Bachelor of Computer Application

Computational Mathematics

LECTURE 6 NOTES

Types Of Relations

- **Types of Relations**

1. **VOID RELATION:**
Let A be a set. Then, $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on set A.

In other words, a relation R on a set A is called void or empty relation, if no element of A is related to any element of A.

Consider the relation R on the set A= (1, 2, 3, 4, 5) defined by R = {(a, b): a-b=12}.

We observe that $a-b \neq 12$ for any two elements of A.

$\therefore$ (a, b) $\notin$R for any a, b A.

$\Longrightarrow$ R does not contain any element of AxA

$\Longrightarrow$ R is empty set

$\Longrightarrow$ R is the void relation on A.

2. **UNIVERSAL RELATION:**
Let A be a set. Then, $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

In other words, a relation R on a set is called universal relation, if each element of A is related to every element of A.

Consider the relation R on the set A={1, 2, 3, 4, 5, 6} defined by R= {(a, b)R $|a-b| \geq 0$}

We observe that
$|a-b| \geq 0$ for all a, b$\in$A

$\Longrightarrow$ (a, b) $\in$R for all (a, b) $\in$ AxA

$\Longrightarrow$ Each element of set A is related to every element of set A

$\Longrightarrow$ $R=A \times A$

$\Longrightarrow$ R is universal relation on set A

- **NOTE-** It is to note here that the void relation and the universal relation on a set A are respectively the smallest and the largest relations on set A. Both the empty (or void) relation and the universal relation are sometimes called trivial relations.

- **Example: Let A be the set of all students of a boy's school. Show that the relation R on A given by R={(a,b): a is sister of b} is empty relation and R'= {(a, b): the difference between the heights of a and b is less than 5 meters} is the universal relation**

- **Solution:**
  Since the school is boy's school. Therefore, no student of the school can be sister of any student of the school.

  Thus, $(a, b) \notin R$ for any $a, b \in A$.

  Hence, $R=\phi$ i.e., R is the empty or void relation on A.

  It is obvious that the difference between the heights of any two students of the school has to be less than 5 meters.

  $\therefore$ $(a,b) \in R$ for all $a, b \in A$.

  $\Rightarrow$ $R= A \times A$
  $\Rightarrow$ R is the universal relation on set A.

3. IDENTITY RELATION:

Let A be a set. Then, the relation $IA=\{(a,a): a \in A\}$ on A is called the identity relation on A.

In other words, a relation IA on A is called the identity relation if every element of A is related to itself only.

If $A= \{1,2,3\}$, then the relation $IA =\{(1,1), (2,2), (3,3)\}$ is the identity relation on set A. But, relations $R_1 = \{(1,1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R1$, and in R_2 element 1 is related to elements 1 and 3.

4. REFLEXIVE RELATION:

A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.
A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

- **Example 1:**
  Let $A= \{1,2,3\}$ be a set. Then $R= \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on

A.
But, $R1= \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2,2) \notin R_1$.

- **Example 2:**
  The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A. For example, the relation $R= \{(a,a), (b, b), (c, c), (a, b)\}$ is a reflexive relation on set $A= \{a, b, c\}$ but it is not the identity relation on A.

- **Example 3:**
  The universal relation on a non-void set A is reflexive.

- **Example 4:**
  A relation R on N defined by $(x, y) \in R \Leftrightarrow x \geq y$ is a reflexive relation on N. because every natural number is greater than or equal to itself.

- **Example 5:**
  Let X be a non-void set and P(X) be the power set of X. A relation R on P(X) defined by $(A, B) \in R \Leftrightarrow A \subseteq B$ is a reflexive relation since every set is subset of itself.

- **Example 6:**
  Let L be the set of all lines in a plane. Then relation R on L defined by $(l1,l2) \in R \Leftrightarrow l1$ is parallel to l2 is reflexive, since every line is parallel to itself.

5. SYMMETRIC RELATION:

A relation R on a set A is said to be a symmetric relation if:

$(a, b) \in R \Longrightarrow (b, a) \in R$ for all $a, b \in A$
i.e., $aRb \Longrightarrow bRa$ for all $a, b \in A$.

- **Example 1:**
  The identity and the universal relations on a non-void set are symmetric relations.

- **Example 2:**
  Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow$ x is perpendicular to y. Then, R is a symmetric relation on L. because $L1 \perp L2 \Longrightarrow L2 \perp L1$ i.e., $(L1,L2) \in R \Longrightarrow (L2,L1) \in R$.

- **Example 3:**
  Let S be a non-void set and R be a relation defined on power set P(S) by (A, B)∈R⇔ A⊆B for all A, B∈P(S). Then, R is not a symmetric relation

- **NOTE:** A relation R on a set A is not a symmetric relation if there are at least two elements a, b∈A such that (a, b)∈R but (b,a)∉R.

6. TRANSITIVE RELATION:

Let A be any set. A relation R on A is said to be a transitive relation if:

(a, b)∈R and (b, c)∈R⟹ (a, c)∈R for all a, b, c ∈A.

i.e., aRb and bRc ⟹ aRc for all a,b,c∈A.

- **Example 1:**
  The identity and the universal relations on a non-void set are transitive.

- **Example 2:**
  The relation R on the set N of all natural numbers defined by
  (x, y)∈R cox divides y, for all x, y∈N is transitive.

- **Solution:**
  Let x, y, z∈N be such that (x, y)∈R and (y, z)∈R. Then, (x, y)∈R and (y, z)∈R

⇒ x divides y and, y divides z
⇒ There exist p, q∈N such that y=xp and z = yq
⇒ z = (xp) q
⇒ z= x(pq)
⇒ x divides z
⇒ (x,z)∈R [: pq € N]

Thus, (x, y)∈R, (y, z)∈R⟹(x, z)∈R for all x, y, z∈N.
Hence, R is a transitive relation on N.

7. ANTI-SYMMETRIC RELATION:

Let A be any set. A relation R on set A is said to be an anti-symmetric relation if:

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

- **NOTE** It follows from this definition that if $(a, b) \in R$ but $(b, a) \notin R$, then also R is an antisymmetric relation.

- **Example 1:**
  The identity relation on a set A is an antisymmetric relation.

- **Example 2:**
  The universal relation on a set A containing at least two elements is not antisymmetric, because if $a \neq b$ are in A, then a is related to b and b is related to a under the universal relation will imply that $a = b$ but $a \neq b$.

- **Example 3:**
  Let S be a non-void set and R be a relation on the power set P(S) defined by

  $(A, B) \in R \Leftrightarrow A \subseteq B$ for all $A, B \in P(S)$

  Then, R is an antisymmetric relation on P(S), because

  $(A, B) \in R$ and $(B, A) \in R \Rightarrow A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

❖ EXAMPLES:

- **Example 1:**
  Three relations R1, R2 and R3 are defined on set A-la,b,c) as follows:
  (i) R1= {(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)}
  (ii) R_2= {(a, b), (b, a), (a, c), (c, a)}
  (iii) R3={(a, b), (b, c), (c, a)}

  Find whether each of R1, R2 and R3 is reflexive, symmetric and transitive.

- **Solution:**

(i) **Reflexive:** Clearly (a, a), (b, b), (c, c) $\in R_1$. So, R_1 is reflexive on A.
Symmetric: We observe that (a, b)∈R1, but (b, a)∈R1. So, R_1 is not a symmetric relation on A.
Transitive: We find that (b, c)∈R1 and (c, a)∈R1 but (b, a)∈R1. So, R_1 is not a transitive relation on A.

(ii) **Reflexive:** Since (a, a), (b, b) and (c, c) are not in R2. So, it is not a reflexive relation on A.

Symmetric: We find that the ordered pairs obtained by interchanging the components of ordered pairs in R2 are also in R2. So, R2 is a symmetric relation on A.

Transitive: Clearly (a, b)$\in R_2$ and (b, a)$\in R_2$ but (a, a)∈R2.

So, it is not a transitive relation on R2.

(iii) **Reflexive:** Since none of (a,a),(b,b) and (c,c) is an element of R3. So, R3 is not reflexive on A.

Symmetric: Clearly, (b, c)∈R3 but (c, b)∈R3. So, R3 is not a symmetric relation on A.

Transitive: Clearly, (a, b)∈R3 and (b, c)∈R3 but (a, c)∈R3. So, R3 is not a transitive relation on A.

- **Example 2:**
  Let a relation R, on the set R of real numbers be defined as (a, b)∈R⇔1+ ab >0 for all a,b∈R.
  Show that R1 is reflexive and symmetric but not transitive.

- **Solution:**
  We observe the following properties:

  Reflexivity: Let a be an arbitrary element of R. Then,

	a∈R	
⟹	1+a.a=1+a2>0	[∵ a2>0 for all a∈R]
⟹	(a, a)∈R1	[By definition of R_1]

  Thus, (a, a)$\in R_1$ for all a∈R. So, R_1 is reflexive on R.

- **Symmetry**: Let (a, b)$\in$R. Then,

  (a, b)$\in$R1
  $\Rightarrow$ 1+ab >0
  $\Rightarrow$ 1+ba > 0 [$\because$ ab=ba for all a, b$\in$R]
  $\Rightarrow$ (b, a)$\in$R1 [By definition of R_1]

  Thus, (a, b)$\in$R1$\Rightarrow$(b, a)$\in$R1 for all a, b$\in$R. So, R1 is symmetric on R.

- **Transitivity:** We observe that (1,1/2)$\in$R1 and (1/2,-1)$\in$R1, but (1, -1)$\notin$R, because 1+1x(-1)=0$\ngtr$ 0. So, R1 is not transitive on R.

- **Example 3:**
  Show that the relation R on R defined as R= {(a, b): a≤b}, is reflexive and transitive but not symmetric.

- **Solution:**
  We have, R= {(a, b): a≤b}, where a, b$\in$R

  Reflexivity: For any a$\in$R

  a≤a
  $\Rightarrow$ (a, a)$\in$R for all a$\in$R
  $\Rightarrow$ R is reflexive.

- **Symmetry:** We observe that (2, 3)$\in$R but (3, 2)$\notin$R. So, R is not symmetric.

- **Transitivity:** Let (a, b)$\in$R and (b, c)$\in$R. Then,

  (a, b)$\in$R and (b, c)$\in$R

  $\Rightarrow$ a≤b and b≤c
  $\Rightarrow$ a≤c
  $\Rightarrow$ (a, c) eR

  So, R is transitive.

- **Example 4:**
  Let X= {1, 2, 3, 4, 5, 6, 7, 8, 9}. Let R1 be a relation on X given by R1={(x, y):x-y is divisible by 3} and R2 be another relation on X given by R2={(x, y):(x,y)⊂{1,4,7} or {x,y}⊂{2,5,8} or {x, y}⊂{3, 6, 9}. Show that R_1 = R2.

- **Solution:**
  Clearly, R1 and R2 are subsets of X×X. In order to prove that R1=R2, it is sufficient to show that R1⊂R2 and R2⊂R1

  We observe that the difference between any two elements of each of the sets {1,4,7} ,{2, 5, 8} and {3, 6, 9} is a multiple of 3.

  Let (x, y) be an arbitrary element of R_1. Then,

  (x, y)∈R1

  ⟹ x-y is divisible by 3.
  ⟹ x-y is a multiple of 3.
  ⟹ {x, y}⊂{1,4,7} or {x, y}⊂{2,5, 8} or {x, y}⊂{3,6,9}
  ⟹ (x, y)∈R2

  Thus, (x, y)∈R1 = (x, y)∈R2.

  So, R1⊂R2 .....(i)

  Now, let (a, b) be an arbitrary element of R2. Then,

  (a, b)∈R2.

  ⟹ {a, b}⊂{1,4,7} or {a, b}⊂{2,5, 8} or {a,b}⊂{3, 6, 9}
  ⟹ a-b is divisible by 3
  ⟹ (a, b)∈R1

  Thus, (a, b)∈R2 ⟹ (a, b)∈R1

  So, $R_2 \subset R_1$ .....(ii)

  From (i) and (ii), we get: $R_1 = R_2$

- **Example 5:**

Let A = (1, 2, 3). Then, show that the number of relations containing (1, 2) and (2,3) which are reflexive and transitive but not symmetric is three.

- **Solution:**

The smallest reflexive relation on set A containing (1, 2) and (2, 3) is

R={(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}

Since (1, 2)∈R and (2, 3)∈R but (1, 3)∉R. So, R is not transitive. To make it transitive we have to include (1, 3) in R. Including (1, 3) in R, we get

R1= {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)}

This is reflexive and transitive but not symmetric as (1, 3)∈R, but (3, 1)∉R1.

Now, if we add the pair (2, 1) to R1 to get R2 =((1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3),(2, 1)).

The relation R2 is reflexive and transitive but not symmetric. Similarly, by adding (3,2) and (3, 1) respectively to R1, we get

R3={(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 2)}

R3={(1,1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 1), (3,2)}

These relations are reflexive and transitive but not symmetric.

We observe that out of ordered pairs (2, 1), (3, 2) and (3, 1) at a time if we add any two ordered pairs at a time to R_1, then to maintain the transitivity we will be forced to add the remaining third pair and in this process the relation will become symmetric also which is not required. Hence, the total number of reflexive, transitive but not symmetric relations containing (1, 2) and (2, 3) is three.

❖ EXERCISES:

1: Let A be the set of all human beings in a town at a particular time. Determine whether each of the following relations are reflexive, symmetric and transitive:

(i) R = {(x, y): x and y work at the same place}
(ii) R= {(x, y): x and y live in the same locality)
(iii) R= ((x, y): x is wife of y)
(iv) R= ((x, y): x is father of y}

2: Let R be a relation defined on the set of natural numbers N as
R={(x, y): x, y∈N, 2x + y = 41}

Find the domain and range of R. Also, verify whether R is (i) reflexive, (ii) symmetric (iii) transitive.

3: An integer m is said to be related to another integer n if m is a multiple of n. Check if the relation is symmetric, reflexive and transitive.

4: Let A= {a, b, c} and the relation R be defined on A as follows: R = {(a, a), (b, c), (a, b)}. Then, write minimum number of ordered pairs to be added in R to make it reflexive and transitive.

5: Each of the following defines a relation on N:

(i) x>y, x, y∈N
(ii) x+y=10, x, y∈N
(iii) xy is square of an integer, x, y∈N
(iv) x+4y=10, x, y∈N

Determine which of the above relations are reflexive, symmetric and transitive.

- **Equivalence Relation**

A relation R on a set A is said to be an equivalence relation on A if it is

(i) reflexive i.e. (a, a)∈R for all a∈A.
(ii) symmetric i.e. (a, b)∈R⟹(b, a)∈R for all a, b∈A.
and, (iii) transitive i.e. (a, b)∈R and (b, c)∈R⟹(a, c)∈R for all a, b, c∈A.

- An equivalence relation R defined on a set A partitions the set A into pair wise disjoint subsets. These subsets are called equivalence classes determined by relation R. The set of all elements of A related to an element a∈A is denoted by [a] i.e. [a] = {x∈A: (x, a)∈R). This is an equivalence class. Corresponding to every element in A there is an equivalence class. Any two equivalence classes are either identical or disjoint. The collection of all equivalence classes forms a partition of set A.
- **Example 1:**
Show that the relation R on the set A of all the books in a library of a college given by
R-={(x, y): x and y have the same number of pages}, is an equivalence relation.

- **Solution:**
  We observe the following properties of relation R.

  Reflexivity: For any book x in set A, we observe that x and x have the same number of pages.

  $\Rightarrow$ (x, x) R

  Thus, (x, x) R for all $x \in A$.

  So, R is reflexive.

  Symmetry: Let $(x, y) \in R$. Then,

  $(x, y) \in R$

  $\Rightarrow$ x and y have the same number of pages
  $\Rightarrow$ y and x have the same number of pages
  $\Rightarrow$ $(y, x) \in R$

  Thus, $(x, y) \in R \Rightarrow (y, x) \in R$

  So, R is symmetric.

  Transitivity: Let $(x, y) \in R$ and $(y, z) \in R$. Then,

  $(x, y) \in R$ and $(y, z) \in R$

  $\Rightarrow$ (x and y have the same number of pages) and (y and z have the same number of pages)
  $\Rightarrow$ x and z have the same number of pages.
  $\Rightarrow$ $(x, z) \in R$

  So, R is transitive.

  Thus, R is reflexive, symmetric and transitive.
  Hence, R is an equivalence relation.

- **Example 2:**
  Let A ={1, 2, 3,...,9} and R be the relation on Ax A defined by (a, b) R (c, d) if a+d=b+c for all (a, b), (c, d)∈Ax A. Prove that R is an equivalence relation and also obtain the equivalence class [(2,5)].

- **Solution:** We observe the following properties of relation R.

Reflexivity: Let (a, b) be an arbitrary element of $A \times A$. Then,

$(a, b) \in A \times A$

$\Rightarrow$ $a, b \in A$

$\Rightarrow$ $a+b=b+a$ [By commutativity of addition on N]

$\Rightarrow$ $(a, b)\ R\ (a, b)$

Thus, (a, b) R (a, b) for all $(a, b) \in A \times A$. So, R is reflexive on $A \times A$.

Symmetry: Let $(a, b), (c, d) \in A \times A$ be such that (a, b) R (c, d). Then,

$(a, b)\ R\ (c, d)$

$\Rightarrow$ $a+d=b+c$

$\Rightarrow$ $c+b=d+a$ [By commutativity of addition on N]

$\Rightarrow$ $(c, d)\ R\ (a, b)$

Thus, (a, b) R (c, d) $(c, d) \Rightarrow R$ (a, b) for all $(a, b), (c, d) \in A \times A$.

So, R is symmetric on $A \times A$.

Transitivity: Let $(a, b), (c, d), (e, f) \in A \times A$ such that (a, b) R (c, d) and (c, d) R (e, f). Then,

$$\left.\begin{aligned} (a, b)\ R(c, d) &\Rightarrow a+d = b+c \\ (c, d)\ R(e, f) &\Rightarrow c+f=d+e \end{aligned}\right\} \Rightarrow (a+d)+(c+f)=(b+c)+(d+e)$$

$\Rightarrow$ $a+f=b+e$

$\Rightarrow$ $(a, b)\ R\ (e, f)$

Thus, (a, b) R (c, d) and (c, d) R (e, f) (a, b) R (e, f) for all $(a, b), (c, d), (e, f) \in A \times A$.

So, R is a transitive relation on $A \times A$.

Now, Hence, R is an equivalence relation on Ax A.

$[(2,5)]=\{(x,y)\in Ax\ A:(x, y)\ R\ (2,5)\}$

$=\{(x, y)\in AxA: x+5=y+2)\}= \{(x, y)\in A\times A:y=x+3\}$
$=\{(x, x+3): x\in A \text{ and } x+3\in A\} = \{(1, 4), (2,5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

$=\{(x, y)\in AxA: x+5=y+2)\}= \{(x, y)\in A\times A:y=x+3\}$
$=\{(x, x+3): x\in A \text{ and } x+3\in A\} = \{(1, 4), (2,5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

- **Example 3:**
  Let N be the set of all natural numbers and let R be a relation on Nx N, defined by
  (a, b) R (c, d)⇔ ad=bc for all (a, b), (c, d)∈Nx N.

  Show that R is an equivalence relation on Nx N. Also, find the equivalence class [(2,6)].

- **Solution:**

  We observe the following properties of relation R.

  Reflexivity: Let (a, b) be an arbitrary element of Nx N. Then,

  $(a, b)\in N\times N$

  $\Rightarrow$ $a,b\in N$
  $\Rightarrow$ $ab=ba$ [By commutativity of multiplication on N]
  $\Rightarrow$ $(a, b)\ R\ (a, b)$

  Thus, (a, b) R (a, b) for all (a, b)∈Nx N.

  So, R is reflexive on Nx N.

  Symmetry: Let (a, b), (c, d)∈Nx N be such that (a, b) R (c, d). Then,

  $(a, b)\ R\ (c, d)$

$\Rightarrow$ ad=bc

$\Rightarrow$ cb=da [By commutativity of multiplication on N]

$\Rightarrow$ (c, d) R (a, b)

Thus, (a, b) R (c, d)$\Rightarrow$ (c, d) R (a, b) for all (a, b), (c, d)$\in$NxN.

So, R is symmetric on Nx N.

Transitivity: Let (a, b), (c, d), (e, f) Nx N such that (a, b) R (c, d) and (c, d) R (e, f). Then,

$$\left.\begin{array}{l}(a, b)\ R\ (c, d) \Rightarrow ad = bc \\ (c, d)\ R\ (e, f) \Rightarrow cf = de\end{array}\right\} \Rightarrow (ad)\ (cf)=(bc)\ (de) \Rightarrow af=be \Rightarrow (a, b)\ R\ (e,f)$$

Thus, (a, b) R (c, d) and (c, d) R (e, f)$\Rightarrow$(a, b) R (e, f) for all (a, b), (c, d), (e, f)$\in$Nx N.

So, R is transitive on Nx N.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on Nx N.

[(2, 6)]= {(x, y)$\in$Nx N:(x, y) R (2, 6)}

={(x, y)$\in$Nx N: 3x=y}

={(x, 3x): x$\in$N}={(1, 3), (2, 6), (3, 9), (4, 12),..}

- **Some Useful Results on Relations**

  - **Theorem 1: If R & S are two equivalence relations on a set A, then R$\cap$S is also an equivalence relation on A.**

Or

The intersection of two equivalence relations on a set is an equivalence relation on the set.

Proof: It is given that R and Sare relations on set A..

$\therefore$ $R \subseteq A \times A$ and $S \subseteq A \times A$

$\Rightarrow$ $R \cap S \subseteq A \times A$

$\Rightarrow$ $R \cap S$ is also a relation on A.

Now, we shall show that it is an equivalence relation on A.
We observe the following properties of relation $R \cap S$.

Reflexivity: Let a be an arbitrary element of A. Then,

$a \in A$

$\Rightarrow$ $(a,a) \in R$ and $(a, a) \in S$ [$\because$ R and S are reflexive]

$\Rightarrow$ $(a,a) \in R \cap S$

Thus, $(a, a) \in R \cap S$ for all $a \in A$. So, $R \cap S$ is a reflexive relation on A.

Symmetry: Let $a, b \in A$ such that $(a, b) \in R \cap S$. Then,

$(a,b) \in R \cap S$

$\Rightarrow$ $(a, b) \in R$ and $(a, b) \in S$
$\Rightarrow$ $(b, a) \in R$ and $(b, a) \in S$ [$\because$ R and S are symmetric]
$\Rightarrow$ $(b,a) \in R \cap S$

Thus, $(a,b) \in R \cap S \Rightarrow (b,a) \in R \cap S$ for all $(a, b) \in R \cap S$

So, $R \cap S$ is symmetric on A.

Transitivity: Let $a, b, c \in A$ such that $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$. Then,

$(a, b) \in R \cap S$ and $(b, c) \in R \cap S$

$\Rightarrow$ $\{(a, b) \in R$ and $(a, b) \in S)\}$ and $\{(b, c) \in R$ and $(b, c) \in S)\}$

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$\Rightarrow$ $\{(a, b)\in R, (b, c)\in R\}$ and $\{(a, b)\in S, (b, c)\in S\}$

$\Rightarrow$ $(a, c)\in R$ and $(a, c)\in S$

$\Rightarrow$ $(a, c)\in R\cap S$

$\because$ R and S are transitive
$\therefore$ $(a,b)\in R$ and $(b, c)\in R \Rightarrow (a, c)\in R$
$(a,b)\in S$ and $(b, c)\in S \Rightarrow (a, c)\in S$

Thus, $(a,b)\in R\cap S$ and $(b, c)\in R\cap S \Rightarrow (a, c)\in R\cap S$

So, $R\cap S$ is transitive on A.

Hence, $R\cap S$ is an equivalence relation on A.

- **Theorem 2: The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.**

Proof: Let $A = \{a, b, c\}$ and let R and S be two relations on A, given by

R=$\{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ and, **S=** $\{(a, a), (b, b), (c, c), (b, c), (c, b)\}$

It can be easily seen that each one of R and S is an equivalence relation on A. But $R\cup S$ is not transitive, because $(a, b) \in R\cup S$ and $(b, c) \in R\cup S$ but $(a, c) \notin R\cup S$

Hence, $R\cup S$ is not an equivalence relation on A.

- **Theorem 3: If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.**

OR

The inverse of an equivalence relation is an equivalence relation.

Proof: Since R is a relation on A.

$\therefore$ $R\subseteq A\times A \Rightarrow R\text{-}1 \subseteq A\times A \Rightarrow R\text{-}1$ is also a relation on A.
Now, we shall show that R-1 is an equivalence relation on A.
We observe the following properties of relation R^{1}.

Reflexivity: Let a be an arbitrary element of A. Then,

$a \in A$

$\Rightarrow (a, a) \in R$ [∵ R is reflexive]
$\Rightarrow (a, a) \in R^{-1}$ [By definition of R-1]

Thus, $(a, a) \in R^{-1}$ for all $a \in A$.

So, R-1 is reflexive on A.

Symmetry: Let $(a, b) \in R^{-1}$. Then,

$(a, b) \in R^{-1}$

$\Rightarrow ((b, a) \in R$ [By definition of R-1]
$\Rightarrow ((a, b) \in R$ [∵ R is symmetric]
$\Rightarrow ((b, a) \in R^{-1}$ [By definition of R-1]

Thus, $(a, b) \in R^{-1} \Rightarrow (b, a) \in R^{-1}$ for all $a, b \in A$.

So, R^1 is symmetric on A.

- **Transitivity:** Let $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. Then,

$(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$

$\Rightarrow (b, a) \in R$ and $(c, b) \in R$ [By definition of R-1]
$\Rightarrow (c, b) \in R$ and $(b, a) \in R$
$\Rightarrow (c, a) \in R$ [∵ R is transitive]
$\Rightarrow (a, c) \in R^{-1}$ [By definition of R-1]

Thus, $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1} \Rightarrow (a, c) \in R^{-1}$ for all $a, b, c \in A$.

Hence, R-1 is an equivalence relation on A.

❖ EXERCISES:

1. Show that the relation R defined by R= {(a, b): a-b is divisible by 3; a, b∈Z} is an equivalence relation.

2. Show that the relation R on the set Z of integers, given by
   R={(a, b): 2 divides a - b}, is an equivalence relation.

3. Let R be the relation defined on the set A= {1, 2, 3, 4, 5, 6, 7} by R = {(a, b): both a and b are either odd or even}. Show that R is an equivalence relation. Further, show that all the elements of the subset {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of the subset {1,3,5,7} is related to any element of the subset {2, 4, 6}.

4. If R and S are relations on a set A, then prove the following:

   (i) R and S are symmetric⟹ R∪S and R∩S are symmetric

   (ii) R is reflexive and S is any relation⇒ R∪S is reflexive.

5. If R and S are transitive relations on a set A, then prove that R∪S may not be a transitive relation on A.