SUNSTONE

Bachelor of Computer Application

Computational Mathematics

LECTURE 23 NOTES

Adjoint of a Matrix

Adjoint of a Matrix

- The adjoint of a square matrix A = $[a_{ij}]$ n × n is defined as the transpose of the matrix $[A_{ij}]$ n × n, where A_{ij} is the cofactor of the element a_{ij} . Adjoint of the matrix A is denoted by adj A.

$$Let \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Then \qquad adj\ A = Transpose\ of \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

- **Example:**

  Find adj A for A= $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

- **Solution:**

  We have $A_{11} = 4$, $A_{12} = -1$, $A_{21} = -3$, $A_{22} = 2$

- Hence, $adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

THEOREM 1:

If A be any given square matrix of order n, then

A (adj A) = (adj A) A = A I ,

where I is the identity matrix of order n.

- **Verification:**

$$Let \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to | A| and otherwise zero, we have

$$A\ (adj\ A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I$$

Similarly, we can show (adj A) A = A I

Hence A (adj A) = (adj A) A = A I

- **Singular Matrix**

Definition: A square matrix A is said to be singular if |A| = 0.

For example, the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero.

Hence A is a singular matrix.

- **Non-Singular Matrix**

Definition: A square matrix A is said to be non-singular if A ≠ 0

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then |A| = $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ = 4 – 6 = –2 ≠ 0.

Hence A is a non-singular matrix

Theorem 2:

- If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

THEOREM 3:

The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A|\,|B|$, where A and B are square matrices of the same order

Remark We know that (adj A) A = $|A|I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$,$|A| \neq 0$

Writing determinants of matrices on both sides, we have

$$|(\text{adj } A)\, A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

i.e., $$|(\text{adj } A)|\, |A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

i.e., $|(\text{adj } A)|\, |A| = |A|^3$ (1)

i.e., $|(\text{adj } A)| = |A|^2$

In general, if A is a square matrix of order n, then $|\text{adj}(A)| = |A|^{n-1}$

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THEOREM 4:

A square matrix A is invertible if and only if A is non-singular matrix.

- **Proof:**

Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix B of order n such that AB = BA = I

Now AB = I. So |AB| = |I| or |A| |B| = 1 (since |I|= 1, |AB|= |A| |B|)

This gives |A| ≠ 0. Hence A is non-singular.

Conversely, let A be non-singular. Then |A| ≠ 0

Now A (adj A) = (adj A) A = |A| I (Theorem 1)

or $$A\left(\frac{1}{|A|}adj\ A\right) = \left(\frac{1}{|A|}adj\ A\right)A = I$$

or AB = BA = I, where B = $\frac{I}{|A|}adj\ A$

Thus, A is invertible and $A^{-1} = \frac{I}{|A|}adj\ A$

❖ EXAMPLES:

- **Example 1:**

If A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A adj A = | A| I. Also find A^{-1}.

- **Solution:**

We have |A| = 1 (16 – 9) –3 (4 – 3) + 3 (3 – 4) = 1 ≠ 0

Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

$$Therefore \qquad adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$Now \qquad A\ (adj\ A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0=4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.I$$

$$Also \qquad A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- **Example 2:**

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$

- **Solution:**

We have $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since $|AB| = -11 \neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|} adj\,(AB) = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, \; B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$Therefore \quad B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$

- **Example 3:**

Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using this equation, find A^{-1}.

- **Solution:**

We have, $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $AA - 4A = -I$

or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$ $\quad$ (*Post multiplying by* A^{-1} *because* $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

❖ EXERCISE:

1: Find adjoint of each of the matrices

$(i)\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad$ $(ii)\ \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

2: Verify A (adj A) = (adj A) A = | A| I

$(i)\ \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ $\qquad$ $(ii)\ \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

3: For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$

4: For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$., Show that A3–6A2 + 5A + 11 I = O. Hence, find A^{-1}.

5: If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$., verify that A3 – 6A2 + 9A – 4I = O and hence find A^{-1}.

6: Let A be a non-singular square matrix of order 3 × 3. Then |adj A| is equal to

(A) | A | $\qquad$ **(B)** $| A |^2$ $\qquad$ **(C)** $| A |^3$ $\qquad$ **(D)** 3|A|

7: If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det (A) $\qquad$ **(B)** 1 / det (A) $\qquad$ **(C)** 1 $\qquad$ **(D)** 0