SUNSTONE

Bachelor of Computer Application

Computational Mathematics

LECTURE 7 NOTES

Functions-Introduction

- **Function As a Special Kind of Relation**

  - Let A and B be two non-empty sets. A relation f from A to B, to, a sub-set of A×B, is called a function (or a mapping or a map) from A to B, if

    (i) for each a∈A there exists be B such that (a, b)∈f
    (ii) (a, b)∈f and (a, c)∈f ⟹b=c.

    Thus, a non-void subset f of A×B is a function from A to B if each element of A appears in some ordered pair in *f* and no two ordered pairs in f have the same first element.

    If (a, b)∈f, then b is called the image of a under f.

  - **Example 1:** Let A= {1,2,3}, B= {1,2, 3, 4} and f_1 f_2 and f_3 are three subsets of Ax Bas given below:

    f1={(1, 2), (2, 3), (3, 4)} f2={(1, 2), (1, 3), (2, 3), (3, 4)}. f3 = {(1, 3), (2, 4)}.

  - Then, f1 is a function from A to B but f2 and f3 are not functions from A to B. f is not a function from A to B, because 1∈A has two images 2 and 3 in B and f3 is not a function from A to B because 3∈A has no image in B.

    If a function f is expressed as the set of ordered pairs, the domain of f is the set of all first components of members of f and the range of f is the set of second components of members of f

    i.e., Domain of f={a: (a, b)∈f), and Range of f={b: (a,b)∈f}

  - **Example 2: If x, y∈ {1, 2, 3, 4}, then which of the following are functions in the given set?**
    (a) f1= {(x,y):y =x+1} **(b) f2= {(x,y): x+y>4}**
    (c) f3= {(x, y): y<x} **(d) f4= {(x, y): x+y=5}**

    Also, in case of a function give its range.

  - **Solution:**
    If we express f1 ,f2, f3 and f4 as sets of ordered pairs, then we have

    f1 = {(1, 2), (2, 3), (3, 4)}

f2 = {(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4,3)}

f3 = {(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)} and f4 = {(1, 4), (2, 3), (3, 2), (4, 1)}

(a) We have, f1 ={(1, 2), (2, 3), (3, 4)}

We observe that an element 4 of the given set has not appeared in first place of any ordered pair of f1 . So, f1 is not a function from the given set to itself.

(b) We have, f2 ={(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)}

We observe that 2, 3, 4 have appeared more than once as first components of the ordered pairs in f2. So, f_2 is not a function.

(c) We have, f3= {(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)}.

We observe that 3 and 4 have appeared more than once as first components of the ordered pairs in f3. So, f3 is not a function.

(d) We have, f4={(1, 4), (2, 3), (3,2), (4, 1)}

We observe that each element of the given set has appeared as first components in same and one ordered pair of f4. So, f4 is a function in the given set. In this case, Range of f ={1, 2, 3,4 }

- **Example 3: Let f be a relation on the set N of natural numbers defined by f = {(n,3n):n∈N}. Is f a function from N to N. If so, find the range of f.**

- **Solution**:
  Since for each n∈N, there exists a unique 3n∈N such that (n, 3n)∈f. Therefore, function from N to N.

  Clearly, Range of f= {f(n): n∈N}= {3n:n∈N}

Function As a Correspondence

- Let A and B be two non-empty sets. Then a function “f” from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that:

  (i) all elements of set A are associated to elements in set B.,

(ii) an element of set A is associated to a unique element in set B.

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function" f. If f is a function from a set A to a set B, then we write f: A → B or A→B, which is read as f is function from A to B or f maps A to B.

If an element a∈A is associated to an element be b∈B, then b is called 'the f-image of a' or 'image of a under f or 'the value of the function f at a'. Also, a is called the pre-image of b under the function.

We write it as: b = f(a)

❖ Exercise:

1. Let A= {-2, -1, 0, 1, 2} and f: A→Z be a function defined by $f(x)=x^2-2x-3$. Find:

   (i) range of f ie. f (A) (ii) pre-images of 6,-3 and 5.

2. If a function f: R→R be defined by

$$f(x)=\begin{cases} 3x-2, & x<0 \\ 1, & x=0 \\ 4x+1, & x>0 \end{cases}$$

   Find: f (1), f (-1). f (0), f (2).

3. A function f: R→R is defined by $f(x)=x^2$. Determine
   (i) range of f (ii) {x:f(x)=4} (iii) {y:f(y)=-1}

4. Let f: R+ →R, where R+ is the set of all positive real numbers, be such that $f(x)=\log_e x$. Determine

   (i) the image set of the domain of f
   (ii) {x:f(x)= -2}
   (iii) whether f (xy) = f (x) + f (y) holds.

- **Domain, Co-Domain and Range of a Function**

  - Let f: A→B Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under fand is denoted by (A).

    Thus, $f(A)=\{f(x):x \in A\}$ Range of f

    Clearly, $f(A)\subseteq B$

  - **Example 1:**
    Let A= {-2, -1, 0 , 1 ,2) and B={0, 1, 2, 3, 4, 5, 6}. Consider a rule f (x) =x^2. Under this rule, we obtain

    $f(-2) = (-2)^2=4$, $f(-1)=(-1)^2=1$, $f(0)=0^2=0$, $f(1)=1^2=1$ and $f(2)=2^2=4$.

    We observe that each element of A is associated to a unique element of B.

    So, f:A→B given by $f(x)=x^2$ is a function.

    Clearly, domain (f)=A={-2,-1,0,1,2} and range (f)={0,1,4,1}

  - **Example 2:**
    Consider a rule f (x)=2x-3 associating elements of N (set of natural numbers) to elements of N.
    This rule does not define a function from N to itself, because f (1)=2×1-3=-1 ∈N i.e., 1∈N (domain) is not associated to any element of N (co-domain).

  - **Example 3:**
    Let A={-2, -1, 0, 1, 2} and f: A→Z be given by f (x)=x^2-2x-3 Find:
    (i) the range of f **(ii) pre-images of 6, -3 and 5.**

  - **Solution:**

(i) We have, $f(x)=x^2-2x-3$.

∴ $f(-2)=(-2)^2-2(-2)-3= 5$, $f(-1)= (-1)^2-2(-2)-3= 0$, $f(0)= -3$
$f(1)= 1^2-2\times1-3= -4$ and $f(2)= 2^2-2\times2-3= -3$

So, range (f)={f(-2), f(-1), f(0), f(1), f(2)}= {0,5,-3,-4}

(ii) Let x be a pre-image of 6. Then,

$f(x)= 6 \Longrightarrow x^2-2x-3-6 \Longrightarrow x^2-2x-9=0 \Longrightarrow x=1\pm\sqrt{10}$

Since $x=1\pm\sqrt{10} \notin A$. So, there is no pre-image of 6. Let x be a pre-image of -3. Then,

$f(x)=-3 \Longrightarrow x^2-2x-3=-3 \Longrightarrow x^2-2x=0 \Rightarrow x=0,2$

Clearly, 0, 2∈A. So, 0 and 2 are pre-images of -3.Let x be a pre-image of 5. Then

$f(x)=5 \Longrightarrow x^2-2x-3= 5 \Rightarrow x^2-2x-8-0 \Rightarrow (x-4)(x+2)=0 \Rightarrow x=4,-2$

Since -2 ∈A bet ∉4 A. So, -2 is a pre-image of 5.

- **Equal Functions**

  - The functions f and g are said to be equal if

    (i) domain of = domain of g

    (ii) co-domain of f = co-domain of g and,

    (iii) f(x)=g(x) for every x belonging to their common domain.

    If two functions f and g are equal, then we write f=g.

  - **Example 1:**
    Let A = {1, 2}, B = {3, 6} and f: A → B given by f (x) = x^2+2 and g: A → B given by g(x)=3x. Then, we observe that f and g have the same domain and co-domain. Also we have f(1)=3=g(1) and f (2)=6=g(2). Hence, f=g

  - **Example 2:**
    Let f:R-{2}→R be defined by $f(x)=\frac{x^2-4}{x-2}$ and g: R→R be defined by

    g(x)=x+2.Find whether f=g or not.

- **Solution:**
- We have, f(x) = $\frac{x^2-4}{x-2}$, x≠2

⟹ f(x)= $\frac{(x-2)(x+2)}{x-2}$ =x+2 for all x≠2

⟹ f(x)=g(x) for all x≠ 2

Thus, f (x)=g (x) for all x∈R-[2]. But f (x) and g (x) have different domains.
Infact, domain of f-R-[2] and domain of g= R. Therefore, f≠g.

- **Real Functions**

  - A function f: A→B is called a real valued function, if B is a subset of R (set of all real numbers).

    If A and B both are subsets of R, then f is called a real function.

  - **Example:**
    Let N be the set of natural numbers. Define a real valued function
    f : N→ N by f (x) = 2x + 1. Using this definition, complete the table given below.

x	1	2	3	4	5	6	7
y	f(1)=...	f(2)=...	f(3)=...	f(4)=...	f(5)=...	f(6)=...	f(7)=...

  - **Solution:** The completed table is given by

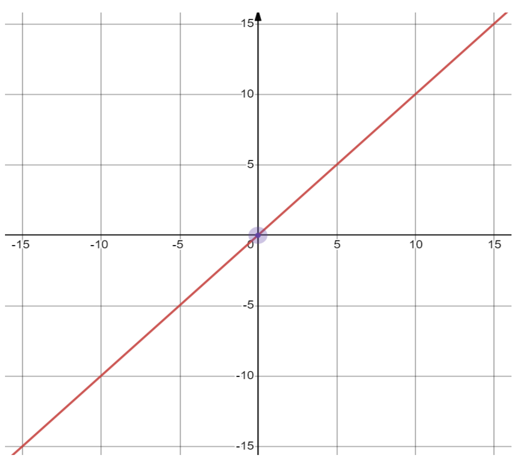
x	1	2	3	4	5	6	7
y	f(1)= 3	f(2)= 5	f(3)= 9	f(4)=1 1	f(5)=1 3	f(6)=1 5	f(7)=1 7

- **Some Functions and Their Graphs**

1. Identity function:

Let R be the set of real numbers. Define the real valued function f : R → R by y = f(x) = x for each x ∈ R. Such a function is called the identity function. Here the domain and range of f are R.

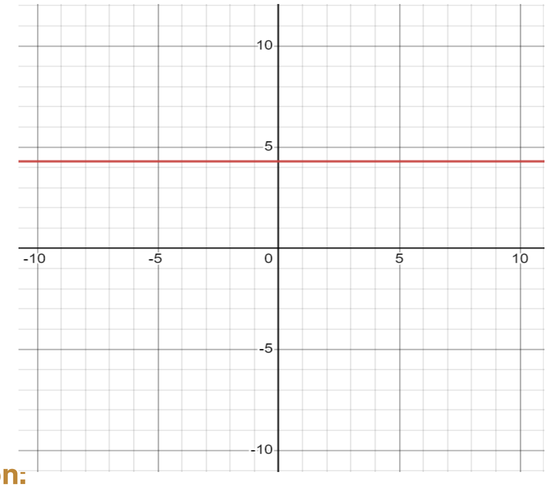
The graph is a straight line as shown in Fig. It passes through the origin.



2. Constant function:

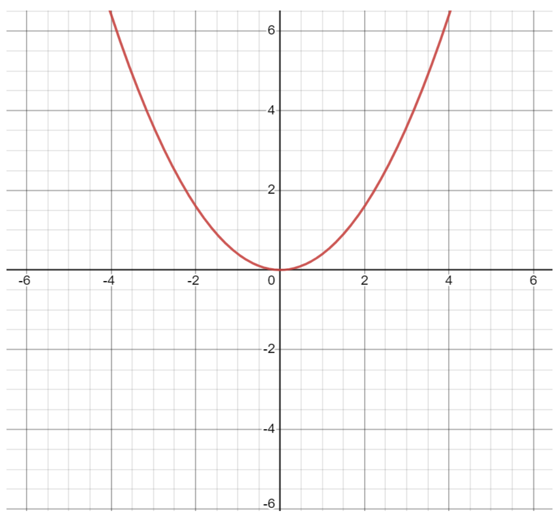
Define the function f: R → R by y = f (x) = c, x ∈ R where c is a constant and each x ∈ R. Here domain of f is R and its range is {c}.

The graph is a line parallel to x-axis. For example, if f(x)=3 for each x∈R, then its graph will be a line as shown in the Fig.



3. Polynomial function:

A function $f : R \rightarrow R$ is said to be polynomial function if for each x in R, $y = f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n \in R$.



- **Example:** The functions defined by $f(x) = x^3 - x^2 + 2$, and $g(x) = x^4 + \sqrt{2}x$ are some examples of polynomial functions, whereas the function h defined by $h(x) = x^{2/3} + 2x$ is not a polynomial function

- **Example:**
  Define the function $f: R \rightarrow R$ by $y = f(x) = x^2, x \in R$. Complete the Table given below by

using this definition. What is the domain and range of this function? Draw the graph of f.

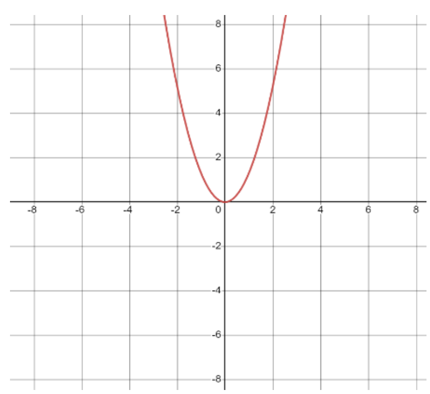
x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

- **Solution:** The completed Table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

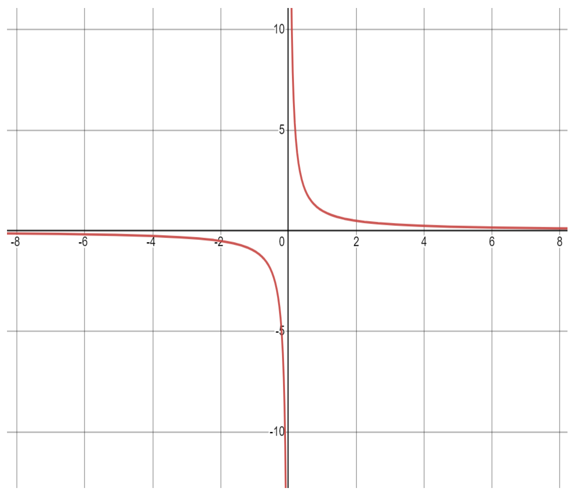
Domain of f = $\{x : x \in R\}$. Range of f = $\{x^2 : x \in R\}$.

The graph of f is given by Fig.



4. **Rational functions:**

Rational Functions are functions of the type, where f(x) and g(x) are polynomial functions of x defined in a domain, where $g(x) \neq 0$.



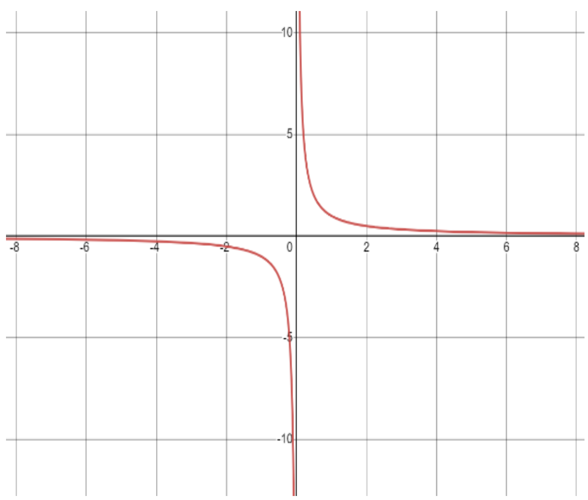
- **Example:**
Define the real valued function $f : R - \{0\} \rightarrow R$ defined by $f(x) = 1/x$, $x \in R - \{0\}$. Complete the Table given below using this definition. What is the domain and range of this function?

x	**-2**	**-1.5**	**-1**	**-0.5**	**0.25**	**0.5**	**1**	**1.5**	**2**
y = 1/x									

- **Solution:** The completed Table is given by

x	**-2**	**-1.5**	**-1**	**-0.5**	**0.25**	**0.5**	**1**	**1.5**	**2**
y = 1/x	**-0.5**	**-0.67**	**-1**	**-2**	**4**	**2**	**1**	**0.67**	**0.5**

- The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig.

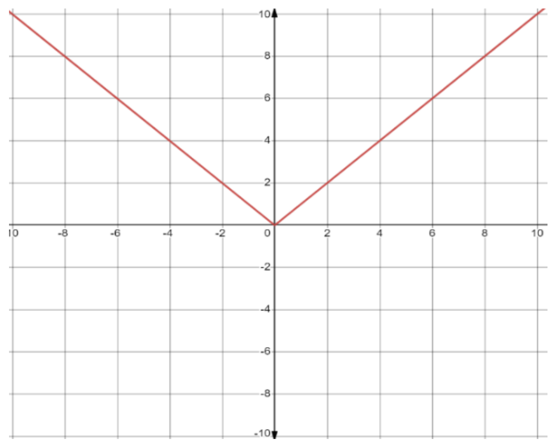


5. **Modulus Function:**
   The Modulus function f: R→R defined by f(x) = |x| for each x ∈R is called modulus function. For each non-negative value of x, f(x) is equal to x.

   But for negative values of x, the value of f(x) is the negative of the value of x, i.e.,

$$f(x)=\begin{cases} x\ ,\ x\geq 0 \\ -x\ ,x<0 \end{cases}$$

The graph of the modulus function is given in Fig.

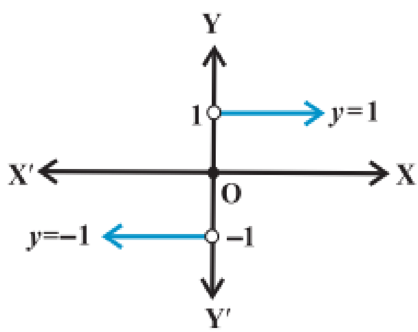


6. Signum function:

The function f:R→R defined by

$$f(x)=\begin{cases} 1, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ -1, & \text{if } x<0 \end{cases}$$

is called the signum function. The domain of the signum function is R and the range is the set {–1, 0, 1}. The graph of the signum function is given by the Fig.



7. Greatest integer fu

$$f(x)=\frac{|x|}{x}, x \neq 0 \text{ and } 0 \text{ for } x=0$$

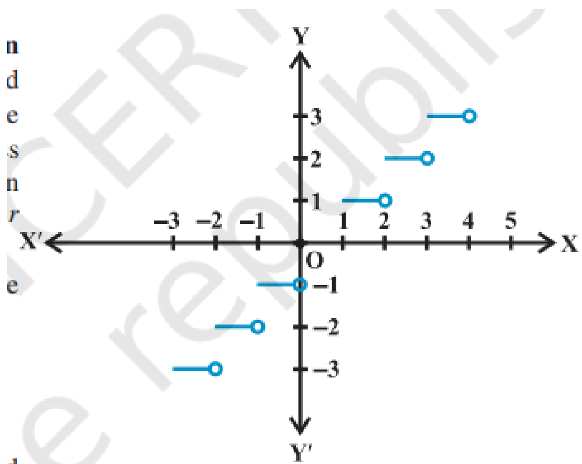
The function f: R → R defined by f(x) = [x], x ∈R assumes the value of the greatest integer, less than or equal to x. Such a function is called the greatest integer function.

From the definition of [x], we can see that

[x] = –1 for –1 ≤ x < 0
[x] = 0 for 0 ≤ x < 1

[x] = 1 for $1 \le x < 2$
[x] = 2 for $2 \le x < 3$ and so on.

The graph of the function is shown in Fig



$f(x) = [x]$

- **Algebra of Real Functions**

1. **Addition of two real functions**

   Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$.

   Then, we define $(f + g): X \to R$ by

   $$\mathbf{(f + g)(x) = f(x) + g(x), \text{ for all } x \in X.}$$

2. **Subtraction of a real function from another**

   Let $f : X \to R$ and $g: X \to R$ be any two real functions,
   where $X \subset R$. Then, we define $(f - g) : X \to R$ by

   $$\mathbf{(f - g)(x) = f(x) - g(x), \text{ for all } x \in X.}$$

3. **Multiplication by a scalar**

Let f : X→R be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product α f is a function from X to R defined by

$$(\alpha f)(x) = \alpha f(x),\ x \in X.$$

4. **Multiplication of two real functions**

The product (or multiplication) of two real functions f:X→R and g:X→R is a function fg:X→R defined by

$$(fg)(x) = f(x)\, g(x), \text{ for all } x \in X.$$

This is also called pointwise multiplication.

5. **Quotient of two real functions**

Let f and g be two real functions defined from X→R, where X ⊂ R.
The quotient of f by g denoted by f/g is a function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ provided\ g(x) \neq 0,\ x \in X$$

❖ **Examples:**

- **Example 1:**
  Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions.
  Find
  (f + g) (x), (f –g) (x), (fg) (x), $\left(\frac{f}{g}\right)(x)$

- **Solution:** We have,
  $(f + g)(x) = x^2 + 2x + 1$, $(f - g)(x) = x^2 - 2x - 1$,

  $(fg)(x) = x^2 (2x + 1) = 2x^3 + x^2$, $\left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1},\ x \neq -\frac{1}{2}$

- **Example 2:**
  Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of nonnegative real numbers.

Find (f + g) (x), (f – g) (x), (fg) (x) and $\left(\frac{f}{g}\right)(x)$

- **Solution:**

We have

$$(f + g)(x) = \sqrt{x + x},\ (f - g)(x) = x - x,$$

$$(fg)\ x = \sqrt{x(x)} = x^{\frac{3}{2}} \text{ and } \left(\frac{f}{g}\right)(x) = x^{-\frac{1}{2}},\ x \neq 0$$

❖ Exercise:

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

   (i) {(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)}

   (ii) {(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)}

   (iii) {(1,3), (1,5), (2,5)}

2. Find the domain and range of the following real functions:

   (i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{9} - x^2$

3. A function f is defined by $f(x) = 2x - 5$. Write down the values of

   (i) f (0), (ii) f (7), (iii) f (–3).

4. The function ‘t’ which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by t(C) =9C5 + 32. Find

   (i) t(0) (ii) t(28) (iii) t(–10) (iv) The value of C, when t(C) =

212.

5. Find the range of each of the following functions.

(i) $f(x) = 2 - 3x$, $x \in R$, $x > 0$.

(ii) $f(x) = x2 + 2$, x is a real number.

(iii) $f(x) = x$, x is a real number.