Bachelor of Computer Application

Computational Mathematics

LECTURE 8 NOTES

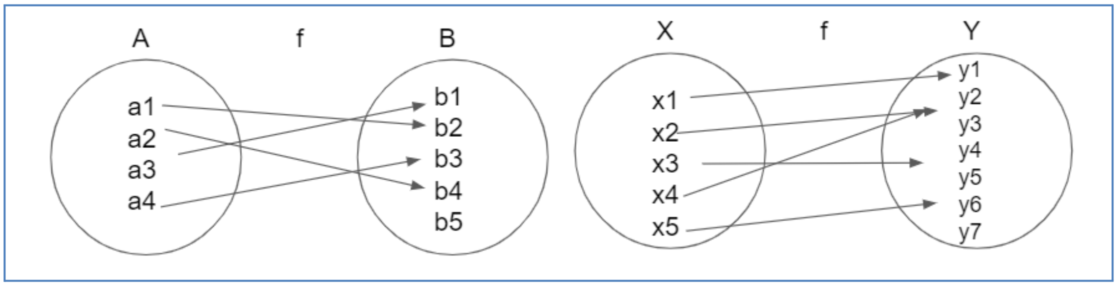
Functions-Types of Functions

- ## Types of Functions

  - ### One- one function (injection):

    A function f : X → Y is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every x_1 , $x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

    Example: let f: A→B and g: X→Y be two functions represented by the following diagrams.



    Clearly, f: A→B is a one-one function. But g:X→Y is not one-one because two distinct elements x_1 and x_3 have the same image under function g.

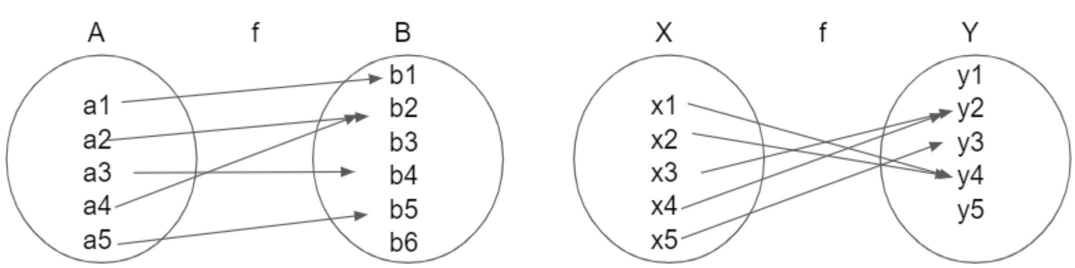
  - ### Many- one function:

    A function f : A→B is said to be many-one function if two or more elements of set A have the same image in B.

    Thus, f:A→B is a many-one function if there exists x,y∈A such that x≠y but f(x)=f(y).

    In other words, f:A→B is a many-one function if it is not a one-one function.

    Example: Let f: A→B and g: X→Y be two functions represented by the following diagrams.



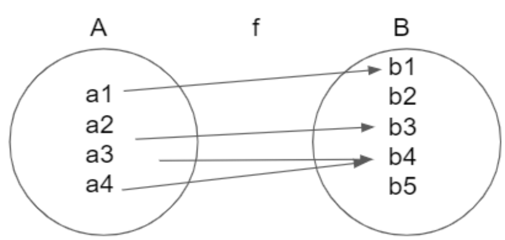
    Clearly $a_2 \neq a_4$ but $f(a_2)= f(a_4)$ and $x_1 \neq x_2$ but $g(x_1)=g(x_2)$. So, f and g are many-one functions.

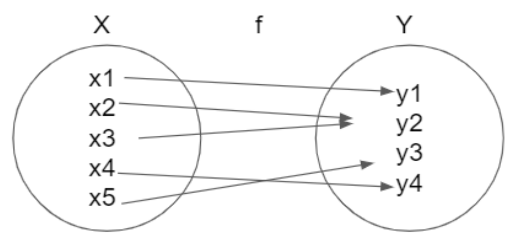
- **Onto function (surjection):**

A function f : X → Y is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every y ∈ Y, there exists an element x in X such that
f(x) = y.

Thus, f:A→B is a surjection iff for each b∈B, there exists a∈A such that f(a)=b.

Example: Let f: A→B and g: X→Y be two functions represented by the following diagrams.





Under function g every element in Y has its pre-image X. So, g: X → Y is an onto function.

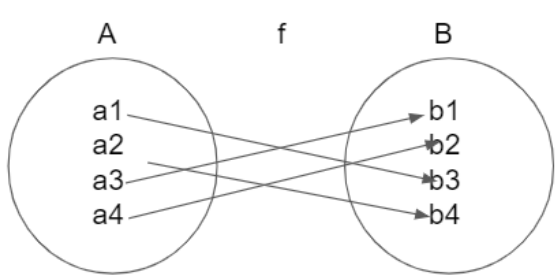
- **One-One Onto function (bijection):**

A function f:A→B is a bijection if it is one-one as well as onto.

In other words, a function f:A→B is a bijection , if it is

(i) one-one i.e., f(x)= f(y) ⟹x=y for all x,y∈A
(ii)onto i.e., for all y∈B, there exists x∈A such that f(x)=y

Example: Let f: A→B be a function represented by the following diagram:



Clearly, f is a bijection since it is both injective as well as surjective.

❖ EXAMPLES:

- **Example 1:**
  Let A be the set of all 50 students of Class X in a school. Let $f : A \to N$ be function defined by f(x) = roll number of the student x. Show that f is one-one but not onto.

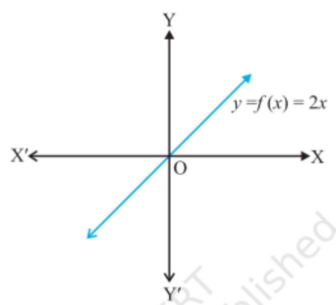
- **Solution:**
  No two different students of the class can have same roll number. Therefore, f must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in N is not roll number of any student of the class, so that 51 cannot be image of any element of X under f. Hence, f is not onto.

- **Example 2:**
  Show that the function $f : N \to N$, given by $f(x) = 2x$, is one-one but not onto.

- **Solution:**
  The function f is one-one, for $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
  Further, f is not onto, as for $1 \in N$, there does not exist any x in N such that $f(x) = 2x = 1$.

- **Example 3:**
  Prove that the function $f : R \to R$, given by $f(x) = 2x$, is one-one and onto.

- **Solution:**
  f is one-one, as $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
  Also, given any real number y in R, there exists y/2 in R such that $f(y/2) = 2 . (y/2) = y$.
  Hence, f is onto.



- **Example 4:**
  Show that the function $f : N \to N$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$, is onto but not One-one.

- **Solution:**
  f is not one-one, as $f(1) = f(2) = 1$. But f is onto, as given any $y \in N$, $y \neq 1$,
  we can choose x as $y + 1$ such that $f(y + 1) = y + 1 - 1 = y$. Also for $1 \in N$, we have $f(1) = 1$.

- **Composition of Functions and Invertible Function**

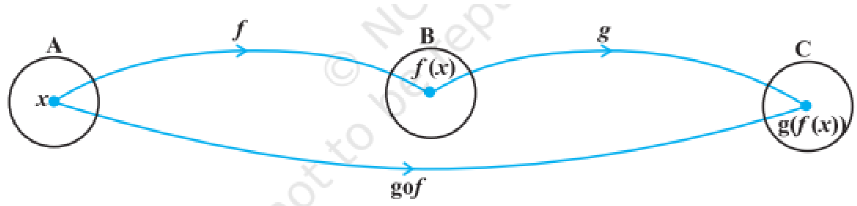
❖ **Composition of a Function:**

- Consider the set A of all students, who appeared in Class X of a Board Examination in 2006. Each student appearing in the Board Examination is assigned a roll number by the Board which is written by the students in the answer script at the time of examination. In order to have confidentiality, the Board arranges to deface the roll numbers of students in the answer scripts and assigns a fake code number to each roll number. Let $B \subset N$ be the set of all roll numbers and $C \subset N$ be the set of all code numbers. This gives rise to two functions $f : A \to B$ and $g : B \to C$ given by f(a) = the roll number assigned to the student a and g(b) = the code number assigned to the roll number b. In this process each student is assigned a roll number through the function f and each roll number is assigned a code number through the function g. Thus, by the combination of these two functions, each student is eventually attached a code number.

  This leads to the following definition:

  Let $f : A \to B$ and $g : B \to C$ be two functions.
  Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$\mathbf{gof\,(x) = g(f\,(x)),\ \forall\, x \in A.}$$



❖ **Examples:**

- **Example 1:**

  Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as f(2) = 3, f(3) = 4, f(4) = f(5) = 5 and g (3) = g (4) = 7 and g (5) = g (9) = 11. Find gof.

- **Solution:**

  We have gof(2) = g (f(2)) = g (3) = 7, gof (3) = g (f(3)) = g (4) = 7,

  gof(4) = g (f(4)) = g (5) = 11 and gof(5) = g (5) = 11.

- **Example 2:**

  Find gof and fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that gof ≠ fog.

- **Solution:**

  We have $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$.

  Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, gof ≠ fog.

- **Example 3:**

  Example 18 Show that if $f : A \to B$ and $g : B \to C$ are one-one, then $gof : A \to C$ is also one-one.

- **Solution:**

  Suppose $gof(x_1) = gof(x_2)$

  $\Rightarrow \quad g(f(x_1)) = g(f(x_2))$
  $\Rightarrow \quad f(x_1) = f(x_2)$, as g is one-one
  $\Rightarrow \quad x_1 = x_2$, as f is one-one

  Hence, gof is one-one.

❖ Invertible function:

- A function $f : X \to Y$ is defined to be invertible, if there exists a function

  $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$.

  The function g is called the inverse of f and is denoted by f^{-1}.

- Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, especially when the actual inverse of f is not to be determined.

❖ EXAMPLES:

- **Example 1:**

  Let $f : N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where,
  $Y = \{y \in N: y = 4x + 3 \text{ for some } x \in N\}$. Show that f is invertible. Find the inverse.

- **Solution:**

  Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$,

  for some x in the domain N. This shows that $x=(y-3)/4$.

  Define $g : Y \rightarrow N$ by $g(y)=(y-3)/4$. Now, $gof(x) = g (f(x)) = g (4x + 3) = (4x+3-3)/4 = x$

  and $fog (y) = f(g (y)) = f((y-3)/4) =(4(y-3)/4)+3 = y-3+3= y$.

  This shows that $gof = I_N$ and $fog = I_Y$, which implies that f is invertible, & g is the inverse of f.

- **Example 2:**

  Consider $f : N \rightarrow N$, $g : N \rightarrow N$ and $h : N \rightarrow R$ defined as $f(x) = 2x$, $g (y) = 3y + 4$ and $h (z) = \sin z$, $\forall$ x, y and z in N. Show that $ho(gof) = (hog) of$.

- **Solution:**

  We have, $ho(gof) (x) = h(gof (x)) = h(g(f(x))) = h(g (2x))$

  $= h(3(2x) + 4) = h(6x + 4) = \sin (6x + 4) \ \forall \in x \ N.$

  Also, $((hog) o f) (x) = (hog) (f(x)) = (hog) (2x) = h (g (2x))$

  $= h(3(2x) + 4) = h(6x + 4) = \sin (6x + 4), \forall x \in N.$

  This shows that $ho(gof) = (hog)of$.

  This result is true in general situation as well.

❖ SOME IMPORTANT THEOREMS

1. **THEOREM 1:** If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow S$ are functions, then

   ho(gof) = (hog)of.

- **Proof:**
  We have $ho(gof)(x) = h(gof(x)) = h(g(f(x)))$, $\forall$ x in X

  and $(hog) of (x) = hog(f(x)) = h(g(f(x)))$, $\forall$ x in X.

  Hence, $ho(gof) = (hog) o f$

- **Example:**
  Consider $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ and $g : \{a, b, c\} \rightarrow \{apple, ball, cat\}$ defined as
  $f(1) = a, f(2) = b, f(3) = c, g(a) = apple, g(b) = ball$ and $g(c) = cat$.

  Show that f, g and gof are invertible.
  Find out f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1} = f^{-1}og^{-1}$.

- **Solution:** Note that by definition, f and g are bijective functions. Let
  $f^{-1}: \{a, b, c\} \rightarrow (1, 2, 3\}$ and $g^{-1} : \{apple, ball, cat\} \rightarrow \{a, b, c\}$ be defined as

  $f^{-1}\{a\} = 1, f^{-1}\{b\} = 2, f^{-1}\{c\} = 3, g^{-1}\{apple\} = a, g^{-1}\{ball\} = b$ and $g^{-1}\{cat\} = c$.

  It is easy to verify that $f^{-1}o f = I_{\{1, 2, 3\}}$, $f o f^{-1} = I_{\{a, b, c\}}$, $g^{-1}og = I_{\{a, b, c\}}$ and $go g^{-1} = I_D$,

  where, D = {apple, ball, cat}. Now, $gof : \{1, 2, 3\} \rightarrow \{apple, ball, cat\}$ is given by gof(1) = apple, gof(2) = ball, gof(3) = cat. We can define

  $(gof)^{-1} : \{apple, ball, cat\} \rightarrow \{1, 2, 3\}$ by $(gof)^{-1}(apple) = 1, (gof)^{-1}(ball) = 2$ and

  $(gof)^{-1}(cat) = 3$. It is easy to see that $(g o f)^{-1} o (g o f) = I_{\{1, 2, 3\}}$ and $(gof) o (gof)^{-1} = I_D$.

  Thus, we have seen that f, g and gof are invertible.

  Now, $f^{-1}og^{-1}(apple)= f^{-1}(g^{-1}(apple)) = f^{-1}(a) = 1 = (gof)^{-1}(apple)f^{-1}og^{-1}(ball) = f^{-1}(g^{-}1(ball)) = f^{-1}(b) = 2 = (gof)^{-1}(ball)$ and

  $f^{-1}og^{-1}(cat) = f^{-1}(g^{-1}(cat)) = f^{-1}(c) = 3 = (gof)^{-1}(cat)$.

  Hence $(gof)^{-1} = f^{-1}og^{-1}$.
  The above result is true in general situation also

2. **THEOREM 2:** Let $f:X\rightarrow Y$ and $g:Y \rightarrow Z$ be two invertible functions. Then gof is also

invertible with

$$(gof)^{-1} = f^{-1}og^{-1}$$

- **Proof:**

To show that gof is invertible with $(gof)^{-1} = f^{-1}og^{-1}$, it is enough to show that

$(f^{-1}og^{-1})o(gof) = I_X$ and $(gof)o(f^{-1}og^{-1}) = I_Z$.

Now, $(f^{-1}og^{-1})o(gof) = ((f^{-1}og^{-1})\ og)$ of, by Theorem 1

$= (f^{-1}o(g^{-1}og))$ of, by Theorem 1= $(f^{-1}$ o IY) of, by definition of g^{-1}

= IX.

Similarly, it can be shown that (gof) o $(f^{-1}og^{-1}) = I_Z$.

- **Example:**

Let S = {1, 2, 3}. Determine whether the functions f : S → S defined as below have inverses.
Find f^{-1}, if it exists.

(a) f = {(1, 1), (2, 2), (3, 3)}
(b) f = {(1, 2), (2, 1), (3, 1)}
(c) f = {(1, 3), (3, 2), (2, 1)}.

- **Solution:**

(a) It is easy to see that f is one-one and onto, so that f is invertible with the inverse f^{-1} of f given by

f^{-1} = {(1, 1), (2, 2), (3, 3)} = f

(b) Since f (2) = f (3) = 1, f is not one-one, so that f is not invertible.

(c) It is easy to see that f is one-one and onto, so that f is invertible with

f^{-1} = {(3, 1), (2, 3), (1, 2)}

❖ EXERCISES:

1. Let f : {1, 3, 4} → {1, 2, 5} and g : {1, 2, 5} → {1, 3} be given by
   f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Write down gof.

2. Find gof and fog, if
   (i) $f(x) = |x|$ and $g(x) = |5x - 2|$
   (ii) $f(x) = 8x^3$ and $g(x) = x^{1/3}$.

3. State with reason whether following functions have inverse
   (i) f : {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}
   (ii) g : {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}
   (iii) h : {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}

4. If f: R → R be given by $f(x) = (3-x^3)^{1/3}$, then fof(x) is
   (A) $x^{1/3}$ (B) x^3 (C) x (D) $(3 - x^3)$.

5. Let f: X → Y be an invertible function. Show that the inverse of f^{-1} is f, i.e.,$(f^{-1})^{-1} = f$.

- **Binary Operations**

  - **A binary operation ∗ on a set A is a function ∗ : A × A → A. We denote ∗ (a, b) by a ∗ b.**

  - **Example 1:**
    Show that addition, subtraction and multiplication are binary operations on R, but division is not a binary operation on R. Further, show that division is a binary operation on the set R∗ of nonzero real numbers.

  - **Solution:** + : R × R → R is given by
    (a, b) → a + b

    –: R × R → R is given by
    (a, b) → a – b

    × : R × R → R is given by
    (a, b) → ab

    Since '+', '–' and '×' are functions, they are binary operations on R.

    But ÷: R × R → R, given by (a, b) →a/b , is not a function and hence not a binary operation, as for b = 0,a/b is not defined.
    However, ÷ : R∗ × R∗ → R∗, given by (a, b) →a/b is a function and hence a binary operation on R∗.

  - **Example 2:**

Show that $*: R \times R \to R$ given by $(a, b) \to a + 4b^2$ is a binary operation.

- **Solution:**
  Since $*$ carries each pair (a, b) to a unique element $a + 4b^2$ in R, $*$ is a binary operation on R.

- **Example 3:**
  Let P be the set of all subsets of a given set X. Show that $\cup : P \times P \to P$ given by $(A, B) \to A \cup B$ and $\cap : P \times P \to P$ given by $(A, B) \to A \cap B$ are binary operations on the set P.

- **Solution:**
  Since union operation $\cup$ carries each pair (A, B) in $P \times P$ to a unique element $A \cup B$ in P, $\cup$ is binary operation on P. Similarly, the intersection operation $\cap$ carries each pair (A, B) in $P \times P$ to a unique element $A \cap B$ in P, $\cap$ is a binary operation on P.

- **Example 4:**
  Show that the $\vee : R \times R \to R$ given by $(a, b) \to \max \{a, b\}$ and the $\wedge : R \times R \to R$ given by $(a, b) \to \min \{a, b\}$ are binary operations.

- **Solution:**
  Since $\vee$ carries each pair (a, b) in $R \times R$ to a unique element namely maximum of a and b lying in R, $\vee$ is a binary operation. Using the similar argument, one can say that $\wedge$ is also a binary operation.

❖ PROPERTIES OF BINARY OPERATIONS:

1. Commutative:

A binary operation $*$ on the set X is called commutative, if $a * b = b * a$, for every $a, b \in X$.

- **Example 1:**
  Show that $+ : R \times R \to R$ and $\times : R \times R \to R$ are commutative binary operations, but $- : R \times R \to R$ and $\div : R* \times R* \to R*$ are not commutative.

- **Solution:** Since $a + b = b + a$ and $a \times b = b \times a$, $\forall\ a, b \in R$, '+' and '×' are commutative binary operation. However, '–' is not commutative, since $3 - 4 \neq 4 - 3$.

  Similarly, $3 \div 4 \neq 4 \div 3$ shows that '÷' is not commutative.

- **Example 2:**
  Show that $*: R \times R \to R$ defined by $a * b = a + 2b$ is not commutative.

- **Solution:** Since $3 * 4 = 3 + 8 = 11$ and $4 * 3 = 4 + 6 = 10$, showing that the operation $*$is not commutative.

2. **Associative:**

- If we want to associate three elements of a set X through a binary operation on X,we encounter a natural problem. The expression $a * b * c$ may be interpreted as $(a * b) * c$ or $a * (b * c)$ and these two expressions need not be same.

  For example, $(8 – 5) – 2 \neq 8 – (5 – 2)$. Therefore, association of three numbers 8, 5 and 3 through the binary operation 'subtraction' is meaningless, unless bracket is used. But in case of addition, $8 + 5 + 2$ has the same value whether we look at it as $(8 + 5) + 2$ or as $8 + (5 + 2)$. Thus, association of 3 or even more than 3 numbers through addition is meaningful without using bracket.
  This leads to the following:

- **Definition:**

  A binary operation $* : A \times A \to A$ is said to be associative if $(a * b) * c = a * (b * c)$, $\forall$ a, b, c, $\in$ A.

- **Example:**
  Show that addition and multiplication are associative binary operation on R. But subtraction is not associative on R. Division is not associative on $R*$.

- **Solution:** Addition and multiplication are associative,

  since $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$ $\forall$ a, b, c $\in$ R.
  However, subtraction and division are not associative,
  as $(8 – 5) – 3 \neq 8 – (5 – 3)$ and $(8 \div 5) \div 3 \neq 8 \div (5 \div 3)$.

- **Example 2:**
  Show that $* : R \times R \to R$ given by $a * b \to a + 2b$ is not associative.

- **Solution:** The operation $*$ is not associative,

  since $(8 * 5) * 3 = (8 + 10) * 3 = (8 + 10) + 6 = 24$,

  while $8 * (5 * 3) = 8 * (5 + 6) = 8 * 11 = 8 + 22 = 30$.

- Associative property of a binary operation is very important in the sense that with this property of a binary operation, we can write $a_1 * a_2 * ... * a_n$ which is not ambiguous.

- But in absence of this property, the expression $a_1 * a_2 * ... * a_n$ is ambiguous unless brackets are used. Recall that in the earlier classes brackets were used whenever subtraction or division operations or more than one operation occurred.

3. Identity:

- For the binary operation '+' on R, the interesting feature of the number zero is that a + 0 = a = 0 + a, i.e., any number remains unaltered by adding zero. But in case of multiplication, the number 1 plays this role, as $a \times 1 = a = 1 \times a$, ∀ a in R. This leads to the following definition:

- **Definition:**

  Given a binary operation $*$: $A \times A \to A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if

  $$\mathbf{a * e = a = e * a,\ \forall a \in A.}$$

- **Example 1:**
  Show that zero is the identity for addition on R and 1 is the identity for multiplication on R. But there is no identity element for the operations:

  $$R \times R \to R \text{ and } \div : R_* \times R_* \to R_*.$$

- **Solution:**
  $a + 0 = 0 + a = a$ and $a \times 1 = a = 1 \times a$, ∀ a ∈ R implies that 0 and 1 are identity elements for the operations '+' and '×' respectively. Further, there is no element e in R with $a - e = e - a$, ∀ a. Similarly, we cannot find any element e in R_* such that $a \div e = e \div a$, ∀ a in R_*. Hence, '–' and '÷' do not have identity element.

4. Inverse:

- Zero is identity for the addition operation on R but it is not identity for the addition operation on N, as $0 \notin N$. In fact the addition operation on N does not have any identity.

  One further notices that for the addition operation $+ : R \times R \to R$, given any a ∈ R, there exists $-a$ in R such that $a + (-a) = 0$ (identity for '+') $= (-a) + a$.

  Similarly, for the multiplication operation on R, given any $a \neq 0$ in R, we can choose 1/a in R such that $a \times 1/a = 1$(identity for '×') $= 1/a \times a$. This leads to the following definition:

- **Definition:**

  Given a binary operation $*$: $A \times A \to A$ with the identity element e in A, an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b

in A such that a ∗ b =e= b ∗ a and b is called the inverse of a and is denoted by a^{-1}.

- **Example 1:**
  Show that **– a** is the inverse of a for the addition operation '+' on R and 1 a is the inverse of
  a ≠ 0 for the multiplication operation '×' on R.

- **Solution:**

  As a + (–a) = a – a = 0 and (–a) + a = 0, – a is the inverse of a for addition. Similarly, for a ≠ 0, a × 1 a = 1 = 1 a × a implies that 1 a is the inverse of a for multiplication.

- **Example 2:**
  Show that – a is not the inverse of a ∈ N for the addition operation + on N and 1/a is not the inverse of a ∈ N for multiplication operation × on N, for a ≠ 1.

- **Solution:**

  Since – a ∉ N, – a cannot be inverse of a for addition operation on N, although – a satisfies a + (–a) = 0 = (–a) + a.

  Similarly, for a ≠ 1 in N, 1/a ∉ N, which implies that other than 1 no element of N has inverse for multiplication operation on N.

❖ EXERCISES:

1. Determine whether or not each of the definition of ∗ given below gives a binary operation. In the event that ∗ is not a binary operation, give justification for this.

   (i) On Z^+, define ∗ by a ∗ b = a – b

   (ii) On Z^+, define ∗ by a ∗ b = ab

   (iii) On R, define ∗ by a ∗ b = ab^2

   (iv) On Z^+, define ∗ by a ∗ b = | a – b |

   (v) On Z^+, define ∗ by a ∗ b = a

2. For each operation $*$ defined below, determine whether $*$ is binary, commutative or associative.

(i) On Z, define $a * b = a - b$

(ii) On Q, define $a * b = ab + 1$

(iii) On Q, define $a * b = ab/2$

(iv) On Z+, define $a * b = 2ab$

(v) On Z+ , define $a * b = ab$

(vi) On R – {–1}, define $a * b = a/(b+1)$

3. Let $*$ be the binary operation on N given by $a * b$ = L.C.M. of a and b. Find

(i) $5 * 7$, $20 * 16$

(ii) Is $*$ commutative?

(iii) Is $*$ associative?

(iv) Find the identity of $*$ in N

(v) Which elements of N are invertible for the operation $*$?

4. State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation $*$ on a set N, $a * a = a \ \forall a \in N$.

(ii) If $*$ is a commutative binary operation on N, then $a * (b * c) = (c * b) * a$

5. Consider a binary operation $*$ on N defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is $*$ both associative and commutative?

(B) Is $*$ commutative but not associative?

(C) Is $*$ associative but not commutative?

(D) Is $*$ neither commutative nor associative?

SUNSTONE