SUNSTONE

Bachelor of Computer Application

Computational Mathematics

LECTURE 16 NOTES

Matrices - Matrix

Introduction

- The matrix has a long history of application in solving linear equations. It was known as arrays until the 1800s. The term ‘matrix’ (Latin for ‘womb’, derived from mater — mother) was coined by James Joseph Sylvester in 1850, who understood a matrix as an object which gives rise to a number of determinants, known today as minors. Which means determinants of smaller matrices that are derived from the original one by removing columns and rows.

- An English mathematician named C.E. Cullis was the first to use modern bracket notation for matrices in 1913 and he simultaneously demonstrated the first significant use of the notation A=aij to represent a matrix where aij refers to the element found in the ith row and the jth column.

- Matrices can be used to write compactly and at the same time work with multiple linear equations, referred to as a system of linear equations. Matrices and matrix multiplication reveal their essential features when related to linear transformations, also known as linear maps.

What is a Matrix?

- A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

- **Example:**
  Suppose we want to say that Radha has 15 notebooks. We may say it as [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has. Now, if we have to say that Radha has 15 notebooks and six pens, we may write it as [15 6] with the understanding that the first number inside [ ] is the number of notebooks while the other one is the number of pens owned by Radha.

  Let us now suppose that we want to write about the notebooks and pens owned by Radha and her two friends Fauzia and Simran. We will write it as follows:

Radha	has	15	notebooks	and	6	pens
Fauzia	has	10	notebooks	and	2	pens
Simran	has	13	notebooks	and	5	pens

  But we can make it simpler and arrange it in a tabular form as follows:

	Notebooks	Pens
Radha	15	6
Fauzia	10	2
Simran	13	5

and this can be expressec

$$\begin{array}{ccl} \left[\begin{matrix}15 & 6\\ 10 & 2\\ 13 & 5\end{matrix}\right] & & \begin{matrix}\text{First row}\\ \text{Second row}\\ \text{Third row}\end{matrix}\\ \begin{matrix}\text{First Column} & \text{Second Column}\end{matrix} & & \end{array}$$

or

	Radha	Fauzia	Simran
Notebooks	15	10	13
Pens	6	2	5

which can be expressed ε

$$\begin{array}{ccl} \left[\begin{matrix}15 & 10 & 13\\ 6 & 2 & 5\end{matrix}\right] & & \begin{matrix}\text{First row}\\ \text{Second row}\end{matrix}\\ \begin{matrix}\text{First Column} & \text{Second Column} & \text{Third Column}\end{matrix} & & \end{array}$$

- **Order of a Matrix**
  - A matrix having m rows and n columns is called a matrix of order m × n or simply m × n matrix (read as an m by n matrix). So referring to the above examples of matrices, we have A as 3 × 2 matrix, B as 3 × 3 matrix and C as 2 × 3 matrix. We observe that A has 3 × 2 = 6 elements, B and C have 9 and 6 elements, respectively.
    In general, an m × n matrix has the following rectangular array:

$$\begin{bmatrix} A11 & A12 & A13\ldots & A1j\ldots & A1n\ldots \\ A21 & A22 & A23\ldots & A2j\ldots & A2n\ldots \\ Ai1 & Ai2 & Ai3\ldots & Aij\ldots & Ain\ldots \\ Am1 & Am2 & Am3\ldots & Amj\ldots & Amn\ldots \end{bmatrix}_{m \times n}$$

or $A = [A_{ij}]_{m \times n}$, $1 \le i \le m$, $1 \le j \le n$ $i, j \in N$

Thus the i^{th} row consists of the elements $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$,

while the j^{th} column consists of the elements $a_{1j}, a_{2j}, a_{3j}, \ldots, a_{mj}$,

In general, aij, is an element lying in the i^{th} row and j^{th} column. We can also call it as the
$(i, j)^{th}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or [x, y]).

For example point P(0, 1) as a matrix representation may be given as P= $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or [0,1]

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix} \\ \begin{array}{cccc} A & B & C & D \end{array} \end{array} \quad \text{OR} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}$$

- **Remarks:**
  We shall follow the notation, namely A = [aij]m × n to indicate that A is a matrix of order
  m × n.

  We shall consider only those matrices whose elements are real numbers or functions which take real values.

- **Types of Matrices**

1. **Column Matrix:**
   A matrix is said to be a column matrix if it has only one column.

   For example, A= $\begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ \frac{1}{2} \end{bmatrix}$ is a column matrix of order 4 × 1.

In general, A = $[a_{ij}]_{m*1}$ is a column matrix of order m × 1.

2. **Row Matrix:**

A matrix is said to be a row matrix if it has only one row.

For example, B= $\begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}$ is a row matrix.

In general, B = $[b_{ij}]_{1*n}$ is a row matrix of order 1 × n.

3. **Square Matrix:**

A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix. Thus, an m × n matrix is said to be a square matrix if m = n and is known as a square matrix of order 'n'.

For example, $\begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

In general, A = $[a_{ij}]_{m*m}$ is a square matrix of order m.

4. **Diagonal Matrix:**

A square matrix B = $[b_{ij}]_{m*m}$ is said to be a diagonal matrix if all its non diagonal elements are zero, that is a matrix B = $[b_{ij}]_{m*m}$ is said to be a diagonal matrix if b_{ij} = 0, when i ≠ j.

For example, A = [4], B= $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ C= $\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

are diagonal matrices of order 1, 2, 3, respectively.

5. **Scalar Matrix:**

A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is, a square matrix B = $[b_{ij}]$n × n is said to be a scalar matrix if

b_{ij} = 0, when i ≠ j

b_{ij} = k, when i = j, for some constant k.

For example,
A = [3], B = $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ C = $\begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

are scalar matrices of order 1, 2 and 3, respectively.

6. **Identity Matrix:**

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix. In other words, the square matrix A = [aij]n*n is an identity matrix,

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We denote the identity matrix of order n by In. When order is clear from the context, we simply write it as **I**.

For example [1], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are identity matrices of order 1, 2 and 3, respectively.

Observe that a scalar matrix is an identity matrix when k = 1. But every identity matrix is clearly a scalar matrix.

7. **Zero Matrix:**

A matrix is said to be zero matrix or null matrix if all its elements are zero.

For example, [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices.

We denote zero matrix by **O**. Its order will be clear from the context.

- **Equality of Matrices**

  - Two matrices A = [aij] and B = [bij] are said to be equal if

    (i) They are of the same order

    (ii) Each element of A is equal to the corresponding element of B, that is aij = bij for all i & j.

- **For example** $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices

  But $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices.

  Symbolically, if two matrices A and B are equal, we write A = B.

- If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then x = –1.5, y = 0, z = 2, a = 6 , b = 3, c = 2.

❖ **Examples:**

1. **A=** $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$ is a matrix having 2 rows and 3 columns and so it is a matrix of order 2×3 such that $a_{11}=2$, $a_{12}=1$, $a_{13}=-1$, $a_{21}=1$, $a_{22}=3$, $a_{23}=2$.

2. **B=** $\begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$ is a matrix having 2 rows and 2 columns and so it is a matrix of order 2×2 such that $b_{11}=\sin x$, $b_{12}=\cos x$, $b_{21}=\cos x$, $b_{22}=-\sin x$.

3. Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	Women workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3 × 2 matrix. What does the entry in the third row and second column represent?

- **Solution:**
The information is represented in the form of a 3 × 2 matrix as follows:

$$A=\begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

4. If a matrix has eight elements, what are the possible orders it can have?

- **Solution:**
We know that if a matrix is of order m × n, it has mn elements. Thus, to find all possible orders of a matrix with eight elements, we will find all ordered pairs of natural numbers, whose product is eight.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1 × 8, 8 ×1, 4 × 2, 2 × 4

5. Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix}=\begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

- **Solution:**
By equality of two matrices, equating the corresponding elements, we get

$2a + b = 4$ $\quad$ $5c - d = 11$
$a - 2b = -3$ $\quad$ $4c + 3d = 24$

Solving these equations, we get

a = 1, b = 2, c = 3 and d = 4

❖ **Exercise:**

1. If a matrix has 24 elements, what are the possible orders it can have? What if it has 13 elements?

2. Construct a 3 × 4 matrix, whose elements are given by:

(i) $a_{ij} = \frac{1}{2}\left|-3i + j\right|$ (ii) $a_{ij} = 2i - j$

3. Find the values of x, y and z from the following equations:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$