2.4 Newton-Raphson and Secant Methods

Slope Methods for Finding Roots

If $f(x)$, $f'(x)$, and $f''(x)$ are continuous near a root p, then this extra information regarding the nature of $f(x)$ can be used to develop algorithms that will produce sequences $\{p_k\}$ that converge faster to p than either the bisection or false position method. The Newton-Raphson (or simply Newton's) method is one of the most useful and best known algorithms that relies on the continuity of $f'(x)$ and $f''(x)$. We shall introduce it graphically and then give a more rigorous treatment based on the Taylor polynomial.

Assume that the initial approximation p_0 is near the root p. Then the graph of $y = f(x)$ intersects the x-axis at the point $(p, 0)$, and the point $(p_0, f(p_0))$ lies on the curve near the point $(p, 0)$ (see Figure 2.13). Define p_1 to be the point of intersection of the x-axis and the line tangent to the curve at the point $(p_0, f(p_0))$. Then Figure 2.13 shows that p_1 will be closer to p than p_0 in this case. An equation relating p_1 and p_0 can be found if we write down two versions for the slope of the tangent line L:

$$m = \frac{0 - f(p_0)}{p_1 - p_0}, \tag{1}$$

which is the slope of the line through $(p_1, 0)$ and $(p_0, f(p_0))$, and

$$m = f'(p_0), \tag{2}$$

which is the slope at the point $(p_0, f(p_0))$. Equating the values of the slope m in equations (1) and (2) and solving for p_1 results in

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}. \tag{3}$$

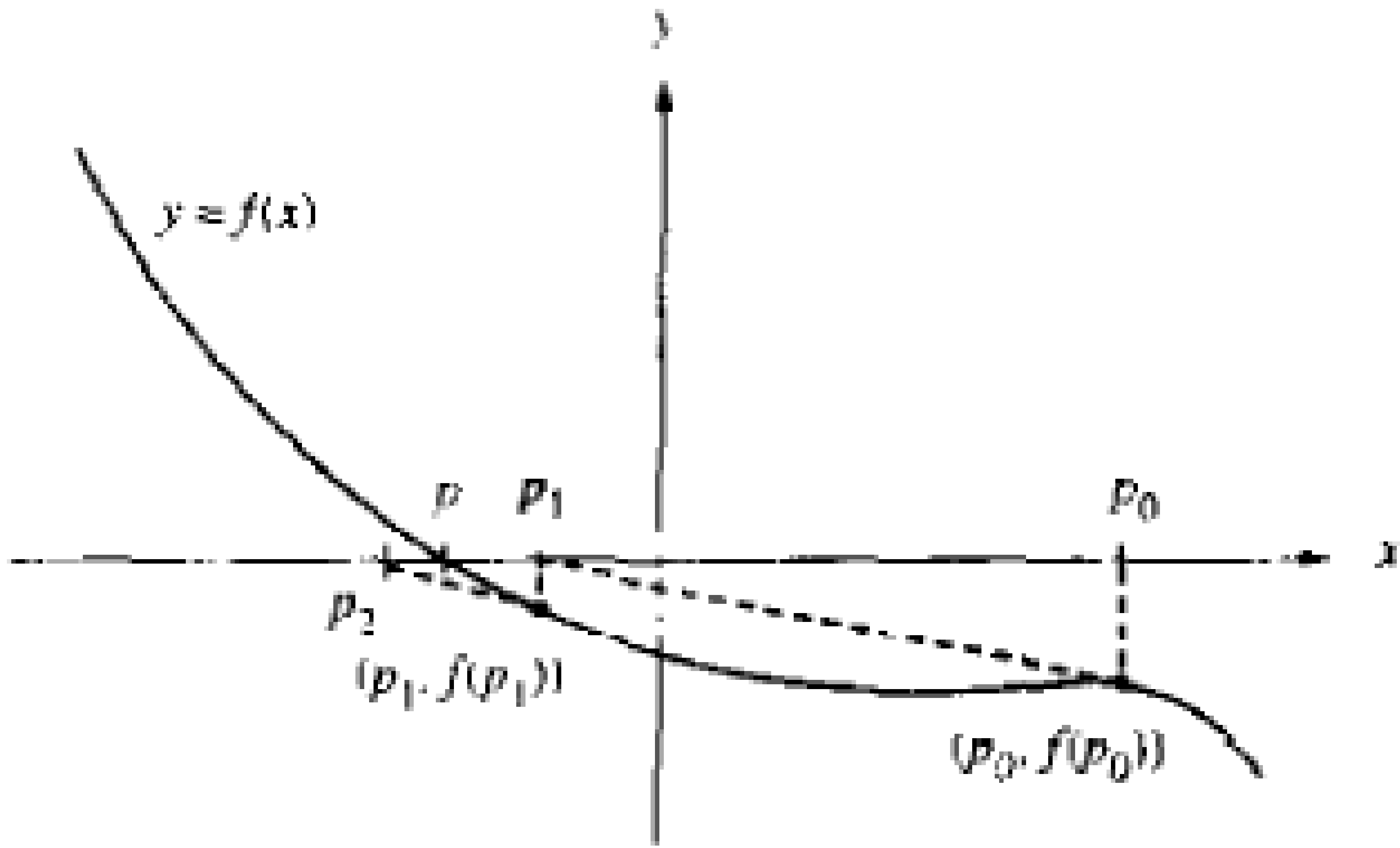

Figure 2.13 The geometric construction of p_1 and p_2 for the Newton-Raphson method.

The process above can be repeated to obtain a sequence $\{p_k\}$ that converges to p. We now make these ideas more precise.

Theorem 2.5 (Newton-Raphson Theorem). Assume that $f \in C^2[a, b]$ and there exists a number $p \in [a, b]$, where $f(p) = 0$. If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \quad \text{for } k = 1, 2, \ldots \tag{4}$$

will converge to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Remark. The function $g(x)$ defined by formula

$$g(x) = x - \frac{f(x)}{f'(x)} \tag{5}$$

is called the ***Newton-Raphson iteration function***. Since $f(p) = 0$, it is easy to see that $g(p) = p$. Thus the Newton-Raphson iteration for finding the root of the equation $f(x) = 0$ is accomplished by finding a fixed point of the function $g(x)$.

Corollary 2.2 (Newton's Iteration for Finding Square Roots). Assume that $A > 0$ is a real number and let $p_0 > 0$ be an initial approximation to $\sqrt{A}$. Define the sequence $\{p_k\}_{k=0}^{\infty}$ using the recursive rule

$$p_k = \frac{p_{k-1} + \dfrac{A}{p_{k-1}}}{2} \quad \text{for } k = 1, 2, \ldots. \tag{11}$$

Then the sequence $\{p_k\}_{k=0}^{\infty}$ converges to $\sqrt{A}$; that is, $\lim_{n\to\infty} p_k = \sqrt{A}$.

Outline of Proof. Start with the function $f(x) = x^2 - A$, and notice that the roots of the equation $x^2 - A = 0$ are $\pm\sqrt{A}$. Now use $f(x)$ and the derivative $f'(x)$ in formula (5) and write down the Newton-Raphson iteration formula

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - A}{2x}. \tag{12}$$

This formula can be simplified to obtain

$$g(x) = \frac{x + \frac{A}{x}}{2}. \tag{13}$$

When $g(x)$ in (13) is used to define the recursive iteration in (4), the result is formula (11). It can be proved that the sequence that is generated in (11) will converge for any starting value $p_0 > 0$. The details are left for the exercises. •

Example 2.11. Use Newton's square-root algorithm to find $\sqrt{5}$.

Starting with $p_0 \approx 2$ and using formula (11), we compute

$$
\begin{aligned}
p_1 &\approx \frac{2 + 5/2}{2} = 2.25 \\
p_2 &\approx \frac{2.25 + 5/2.25}{2} = 2.236111111 \\
p_3 &\approx \frac{2.236111111 + 5/2.236111111}{2} = 2.236067978 \\
p_4 &\approx \frac{2.36067978 + 5/2.236067978}{2} = 2.236067978.
\end{aligned}
$$

Further iterations produce $p_k \approx 2.236067978$ for $k > 4$, so we see that convergence *accurate to nine decimal places has been achieved.* ■

1. Let $f(x) = x^2 - x + 2$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) Start with $p_0 = -1.5$ and find p_1, p_2, and p_3.

2. Let $f(x) = x^2 - x - 3$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) Start with $p_0 = 1.6$ and find p_1, p_2, and p_3.

(c) Start with $p_0 = 0.0$ and find p_1, p_2, p_3, and p_4. What do you conjecture about *this sequence?*

3. Let $f(x) = (x - 2)^4$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) Start with $p_0 = 2.1$ and find p_1, p_2, p_3, and p_4.

(c) Is the sequence converging quadratically or linearly?

4. Let $f(x) = x^3 - 3x - 2$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) Start with $p_0 = 2.1$ and find p_1, p_2, p_3, and p_4.

(c) Is the sequence converging quadratically or linearly?

Definition 2.4 (Order of a Root). Assume that $f(x)$ and its derivatives $f'(x)$, $\ldots$, $f^{(M)}(x)$ are defined and continuous on an interval about $x = p$. We say that $f(x) = 0$ has a root of order M at $x = p$ if and only if

(17)
$$f(p) = 0, \quad f'(p) = 0, \quad \ldots, \quad f^{(M-1)}(p) = 0, \quad \text{and} \quad f^{(M)}(p) \neq 0.$$

A root of order $M = 1$ is often called a ***simple root***, and if $M > 1$, it is called a ***multiple root***. A root of order $M = 2$ is sometimes called a ***double root***, and so on. The next result will illuminate these concepts. ▲

Lemma 2.1. If the equation $f(x) = 0$ has a root of order M at $x = p$, then there exists a continuous function $h(x)$ so that $f(x)$ can be expressed as the product

$$f(x) = (x - p)^M h(x), \quad \text{where } h(p) \neq 0. \tag{18}$$

Example 2.13. The function $f(x) = x^3 - 3x + 2$ has a simple root at $p = -2$ and a double root at $p = 1$. This can be verified by considering the derivatives $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. At the value $p = -2$, we have $f(-2) = 0$ and $f'(-2) = 9$, so $M = 1$ in Definition 2.4; hence $p = -2$ is a simple root. For the value $p = 1$, we have $f(1) = 0$, $f'(1) = 0$, and $f''(1) = 6$, so $M = 2$ in Definition 2.4; hence $p = 1$ is a double root. Also, notice that $f(x)$ has the factorization $f(x) = (x + 2)(x - 1)^2$. ■

Speed of Convergence

The distinguishing property we seek is the following. If p is a simple root of $f(x) = 0$, Newton's method will converge rapidly, and the number of accurate decimal places (roughly) doubles with each iteration. On the other hand, if p is a multiple root, the error in each successive approximation is a fraction of the previous error. To make this precise, we define the ***order of convergence***. This is a measure of how rapidly a sequence converges.

Definition 2.5 (Order of Convergence). Assume that $\{p_n\}_{n=0}^{\infty}$ converges to p and set $E_n = p - p_n$ for $n \geq 0$. If two positive constants $A \neq 0$ and $R > 0$ exist, and

$$\lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A, \tag{19}$$

then the sequence is said to converge to p with order of convergence R. The number A is called the asymptotic error constant. The cases $R = 1, 2$ are given special consideration.

(20) If $R = 1$, the convergence of $\{p_n\}_{n=0}^{\infty}$ is called ***linear***.

(21) If $R = 2$, the convergence of $\{p_n\}_{n=0}^{\infty}$ is called ***quadratic***. ▲

If R is large, the sequence $\{p_n\}$ converges rapidly to p; that is, relation (19) implies that for large values of n we have the approximation $|E_{n+1}| \approx A|E_n|^R$. For example, suppose that $R = 2$ and $|E_n| \approx 10^{-2}$; then we would expect that $|E_{n+1}| \approx A \times 10^{-4}$.

Some sequences converge at a rate that is not an integer, and we will see that the order of convergence of the secant method is $R = (1 + \sqrt{5})/2 \approx 1.618033989$.

Theorem 2.6 (Convergence Rate for Newton-Raphson Iteration). Assume that Newton-Raphson iteration produces a sequence $\{p_n\}_{n=0}^{\infty}$ that converges to the root p of the function $f(x)$. If p is a simple root, convergence is quadratic and

$$(23) \qquad |E_{n+1}| \approx \frac{|f''(p)|}{2|f'(p)|}|E_n|^2 \quad \text{for } n \text{ sufficiently large.}$$

If p is a multiple root of order M, convergence is linear and

$$(24) \qquad |E_{n+1}| \approx \frac{M-1}{M}|E_n| \quad \text{for } n \text{ sufficiently large.}$$

Example 2.14 (Quadratic Convergence at a Simple Root). Start with $p_0 = -2.4$ and use Newton-Raphson iteration to find the root $p = -2$ of the polynomial $f(x) = x^3 - 3x + 2$. The iteration formula for computing $\{p_k\}$ is

$$(22) \qquad p_k = g(p_{k-1}) = \frac{2p_{k-1}^3 - 2}{3p_{k-1}^2 - 3}.$$

Using formula (21) to check for quadratic convergence, we get the values in Table 2.5. ■

Table 2.5 Newton's Method Converges Quadratically at a Simple Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\dfrac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$
0	−2.400000000	0.323809524	0.400000000	0.476190475
1	−2.076190476	0.072594465	0.076190476	0.619469086
2	−2.003596011	0.003587422	0.003596011	0.664202613
3	−2.000008589	0.000008589	0.000008589	
4	−2.000000000	0.000000000	0.000000000	

A detailed look at the rate of convergence in Example 2.14 will reveal that the error in each successive iteration is proportional to the square of the error in the previous iteration. That is,

$$|p - p_{k+1}| \approx A|p - p_k|^2,$$

where $A \approx 2/3$. To check this, we use

$$|p - p_3| = 0.000008589 \quad \text{and} \quad |p - p_2|^2 = |0.003596011|^2 = 0.000012931$$

and it is easy to see that

$$|p - p_3| = 0.000008589 \approx 0.000008621 = \frac{2}{3}|p - p_2|^2.$$

Table 2.6 Newton's Method Converges Linearly at a Double Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert}$
0	1.200000000	−0.096969697	−0.200000000	0.515151515
1	1.103030303	−0.050673883	−0.103030303	0.508165253
2	1.052356420	−0.025955609	−0.052356420	0.496751115
3	1.026400811	−0.013143081	−0.026400811	0.509753688
4	1.013257730	−0.006614311	−0.013257730	0.501097775
5	1.006643419	−0.003318055	−0.006643419	0.500550093
⋮	⋮	⋮	⋮	⋮

Example 2.15 ***(Linear Convergence at a Double Root).*** Start with $p_0 = 1.2$ and use Newton-Raphson iteration to find the double root $p = 1$ of the polynomial $f(x) = x^3 - 3x + 2$.

Using formula (20) to check for linear convergence, we get the values in Table 2.6. ■

5. Consider the function $f(x) = \cos(x)$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) We want to find the root $p = 3\pi/2$. Can we use $p_0 = 3$? Why?

(c) We want to find the root $p = 3\pi/2$. Can we use $p_0 = 5$? Why?

6. Consider the function $f(x) = \arctan(x)$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) If $p_0 = 1.0$, then find p_1, p_2, p_3, and p_4. What is $\lim_{n\to\infty} p_k$?

(c) If $p_0 = 2.0$, then find p_1, p_2, p_3, and p_4. What is $\lim_{n\to\infty} p_k$?

7. Consider the function $f(x) = xe^{-x}$.

(a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.

(b) If $p_0 = 0.2$, then find p_1, p_2, p_3, and p_4. What is $\lim_{n\to\infty} p_k$?

(c) If $p_0 = 20$, then find p_1, p_2, p_3, and p_4. What is $\lim_{n\to\infty} p_k$?

(d) What is the value of $f(p_4)$ in part (c)?

12. Consider $f(x) = x^N - A$, where N is a positive integer.

(a) What real values are the solution to $f(x) = 0$ for the various choices of N and A that can arise?

(b) Derive the recursive formula

$$p_k = \frac{(N-1)p_{k-1} + A/p_{k-1}^{N-1}}{N} \quad \text{for } k = 1, 2, \ldots$$

for finding the Nth root of A.

13. Can Newton-Raphson iteration be used to solve $f(x) = 0$ if $f(x) = x^2 - 14x + 50$? Why?

14. Can Newton-Raphson iteration be used to solve $f(x) = 0$ if $f(x) = x^{1/3}$? Why?

15. Can Newton-Raphson iteration be used to solve $f(x) = 0$ if $f(x) = (x-3)^{1/2}$ and the starting value is $p_0 = 4$? Why?