The Secant Method

The Newton-Raphson algorithm requires the evaluation of two functions per iteration, $f(p_{k-1})$ and $f'(p_{k-1})$. Traditionally, the calculation of derivatives of elementary functions could involve considerable effort. But, with modern computer algebra software packages, this has become less of an issue. Still many functions have nonelementary forms (integrals, sums, etc.), and it is desirable to have a method that converges almost as fast as Newton's method yet involves only evaluations of $f(x)$ and not of $f'(x)$. The secant method will require only one evaluation of $f(x)$ per step and at a simple root has an order of convergence $R \approx 1.618033989$. It is almost as fast as Newton's method, which has order 2.

Two initial points $(p_0, f(p_0))$ and $(p_1, f(p_1))$ near the point $(p, 0)$ are needed, as shown in Figure 2.16. Define p_2 to be the abscissa of the point of intersection of the line through these two points and the x-axis;
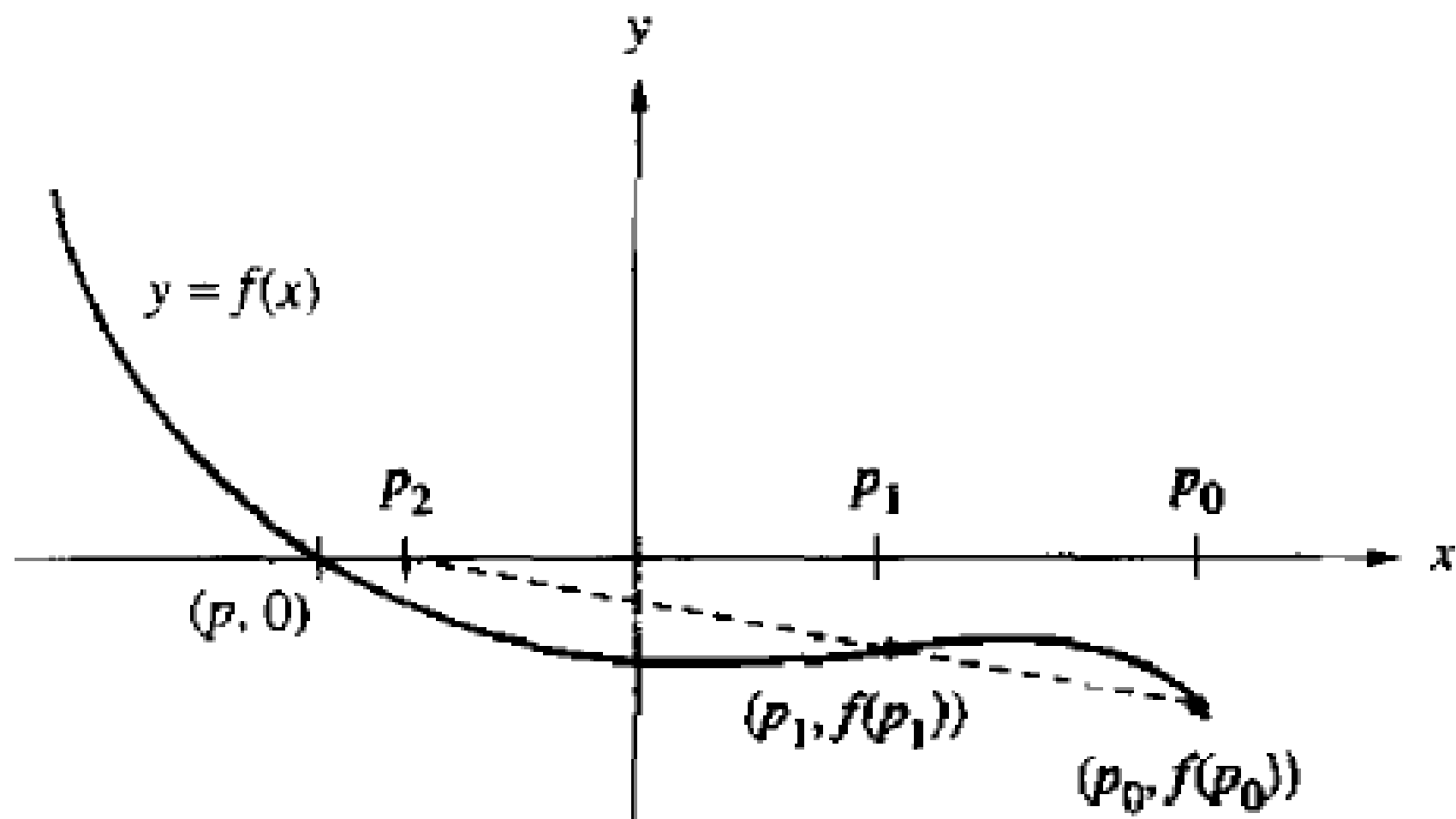
Figure 2.16 The geometric construction of p_2 for the secant method.

Figure 2.16 shows that p_2 will be closer to $\bar{p}$ than to either p_0 or p_1. The equation relating p_2, p_1, and p_0 is found by considering the slope

$$(25) \qquad m = \frac{f(p_1) - f(p_0)}{p_1 - p_0} \quad \text{and} \quad m = \frac{0 - f(p_1)}{p_2 - p_1}.$$

The values of m in (25) are the slope of the secant line through the first two approximations and the slope of the line through $(p_1, f(p_1))$ and $(p_2, 0)$, respectively. Set the right-hand sides equal in (25) and solve for $p_2 = g(p_1, p_0)$ and get

$$(26) \qquad p_2 = g(p_1, p_0) = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}.$$

The general term is given by the two-point iteration formula

$$(27) \qquad p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})}.$$

Example 2.16 (Secant Method at a Simple Root). Start with $p_0 = -2.6$ and $p_1 = -2.4$ and use the secant method to find the root $p = -2$ of the polynomial function $f(x) = x^3 - 3x + 2$.

In this case the iteration formula (27) is

$$(28) \qquad p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{(p_k^3 - 3p_k + 2)(p_k - p_{k-1})}{p_k^3 - p_{k-1}^3 - 3p_k + 3p_{k-1}}.$$

This can be algebraically manipulated to obtain

$$(29) \qquad p_{k+1} = g(p_k, p_{k-1}) = \frac{p_k^2 p_{k-1} + p_k p_{k-1}^2 - 2}{p_k^2 + p_k p_{k-1} + p_{k-1}^2 - 3}.$$

The sequence of iterates is given in Table 2.7. ∎

Table 2.7 Convergence of the Secant Method at a Simple Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\dfrac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^{1.618}}$
0	-2.600000000	0.200000000	0.600000000	0.914152831
1	-2.400000000	0.293401015	0.400000000	0.469497765
2	-2.106598985	0.083957573	0.106598985	0.847290012
3	-2.022641412	0.021130314	0.022641412	0.693608922
4	-2.001511098	0.001488561	0.001511098	0.825841116
5	-2.000022537	0.000022515	0.000022537	0.727100987
6	-2.000000022	0.000000022	0.000000022	
7	-2.000000000	0.000000000	0.000000000	

Proofs about the rate of convergence of the secant method can be found in advanced texts on numerical analysis. Let us state that the error terms satisfy the relationship

$$(30) \qquad |E_{k+1}| \approx |E_k|^{1.618} \left| \frac{f''(p)}{2f'(p)} \right|^{0.618}$$

where the order of convergence is $R = (1 + \sqrt{5})/2 \approx 1.618$ and the relation in (30) is valid only at simple roots.

To check this, we make use of Example 2.16 and the specific values

$$|p - p_5| = 0.000022537$$
$$|p - p_4|^{1.618} = 0.001511098^{1.618} = 0.000027296,$$

and

$$A = |f''(-2)/2f'(-2)|^{0.618} = (2/3)^{0.618} = 0.778351205.$$

Combine hese and it is easy to see that

$$|p - p_5| = 0.000022537 \approx 0.000021246 = A|p - p_4|^{1.618}.$$

Accelerated Convergence

We could hope that there are root-finding techniques that converge faster than linearly when p is a root of order M. Our final result shows that a modification can be made to Newton's method so that convergence becomes quadratic at a multiple root.

Theorem 2.7 (Acceleration of Newton-Raphson Iteration). Suppose that the Newton-Raphson algorithm produces a sequence that converges linearly to the root $x = p$ of order $M > 1$. Then the Newton-Raphson iteration formula

$$(31) \qquad\qquad p_k = p_{k-1} - \frac{Mf(p_{k-1})}{f'(p_{k-1})}$$

will produce a sequence $\{p_k\}_{k=0}^{\infty}$ that converges quadratically to p.

Example 2.17 (Acceleration of Convergence at a Double Root).   Start with $p_0 = 1.2$ and use accelerated Newton-Raphson iteration to find the double root $p = 1$ of $f(x) = x^3 - 3x + 2$.

Since $M = 2$, the acceleration formula (31) becomes

$$p_k = p_{k-1} - 2\frac{f(p_{k-1})}{f'(p_{k-1})} = \frac{p_{k-1}^3 + 3p_{k-1} - 4}{3p_{k-1}^2 - 3},$$

and we obtain the values in Table 2.8.   ∎

Table 2.8   Acceleration of Convergence at a Double Root

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\dfrac{\|E_{k+1}\|}{\|E_k\|^2}$
0	1.200000000	−0.193939394	−0.200000000	0.151515150
1	1.006060606	−0.006054519	−0.006060606	0.165718578
2	1.000006087	−0.000006087	−0.000006087	
3	1.000000000	0.000000000	0.000000000	

Table 2.9 Comparison of the Speed of Convergence

Method	Special considerations	Relation between successive error terms
Bisection		$E_{k+1} \approx \frac{1}{2}|E_k|$
Regula falsi		$E_{k+1} \approx A|E_k|$
Secant method	Multiple root	$E_{k+1} \approx A|E_k|$
Newton-Raphson	Multiple root	$E_{k+1} \approx A|E_k|$
Secant method	Simple root	$E_{k+1} \approx A|E_k|^{1.618}$
Newton-Raphson	Simple root	$E_{k+1} \approx A|E_k|^2$
Accelerated Newton-Raphson	Multiple root	$E_{k+1} \approx A|E_k|^2$

In Exercises 8 through 10, use the secant method and formula (27) and compute the next two iterates p_2 and p_3.

8. Let $f(x) = x^2 - 2x - 1$. Start with $p_0 = 2.6$ and $p_1 = 2.5$.

9. Let $f(x) = x^2 - x - 3$. Start with $p_0 = 1.7$ and $p_1 = 1.67$.

10. Let $f(x) = x^3 - x + 2$. Start with $p_0 = -1.5$ and $p_1 = -1.52$.

5. Modify Program 2.5 to use the accelerated Newton-Raphson algorithm in Theorem 2.7 to find the root p of order M of each of the following functions.

    (a) $f(x) = (x - 2)^5$, $M = 5$, $p = 2$; start with $p_0 = 1$.

    (b) $f(x) = \sin(x^3)$, $M = 3$, $p = 0$; start with $p_0 = 1$.

    (c) $f(x) = (x - 1)\ln(x)$, $M = 2$, $p = 1$; start with $p_0 = 2$.