Digital Signal Processing

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Circuits and Transforms

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7 Oct 2022

Contents

1. Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2. Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.
2. Draw the circuit using latex-tikz.
Solution: The following code yields Fig.2.2

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/Tikz
    %20Circuits/2.2.tex
```

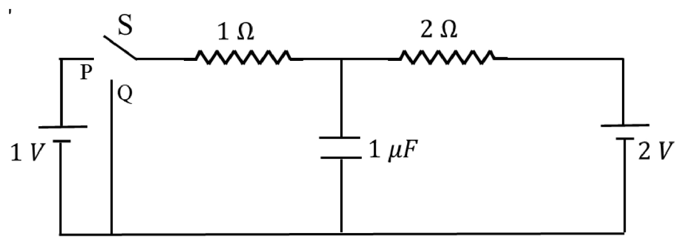

Fig. 2.1.

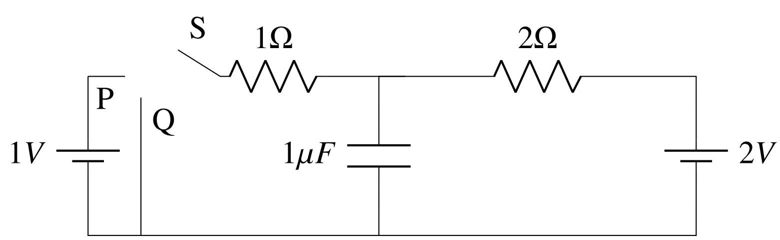

Fig. 2.2. Given Circuit

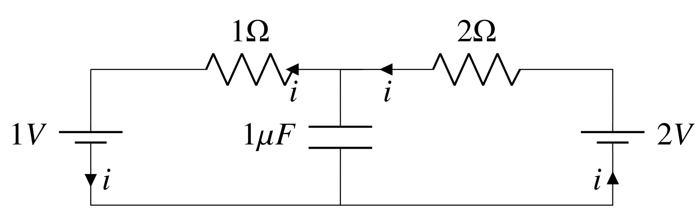

Fig. 2.3. Before switching S to Q

3. Find q_1.
Solution: Before switching S to Q: Calculating

current,

$$2 - 2i - i - 1 = 0 \tag{2.1}$$
$$\Rightarrow i = \frac{1}{3} \tag{2.2}$$

Potential Difference between capacitor at steady state is

$$V_c = 2 - \left(\frac{2}{3}\right) = \frac{4}{3} \tag{2.3}$$
$$q_1 = \frac{4}{3} \cdot 1 \tag{2.4}$$
$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We know that from definition of Laplace Transform,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt U(s) \quad = \int_0^\infty u(t)e^{-st}\,dt \tag{2.6}$$

Using (1.1),

$$U(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.7}$$
$$= \int_0^\infty e^{-st}\,dt \tag{2.8}$$
$$= -\left(0 - \frac{1}{s}\right) \tag{2.9}$$
$$= \frac{1}{s} \tag{2.10}$$

ROC is $Re(s) > 0$ since $e^{-st} < \infty$ for $t \to \infty$ The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/codes
    /2.4.py
python3 2.4.py
```

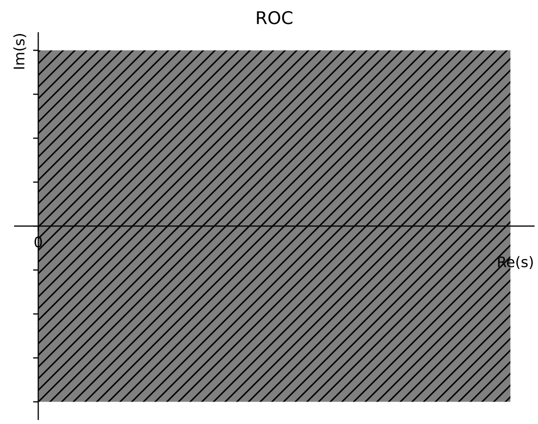

Fig. 2.4.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.11}$$

and find the ROC. **Solution:** From (2.6),

$$F(s) = \int_0^\infty u(t)e^{-at}e^{-st}\,dt \tag{2.12}$$
$$= \int_0^\infty u(t)e^{-(s+a)t}\,dt \tag{2.13}$$
$$= \int_0^\infty e^{-(s+a)t}\,dt \tag{2.14}$$
$$= -\left(0 - \frac{1}{s+a}\right) \tag{2.15}$$
$$= \frac{1}{s+a} \tag{2.16}$$

ROC is

$$s + a > 0 \Rightarrow s > -a \tag{2.17}$$

The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/codes
    /2.5.py
python3 2.5.py
```

6. Now consider the following resistive circuit transformed from Fig. 2.8 where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.18}$$
$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.19}$$

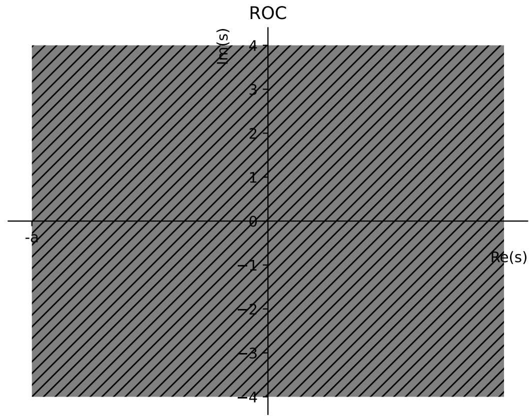

Fig. 2.5.

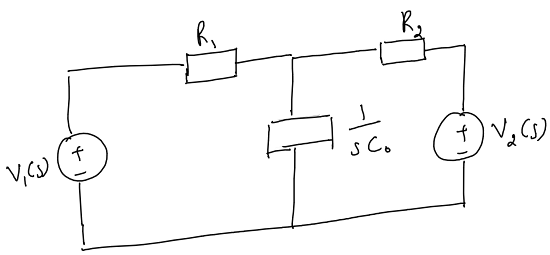

Fig. 2.6.

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$V_1(s) = \frac{1}{s} \tag{2.20}$$

$$V_2(s) = \frac{2}{s} \tag{2.21}$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.22}$$

$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.23}$$

$$\implies V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.24}$$

$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.25}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right), \Re(s) > 0 \tag{2.26}$$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{2.27}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{2.28}$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\text{F}$

$$v_c(t) = \frac{4}{3}\left(1 - e^{-\frac{3}{2}\times 10^6 t}\right)u(t)\ \text{V} \tag{2.29}$$

The following command plots the above equation.

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/codes
    /2.7.py
python3 2.7.py
```

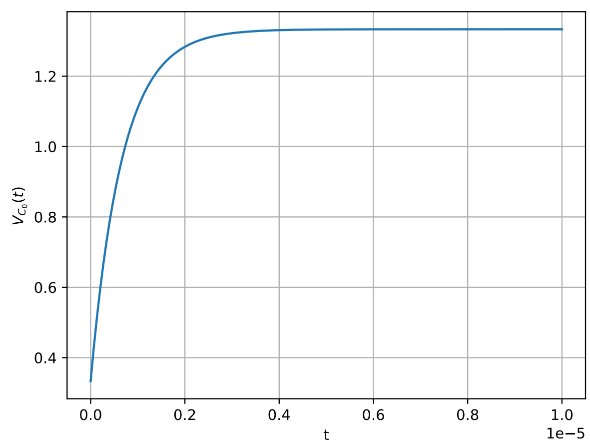

Fig. 2.7. Plot of $V_{C_0}(t)$

8. Verify your result using ngspice.
Solution: The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/codes
    /2.8.cir
ngspice 2.8.cir
python3 2.8.py
```

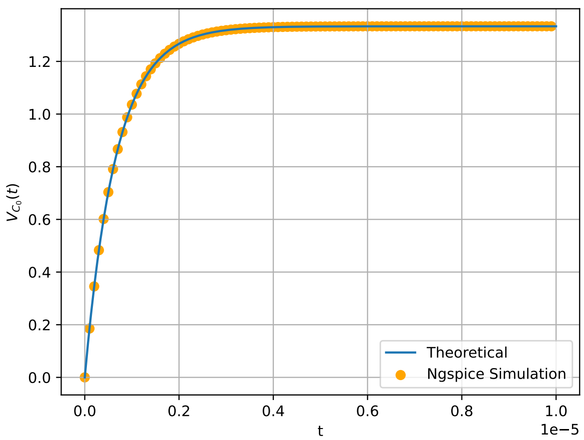

Fig. 2.8.

3. Initial Conditions

1. Find q_2 in Fig. 2.8.
Solution: At steady state, $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1+2} = \frac{2}{3}$$
$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution: The following command downloads tex file of circuit.

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/Tikz
    Circuits/3.2.tex
```

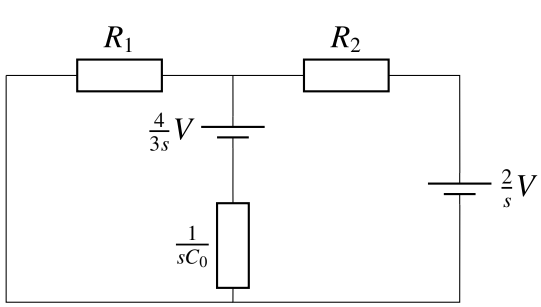

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$
Solution: Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.1}$$
$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.2}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: From (3.2),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.3}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.4}$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-(1.5\times10^6)t}\right)u(t) \tag{3.5}$$

The following command plots the above equation.

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/codes
    /3.4.py
python3 3.4.py
```

5. Verify your result using ngspice.
Solution: The following command plots Fig.3.3

```
wget https://github.com/mohilmukundareddy/
    Signal−processing/blob/main/cktsig/codes
    /3.5.cir
ngspice 3.5.cir
python3 3.5.py
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \tag{3.6}$$

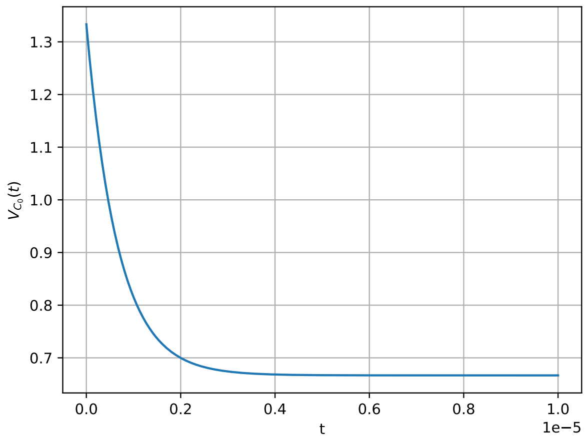

Fig. 3.2. Plot of $V_{C_0}(t)$

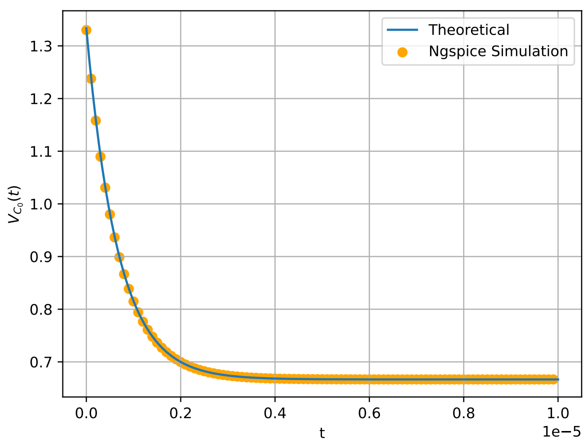

Fig. 3.3.

Using (3.5),

$$v_{C_0}(0+) = \lim_{t \to 0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.7}$$

$$v_{C_0}(\infty) = \lim_{t \to \infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.8}$$

7. Obtain Fig. 3.2 using the equivalent differential equation
Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{3.9}$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sQ(s) - q(0^-)\right) = 0 \tag{3.10}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.11}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.12}$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.13}$$

$$\implies \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.14}$$

which is the same equation as the one we obtained from Fig. 3.2

4. Bilinear Transform

4.1. In Fig. 2.8, consider the case when S is switched to Q right in the beginning. Formulate the differential equation
Solution: The differential equation is the same as before

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.2}$$

$$q(0^-) = q(0) = 0 \tag{4.3}$$

4.2. Find $H(s)$ considering the ouput voltage at the capacitor
Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - 0 = 0 \tag{4.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + sC_0V_c(s) = \frac{V_2(s)}{R_2} \tag{4.5}$$

$$\implies \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.6}$$

$$H(s) = \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2C_0}}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.7}$$

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.8}$$

4.3. Plot $H(s)$. What kind of filter is it?
Solution: Download the following Python code that plots Fig. 4.1

```
https://github.com/mohilmukundareddy/Signal
    −processing/blob/main/cktsig/codes/4.3.py
python 4.3.py
```

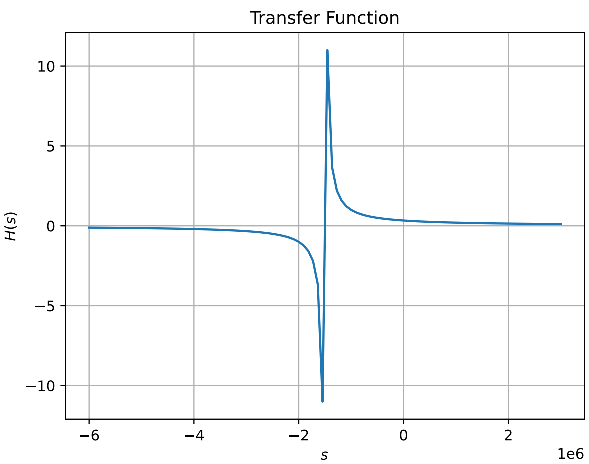

Fig. 4.1. Plot of $H(s)$

$$H(\jmath\omega) = \frac{5\times 10^5}{\jmath\omega + 1.5\times 10^6} \tag{4.9}$$

$$\implies \left|H(\jmath\omega)\right| = \frac{5\times 10^5}{\sqrt{\omega^2 + 2.25\times 10^{12}}} \tag{4.10}$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases.
In other words, the amplitude of high-frequency signals gets diminished and they get filtered out.
Therefore, this is a low-pass filter.

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.11}$$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.12}$$

$$\implies C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \tag{4.13}$$

$$\implies v_c(t)|_{t=n}^{n+1} = \int_n^{n+1}\left(\frac{2u(t) - v_c(t)}{R_2C_0} - \frac{v_c(t)}{R_1C_0}\right)\mathrm{d}t \tag{4.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)\mathrm{d}t \approx \frac{b-a}{2}(f(a) + f(b)) \tag{4.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n) + u(n+1)) - \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \tag{4.16}$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n) + u(n+1)) \tag{4.17}$$

4.5. Find $H(z)$
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \tag{4.18}$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \tag{4.19}$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \tag{4.20}$$

$$\implies x(n) = 2u(n) \tag{4.21}$$

$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \tag{4.22}$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.23}$$

$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \tag{4.24}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.25}$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5 (1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1) z^{-1}} \tag{4.26}$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right) \tag{4.27}$$

$$\implies |z| > 1 \tag{4.28}$$

4.6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.29}$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2 C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \tag{4.30}$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0}\right)(1 + z^{-1})} \tag{4.31}$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \tag{4.32}$$

$$= \frac{2.5 \times 10^5 (1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1) z^{-1}} \tag{4.33}$$

which is the same as what we obtained earlier