Derivation of the differential equation, interpretation and Applications of a Generalized Series RLC Circuit

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Introduction

In the context of circuit analysis, one prevalent and important system is the RLC circuit.

The RLC circuit is a type of circuit in physics and electrical engineering that is commonly used to model signal response. Understanding this circuit's behaviour provides insight into energy storage, and transfer within the system through the components over time.

The circuit consists of 3 key components apart from the alternating voltage source, being a resistor, an inductor, and a capacitor. A resistor restricts the flow of electrical current, an inductor stores energy in the form of a magnetic field as current passes through it, and a capacitor temporarily stores energy by holding opposite charges on its separate plates.

When these components are placed in a circuit, analysis leads to a differential equation that models the changes in current, or charge over time. In this paper, we derive the RLC differential equation for one capacitor, inductor and resistor. Then, using the approach utilized in the first derivation, we generalize this base case to an arbitrary number of resistors, capacitors, and inductors respectively.

Finally, we interpret this differential equation, mathematically and physically, as solving the equation does not provide conceptual insight. We also look at different applications of these generalized circuit cases. These applications are either conceptual, with how the circuit can be used in mechanisms, or/and mathematical with how the generalized cases simplify calculations.

This methodology enables investigation of signal response in generalized series RLC networks by reducing circuits with a large number of components to a singular governing differential equation.

Base Case - Circuit Analysis

In Figure 1, the base case RLC circuit with an alternating voltage source can be seen. The base case is where there is one of each component, or where n, m, and p, the number of resistors, inductors, and capacitors respectively, are all 1.

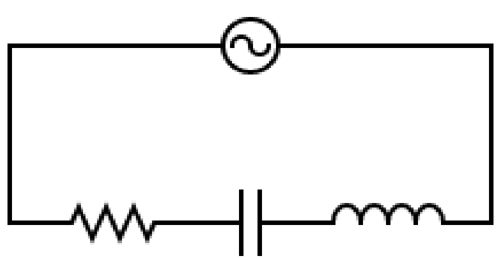

Figure 1: RLC Circuit in Series: circuitdiagram.org

We will derive the differential equation that governs the base case to define an approach, and then derive the generalized formula. Keep in mind, we will not be solving either differential equation, as we are only interested in the equations formed

We begin by applying Kirchhoff's Voltage Law, or KVL, which states that around a loop, defined as a path in a circuit that starts and ends at one point, the sum of the voltage increases in the circuit must be equal to the sum of the voltage decreases (Kirchhoff's Voltage Law - Digilent Reference.). In this case, the voltage increase is provided by the alternating voltage source, and the voltage decreases, or drops, occur due to the resistor, inductor, and the capacitor. We assume that the components all have a uniform shape and geometry. We also assume passive sign convention, where the current enters the positive terminals of all the components, so the voltage is positive, and hence, a drop.

Base Case Derivation

Algebraically, KVL in this circuit would be

$$\sum V_{increases} = \sum V_{decreases}$$

$$V_{Battery} = V_{Resistor} + V_{Inductor} + V_{Capacitor}$$

The voltage provided by a battery source is alternating, so it depends on time. Also, since it is alternating, we can express it in terms of a periodic function. We assume that at time t=0, the voltage is 0, so we use a sinusoidal function. We also assume that since the voltage is 0 at t=0, there is no phase shift. Since we are assuming a sinusoidal function, there must be a max voltage, represented by the amplitude, and a period, or time interval that the battery takes to return to the corresponding voltage source, also known as the angular frequency ω.

$$V_{Battery}(t) = V_{max} sin(\omega t)$$

Our equation is then

$$V_{max} sin(\omega t) = V_{Resistor} + V_{Inductor} + V_{Capacitor}$$

One more key aspect to remember is that we clearly assume a non zero constant resistance, so damping is present. However, since we have an unspecified resistance, the derived differential equation can be used for whatever damping specification we place on the circuit, whether it be underdamped, or overdamped.

To begin, the voltage across a resistor is given by Ohm's Law, which is

$$V_{resistor} = IR$$

We can express this voltage in terms of current, by noting that current is defined as the derivative of charge.

$$V_{resistor} = IR = R\frac{dq}{dt}$$

Now that we have our first term, we move on to the next term in our algebraic equation, the voltage across an inductor. We can derive this quantity using Faraday's Law of Induction. As we know, an inductor is a coil of wire that is wound a certain number of times. So we can use Faraday's Law of Induction with a coil, and derive the voltage across an inductor.

Faraday's Law of Induction says that a changing magnetic field induces a voltage across a coil of wire, and in this, the inductor.

$$\varepsilon = V_{inductor} = -N\frac{d\Phi_B}{dt}$$

Where ε is the induced emf, or voltage across the inductor, N is the number of loops in the inductor, and Φ_b is the magnetic flux, conceptually being the number of field lines cutting each part of the coil.

However, since we assume passive sign convention, the minus sign can be omitted.

Since our inductor has fixed geometry as we assumed, magnetic flux is related to the previously stated variables, and current and inductance through the magnetic flux linkage equation shown below.

$$L = N\frac{\Phi_B}{I}$$

Upon rearranging, we express the voltage induced across the component of the inductor in terms of current and inductance.

$$IL = N\Phi_B$$

We now replace the term inside the derivative operator with the product of current and inductance.

$$\varepsilon = N\frac{d\Phi_B}{dt} = \frac{d(N\Phi_B)}{dt} = \frac{d(IL)}{dt}$$

Inductance does not depend on time, so we can remove it from the derivative operator.

$$V_{inductor} = \frac{d(IL)}{dt} = L\frac{dI}{dt}$$

However, we want the RLC circuit differential equation in terms of charge, so we express the derivative of current as the second derivative of charge.

$$V_{inductor} = L\frac{dI}{dt} = L\frac{d^2q}{dt^2}$$

Now, we only need the voltage of the capacitor to complete the differential equation.

One quantity that will aid us in finding the voltage of the capacitor is capacitance. Capacitance is defined as the measure of the ability of an object to store electrical charge, measured in charge per volt. We can use capacitance for the capacitor, and the equation can be seen below.

$$C = \frac{Q}{V}$$

In the equation, C refers to the capacitance, Q refers to the total charge within the capacitor, and V refers to the voltage across the capacitor. We use a lowercase q in this paper, though this is Q in standard notation.

So the voltage across a capacitor is

$$C = \frac{q}{V_{capacitor}}$$

$$V_{capacitor} = \frac{q}{C}$$

Now that we have derived all of the voltages across the components, we solidify a differential equation for the base RLC circuit by plugging all the relevant quantities into the algebraic equation that was introduced, which can be seen below.

$$V_{Resistor} + V_{Inductor} + V_{Capacitor} = V_{max} sin(\omega t)$$

$$(R\frac{dq}{dt}) + (L\frac{d^2q}{dt^2}) + (\frac{q}{C}) = V_{max} sin(\omega t)$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_{max} sin(\omega t)$$

Now that we have derived a differential equation for the base case using the procedure defined in the introduction, and have verified that it is successful, we now move to the generalization of the series RLC circuit, with n resistors, m inductors, and p capacitors.

Assumptions

To begin, we once again assume passive sign convention, and that current flows into the positive terminal of the component resulting in a voltage drop. We then understand that if we want to calculate the voltage drops for each and every component, we can simplify every resistor, inductor, and capacitance into its equivalent component. We also say that since all the components are in series, the order of the components can vary, and the equivalence can still be used as the current flowing through each component is the same. An example of these properties in a circuit can be seen below.

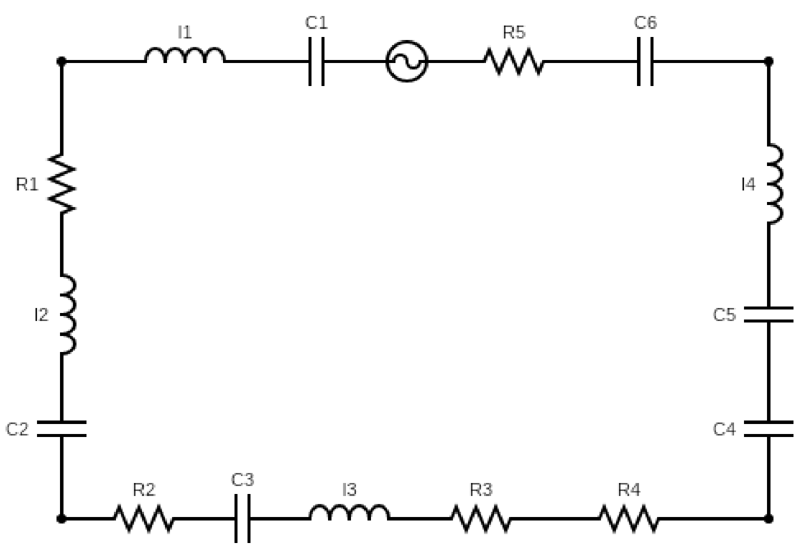

Figure 2: Series RLC circuit with component order varying, and (n, m, p) = (5, 4, 6). (The figure is illustrative only, and the analysis that follows applies to any series configuration, regardless of the number of components or their placement along the loop.)

Generalized Case Derivation

Now, we start our circuit analysis for the new RLC circuit. As has been said, we use the equivalence formulas for resistance, inductance, and capacitance. However, one assumption must be made before. We must clarify that mutual inductance, or the phenomena in which the changing current in one coil of wire induces a voltage in another nearby coil of wire is negligible, as we keep the condition that no 2 or more inductors are next to each other. The reason that mutual inductance is assumed to be negligible is because with that additional voltage induced across the nearby coil, there are coupled terms for the voltage, increasing the complexity of the problem.

Since Kirchhoff's Voltage Law is linear, the base case equation retains the same form when individual components are replaced by their series equivalents with n, m and p resistors, inductors, and capacitors.

$$V_{resistor} + V_{inductor} + V_{capacitor} = V_{max} \, sin(\omega t)$$

Since we are in the generalized series circuit, we can express the voltage in terms of its respective component value, be it resistance, inductance or capacitance.

We now begin with equivalent resistance. The formula for the equivalent resistance of n resistors in series is defined as

$$R_{eq} = \sum_{i=1}^{n} R_i$$

As has been noted before, the voltage across a resistor is V=IR. Since we are combining all the resistors into one equivalent resistor, we use the equivalent resistance for n resistors, and the differential form of current.

$$V = \frac{dq}{dt} \sum_{i=1}^{n} R_i$$

Similarly, we assume the same for the inductors, but as has been said before, only if there is negligible mutual inductance, which occurs when there is spatial separation between the inductors.

The formula for equivalent inductance for m inductors in series is defined as

$$L_{eq} = \sum_{j=1}^{m} L_i$$

Once again, we replace the inductance in the base case with the equivalent inductance as we treat the m inductors as one again.

$$V_{inductor} = \frac{d^2q}{dt^2}\sum_{j=1}^{m} L_i$$

Finally, we repeat the procedure for the capacitance. We treat p capacitors as a singular capacitance with an equivalent capacitance, defined as

$$\frac{1}{C_{eq}} = \sum_{k=1}^{p} \frac{1}{C_k}$$

To finish the derivation of the generalized RLC circuit, we algebraically manipulate the formula for capacitance, using the equivalent capacitance in place of capacitance, using a lower case q instead of an upper case q.

$$C = \frac{Q}{V}$$

$$V = q \times \frac{1}{C_{eq}}$$

$$V_{capacitor} = q\sum_{k=1}^{p} \frac{1}{C_k}$$

Now, we replace all of the terms in the Voltage Law equation.

$$V_{Resistor} + V_{Inductor} + V_{Capacitor} = V_{max} sin(\omega t)$$

$$\frac{dq}{dt}\sum_{i=1}^{n} R_i + \frac{d^2q}{dt^2}\sum_{j=1}^{m} L_i + q\sum_{k=1}^{p} \frac{1}{C_k} = V_{max} sin(\omega t)$$

$$\frac{d^2q}{dt^2}\sum_{j=1}^{m} L_i + \frac{dq}{dt}\sum_{i=1}^{n} R_i + q\sum_{k=1}^{p} \frac{1}{C_k} = V_{max} sin(\omega t)$$

Now that we have derived the differential equation, we turn our eyes to the physical and mathematical interpretation of this equation.

Interpretation and Practical Applications

We begin our interpretation with noticing the similarities of the base case and generalized case differential equation. It is clear that when n, m and p are all equal to 1, the equation collapses to the base case. The behaviour of the generalized case is similar to the base RLC circuit, as no matter how many resistors, inductors, or capacitors there are, the individual component values will converge to a constant resistance, inductance and capacitance.

Since both the generalized and the base have similar behaviour, with energy transfer being in the same direction, but at different rates, as there is at least 1 more component, be it a resistor, inductor or capacitor, and as the possibility of more resistors in the generalized case results in higher energy dissipation, we say that the degrees of freedom, or the ways that the system can move or store energy are 2, as energy can be stored in the electric field of the capacitor, or the magnetic field of the inductor.

Though there are 2 degrees of freedom, or DOF, there is still just one oscillatory mode, or a specific pattern that the system oscillates at. Though the modes are the same for each DOF, the rates at which energy is transferred, as has been said, are different.

Now, we can also notice that when any of the individual numbers of components are 0, the equation collapses correctly into the generalized form of the different cases. For instance, when n is 0, then it is the generalized form for an LC circuit, and likewise, for m = 0, and p = 0, they collapse respectively into the generalized forms for the RC and RL circuits. We now investigate deeper into the applications and analysis of the individual cases.

Generalized LC case

When n = 0, the equivalent resistance is 0. One quantity that depends on the resistance in an RLC circuit is the dimensionless damping factor, or the rate of energy loss in an oscillating system, which fits the description of our base and generalized case as the voltage alternates. The damping factor mathematically is

$$\zeta = \frac{R_{eq}}{2}\sqrt{\frac{C_{eq}}{L_{eq}}}$$

The damping factor is 0 when there are no resistors, which conceptually means that the rate of energy loss is 0. This means the circuit will indefinitely oscillate, meaning without the resistor, the LC circuit is ideal. However, though this theoretically occurs, in the real world, there is energy loss to heat, which can come in the form of capacitor plate heating or coil heating.

Though we cannot create a completely ideal LC circuit, the LC circuit is still a powerful tool. One application of such a tool is in frequency detection. This can be done by noticing that the LC circuit can oscillate at its resonant frequency, which is the frequency that a system vibrates at when a net external force acts on the system at its natural frequency, resulting in maximum amplitude.

Resonant frequency is defined as

$$f_{res} = \frac{1}{2\pi\sqrt{LC}}$$

In a theoretically ideal software model, we could simulate a net external force on the circuit, or resonance, to have the system oscillate with a resonant frequency. We could then match the frequency of an incoming signal using detecting software, or we could alter the online circuit's component values to achieve the same detected frequency, by either increasing or decreasing the capacitance. The reason for not doing this with the inductance will be explained shortly.

This allows signal selection in transmission systems and interfaces, which greatly simplifies communication. Now this is for the base case, but if we extend this idea to the generalized LC circuit case with m and p inductors and capacitors respectively, we get

$$f_{res} = \frac{1}{2\pi\sqrt{\sum_{j=1}^{m} L_i \sum_{k=1}^{p} \frac{1}{C_k}}}$$

The reason this resonant frequency formula for our generalized LC circuit is important, is because in simulations, splitting a large capacitance into multiple smaller capacitors can improve numerical accuracy, since small changes in individual values allow for more precise fine tuning of the equivalent capacitance and the resulting resonant frequency, and hence, improving signal detection and communication capability. We do not do this with the inductance, as it is combined linearly and does not provide improved preciseness.

Generalized RC case

Then, when m = 0, or when there are no inductors, there is one condition that is removed from the circuit, because when the inductors are removed, there is no worry of mutual inductance. Additionally, there is no magnetic field storage mechanism, so there is only one degree of freedom.

With this RC circuit, when current flows through the capacitor, it'll begin charging one plate, and removing charge from the other plate. However, once the voltage reverses direction, charge will be put on the opposite plate, and the charged plate will have charge removed. This means that charge is constantly added and removed from the capacitor's plates. But, with the presence of a resistor, not only does the resistor limit the peak current, but it also determines how quickly the capacitor charges and discharges.

This is essential in electrical systems whose mechanism is driven by the time taken for the capacitor to fully charge, which is why it is necessary to notice this.

One key example of this idea in practice is in a timing circuit, in which the time required for a capacitor to charge through a resistor determines the interval between subsequent actions, such as a light turning on or a car's turn indicator blinking, occurring each time a plate on the capacitor fully charges.

From an engineering perspective, using multiple components is economically beneficial, as the price of many low resistance resistors, or low capacitance capacitors is far lower than the prices for the respective large component valued circuit parts. This can be seen with an example like a digital alarm blinking every 5 minutes, where more components with lower values results in a cheaper price, whilst offering more precisely controlled timing intervals.

Generalized RL case

Finally, when p = 0, or when there are no capacitors, the system reverts to the RL circuit, in which the key aspect is that there is no electric field storage mechanism, once again resulting in one degree of freedom. Though there is no capacitor, a similar mechanism for the timing circuit can be implemented, but with an inductor. We refer back to the magnetic flux linkage equation shown in the derivation of the voltage across an inductor, with the same quantities as before, saying that

$$\Phi_B = \frac{IL}{N}$$

As can be seen, magnetic flux and current have a direct linear relationship. Therefore, we can use a device on the inductors, known as a digital flux meter, which measures magnetic flux, to detect the peak magnetic flux, and connect a signal module to it to trigger some sort of mechanism.

This can also be done with current, using an ammeter and once again, a signal module to trigger a mechanism, be it the examples shown with the RC circuit, or something completely different. Depending on the mechanism triggered by peak current or peak magnetic flux, it is redundant to do so, as the direct relation means that the mechanisms would be triggered at the same time.

However, this could also be productive depending on its application. One possibility is with regards to toy racecar driving competitions, where the starting green light, and a mechanical flag to signal the start of the race could be used. The green light could turn on at peak flux while the flag releases at peak current.

The resistor also plays a key role, as it delays the time for the peak current to be reached, so varying the resistance depending on the situation is beneficial. In the race car example, the resistance would be lowered, as there needs to be a short time interval between the cars being advised to be ready, and the green light or mechanical flag. Therefore, by varying the resistance with a system like the aforementioned one, RL circuits can be used as precise time delay elements.

The generalized circuit aids even with this, as multiple mechanisms can be triggered at the same time using multiple flux meters and ammeters with the inductors, and more resistors could be employed to increase the time delay.

Conclusion

Having investigated the generalized RLC circuit, and its 3 subcases, we understand that it is a powerful tool, providing us insight into how electrical networks with multiple components can be simplified. However, with the subcases of the RLC circuit, we see that it can be widely applied for various mechanisms, for example actions that require a time delay upon activation of the circuit, or actions that require frequency matching.

Moreover, the generalized RLC framework not only allows us to understand the circuit, but also offers practical advantages. With the generalized framework, there is precision in frequency detection for LC circuits, and cost efficiency and precise timing for RC and RL circuits respectively.

Furthermore, we also recognize the individual roles of the resistor, inductor, and capacitor, and the effect of n, m and p, the number of resistors, inductors and capacitors, and what their effects on the generalized circuits are.

Finally, though we did not solve the differential equations for the base and generalized cases of the series RLC circuits, we have interpreted the equations and concepts to gain an understanding in factors that affect the generalized cases of the RL, LC, and RC circuits, which ultimately provides us a framework to perhaps conduct numerical research into more complicated phenomena related to the RLC circuit.

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