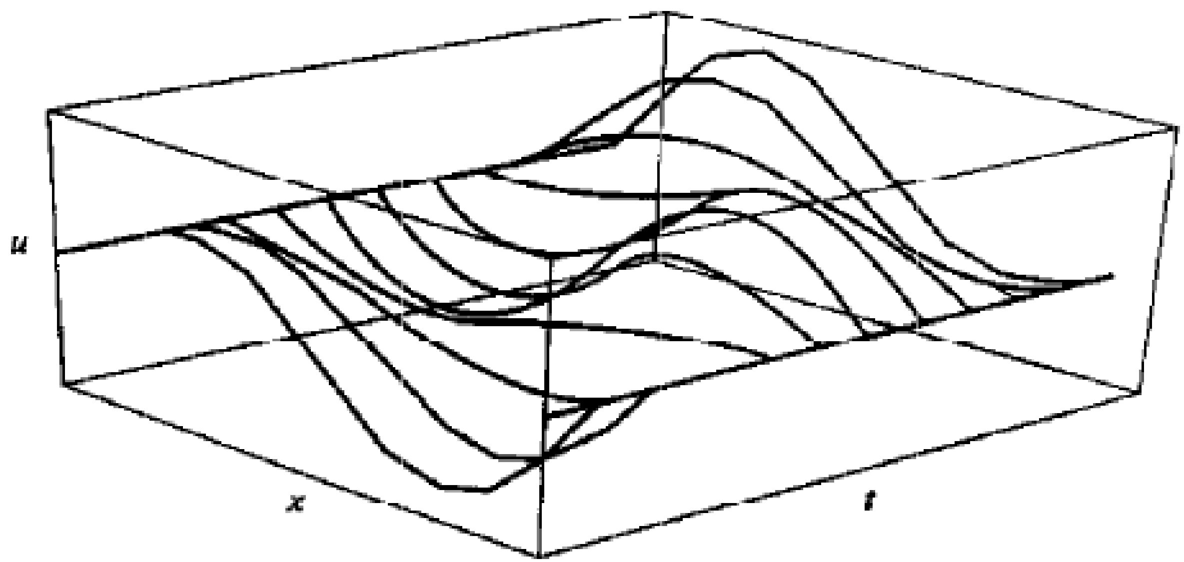
Applying formulas (21) and (22) successively to generate rows will produce the approximations to $u(x,t)$ given in Table 10.1 for $0 < x_i < 1$ and $0 \le t_j \le 0.50$.

The numerical values in Table 10.1 agree to more than six decimal places of accuracy with those obtained with the analytic solution

$$u(x,t) = \sin(\pi x)\cos(2\pi t) + \sin(2\pi x)\cos(4\pi t).$$

Table 10.1 Solution of the Wave Equation (19) with Boundary Conditions (20)

t_j	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.00	0.896802	1.538842	1.760074	1.538842	1.000000	0.363271	−0.142040	−0.363271	−0.278768
0.05	0.769421	1.328438	1.538842	1.380037	0.951056	0.428980	0.000000	−0.210404	−0.181636
0.10	0.431636	0.769421	0.948401	0.951056	0.809017	0.587785	0.360616	0.181636	0.068364
0.15	0.000000	0.051599	0.181636	0.377381	0.587785	0.740653	0.769421	0.639384	0.363271
0.20	−0.380037	−0.587785	−0.519421	−0.181636	0.309017	0.769421	1.019421	0.951056	0.571020
0.25	−0.587785	−0.951056	−0.951056	−0.587785	0.000000	0.587785	0.951056	0.951056	0.587785
0.30	−0.571020	−0.951056	−1.019421	−0.769421	−0.309017	0.181636	0.519421	0.587785	0.380037
0.35	−0.363271	−0.639384	−0.769421	−0.740653	−0.587785	−0.377381	−0.181636	−0.051599	0.000000
0.40	−0.068364	−0.181636	−0.360616	−0.587785	−0.809017	−0.951056	−0.948401	−0.769421	−0.431636
0.45	0.181636	0.210404	0.000000	−0.428980	−0.951056	−1.380037	−1.538842	−1.328438	−0.769421
0.50	0.278768	0.363271	0.142040	−0.363271	−1.000000	−1.538842	−1.760074	−1.538842	−0.896802



Example 10.2. Use the finite-difference method to solve the wave equation for a vibrating string:

(23) $$u_{tt}(x,t) = 4u_{xx}(x,t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.5,$$

with the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 \le t \le 1,$$

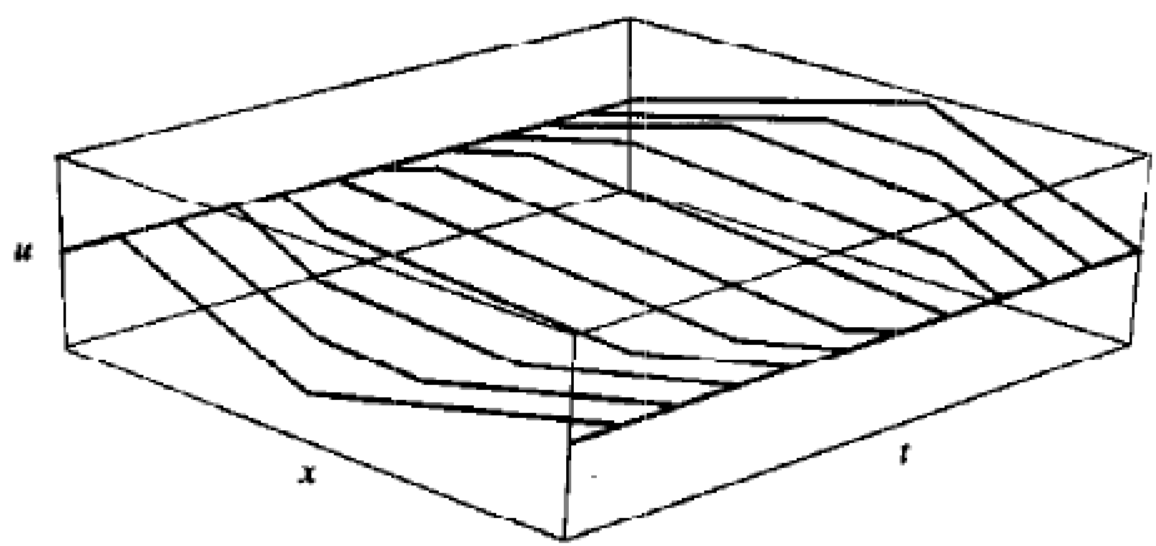
(24) $$u(x,0) = f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{3}{5} \\ 1.5 - 1.5x & \text{for } \frac{3}{5} \le x \le 1, \end{cases}$$

$$u_t(x,0) = g(x) = 0 \quad \text{for } 0 < x < 1.$$

For convenience we choose $h = 0.1$ and $k = 0.05$. Since $c = 2$, this again yields $r = 1$. Applying formulas (21) and (22) successively to generate rows will produce the approximations to $u(x,t)$ given in Table 10.2 for $0 \le x_i \le 1$ and $0 \le t_j \le 0.50$. A three-dimensional presentation of the data in Table 10.2 is given in Figure 10.7. ∎

Table 10.2 Solution of the Wave Equation (23) with Boundary Conditions (24)

t_j	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.00	0.100	0.200	0.300	0.400	0.500	0.600	0.450	0.300	0.150
0.05	0.100	0.200	0.300	0.400	0.500	0.475	0.450	0.300	0.150
0.10	0.100	0.200	0.300	0.400	0.375	0.350	0.325	0.300	0.150
0.15	0.100	0.200	0.300	0.275	0.250	0.225	0.200	0.175	0.150
0.20	0.100	0.200	0.175	0.150	0.125	0.100	0.075	0.050	0.025
0.25	0.100	0.075	0.050	0.025	0.000	−0.025	−0.050	−0.075	−0.100
0.30	−0.025	−0.050	−0.075	−0.100	−0.125	−0.150	−0.175	−0.200	−0.100
0.35	−0.150	−0.175	−0.200	−0.225	−0.250	−0.275	−0.300	−0.200	−0.100
0.40	−0.150	−0.300	−0.325	−0.350	−0.375	−0.400	−0.300	−0.200	−0.100
0.45	−0.150	−0.300	−0.450	−0.475	−0.500	−0.400	−0.300	−0.200	−0.100
0.50	−0.150	−0.300	−0.450	−0.600	−0.500	−0.400	−0.300	−0.200	−0.100



Exercises for Hyperbolic Equations

1. (a) Verify by direct substitution that $u(x, t) = \sin(n\pi x)\cos(2n\pi t)$ is a solution to the wave equation $u_{tt}(x, t) = 4u_{xx}(x, t)$ for each positive integer $n = 1, 2, \ldots$.
   (b) Verify by direct substitution that $u(x, t) = \sin(n\pi x)\cos(cn\pi t)$ is a solution to the wave equation $u_{tt}(x, t) = c^2 u_{xx}(x, t)$ for each positive integer $n = 1, 2, \ldots$.

2. Assume that the initial position and velocity are $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$, respectively. Show that the d'Alembert solution for this case is

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2}.$$

3. Obtain a simplified form of the difference equation (7) in the case $h = 2ck$.

In Exercises 4 and 5, use the finite-difference method to calculate the first three rows of the approximate solution for the given wave equation. Carry out your calculations by hand (calculator).

4. $u_{tt}(x, t) = 4u_{xx}(x, t)$, for $0 \le x \le 1$ and $0 \le t \le 0.5$, with the boundary conditions

$$\begin{array}{lll} u(0, t) = 0 \quad \text{and} \quad u(1, t) = 0 & & \text{for } 0 \le t \le 0.5, \\ u(x, 0) = f(x) = \sin(\pi x) & & \text{for } 0 \le x \le 1, \\ u_t(x, 0) = g(x) = 0 & & \text{for } 0 \le x \le 1. \end{array}$$

Let $h = 0.2$, $k = 0.1$, and $r = 1$.