SUNSTONE

Bachelor of Computer Application

Computational Mathematics

LECTURE 24 NOTES

Solution of Simultaneous Linear Equations

- **Introduction**

  - Consider the following system of **'m'** linear equations in **'n'** unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots\ldots. + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots\ldots. + a_{2n}x_n &= b_2 \\
\ldots.. \quad \ldots.. \quad \ldots\ldots. \quad \ldots. \quad & \ldots. \qquad \ldots\ldots.(i) \\
\ldots.. \quad \ldots.. \quad \ldots\ldots. \quad \ldots. \quad & \ldots. \\
a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots. + a_{mn}x_n &= b_m
\end{aligned}$$

  - This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots. & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$or \quad AX = B, \text{ where } A = \begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots. & a_{mn} \end{bmatrix}_{m \times n}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$$

  - The m×n matrix A is called the coefficient matrix of the system of linear equations.

- **Example:**
Express the following system of simultaneous linear equations as a matrix equation:

$$2x + 3y - z = 1$$
$$x + y + 2z = 2$$
$$2x - y + z = 3$$

- **Solution:** We have,

$$2x + 3y - z = 1$$
$$x + y + 2z = 2$$
$$2x - y + z = 3$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

or $$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} and\ B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Consistent System:

A system of equations is said to be consistent if its solution (one or more) exists.

- **For example:** The system of linear equations
2x + 3y = 5
4x + 6y = 10

Is consistent, because x=1, y=1 and x=2, y=⅓ are solution of it.

Inconsistent system:

A system of equations is said to be inconsistent if its solution does not exist.

- **For example:** The system of linear equations
  $2x + 3y = 5$
  $4x + 6y = 9$

  Is inconsistent, because there is no set values of x,y which satisfy the two equations simultaneously.

Homogenous System:

A system of equations AX=B is called a homogenous system if B=0.

- **For Example:** The system of equations
  $2x+3y = 0$
  $3x-y = 0$

  Is a homogeneous system of linear equations.

Non-homogeneous System:

A system of equations AX=B is called a non-homogeneous system if $B\neq 0$.

- **For Example:** The system of equations
  $2x+3y = 1$
  $3x-y = 5$

  Is a non-homogeneous system of linear equations.

- **Matrix Method for The Solution of Non- Homogenous System**

- **THEOREM 1:**
  If A is non-singular matrix, then the system of equations given by AX= B has the unique solution given by $\mathbf{X = A^{-1}B}$

- **Proof:** We have, AX= B, where $|A| \neq 0$.

  Now, $|A| \neq 0$. So, A^{-1} exists.

  Pre-multiplying both sides of AX = B by A^{-1}, we get

  $A^{-1}(AX) = A^{-1}B$

  $\Rightarrow (A^{-1}A)X = A^{-1}B$

  $\Rightarrow IX = A^{-1}B$

  $\Rightarrow X = A^{-1}B$

  Thus, the system of equations AX=B has a solution given by $X = A^{-1}B$

- **Uniqueness:** If possible, let X_1 and X_2 be two solutions of AX = B. Then,

  AX1= B and AX2= B

  $\Rightarrow AX_1 = AX_2$

  $\Rightarrow A^{-1}(AX_1) = A^{-1}(AX_2)$

  $\Rightarrow (A^{-1}-A)X_1 = (A^{-1}-A)X_2$

  $\Rightarrow IX_1 = IX_2$

  $\Rightarrow X_1 = X_2$

  Hence, the given system of equations has the unique solution given by $X = A^{-1}B$.

- **When A is a singular matrix?**

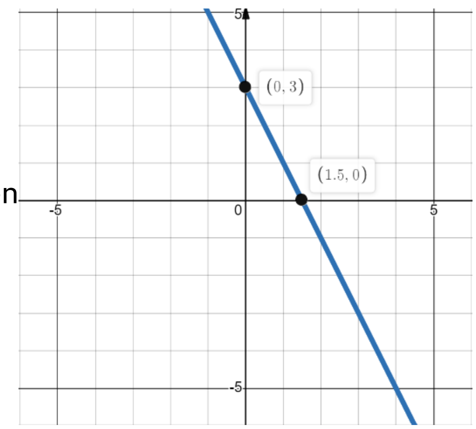
  Let us consider the following system of equations:

2x+y = 3
4x+2y = 6

This system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix},\ or,\ AX = B,\ where\ A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix},\ x = \begin{bmatrix} x \\ y \end{bmatrix}\ and\ B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

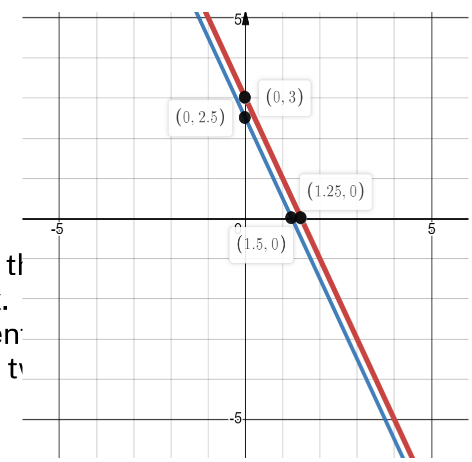
Clearly, |A| = 0. Also, the system of equations has finitely many solutions as the two equations represen coincident lines in xy-plane.



Now consider the following system of equations:

2x+ y = 3
4x+ 2y = 5

For this system of equations also the determinant of tl coefficient matrix A is zero i.e., A is a singular matrix. But the system has no solution i.e., it is an inconsisten system of equations, as the lines represented by the t equations are non-coincident parallel lines.



- **THEOREM 2 (Criterion of consistency):**

Let AX = B be a system of n-linear equations in n unknowns.

(i) If |A|≠0, then the system is consistent and has the unique solution given by X= A-1B.
(ii) If |A|=0 and (adj A)B=0, then the system is consistent and has finally many solutions.
(iii) If |A|=0 and (adj A)B≠0, then the system is inconsistent.

- **Solution:**

  (i) We have, $AX = B$, where $|A| \neq 0$.

  Now, $|A| \neq 0$. So, A^{-1} exists.

  Pre-multiplying both sides of $AX = B$ by A^{-1}, we get

  $A^{-1}(AX) = A^{-1}B$
  $\Rightarrow (A^{-1}A)X = A^{-1}B$
  $\Rightarrow IX = A^{-1}B$
  $\Rightarrow X = A^{-1}B$

  Thus, the system of equations $AX=B$ has a solution given by $X = A^{-1}B$

  (ii) We have,

  $AX = B$, where $|A|=0$

  $\Rightarrow$ (adj)A (AX) = (adj A) B

  $\Rightarrow$ ((adj)A A) X) = (adj A) B

  $\Rightarrow$ $(|A|\ I_n)\ X$ = (adj A) B [$\because$ (adj A)A= $|A|I_n$]

  $\Rightarrow$ $|A|\ X$ = (adj A) B

  If $|A|=0$ and (adj A) B = 0, then $|A|X$ = (adj A) B is true for every value of X.

  So, the system of equations $AX = B$ is consistent, and it has finitely many solutions.

  (iii) If $|A| = 0$ and (adj A) B$\neq$0, then the equation $|A|X$ = (adj A) B is not true because its LHS is always a null matrix whereas the RHS is non-null matrix.

  So, the system is inconsistent.

- **STEPS TO SOLVE A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS.**

  - **Step 1:** Obtain the system of equations and express it in the matrix equation form $AX=B$.

  - **Step 2:** Find $|A|$.

- **Step 3:** If $|A| \neq 0$, then the given system of equations is consistent with unique solution.

  To obtain the solution compute A^{-1} by using $A^{-1} = (1/|A|)\text{adj } A$ and use the formula $X = A^{-1}B$.

- **Step 4:** If $|A| = 0$, then the given system of equations is either inconsistent or it has infinitely

  many solutions. To distinguish these two proceed as follows:

  Compute (adj A)B.

  If (adj A)$B \neq 0$, then the given system of equations is inconsistent i.e., it has no solutions.

  If (adj A)$B = 0$, then the given system of equations is consistent with infinitely many solutions.

  In order to find these infinitely many solutions, replace one of the variables by some real number. This will reduce the number of variables by one. Now, take any two out of the three equations and solve them by matrix method.

TYPE 1: SOLVING THE GIVEN SYSTEM OF LINEAR EQUATIONS WHEN THE COEFFICIENT MATRIX IS NON-SINGULAR.

- **Example 1:**
  Use matrix method to solve the following system of equations:

  $5x-7y = 2,\ 7x-5y = 3.$

- **Solution:** The given system of equations can be written as

  $5x-7y = 2$
  $7x-5y = 3$

or, $$\begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or, $AX = B$, where $A = \begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Continued........

Now, $|A|= \begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}$ = -25+49= 24 ≠0

So, the given system has a unique solution given by X= A^{-1}B.

Let C_{ij} be the cofactors of elements a_{ij} in A= [a_{ij}]. Then,

$C_{11} = (-1)^{1+1} (-5) = -5$, $C_{12} = (-1)^{1+2}\, 7 = -7$ $C_{21} = (-1)^{2+1} (-7) = 7$ and $C_{22} = (-1)^{2+2}$ 5 = 5.

∴ adj A = $\begin{bmatrix} 5 & -7 \\ 7 & -5 \end{bmatrix}^T = \begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}$

So, $A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{24}\begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}$

∴ $X = A^{-1}B = \frac{1}{24}\begin{bmatrix} -5 & 7 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} -10+21 \\ -14+15 \end{bmatrix} = \begin{bmatrix} \frac{11}{24} \\ \frac{1}{24} \end{bmatrix}$

⇒ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{11}{24} \\ \frac{1}{24} \end{bmatrix}$

⇒ $x = \frac{11}{24}$ and $y = \frac{1}{24}$

Hence, x= 11/24 and y= 1/24 is the required solution.

- **Example 2:** Solve the following system of equations, using matrix method:

  x+2y+z = 7, **x+3z = 11,** **2x-3y = 1**

- **Solution:** The given system of equations is

x+2y+z = 7
x+0y+3z = 11
2x-3y+0z = 1

$$or,\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$or,\ \ AX = B\ ,\ where\ A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ x \end{bmatrix}\ and\ B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$Now,\ |A| = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = 1(0+9) - 2(0-6) + 1(-3-0)\ = 9+12-3\ = 18 \neq 0$$

$So,\ the\ given\ system\ of\ equations\ has\ a\ unique\ solution\ given\ by\ X = A^{-1}B.$

$Let\ C_{ij}\ be\ the\ co-factors\ of\ elements\ a_{ij}\ in\ A = [a_{ij}].\ Then,$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 9, \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -3 \qquad C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$$

$$and\ C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$\therefore\ \ adj\ A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\Rightarrow\ \ A^{-1} = \frac{1}{|A|}adj\ A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Continued........

$Now,\ \ X = A^{-1}B$

$$\Rightarrow\ \ X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 & -33 & 6 \\ 42 & -22 & -2 \\ -21 & 77 & -2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow\ \ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\ \ \Rightarrow\ \ x = 2,\ y = 1\ and\ z = 3$$

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TYPE 2: SOLVING THE GIVEN SYSTEM OF EQUATIONS WHEN THE COEFFICIENT MATRIX IS SINGULAR.

- **Example 1:** Use matrix method to examine the following system of equations for consistency or inconsistency:

  4x-2y = 3, 6x-3y = 5.

- **Solution:** The given system of equations can be written as

$$AX = B, \text{ where } A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

$$Now, \ |A| = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix} = -12 + 12 = 0$$

So, the given system of equations is inconsistent or it has infinitely many solutions according as (adj A)B≠ 0 or, (adj A) B = 0 respectively.

Let C_{ij} be the co-factors of elements a_{ij} in A= [a_{ij}]. Then,

$$C_{11} = (-1)^{1+1}(-3) = \qquad C_{12} = (-1)^{1+2} 6 = -6 \qquad C_{21} = (-1)^{2+1}(-2) = 2$$

Continued........

$$and, \ C_{22} = (-1)^{2+2}(4) = 4$$

$$\therefore \quad adj \ A = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

$$So, \quad (adj \ A) B = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 & +10 \\ -18 & +20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 0$$

- **Example 2:** Show that the following system of equations is consistent.

2x-y+3z = 5, 3x+2y-z = 7, 4x+5y-5z = 9

- **Solution:**

The given system of equation can be written in matrix form as

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$or,\ AX = B,\ where\ A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\ and\ B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$Now,\ |A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{vmatrix} = 2(-10+5) + 1(-15+4) + 3(15-8) = 0$$

So, A is singular. Thus, the given system of equations is either either inconsistent or

it is consistent with infinitely many solutions according as $(adj\ A)B \neq 0$ *or,* $(adj\ A)B = 0$ *respectively.*

Let C_{ij} *be the co-factors of elements* a_{ij} *in* $A = [a_{ij}]$*. Then,*

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} = -5 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ 4 & -5 \end{vmatrix} = 11$$

Continued........

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 7 \qquad C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 3 \\ 5 & -5 \end{vmatrix} = 10$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -22 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} = -14$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 3 \end{vmatrix} = -5 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \end{vmatrix} = 11$$

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Continued........

$$Now,\ |A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0 \text{ and } adj\ A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$Now,\ X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 & -3k \\ 7 & +k \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 10 - 6k + 7 + k \\ -15 + 9k + 2k + 14 \end{bmatrix} = \begin{bmatrix} \dfrac{17-5k}{7} \\ \dfrac{11k-1}{7} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{17-5k}{\ }, \ y = \frac{11k-1}{\ }$$

❖ EXERCISE:

1:

$x + y + z = 1$
$2x + 3y + 2z = 2$
$ax + ay + 2az = 4$

2:

$3x - y - 2z = 2$
$2y - z = -1$
$3x - 5y = 3$

3:

$5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$

4:

$2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$

5:

$2x + y + z = 1$
$x - 2y - z = 3/2$
$3y - 5z = 9$

6: The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

7: The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.