Digital Signal Processing

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Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script

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in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

use the following command in terminal to run the code find the code required below

```
https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/xnyn.c
https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/3−2.py
```

```
gcc xnyn.c
python3 3−2.py
```

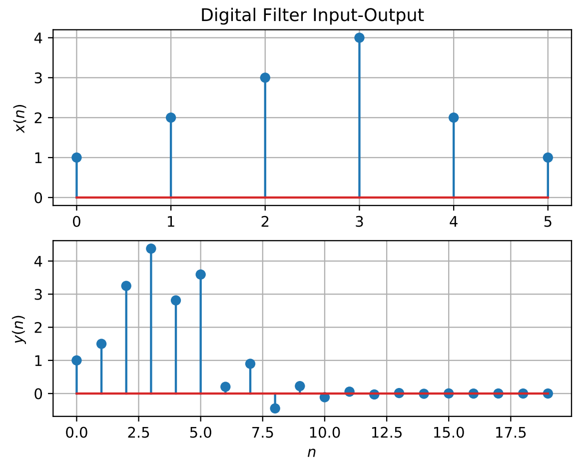

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 4.1?

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$X(z) = ...0 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$
$$+ x(4)z^{-4} + x(5)z^{-5} + 0...$$
$$= 1 + 2z^{-}1 + 3z^{-}2 + 4z^{-}3 + 2z^{-}4 + 1z^{-5}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.7}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.8}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.9}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.12}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.13}$$

and from (4.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.14}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.16}$$

Solution: $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$

$$X(z) = \sum_{n=-\infty}^{-1} 0 + \sum_{n=0}^{\infty} a^n(1)z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^-1}$$

also the series is convergent for $|z| > |a|$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.17}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6.

```
https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/dtft.py
```

run the command

```
python3 dtft.py
```

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.18}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.19}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.20}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.21}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.22}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.23}$$

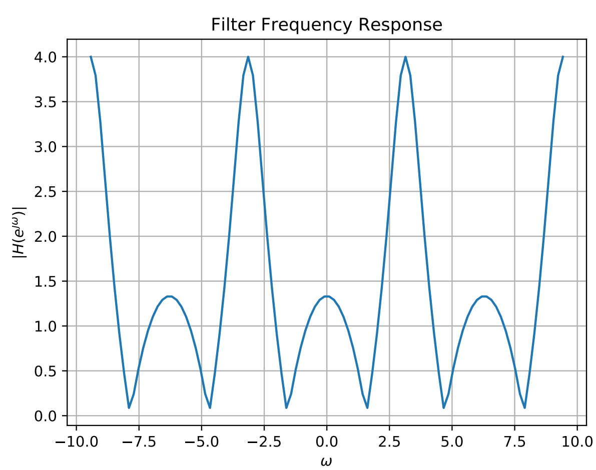

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

The period of the numerator is π and the denominator is 2π taking the L.C.M of numerator and the denominator we get period as 2π

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$

Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n} \tag{4.24}$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k}e^{\jmath\omega n} \tag{4.25}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} \tag{4.26}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} = \begin{cases} \int_{-\pi}^{\pi} & n-k=0 \\ \left.\frac{\exp(\jmath\omega(n-k))}{\jmath(n-k)}\right|_{-\pi}^{\pi} & n-k \neq 0 \end{cases} \tag{4.27}$$

$$= \begin{cases} 2\pi & n-k=0 \\ 0 & n-k\neq 0 \end{cases} \tag{4.28}$$

$$= 2\pi\delta(n-k) \tag{4.29}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n} = 2\pi \sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.30}$$

$$= 2\pi h(n) * \delta(n) \tag{4.31}$$

$$= 2\pi h(n) \tag{4.32}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n} \tag{4.33}$$

5 Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: The $H(z)$ can be written as,

$$H(z) = \frac{1}{1+\frac{z^{-1}}{2}} + \frac{z^{-2}}{1+\frac{z^{-1}}{2}} \tag{5.2}$$

From (4.16) we can write it as,

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.3}$$

5.2 Sketch $h(n)$. Is it bounded? Justify Theoritically.

Solution: Download the code for the plot 5.2 from the below link,

```
wget https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/5_2.py
```

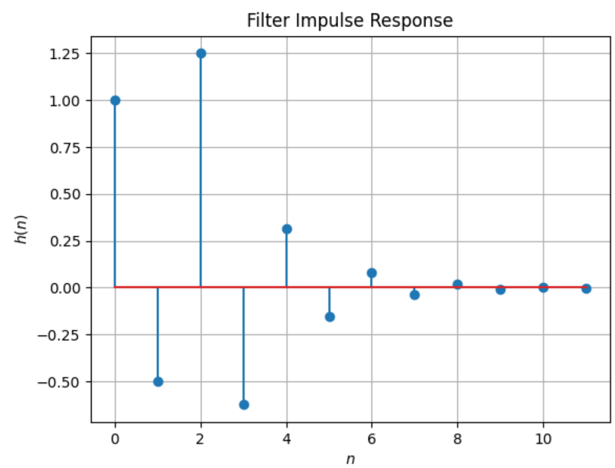

Fig. 5.2: $h(n)$ as inverse of $H(n)$

From the plot it seems like $h(n)$ is bounded and becomes smaller in magnitude as n increases. Using (5.3), we can get theoritical expression as,

$$h(n) = \begin{cases} 0 & ,n<0 \\ \left(\frac{-1}{2}\right)^n & ,0\leq n<2 \\ 5\left(\frac{-1}{2}\right)^n & ,n\geq 2 \end{cases} \tag{5.4}$$

A sequence $\{x_n\}$ is said to be bounded if and only if there exist a positive real number M such that,

$$|x_n| \leq M, \forall n \in \mathcal{N} \tag{5.5}$$

So to say $h(n)$ is bounded we should able to find the M which satisfies (5.5).

For $n<0$,

$$|h(n)| \leq 0 \tag{5.6}$$

For $0 \leq n < 2$,

$$|h(n)| = \left|\frac{-1}{2}\right|^n \tag{5.7}$$

$$= \left(\frac{1}{2}\right)^n \leq 1 \tag{5.8}$$

And for $n \geq 2$,

$$|h(n)| = \left|5\left(\frac{-1}{2}\right)\right|^n \tag{5.9}$$
$$= \left(\frac{5}{2}\right)^n \leq \frac{5}{4} \tag{5.10}$$

From above three cases, we can get M as,

$$M = max\left\{0, 1, \frac{5}{4}\right\} \tag{5.11}$$
$$= \frac{5}{4} \tag{5.12}$$

Therefore, $h(n)$ is bounded using (5.5) with $M = \frac{5}{4}$ i.e.,

$$|h(n)| \leq \frac{5}{4} \forall n \in \mathcal{N} \tag{5.13}$$

5.3 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.14}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^n}\right| \tag{5.15}$$
$$= \lim_{n\to\infty}\left|\frac{-1}{2}\right| \tag{5.16}$$
$$= \frac{1}{2} \tag{5.17}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.4 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.18}$$

Is the system defined by (3.2) stable for the impulse response in (5.1)?

Solution: From (5.3),

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(\left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)\right) \tag{5.19}$$
$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.20}$$
$$= \frac{4}{3} \tag{5.21}$$

$\therefore$ the system is stable.

5.5 Verify the above result using a python code.
Solution: Download the python code from the below link

```
https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/5_5.py
```

Then run the following command,

```
python3 5_5.py
```

5.6 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.22}$$

This is the definition of $h(n)$.
Solution: Download the code for the plot 5.6 from the below link,

```
wget https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/5_6.py
```

Note that this is same as 5.2.
For $n < 0$, $h(n) = 0$ and,

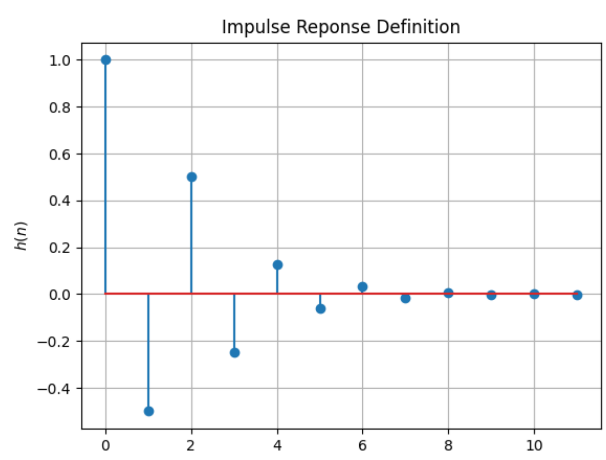

Fig. 5.6: From the definition of $h(n)$

$$h(0) = \delta(0) \tag{5.23}$$
$$= 1 \tag{5.24}$$

For $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) \tag{5.25}$$
$$\implies h(1) = -\frac{1}{2}h(0) \tag{5.26}$$
$$= -\frac{1}{2} \tag{5.27}$$

$n = 2$,

$$h(2) + \frac{1}{2}h(1) = 0 + \delta(0) \tag{5.28}$$
$$h(2) = 1 + \frac{1}{4} \tag{5.29}$$
$$= \frac{5}{4} \tag{5.30}$$

And for $n > 2$ RHS will be 0 so,

$$h(n) = -\frac{1}{2}h(n-1) \tag{5.31}$$

Overall

$$h(n) = \begin{cases} 0 & , n < 0 \\ 1 & , n = 0 \\ -\frac{1}{2} & , n = 1 \\ \frac{5}{4} & , n = 2 \\ -\frac{1}{2}h(n-1) & , n > 2 \end{cases} \tag{5.32}$$

5.7 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.33}$$

Comment. The operation in (5.33) is known as *convolution*.

Solution: Download the code for plot 5.7 from the below link

```
wget https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/ynconv.py
```

5.8 Express the above convolution using a Toeplitz matrix.

Solution: Download the python code from the below link for the plot 5.8,

```
wget https://github.com/mohilmukundareddy/
    EE3900/blob/main/a1/codes/5_8.py
```

Then run the following command,

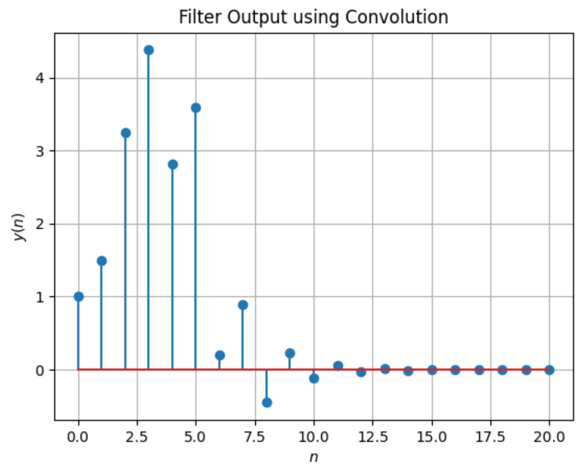

Fig. 5.7: $y(n)$ using the convolution definition

```
python3 5_8.py
```

From (5.33),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.34}$$

To understand how we can use a Toeplitz matrix, we will see what we are doing in (5.33)

$$y(0) = x(0)h(0) \tag{5.35}$$
$$y(1) = x(0)h(1) + x(1)h(0) \tag{5.36}$$
$$y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) \tag{5.37}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.38}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.39}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.40}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.41}$$

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.42}$$

And from (5.4)

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.43}$$

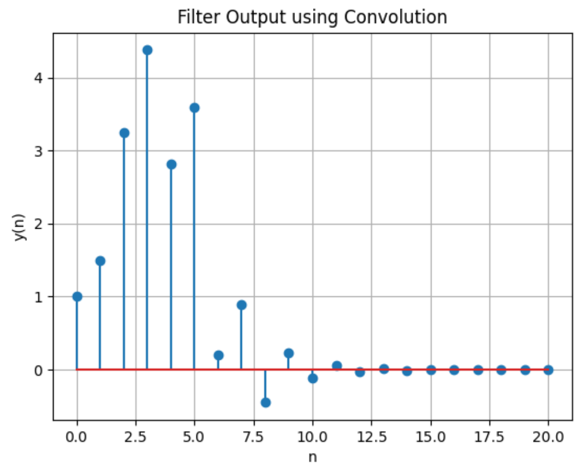

Fig. 5.8: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

Now using (5.41),

$$y(n) = x(n) * h(n) \tag{5.44}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.45}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.46}$$

5.9 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.47}$$

Solution: Substitute $k = n - k$ in (5.33), we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.48}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.49}$$

$$= \sum_{k-\infty}^{\infty} x(n-k)h(k) \tag{5.50}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N} \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$

Solution: Download the following Python code that plots Fig. 6.1.

```
https://github.com/
    mohilmukundareddy/
    EE3900/blob/main/a1/
    codes/6_1.py
```

Run the code by executing

```
python 6_1.py
```

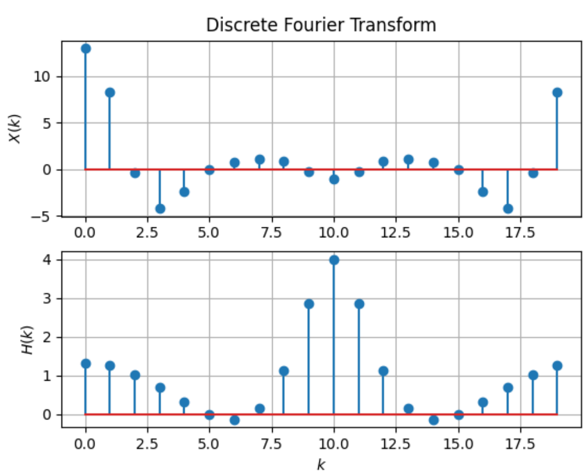

Fig. 6.1: Plots of the real parts of the discrete Fourier transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Download the following Python code that plots Fig. 6.2.

```
https://github.com/
    mohilmukundareddy/
    EE3900/blob/main/a1/
    codes/6_2.py
```

Run the code by executing

```
python 6_2.py
```

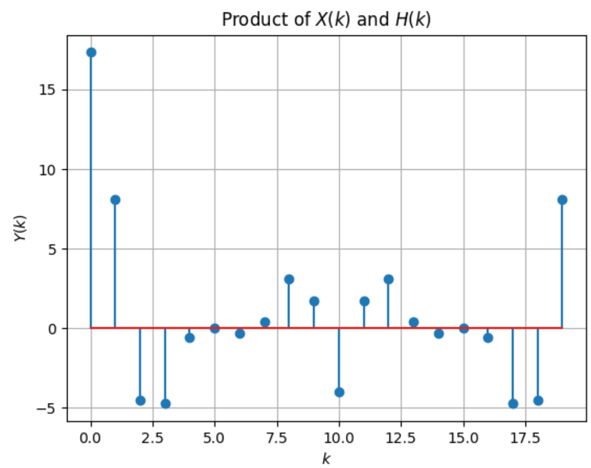

Fig. 6.2: Plot of $Y(k)$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\jmath 2\pi kn/N} \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: Download the following Python code that plots Fig. 6.3.

```
https://github.com/
    mohilmukundareddy/EE3900/
    blob/main/a1/codes/6_3.py
```

Run the code by executing

```
python 6_3.py
```

The plot is exactly the same as that obtained in Fig. **??**. Therefore, we conclude that

$$y(n) = x(n) * h(n) \tag{6.4}$$

$$\Longleftrightarrow Y(k) = X(k)H(k) \tag{6.5}$$

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

Solution: Download the following Python code that plots Fig. 6.4.

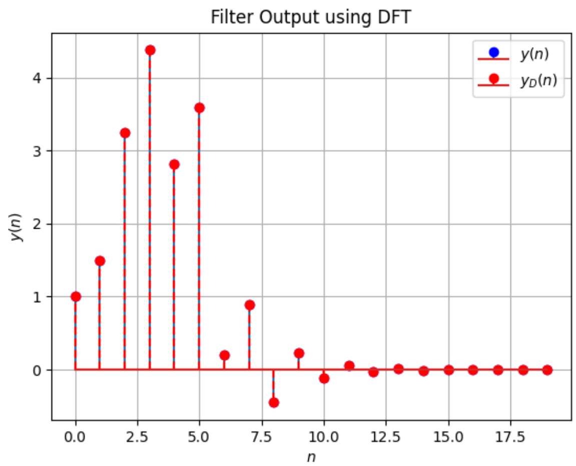

Fig. 6.3: Plot of the inverse discrete Fourier transform of $Y(k)$

https://github.com/mohilmukundareddy/EE3900/blob/main/a1/codes/6_4.py

Run the code by executing

```
python 6_4.py
```

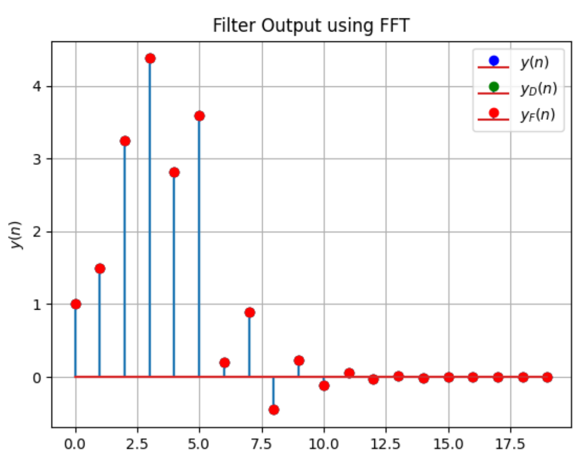

Fig. 6.4: Plot of $y(n)$ by fast Fourier transform

The plot is exactly the same as that obtained in Fig.3-2

7 FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\vec{F}_N$.
3. Let
$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: Given
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$W_N^2 = e^{2(-j2\pi/N)} \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$
Solution: $\vec{I}_2$ is 2×2 identity matrix
$$\begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} = \begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} \tag{7.13}$$
Given
$$\vec{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.14}$$
$$\vec{D}_2 = diag(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.15}$$

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.16}$$

$$\vec{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.17}$$

$$\begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \tag{7.18}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.19}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.20}$$

which is same as $\vec{F}_4$.

$$\therefore \vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \tag{7.21}$$

7. Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.22}$$

8. Find

$$\vec{P}_4\vec{x} \tag{7.23}$$

Solution: From (7.16),

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.24}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.25}$$

After proper zero padding of $\vec{P}_4$,

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.26}$$

$$\vec{P}_4\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.27}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7.28}$$

9. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.29}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Given $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.30}$$

$$\vec{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{7.31}$$

$$\vec{F}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.32}$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \tag{7.33}$$

Upon linear transformation over k,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.34}$$

$$\therefore \vec{X} = \vec{F}_N \vec{x} \tag{7.35}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.37}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.40}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.41}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.42}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.43}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.44}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.47}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.48}$$

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.49}$$

compte the DFT using (7.29) **Solution:**

$$\vec{F}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \tag{7.50}$$

Using (7.49),

$$\vec{X} = \vec{F}_6 \vec{x} \tag{7.51}$$

$$\vec{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.52}$$

$$= \begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \tag{7.53}$$

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution: $\vec{x}$ after padding is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.54}$$

Using ,

$$\vec{F}_8 = \begin{bmatrix} \vec{I}_4 & \vec{D}_4 \\ \vec{I}_4 & -\vec{D}_4 \end{bmatrix} \begin{bmatrix} \vec{F}_4 & 0 \\ 0 & \vec{F}_4 \end{bmatrix} \vec{P}_8 \tag{7.55}$$

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.56}$$

$$\vec{F}_2 = \begin{bmatrix} \vec{I}_1 & \vec{D}_1 \\ \vec{I}_1 & -\vec{D}_1 \end{bmatrix} \begin{bmatrix} \vec{F}_1 & 0 \\ 0 & \vec{F}_1 \end{bmatrix} \vec{P}_2 \tag{7.57}$$

$$\vec{F}_1 = [1] \tag{7.58}$$

Calculating $\vec{F}_2$,

$$\vec{F}_2 = \begin{bmatrix} \vec{F}_1 & D_1\vec{F}_1 \\ \vec{F}_1 & -D_1\vec{F}_1 \end{bmatrix} \vec{P}_2 \tag{7.59}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.60}$$

Calculating $\vec{F}_4$,

$$\vec{D}_2 = diag(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.61}$$

$$D_2\vec{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.62}$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \tag{7.63}$$

$$\vec{F}_4 = \begin{bmatrix} \vec{F}_2 & D_2\vec{F}_2 \\ \vec{F}_2 & -D_2\vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.64}$$

$$\vec{F}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -j & j \\ 1 & 0 & -1 & -1 \\ 0 & 1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.65}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.66}$$

Calculating $\vec{F}_8$,

$$\vec{D}_4 = diag\left(1, W_8, W_8^2, W_8^3\right) \tag{7.67}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \tag{7.68}$$

$$D_4\vec{F}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.69}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ -1 & 1 & 0 & -j \\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \end{bmatrix} \tag{7.70}$$

$F_8 = A\vec{B}P_8$ where

$$\vec{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1+j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & j \\ 0 & 0 & 0 & 1 & 0 & \frac{-1+j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} \end{bmatrix} \tag{7.71}$$

$$\vec{B} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -j & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & j & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & j & -1 & j \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & -j & -1 & j \end{bmatrix} \tag{7.72}$$

$$\vec{F_8} = \begin{bmatrix} 13 \\ -3.12 - 6.53j \\ j \\ 1.12 - 0.53j \\ -1 \\ 1.12 + 0.53j \\ -j \\ -3.12 + 6.5355 \end{bmatrix} \tag{7.73}$$

And $\vec{P_8}$ is a permutation matrix.

$$\vec{X} = \begin{bmatrix} 13 \\ -4 - 8j \\ j \\ 2 - 2j \\ -1 \\ 2 + 2j \\ -j \\ -4 + 8j \end{bmatrix} \tag{7.74}$$

13. Write a C program to compute the 8-point FFT.

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.