MOMENTUM EQUATION

Fluid Mechanics

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Momentum Equation

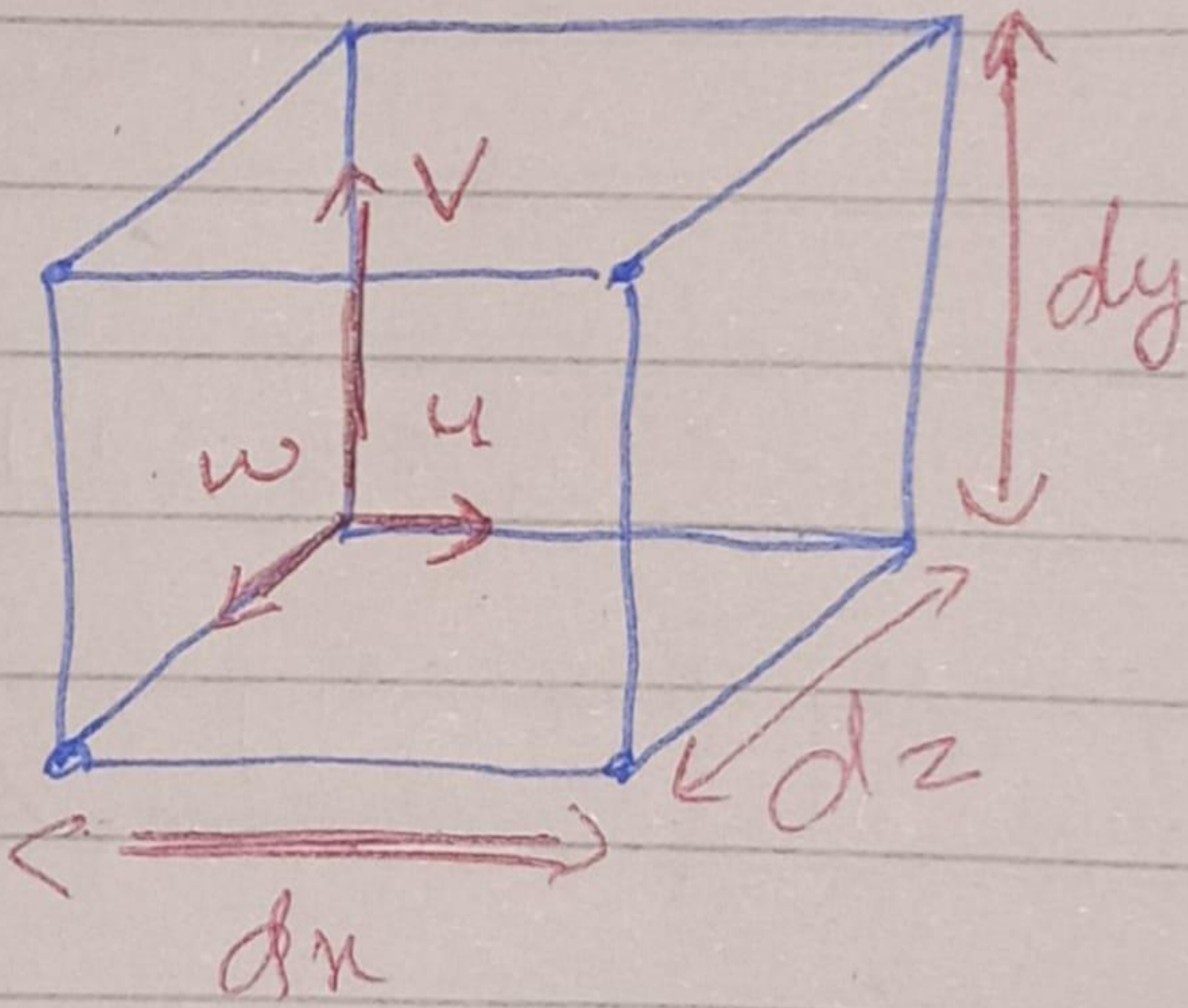
Navier-Stokes Equation

$$F = ma$$

for x-direction

$$F = ma_x$$

$$\because \quad a = \frac{\Delta v}{\Delta t} \quad = \quad F_x = m\frac{\Delta u}{\Delta t}$$


v
u
w
dy
dz
dx

$$F_x = m \frac{\Delta u}{\Delta t}$$

as $\Delta t \longrightarrow 0$

$$F_x = m \frac{du}{dt}$$ (u - velocity in x-direction)

$$u = u(x, y, z, t)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot u + \frac{\partial u}{\partial y} \cdot v + \frac{\partial u}{\partial z} \cdot w + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = u \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} + w \cdot \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Putting this in $F_x = m\frac{du}{dt}$

$$F_x = m\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w.\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$F_x = \rho\, dv\left[u.\frac{\partial u}{\partial x} + v.\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$F_x = \underbrace{\rho.dx\,dy\,dz}_{\text{Mass Term}}\left[\underbrace{u.\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}}_{\text{acceleration Term}}\right]$$

For the Force;

Force;

1. Body Forces ⟶
2. Surface forces

Body force = weight

Surface force = (i) Normal Force
(ii) Shear Force

$$\text{Weight} = mg$$

$$W = m g_x$$

$$W = \rho dV g_x$$

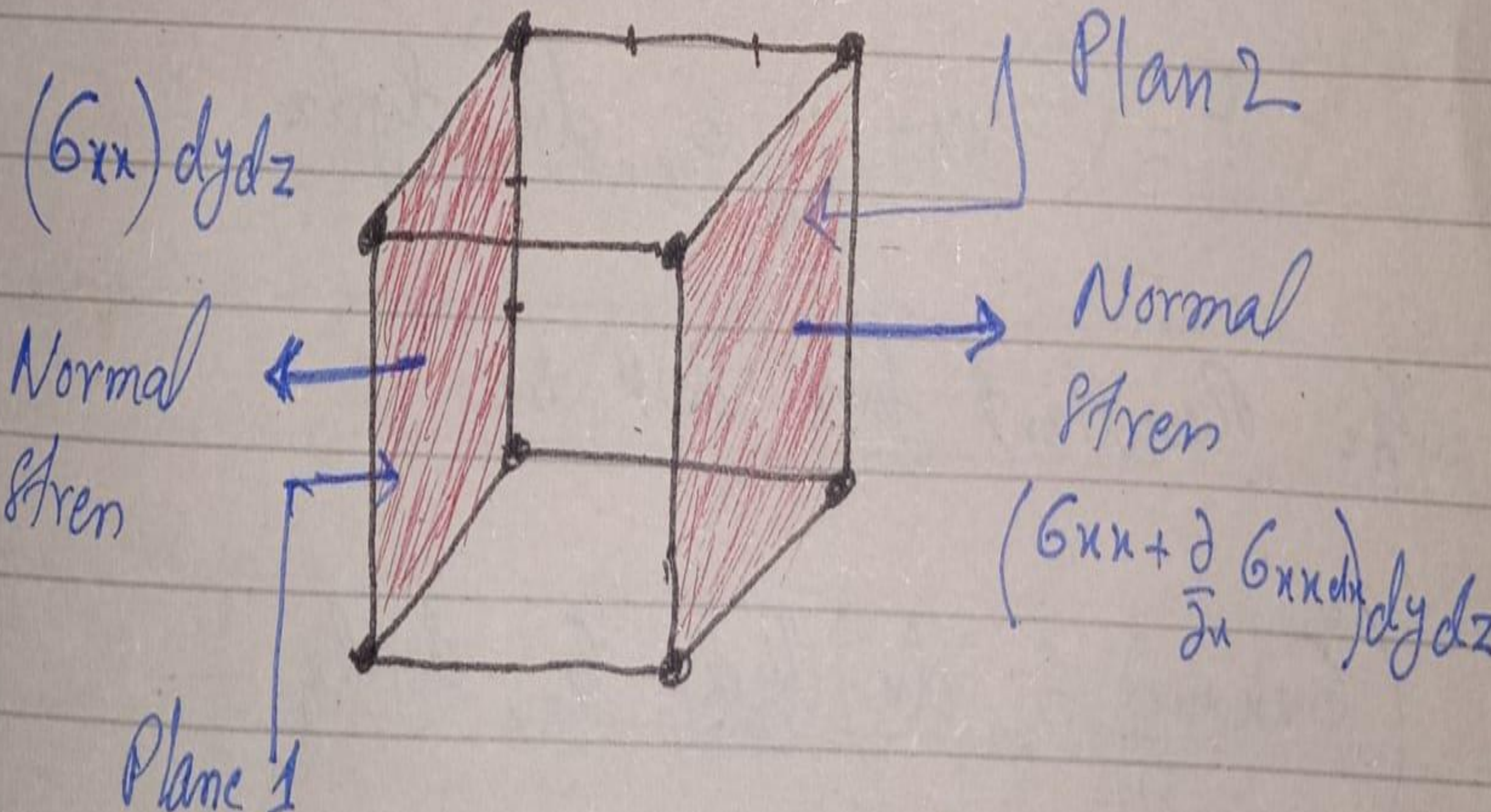
$$W = \rho g_x \cdot dx\,dy\,dz$$

Normal Force = Normal Stress × Area

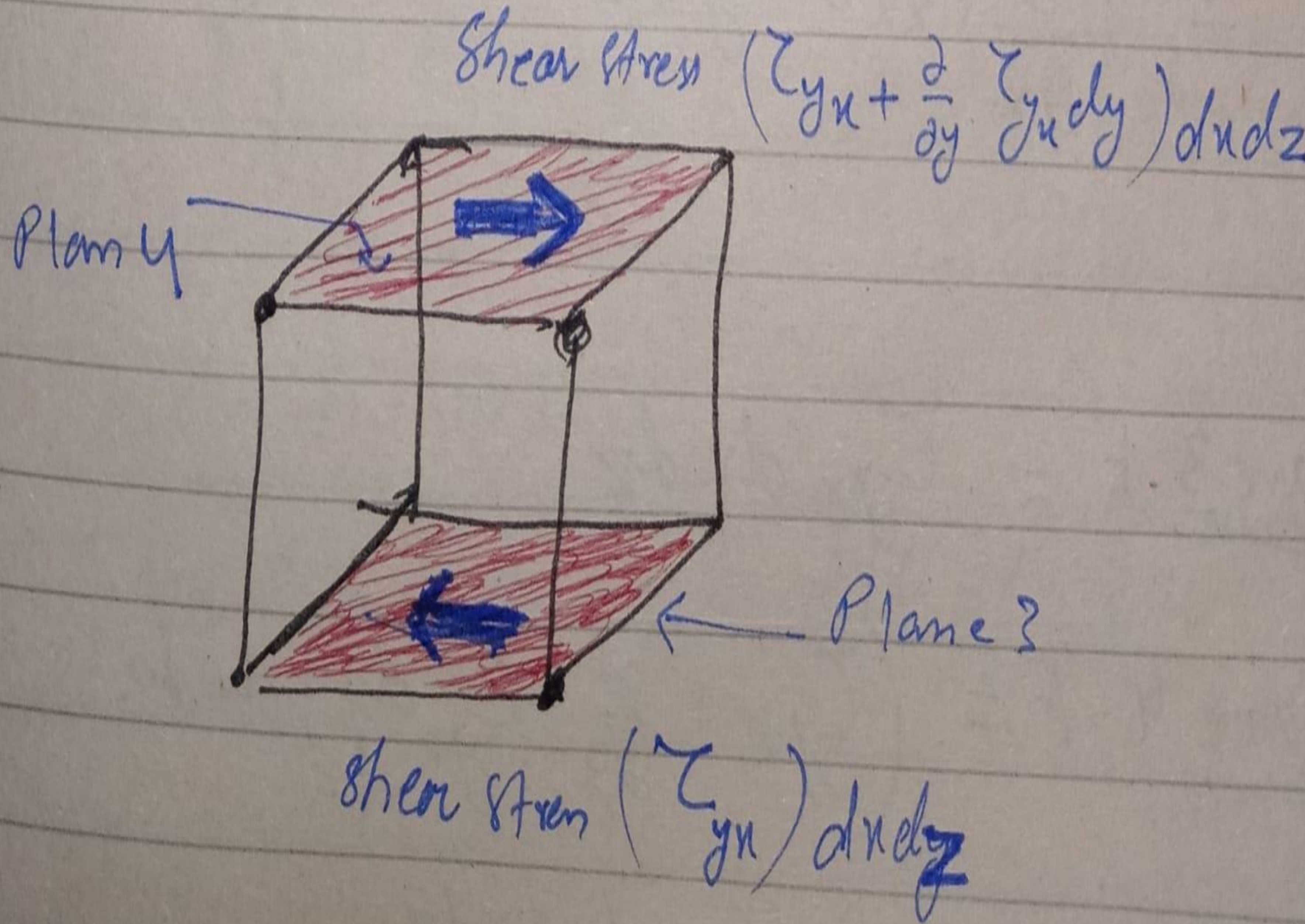
$$\text{Stress} = \frac{\text{Force}}{\text{Area}}$$

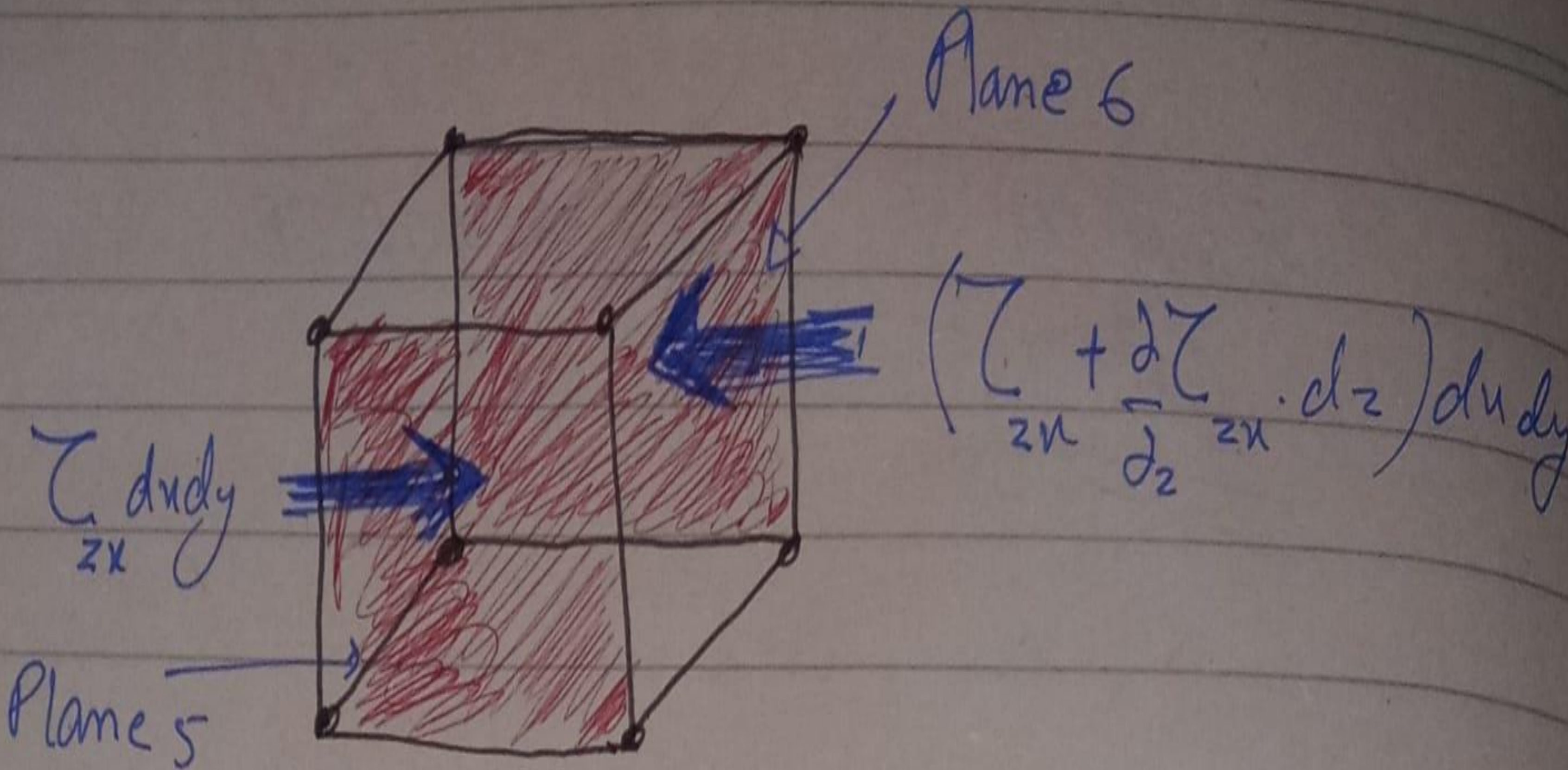
$$F = \frac{\sigma}{A}$$

$$\sigma = F \times A$$



$(\sigma_{xx})\, dy\, dz$
Normal Stress
Plane 1
Plan 2
Normal Stress
$\left(\sigma_{xx} + \frac{\partial}{\partial x}\sigma_{xx}\, dx\right) dy\, dz$


Shear stress $\left(\tau_{yx} + \frac{\partial}{\partial y} \tau_{yx} dy\right) dx dz$
Plane 4
Plane 3
Shear stress $\left(\tau_{yx}\right) dx dz$


Plane 6
$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot dz\right) dx\,dy$
$\tau_{zx}\, dx\,dy$
Plane 5

Force on Plane 1 in x-direction will be normal force

$$F = \sigma_{xx} \cdot A$$

$$F = \sigma_{xx} \cdot dy\,dz$$

on Plane-2

$$F = \left(\sigma_{xx} + \frac{\partial}{\partial x}\sigma_{xx}\,dx\right)dy\,dz$$

The Resultant force will be

$$\left(\sigma_{xx} + \frac{\partial}{\partial x}\sigma_{xx}\,dx\right)dy\,dz - \sigma_{xx} \cdot dy\,dz$$

$$= \frac{\partial}{\partial x}\sigma_{xx}\,dx\,dy\,dz$$

Plane 3 $F = \tau_{yx}\, dx\, dz$

Plane 4 $F = \left(\tau_{yx} + \frac{\partial}{\partial y} \tau_{yx}\, dy\right) dx\, dz$

Resultant $F = \frac{\partial}{\partial y} \tau_{yx}\, dx\, dy\, dz$

Plane 5 $F = \tau_{zx}\, dx\, dy$

Plane 6 $F = \left(\tau_{zx} + \frac{\partial}{\partial z} \tau_{zx}\, dz\right) dx\, dy\, dz$

Resultant $F = \frac{\partial}{\partial z} \tau_{zx}\, dx\, dy\, dz$

Total $= \frac{\partial}{\partial x} \sigma_{xx}\, \partial x\, \partial y\, dz + \frac{\partial}{\partial y} \tau_{yx}\, dx\, dy\, dz + \frac{\partial}{\partial z} \tau_{zx}\, dx\, dy\, dz$

$$F_{sx} = \left[\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx}\right] dx\, dy\, dz$$

<u>Now</u>

$$F = m\,a$$

$$\text{weight} + \text{Surface forces} = m\,a$$

$$\text{weight} = \rho g_x \, dx\,dy\,dz$$

$$F_{Sx} = \left[\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right] dx\,dy\,dz$$

$$ma = \rho\left[u\cdot\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right] dx\,dy\,dz$$

$$F = m\,a$$

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + \rho g_x = \rho\left[u\cdot\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$\boxed{\rho g_x + \frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] \rightarrow (i)}$$

As Per Stokes Theorem

$$\sigma_{xx} = -P + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\right)$$

ρ The L.H.S of eq (i)

$$L.H.S = \rho g_x + \frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}$$

$$= \rho g_x + \frac{\partial}{\partial x}\left[-P + 2\mu\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\Rightarrow \rho g_x - \frac{\partial P}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y}$$

$$+ \mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 w}{\partial x \partial z}$$

$$\Rightarrow \rho g_x - \frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 u}{\partial x^2}$$

$$+ \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 w}{\partial x \partial z}$$

$$\text{L.H.S} = \rho g_x - \frac{\partial P}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

$$+ \mu \frac{\partial}{\partial x} \underbrace{\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right]}_{\substack{\text{continuity eq:} \\ \text{Incomp fluids} \\ = 0}}$$

$$L.H.S = \rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\boxed{\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]}$$

Final Momentum Eq: