E262 - Amrut Katkar

E264 - Mukund Chavan

MATHEMATICAL FOUNDATION OF URBAN TRAFFIC MANAGEMENT OPTIMIZATION

E268 - Krishna Kedar

E270 - Tushar Ingle

Today’s Focus

What is Traffic Flow?

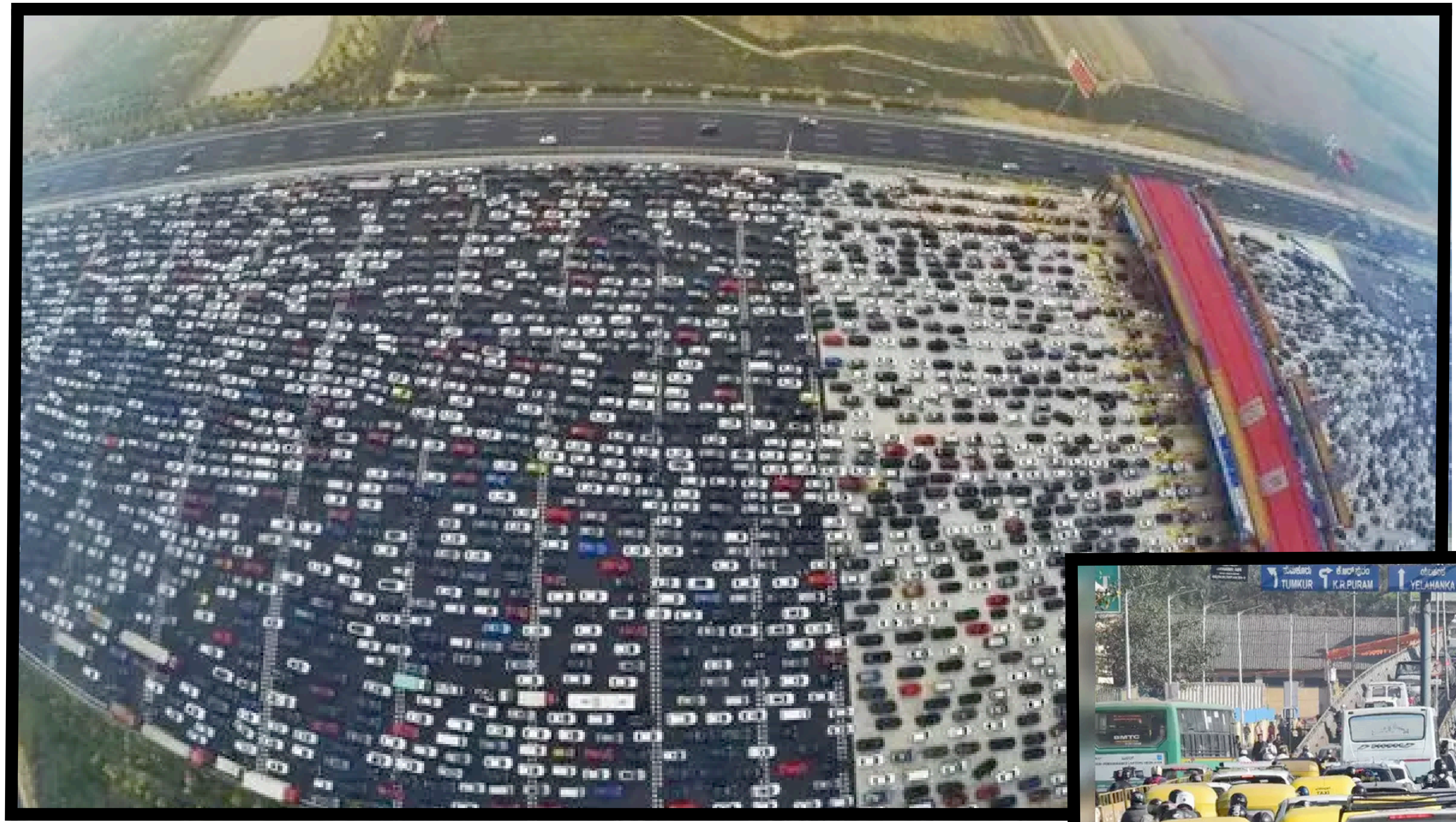
- It is the flow of vehicles on highways (can be a road in cities or towns)

Why does it need optimization?

- In developing countries like India, the amount of time wasted by a traveler in a traffic jam is quite high and this is not only a waste of time but this is a waste of the economy and causes a huge loss to the economy.

- In 2015, a 10 km long traffic jam was witnessed on the Beijing-Hong Kong-Macau express highway.

- In the largest urban areas, an average traveler wastes 62 hours per year just because of traffic jams.



Beijing-Hong Kong-Macau Express highway, China

Traffic at Outer Ring Road, Bangalore

Objectives

The vehicular traffic is studied with the following primary objectives:

- To optimize the flow
- to reduce traffic congestion
- to help engineers decide on planning of road or highway network
- to understand the reason for the occurrence of traffic jams
- to develop intelligent transportation systems like driverless cars
- to reduce traffic accidents

Assumptions

Few General Assumptions:

- Drivers always obey traffic rules.
- Each vehicle's length is the same.
- Each vehicle is called a car. (Here, the car can be any vehicle)
- Conservation of cars.

Fundamental Variables

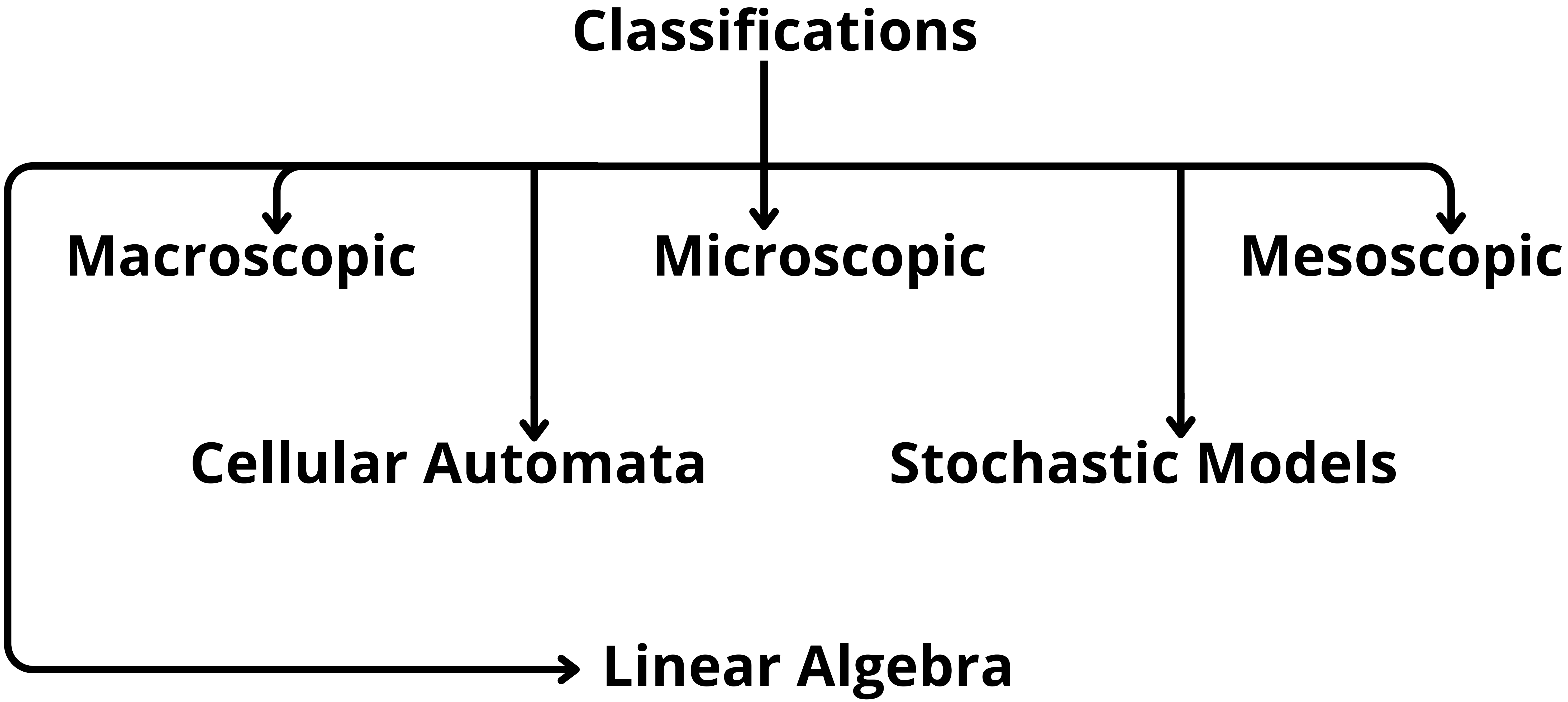
Traffic Density: It is the number of cars per unit distance at a given time. It is denoted by ρ

Traffic Flow rate: It is the number of cars passing through a particular position on the road per unit of time. It is denoted by Q. It can be expressed as $Q=\rho v$

Position Function: The position $x(t)$ of a vehicle over time t can be represented as a function. The velocity $v(t)$ of the vehicle is the derivative of the position function with respect to time: $v(t)=dx/dt$

Acceleration: The acceleration $a(t)$ of the vehicle is the derivative of the velocity function with respect to time: $a(t)=(dv)/(dt)=((d^{2}x))/(dt^{2})$. It indicates how quickly the vehicle's speed is changing.

Classifications of Traffic Flow Model



Use of Linear Algebra in Traffic Flow

Purpose of Study:

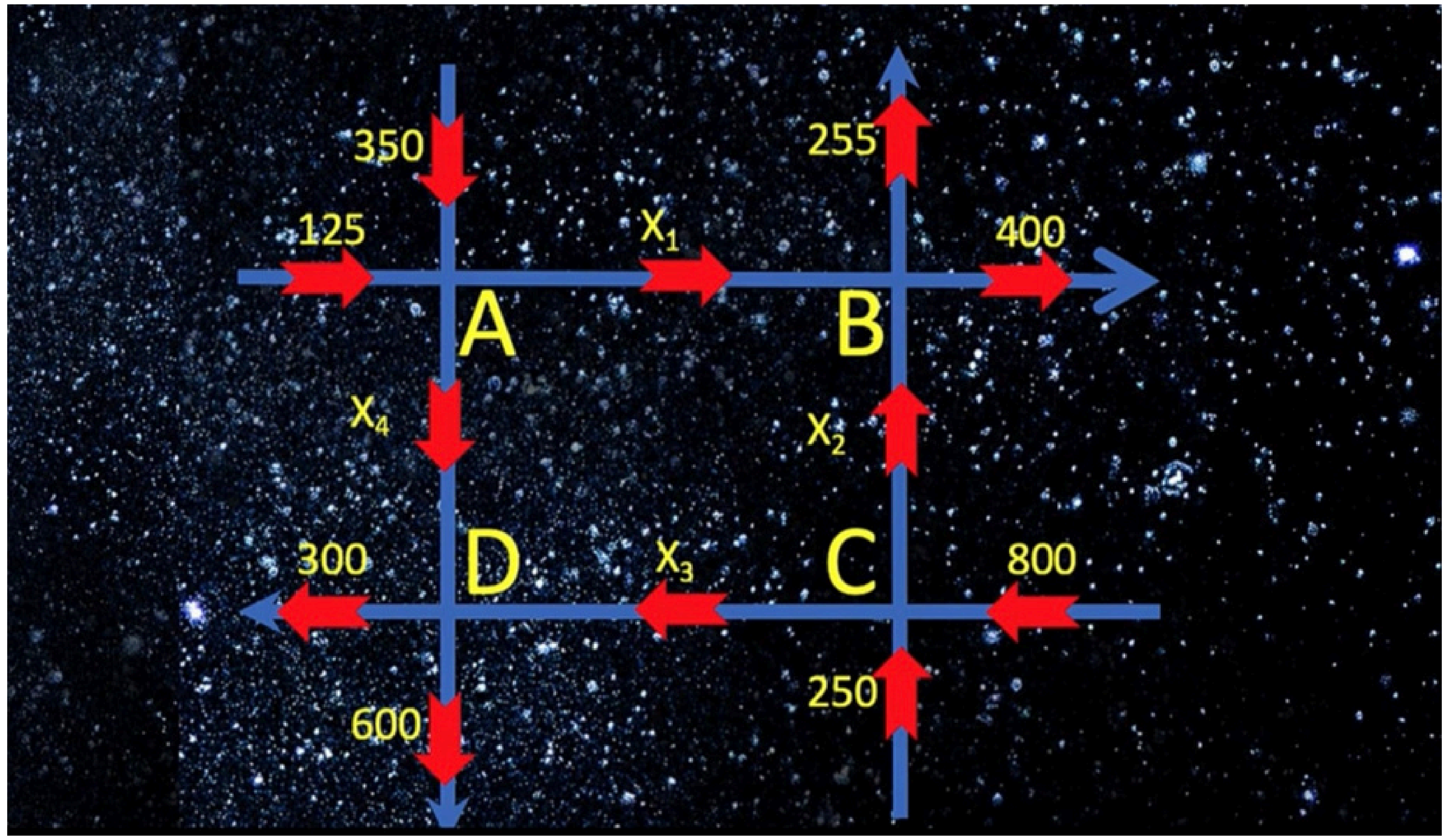
The main purpose of this study is to show how linear algebra is applied in the analysis of traffic flow through a road network including how systems of linear equations with many solutions can arise in practice.

The following assumptions were made in order to ensure the smooth flow of the traffic;

I. Vehicles entering each intersection should always be equal to the number of vehicles leaving the intersection.

II. the streets must all be one way with the arrows indicating the direction of traffic flow.

Let's say we have this situation, the flow of traffic through a network of streets is shown below. So, we have these four streets that are all one-way streets and there are four intersections corresponding to the nodes of the network.



Let’s consider from the **North 350 cars/hr.**

West 125 cars/hr.

East 125 cars/hr.

South 250 cars/hr.

going to intersection A,B,C,D

So, the question is to find the values of X1, X2, X3, X4?

Systems of Equations

NODE	FLOW IN	FLOW OUT	EQUATIONS
A	125+350	X1+X4	X1+X4=475
B	X1+X2	225+400	X1+X2=625
C	250+800	X2+X3	X2+X3=1050
D	X3+X4	300+600	X3+X4=900

Systems of Equations

$$\begin{aligned} x_1 \quad\quad\quad\quad + x_4 &= 475 \\ x_1 + x_2 \quad\quad\quad\quad &= 625 \\ x_2 + x_3 \quad\quad &= 1050 \\ x_3 + x_4 &= 900 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 1 & 1 & 0 & 0 & 625 \\ 0 & 1 & 1 & 0 & 1050 \\ 0 & 0 & 1 & 1 & 900 \end{array}\right]$$

Three types of elementary row operations:

1. **Swap rows**
2. **Multiply a row by a non-zero number**
3. **Add a multiple of one row to another**
4. **Convert this matrix into it’s row Echelon form**

Systems of Equations

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 1 & 0 & 1050 \\ 0 & 0 & 1 & 1 & 900 \end{array}\right]$$

$R2 - R1 \rightarrow R2$

$R4 \leftrightarrow R3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 1 & 1 & 0 & 1050 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & -1 & -1 & -900 \end{array}\right]$$

$R2 - R4 \rightarrow R4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

R3+R4 → R4

Systems of Equations

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad\qquad + x_4 &= 475 \\ x_2 \qquad - x_4 &= 150 \\ x_3 + x_4 &= 900 \end{aligned}$$

The system of equations that corresponds to this reduced row-echelon form is:

$$x_1 = 475 - x_4$$
$$x_2 = 150 + x_4$$
$$x_3 = 900 - x_4$$

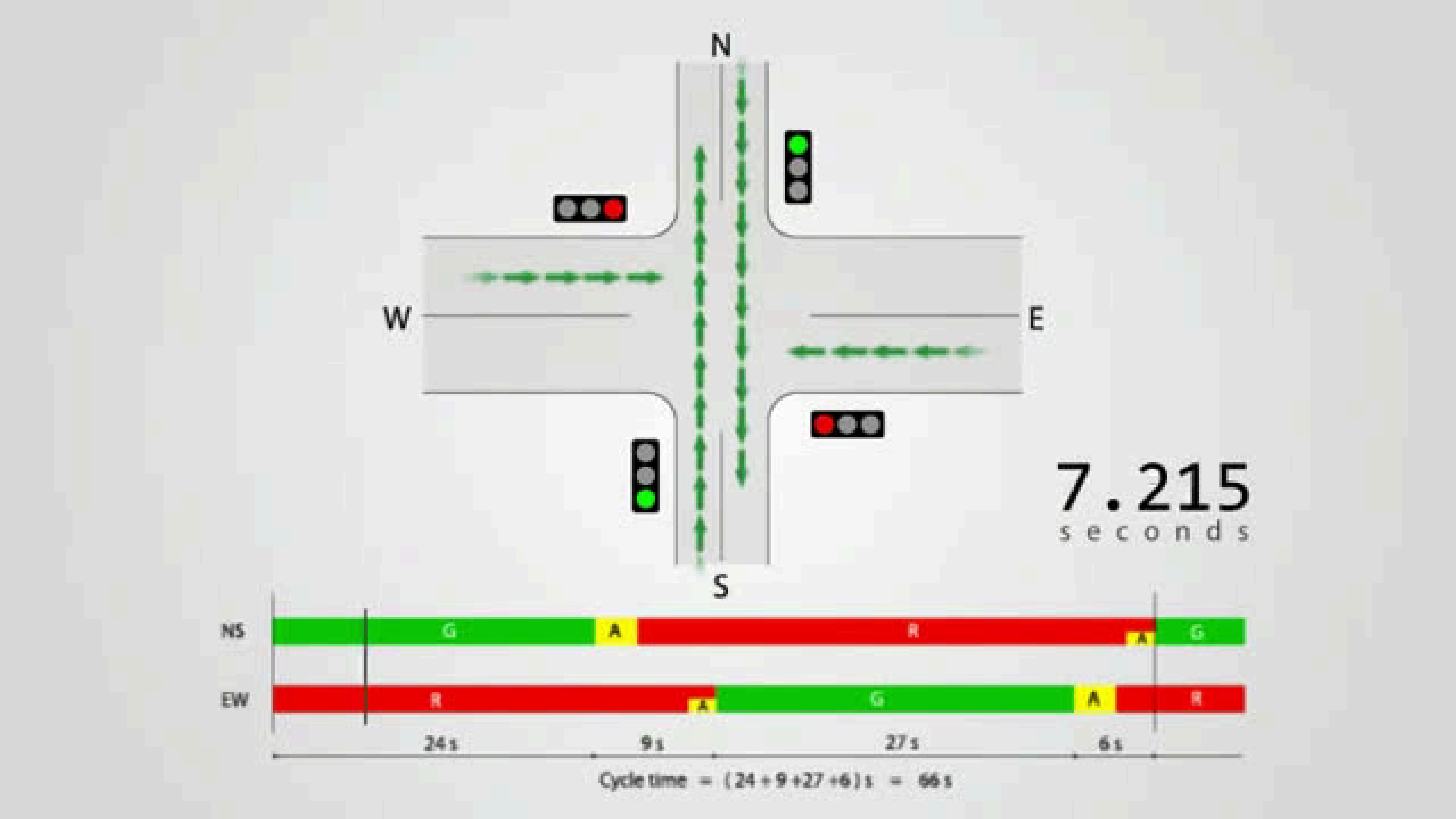
x_4 is a free variable

$$x_1 = 475 - x_4$$
$$x_4 \leq 475$$
$$x_2 = 150 + x_4$$
$$x_4 \geq -150$$
$$x_4 \geq 0$$

Since X4 is a free variable, this problem has many possible solutions, but -

$$\boxed{0 \leq x_4 \leq 475}$$

Traffic Signal Design



Traffic Signal Design

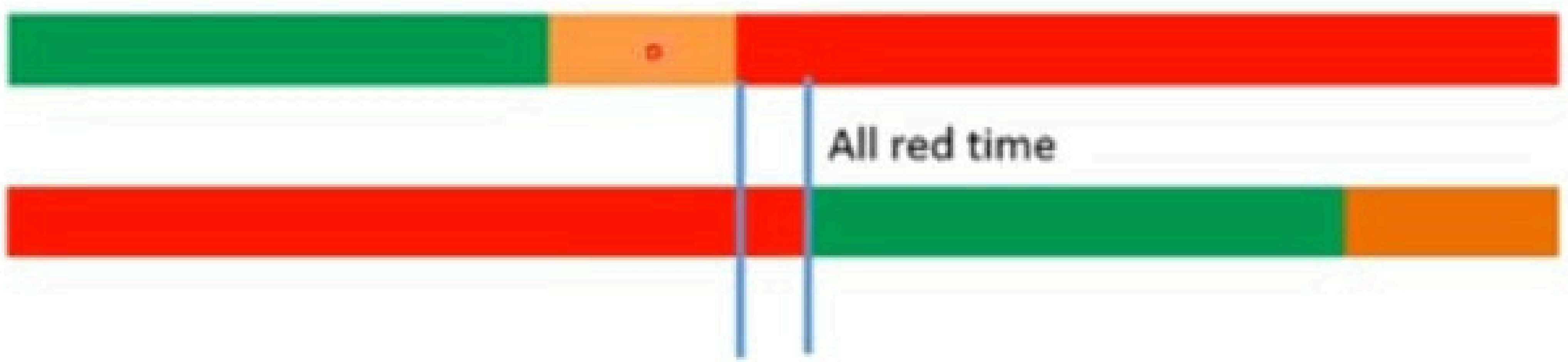
- To provide orderly movement of traffic
- To increase traffic handling capacity of the intersection
- Signals can reduce frequency of certain types of accidents
- Signals can replace traffic police

Three colors are used

- The British practice is R → (A+Red) → G → A
- The Indian practice R → G → A

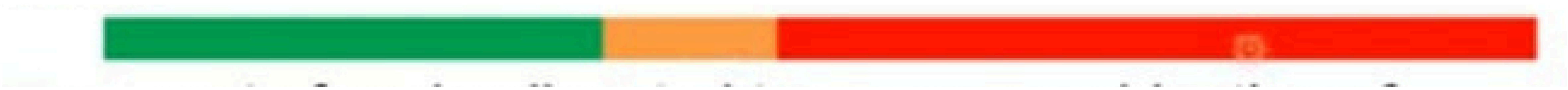
Traffic Signal Design

- When Amber is between termination of green and start of the red -- clearance amber
- When Amber is between termination of red and start of the green -- initial amber
- A signal phase - part of the cycle length allocated to a traffic movement receiving the right of way



Definitions

- **Cycle** - the time for one complete sequence of signal indications



- **Phase** - part of cycle allocated to one or combination of movements
- **Interval** - a discrete portion of the signal cycle during which an indication remains unchanged

Webster's Method of Signal Design

$$C_o = \frac{1.5L + 5}{1 - \sum_i y_i}$$

C_o – optimum cycle length, sec
L – lost time per cycle = n.*l* + R

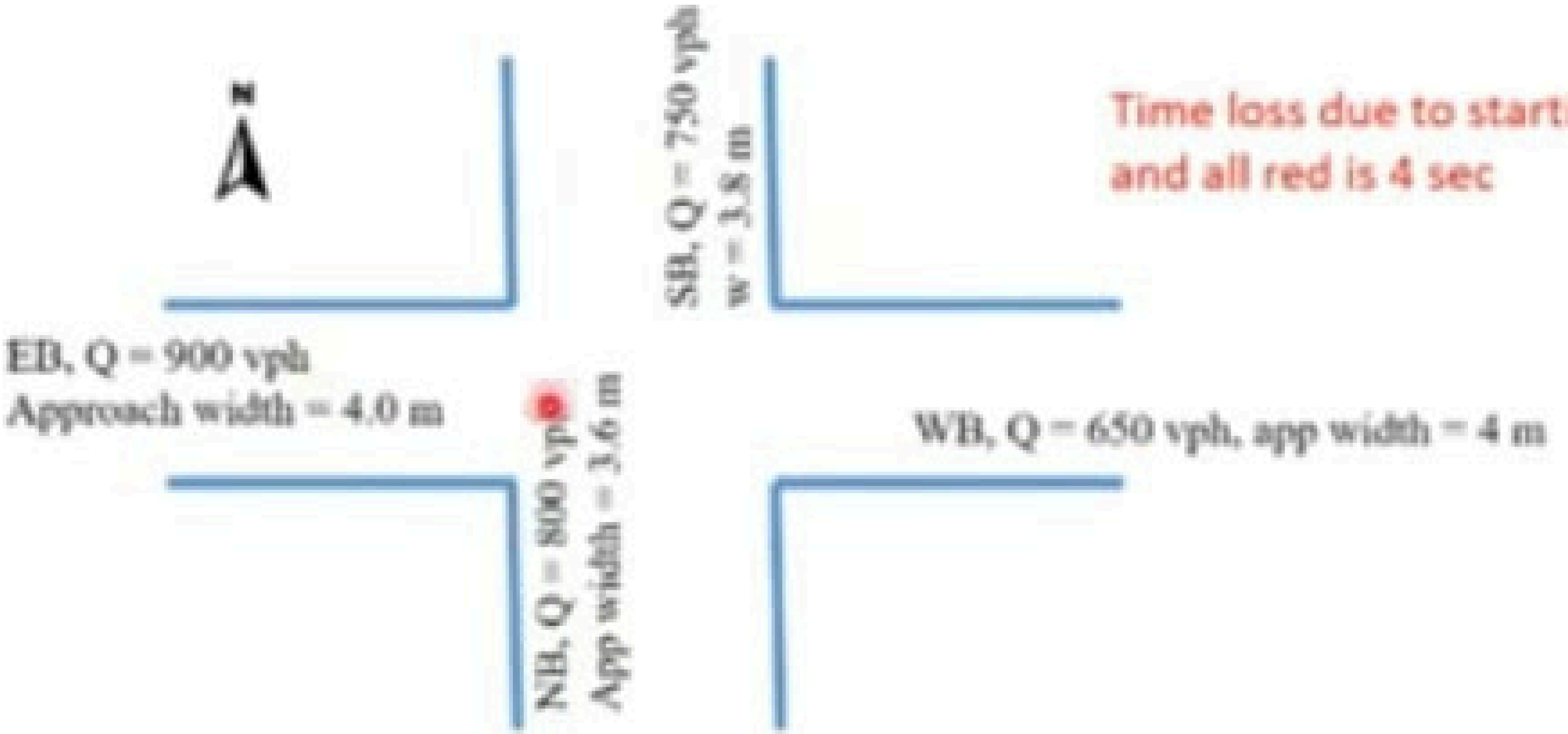
n – number of phases
l = average lost time per phase
R = total red time per cycle

$$y_i = \frac{\text{critical lane volume (in i}^{\text{th}}\text{ phase) } q_i}{\text{Saturation Flow } S_i}$$

S = 525 W

Example – 1

A fixed time two phase signal is to be provided at an intersection having four arms.



Time loss due to starting delay is 2 s and all red is 4 sec

Solutions

(i) Estimation of Saturation flow and y values.

approach	EB	WB	NB	SB
Width, m	4.0	4.0	3.6	3.8
SF, S = 525 w	2100	2100	1890	1995
Q	900	650	800	750
Y = S/Q	0.428	0.309	0.423	0.375
Critical y for the phase	0.428		0.423	

$$\sum y_i = 0.428 + 0.423 = 0.851$$

Time loss due to starting delay is 2 s and all red is 4 sec

Loss time per cycle = n.l+R = 2x2+4 = 8 sec

$$C_o = \frac{1.5L + 5}{1 - \sum_i y_i} \qquad C_o = \frac{1.5 \times 8 + 5}{1 - 0.851} = 114 \text{ sec}$$

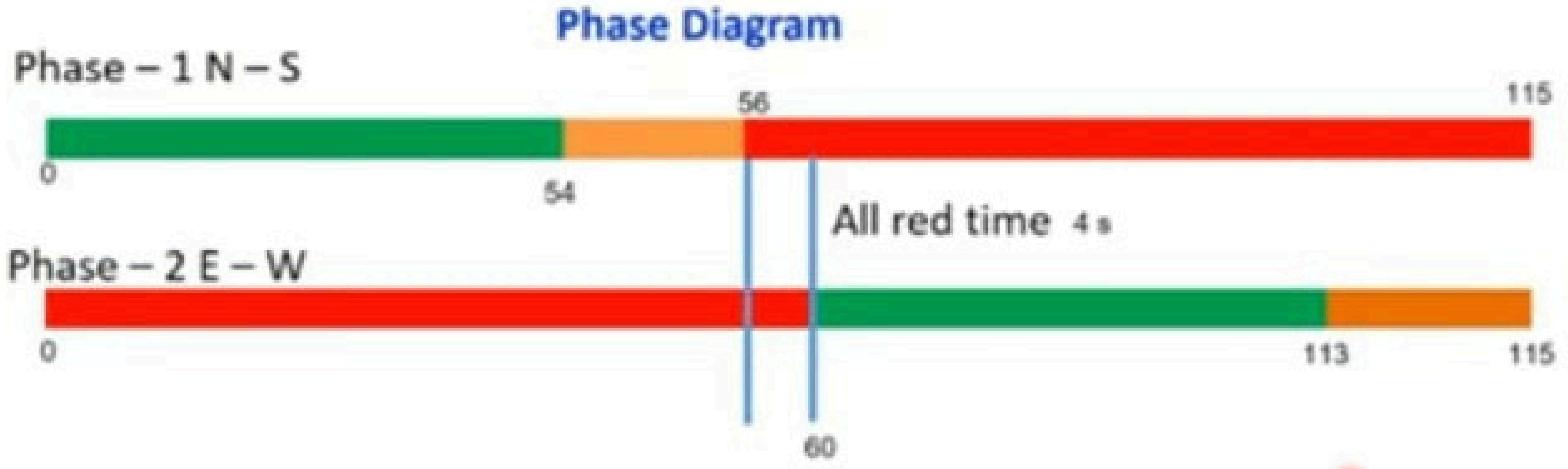
Let us say 115 sec

Effective green time available = 115 – 8 = 107 sec

Green time for phase 1 (N – S)

$$g_{NS} = \frac{y_{NS}}{y}(C_0 - L) \qquad g_{NS} = \frac{0.428}{0.851} \times 107 = 53.8 = 54s$$

$$g_{EW} = \frac{y_{EW}}{y}(C_0 - L) \qquad g_{EW} = \frac{0.423}{0.851} \times 107 = 53s$$



Ø1 N – S
Green = 54 s. A = 2

Ø2 E – W
All red = 4 Green = 53 s. A = 2

Microscopic Model

- Microscopic modeling gives attention to the details of traffic flow and the interactions taking place within it.
- Based on leader and follower interactions
- A model is a system of a large number of ordinary differential equations.
- Computationally expensive.

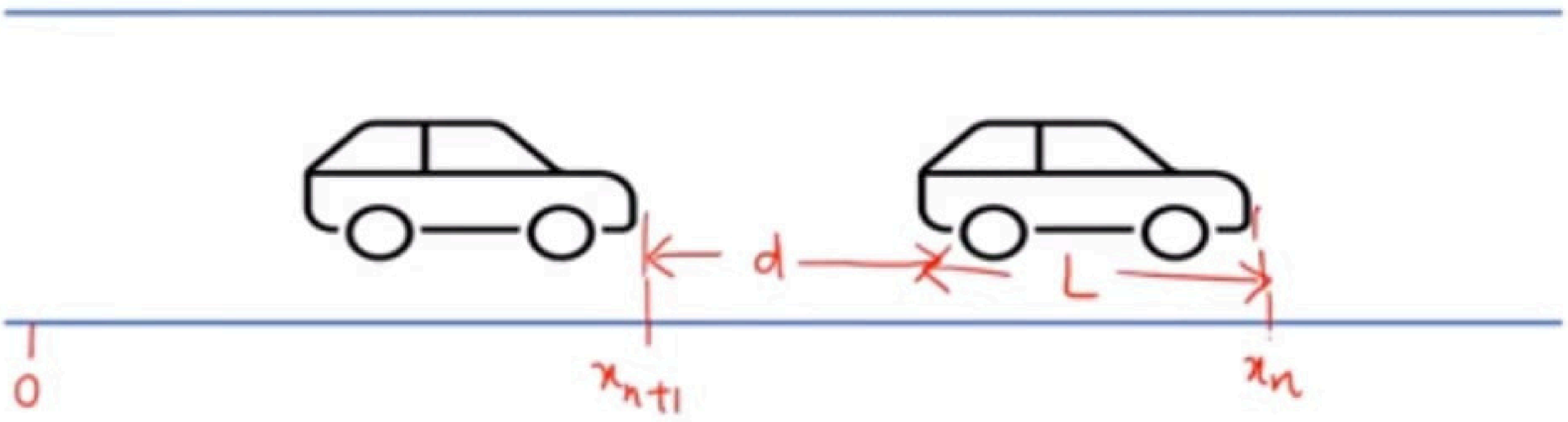
- **Ex:** Car-following model, cellular automata model.

Car-following Model

Based on "Follow-the-leader" theory

Following another vehicle on a single lane highway with no passing allowed has been characterized as one of the driving tasks by R.W. Rothery.

Car-following Model



$x_n - x_{n+1} = L + d$ — (1)

If the length of road = L_R

and no. of cars on road = N_R

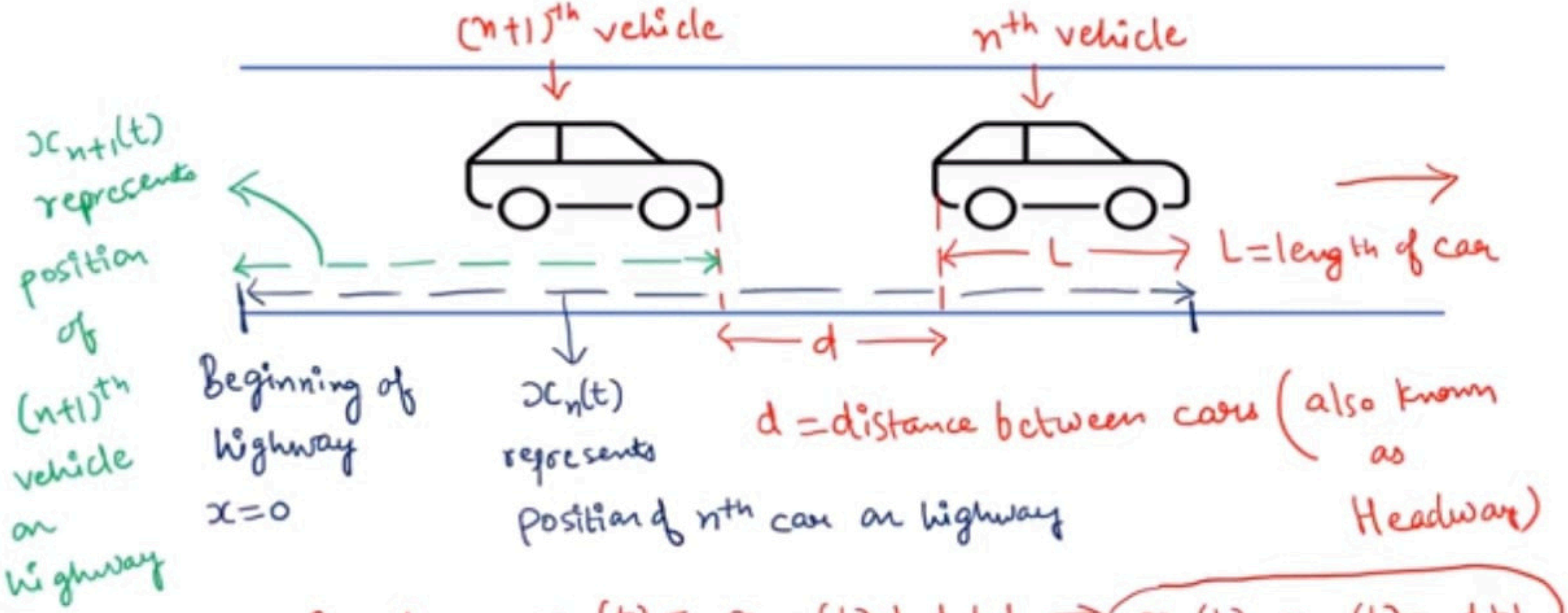
then $L_R = (L+d)N_R$

$\Rightarrow N_R = L_R / L+d$ — (2)

$\Rightarrow \frac{N_R}{L_R} = \frac{1}{L+d} = \rho$ = density

Dr. Isha Dhiman, TIET UMC 511- Lecture 15

Car-following Model



Clearly $x_n(t) = x_{n+1}(t) + d + L \Rightarrow x_n(t) - x_{n+1}(t) = d + L$

Car-following Model

$$\frac{d^2 x_{n+1}(t)}{dt^2} = -k_p \left(\frac{dx_{n+1}}{dt} - \frac{dx_n}{dt} \right)$$

(i) If $\frac{dx_n}{dt} > \frac{dx_{n+1}}{dt}$ i.e. leader is fast

then follower would naturally accelerate

$\Rightarrow \frac{d^2 x_{n+1}}{dt^2} > 0 \Rightarrow \underline{k_p < 0}$

(ii) If $\frac{dx_n}{dt} < \frac{dx_{n+1}}{dt}$ i.e. leader is slow

then $\frac{d^2 x_{n+1}}{dt^2} < 0 \Rightarrow \underline{k_p > 0}$

So, k_p keeps on changing

Car-following Model

Mathematically, if $x_n(t)$ is position of a n^{th} vehicle at time t

then $\frac{dx_n}{dt}$ is speed " " " " " "

and $\frac{d^2x_n}{dt^2}$ is acceleration of " " "

$$\underbrace{\frac{d^2x_{n+1}(t)}{dt^2}}_{\text{Acceleration/deceleration of follower vehicle}} = \underbrace{-k_p}_{\text{Sensitivity parameter}} \underbrace{\left(\frac{dx_{n+1}}{dt}(t) - \frac{dx_n}{dt}(t) \right)}_{\text{Stimulus} \rightarrow \text{speed difference b/w } (n+1)^{th} \text{ and } n^{th} \text{ vehicle}}$$

Acceleration/deceleration of follower vehicle

Sensitivity parameter (dimension of k_p is per unit time)

Stimulus → speed difference b/w $(n+1)^{th}$ and n^{th} vehicle

Car-following Model

Follow the leader concept:

Response = Stimulus . $\underbrace{\text{sensitivity}}_{}$

parameter (vary as per model)

Response → acceleration/deceleration

Stimulus → relative distance b/w follower and leader
or
Difference in speed
or
Speed of leading car

Example

If the average length of a car (in pre-*Expedition* days) is 5 m, what is the density of traffic in a line when its cars are maintaining a distance of two car lengths between themselves. What is the traffic flow if the line is moving at 80 km/hr (50 mph)? (*Hint*: You may ignore the fact that the data given ignores both AAA recommendations and your own experience on a freeway or turnpike.)

$L = 5\ m$

d = distance b/w cars = 2 car lengths $= 2L = 10\ m$

$$\text{Density} = \frac{1}{L+d} = \frac{1}{5+10} = \frac{1}{15}\ m^{-1} = \frac{1000}{15}\ km^{-1} = 66.66\ km^{-1}$$

$$q = \rho v = \frac{1000}{15} \cdot 80\ /hr = \frac{80000}{15}\ /hr = 5333.33\ \text{cars}/hr.$$

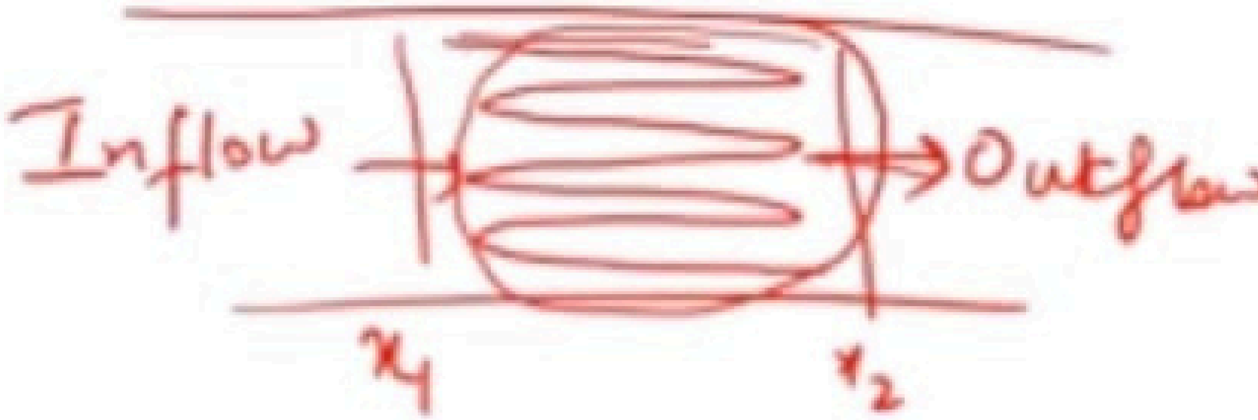
Macroscopic Model

- Macroscopic modeling looks at traffic flow from a global perspective, analogous to a fluid flow.
- Based on the continuum hypothesis and conservation laws.
- Model is a system of one or two partial differential equations.
- Computationally less expensive.

Conservation Law

Recall the derivation of conservation Law for one-dimensional

flow of fluid in a pipe



Using analogy with fluid flow,

the same equation can work for macroscopic traffic flow model

ρ = no. of cars / distance

q = no. of cars / time

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

; $\rho(x,t)$ is density

$q(x,t)$ is flow rate

Continuum / Macroscopic model derived is

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$q = \rho v$

ρ_0 can be constant / function of space x.

Using the fundamental relationship,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

with some initial condition $\underbrace{\rho(0) = \rho_0}_{\text{initial density}}$

LWR Model

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \rightarrow \text{LWR model}$$

This is first order model (∵ derivatives occuring are of first order)

and it is a PDE (Partial differential Equation)

* Famously known as LWR model (LightHill-Whitham-Richard)

LWR Model Solution

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0$$ → can be solved if v is a function of ρ

[∵ one equation and 2 unknowns, ρ and v]

Therefore, assume that

$$v = v(\rho)$$

Speed depends on density [Sounds Reasonable]

Furthermore, Method of Characteristics is used to solve above model.

→ Analytic technique

" Alternatively, numerical technique can be used "

Example

A highway section has an average spacing of 25ft under jam conditions and a free-flow speed of 55mph. Assuming that the relationship between speed and density is linear, determine the jam density, the maximum flow, the density at maximum flow, and the speed at maximum flow.

Average spacing = 25 ft under jam conditions

$$\rho = \frac{1}{\text{av. spacing}} = \frac{1}{25} \text{ ft}^{-1}$$

$$= \frac{5280}{25} \text{ mile}^{-1} = 211.2 \text{ vehicles/mile}$$

$$\Rightarrow \rho_{jam} = 211.2 \text{ vehicles/mile}$$

$$q_{max} = \frac{\rho_{max} v_{max}}{4}$$

$$= \frac{211.2 \times 55}{4}$$

$$= 2904 \text{ vehicles/hr.}$$

$$\rho = \frac{\rho_{max}}{2} \text{ at } q = q_{max}$$

$$\Rightarrow \rho = \frac{211.2}{2} = 105.6 \text{ vehicles/mile}$$

$$v = v_{max}/2 = 55/2 = 27.5 \text{ mph}$$