Random Numbers

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex1/1.1/exrand.c
we get https://github.com/mohilmukundareddy
    /Assignment1/blob/main/ex1/1.1/coeffs.h
```

Use the below command in the terminal to run the code

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The graph 1.2 is obtained by running the below code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex1/1.2/main.py
```

Run the following command in the terminal to run the code.

```
python3 main.py
```

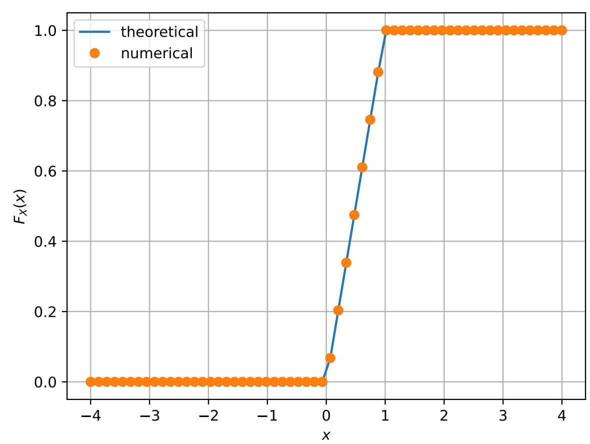

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is uniform random variable so

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(x)dx \\ &= 0 + \int_0^x 1dx \\ &= \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases} \end{aligned}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.3}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex1/1.4/exrand.c
weget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex1/1.4/coeffs.h
```

Use below command to run file,

```
gcc exrand.c −lm
./a.out
```

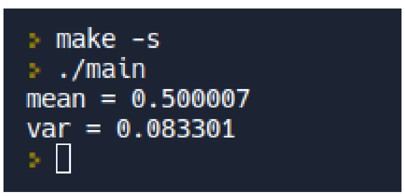

Fig. 1.4: Caption

1.5 Verify your result theoretically given that

Given

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_X(x)$$

$$\begin{aligned} E[U] &= \int_{-\infty}^{\infty} x^k f_X(x)dx \\ &= \int_0^1 x \times 1dx \\ &= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} \end{aligned}$$

if k=2

$$\begin{aligned} E[U^2] &= \int_0^1 x^2 \times 1dx \\ &= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} variance &= E[u - E[u]]^2 \\ &= E[U^2] - E^2[U] \\ &= \frac{1}{3} - \frac{1}{4} = 0.0833 \end{aligned}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex2/2.1/exrand.c
wget hhttps://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex2/2.1/coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
gcc exrand.c −lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in plot,Properties of the CDF:
- $F_X(x) = P(X \le x)$
- $Q_X(x) = P(X > x)$
- $F_X(x) = 1 - Q_X(x)$ This can be used to calculate F (x).

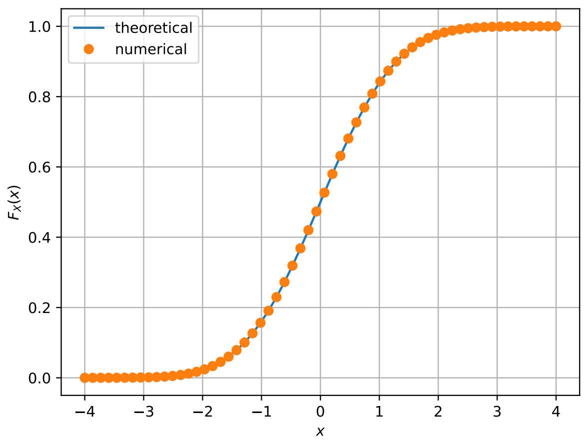

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?

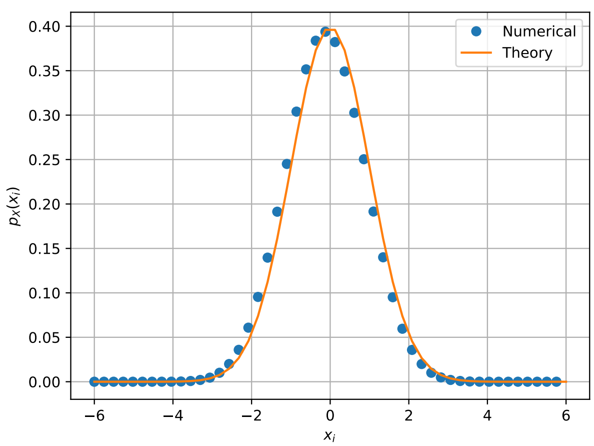

Fig. 2.3: The PDF of X

Solution: The PDF of X is plotted using the code below

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex2/2.3/main.py
```

Use the below command to run the code:

```
python3 main.py
```

Properties of PDF:

- PDF is symmetric about $x \approx 0$
- graph is similar to bell shaped
- mean of graph is situated at the symmetrical point

2.4 Find the mean and variance of X by writing a C program.
Solution: Running the below code gives
Mean = 0.000326 Variance= 1.000906

```
wget https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex2/2.4/exrand.c
```

Command used:

```
gcc exrand.c −lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$$

by property of probability

$$\int_{-\infty}^{\infty} p_X(x)dx = 1$$

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx$$
$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$E[X] = \int_{-\infty}^{\infty} xp(x)dx$$
$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

$\frac{x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ is an odd function so integral is zero i.e $E[X] = 0$.

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p(x)dx$$
$$= x\int_{-\infty}^{\infty} xp(x)dx - \int_{-\infty}^{\infty}\left(\int xp(x)dx\right)dx$$
$$= \left[-x\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx$$
$$= 0 - (-1)$$
$$=$$

we know that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$$

by series expansion $\frac{x}{e^{\frac{x^2}{2}}} = \frac{x}{1+\frac{x^2}{2}+\frac{x^4}{8}+...}$
putting $x = \infty$,we get $\frac{1}{\infty} = 0$
Similarly when $x = -\infty$ we get 0
$var(x) = E[x^2] - E[x] = 1 - 0 = 1$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:
Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex3/main.py
```

Use the below command in the terminal to run the code:

```
python3 main.py
```

Now these samples are used to plot by running the below code

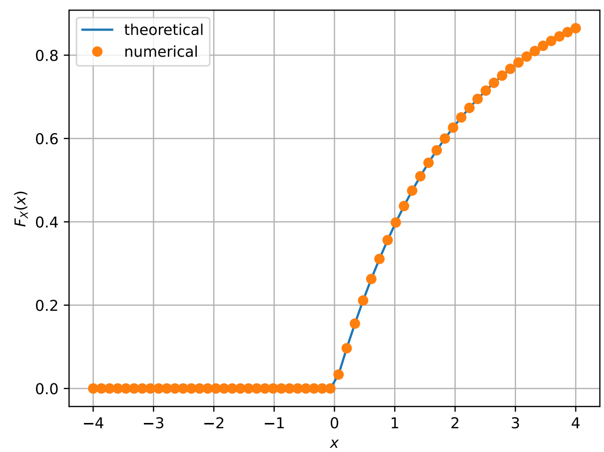

Fig. 3.5: CDF for (3)

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex3/cdf.py
```

Use the below command to run the code:

```
python3 cdf.py
```

3.2 Theorotical expression for $F_V(x)$

$$F_V(x) = P\{V \leq x\}$$
$$= P\{-2 \times \ln(1-U) \leq x\}$$
$$= P\{U \leq 1 - e^{(-\frac{x}{2})}\}$$
$$= F_U\{1 - e^{(-\frac{x}{2})}\}$$
$$= \begin{cases} 1 - e^{(-\frac{x}{2})} & 0 \leq x < \infty \\ 0 & x < 0 \end{cases}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

we get the code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/1/main.
    py
```

run the command

```
python3 main.py
```

4.2 Find the CDF of T. we have code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/2/
    main1.py
```

run the command

```
python3 main1.py
```

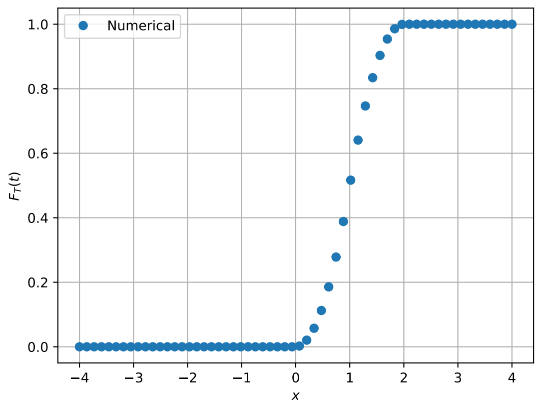

Fig. 4.5: numerical cdf

4.3 Find the PDF of T we have code

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/3/
    main2.py
```

run the command

```
python3 main2.py
```

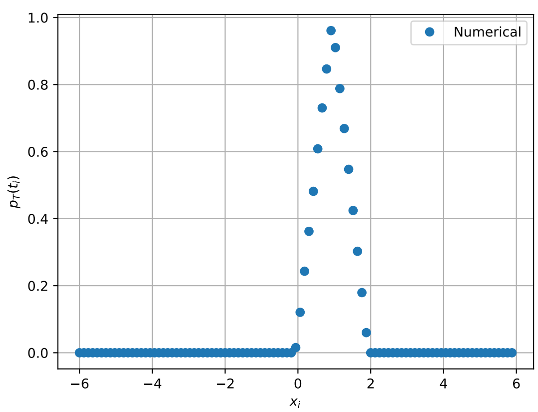

Fig. 4.5: Numerical cdf

4.4 Find the theoretical expressions for the PDF and CDF of T. **Solution:**

$$
\begin{aligned}
F_T(t) &= P\{T \le t\} \\
&= P\{U1 + U2 \le t\}
\end{aligned}
$$

let us take two cases if $0 \le t \le 1$ and $1 < t \le 2$

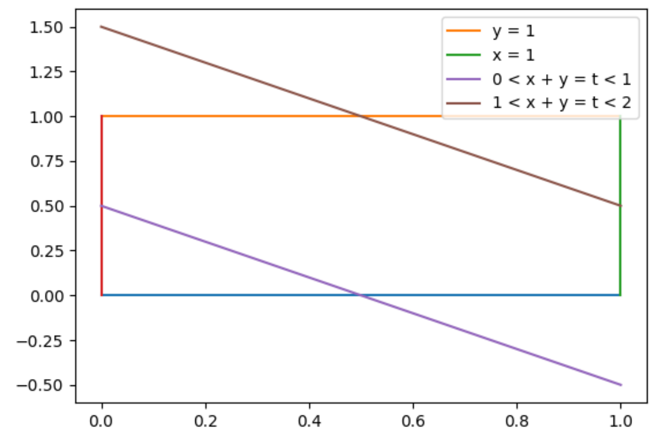

Fig. 4.5: def plot

The above graph is produced by

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex4/2/find.
    py
```

Run the code in terminal

```
python3 find.py
```

from the figures it is evident that $P(U1 + U2 < t, 0 \le t < 1) = \frac{t^2}{2}$
$P(U1 + U2 < t, 1 \le t \le 2) = 1 - \frac{(2-t)^2}{2}$

$$
F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}
$$

$$
P_T(t) = \frac{d(F_T(t))}{dt}
$$

$$
\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t \le 1 \\ 2 - t & 0 < t \le 2 \\ 0 & t > 2 \end{cases}
$$

4.5 Verify your results through a plot Take the code for cdf https://github.com/mohilmukundareddy/Assignment1/blob/main/ex4/5/main1.py

Run in terminal

```
python3 main1.py
```

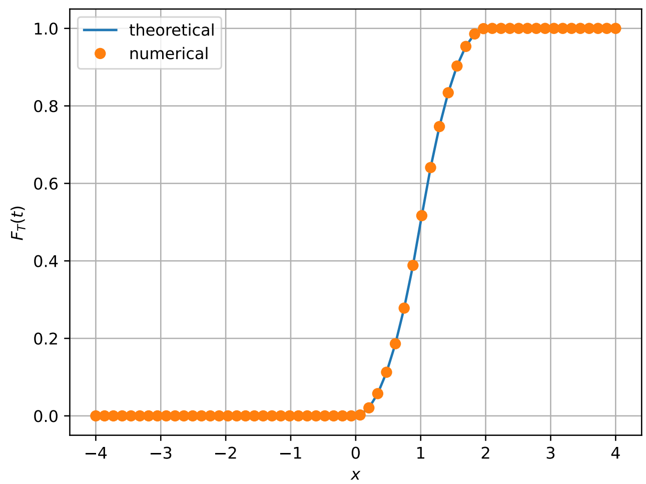

Fig. 4.5: t-cdf

Take the code for pdf

https://github.com/mohilmukundareddy/Assignment1/blob/main/ex4/5/main2.py

Run in terminal

```
python3 main2.py
```

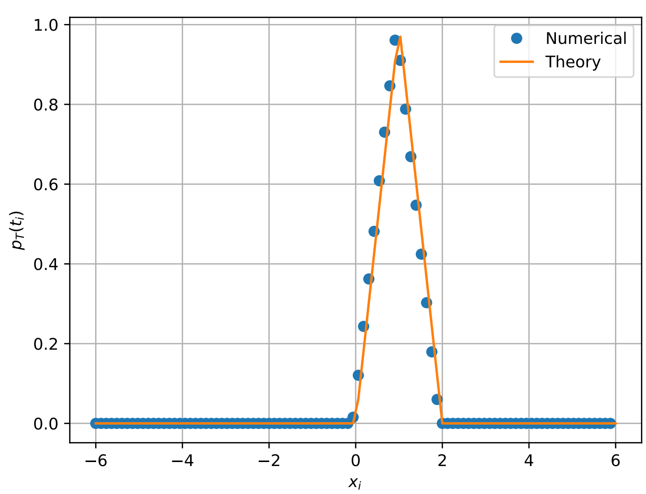

Fig. 4.5: t-pdf

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X1\{1, -1\}$, is Bernoulli and $N \sim 01$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: The sum of squares of n independent standard random normal variables is χ^2 distribution with n degrees of freedom.

$$P_{\chi^2}(x|n) = \frac{x^{\frac{n}{2}-1}e^{\frac{-x}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}, \forall x \geq 0$$

Here k=2,

$$P_{\chi^2}(x|2) = P_V(v) = \frac{e^{\frac{-v}{2}}}{2}$$

For the cummulative distribution

$$F_V(v) = \int_0^v \frac{e^{\frac{-v}{2}}}{2} dv$$
$$= 1 - e^{\frac{-v}{2}}$$

To generate data for V , run the following code,

https://github.com/mohilmukundareddy/Assignment1/blob/main/ex6/1/main.py

Run the below command in terminal,

```
python3 main.py
```

The PDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/1/
    main1.py
```

Use the following command in the terminal to run the code

```
python3 main1.py
```

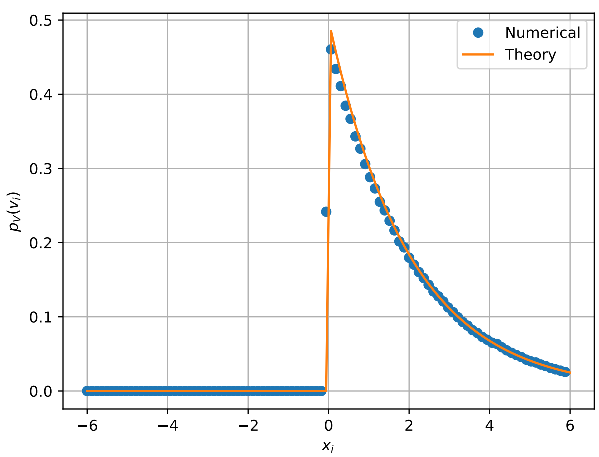

Fig. 6.5: PDF plot

The CDF plot of the $\chi^2(2)$ can be obtained by running the code below,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/1/
    main2.py
```

Use the following command in the terminal to run the code

```
python3 main2.py
```

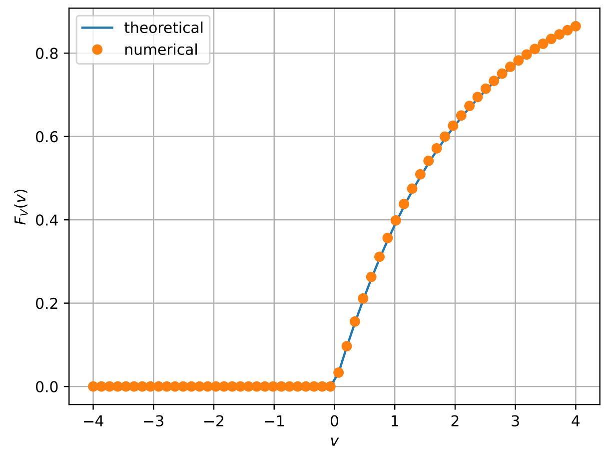

Fig. 6.5: CDF plot

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases}$$

find α.

Solution: From 6.1 we know that $\alpha = 0.5$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V}$$

Solution:

To generate data for A , run the following code,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/3/main.
    py
```

Run the below command in terminal,

```
python3 main.py
```

The PDF plot of A can be obtained by running the code below,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/3/
    main1.py
```

Use the following command in the terminal to run the code

```
python3 main1.py
```

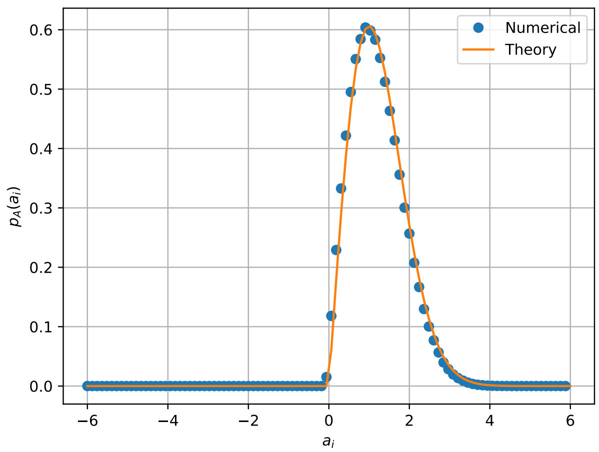

Fig. 6.5: PDF

The CDF plot of the A can be obtained by running the code below,

```
https://github.com/mohilmukundareddy/
    Assignment1/blob/main/ex6/3/
    main2.py
```

Use the following command in the terminal to run the code

```
python3 main2.py
```

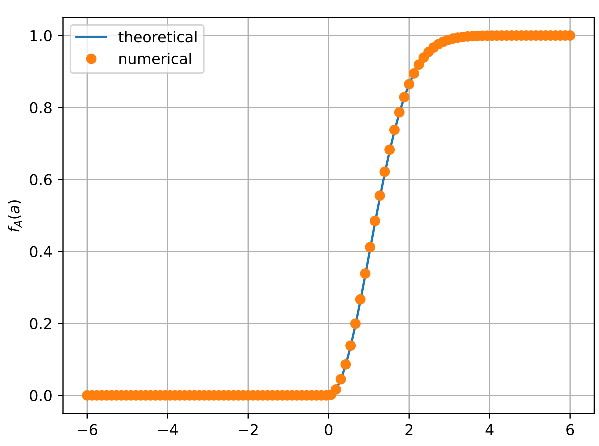

Fig. 6.5: CDF

7 Conditional Probability

7.1

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.5 Plot P_e in problems 7.7.5 and 7.7.5 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.