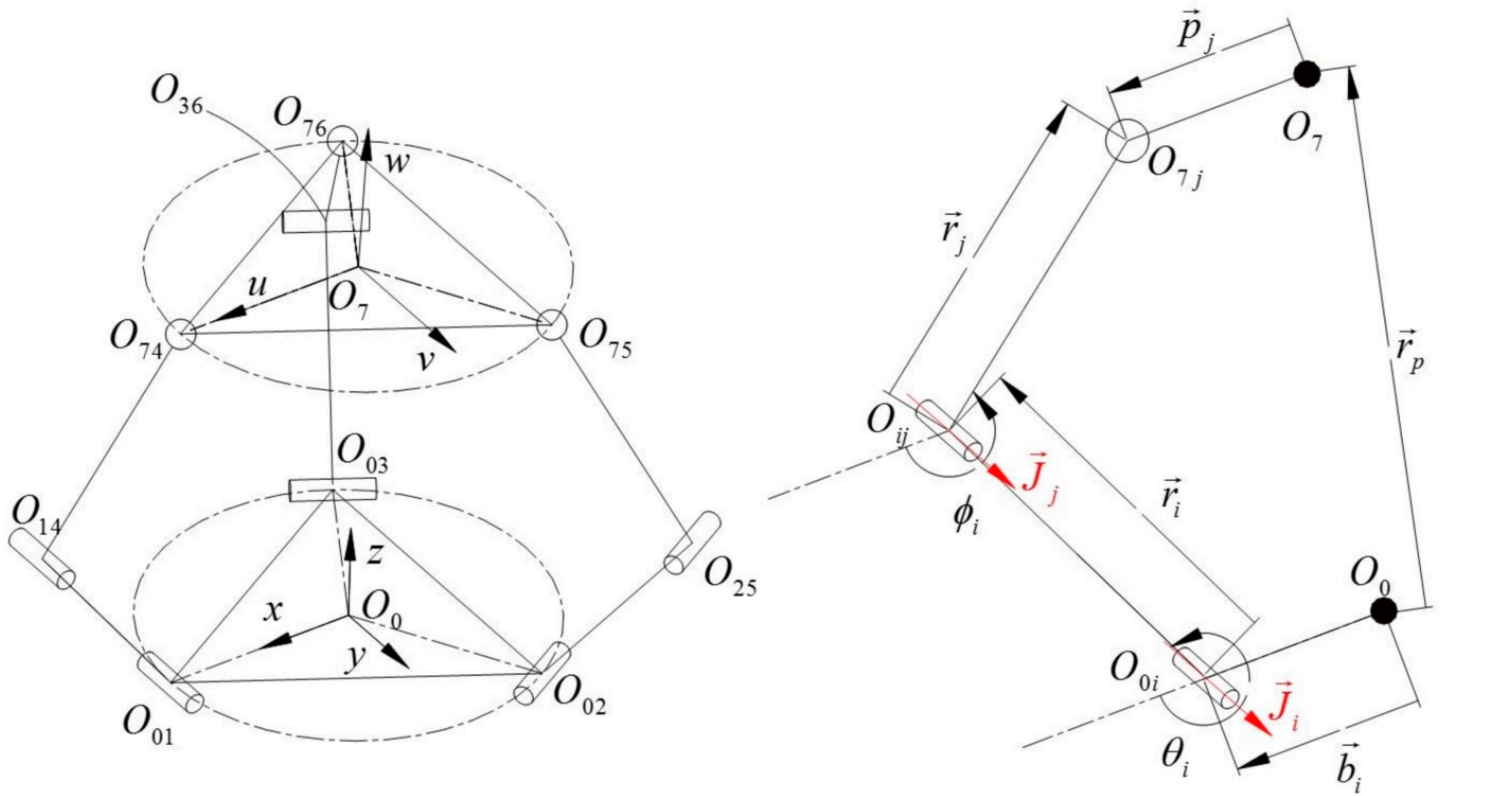

Figure 3.1. Kinematic Diagram of the 3-RRS PM

3.3.1. Manipulator Constraints

Since O_{74} is constrained on xz plane:

$$\overrightarrow{O_0 O_{74}} = \vec{r}_P + \vec{p}_1 \Rightarrow \begin{bmatrix} O_{74,x} \\ O_{74,y} \\ O_{74,z} \end{bmatrix} = \begin{bmatrix} O_{74,x} \\ 0 \\ O_{74,z} \end{bmatrix} = \begin{bmatrix} O_{7,x} \\ O_{7,y} \\ O_{7,z} \end{bmatrix} + \mathbf{R} \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} O_{7,x} + pu_x \\ O_{7,y} + pu_y \\ O_{7,z} + pu_z \end{bmatrix} \tag{3.4}$$

Since O_{75} is constrained on the $y = \tan(120°)\, x$ plane:

$$\begin{gathered} \overrightarrow{O_0 O_{75}} = \vec{r}_P + \vec{p}_2 \Rightarrow \begin{bmatrix} O_{75,x} \\ O_{75,y} \\ O_{75,z} \end{bmatrix} = \begin{bmatrix} O_{75,x} \\ -\sqrt{3} O_{75,x} \\ O_{75,z} \end{bmatrix} = \\ \begin{bmatrix} O_{7,x} \\ O_{7,y} \\ O_{7,z} \end{bmatrix} + \mathbf{R} \cdot \mathbf{Z}(\alpha_{45}) \cdot \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} O_{7,x} - \frac{pu_x}{2} + \frac{\sqrt{3} p v_x}{2} \\ O_{7,y} - \frac{pu_y}{2} + \frac{\sqrt{3} p v_y}{2} \\ O_{7,z} - \frac{pu_z}{2} + \frac{\sqrt{3} p v_z}{2} \end{bmatrix} \end{gathered} \tag{3.5}$$

where $\mathbf{Z}(\cdot)$ represents the rotation matrix around the z-axis. Since O_{76} is constrained on the $y = \tan(240°)\, x$ plane:

$$\begin{gathered} \overrightarrow{O_0 O_{76}} = \vec{r}_P + \vec{p}_3 \Rightarrow \begin{bmatrix} O_{76,x} \\ O_{76,y} \\ O_{76,z} \end{bmatrix} = \begin{bmatrix} O_{76,x} \\ \sqrt{3} O_{76,x} \\ O_{76,z} \end{bmatrix} = \\ \begin{bmatrix} O_{7,x} \\ O_{7,y} \\ O_{7,z} \end{bmatrix} + \mathbf{R} \cdot \mathbf{Z}(\alpha_{46}) \cdot \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} O_{7,x} - \frac{pu_x}{2} - \frac{\sqrt{3} p v_x}{2} \\ O_{7,y} - \frac{pu_y}{2} - \frac{\sqrt{3} p v_y}{2} \\ O_{7,z} - \frac{pu_z}{2} - \frac{\sqrt{3} p v_z}{2} \end{bmatrix} \end{gathered} \tag{3.6}$$

From equations 3.4, 3.5 and 3.6:

$$O_{7,y} = -u_y p \tag{3.7}$$

$$O_{7,y} = \frac{pu_y}{2} - \frac{\sqrt{3}pv_y}{2} - \sqrt{3}\left(O_{7,x} - \frac{pu_x}{2} + \frac{\sqrt{3}pv_x}{2}\right) \tag{3.8}$$

$$O_{7,y} = \frac{pu_y}{2} + \frac{\sqrt{3}pv_y}{2} + \sqrt{3}\left(O_{7,x} - \frac{pu_x}{2} - \frac{\sqrt{3}pv_x}{2}\right) \tag{3.9}$$

Adding up Equations 3.8 and 3.9 and subtracting from 2 times of Equation 3.7:

$$\begin{pmatrix} \frac{pu_y}{2} - \frac{\sqrt{3}pv_y}{2} - \sqrt{3}\left(O_{7,x} - \frac{pu_x}{2} + \frac{\sqrt{3}pv_x}{2}\right) + \\ \frac{pu_y}{2} + \frac{\sqrt{3}pv_y}{2} + \sqrt{3}\left(O_{7,x} - \frac{pu_x}{2} - \frac{\sqrt{3}pv_x}{2}\right) \end{pmatrix} = -2u_y p \rightarrow v_x = u_y \tag{3.10}$$

Equating Equations. 3.8 and 3.9:

$$O_7{}^x = \frac{p\left(u_x - v_y\right)}{2} \tag{3.11}$$

Equations 3.7 and 3.11 constitute the constraint equations for the position of the moving platform and Equation 3.10 is the constraint equation for the orientation of the moving platform due to the geometry of the manipulator. The rotation matrix elements in Equations 3.7 and 3.11 are found using ψ_x and ψ_y as explained in the following subsection.

3.3.2. Rotation Matrix Constraints

$\mathbf{R}$ is defined by using $x-y-z-$Euler rotation sequence:

$$\mathbf{R} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \begin{bmatrix} \mathrm{c}\psi_y \mathrm{c}\psi_z & -\mathrm{c}\psi_y \mathrm{s}\psi_z & \mathrm{s}\psi_y \\ \mathrm{c}\psi_z \mathrm{s}\psi_x \mathrm{s}\psi_y + \mathrm{c}\psi_x \mathrm{s}\psi_z & \mathrm{c}\psi_x \mathrm{c}\psi_z - \mathrm{s}\psi_x \mathrm{s}\psi_y \mathrm{s}\psi_z & -\mathrm{s}\psi_x \mathrm{c}\psi_y \\ \mathrm{s}\psi_x \mathrm{s}\psi_z - \mathrm{c}\psi_x \mathrm{c}\psi_z \mathrm{s}\psi_y & \mathrm{s}\psi_x \mathrm{c}\psi_z + \mathrm{c}\psi_x \mathrm{s}\psi_y \mathrm{s}\psi_z & \mathrm{c}\psi_x \mathrm{c}\psi_y \end{bmatrix} \tag{3.12}$$

where s and c stand for $\sin$ and $\cos$ respectively. Since ψ_x and ψ_y are selected as independent orientation parameters, the only unknown parameter of the rotation matrix is ψ_z. By making use of Equations 3.10 and 3.12:

$$\psi_z = \tan^{-1}\left(\frac{-\,\mathrm{s}\,\psi_x\,\mathrm{s}\,\psi_y}{\mathrm{c}\,\psi_x + \mathrm{c}\,\psi_y}\right) \tag{3.13}$$

for $\mathrm{c}_x + \mathrm{c}_y \neq 0$. All the unknown parameters in the rotation matrix can be represented in terms of ψ_x, ψ_y and $\psi_z$by using Equations 3.12 and 3.13.

3.4. Inverse Kinematics

Inverse kinematics problem is to find the input angles θ_i for given independent pose parameters of moving platform: $\bar{x}_i = \begin{bmatrix} O_{7,z} & \psi_x & \psi_y \end{bmatrix}^T$. For each limb (see Figure 3.1:

$$\vec{r}_P + \vec{p}_j = \vec{b}_i + \vec{r}_i + \vec{r}_j \tag{3.14}$$

First, the positions of O_{7j} points in terms of given pose parameters are determined by using the left-hand side of Equation 3.14. Then using the right-hand side of the Equation 3.14 and resolving into x, y and z components:

$$x: \quad l_2 \mathrm{c}\phi_i \mathrm{c}\alpha_{1i} = O_{7j,x} - \mathrm{c}\alpha_{1i}\left(b + l_1 \mathrm{c}\theta_i\right) \tag{3.15}$$

$$y: \quad l_2 \mathrm{c}\phi_i \mathrm{s}\alpha_{1i} = O_{7j,y} - \mathrm{s}\alpha_{1i}\left(b + l_1 \mathrm{c}\theta_i\right) \tag{3.16}$$

$$z: \quad l_2 \mathrm{s}\phi_i = -O_{7j,z} - l_1 \mathrm{s}\theta_i \tag{3.17}$$

Multiplying Equation 3.17 with $\mathrm{c}\,\alpha_{1i}$ and adding up the square of Equation 3.17 with the square of Equation 3.15:

$$\begin{array}{r} \left\{l_2 \mathrm{c}\alpha_{1i} \mathrm{c}\phi_i = O_{7j,x} - \mathrm{c}\alpha_{1i}\left(b + l_1 \mathrm{c}\theta\right)\right\}^2 \\ + \quad \left\{l_2 \mathrm{c}\alpha_{1i} \mathrm{s}\phi_i = \mathrm{c}\alpha_{1i}\left(-O_{7j,z} - l_1 \mathrm{s}\theta_i\right)\right\}^2 \\ \hline A_i \mathrm{c}\theta_i + B_i \mathrm{s}\theta_i + C_i = 0 \end{array} \tag{3.18}$$

where

$$A_i = 2 l_1 \mathrm{c}\alpha_{1i}\left(b \mathrm{c}\alpha_{1i} - O_{7j,x}\right)$$

$$B_i = 2 l_1 O_{7j,z} \mathrm{c}^2 \alpha_{1i}$$

$$C_i = {O_{7j,x}}^2 - 2b O_{7j,x} \mathrm{c}\alpha_{1i} + \mathrm{c}^2 \alpha_{1i}\left(b^2 + {l_1}^2 - {l_2}^2 + {O_{7j,z}}^2\right)$$

for $C_i - A_i \neq 0$, $i = 1, 2, 3$ and $j = i + 3$. Applying tangent of the half angle substitution to Equation 3.18 and solving for θ_i:

$$\theta_i = 2\tan^{-1}\left(\frac{-B_i \pm \sqrt{{A_i}^2 + {B_i}^2 - {C_i}^2}}{C_i - A_i}\right) \tag{3.19}$$

Once the input angles θ_i are found, corresponding passive joint angles can be solved uniquely by using Equations 3.15 and 3.17:

$$
\begin{gathered}
\mathrm{c}\phi_i = \frac{O_{7j,x} - \mathrm{c}\alpha_{1i}(b + l_1 \mathrm{c}\theta_i)}{l_2 \mathrm{c}\alpha_{1i}} \quad \mathrm{s}\phi_i = -\frac{O_{7j,z} + l_1 \mathrm{s}\theta_i}{l_2} \\
\phi_i = \mathrm{atan2}\left(\mathrm{c}\phi_i, \mathrm{s}\phi_i\right)
\end{gathered} \tag{3.20}
$$

As a case study, a set of independent pose parameters are chosen as $\bar{x}_i = \begin{bmatrix} O_{7,z} & \psi_x & \psi_y \end{bmatrix}^T = \begin{bmatrix} 900 & -10^\circ & 15^\circ \end{bmatrix}^T$ for $b = 500$ mm, $p = 250$ mm, $l_1 = 750$ mm and $l_2 = 850$ mm . The formulation presented above is implemented into Mathematica® software and inverse kinematics solution is sought. Firstly the task space parameters are found as $\bar{t} = \begin{bmatrix} O_{7,x} & O_{7,y} & O_{7,z} & \psi_x & \psi_y & \psi_z \end{bmatrix}^T =$