Digital Signal Processing

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1. Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900-Linear-Systems-
    and-Signal-Processing/main/charger/
    codes/1.1.py
python3 1.1.py
```

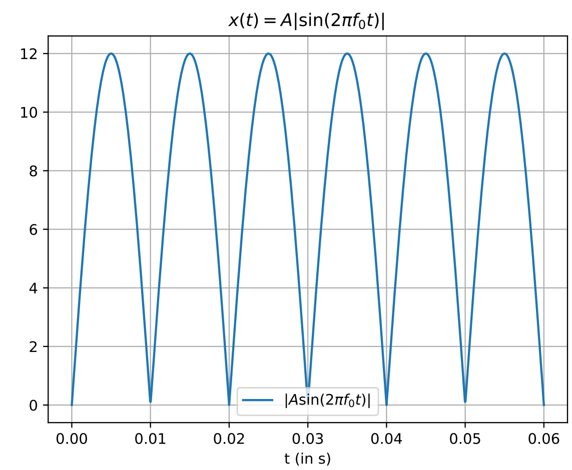

Fig. 1.1.

1.2 Show that $x(t)$ is periodic and find its period.
Solution: If a signal $x(t)$ is periodic then

$$x(t+T) = x(t) \tag{1.2}$$

where T is known as fundamental period. Since $|sin\theta|$ function is periodic, x(t) is also periodic.

$$\text{Fundamental Period} = T = \frac{1}{2}\left(\frac{2\pi}{2\pi f_0}\right) \tag{1.3}$$

$$= \frac{1}{2f_0} \tag{1.4}$$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.3}$$

Mulitply $e^{-\jmath 2\pi l f_0 t}$ on both sides

$$x(t)e^{-\jmath 2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} e^{-\jmath 2\pi l f_0 t} \tag{2.4}$$

Integrate on both sides with respect to 't' between $-T$ to T where T is fundamental time period of x(t).
Using (1.4),

$$T = \frac{1}{2f_0} \tag{2.5}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi (k-l) f_0 t}\, dt \tag{2.6}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (k-l) f_0 t}\, dt \tag{2.7}$$

The above integral:

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (k-l) f_0 t}\, dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt & k = l \end{cases} \tag{2.8}$$

$$\therefore \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\, dt = \left(\frac{1}{f_0}\right) c_k \tag{2.9}$$

$$\therefore c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.10}$$

2.2 Find c_k for (1.1)
Solution: c_k can be calculated even simpler by using

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.11}$$

$x(t) = A_0 \sin(2\pi f_0 t)$ in 0 to $\frac{1}{2f_0}$ region.
Also,

$$\sin\theta = \frac{e^{\jmath\theta} - e^{-\jmath\theta}}{2\jmath} \tag{2.12}$$

Using (2.12),

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \left(\frac{e^{\jmath 2\pi f_0 t} - e^{-\jmath 2\pi f_0 t}}{2\jmath}\right) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.13}$$

$$= A_0 f_0 \int_0^{\frac{1}{2f_0}} \left(\frac{e^{\jmath 2\pi (1-k) f_0 t} - e^{\jmath 2\pi (-1-k) f_0 t}}{\jmath}\right) dt \tag{2.14}$$

$$= A_0 f_0 \left[\frac{e^{\jmath 2\pi (1-k) f_0 t}}{-2\pi(1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} - \frac{e^{\jmath 2\pi (-1-k) f_0 t}}{-2\pi(-1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}}\right] \tag{2.15}$$

$$= A_0 \left[\frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)}\right] \tag{2.16}$$

$$= \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = even \\ 0 & k = odd \end{cases} \tag{2.17}$$

2.3 Verify (1.1) using python.
Solution:

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900−Linear−Systems−
    and−Signal−Processing/main/charger/
    codes/2.3.py
python3 2.3.py
```

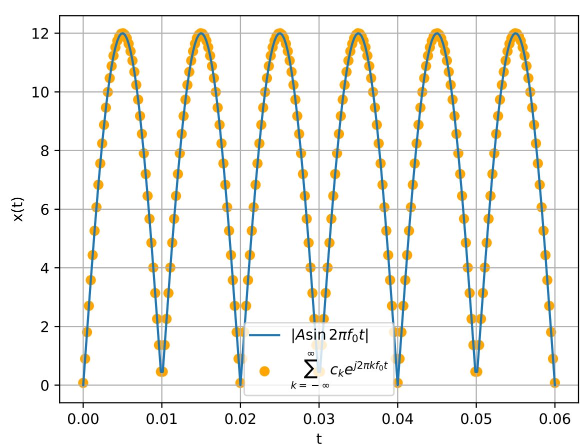

Fig. 2.3.

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t\right) \tag{2.18}$$

and obtain the formulae for a_k and b_k.
Solution: Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.19}$$

As,

$$e^{\jmath 2\pi k f_0 t} = \cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t) \tag{2.20}$$

Substituting leads to

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \left[\cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t)\right] \tag{2.21}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t) \tag{2.22}$$

$$= \sum_{k=-\infty}^{-1} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] + c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.23}$$

$$= \sum_{k=1}^{\infty} \left[c_{-k} \cos(2\pi k f_0 t) - j c_{-k} \sin(2\pi k f_0 t)\right] + c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.24}$$

$$= c_0 + \sum_{k=1}^{\infty} \left[(c_k + c_{-k}) \cos(2\pi k f_0 t) + j (c_k - c_{-k}) \sin(2\pi k f_0 t)\right] \tag{2.25}$$

Replacing $(c_k + c_{-k}) \to a_k$ and $j(c_k - c_{-k}) \to b_k$,

$$= c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.26}$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.27}$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \tag{2.28}$$

$$b_k = j(c_k - c_{-k}) \tag{2.29}$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.30}$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{\jmath 2\pi k f_0 t}\, dt \tag{2.31}$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \left[e^{-\jmath 2\pi k f_0 t} + e^{\jmath 2\pi k f_0 t}\right] dt \tag{2.32}$$

$$= 2 f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t)\, dt \tag{2.33}$$

Parallely,

$$b_k = -2 j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin(2\pi k f_0 t)\, dt \tag{2.34}$$

2.5 Find a_k and b_k for (1.1)
Solution: Using (2.28) and (2.29) with (2.17),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi(1-k^2)} & k = even \\ \frac{2A_0}{\pi} & k = 0 \\ 0 & k = odd \end{cases} \tag{2.35}$$

$$b_k = j(c_k - c_{-k}) = 0 \tag{2.36}$$

2.6 Verify (2.18) using python.
Solution:

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900−Linear−Systems−
    and−Signal−Processing/main/charger/
    codes/2.6.py
python3 2.3.py
```

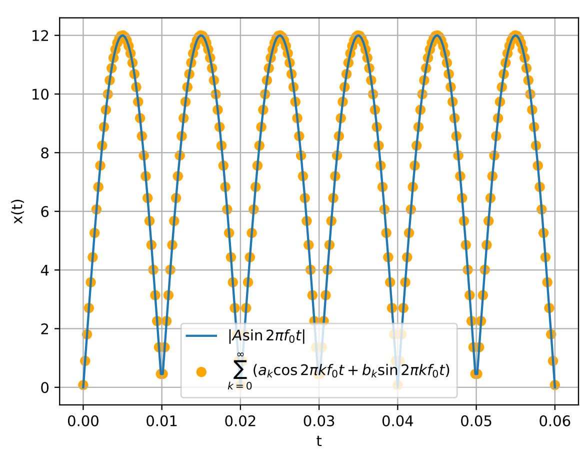

Fig. 2.6.

3. Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

$$\tag{3.5}$$

Solution:

$$\mathcal{F}\{g(t - t_0)\} = \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi ft}\, dt \tag{3.6}$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f (t - t_0)}\, dt \tag{3.7}$$

$$= G(f) e^{-j2\pi f t_0} \tag{3.8}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.9}$$

Solution: From the definition of Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df \tag{3.10}$$

Replace t → f,

$$g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi ft}\, dt \tag{3.11}$$

Replace f → -f,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi ft}\, dt \tag{3.12}$$

$$= \mathcal{F}\{G(t)\} \tag{3.13}$$

$$\therefore G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.14}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$

Solution:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt \tag{3.15}$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt \tag{3.16}$$

$$\tag{3.17}$$

Since $e^{-j2\pi ft} = 1$ for t=0 and remaining integrand is zero for t $\neq$ 0.

$$= \int_{-\infty}^{\infty} \delta(t)\, dt \tag{3.18}$$

$$= 1 \tag{3.19}$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$

Solution:

$$\mathcal{F}\left\{e^{-j2\pi f_0 t}\right\} = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi (f + f_0) t}\, dt \tag{3.20}$$

$$= \int_{-\infty}^{\infty} \mathcal{F}\{\delta(t)\} e^{-j2\pi (f + f_0) t}\, dt \tag{3.21}$$

Using (3.9),

$$= \delta(-(f + f_0)) = \delta(f + f_0) \tag{3.22}$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$

Solution:

$$\mathcal{F}\{\cos(2\pi f_0 t)\} = \mathcal{F}\left\{\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right\} \tag{3.23}$$

Using (3.22),

$$= \frac{\delta(f + f_0) + \delta(f - f_0)}{2} \tag{3.24}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python. Using (2.1),

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}2\pi k f_0 t}\right\} \tag{3.25}$$

$$X(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \tag{3.26}$$

3.9 Show that

$$\mathrm{rect}\,(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}\,(t) \tag{3.27}$$

Verify using python.

Solution:

$$\mathcal{F}\{\mathrm{rect}\,(t)\} = \int_{-\infty}^{\infty} \mathrm{rect}\,(t) e^{-j2\pi ft}\, dt \tag{3.28}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 \cdot e^{-j2\pi ft}\, dt \tag{3.29}$$

$$= \left.\frac{e^{-j2\pi ft}}{-j2\pi f}\right|_{-\frac{1}{2}}^{\frac{1}{2}} \tag{3.30}$$

$$= \mathrm{sinc}\,(t) \tag{3.31}$$

3.10 $\mathrm{sinc}\,(t) \xleftrightarrow{\mathcal{F}} ?$. Verify using python.

Solution: Using (3.31), (3.9) and even property of rect function,

$$\mathrm{sinc}\,(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}\,(t) \tag{3.32}$$

4. Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution:

$$x[t] \longrightarrow \boxed{h[t]} \longrightarrow x[t] * h[t]$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow X(f)H(f)$$

$$X(f)H(f) = V_0\delta(f) \tag{4.1}$$

Above equation indicates that H(f) will pass X(f) for f=0.
$\therefore$ H(f) should be a low pass filter.

$$|H(f)| = \frac{V_0}{\left(\frac{2A_0}{\pi}\right)} = \frac{V_0\pi}{2A_0} \tag{4.2}$$

$$H(f) = \frac{V_0\pi}{2A_0} \quad in \quad -2f_0 \le f \le 2f_0 \tag{4.3}$$

$$H(f) = \frac{V_0\pi}{2A_0}\text{rect}\left(\frac{f}{4f_0}\right) \tag{4.4}$$

4.2 Find $h(t)$.
Solution: Using (4.4) and (3.32),

$$h(t) = \frac{2V_0\pi f_0}{A_0}\text{sinc}\,(4f_0t) \tag{4.5}$$

4.3 Verify your result using through convolution.

5. Filter Design

5.1 Design a Butterworth filter for $H(f)$.
5.2 Design a Chebyschev filter for $H(f)$.
5.3 Design a circuit for your Butterworth filter.
5.4 Design a circuit for your Chebyschev filter.