CSL 3010: Design and Analysis of Algorithms July-November 2022

Lecture 02: Non-Trivial Algorithms - II

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Last Class

1. Find Max or Min - through (n - 1) comparisons.
2. Find Max and Min together - through $\frac{3n}{2}$-2 comparisons.
3. How to multiply 2n digit number in - O($n^{log_2 3}$).

Today

1 Graphs

$G = (V, E)$; where V - Vertex Set and E - Edge Set
such that $E \subseteq V \times V$

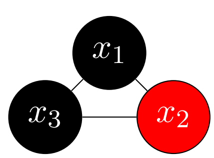
1.1 Reresentation of Graphs

- Adjcency Linked List
- Adjcency Matrix

1.2 2 - Colouring Graph

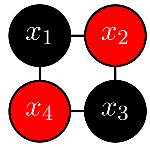
Input: G
Given: Graph G = **(V, E)** & **E** $\subset$ **(U, V)**
Question: Could we colour the vertices of G with 2 colours say *Red* and *Black*. Such that there is no edge with both endpoints being *Red* or *Black*.

1. **Example 1:**



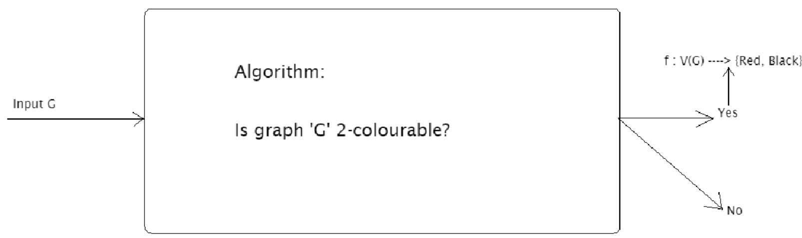
The above graph isn't 2-colourable.

2. **Example 2:**



The above graph is 2-colorable with set=$\{1, 4\}$ as Black *and* set=$\{2, 3\}$ as Red.

3.



$$\forall u, v \in E(G); f(u) \neq f(v)$$

4. **Message:** Do not trust your algorithm blindly.
   - Need Proof of Correctness.
     Which means:
     - **Yes** instance reports Yes.
     - **No** instance reports No.

5. **Algorithm** (proposed):
   G is 2 - colourable **iff** G has no odd length cycles.

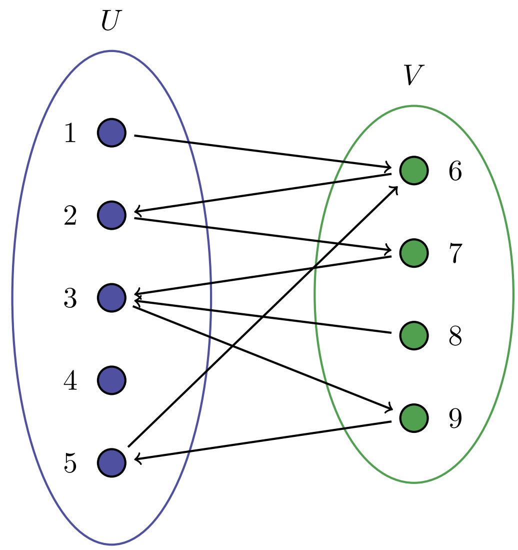
   Algorithm: Test weather G has odd cycle or not.
   Proof: ...?

6. **Solution:**
   Design a function **f**

$$\text{f : V (G)} \rightarrow \{\text{Red, Black}\}$$

$$\forall U, V \in E(G), f(U) \neq f(V)$$

The graph is 2-colourable if it can be partitioned into 2 independent sets.



7. **Algorithm:**
   Grapg G must not have odd length cycles.
   Check for odd cycles using BFS(Breadth-First Search) $\rightarrow$ O(n + m)
   where

$$\text{n} \rightarrow \text{number of vertices}$$

$$\text{m} \rightarrow \text{number of edges}$$

Learning Outcomes

O(n + m) time algorithm.
Importance of correctness.
Here we are only finding small no. of odd cycles.
Odd cycles formed from more than one non-tree edges are not considered, i.e. why we ain't ennumerating all odd cycles.
We are never going over every odd cycles.
Proposed algorithm can work even when only one odd cycles exist.

1.3 3-Colouring

Input: G
Given: Graph G = **(V, E)** & **E** $\subset$ **(U, V)**
Question: Is G 3 colourable?

1. G is K colourable iff $\exists$ a function f
   such that:

$$f : V(G) \rightarrow \{1, 2, 3, ..., K\}$$

$$\text{s.t.:} \qquad \forall \text{ U, V} \in E(G)$$

2. **Brute Force:**
   - Ennumerate all $f : V(G) \rightarrow \{1, 2, 3\}$
   - $3 \times 3 \times 3 \times ... \times = 3^n$
   - Check if f is correct assignment - $O(3^n.n^2)$
3. **K-colourable:** G is K colourable iff $v(G)$ can be partitioned into K independent sets.

$$V(G) = \{P_1, P_2, ..., P_k\} \text{ is a partition } iff$$

$$\forall \ O_i j P_i \cap P_j = \phi$$

$$\bigcup_{i=1}^{k} P_i = V(G)$$

   - $2^n.(n+m)^{O(1)}$
4. **Take 2: (Improvisation)**
   - Guess V_1 (& check if it is independent)
   - Check weather $G - V$ is independent and 2 - colourable.
   - Time complexity: $O(2^n.(n+m))$
   - **Algorithm:**
     - Taking $V_1, V_2 \& V_3$ as subsets of G.
       Either $|V_1| \leqslant \frac{n}{3}$
       OR $|V_2| \leqslant \frac{n}{3}$
       OR $|V_3| \leqslant \frac{n}{3}$
     - Ennumerate all set $X \subseteq V(G)$ of size $\leq \frac{n}{3}$
       * Check if is an independent set.
       * If yes check weather G -X is 2-colourable.
     - If for any X returns true/
       return true;

$$O(\textstyle\sum_{i=1}^{\frac{n}{3}} \binom{n}{i}.(n+m))$$

$$\because \binom{n}{i} \leqslant n.\binom{n}{\frac{n}{2}}$$

$$\therefore \textstyle\sum_{i=1}^{\frac{n}{3}} \binom{n}{i} \leqslant \frac{n}{3}\binom{n}{\frac{n}{3}}$$

$$\implies \binom{n}{\alpha n} \leqslant \frac{n^n}{(\alpha n)^{\alpha n}(n-\alpha n)^{n-\alpha n}}$$

$$\implies \binom{n}{\alpha n} \leqslant [[\frac{n}{\alpha n}]^{\alpha n}[\frac{n}{n-\alpha n}]^{n-\alpha n}]$$

$$\alpha \leqslant 1$$

$$\implies \binom{n}{\alpha n} \leqslant [[\frac{1}{\alpha}]^{\alpha}[\frac{1}{1-\alpha}]^{1-\alpha}]$$

$$\text{Set } \alpha = \frac{1}{3}$$

$$\therefore \binom{n}{\alpha n} \leqslant [3^{\frac{1}{3} \times \frac{3}{2}^{\frac{2}{3}}}]^n = 1.889^n$$

$$\text{better than } 2^n$$

1.4 K-colourable

Input: G
Given: as mentioned 2 and/or 3 colouring.
Ouestion: is G K-colourable?
Trivial: $T = O(k^n)$
Using 3-colouring improvised algorithm $T = O((k-1)^n)$
fig

- G is K-colourable $iff \ \exists \ V_1$
  s.t. V_1 is an independent set & $G - V_1$ is K-colourable.

1.5 Chromatic Number $[\chi(G)]$

It is the minimum number od K for which G is K-colourable.

$$\chi(G) = min\{1 + \chi(G - V_1)\}$$

$$-V_1 \subseteq V(G)$$

$$-V_1 \text{ is an independent set.}$$

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Suppose an array

$$A = [\qquad ..... \qquad]$$

$$\text{index } X \subseteq V(G)$$

$$A[X] \text{ is defined as } \chi(G[X])$$

$$A[\phi] = 0$$

$$A[v] = 1 \text{ for some v} \in \text{V(G)}$$

$$\text{fill the array A in increasing order of } X \text{ \&}$$

$$\text{A[X]} = min_{X` \subseteq X} \ \{1 + \text{A[X - X`]}\}$$

$$|X| \geqslant 2$$

$\therefore$ Each entry will take :

$$O(2^{X`}.n)$$

where n $\rightarrow$ arbitrary constant.

$$O(\sum_{X` \subseteq V(G)} 2^{X`}.n)$$

$$O(\sum_{i=1} n \binom{n}{i}.2^{i}.n)$$

$$O(3^{n}.n)$$

Best Known Algorithm:
$O(2^{n}.n^{O(1)})$

Probably Chromatic number dosen't have $n^{O(1)}$ time algorithm.