Random Numbers

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

> wget https://github.com/mohilmukundareddy/
> Assignment1/blob/main/ex1/1.1/exrand.c
> we get https://github.com/mohilmukundareddy
> /Assignment1/blob/main/ex1/1.1/coeffs.h

Use the below command in the terminal to run the code

> gcc exrand.c −lm
> ./a.out

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The graph 1.2 is obtained by running the below code

> https://github.com/mohilmukundareddy/
> Assignment1/blob/main/ex1/1.2/main.py

Run the following command in the terminal to run the code.
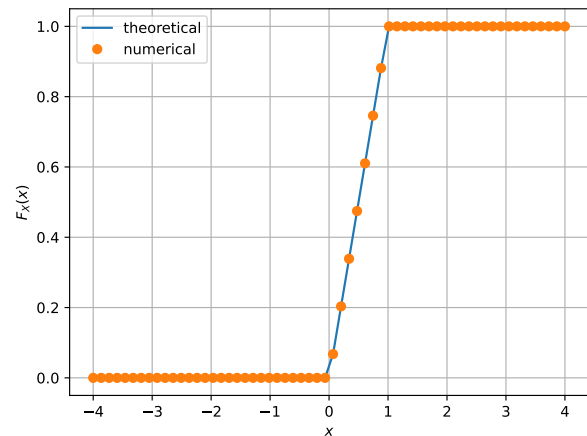
> python3 main.py

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is uniform random variable so

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$= 0 + \int_{0}^{x} 1dx$$

$$= \begin{cases} 1 & x > 1 \\ x & 0 \le x \le 1 \\ 0 & x < 0 \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.2)$$

and its variance as
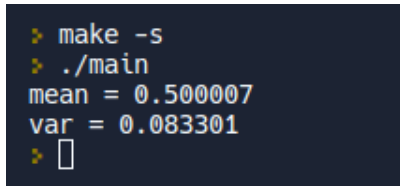
$$\text{var}[U] = E[U - E[U]]^2 \qquad (1.3)$$

Write a C program to find the mean and variance of U.

Solution:

```
wget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex1/1.4/exrand.c
weget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex1/1.4/coeffs.h
```

Use below command to run file,

```
gcc exrand.c −lm
./a.out
```

```
> make -s
> ./main
mean = 0.500007
var = 0.083301
>
```

Fig. 1.4: Caption

1.5 Verify your result theoretically given that

Given

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_X(x)$$

$$E[U] = \int_{-\infty}^{\infty} x^k f_X(x)dx$$

$$= \int_0^1 x \times 1 dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

if k=2

$$E[U^2] = \int_0^1 x^2 \times 1 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\begin{aligned}
\text{variance} &= E[u - E[u]]^2 \\
&= E[U^2] - E^2[U] \\
&= \frac{1}{3} - \frac{1}{4} = 0.0833
\end{aligned}$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
wget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex2/2.1/exrand.c
wget hhttps://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex2/2.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
gcc exrand.c −lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in plot,Properties of the CDF:

- $F_X(x) = P(X \le x)$
- $Q_X(x) = P(X > x)$
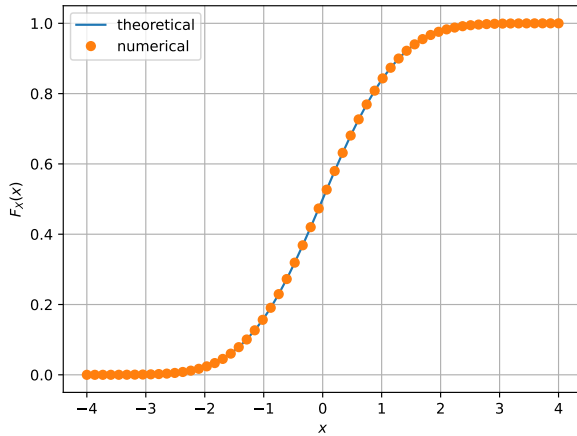- $F_X(x) = 1 - Q_X(x)$ This can be used to calculate F (x).

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \qquad (2.2)$$
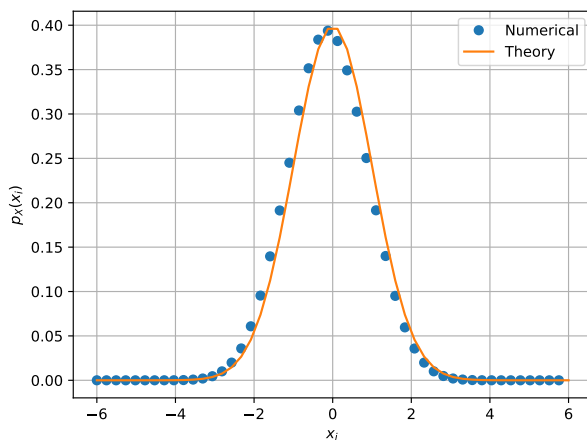
What properties does the PDF have?

Fig. 2.3: The PDF of X

Solution: The PDF of X is plotted using the code below

https://github.com/mohilmukundareddy/
　　Assignment1/blob/main/ex2/2.3/main.py

Use the below command to run the code:

python3 main.py

Properties of PDF:

- PDF is symmetric about $x \approx 0$

- graph is similar to bell shaped

- mean of graph is situated at the symmetrical point

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives
Mean = 0.000326 Variance= 1.000906

wget https://github.com/mohilmukundareddy/
　　Assignment1/blob/main/ex2/2.4/exrand.c

Command used:

gcc exrand.c −lm
./a.out

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

by property of probability

$$\int_{-\infty}^{\infty} p_X(x)dx = 1$$

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$E[X] = \int_{-\infty}^{\infty} xp(x)dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx$$

$\frac{x}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$ is an odd function so integral is zero i.e $E[X] = 0$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p(x)dx$$

$$= x \int_{-\infty}^{\infty} xp(x)dx - \int_{-\infty}^{\infty} \left(\int xp(x)dx \right) dx$$

$$= \left[-x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

$$= 0 - (-1)$$

we know that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$$

by series expansion $\frac{x}{e^{\frac{x^2}{2}}} = \frac{x}{1+\frac{x^2}{2}+\frac{x^4}{8}+...}$

putting $x = \infty$, we get $\frac{1}{\infty} = 0$

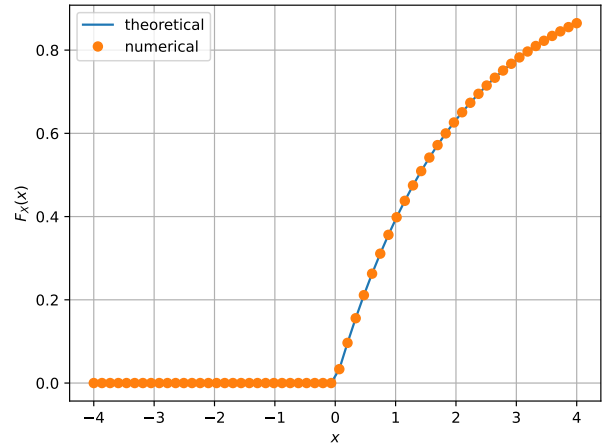Similarly when $x = -\infty$ we get 0

$var(x) = E[x^2] - E[x] = 1 - 0 = 1$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \qquad (3.1)$$

and plot its CDF.

Solution:

Running the below code generates samples of V from file uni.dat(U).

https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex3/main.py

Use the below command in the terminal to run the code:

python3 main.py

Now these samples are used to plot by running the below code

Fig. 3.1: CDF for (3)

https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex3/cdf.py

Use the below command to run the code:

python3 cdf.py

3.2 Theorotical expression for $F_V(x)$

$$F_V(x) = P\{V \le x\}$$

$$= P\{-2 \times \ln(1 - U) \le x\}$$

$$= P\{U \le 1 - e^{(-\frac{x}{2})}\}$$

$$= F_U\{1 - e^{(-\frac{x}{2})}\}$$

$$= \begin{cases} 1 - e^{(-\frac{x}{2})} & 0 \le x < \infty \\ 0 & x < 0 \end{cases}$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \qquad (4.1)$$

we get the code

https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex4/1/main.py

run the command

python3 main.py

4.2 Find the CDF of T. we have code

https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex4/2/main1.py

run the command

```
python3 main1.py
```

let us take two cases if $0 \le t \le 1$ and $1 < t \le 2$
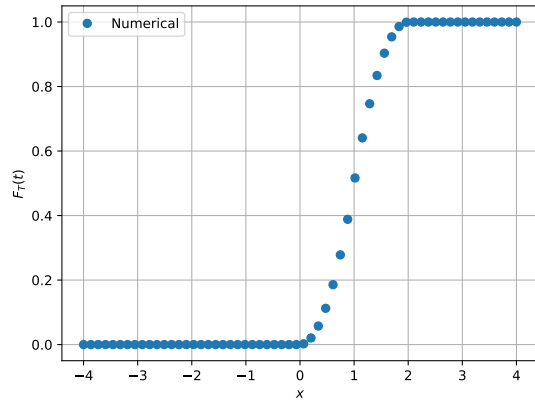

Fig. 4.2: numerical cdf

Fig. 4.4: def plot

4.3 Find the PDF of T we have code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/3/main2.py
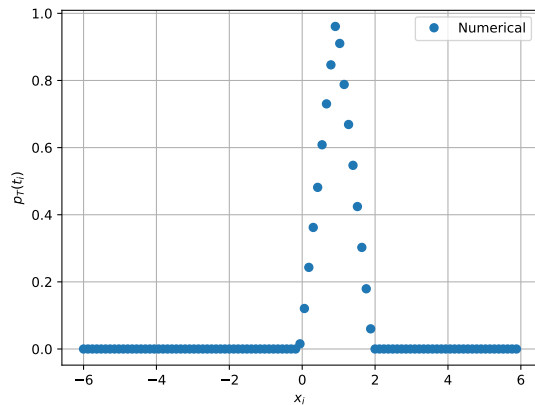```
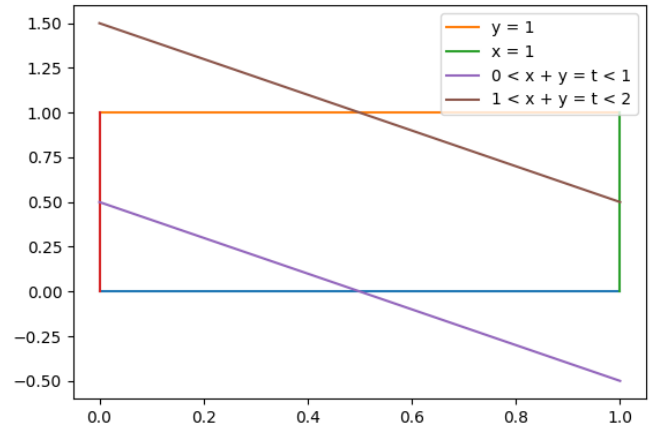
run the command

```
python3 main2.py
```

The above graph is produced by

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/2/find.py
```

Run the code in terminal

```
python3 find.py
```

from the figures it is evident that
$P(U1 + U2 < t, 0 \le t < 1) = \frac{t^2}{2}$
$P(U1 + U2 < t, 1 \le t \le 2) = 1 - \frac{(2-t)^2}{2}$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}$$

$$P_T(t) = \frac{d(F_T(t))}{dt}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases}$$

Fig. 4.3: Numerical cdf

4.4 Find the theoretical expressions for the PDF and CDF of T. **Solution:**

$$F_T(t) = P\{T \le t\}$$
$$= P\{U1 + U2 \le t\}$$

4.5 Verify your results through a plot Take the code for cdf

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/5/main1.py
```

Run in terminal
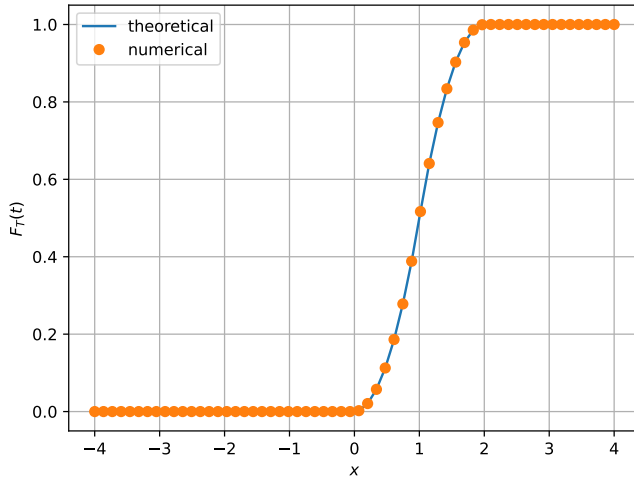
```
python3 main1.py
```


Fig. 4.5: t-cdf

Take the code for pdf

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/5/main2.py
```
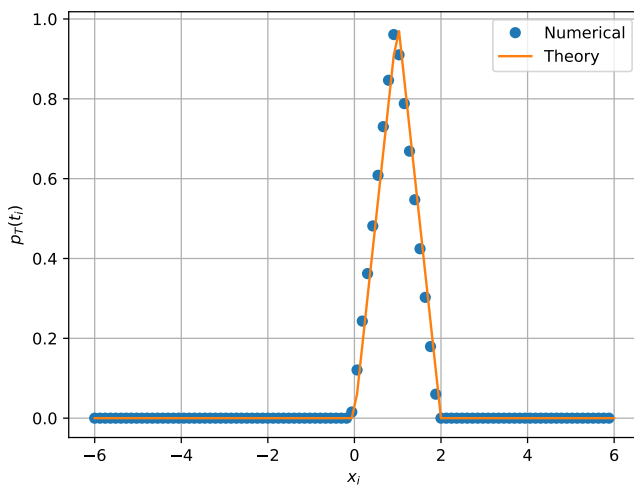
Run in terminal

```
python3 main2.py
```


Fig. 4.5: t-pdf

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \qquad (5.1)$$

where $A = 5$ dB, $X_1\{1, -1\}$ **Solution:** use bernouli function from exrand.c find the code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex5/1/exrand.c
```

run the terminal command

```
gcc exrand.c −lm
./a.out
```

5.2 Generate

$$Y = AX + N, \qquad (5.2)$$

where $A = 5$ dB, and $N \sim N(0, 1)$. find the code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex5/2/main.py
```

run the command

```
python3 main.py
```

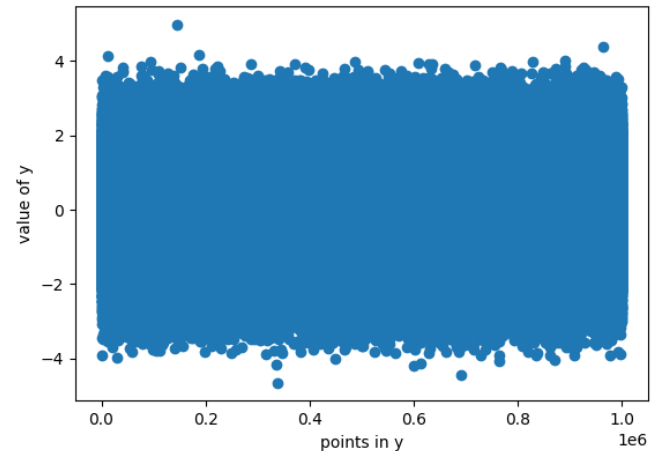### 5.3 Plot Y using a scatter plot. find the scatter plot

Fig. 5.3: scatter plot

5.4 Guess how to estimate X from Y. **Solution:** To estimate X from Y, consider function:

$$\text{sgn}(y) = \begin{cases} -1, & y \in (-\infty, 0] \\ 1, & y \in (0, \infty) \end{cases} \qquad (5.3)$$

Using $\text{sgn}\, y$, we can operate on Y to find corresponding values of X.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \qquad (5.4)$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \qquad (5.5)$$

find the code below

run the code

```
python3 main.py
```

we get the values as $P_{(e|0)} = 0.310007$
$P_{(e|1)} = 0.310142$

5.6 Find P_e assuming that X has equiprobable symbols.

Solution: Assume a general value of A.
Our estimation function predicts the data above the x axis correspond to $X = 1$, and the data points below the x-axis correspond to $X = -1$. We have:

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right)$$
$$= \Pr\left(AX + N < 0 | X = 1\right)$$
$$= \Pr\left(N < -A\right)$$
$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$
$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$
$$= Q_N(A)$$

where Q_N is the Qfunction of the normal distribution. Similarly,

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right)$$
$$= \Pr\left(AX + N > 0 | X = -1\right)$$
$$= \Pr\left(N > A\right)$$
$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$
$$= Q_N(A)$$

Given X is equiprobable so we have

$$P_e = P_{e|0}\Pr\left(X = 1\right) + P_{e|1}\Pr\left(X = -1\right)$$
$$= \frac{1}{2}P_{e|0} + \frac{1}{2}P_{e|1}$$
$$= \frac{1}{2}Q_N(A) + \frac{1}{2}Q_N(A)$$
$$= Q_N(A)$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB. find the code below
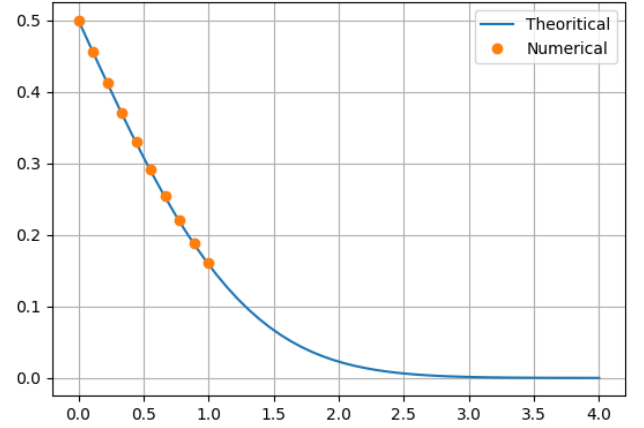
run the command

```
python3 main.py
```


Fig. 5.7: verification of p_e

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution: To estimate X from Y, we consider

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases}$$

so we have

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right)$$
$$= \Pr\left(AX + N < \delta | X = 1\right)$$
$$= \Pr\left(N < \delta - A\right)$$
$$= \int_{-\infty}^{\delta - A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= Q_N(A - \delta)$$

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right)$$
$$= \Pr\left(AX + N > \delta | X = -1\right)$$
$$= \Pr\left(N > \delta + A\right)$$
$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= Q_N(A + \delta)$$
$$P_e = P_{e|0}\Pr\left(X = 1\right) + P_{e|1}\Pr\left(X = -1\right)$$
$$= \frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta))$$

To minimise P_e, use differentiate wrt δ:

$$\frac{d}{d\delta}\left(\frac{1}{2}(Q_N(A - \delta) + Q_N(A + \delta))\right) = 0$$
$$\frac{1}{2}(\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}}) = 0$$
$$(\delta - A)^2 = (\delta + A)^2$$
$$\delta = 0$$

so $\delta = 0$ maximize P_e

5.9 Repeat the above exercise when $p_X(0) = p$
Solution: we have:

$$P_e = P_{e|0}p + P_{e|1}(1 - p)$$
$$= pQ_N(A - \delta) + (1 - p)Q_N(A + \delta)$$

Differentiating wrt δ we get

$$p\frac{1}{\sqrt{2\pi}}e^{-\frac{(\delta-A)^2}{2}} - (1 - p)\frac{1}{\sqrt{2\pi}}e^{-\frac{(A+\delta)^2}{2}} = 0$$

Taking ln on both sides and find δ :

$$\ln p - \frac{(\delta - A)^2}{2} = \ln(1 - p) - \frac{(\delta + A)^2}{2}$$
$$\delta = \frac{1}{2A}\ln\frac{1 - p}{p}$$

if $p = \frac{1}{2}$ then $\delta = 0$ which verifies with above result.

5.10 Repeat the above exercise using the MAP criterion.
Solution: Assume that $\Pr\left(X = -1\right) = p$, and $\Pr\left(X = 1\right) = (1 - p)$. Then, using the Law of Total Probability, we have:

$$p_Y(y) = p_{Y|X=-1}(y|-1)\Pr\left(X = -1\right)$$
$$+ p_{Y|X=1}(y|1)\Pr\left(X = 1\right)$$
$$= p \times p_{(N-A)}(y)$$
$$+ (1 - p) \times p_{(N+A)}(y)$$

where $p_Y(y)$ is the pdf of Y. Now, $p_{(N-A)}$ is the pdf of a shifted normal distribution, so

$$p_Y(y) = p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}$$

MAP criterion, find $p_{X|Y}(x|y)$. we use the Theorem of Conditional Probability:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)}$$

When $X = 1$, we have:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)}$$
$$= \frac{(1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}$$
$$= \frac{(1 - p)e^{2yA}}{p + (1 - p)e^{2yA}}$$

Similarly, when $X = -1$, we have:

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y|-1) \times p_X(-1)}{p_Y(y)}$$
$$= \frac{(p)\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}$$
$$= \frac{p}{p + (1 - p)e^{2yA}}$$

Therefore, when $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$, we have:

$$\frac{(1 - p)e^{2yA}}{p + (1 - p)e^{2yA}} > \frac{p}{p + (1 - p)e^{2yA}}$$
$$e^{2yA} > \frac{p}{(1 - p)}$$
$$y > \frac{1}{2A}\ln\frac{p}{(1 - p)}$$

Therefore, we can assert that $X = 1$, and $X = -1$ otherwise. Now, consider when $p = \frac{1}{2}$. We

have:

$$y > \frac{1}{2A} \ln \frac{p}{(1-p)}$$
$$= \frac{1}{2A} \ln 1$$
$$= 0$$

Therefore, when $y > 0$, we choose $X = 1$, and we choose $X = -1$ otherwise.

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim N(0, 1)$ and $X_2 \sim N(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: The sum of squares of n independent standard random normal variables is χ^2 distribution with n degrees of freedom.

$$P_{\chi^2}(x|n) = \frac{x^{\frac{n}{2}-1}e^{\frac{-x}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}, \forall x \geq 0$$

Here k=2,

$$P_{\chi^2}(x|2) = P_V(v) = \frac{e^{\frac{-v}{2}}}{2}$$

For the cummulative distribution

$$F_V(v) = \int_0^v \frac{e^{\frac{-v}{2}}}{2} dv$$
$$= 1 - e^{\frac{-v}{2}}$$

To generate data for V , run the following code,

https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/1/exrand.c
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/1/coeffs.h

Run the below command in terminal,

gcc exrand.c −lm

The PDF plot of the $\chi^2(2)$ can be obtained by running the code below,

https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/1/main1.py

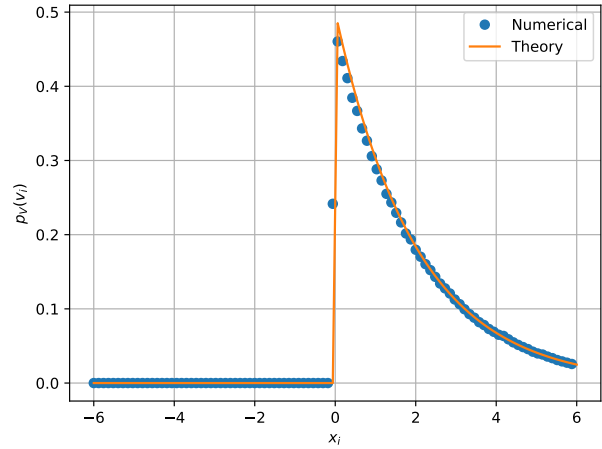Use the following command in the terminal to run the code

python3 main1.py

Fig. 6.1: PDF plot

The CDF plot of the $\chi^2(2)$ can be obtained by running the code below,

https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/1/main2.py

Use the following command in the terminal to run the code

python3 main2.py

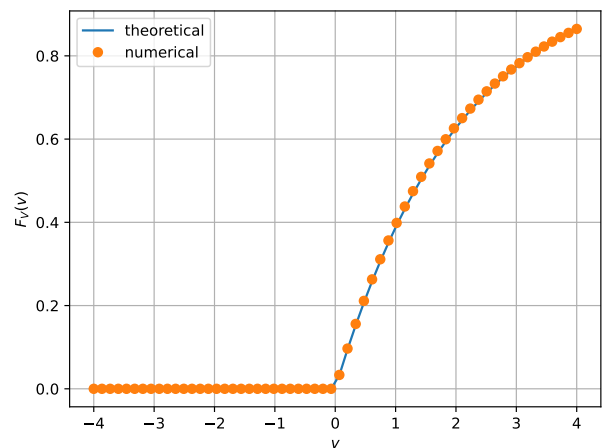Fig. 6.1: CDF plot

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases}$$

find α.

Solution: We will assume that X_1 and X_2 are i.i.d. Let

$$X_1 = r\cos\theta$$
$$X_2 = r\sin\theta$$

The Jacobian Matrix is then defined as:

$$J(r, \theta) = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\delta r\cos\theta}{\delta r} & \frac{\delta r\cos\theta}{\delta \theta} \\ \frac{\delta r\sin\theta}{\delta r} & \frac{\delta r\sin\theta}{\delta \theta} \end{pmatrix}$$

$$J = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix}$$

$$|J(r, \theta)| = R$$

Then as X_1, X_2 are independent we have,

$$p_{X_1, X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$
$$= \frac{1}{2\pi}e^{\frac{-(x_1^2 + x_2^2)}{2}}$$
$$= \frac{1}{2\pi}e^{\frac{-r^2}{2}}$$

Now, since

$$p_{r,\theta}(r, \theta) = |J(r, \theta)|p_{X_1, X_2}(x_1, x_2)$$

we have:

$$p_{R,\theta}(r, \theta) = \frac{r}{2\pi}e^{-\frac{r^2}{2}}$$
$$p_R(r) = \int_0^{2\pi} p_{R,\theta}(r, \theta)$$
$$= \int_0^{2\pi} \frac{r}{2\pi}e^{-\frac{r^2}{2}} d\theta$$
$$= re^{-\frac{r^2}{2}}$$
$$F_R(r) = \Pr(R \le r)$$
$$= \int_0^r f_R(r)dr$$
$$= 1 - e^{-\frac{r^2}{2}}$$

$F_V(x)$ is given by:

$$F_V(x) = F_{X_1^2 + X_2^2}(x)$$
$$= F_{R^2}x$$
$$= \Pr\left(R^2 \le x\right)$$
$$= \Pr\left(R \le \sqrt{x}\right)$$

Therefore,

$$F_V(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}}, & x \ge 0 \end{cases}$$

by Comparing we get $\alpha = \frac{1}{2}$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V}$$

Solution: To generate data for A , run the following code,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/3/exrand.c
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/3/coeffs.h
```

Run the below command in terminal,

```
gcc exrand.c −lm
```

The PDF plot of A can be obtained by running the code below,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/3/main1.py
```

Use the following command in the terminal to run the code
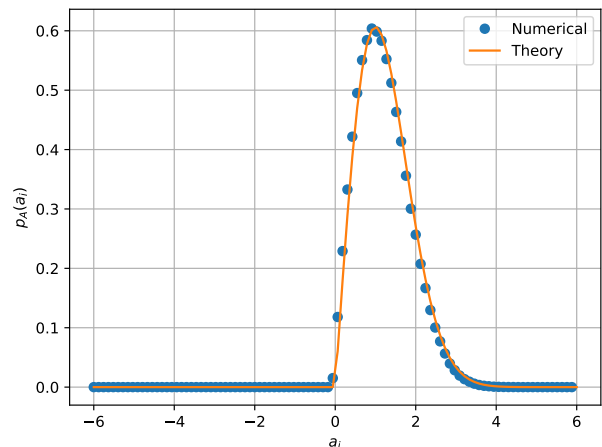
```
python3 main1.py
```


Fig. 6.3: PDF

The CDF plot of the A can be obtained by running the code below,

Use the following command in the terminal to run the code
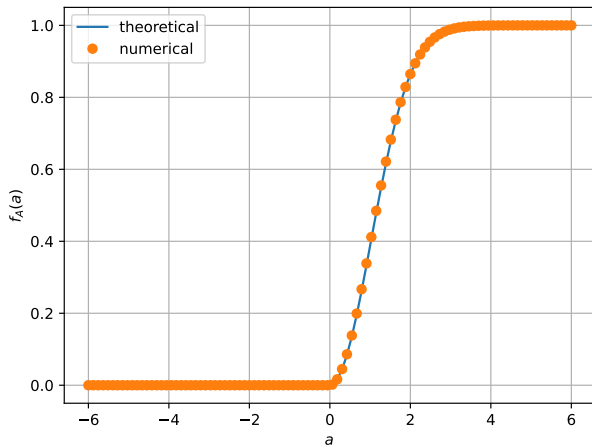
```
python3 main2.py
```


Fig. 6.3: CDF

The CDF of A for $x \geq 0$ is given by

$$
\begin{aligned}
F_A(x) &= \Pr(A \leq x) \\
&= \Pr\left(\sqrt{V} \leq x\right) \\
&= \Pr\left(V \leq x^2\right) \\
&= F_V(x^2) \\
&= 1 - \exp\left(-\frac{x^2}{2}\right)
\end{aligned}
$$

The PDF of A is given by

$$
\begin{aligned}
p_A(x) &= \frac{}{x} F_A(x) \\
&= \frac{}{x}\left(1 - \exp\left(-\frac{x^2}{2}\right)\right) \\
&= x \exp\left(-\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore,

$$
F_A(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
p_A(x) = \begin{cases} x \exp\left(-\frac{x^2}{2}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

7 Conditional Probability

7.1 Plot

$$
P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}
$$

for

$$
Y = AX + N, \tag{7.2}
$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim (0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB. **Solution:** Find the c code below

Run the command

```
gcc exrand.c -lm
```

For the plot
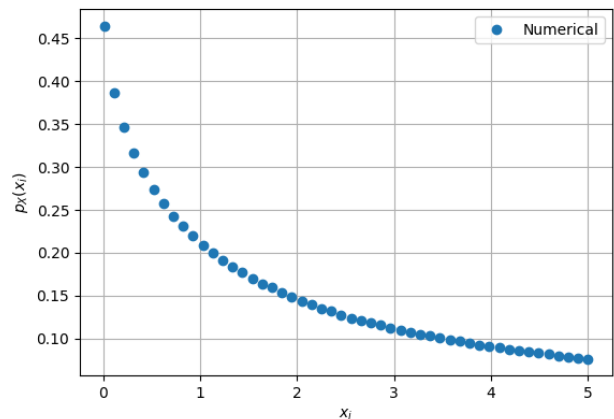
run the command

```
python3 main1.py
```


Fig. 7.1: P_e

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Solution: We use the signum function for estimation. We need to find:

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.3}$$
$$= \Pr\left(AX + N < 0|X = 1\right) \tag{7.4}$$
$$= \Pr\left(A < -N\right) \tag{7.5}$$

Assuming N is a constant, we have:

$$P_e = \Pr\left(A < -N\right) \tag{7.6}$$
$$= F_A(-N) \tag{7.7}$$
$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}}, & N \leq 0 \\ 0, & N > 0 \end{cases} \tag{7.8}$$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{7.9}$$

Find $P_e = E\left[P_e(N)\right]$.
Solution:

$$P_e = \int_{-\infty}^{0} P_e(x)p_N(x)x$$

$$= \int_{-\infty}^{0} \left(1 - \exp\left(-\frac{x^2}{\gamma}\right)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) x$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right) x$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-x^2\left(\frac{1}{2} + \frac{1}{\gamma}\right)\right) x \tag{7.10}$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right) x$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{(2+\gamma)x^2}{2\gamma}\right) x \tag{7.11}$$

Now,

$$\int_{-\infty}^{0} \exp\left(-\frac{x^2}{2a^2}\right) x = \int_{0}^{\infty} \exp\left(-\frac{x^2}{2a^2}\right) x$$
$$= a \sqrt{\frac{\pi}{2}}$$

Therefore,

$$P_e = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} - \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \sqrt{\frac{\gamma}{\gamma + 2}}$$
$$= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma}{\gamma + 2}}$$

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.
Solution: find the code below

https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex7/4/pverify.png

run the command

python3 pverify.png
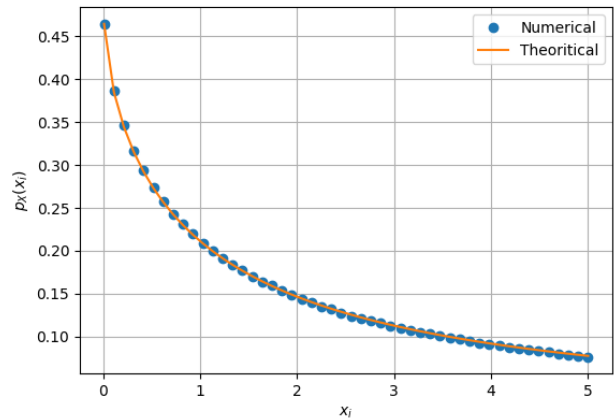

Fig. 7.4: P_e

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.