

Random Numbers

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex1/1.1/exrand.c
we get https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex1/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
gcc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex1/1.2/main.py
```

Run the following command in the terminal to run the code.

```
python3 main.py
```

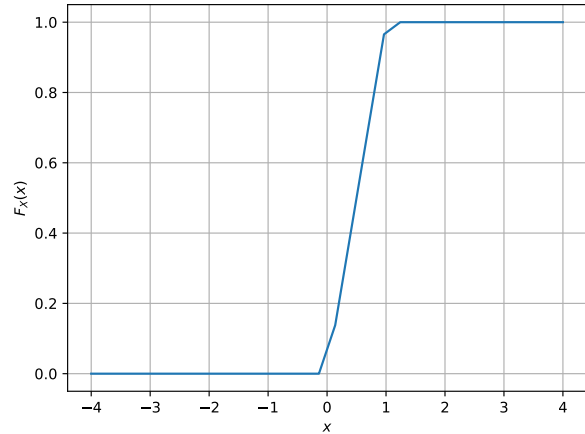


Fig. 1.2: The CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given U is uniform random variable so

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x f_X(x) dx \\ &= 0 + \int_0^x 1 dx \\ &= \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases} \end{aligned}$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.2)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.3)$$

Write a C program to find the mean and variance of U .

Solution:

```
wget https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex1/1.4/exrand.c
wget https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex1/1.4/coeffs.h
```

Use below command to run file,

```
gcc exrand.c -lm
./a.out
```

```
➤ make -s
➤ ./main
mean = 0.500007
var = 0.083301
➤ □
```

Fig. 1.4: Caption

1.5 Verify your result theoretically given that
Given

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_X(x)$$

$$\begin{aligned} E[U] &= \int_{-\infty}^{\infty} x^k f_X(x) dx \\ &= \int_0^1 x \times 1 dx \\ &= \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} \end{aligned}$$

if $k=2$

$$\begin{aligned} E[U^2] &= \int_0^1 x^2 \times 1 dx \\ &= \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{variance} &= E[u - E[u]]^2 \\ &= E[U^2] - E^2[U] \\ &= \frac{1}{3} - \frac{1}{4} = 0.0833 \end{aligned}$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

```
wget https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex2/2.1/exrand.c
wget https://github.com/mohilmukundareddy/
Assignment1/blob/main/ex2/2.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
cc exrand.c -lm
./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in plot, Properties of the CDF:

- $F(x) = P(X \leq x) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du$
- $Q(x) = 1 - F(x)$ This can be used to calculate $F(x)$.

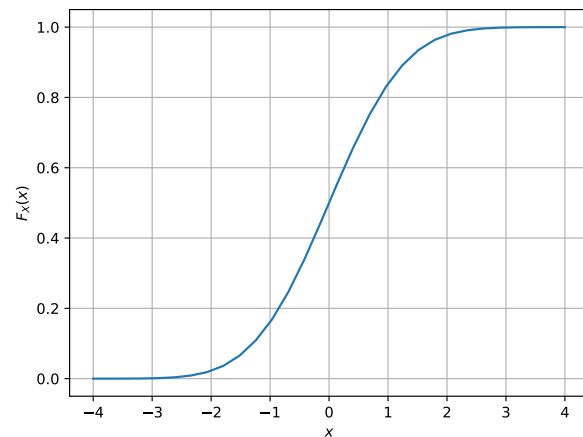


Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

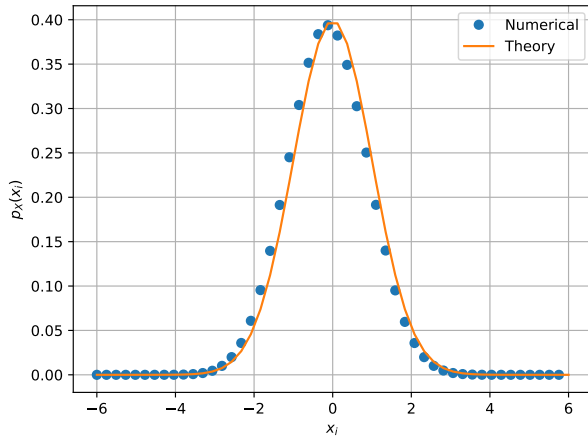


Fig. 2.3: The PDF of X

Solution: The PDF of X is plotted using the code below

<https://github.com/mohilmukundareddy/Assignment1/blob/main/ex2/2.3/main.py>

Use the below command to run the code:

`python3 main.py`

Properties of PDF:

- PDF is symmetric about $x \approx 0$
- graph is similar to bell shaped
- mean of graph is situated at the symmetrical point

2.4 Find the mean and variance of X by writing a C program.

Solution: Running the below code gives
Mean = 0.000326 Variance= 1.000906

`wget https://github.com/mohilmukundareddy/Assignment1/blob/main/ex2/2.4/exrand.c`

Command used:

`gcc exrand.c -lm
./a.out`

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

by property of probability

$$\int_{-\infty}^{\infty} p_X(x) dx = 1$$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^x p_X(x) dx \\ &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \end{aligned}$$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x p(x) dx \\ &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \end{aligned}$$

$\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is an odd function so integral is zero
i.e $E[X] = 0$.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 p(x) dx \\ &= x \int_{-\infty}^{\infty} x p(x) dx - \int \left(\int x p(x) dx \right) dx \\ &= -x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} - \int -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= 1 \end{aligned}$$

$$\text{var}(x) = E[x^2] - E[x]^2 = 1 - 0 = 1$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution:

Running the below code generates samples of V from file uni.dat(U).

<https://github.com/mohilmukundareddy/Assignment1/blob/main/ex3/main.py>

Use the below command in the terminal to run the code:

`python3 main.py`

Now these samples are used to plot by running the below code,

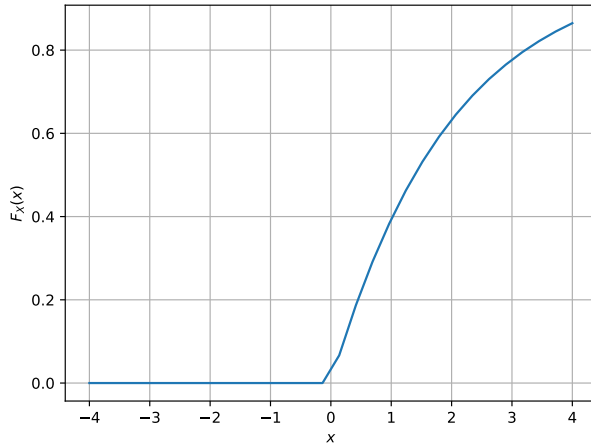


Fig. 3.5: CDF for (3)

<https://github.com/mohilmukundareddy/Assignment1/blob/main/ex3/cdf.py>

Use the below command to run the code:

```
python3 cdf.py
```

3.2 Theoretical expression for $F_V(x)$

$$\begin{aligned}
 F_V(x) &= P\{V \leq x\} \\
 &= P\{-2 \times \ln(1 - U) \leq x\} \\
 &= P\{U \leq 1 - e^{(-\frac{x}{2})}\} \\
 &= F_U\{1 - e^{(-\frac{x}{2})}\} \\
 &= \begin{cases} 1 - e^{(-\frac{x}{2})} & 0 \leq x < \infty \\ 0 & x < 0 \end{cases}
 \end{aligned}$$