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Tropical Optimization Problems with Application to Project Scheduling with Minimum Makespan

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Abstract

We consider multidimensional optimization problems in the framework of tropical mathematics. The problems are formulated to minimize a nonlinear objective function that is defined on vectors in a finite-dimensional semimodule over an idempotent semifield and calculated by means of multiplicative conjugate transposition. We start with an unconstrained problem and offer two complete direct solutions, which follow different argumentation schemes. The first solution consists of the derivation of a sharp lower bound for the objective function and the solving of an equation to find all vectors that yield the bound. The second is based on extremal properties of the spectral radius of matrices and involves the evaluation of this radius for a certain matrix. The second solution is then extended to problems with boundary constraints that specify the feasible solution set by a double inequality, and with a linear inequality constraint given by a matrix. We apply the results obtained to solve problems in project scheduling under the minimum makespan criterion subject to various precedence constraints imposed on the time of initiation and completion of activities in the project. To illustrate the solutions, simple numerical examples are also included.

Key-Words: idempotent semifield; tropical mathematics; constrained optimization problem; direct solution project scheduling minimum makespan.

MSC (2010): 65K10, 15A80, 65K05, 90C48, 90B35

1 Introduction

Tropical (idempotent) mathematics, which focuses on the theory and applications of semirings with idempotent addition, offers a useful framework for the formulation and solution of real-world optimization problems in various

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fields, including project scheduling. Even the early works by Cuninghame-Green [1] and Giffler [2] on tropical mathematics used optimization problems drawn from machine scheduling to motivate and illustrate the study.

In the last few decades, the theory and methods of tropical mathematics have received much attention, which resulted in many published works, such as recent monographs by Golan [3], Heidergott et al. [4], Gondran and Minoux [5], Butkovič [6], McEneaney [7] and a great many contributed papers. Tropical optimization forms an important research domain within the field, which mainly concentrates on new solutions for problems in operations research. Applications to scheduling problems remain of great concern in a number of researches, such as the works by Zimmermann [8, 9], Butkovič et al. [10, 11, 12] and Krivulin [13, 14, 15, 16, 17]. There are also applications in other areas, including applications in location analysis developed by Cuninghame-Green [18, 19] and Krivulin [20, 21, 22, 23], in decision making by Elsner and van den Driessche [24, 25], Akian et al. [26], Gaubert et al. [27] and Gursoy et al. [28], and in discrete event systems by Gaubert [29], De Schutter [30], and De Schutter and van den Boom [31], to name only a few.

In this paper, we consider multidimensional optimization problems, which are formulated and solved in the tropical mathematics setting. The problems are to minimize a nonlinear objective function defined on vectors in a finite-dimensional semimodule over an idempotent semifield by means of a multiplicative conjugate transposition operator. We start with an unconstrained problem to propose two complete direct solutions to the problem, which offer different representations for the solution set. The first solution follows the approach developed in [13] to derive a sharp lower bound for the objective function and to solve an equation to find all vectors that yield the bound. The other one is based on extremal properties of the spectrum of matrices investigated in [14, 17, 16] and involves the evaluation of the spectral radius of a certain matrix. We show that, although these solutions are represented in different forms, they define the same solution set. The latter solution is then extended to solve the problem under constraints that specify the lower and upper boundaries for the feasible solution set, and the problem under a linear inequality constraint given by a matrix.

To illustrate the application of the results obtained, we provide new exact solutions to problems in project scheduling under the minimum makespan. The problems are to minimize the overall duration of a project that consists of a number of activities to be performed in parallel subject to temporal precedence constraints, including start-finish, finish-start, early start and due date constraints. The problems under consideration are known to have, in the usual setting, polynomial-time solutions in the form of computational algorithms (see, eg, overviews in Demeulemeester and Herroelen [32], T'kindt and Billaut [33] and Vanhoucke [34]). In contrast to these algorithmic solutions, the new ones are given in a compact vector form of direct general solutions, which are immediately ready for further analysis and straightforward computations.

The paper is organized as follows. In Section 2, we offer a brief overview of key definitions and notation that underlie the development of solutions to the optimization problem and their applications in the subsequent sections. Section 3 includes preliminary results, which provide a necessary prerequisite for the solution of the problems. In Section 4, we first formulate an unconstrained optimization problem and then solve the problem in two different ways. Furthermore, the solution is extended in Section 5 to problems with constraints added. Section 6 presents application of the results to project scheduling. Numerical examples are given in Section 7.

2 Definitions, Notation and General Remarks

We start with a brief overview of main definitions and notation of tropical mathematics to provide a formal framework for the description of solutions to optimization problems in the next sections. The overview is mainly based on the results in [35]. For additional details, insights and references, one can consult [3, 4, 36, 5, 37, 6, 7].

Let $\mathbb{X}$ be a set endowed with two associative and commutative operations, $\oplus$ (addition) and $\otimes$ (multiplication), and equipped with additive and multiplicative neutral elements, $\mathbb{0}$ (zero) and $\mathbb{1}$ (one). Addition is idempotent, which yields $x\oplus x=x$ for every $x\in\mathbb{X}$. Multiplication distributes over addition and is invertible to provide each nonzero $x\in\mathbb{X}$ with its inverse x^{-1} such that $x^{-1}\otimes x=\mathbb{1}$. The system $(\mathbb{X},\oplus,\otimes,\mathbb{0},\mathbb{1})$ is commonly referred to as the idempotent semifield, and retains certain properties of the usual fields.

We assume that the semifield is linearly ordered by an order that is consistent with the partial order induced by idempotent addition to define $x\leq y$ if and only if $x\oplus y=y$. From here on, we use the relation symbols as well as max and min operators in the sense of this definition. Specifically, it follows from the definition that $x\oplus y=\max(x,y)$. Moreover, in terms of the above order, both operations $\oplus$ and $\otimes$ are monotone in each argument.

As usual, the integer power specifies iterated multiplication, and is defined by $x^{p}=x^{p-1}\otimes x$, $x^{-p}=(x^{-1})^{p}$, $x^{0}=\mathbb{1}$ and $\mathbb{0}^{p}=\mathbb{0}$ for each nonzero $x\neq\mathbb{0}$ and integer $p\geq 1$. Moreover, the semifield is taken to be algebraically complete (radicable), which yields the existence of a solution of the equation $x^{m}=a$ for all $a\neq\mathbb{0}$ and integer m, and hence the existence of the root $a^{1/m}$. In the rest of the paper, we omit the multiplication sign $\otimes$ for the sake of simplicity.

Examples of the semifield include $\mathbb{R}_{\max,+}=(\mathbb{R}\cup\{-\infty\},\max,+,-\infty,0)$, $\mathbb{R}_{\min,+}=(\mathbb{R}\cup\{+\infty\},\min,+,+\infty,0)$, $\mathbb{R}_{\max,\times}=(\mathbb{R}_{+}\cup\{0\},\max,\times,0,1)$ and $\mathbb{R}_{\min,\times}=(\mathbb{R}_{+}\cup\{+\infty\},\min,\times,+\infty,1)$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$.

Let $\mathbb{X}^{m \times n}$ be the set of matrices with m rows and n columns over $\mathbb{X}$. A matrix with all entries equal to $\mathbb{0}$ is the zero matrix denoted $\mathbf{0}$. If a matrix has no zero rows (columns), then it is called row-regular (column-regular). A matrix is regular if it is both row- and column-regular.

Addition and multiplication of conforming matrices and scalar multiplication are defined by the standard rules with the scalar operations $\oplus$ and $\otimes$ used in place of the ordinary addition and multiplication. These operations are monotone with respect to the order relations defined component-wise.

For any matrix $\boldsymbol{A} \in \mathbb{X}^{m \times n}$, the transpose of $\boldsymbol{A}$ is the matrix $\boldsymbol{A}^T \in \mathbb{X}^{n \times m}$.

Consider the set $\mathbb{X}^{n \times n}$ of square matrices of order n. A matrix with $\mathbb{1}$ on the diagonal and $\mathbb{0}$ elsewhere is the identity matrix denoted $\boldsymbol{I}$.

The integer powers of any matrix $\boldsymbol{A} \in \mathbb{X}^{n \times n}$ are defined as $\boldsymbol{A}^0 = \boldsymbol{I}$ and $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A}$ for any integer $p \geq 1$.

Tropical analogues of the trace and the norm of a matrix $\boldsymbol{A} = (a_{ij})$ are respectively given by

$$\operatorname{tr} \boldsymbol{A} = \bigoplus_{i=1}^{n} a_{ii}, \qquad \|\boldsymbol{A}\| = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n} a_{ij}.$$

For any matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and a scalar $x \in \mathbb{X}$, we obviously have

$$\operatorname{tr}(\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{BA}), \qquad \operatorname{tr}(x\boldsymbol{A}) = x \operatorname{tr} \boldsymbol{A}, \qquad \|\boldsymbol{A} \oplus \boldsymbol{B}\| = \|\boldsymbol{A}\| \oplus \|\boldsymbol{B}\|.$$

A scalar $\lambda \in \mathbb{X}$ is an eigenvalue of the matrix $\boldsymbol{A}$ if there exists a nonzero vector $\boldsymbol{x} \in \mathbb{X}^n$ such that $\boldsymbol{Ax} = \lambda \boldsymbol{x}$. The maximum eigenvalue is called the spectral radius of $\boldsymbol{A}$ and calculated by the formula

$$\lambda = \bigoplus_{m=1}^{n} \operatorname{tr}^{1/m}(\boldsymbol{A}^m). \tag{1}$$

Furthermore, we define a function that assigns to each matrix $\boldsymbol{A}$ a scalar

$$\operatorname{Tr}(\boldsymbol{A}) = \bigoplus_{m=1}^{n} \operatorname{tr} \boldsymbol{A}^m. \tag{2}$$

Provided the condition $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ holds, the asterate of $\boldsymbol{A}$ (also known as the Kleene star) is the matrix given by

$$\boldsymbol{A}^* = \bigoplus_{m=0}^{n-1} \boldsymbol{A}^m. \tag{3}$$

Under the above condition, the asterate possesses a useful property that takes the form of the inequality (the Carrè inequality)

$$\boldsymbol{A}^m \leq \boldsymbol{A}^*, \qquad m \geq 0.$$

The set of column vectors of size n over $\mathbb{X}$ is denoted by $\mathbb{X}^n$. A vector with all entries equal to $\mathbb{0}$ is the zero vector denoted $\mathbf{0}$. A vector is regular if it has no zero elements. For any nonzero vector $\boldsymbol{x} = (x_i) \in \mathbb{X}^n$, the multiplicative conjugate transpose is the row vector $\boldsymbol{x}^- = (x_i^-)$ with elements $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$ and $x_i^- = \mathbb{0}$ otherwise.

The conjugate transposition possesses certain useful properties, which are not difficult to verify. Specifically, for any nonzero vector $\boldsymbol{x}$, we have the obvious equality $\boldsymbol{x}^-\boldsymbol{x} = \mathbb{1}$. For any regular vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of the same order, the component-wise inequality $\boldsymbol{x}\boldsymbol{y}^- \geq (\boldsymbol{x}^-\boldsymbol{y})^{-1}\boldsymbol{I}$ is valid as well.

Note that for any matrix $\boldsymbol{A}$ we have $\|\boldsymbol{A}\| = \mathbf{1}^T\boldsymbol{A}\mathbf{1}$, where $\mathbf{1} = (\mathbb{1}, \dots, \mathbb{1})^T$. If $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}^T$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, then $\|\boldsymbol{A}\| = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$.

3 Preliminary Results

We now present preliminary results concerning the solution of algebraic and optimization problems in the tropical mathematics setting to be used below.

First, we assume that, given a vector $\boldsymbol{a} \in \mathbb{X}^n$ and a scalar $d \in \mathbb{X}$, we need to obtain vectors $\boldsymbol{x} \in \mathbb{X}^n$ to satisfy the equation

$$\boldsymbol{a}^T\boldsymbol{x} = d. \tag{4}$$

A complete solution to the problem can be described as follows [35].

Lemma 1. *Let $\boldsymbol{a} = (a_i)$ be a regular vector and $d \neq \mathbb{0}$ be a scalar. Then, the solution of equation* (4) *forms a family of solutions each defined for one of $k = 1, \dots, n$ as a set of vectors $\boldsymbol{x} = (x_i)$ with components*

$$x_k = a_k^{-1}d, \qquad x_i \leq a_i^{-1}d, \quad i \neq k.$$

Given a matrix $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ and a vector $\boldsymbol{d} \in \mathbb{X}^m$, consider the problem to find all regular vectors $\boldsymbol{x} \in \mathbb{X}^n$ that satisfy the inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}. \tag{5}$$

The next statement offers a solution that is obtained as a consequence of the solution to the corresponding equation [35], and by independent proof [22].

Lemma 2. *Let $\boldsymbol{A}$ be a column-regular matrix and $\boldsymbol{d}$ a regular vector. Then, all regular solutions to inequality* (5) *are given by*

$$\boldsymbol{x} \leq (\boldsymbol{d}^-\boldsymbol{A})^-.$$

Furthermore, assume that, for a given matrix $\boldsymbol{A} \in \mathbb{X}^{n\times n}$, we need to find regular solutions $\boldsymbol{x} \in \mathbb{X}^n$ to the problem

$$\text{minimize} \quad \boldsymbol{x}^-\boldsymbol{A}\boldsymbol{x}. \tag{6}$$

A complete solution to (6) is provided by the following result [14, 17, 16].

Lemma 3. *Let* $\boldsymbol{A}$ *be a matrix with spectral radius* $\lambda > \mathbb{0}$*. Then, the minimum value in problem* (6) *is equal to* λ *and all regular solutions are given by*

$$\boldsymbol{x} = (\lambda^{-1}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{u} > \boldsymbol{0}.$$

We conclude with solutions obtained in [17] to constrained versions of problem (6). First, we offer a solution to the problem: given a matrix $\boldsymbol{A} \in \mathbb{X}^{n\times n}$ and vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{X}^{n}$, find all regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ that

$$\begin{aligned} &\text{minimize} && \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}. \end{aligned} \tag{7}$$

Theorem 4. *Let* $\boldsymbol{A}$ *be a matrix with spectral radius* $\lambda > \mathbb{0}$ *and* $\boldsymbol{h}$ *be a regular vector such that* $\boldsymbol{h}^{-}\boldsymbol{g} \leq \mathbb{1}$*. Then, the minimum in problem* (7) *is equal to*

$$\theta = \lambda \oplus \bigoplus_{m=1}^{n} (\boldsymbol{h}^{-}\boldsymbol{A}^{m}\boldsymbol{g})^{1/m},$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{h}^{-}(\theta^{-1}\boldsymbol{A})^{*})^{-}.$$

Finally, we present a solution to the following problem. Given matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{g}$, we look for all regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ to

$$\begin{aligned} &\text{minimize} && \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ &\text{subject to} && \boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}. \end{aligned} \tag{8}$$

Theorem 5. *Let* $\boldsymbol{A}$ *be a matrix with spectral radius* $\lambda > \mathbb{0}$ *and* $\boldsymbol{B}$ *be a matrix such that* $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$*. Then, the minimum value in problem* (8) *is equal to*

$$\theta = \bigoplus_{k=1}^{n} \bigoplus_{0\leq i_1+\cdots+i_k\leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k}),$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{g}.$$

4 Unconstrained Optimization Problem

In this section we examine an unconstrained multidimensional optimization problem formulated in the tropical mathematics setting as follows. Given vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{X}^{n}$, the problem is to find regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ such that

$$\text{minimize} \quad \boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p}. \tag{9}$$

Below, we offer two direct complete solutions to the problem under fairly general assumptions. We show that, although these solutions have different forms, both forms determine the same solution set.

4.1 Straightforward Solution

We start with a solution based on the derivation of a lower bound for the objective function and the solution of an equation to find all vectors that yield the bound.

Theorem 6. *Let $\boldsymbol{p}$ be a nonzero vector and $\boldsymbol{q}$ a regular vector. Then, the minimum value in problem* (9) *is equal to*

$$\Delta = \boldsymbol{q}^{-}\boldsymbol{p},$$

and all regular solutions of the problem are given by

$$\alpha\boldsymbol{p} \leq \boldsymbol{x} \leq \alpha\Delta\boldsymbol{q}, \qquad \alpha > \mathbb{0}. \tag{10}$$

Proof. First, we find the minimum of the objective function in the problem by using the properties of the conjugate transposition. With the inequality $\boldsymbol{x}\boldsymbol{x}^{-} \geq \boldsymbol{I}$, which is valid for any regular vector $\boldsymbol{x}$, we derive a lower bound

$$\boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p} \geq \boldsymbol{q}^{-}\boldsymbol{p} = \Delta.$$

Note that, since $\boldsymbol{p}$ is nonzero and $\boldsymbol{q}$ is regular, we have $\Delta > \mathbb{0}$.

It remains to verify that this bound is attained at certain $\boldsymbol{x}$, say $\boldsymbol{x} = \Delta\boldsymbol{q}$. Indeed, substitution into the objective function and the equality $\boldsymbol{q}^{-}\boldsymbol{q} = \mathbb{1}$ yield

$$\boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p} = \Delta(\boldsymbol{q}^{-}\boldsymbol{q})\Delta^{-1}(\boldsymbol{q}^{-}\boldsymbol{p}) = \boldsymbol{q}^{-}\boldsymbol{p} = \Delta.$$

To obtain all regular vectors $\boldsymbol{x}$ that solve the problem, we examine the equation

$$\boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p} = \Delta.$$

It is clear that, if $\boldsymbol{x}$ is a solution, then so is $\alpha\boldsymbol{x}$ for any $\alpha > \mathbb{0}$, and thus all solutions of the equation are scale-invariant.

Furthermore, we take an arbitrary $\alpha > \mathbb{0}$ and rewrite the equation in an equivalent form as the system of two equations

$$\boldsymbol{q}^{-}\boldsymbol{x} = \alpha\Delta, \qquad \boldsymbol{x}^{-}\boldsymbol{p} = \alpha^{-1}.$$

Taking into account that all solutions are scale-invariant, we put $\alpha = \mathbb{1}$ to further reduce the system as

$$\boldsymbol{q}^{-}\boldsymbol{x} = \Delta, \qquad \boldsymbol{x}^{-}\boldsymbol{p} = \mathbb{1}. \tag{11}$$

According to Lemma 1, the solutions to the first equation form a family of solutions $\boldsymbol{x} = (x_i)$, each defined for one of $k = 1, \ldots, n$ by the conditions

$$x_k = \Delta q_k, \qquad x_i \leq \Delta q_i, \quad i \neq k.$$

We now find those solutions from the family which satisfy the second equation at (11). Note that $\boldsymbol{x} = \Delta\boldsymbol{q}$ solves the problem and thus this equation.

Consider the minimum value of the problem and write

$$\Delta = \boldsymbol{q}^{-}\boldsymbol{p} = \bigoplus_{i=1}^{n} q_i^{-1} p_i = q_k^{-1} p_k,$$

where k is an index that yields the maximum of $q_i^{-1}p_i$ over all $i = 1, \dots, n$.

We denote by K the set of all such indices that produce Δ, and then verify that all solutions to the second equation must have $x_k = \Delta q_k$ for each $k \in K$.

Assuming the contrary, let k be an index in K to satisfy the condition

$$x_k < \Delta q_k = (\boldsymbol{q}^{-}\boldsymbol{p}) q_k = q_k^{-1} p_k q_k = p_k.$$

Then, for the left hand side of the second equation at (11), we have

$$\boldsymbol{x}^{-}\boldsymbol{p} = x_1^{-1} p_1 \oplus \cdots \oplus x_n^{-1} p_n \geq x_k^{-1} p_k > p_k^{-1} p_k = \mathbb{1},$$

and thus the equation is not valid anymore and becomes a strict inequality.

Furthermore, for all $i \notin K$ if any, we can take $x_i \leq \Delta q_i$ but not too small to keep the condition $\boldsymbol{x}^{-}\boldsymbol{p} \leq \mathbb{1}$. It follows from this condition that

$$\mathbb{1} \geq \boldsymbol{x}^{-}\boldsymbol{p} = x_1^{-1} p_1 \oplus \cdots \oplus x_n^{-1} p_n \geq x_i^{-1} p_i.$$

To satisfy the condition when $p_i \neq \mathbb{0}$, we have to take $x_i \geq p_i$. With $p_i = \mathbb{0}$, the term $\boldsymbol{x}^{-}\boldsymbol{p}$ does not depend on x_i, and we can write $x_i \geq \mathbb{0} = p_i$.

We can summarize the above consideration as follows. All solutions to the problem are vectors $\boldsymbol{x} = (x_i)$ that satisfy the conditions

$$\begin{aligned} x_i &= \Delta q_i, & i &\in K \\ p_i \leq x_i &\leq \Delta q_i, & i &\notin K. \end{aligned}$$

Since we have $x_i = \Delta q_i = q_i^{-1} p_i q_i = p_i$ for all $i \in K$, the solution can be written as one double inequality $p_i \leq x_i \leq \Delta q_i$ for all $i = 1, \dots, n$, or, in the vector form, as the inequality

$$\boldsymbol{p} \leq \boldsymbol{x} \leq \Delta\boldsymbol{q}.$$

Considering that each solution is scale-invariant, we arrive at (10). □
□

4.2 Solution Using Spectral Radius

To provide another solution to problem (9), we first put the objective function in the equivalent form

$$\boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p} = \boldsymbol{x}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{x},$$

and then rewrite the problem as

$$\text{minimize} \quad \boldsymbol{x}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{x}. \tag{12}$$

The problem now becomes a special case of problem (6) with $\boldsymbol{A} = \boldsymbol{p}\boldsymbol{q}^{-}$, and can therefore be solved using Lemma 3.

Theorem 7. *Let $\boldsymbol{p}$ be a nonzero vector and $\boldsymbol{q}$ a regular vector. Then, the minimum value in problem* (9) *is equal to*

$$\Delta = \boldsymbol{q}^{-}\boldsymbol{p},$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u}, \qquad \boldsymbol{u} > \boldsymbol{0}. \tag{13}$$

Proof. We examine the problem in the form of (12). To apply Lemma 3, we take the matrix $\boldsymbol{A} = \boldsymbol{p}\boldsymbol{q}^{-}$ and calculate

$$\boldsymbol{A}^{m} = (\boldsymbol{q}^{-}\boldsymbol{p})^{m-1}\boldsymbol{p}\boldsymbol{q}^{-}, \qquad \operatorname{tr}\boldsymbol{A}^{m} = (\boldsymbol{q}^{-}\boldsymbol{p})^{m}, \qquad m = 1, \ldots, n.$$

Let Δ be the spectral radius of the matrix $\boldsymbol{A}$. Using formula (1), we obtain $\Delta = \boldsymbol{q}^{-}\boldsymbol{p}$, which, due to Lemma 3, presents the minimum value in the problem.

To describe the solution set, we calculate $(\Delta^{-1}\boldsymbol{A})^{m} = \Delta^{-m}\boldsymbol{A}^{m} = \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}$, and then employ (3) to derive the matrix

$$(\Delta^{-1}\boldsymbol{A})^{*} = \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}.$$

Finally, the application of Lemma 3 gives solution (13). □ □

Note that, although the solution sets offered by Theorems 6 and 7 look different, it is not difficult to see that they are the same.

Let us take any vector $\boldsymbol{u}$ and verify that $\boldsymbol{x}$ which is given by (13) satisfies inequality (10) for some α. Indeed, if we put $\alpha = \Delta^{-1}(\boldsymbol{q}^{-}\boldsymbol{u})$, then we have

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u} \geq \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{u} = \Delta^{-1}(\boldsymbol{q}^{-}\boldsymbol{u})\boldsymbol{p} = \alpha\boldsymbol{p},$$

which yields the left inequality at (10).

Since $\boldsymbol{q}\boldsymbol{p}^{-} \geq (\boldsymbol{q}^{-}\boldsymbol{p})^{-1}\boldsymbol{I} = \Delta^{-1}\boldsymbol{I}$, we obtain the right inequality as follows

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u} \leq (\boldsymbol{I} \oplus \boldsymbol{q}\boldsymbol{p}^{-}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u} = (\boldsymbol{I} \oplus \boldsymbol{q}\boldsymbol{q}^{-})\boldsymbol{u} = (\boldsymbol{q}^{-}\boldsymbol{u})\boldsymbol{q} = \alpha\Delta\boldsymbol{q}.$$

Now assume that the vector $\boldsymbol{x}$ satisfies (10) and then show that it can be written as (13). From the right inequality at (10), it follows that

$$\Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{x} \leq \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}(\alpha\Delta\boldsymbol{q}) = \alpha\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{q} = \alpha\boldsymbol{p}.$$

Considering the left inequality, we have $\boldsymbol{x} \geq \alpha\boldsymbol{p} \geq \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{x}$. Finally, by setting $\boldsymbol{u} = \boldsymbol{x}$, we can write

$$\boldsymbol{x} = \boldsymbol{x} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{x} = \boldsymbol{u} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{u} = (\boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u},$$

which gives a representation of the vector $\boldsymbol{x}$ in the form of (13).

5 Constrained Optimization Problems

We now add lower and upper boundary constraints on the feasible solutions. Given vectors $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{g}, \boldsymbol{h} \in \mathbb{X}^{n}$, consider the problem to find all regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ that

$$\begin{aligned} &\text{minimize} && \boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p}, \\ &\text{subject to} && \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}. \end{aligned} \tag{14}$$

The next theorem provides a complete direct solution to the problem.

Theorem 8. *Let $\boldsymbol{p}$ be a nonzero vector, $\boldsymbol{q}$ a regular vector, and $\boldsymbol{h}$ a regular vector such that $\boldsymbol{h}^{-}\boldsymbol{g} \leq \mathbb{1}$. Then, the minimum in problem* (14) *is equal to*

$$\theta = \boldsymbol{q}^{-}(\boldsymbol{I} \oplus \boldsymbol{g}\boldsymbol{h}^{-})\boldsymbol{p},$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{p}\boldsymbol{q}^{-})\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{h}^{-}(\boldsymbol{I} \oplus \theta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}))^{-}. \tag{15}$$

Proof. As in the previous proof, we rewrite the objective function in the form $\boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p} = \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A} = \boldsymbol{p}\boldsymbol{q}^{-}$, and thus reduce the problem to (7).

Furthermore, we obtain the spectral radius of $\boldsymbol{A}$ in the form $\Delta = \boldsymbol{q}^{-}\boldsymbol{p}$, write $\boldsymbol{A}^{m} = \Delta^{m-1}\boldsymbol{p}\boldsymbol{q}^{-}$, and calculate $\boldsymbol{h}^{-}\boldsymbol{A}^{m}\boldsymbol{g} = \Delta^{m-1}\boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g}$.

Then, we apply Theorem 4 to write the minimum value in the form

$$\theta = \Delta \oplus \bigoplus_{m=1}^{n}(\Delta^{m-1}\boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g})^{1/m} = \Delta\left(\mathbb{1} \oplus \bigoplus_{m=1}^{n}(\Delta^{-1}\boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g})^{1/m}\right).$$

To simplify the last expression, consider two cases. First, suppose that $\Delta \geq \boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g}$. It follows immediately from this condition that the inequality $(\Delta^{-1}\boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g})^{1/m} \leq \mathbb{1}$ holds for every m, and therefore, $\theta = \Delta$.

Otherwise, if the opposite inequality $\Delta < \boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g}$ is valid, we see that $\Delta^{-1}\boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g} \geq (\Delta^{-1}\boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g})^{1/m} > \mathbb{1}$, which gives $\theta = \boldsymbol{h}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{g}$.

By combining both results and considering $\Delta = \boldsymbol{q}^{-}\boldsymbol{p}$, we obtain the minimum value

$$\theta = \Delta \oplus \boldsymbol{q}^{-}\boldsymbol{g}\boldsymbol{h}^{-}\boldsymbol{p} = \boldsymbol{q}^{-}(\boldsymbol{I} \oplus \boldsymbol{g}\boldsymbol{h}^{-})\boldsymbol{p}.$$

We now define the solution set according to Theorem 4. We first calculate $(\theta^{-1}\boldsymbol{A})^{m} = \theta^{-m}\Delta^{m-1}\boldsymbol{p}\boldsymbol{q}^{-}$, and then apply (3) to write

$$(\theta^{-1}\boldsymbol{A})^{*} = \boldsymbol{I} \oplus \theta^{-1}\left(\bigoplus_{m=1}^{n-1}(\theta^{-1}\Delta)^{m-1}\right)\boldsymbol{p}\boldsymbol{q}^{-}.$$

Since $\theta \geq \Delta$, the inequality $(\theta^{-1}\Delta)^{m-1} \leq \mathbb{1}$ is valid for all m and becomes an equality if $m = 1$. Therefore, the term in the parenthesis on the right-hand side is equal to $\mathbb{1}$, and hence

$$(\theta^{-1}\boldsymbol{A})^{*} = \boldsymbol{I} \oplus \theta^{-1}\boldsymbol{p}\boldsymbol{q}^{-}.$$

Substitution into the solution provided by Theorem 4 leads to (15). $\square$

$\square$

Suppose that we replace the simple boundary constraints in the above problem by a linear inequality constraint given by a matrix $\boldsymbol{B} \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{g} \in \mathbb{X}^{n}$. Consider the problem to find regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ that

$$\begin{aligned} &\text{minimize} && \boldsymbol{q}^{-}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{p}, \\ &\text{subject to} && \boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}. \end{aligned} \tag{16}$$

A solution to the problem can be obtained as follows.

Theorem 9. *Let* $\boldsymbol{p}$ *be a nonzero vector,* $\boldsymbol{q}$ *a regular vector, and* $\boldsymbol{B}$ *be a matrix such that* $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$*. Then, the minimum value in problem* (16) *is equal to*

$$\theta = \boldsymbol{q}^{-}\boldsymbol{B}^{*}\boldsymbol{p},$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{p}\boldsymbol{q}^{-} \oplus \boldsymbol{B})^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{g}. \tag{17}$$

Proof. To solve the problem, we again represent the objective function as $\boldsymbol{x}^{-}\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{x}$ and then apply Theorem 5 with $\boldsymbol{A} = \boldsymbol{p}\boldsymbol{q}^{-}$. First, we examine the minimum value provided by Theorem 5. This minimum now takes the form

$$\theta = \bigoplus_{k=1}^{n}\bigoplus_{0\leq i_1+\cdots+i_k\leq n-k}\mathrm{tr}^{1/k}(\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{B}^{i_1}\cdots\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{B}^{i_k}),$$

where the properties of the trace allow us to write

$$\mathrm{tr}(\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{B}^{i_1}\cdots\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{B}^{i_k}) = \boldsymbol{q}^{-}\boldsymbol{B}^{i_1}\boldsymbol{p}\cdots\boldsymbol{q}^{-}\boldsymbol{B}^{i_k}\boldsymbol{p}.$$

By truncating the sum at $k = 1$, we bound from below the value of θ as

$$\theta \geq \bigoplus_{i=0}^{n-1} \mathrm{tr}(\boldsymbol{p}\boldsymbol{q}^{-}\boldsymbol{B}^{i}) = \bigoplus_{i=0}^{n-1} \boldsymbol{q}^{-}\boldsymbol{B}^{i}\boldsymbol{p} = \boldsymbol{q}^{-}\boldsymbol{B}^{*}\boldsymbol{p}.$$

On the other hand, since $\boldsymbol{B}^{m} \leq \boldsymbol{B}^{*}$ for any integer $m \geq 0$, we obtain

$$\boldsymbol{q}^{-}\boldsymbol{B}^{i_1}\boldsymbol{p}\cdots\boldsymbol{q}^{-}\boldsymbol{B}^{i_k}\boldsymbol{p} \leq \boldsymbol{q}^{-}\boldsymbol{B}^{*}\boldsymbol{p}\cdots\boldsymbol{q}^{-}\boldsymbol{B}^{*}\boldsymbol{p} = (\boldsymbol{q}^{-}\boldsymbol{B}^{*}\boldsymbol{p})^{k}.$$

Consequently, we have the inequality

$$\theta = \bigoplus_{k=1}^{n} \bigoplus_{0\leq i_1+\cdots+i_k\leq n-k} (\boldsymbol{q}^{-}\boldsymbol{B}^{i_1}\boldsymbol{p}\cdots\boldsymbol{q}^{-}\boldsymbol{B}^{i_k}\boldsymbol{p})^{1/k} \leq \boldsymbol{q}^{-}\boldsymbol{B}^{*}\boldsymbol{p},$$

which together with the opposite inequality yields the desired result.

Finally, we use Theorem 5 to write the solution in the form of (17), and thus complete the proof. $\square$ $\square$

6 Application to Project Scheduling

In this section, we show how the results obtained can be applied to solve real-world problems that are drawn from project scheduling under the minimum makespan criterion (see, e.g., [32, 33, 34] for further details).

Consider a project that involves n activities operating under start-finish, finish-start, early start, and late finish (due date) temporal constraints. The start-finish constraints define the lower limit for the allowed time lag between the initiation of one activity and the completion of another. The activities are assumed to be completed as early as possible within the start-finish constraints. The finish-start constraints determine the minimum time lag between the completion of one activity and the initiation of another. The early start and late finish constraints specify, respectively, the earliest possible initiation time and latest possible completion time for every activity.

Below, we first examine a problem that has only start-finish constraints, and then extend the result obtained to problems with additional constraints.

For each activity $i = 1, \ldots, n$, we denote the initiation time by x_i and the completion time by y_i. Let c_{ij} be the minimum time lag between the initiation of activity $j = 1, \ldots, n$ and the completion of i. If c_{ij} is not given for some j, we put $c_{ij} = -\infty$. The completion time of activity i must satisfy the start-finish relations written in terms of the usual operations as

$$x_j + c_{ij} \leq y_i, \qquad j = 1, \ldots, n,$$

where at least one inequality holds as equality. Combining the relations gives

$$y_i = \max_{1\leq j\leq n}(x_j + c_{ij}).$$

The makespan is defined as the duration between the earliest initiation time and the latest completion time in the project, and takes the form

$$\max_{1\leq i\leq n} y_i - \min_{1\leq i\leq n} x_i = \max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-x_i).$$

After substitution of y_i, the problem of scheduling under the start-finish constraints and the minimum makespan criterion can be formulated in the conventional form as follows: given c_{ij} for $i,j=1,\ldots,n$, find $x_1,\ldots,x_n$ that

$$\text{minimize} \quad \max_{1\leq i\leq n}\max_{1\leq j\leq n}(x_j + c_{ij}) + \max_{1\leq i\leq n}(-x_i).$$

In terms of the operations in the semifield $\mathbb{R}_{\max,+}$, the problem becomes

$$\text{minimize} \quad \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n} c_{ij}x_j \bigoplus_{k=1}^{n} x_k^{-1}.$$

Furthermore, we introduce the matrix $\boldsymbol{C}=(c_{ij})$ and the vector $\boldsymbol{x}=(x_i)$. Using this notation and considering that $\mathbf{1}=(0,\ldots,0)^T$ in $\mathbb{R}_{\max,+}$, we put the problem in the vector form

$$\text{minimize} \quad \mathbf{1}^T\boldsymbol{C}\boldsymbol{x}\boldsymbol{x}^{-}\mathbf{1}. \tag{18}$$

It is easy to see that the last problem is a special case of problem (9), where we take $\boldsymbol{p}=\mathbf{1}$ and $\boldsymbol{q}^{-}=\mathbf{1}^T\boldsymbol{C}$.

As a consequence of Theorems 6 and 7, we obtain the following result.

Theorem 10. *Let $\boldsymbol{C}$ be a row-regular matrix. Then, the minimum value in problem* (18) *is equal to*

$$\Delta = \mathbf{1}^T\boldsymbol{C}\mathbf{1} = \|\boldsymbol{C}\|,$$

and all regular solutions of the problem are given by

$$\alpha\mathbf{1} \leq \boldsymbol{x} \leq \alpha\Delta(\mathbf{1}^T\boldsymbol{C})^{-}, \qquad \alpha\in\mathbb{R}, \tag{19}$$

or, equivalently, by

$$\boldsymbol{x} = (\boldsymbol{I}\oplus\Delta^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{C})\boldsymbol{u}, \qquad \boldsymbol{u}\in\mathbb{R}^n. \tag{20}$$

We now consider the problem with the early start and late finish constraints added. For each activity $i=1,\ldots,n$, let g_i be the earliest possible time to start, and f_i the latest possible time to finish (the due date) for activity i. The early start and late finish constraints imply the inequalities

$$g_i \leq x_i, \qquad y_i = \max_{1\leq j\leq n}(x_j + c_{ij}) \leq f_i,$$

which, combined with the objective function, yields a problem that is written in the usual form as

$$\begin{aligned}
&\text{minimize} && \max_{1\leq i\leq n}\max_{1\leq j\leq n}(x_j + c_{ij}) + \max_{1\leq i\leq n}(-x_i),\\
&\text{subject to} && \max_{1\leq j\leq n}(x_j + c_{ij}) \leq f_i,\\
& && g_i \leq x_i, \qquad i = 1,\dots,n.
\end{aligned}$$

As before, we rewrite the problem in terms of the semifield $\mathbb{R}_{\max,+}$ and use the vector notation $\boldsymbol{g} = (g_i)$ and $\boldsymbol{f} = (f_i)$ to extend the unconstrained problem at (18) to the problem

$$\begin{aligned}
&\text{minimize} && \mathbf{1}^T\boldsymbol{C}\boldsymbol{x}\boldsymbol{x}^{-}\mathbf{1},\\
&\text{subject to} && \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \quad \boldsymbol{g} \leq \boldsymbol{x}.
\end{aligned} \tag{21}$$

It follows from Lemma 2 that the first inequality constraint can be solved in the form $\boldsymbol{x} \leq (\boldsymbol{f}^{-}\boldsymbol{C})^{-}$. Then, the problem reduces to (14) with $\boldsymbol{p} = \mathbf{1}$, $\boldsymbol{q}^{-} = \mathbf{1}^T\boldsymbol{C}$ and $\boldsymbol{h} = (\boldsymbol{f}^{-}\boldsymbol{C})^{-}$. By applying Theorem 8 and using properties of the norm to represent the minimum value, we come to the following result.

Theorem 11. *Let $\boldsymbol{C}$ be a regular matrix, and $\boldsymbol{f}$ a regular vector such that $\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{g} \leq \mathbb{1}$. Then, the minimum value in problem* (21) *is equal to*

$$\theta = \mathbf{1}^T\boldsymbol{C}(\boldsymbol{I} \oplus \boldsymbol{g}\boldsymbol{f}^{-}\boldsymbol{C})\mathbf{1} = \|\boldsymbol{C}\| \oplus \|\boldsymbol{C}\boldsymbol{g}\|\|\boldsymbol{f}^{-}\boldsymbol{C})\|,$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\boldsymbol{I} \oplus \theta^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{C})\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^{-}\boldsymbol{C}(\boldsymbol{I} \oplus \theta^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{C}))^{-}. \tag{22}$$

Finally, suppose that, in the project under consideration, the late finish constraints are replaced by finish-start constraints. For each activity $i = 1,\dots,n$, we denote by d_{ij} the minimum allowed time lag between the completion of activity j and initiation of i. We take $d_{ij} = -\infty$ if the time lag is not specified.

The finish-start constraints are given in terms of the usual operations by the inequalities

$$y_j + d_{ij} \leq x_i, \qquad j = 1,\dots,n.$$

Furthermore, we substitute y_j from the start-finish constraints and combine the inequalities into one to write

$$\max_{1\leq j\leq n}\big(\max_{1\leq k\leq n}(x_k + c_{jk}) + d_{ij}\big) \leq x_i.$$

The scheduling problem under finish-start and early start constraints can now be formulated as

$$\begin{aligned}
&\text{minimize} && \max_{1\leq i\leq n}\max_{1\leq j\leq n}(x_j + c_{ij}) + \max_{1\leq i\leq n}(-x_i),\\
&\text{subject to} && \max_{1\leq j\leq n}\big(\max_{1\leq k\leq n}(x_k + c_{jk}) + d_{ij}\big) \leq x_i,\\
& && g_i \leq x_i, \qquad i = 1,\dots,n.
\end{aligned}$$

We introduce the matrix $\boldsymbol{D} = (d_{ij})$, and then represent the problem in terms of the semifield $\mathbb{R}_{\max,+}$ to write

$$\begin{aligned} &\text{minimize} && \boldsymbol{1}^T \boldsymbol{C}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{1}, \\ &\text{subject to} && \boldsymbol{D}\boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}. \end{aligned} \tag{23}$$

Application of Theorem 9 with $\boldsymbol{p} = \boldsymbol{1}$, $\boldsymbol{q}^{-} = \boldsymbol{1}^T \boldsymbol{C}$ and $\boldsymbol{B} = \boldsymbol{D}\boldsymbol{C}$ yields the next result.

Theorem 12. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be matrices such that $\mathrm{Tr}(\boldsymbol{D}\boldsymbol{C}) \leq \mathbb{1}$. Then, the minimum value in problem* (23) *is equal to*

$$\theta = \boldsymbol{1}^T \boldsymbol{C}(\boldsymbol{D}\boldsymbol{C})^* \boldsymbol{1} = \|\boldsymbol{C}(\boldsymbol{D}\boldsymbol{C})^*\|,$$

and all regular solutions of the problem are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{1}\boldsymbol{1}^T \boldsymbol{C} \oplus \boldsymbol{D}\boldsymbol{C})^* \boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{g}. \tag{24}$$

Note that the solutions obtained are not unique. In the context of project scheduling, this leaves the freedom to account for additional temporal constraints.

7 Numerical Examples

In this section, we show how the results obtained can be applied to solve simple example problems. The main aim of this section is to provide a transparent and detailed illustration of the computational technique in terms of the semifield $\mathbb{R}_{\max,+}$, which is used to obtain the solution. To this end, we examine relatively artificial low-dimensional problems which, however, clearly demonstrate the ability the approach to solve real-world problems of high dimensions.

Consider an example project that involves $n = 3$ activities and operates under the constraints given by

$$\boldsymbol{C} = \begin{pmatrix} 4 & 0 & \mathbb{0} \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 0 & \mathbb{0} & \mathbb{0} \\ 2 & 1 & \mathbb{0} \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 8 \\ 7 \\ 4 \end{pmatrix},$$

where the symbol $\mathbb{0} = -\infty$ is employed to save writing.

We start with problem (18), where only the start-finish constraints are defined. Application of Theorem 10 gives the minimum makespan

$$\Delta = \|\boldsymbol{C}\| = 4.$$

To represent the solution to the problem in the form of (19), we take the vector $\boldsymbol{1} = (0, 0, 0)^T$ and calculate

$$\boldsymbol{1}^T \boldsymbol{C} = \begin{pmatrix} 4 & 3 & 2 \end{pmatrix}, \qquad (\boldsymbol{1}^T \boldsymbol{C})^{-} = \begin{pmatrix} -4 \\ -3 \\ -2 \end{pmatrix}, \qquad \Delta(\boldsymbol{1}^T \boldsymbol{C})^{-} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Then, the solution vector $\boldsymbol{x} = (x_1, x_2, x_3)^T$ provided by the theorem is given by

$$\alpha \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \boldsymbol{x} \leq \alpha \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \alpha \in \mathbb{R},$$

and can be written in terms of the usual operations as

$$x_1 = \alpha \qquad \alpha \leq x_2 \leq \alpha + 1, \qquad \alpha \leq x_3 \leq \alpha + 2, \qquad \alpha \in \mathbb{R}.$$

Furthermore, we derive the solution represented as (20). After calculating the matrices

$$\boldsymbol{1}\boldsymbol{1}^T \boldsymbol{C} = \begin{pmatrix} 4 & 3 & 2 \\ 4 & 3 & 2 \\ 4 & 3 & 2 \end{pmatrix}, \qquad \boldsymbol{I} \oplus \Delta^{-1}\boldsymbol{1}\boldsymbol{1}^T \boldsymbol{C} = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix},$$

we obtain the solution

$$\boldsymbol{x} = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^3.$$

In the ordinary notation, assuming u_1, u_2, u_3 to be real numbers, we have

$$\begin{aligned} x_1 &= \max(u_1, u_2 - 1, u_3 - 2), \\ x_2 &= \max(u_1, u_2, u_3 - 2), \\ x_3 &= \max(u_1, u_2 - 1, u_3). \end{aligned}$$

Suppose that, in addition to the start-finish constraints, both early start and late finish constraints are also imposed. To check whether Theorem 11 can be applied, we first obtain

$$\boldsymbol{f}^- \boldsymbol{C} = \begin{pmatrix} -4 & -2 & -2 \end{pmatrix}, \qquad \boldsymbol{C}\boldsymbol{g} = \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}, \qquad \boldsymbol{f}^- \boldsymbol{C}\boldsymbol{g} = 0.$$

Since $\boldsymbol{f}^- \boldsymbol{C}\boldsymbol{g} = 0 = \mathbb{1}$, we see that the conditions of Theorem 11 are fulfilled. Considering that

$$\|\boldsymbol{C}\| = 4, \qquad \|\boldsymbol{f}^- \boldsymbol{C}\| = -2, \qquad \|\boldsymbol{C}\boldsymbol{g}\| = 7,$$

we evaluate the minimum makespan

$$\theta = \|\boldsymbol{C}\| \oplus \|\boldsymbol{C}\boldsymbol{g}\| \|\boldsymbol{f}^- \boldsymbol{C}\| = 5.$$

Furthermore, we successively calculate the matrices

$$\theta^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{C} = \begin{pmatrix} -1 & -2 & -3 \\ -1 & -2 & -3 \\ -1 & -2 & -3 \end{pmatrix}, \qquad \boldsymbol{I} \oplus \theta^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{C} = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 0 & -3 \\ -1 & -2 & 0 \end{pmatrix},$$

and the vector

$$\boldsymbol{f}^{-}\boldsymbol{C}(\boldsymbol{I} \oplus \theta^{-1}\mathbf{1}\mathbf{1}^T\boldsymbol{C}) = \begin{pmatrix} -3 & -2 & -2 \end{pmatrix}.$$

The solution given by Theorem 11 at (22) takes the form

$$\boldsymbol{x} = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 0 & -3 \\ -1 & -2 & 0 \end{pmatrix} \boldsymbol{u}, \qquad \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \leq \boldsymbol{u} \leq \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

Scalar representation in the ordinary notation gives the equalities

$$\begin{aligned} x_1 &= \max(u_1, u_2 - 2, u_3 - 3), \\ x_2 &= \max(u_1 - 1, u_2, u_3 - 3), \\ x_3 &= \max(u_1 - 1, u_2 - 2, u_3), \end{aligned}$$

where the numbers u_1, u_2 and u_3 satisfy the conditions

$$u_1 = 3, \qquad u_2 = 2, \qquad 1 \leq u_3 \leq 2.$$

By combining these conditions with the above equalities, we obtain the single solution

$$x_1 = 3, \qquad x_2 = 2, \qquad x_3 = 2.$$

Finally, we assume that the start-finish constraints given by the matrix $\boldsymbol{D}$ are defined in the above project instead of the late finish constraints. To solve the problem, we use the result provided by Theorem 12.

First, we calculate the matrices

$$\boldsymbol{D}\boldsymbol{C} = \begin{pmatrix} \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 4 & 0 & \mathbb{0} \\ 6 & 4 & 0 \end{pmatrix}, \qquad (\boldsymbol{D}\boldsymbol{C})^2 = \begin{pmatrix} \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 4 & 0 & \mathbb{0} \\ 8 & 4 & 0 \end{pmatrix}, \qquad (\boldsymbol{D}\boldsymbol{C})^3 = \begin{pmatrix} \mathbb{0} & \mathbb{0} & \mathbb{0} \\ 4 & 0 & \mathbb{0} \\ 8 & 4 & 0 \end{pmatrix}.$$

Application of formula (2) yields

$$\mathrm{Tr}(\boldsymbol{D}\boldsymbol{C}) = 0 = \mathbb{1},$$

and hence the conditions of Theorem 12 are satisfied.

Furthermore, we have

$$(\boldsymbol{D}\boldsymbol{C})^{*} = \begin{pmatrix} 0 & \mathbb{0} & \mathbb{0} \\ 4 & 0 & \mathbb{0} \\ 8 & 4 & 0 \end{pmatrix}, \qquad \boldsymbol{C}(\boldsymbol{D}\boldsymbol{C})^{*} = \begin{pmatrix} 4 & 0 & \mathbb{0} \\ 7 & 3 & -1 \\ 10 & 6 & 2 \end{pmatrix},$$

and then find the minimum makespan

$$\theta = \|\boldsymbol{C}(\boldsymbol{D}\boldsymbol{C})^{*}\| = 10.$$

It remains to represent the solution according to Theorem 12. We successfully calculate the matrices

$$\theta^{-1}\mathbf{1}\mathbf{1}^{T}\boldsymbol{C} = \begin{pmatrix} -6 & -7 & -8 \\ -6 & -7 & -8 \\ -6 & -7 & -8 \end{pmatrix}, \qquad \theta^{-1}\mathbf{1}\mathbf{1}^{T}\boldsymbol{C} \oplus \boldsymbol{D}\boldsymbol{C} = \begin{pmatrix} -6 & -7 & -8 \\ 0 & -7 & -8 \\ 2 & 1 & -8 \end{pmatrix},$$
$$(\theta^{-1}\mathbf{1}\mathbf{1}^{T}\boldsymbol{C} \oplus \boldsymbol{D}\boldsymbol{C})^{2} = \begin{pmatrix} -6 & -7 & -14 \\ -6 & -7 & -8 \\ 1 & -5 & -6 \end{pmatrix}, \qquad (\theta^{-1}\mathbf{1}\mathbf{1}^{T}\boldsymbol{C} \oplus \boldsymbol{D}\boldsymbol{C})^{*} = \begin{pmatrix} 0 & -7 & -8 \\ 0 & 0 & -8 \\ 2 & 1 & 0 \end{pmatrix}.$$

The solution defined by (24) becomes

$$\boldsymbol{x} = \begin{pmatrix} 0 & -7 & -8 \\ 0 & 0 & -8 \\ 2 & 1 & 0 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} \geq \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

By rewriting the solution in the usual notation, we have

$$\begin{aligned} x_1 &= \max(u_1, u_2 - 7, u_3 - 8), \\ x_2 &= \max(u_1, u_2, u_3 - 8), \\ x_3 &= \max(u_1 + 2, u_2 + 1, u_3), \end{aligned}$$

under the conditions that

$$u_1 \geq 3, \qquad u_2 \geq 2, \qquad u_3 \geq 2.$$

Specifically, with $u_1 = 3$, $u_2 = 2$ and $u_3 = 2$, we obtain the earliest optimal initiation times given by $x_1 = 3$, $x_2 = 3$ and $x_3 = 5$.

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