SUNSTONE

Bachelor of Computer Application

Computational Mathematics

LECTURE 17 NOTES

Operations on Matrices

- ## Addition of Matrices
  - Matrix addition explains the addition of two or more matrices. Unlike arithmetic addition of numbers, matrix addition will follow different rules. The order of matrices should be the same, before adding them.

$$\mathbf{A + B = [aij]_{m*n} + [bij]_{m*n} = [aij + bij]_{m*n}}$$

  - **Example:** Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find A + B.

  - **Solution:** Since A, B are of the same order 2 × 3. Therefore, addition of A and B is defined

    and is given by, $A + B = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 1-1 \\ 2-2 & 3+3 & 0+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$

❖ Properties of Matrix Addition:

- The addition of matrices satisfies the following properties:

(i) **Commutative Law**

If A = $[a_{ij}]$, B = $[b_{ij}]$ are matrices of the same order, say m × n, then A + B = B + A.

Now A + B = $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

= $[b_{ij} + a_{ij}]$ (addition of numbers is commutative) = $([b_{ij}] + [a_{ij}])$ = B + A

(ii) **Associative Law**

For any three matrices A = $[a_{ij}]$, B = $[b_{ij}]$, C = $[c_{ij}]$ of the same order, say m × n, (A + B) + C = A + (B + C).

Now (A + B) + C = $([a_{ij}] + [b_{ij}]) + [c_{ij}]$

= $[a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}] = [a_{ij} + (b_{ij} + c_{ij})]$

= $[a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}])$ = A + (B + C)

(iii) **The Existence of Additive Identity**

Let A = $[a_{ij}]$ be an m × n matrix and O be an m × n zero matrix, then A + O = O + A = A.

In other words, O is the additive identity for matrix addition.

(iv) **The Existence of Additive Inverse**

Let A = $[a_{ij}]_{m*n}$ be any matrix, then we have another matrix as – A = $[-a_{ij}]_{m*n}$

such that A + (– A) = (– A) + A= O.

So, A is the additive inverse of A or negative of A.

- **Example1:** Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ $and\ X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

- **Solution:** We have $(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or (X + X) + (Y – Y) = $\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$ ⇒2X= $\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

or X = $\frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$ = $\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also (X + Y) – (X – Y) = $\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or (X – X) + (Y + Y) = $\begin{bmatrix} 5-3 & 2-6 \\ 0-0 & 9+1 \end{bmatrix}$

or Y = $\frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$ = $\begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

- **Example 2:** Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

- **Solution:** $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$or \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Or $2x + 3 = 7$ and $2y - 4 = 14$

or $2x = 7 - 3$ and $2y = 18$

Or $x =$ and $y = \frac{18}{2}$ i.e. $x = 2$ and $y = 9$.

- **Difference of Matrices**

  - If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference $A - B$ is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j.

  In other words, $D = A - B = A + (-1) B$, that is sum of the matrix A and the matrix $-B$.

- **Example:** If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find $2A - B$.

- **Solution:** $A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

- **Scalar Multiplication of a Matrix**

❖ Properties of Scalar Multiplication of a Matrix

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say m × n, and k and l are scalars, then

(i) **k(A +B) = k A + kB**

▪ **Proof:**
Since A and B are matrices of the same order mxn, A+B is also a matrix of order mx n. Therefore, k (A + B) is also of order mx n. Further, kA and kB are of order mxn. Therefore, kA+ kB is also of order mxn. Thus, k (A + B) and kA + kB are matrices of the same order such that

$(k\ (A+B))_{ij} = k(A+B)_{ij}$ [By definition of scalar multiplication]

$(k\ (A+B))_{ij} = k(a_{ij} + b_{ij})$ [By definition of addition of matrices]

$(k\ (A+B))_{ij} = ka_{ij} + kb_{ij}$ [By distributivity of multiplication over addition]

$(k\ (A+B))_{ij} = (kA)_{ij} + (kB)_{ij}$ [By definition of scalar multiplication]

=> $(k\ (A+B))_{ij} = (kA+kB)_{ij}$ [By definition of matrix addition]

$(k\ (A+B))_{ij} = (kA + kB)_{ij}$ for i=1, 2,..., m and j=1, 2,..., 1

Hence, k (A+B) = kA + kB [By definition of equality of two matrices]

(ii) **(k + l)A = k A + l A**

- **Proof:**
  Since k and l are scalars, k+l is also a scalar. Therefore, (k+l) A is a matrix of order mxn. Also, kA and lA are m x n matrices. Therefore, kA + lA is also an mxn matrix.

  Thus, (k+l) A and kA + lA are two matrices of the same order mxn such that

  $((k+l)\ A)_{ij} = (k+l)\ a_{ij}$ [By definition of scalar multiplication]

  $= ((k+l)\ A)_{ij} = ka_{ij} + la_{ij}$ [By distributivity of multiplication over addition]

  $((k+l)\ A)_{ij} = (kA)_{ij} + (lA)_{ij}$ [By definition of scalar multiplication]

  $((k+l)\ A)_{ij} = (KA + lA)_{ij}$ [By definition of addition of matrices]

  $((k+l)\ A)_{ij} = (kA + lA)_{ij}$ for i=1, 2,..., mm and j=1, 2,...,n

  Hence, (k+l) A = KA+ lA.

- **Example:**
- If, A= $\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B.

- **Solution:**
  We have 2A + 3X = 5B
  or 2A + 3X − 2A = 5B − 2A
  or 2A − 2A + 3X = 5B − 2A (Matrix addition is commutative)
  or O + 3X = 5B − 2A (−2A is the additive inverse of 2A)
  or 3X = 5B − 2A (O is the additive identity)
  or X (5B − 2A)

$$\frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & -2+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 2 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & -\frac{10}{3} \\ 4 & \frac{2}{3} \\ -\frac{31}{3} & -\frac{7}{3} \end{bmatrix}$$

❖ **Exercise:**

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) A + B (ii) A – B (iii) 3A – C (iv) 2A+B (v) A-2B

2. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

3. Find X, if Y= $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and 2X+Y= $\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

- ## Multiplication of a Matrix

  - Matrix multiplication is a binary operation whose output is also a matrix when two matrices are multiplied. In linear algebra, the multiplication of matrices is possible only when the matrices are compatible.

  - Two matrices A and B are said to be compatible if the number of columns in A is equal to the number of rows in B. That means if A is a matrix of order m×n and B is a matrix of order n×p, then we can say that matrices A and B are compatible.

$$A \quad . \quad B$$

$$m \times n \qquad n \times p$$

Equal

Dimensions of AB

  - Suppose we have two matrices A and B, the multiplication of matrix A with Matrix B can be given as (AB). That means, the resultant matrix for the multiplication of for any m × n matrix 'A' with an n × p matrix 'B', the result can be given as matrix 'C' of the order m × p.

❖ How to Multiply Matrices?

- Multiplying matrices can be performed using the following steps:

- **Step 1:** Make sure that the number of columns in the 1st matrix equals the number of rows in the 2nd matrix (compatibility of matrices).

- **Step 2:** Multiply the elements of i^{th} row of the first matrix by the elements of j^{th} column in the second matrix and add the products.

  This would the element that is in the i^{th} row and j^{th} column of the resultant matrix.

- **Step 3:** Place the added products in the respective positions.

- **Example:**

  Find AB, if A= $\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$

- **Solution:**
  The matrix A has 2 columns which is equal to the number of rows of B.
  Hence AB is defined. Now

$$AB = \begin{bmatrix} 6(2)+9(7) & 6(6)+9(9) & 6(0)+9(8) \\ 2(2)+3(7) & 2(6)+3(9) & 2(0)+3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12+63 & 36+81 & 0+72 \\ 4+21 & 12+27 & 0+24 \end{bmatrix}$$

$$= \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

❖ Properties Of Multiplication of Matrices

1. **The Associative Law:**
   For any three matrices A, B and C. We have

(AB) C = A (BC), whenever both sides of the equality are defined.

- **Example:**

If A= $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$ B= $\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and C= $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$

find A(BC), (AB)C and show that (AB)C = A(BC).

- **Solution:**

$$We\ have,\ AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3-0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$$

$$(AB)(C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

$$Now\ BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

$Therefore$

$$A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Clearly, (AB) C = A (BC)

2. **The Distributive Law:**
   For three matrices A, B and C

   (i) A (B+C) = AB + AC

(ii) (A+B) C = AC + BC, whenever both sides of equality are defined.

- **Example:** $If\ A=\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}\ and\ C=\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

Calculate AC, BC and (A + B)C. Also, verify that (A + B)C = AC + BC

- **Solution:** $Now,\ A+B=\begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

$$So \quad (A+B)\,C=\begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0-14+24 \\ -10+0+30 \\ 16+12+0 \end{bmatrix}=\begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

$$Further \quad AC=\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0-12+21 \\ -12+0+24 \\ 14+16+0 \end{bmatrix}=\begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$$

$$and \quad BC=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}=\begin{bmatrix} 0-2+3 \\ 2+0+6 \\ 2-4+0 \end{bmatrix}=\begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$$

$$AC+BC=\begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}+\begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}=\begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

$$Clearly,\ (A+B)\,C=AC+BC$$

3. **The Existence of Multiplicative Identity:**
   For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

- **Example:**

$Let\ A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\ and\ I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\ Proove\ that\ AI=A$

$$IA=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- **Solution:**

4. **Non-Commutativity of Multiplication of Matrices:**

- **Example:** If A= $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then find AB, BA. Show that AB ≠ BA.

- **Solution:**
  Since A is a 2 × 3 matrix and B is 3 × 2 matrix. Hence AB and BA are both defined and are matrices of order 2 × 2 and 3 × 3, respectively.

  Note that $AB = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$

  and

  $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

  Clearly AB ≠ BA

5. **Zero Matrix as The Product of Two Non-Zero Matrices:**

- **Example:** If A= $\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and B= $\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ then find AB.

- **Solution:**

We have,

$$AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

❖ Exercise:

1. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ prove that $A^3 - 6A^2 + 7A + 2I = 0$.

2. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

   Find each of the following:

   (i) A + B (ii) A – B (iii) 3A – C (iv) AB (v) BA

3. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

   (a) ₹ 1800 (b) ₹ 2000