黄克智版张量分析课后习题答案完整版

第一章

1.1 求证：$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})=(\boldsymbol{u}\bullet\boldsymbol{w})\boldsymbol{v}-(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{w}$

并问：$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ 与（$\mathbf{u}\times\mathbf{v}$）$\times\mathbf{w}$是否相等？ $\mathbf{u}$、$\mathbf{v}$、$\mathbf{w}$ 为矢量

证明：因为 u=(u_x,u_y,u_z)； v=(v_x,v_y,v_z)； w=(w_x,w_y,w_z)；

左边=$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})=(u_x,u_y,u_z)\times[(v_x,v_y,v_z)\times(w_x,w_y,w_z)]$

$$=(u_x,u_y,u_z)\times\begin{bmatrix} i & j & k \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

$$=(u_x,u_y,u_z)\times[(v_y w_z-w_y v_z),(w_x v_z-v_x w_z),(v_x w_y-w_x v_y)]$$

$$=[u_y(v_x w_y-w_x v_y)-u_z(w_x v_z-v_x w_z),\ u_z(v_y w_z-w_y v_z)-u_x(u_x w_y-w_x v_y),\ u_x(w_x v_z-v_x w_z)-u_y(v_y w_z-w_y v_z)]$$

右边=$(\boldsymbol{u}\bullet\boldsymbol{w})\boldsymbol{v}-(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{w}$

$$=(u_x w_x+u_y w_y+u_z w_z)\boldsymbol{v}-(u_x w_x+u_y w_y+u_z w_z)\boldsymbol{w}$$

$$=(u_x w_x+u_y w_y+u_z w_z)(v_x,v_y,v_z)-(u_x w_x+u_y w_y+u_z w_z)(w_x,w_y,w_z)$$

$$=[u_y(v_x w_y-w_x v_y)-u_z(w_x v_z-v_x w_z),\ u_z(v_y w_z-w_y v_z)-u_x(u_x w_y-w_x v_y),\ u_x(w_x v_z-v_x w_z)-u_y(v_y w_z-w_y v_z)]$$

所以：$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})=(\boldsymbol{u}\bullet\boldsymbol{w})\boldsymbol{v}-(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{w}$

同理可证：$(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w}=(\boldsymbol{u}\bullet\boldsymbol{w})\boldsymbol{v}-(\boldsymbol{v}\bullet\boldsymbol{w})\boldsymbol{u}$

所以$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\neq$（$\mathbf{u}\times\mathbf{v}$）$\times\mathbf{w}$

1.11 根据上题结果验算公式：$\mathbf{g}_j=g_{ji}\mathbf{g}^i$

由上题结果：$\sqrt{\boldsymbol{g}}=2$，$\boldsymbol{g}_1=\frac{1}{2}(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$，$\boldsymbol{g}_2=\frac{1}{2}(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$，$\boldsymbol{g}_3=\frac{1}{2}(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})$

$$g_{rs}=\begin{cases}2 & \text{当r=s} \\ 1 & \text{当}r\neq s\end{cases}$$

$$g_{11}\mathbf{g}^1+g_{12}\mathbf{g}^2+g_{13}\mathbf{g}^3=2\mathbf{g}^1+\mathbf{g}^2+\mathbf{g}^3$$

$$=\frac{2}{2}(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})+\frac{1}{2}(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})+\frac{1}{2}(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})$$

$$=\boldsymbol{j}+\boldsymbol{k}=\boldsymbol{g}_1$$

及：$\mathbf{g}_1=g_{11}\mathbf{g}^1+g_{12}\mathbf{g}^2+g_{13}\mathbf{g}^3$

同理；$g_{21}\mathbf{g}^1+g_{22}\mathbf{g}^2+g_{23}\mathbf{g}^3=\mathbf{g}^1+2\mathbf{g}^2+\mathbf{g}^3$

$$=\frac{1}{2}(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})+\frac{2}{2}(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})+\frac{1}{2}(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})$$

$$=\boldsymbol{i}+\boldsymbol{k}=\mathbf{g}_2$$

及：$\mathbf{g}_2=g_{21}\mathbf{g}^1+g_{22}\mathbf{g}^2+g_{23}\mathbf{g}^3$

$$g_{31}\mathbf{g}^1+g_{32}\mathbf{g}^2+g_{33}\mathbf{g}^3=\mathbf{g}^1+\mathbf{g}^2+2\mathbf{g}^3$$

$$=\frac{1}{2}(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})+\frac{1}{2}(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})+\frac{2}{2}(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})$$

$$=\boldsymbol{i}+\boldsymbol{j}=\mathbf{g}_3$$

及：$\mathbf{g}_3=g_{31}\mathbf{g}^1+g_{32}\mathbf{g}^2+g_{33}\mathbf{g}^3$

及验证：$\mathbf{g}_j=g_{ji}\mathbf{g}^i$ 正确

1.21 试证明若一张量的所有分量在某一坐标系中为零，则它们在任何其他坐标系中亦必为零。

证明：不妨取三界张量

根据 P24 页所讲的分量表示法和坐标转换关系知识

$T=T^{ijk}g_ig_jg_k=T_{ijk}g^ig^jg^k=T^{ij}_{\cdot\cdot k}g_ig_jg^k=T^{i}_{\cdot jk}g_ig^jg^k=\ldots\ldots$

其分量为：$T^{ijk}T_{ijk}T^{ij}_{\cdot\cdot k}T^{i}_{\cdot jk}\ldots\ldots$

他们满足坐标转变关系，先将 ijk 用 rst 表示，我们可以得到

$$T^{i'j'k'}=\beta^{i'}_r\beta^{j'}_s\beta^{k'}_tT^{rst}$$

$$T_{i'j'k'}=\beta^{r}_{i'}\beta^{s}_{j'}\beta^{t}_{k'}T_{rst}$$

$$T^{i'j'}_{\cdot\cdot k'}=\beta^{i'}_r\beta^{j'}_s\beta^{t}_{k'}T^{rs}_{\cdot\cdot t}$$

$$T^{i'}_{\cdot j'k'}=\beta^{i'}_r\beta^{s}_{j'}\beta^{t}_{k'}T^{r}_{\cdot st}$$

......

$(i',j',k'=1,2,3)$

$\because T^{rst}T_{rst}T^{rs}_{\cdot\cdot t}T^{r}_{\cdot st}$......都为零

$\therefore$ 等式左边在新坐标系下的张量分量都为零

即 $T^{i'j'k'}T_{i'j'k'}T^{i'j'}_{\cdot\cdot k'}T^{i'}_{\cdot j'k'}$......全为零

n阶张量同理可证

$\therefore$ 当一张量在一个坐标系中所有分量都为零时，

则他们在任何坐标系中亦必为零

1.31 已知：v_k为一矢量的协变分量。

（根据 P31 页所讲的张量的对称与反对称知识来证明这个题目。重点 $T_{(n.m)}=-T_{(m.n)}$）

求证：$\dfrac{\partial v_m}{\partial x^n}-\dfrac{\partial v_n}{\partial x^m}$为一反对称二阶张量的协变分量。

证明：

令 $T_{(m.n)}=\dfrac{\partial v_m}{\partial x^n}-\dfrac{\partial v_n}{\partial x^m}$

则由 $v_{m'}=\beta^m_{m'}v_m=\dfrac{\partial x^m}{\partial x^{m'}}v_m$

可知: $\dfrac{\partial v_{m'}}{\partial x^{n'}}=\dfrac{\partial x^m}{\partial x^{m'}}\dfrac{\partial v_m}{\partial x^n}\dfrac{\partial x^n}{\partial x^{n'}}+\dfrac{\partial x^m}{\partial x^{m'}\partial x^{n'}}v_m$

同理可得：$\dfrac{\partial v_{n'}}{\partial x^{m'}}=\dfrac{\partial x^n}{\partial x^{n'}}\dfrac{\partial v_n}{\partial x^m}\dfrac{\partial x^m}{\partial x^{m'}}+\dfrac{\partial x^n}{\partial x^{m'}\partial x^{n'}}v_n$

则 $T_{(m'.n')}=\dfrac{\partial v_{m'}}{\partial x^{n'}}-\dfrac{\partial v_{n'}}{\partial x^{m'}}=\dfrac{\partial x^m}{\partial x^{m'}}\dfrac{\partial v_m}{\partial x^n}\dfrac{\partial x^n}{\partial x^{n'}}+\dfrac{\partial x^m}{\partial x^{m'}\partial x^{n'}}v_m-\dfrac{\partial x^n}{\partial x^{n'}}\dfrac{\partial v_n}{\partial x^m}\dfrac{\partial x^m}{\partial x^{m'}}-\dfrac{\partial x^n}{\partial x^{m'}\partial x^{n'}}v_n$

由于：$\dfrac{\partial x^m}{\partial x^{m'}\partial x^{n'}}v_m=\dfrac{\partial x^n}{\partial x^{m'}\partial x^{n'}}v_n$

所以 $T_{(m'.n')}=\dfrac{\partial v_{m'}}{\partial x^{n'}}-\dfrac{\partial v_{n'}}{\partial x^{m'}}=\dfrac{\partial x^m}{\partial x^{m'}}\dfrac{\partial x^n}{\partial x^{n'}}(\dfrac{\partial v_m}{\partial x^n}-\dfrac{\partial v_n}{\partial x^m})$

即 $T_{(m'.n')}=\beta^m_{m'}\beta^n_{n'}(\dfrac{\partial v_m}{\partial x^n}-\dfrac{\partial v_n}{\partial x^m})=\beta^m_{m'}\beta^n_{n'}T_{(m.n)}$

所以的证：$T_{(m.n)}=\dfrac{\partial v_m}{\partial x^n}-\dfrac{\partial v_n}{\partial x^m}$为二阶张量的协变分量。

当 $m=n$ 时恒有 $T_{(m.n)}=0$

又有 $T_{(n.m)}=-\dfrac{\partial v_m}{\partial x^n}+\dfrac{\partial v_n}{\partial x^m}=-T_{(m.n)}$

综上可知：$\dfrac{\partial v_m}{\partial x^n}-\dfrac{\partial v_n}{\partial x^m}$为一反对称二阶张量的协变分量

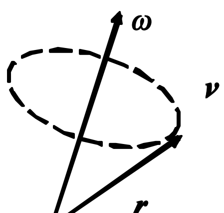
1.41　质量为 m、绕定点 O 以角速度$\boldsymbol{\omega}$转动的质点（见图），其动量矩矢量的定义为$\boldsymbol{L}=m\boldsymbol{r}\times\boldsymbol{v}$，其中，$\boldsymbol{r}$为定点 O 至质点的矢径，$\boldsymbol{v}$为质点的线速度。

求证：$\boldsymbol{L}=\boldsymbol{I}\bullet\boldsymbol{\omega}$，式中$\boldsymbol{I}$为惯性矩张量，$\boldsymbol{I}=m[(\boldsymbol{r}\bullet\boldsymbol{r})\boldsymbol{G}-\boldsymbol{rr}]$

证明：$\boldsymbol{L}=m\boldsymbol{r}\times\boldsymbol{v}=m\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})$

$=m[\boldsymbol{\omega}(\boldsymbol{r}\bullet\boldsymbol{r})-\boldsymbol{r}(\boldsymbol{r}\bullet\boldsymbol{\omega})]$此题为书上 P34 页（1.8）例题

$L^i=m[\omega^i r^m r_m-r^i r_k\omega^k]$



$$= m[\delta^i_k r^m r_m - r^i r_k]\omega^k$$

$$= I^i_{\cdot k}\omega^k$$

所以 $\boldsymbol{L} = m[(\boldsymbol{r}\bullet\boldsymbol{r})\boldsymbol{G} - \boldsymbol{rr}]\bullet\boldsymbol{\omega} = \boldsymbol{I}\bullet\boldsymbol{\omega}$

1.51 已知向量$\boldsymbol{\omega}_1$与二阶反对称张量$\boldsymbol{\Omega}_1$，矢量$\boldsymbol{\omega}_2$与二阶反对称张量$\boldsymbol{\Omega}_2$分别互为反偶。反偶?

求证：$\boldsymbol{\omega}_1\bullet\boldsymbol{\omega}_2 = \dfrac{1}{2}\boldsymbol{\Omega}_1 : \boldsymbol{\Omega}_2$

证明：由已知得

$$\vec{\omega}_1\bullet\vec{\omega}_2 = (-\frac{1}{2}\vec{\in}:\vec{\Omega}_1)\bullet(-\frac{1}{2}\vec{\in}:\vec{\Omega}_2)$$

$$= \frac{1}{4}(\in_{ijk}\ \vec{g}^i\vec{g}^j\vec{g}^k : {\Omega_1}^{lm}\vec{g}_l\vec{g}_m)\bullet(\in^{rst}\ \vec{g}_r\vec{g}_s\vec{g}_t : \Omega_{2\,xy}\vec{g}^x\vec{g}^y)$$

$$= \frac{1}{4}(\in_{ijk}\ {\Omega_1}^{jk}\vec{g}^i)\bullet(\in^{rst}\ \Omega_{2\,st}\vec{g}_r)$$

$$= \frac{1}{4}\in_{ijk}\in^{rst}\ {\Omega_1}^{jk}{\Omega_2}^{st}$$

$$= \frac{1}{4}(\delta^s_j\delta^t_k - \delta^t_j\delta^s_k){\Omega_1}^{jk}\Omega_{2\,st} = \frac{1}{4}({\Omega_1}^{jk}\Omega_{2\,jk} - {\Omega_1}^{jk}\Omega_{2\,kj})$$

已知$\vec{\Omega}_2$为反对称张量，故 ${\Omega_1}^{jk}\Omega_{2\,kj} = -{\Omega_1}^{jk}\Omega_{2\,jk}$

所以 $\vec{\omega}_1\bullet\vec{\omega}_2 = \dfrac{1}{2}{\Omega_1}^{jk}\Omega_{2\,jk}$

而

$$\vec{\Omega}_1 : \vec{\Omega}_2 = {\Omega_1}^{jk}\vec{g}_j\vec{g}_k : \Omega_{2\,lm}\vec{g}^l\vec{g}^m = {\Omega_1}^{jk}\Omega_{2\,lm}\delta^l_j\delta^m_k = {\Omega_1}^{jk}\Omega_{2\,jk} = 2\vec{\omega}_1\bullet\vec{\omega}_2$$

得证

第二章

2.2 已知：二阶张量T与T^T互为转置（$T_{ij} = T^T_{ij}$）

求证：T与S具有相同的主不变量。

证明：对于T：

$$J_1^T = T_{ii} ; J_2^T = tr(T \bullet T) = T \bullet T : G = T_a^{\bullet m} T_m^{\bullet a} ; J_3^T = T \bullet T \bullet T : G = T_a^{\bullet m} T_m^{\bullet p} T_p^{\bullet a}$$

对于 S：

$$J_1^T = T_{jj} ; J_2^T = tr(T^T \bullet T^T) = T^T \bullet T^T : G = T_{\bullet a}^{m} T_{m}^{\bullet a} ; J_3^T = T^T \bullet T^T \bullet T^T : G = T_{\bullet p}^{m} T_{\bullet a}^{p} T_{\bullet m}^{a}$$

得证。

2.3 已知：任意二阶张量 A, B,且 $T = A \bullet B, S = B \bullet A$

求证：T 与 S 具有相同的主不变量。

证明：

$$f_1^{T*} = tr(T) = tr(A \bullet B) = A \bullet B : G = T_{.j}^{i} g_i g^j \bullet T_{.n}^{m} g_m g^n : g^{ab} g_a g_b = T_{am} T^{ma}$$

$$f_1^{S*} = tr(S) = tr(B \bullet A) = B \bullet A : G = T_{.n}^{m} g_m g^n \bullet T_{.j}^{i} g_i g^j : g^{ab} g_a g_b = T_{an} T^{na}$$

$\therefore T$与S具有相同的主不变量。

2.4 求证：（1） $[\mathbf{T} \cdot \mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{T} \cdot \mathbf{w}] + [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{T} \cdot \mathbf{w}] = \phi_1^T [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]$

（2） $[\mathbf{T} \cdot \mathbf{a}\ \ \mathbf{T} \cdot \mathbf{b}\ \ \mathbf{c}] + [\mathbf{a}\ \ \mathbf{T} \cdot \mathbf{b}\ \ \mathbf{T} \cdot \mathbf{c}] + [\mathbf{T} \cdot \mathbf{a}\ \ \mathbf{b}\ \ \mathbf{T} \cdot \mathbf{c}] = \phi_2^T [\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]$

证明：（1）式左边

$$= [T_{\cdot j}^{i} u^j \mathbf{g_i}\ \ v^a \mathbf{g_a}\ \ w^b \mathbf{g_b}] + [u^c \mathbf{g_c}\ \ T_{\cdot j}^{i} \mathbf{v^j g_j}\ \ w^d \mathbf{g_d}] + [u^e \mathbf{g_e}\ \ v^f \mathbf{g_f}\ \ T_{\cdot j}^{i} \mathbf{w^j g_i}]$$

$$= T_{\cdot j}^{i} u^j v^a w^b \varepsilon_{iab} + T_{\cdot j}^{i} u^c v^j w^d \varepsilon_{cid} + T_{\cdot j}^{i} u^e v^f w^j \varepsilon_{cid}$$

$$= \frac{1}{6} T_{\cdot j}^{i} u^j v^a w^b \varepsilon_{iab} \varepsilon_{jab} \varepsilon^{jab} + \frac{1}{6} T_{\cdot j}^{i} u^c v^j w^d \varepsilon_{cid} \varepsilon_{cjd} \varepsilon^{cjd} + \frac{1}{6} T_{\cdot j}^{i} u^e v^f w^j \varepsilon_{efi} \varepsilon_{efj} \varepsilon^{efj}$$

$$= \frac{1}{6} T_{\cdot j}^{i} \left(2\delta_j^i [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + 2\delta_j^i [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + 2\delta_j^i [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]\right)$$

$= T_{\cdot i}^{i} \delta_j^i [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] = T_{\cdot i}^{i} [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] = \phi_1^T [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]$，命题得证。

（2）式左边

$$= [T_{\cdot j}^{i} a^j \mathbf{g_i}\ \ T_{\cdot b}^{a} b^b \mathbf{g_a}\ \ c^c \mathbf{g}_c] + [\mathbf{a}^d \mathbf{g}_d\ \ T_{\cdot j}^{i} b^j \mathbf{g}_i\ \ T_{\cdot b}^{a} c^b \mathbf{g}_a] + [T_{\cdot j}^{i} \mathbf{a}^j \mathbf{g}_i\ \ b^e \mathbf{g}_e\ \ T_{\cdot b}^{a} c^b \mathbf{g}_a]$$

$$= T_{\cdot j}^{i} T_{\cdot b}^{a} a^j b^b c^c \varepsilon_{iac} + T_{\cdot j}^{i} T_{\cdot b}^{a} a^d b^j c^b \varepsilon_{dia} + T_{\cdot j}^{i} T_{\cdot b}^{a} a^j b^e c^b \varepsilon_{iea}$$

$$= \frac{1}{6} T_{\cdot j}^{i} T_{\cdot b}^{a} \left(a^j b^b c^c \varepsilon_{iea} \varepsilon_{jbc} \varepsilon^{jbc} + a^d b^j c^b \varepsilon_{dia} \varepsilon_{djb} \varepsilon^{djb} + a^j b^e c^b \varepsilon_{iea} \varepsilon_{jbb} \varepsilon^{jbb}\right)$$

$$= \frac{1}{6} T_{\cdot j}^{i} T_{\cdot b}^{a} \left\{ (\delta_j^i \delta_a^b - \delta_a^j \delta_i^b) [\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] + (\delta_j^i \delta_a^b - \delta_a^j \delta_i^b) [\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] + (\delta_j^i \delta_a^b - \delta_a^j \delta_i^b) [\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] \right\}$$

$=\frac{1}{2}\left(T_{\cdot j}^{i}\mathrm{T}_{\cdot b}^{a}\delta_{i}^{j}\delta_{a}^{b}-T_{\cdot j}^{i}\mathrm{T}_{\cdot b}^{a}\delta_{a}^{j}\delta_{i}^{b}\right)[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$=\frac{1}{2}\left(T_{\cdot i}^{i}\mathrm{T}_{\cdot a}^{a}-T_{\cdot a}^{i}\mathrm{T}_{\cdot i}^{a}\right)[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ $=\phi_{2}^{T}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$命题得证。

2.5 $N\cdot a_1=\lambda_1\cdot a_1$ $\quad N\cdot a_2=\lambda_2\cdot a_2$

$a_2\cdot N\cdot a_1=a_2\cdot\lambda_1\cdot a_1$ $\quad a_1\cdot N\cdot a_2=a_1\cdot\lambda_1\cdot a_2$

上式左端相等，$a_1\cdot N\cdot a_2=a_2\cdot N\cdot a_1$

故其右端也相等，即$(\lambda_1-\lambda_2)a_1\cdot a_2=0$

注意到$\begin{matrix}\lambda_1-\lambda_2\neq 0\\ a_1a_2=0\end{matrix}$同理可得　　所以$a_1,a_2,a_3$互相正交且唯一

2.6 $\boldsymbol{N}=\boldsymbol{e}_1\boldsymbol{e}_1+2\boldsymbol{e}_2\boldsymbol{e}_2-2(\boldsymbol{e}_1\boldsymbol{e}_2+2\boldsymbol{e}_2\boldsymbol{e}_1)-2(\boldsymbol{e}_1\boldsymbol{e}_3+2\boldsymbol{e}_3\boldsymbol{e}_1)$

$$\left[\boldsymbol{N}_{\cdot j}^{i}\right]=\begin{bmatrix}1&-2&-2\\-2&2&0\\-2&0&0\end{bmatrix}$$

(1)

(2) $\boldsymbol{e}_1',\boldsymbol{e}_2',\boldsymbol{e}_3'$(　　)•

(1) $$\left[\boldsymbol{N}_{\cdot j}^{i}\right]=\begin{bmatrix}1&-2&-2\\-2&2&0\\-2&0&0\end{bmatrix}$$

$$\Delta\lambda=\begin{bmatrix}1-\lambda&-2&-2\\-2&2-\lambda&0\\-2&0&-\lambda\end{bmatrix}=-(\lambda+2)(\lambda-1)(\lambda-4)$$

$\therefore$ 　4 1 -2

(2) $\boldsymbol{N}=4\boldsymbol{e}_1'\boldsymbol{e}_1'+\boldsymbol{e}_2'\boldsymbol{e}_2'-2\boldsymbol{e}_3'\boldsymbol{e}_3'$

$(\boldsymbol{N}-4\boldsymbol{G})\cdot\boldsymbol{e}_1'=0;(\boldsymbol{N}-\boldsymbol{G})\cdot\boldsymbol{e}_2'=0;(\boldsymbol{N}+2\boldsymbol{G})\cdot\boldsymbol{e}_3'=0$

$$\therefore\boldsymbol{e}_1'=-\frac{2}{3}\boldsymbol{e}_1+\frac{2}{3}\boldsymbol{e}_2+\frac{1}{3}\boldsymbol{e}_3,\ \boldsymbol{e}_2'=-\frac{1}{3}\boldsymbol{e}_1-\frac{2}{3}\boldsymbol{e}_2+\frac{2}{3}\boldsymbol{e}_3,\ \boldsymbol{e}_3'=\frac{2}{3}\boldsymbol{e}_1+\frac{1}{3}\boldsymbol{e}_2+\frac{2}{3}\boldsymbol{e}_3$$

2.7已知：$N=10e_1e_1+4(e_1e_2+e_2e_1)+5e_2e_2-2(e_1e_3+e_3e_1)+3(e_2e_3+e_3e_2)-e_3e_3$

$$[N_{\bullet j}^{i}]=\begin{bmatrix}10&4&-2\\4&5&3\\-2&3&-1\end{bmatrix}$$

求：（1）主分量（从大到小排列)

(2)主方向对应的正交标准化基e_1',e_2',e_3'(右手系）。

$$(1)令[N_{\bullet j}^{i}]-\lambda e=\begin{bmatrix}10-\lambda & 4 & -2\\ 4 & 5-\lambda & 3\\ -2 & 3 & -1-\lambda\end{bmatrix}=0$$

解得：$\lambda_1\lambda_2\lambda_3$

2.8 求证对于任意二阶张量 T 有 $\Delta\left(\lambda\right)=\det\left(\lambda\delta_j^i-T_{.j}^{i}\right)=\det\left(\lambda\delta_i^j-T_{.i}^{j}\right)$

证明：①=$\begin{vmatrix}\lambda-T_{.1}^{1} & -T_{.2}^{1} & -T_{.3}^{1}\\ -T_{.1}^{2} & \lambda-T_{.2}^{2} & -T_{.3}^{2}\\ -T_{.1}^{3} & -T_{.2}^{3} & \lambda-T_{.3}^{3}\end{vmatrix}$，②=$\begin{vmatrix}\lambda-T_{1}^{.1} & -T_{1}^{.2} & -T_{1}^{.3}\\ -T_{2}^{.1} & \lambda-T_{2}^{.2} & -T_{2}^{.3}\\ -T_{3}^{.1} & -T_{3}^{.2} & \lambda-T_{3}^{.3}\end{vmatrix}$

2.9

由题给出 $X = T\bullet T^{T}, Y = T^{T}\bullet T$

$$X^{T} = \left(T\bullet T^{T}\right)^{T} = \left(T^{T}\right)^{T}\bullet T^{T} = T\bullet T^{T} = X$$

同理

$$Y^{T} = \left(T^{T}\bullet T\right)^{T} = T^{T}\bullet\left(T^{T}\right)^{T} = T^{T}\bullet T = Y$$

因此 **X, Y** 均为对称张量，两相量分别用分量表示

$$X = T\bullet T^{T} = T^{i}{}_{\cdot j}T^{m}{}_{\cdot i}g^{j}g_{m} = X_{j}{}^{\cdot m}g^{j}g_{m}$$

因 **X** 为对称矩阵 所以 $X^{i}{}_{\cdot j} = X_{i}{}^{\cdot j} = T^{i}{}_{\cdot j}T^{m}{}_{\cdot i} = T^{m}{}_{\cdot i}T^{i}{}_{\cdot j}$

$$Y = T^{i}{}_{\cdot m}T^{m}{}_{\cdot j}g_{i}g^{j} = Y^{i}{}_{\cdot j}g_{i}g^{j} = X^{i}{}_{\cdot j}$$

则可知 XY 的特征多项式相同，特征值相等则显然 $\lambda_X = \lambda_Y = \lambda$

即证得 $\Delta\left(\lambda\right) = \det\left(\lambda\delta_j^i - X^{i}{}_{\cdot j}\right) = \det\left(\lambda\delta_j^i - Y^{i}{}_{\cdot j}\right)$

2.10 已知：任意二阶张量 $\boldsymbol{T}$ 及其转置 $\boldsymbol{T}^{T}$，任意矢量 $\boldsymbol{u}$，求证：$\boldsymbol{T}\bullet\boldsymbol{u}=\boldsymbol{u}\bullet\boldsymbol{T}^{T}$

证：$\boldsymbol{T}\bullet\boldsymbol{u} = T_{ij}\boldsymbol{g}^i\boldsymbol{g}^j\bullet u^k\boldsymbol{g}_k = T_{ij}u^k\delta_k^j\boldsymbol{g}^i = T_{ij}u^j\boldsymbol{g}^i$

$\boldsymbol{u}\bullet\boldsymbol{T}^T = u^k\boldsymbol{g}_k\bullet T_{ij}\boldsymbol{g}^j\boldsymbol{g}^i = T_{ij}u^k\delta_k^j\boldsymbol{g}^i = T_{ij}u^j\boldsymbol{g}^i$

$\therefore$ 原式得证。

2.11 无

2.12 已知：T 为正则的二阶张量，u 为一矢量，$T\cdot u=0$

求证：$u=0$

证明:因为 T^{-1} 存在，且不为零，将 T^{-1}、点积式 $T\cdot u=0$ 的左右两端，得

左端=$T^{-1}\cdot T\cdot u=G\cdot u=u$

右端=$T^{-1}\cdot 0=0$

故 $u=0$

2.13 无

2.14 求证：$\left(\mathbf{T}^T\right)^{-1}=\left(\mathbf{T}^{-1}\right)^T$ （$\mathbf{T}$ 为正则二阶张量)

证明：对于映射量，转置和逆运算可也交换次序

$\left(\mathbf{T}^{-1}\right)^{\mathrm{T}}\bullet\mathbf{T}^{\mathrm{T}}=\left(\mathbf{T}\bullet\mathbf{T}^{-1}\right)^{\mathrm{T}}=\mathbf{I}^{\mathrm{T}}=\mathbf{I}=\left(\mathbf{T}^T\right)^{-1}\bullet\mathbf{T}^T$

从而 $\left[\left(\mathbf{T}^{-1}\right)^T-\left(\mathbf{T}^T\right)^{-1}\right]\bullet\mathbf{T}^T=0$

两边右乘 $\left(T^T\right)^{-1}$ 有：$\left(\mathbf{T}^T\right)^{-1}=\left(\mathbf{T}^{-1}\right)^T$

2.15 已知：A B 为正则的二阶张量。

求证：$\left(A?B\right)^{-1}\quad B^{1}\square A^{1}$

证：$\because\left(A\quad B\right)^{-1}\ \left(A?B\right)\quad G=B^{-1}\quad A^{1}\quad A\square B$

\ $\left(A?B\right)^{-1}\quad B^{-1}A^{-1}\ \left(A?B\right)\quad 0$

$\because$ A和B为正则的二阶张量，

\ $\left(A\quad B\right)\quad [0]$

\ $\left(A?B\right)^{-1}\quad B^{-1}A^{-1}=0$

即 $\left(A\times B\right)^{-1}=B^{-1}A^{-1}$

故命题由此得证

2.16 (1)已知$\boldsymbol{T}$为任意二阶张量。求证：$\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}} \geq 0, \boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{T} \geq 0$

(2)已知：$\boldsymbol{T}$为正则的二阶张量。求证：$\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}} > 0, \boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{T} > 0$

解：设$\boldsymbol{u}$为任一非零矢量，它与二阶张量$\boldsymbol{T}$的点积$\boldsymbol{u}\cdot\boldsymbol{T} = \boldsymbol{v}$
$\boldsymbol{v}$也是一矢量

$\left(\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}}\right) = \boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}}$,所以$\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}}$为对称二阶张量。

$$\boldsymbol{u}\cdot\left(\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{T}\right)\cdot\boldsymbol{u} = \left(\boldsymbol{u}\cdot\boldsymbol{T}^{\mathrm{T}}\right)\cdot\left(\boldsymbol{T}\cdot\boldsymbol{U}\right) = \left(\boldsymbol{T}\cdot\boldsymbol{u}\right)\cdot\left(\boldsymbol{T}\cdot\boldsymbol{u}\right) = |\boldsymbol{v}|^2 \geq 0$$

故由定义$\boldsymbol{u}\cdot\boldsymbol{N}\cdot\boldsymbol{u} = \boldsymbol{N}:\boldsymbol{uu} \geq 0$，$\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{T} \geq 0$。同理可证$\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}} \geq 0$。

若$\boldsymbol{T}$为可逆二阶张量，$\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}}$为对称二阶张量。只有当$\boldsymbol{u}$为零矢量的时候$(\boldsymbol{T}\cdot\boldsymbol{u})$才是零矢量。现在一直$\boldsymbol{u}$为非零矢量，故

$$\boldsymbol{u}\cdot\left(\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{T}\right)\cdot\boldsymbol{u} = \left(\boldsymbol{u}\cdot\boldsymbol{T}^{\mathrm{T}}\right)\cdot\left(\boldsymbol{T}\cdot\boldsymbol{U}\right) = \left(\boldsymbol{T}\cdot\boldsymbol{u}\right)\cdot\left(\boldsymbol{T}\cdot\boldsymbol{u}\right) = |\boldsymbol{v}|^2 > 0$$

由定义$\boldsymbol{u}\cdot\boldsymbol{N}\cdot\boldsymbol{u} = \boldsymbol{N}:\boldsymbol{uu} > 0$，$\boldsymbol{T}^{\mathrm{T}}\cdot\boldsymbol{T} > 0$。同理可证$\boldsymbol{T}\cdot\boldsymbol{T}^{\mathrm{T}} > 0$。

2.17 已知：正交张量Q。

求证：$Q^T = Q^{-1}$亦为正交张量

证明：$\because Q$是正交张量，则满足$Q^T = Q^{-1}$

$$Q^T.(Q^T)^T = Q^{-1}.Q = G$$

$$(Q^{-1}).(Q^{-1})^T = Q^T.Q = G$$

则$Q^T = Q^{-1}$亦为正交张量

2.18 已知：对于任意矢量u,v,均成立（Q·u）·（Q·v）= u·v

求证：$Q^T = Q^{-1}, Q$为正交张量。

证明：

$$(Q\cdot U)\cdot(Q\cdot V)=(Q\cdot V)\cdot(Q\cdot U)^T=(Q\cdot V)\cdot(U\cdot Q^T)$$
$$=Q^{i}_{\cdot j}g_i g^j\cdot V^m g_m\cdot U^n g_n\cdot Q^{\cdot i}_{j}g g^j$$
$$=Q^{i}_{\cdot j}V^j g_i\cdot U^n g_n\cdot Q^{\cdot i}_{j}g_i g^j$$
$$=Q^{i}_{\cdot j}V^j U^n g_{in}\cdot Q^{\cdot i}_{j}g_i g^j$$
$$=Q_{nj}V^j U^n\cdot Q$$
$$(Q\cdot V)\cdot(Q\cdot U)^T=Q^{i}_{\cdot j}V^j g_i\cdot(U_n g^n\cdot Q^{\cdot i}_{j}g_i g^j)$$
$$=Q^{i}_{\cdot j}V^j g_i\cdot U_i Q^{\cdot i}_{j}g^j=Q^{i}_{\cdot j}V^j U_i Q^{\cdot i}_{j}\delta^j_i$$
$$=Q^{i}_{\cdot j}V^i U_i Q^{\cdot i}_{i}=Q^{i}_{\cdot j}V^j\delta^j_i U_i Q^{\cdot i}_{i}=U_i V^i Q^{i}_{\cdot i}Q^{\cdot i}_{i}$$
$$=U\cdot V=U_n g^n V^m g_m=U_n V^n$$

所以：$Q^{i}_{\cdot i}Q^{\cdot i}_{i}$ =1　　即 Q 为正交矩阵

2-19 证明：

$$\left(Q\cdot V\right)\times\left(Q\cdot \mathrm{W}\right)=\left(Q\cdot V\right)\left(Q\cdot \mathrm{W}\right):\in=Q^{\mathrm{i}}_{\mathbf{j}}V^j g_i Q^m_n W^n g_m$$

$$\in_{opq}\ g^o g^p g^q=Q^{\mathrm{i}}_{\mathbf{j}}Q^m_n V^j W^n\in_{opq}\ \delta^o_i\delta^p_m g^q=Q^{\mathrm{i}}_{\mathbf{j}}Q^m_n V^j W^n\in_{imq}\ g^q$$

$$=\frac{Q^{\mathrm{i}}_{\mathbf{j}}Q^m_n Q^q_s}{Q^q_s}V^j W^n\in_{imq}\ g^q=\left(\det\theta\right)^{\in_{jms}}\frac{V^j W^n g^s\delta^q_s}{Q^q_s}$$

$$=\left(\det\theta\right)^{\delta^q_s}\frac{\left(V\times W\right)}{Q^q_s}$$

2.20 已知：矢量 w、v，正则的二阶张量 B。求证：

$$\left(B\bullet v\right)\times\left(B\bullet w\right)=\left(\det B\right)\left(B^{-1}\right)^T\bullet\left(v\times w\right)$$

证明：所证命题等价于

$$\left(B\bullet v\right)\times\left(B\bullet w\right)=\left(\det B\right)\left(B^{-1}\right)^T\bullet\left(v\times w\right)$$

则可得：

$$\left(B\bullet v\right)\times\left(B\bullet w\right)=B^i_{\cdot j}B^m_{\cdot n}V^j W^n B^q_{\cdot b}\varepsilon_{imq}g^b$$
$$=\det B\varepsilon_{jnb}V^j W^n g^b=\det B\left(v\times w\right)$$

即原命题成立。

得证，其他类推

2.21

求证X=T T^T与$Y=T^T\ T$之间互为正交相似张量。

即存在正交张量Q，使X=Q Y Q^T

证明：Q Y $Q^T=QT^TTQ^T=QT^T\left(QT^T\right)^T=QT^T\left(QT^T\right)^T=\left(TQ^T\right)^T\left(QT^T\right)^T=\left(TQ^TQT^T\right)^T=TT^T=X$

注：正交张量存在如下性质：$Q^{1}=Q^T$

故命题得证

2.24已知：二阶张量

$$T=-\frac{1}{2}e_1e_1-\frac{\sqrt{3}}{2}e_1e_2+\sqrt{3}e_2e_1-e_2e_2+e_3e_3$$

求（1） 进行加法分解 （2) 进行乘法分解

加法分解 $T=\begin{matrix}\frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0\\ \sqrt{3} & -1 & 0\\ 0 & 0 & 3\end{matrix}$ $\quad N_i^j=\frac{1}{2}\left(T_i^j+T_j^i\right)$

所以 $N=\begin{matrix}-\frac{1}{2} & \frac{\sqrt{3}}{4} & 0\\ \frac{\sqrt{3}}{4} & -\frac{1}{2} & 0\\ 0 & 0 & \frac{3}{2}\end{matrix}$ （2）乘法分解 $T=\begin{matrix}\frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0\\ \sqrt{3} & -1 & 0\\ 0 & 0 & 3\end{matrix}$ ，其中

$T\ =\ Q\ \square H$

设 正 张 量 H 为 $\begin{matrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{matrix}$ ， 其 中 所 设 H 满 足 正 张 量

所 以 $H^{-1}=\begin{matrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{3}\end{matrix}$

则 $Q=H^{-1}\ T=\begin{matrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{3}\end{matrix}\ \begin{matrix}\frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0\\ \sqrt{3} & -1 & 0\\ 0 & 0 & 3\end{matrix}=\begin{matrix}-1 & -\sqrt{3} & 0\\ \sqrt{3} & -1 & 0\\ 0 & 0 & 1\end{matrix}$

所 求 Q张 量 满 足 $Q^{-1}=Q^T$为 正 交 张 量

2.25 对于以下三种应力状态的应力张量s，将其分解为球形张量和偏斜张量 S。求 J_1^s, J_2^S与J_3^S，以及偏斜张量 S的 v 角 。

（1）单向拉伸：$s_1 = s_2 > 0$，$s_2 = s_3 = 0$

（2）单向压缩；$s_1 = s_2 = 0$，$s_3 = -s_o < 0$

（3）纯剪切： $s_1 = t > 0, s_2 = 0, s_3 = -t$

解：

由题意得应力张量 $s = \begin{matrix} s_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$

所以 $J_1^s = T_1^1 + T_2^2 + T_3^3 = s_0$

$J_2^s = \frac{1}{2}\left(T_i^i T_l^l - T_i^l T_l^i\right) = 0$

$J_3^s = \det T = 0$

又因为N=P+D

$\therefore P_{\times j}^i = \frac{1}{3} J_1^s = \begin{cases} \frac{1}{3}\left(T_1^1 + T_2^2 + T_3^3\right) 当 i = j \\ 0 当 i \neq j \end{cases}$

所以偏斜张量D= $\begin{matrix} \frac{2}{3}s_0 & 0 & 0 \\ 0 & -\frac{1}{3}s_0 & 0 \\ 0 & 0 & -\frac{1}{3}s_0 \end{matrix}$

所以 $J_2^D = J_2^N - \frac{1}{3}\left(J_1^N\right)^2 = -\frac{1}{3}s_0^2$

$J_3^D = J_3^N - \frac{1}{3}J_1^N J_2^N + \frac{2}{27}\left(J_1^N\right)^3 = \frac{2}{27}s^3$，$COS\,3v = -\frac{\sqrt{27}J_3^D}{2\left|J_2^D\right|^{3/2}} =$

（2）由题意得应力张量

$$\sigma=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-\sigma_0\end{bmatrix}，球形张量 P=\begin{bmatrix}-\frac{1}{3}\sigma_0&0&0\\0&-\frac{1}{3}\sigma_0&0\\0&0&-\frac{1}{3}\sigma_0\end{bmatrix}，偏斜张量 D=\begin{bmatrix}\frac{1}{3}\sigma_0&0&0\\0&\frac{1}{3}\sigma_0&0\\0&0&\frac{2}{3}\sigma_0\end{bmatrix}$$

所以

$$J_1^s=T_1^1+T_2^2+T_3^3=-\sigma_0$$

$$J_2^s=\frac{1}{2}\left(T_i^iT_l^l-T_i^lT_l^i\right)=0$$

$$J_3^s=\det T=0$$

$$J_2^D=J_2^N-\frac{1}{3}\left(J_1^N\right)^2=-\frac{1}{3}\sigma_0^2$$

$$J_3^D=J_3^N-\frac{1}{3}J_1^NJ_2^N+\frac{2}{27}\left(J_1^N\right)^3=-\frac{2}{27}\sigma^3，COS3v=-\frac{\sqrt{27}J_3^D}{2\left|J_2^D\right|^{3/2}}=$$

（3）由题意得应力张量

$$\sigma=\begin{bmatrix}0&0&\tau\\0&0&0\\-\tau&0&o\end{bmatrix}，球形张量 P=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}，偏斜张量 D=\begin{bmatrix}0&0&\tau\\0&0&0\\\tau&0&0\end{bmatrix}$$

所以 $J_1^s=T_1^1+T_2^2+T_3^3=0$

$$J_2^s=\frac{1}{2}\left(T_i^iT_l^l-T_i^lT_l^i\right)=\tau$$

$$J_3^s=\det T=0$$

$$J_2^D=J_2^N-\frac{1}{3}\left(J_1^N\right)^2=\tau$$

$$J_3^D=J_3^N-\frac{1}{3}J_1^NJ_2^N+\frac{2}{27}\left(J_1^N\right)^3=0，COS3v=-\frac{\sqrt{27}J_3^D}{2\left|J_2^D\right|^{3/2}}=0$$

2.26 题

证明：

$\because$ 由已知条件 Q为一正交张量得:

$Q^{-1} = Q^{T}$

又 $\because \tilde{T} = Q \bullet T \bullet Q^{T}$得，

$\tilde{T} = Q \bullet T \bullet Q^{-1}$

两边乘以 Q得，

$\tilde{T}Q = Q \bullet T \bullet Q^{-1}Q$

即 $\tilde{T}Q = Q \bullet T$,

又已知 T的特征值为 λ，$\tilde{T}$的特征值为 $\tilde{\lambda}$,

$\therefore \lambda = \tilde{\lambda}$

2.27 $\mathrm{M}^{\mathrm{i}}_{.\mathrm{j}}\vec{a}^{j} = \lambda\vec{a}^{i}$

$$\mathrm{M}^{2} = \mathrm{M}^{\mathrm{i}}_{.\mathrm{j}}\vec{a}^{j} \bullet \mathrm{M}^{\mathrm{i}}_{.\mathrm{j}}\vec{a}^{j} = \lambda\vec{a}^{i} \bullet \lambda\vec{a}^{i}$$

$$= \lambda^{2}\vec{a}^{i}$$

因为$\vec{\mathrm{M}}^{2} = \vec{\mathrm{N}}$,所以有$\mathrm{M}^{\mathrm{i}}_{.\mathrm{j}} \bullet \mathrm{M}^{\mathrm{i}}_{.\mathrm{j}} = \mathrm{N}^{\mathrm{i}}_{.\mathrm{j}}$

$\mathrm{N}^{\mathrm{i}}_{.\mathrm{j}} = \lambda^{2}\vec{a}^{i}$

所以，$\vec{\mathrm{M}}$和$\vec{\mathrm{M}}^{2} = \vec{\mathrm{N}}$有相同的特征向量，

所以，其主方向相同。

2.28 已知：$\boldsymbol{A}$为二阶张量，$\boldsymbol{Q}$为任意正交张量，对于一切$\boldsymbol{Q}$，均有$\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^{T} = \boldsymbol{A}$

求证：$\boldsymbol{A}$为球形张量

证明：设二阶张量$\boldsymbol{A}$在一组正交标准基$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$中的并矢展开式为

$$\begin{aligned}\boldsymbol{A} = &A_{11}\boldsymbol{e}_1\boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1\boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1\boldsymbol{e}_3 + A_{21}\boldsymbol{e}_2\boldsymbol{e}_1\\ &+A_{22}\boldsymbol{e}_2\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\boldsymbol{e}_3 + A_{31}\boldsymbol{e}_3\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\boldsymbol{e}_3\end{aligned}$$

由于$\boldsymbol{Q}$为任意正交张量，取正交张量$\boldsymbol{Q} = -\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3$

则，
$$\begin{aligned}\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^{T} = &A_{11}\boldsymbol{e}_1\boldsymbol{e}_1 - A_{12}\boldsymbol{e}_1\boldsymbol{e}_2 - A_{13}\boldsymbol{e}_1\boldsymbol{e}_3 - A_{21}\boldsymbol{e}_2\boldsymbol{e}_1\\ &+A_{22}\boldsymbol{e}_2\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\boldsymbol{e}_3 - A_{31}\boldsymbol{e}_3\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\boldsymbol{e}_3\end{aligned}$$

由题知$\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^{T} = \boldsymbol{A}$

则有：$A_{12} = -A_{12} = 0$，$A_{13} = -A_{13} = 0$，$A_{21} = -A_{21} = 0$，$A_{31} = -A_{31} = 0$

同理，取正交张量$\boldsymbol{Q} = \boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3$

可得：$A_{23}=A_{32}=0$

则有：$\boldsymbol{A}=A_{11}\boldsymbol{e}_1\boldsymbol{e}_1+A_{22}\boldsymbol{e}_2\boldsymbol{e}_2+A_{33}\boldsymbol{e}_3\boldsymbol{e}_3$

证得 $\boldsymbol{A}$ 为对称张量

取正交张量 $\boldsymbol{Q}=\boldsymbol{e}_2\boldsymbol{e}_1-\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_3\boldsymbol{e}_3$

有 $\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T=A_{11}\boldsymbol{e}_1\boldsymbol{e}_1+A_{22}\boldsymbol{e}_2\boldsymbol{e}_2+A_{33}\boldsymbol{e}_3\boldsymbol{e}_3$

由 $\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^T=\boldsymbol{A}$

得 $A_{11}=A_{22}$

同理，取正交张量 $\boldsymbol{Q}=\boldsymbol{e}_1\boldsymbol{e}_1+\boldsymbol{e}_3\boldsymbol{e}_2-\boldsymbol{e}_2\boldsymbol{e}_3$

可证：$A_{22}=A_{33}$

故：$\boldsymbol{A}=A_{11}\left(\boldsymbol{e}_1\boldsymbol{e}_1+\boldsymbol{e}_2\boldsymbol{e}_2+\boldsymbol{e}_3\boldsymbol{e}_3\right)$ 是球形张量。

2.29 解

$$T = \mathrm{N}+\Omega = \begin{bmatrix} \mathrm{N}_1 & -\omega_3 & \omega_2 \\ \omega_3 & \mathrm{N}_2 & -\omega_1 \\ -\omega_2 & \omega_1 & \mathrm{N}_3 \end{bmatrix}$$

$\mathrm{Tr}(\mathrm{T})=T^{i}{}_{\cdot j}$

$\mathrm{Tr}(T^2)=T^{i}{}_{\cdot j}\,T^{j}{}_{\cdot i}$

$\mathrm{Tr}(T^3)=T^{i}{}_{\cdot j}\,T^{j}{}_{\cdot k}\,T^{k}{}_{\cdot i}$

因主不变量与坐标的变换无关，因此可以将上试与矩阵中的元素分别对应

$\mathrm{Tr}(\mathrm{T})=N_1+N_2+N_3$

$\mathrm{Tr}(\mathrm{T})=N_1{}^2+N_2{}^2+N_3{}^2-2\omega_1{}^2-2\omega_2{}^2-2\omega_3{}^2$

$\mathrm{Tr}(\mathrm{T})=$P83 页

2.23

$$D_i = N_i - \frac{1}{3}\mathcal{J}_1^N \qquad (i=1,\ 2,\ 3)$$

证明： $$\boldsymbol{N}\bullet\boldsymbol{a} = (\boldsymbol{D}+\boldsymbol{P})\bullet\boldsymbol{a} = \lambda^N\boldsymbol{a}$$

即 $$N^i_{\bullet j}a^j = \left(D^i_{\bullet j} + \frac{1}{3}\mathcal{J}_1^N\delta^i_j\right)a^j = \lambda^N a^i$$

或 $$D^i_{\bullet j}a^j = \left(\lambda^N - \frac{1}{3}\mathcal{J}_1^N\right)a^i = \lambda^D a^i$$

即 $$\boldsymbol{D}\bullet\boldsymbol{a} = \lambda^D\boldsymbol{a}$$

偏斜张量D与它对应的对称二阶张量N具有相同的主方向$\boldsymbol{a}$，且其主分量满足

$$D_i = N_i - \frac{1}{3}\mathcal{J}_1^N \qquad (i=1,\ 2,\ 3)$$

第三章

3.1 已知：$\mathbf{v}$为矢量。求：$f = e^{v^2}$是否为$\mathbf{v}$的各向同性函数，并说明理由。

答：是的。

3.2 已知：$\mathbf{T}$为二阶张量。求：下列函数是否为$\mathbf{T}$的各向同性标量函数，并说明理由。

（1）在某一特定的笛卡尔坐标系中

$$f = \sum_{i=1}^{3}\sum_{j=1}^{3}(T_{ij})^2$$

（2）$f = \mathbf{T}^T : \mathbf{T}$

答：（1）是。$f=\mathbf{T}^{T}:\mathbf{T}$是$\mathbf{T}$的不变量。

（2）是。$f=T_{\bullet j}^{i}T_{\bullet i}^{j}=\lambda_{2}^{*}$

3.4 已知：二阶张量$\mathbf{T}$。求：下列张量函数是否为$\mathbf{T}$的各向同性标量函数，并说明理由。

（1）$\mathbf{H}=\mathbf{T}^{T}$

（2）$\mathbf{H}=\mathbf{T}\bullet\mathbf{A}\bullet\mathbf{T}$

答：（1）是。$\left(\tilde{\mathbf{T}}\right)^{*}=\left(\mathbf{Q}\bullet\mathbf{T}\bullet\mathbf{Q}^{*}\right)^{*}=\mathbf{Q}\bullet\mathbf{T}^{*}\bullet\mathbf{Q}^{*}=\left(\widetilde{\mathbf{T}^{*}}\right)$

（2）不是。$\tilde{\mathbf{T}}\bullet\mathbf{A}\bullet\tilde{\mathbf{T}}=\mathbf{Q}\bullet\mathbf{T}\bullet\left(\mathbf{Q}^{*}\bullet\mathbf{A}\bullet\mathbf{Q}\right)\bullet\mathbf{T}\bullet\mathbf{Q}^{*}$，一般$\mathbf{Q}^{*}\bullet\mathbf{A}\bullet\mathbf{Q}\neq\mathbf{A}$

3.4 已知：二阶张量 $\boldsymbol{T}$。

求：下列张量函数是否为$\boldsymbol{T}$的各向同性函数，并说明理由。

解：（1）是 。 $(\tilde{T})^{T}=(Q\cdot T\cdot Q^{T})^{T}=Q\cdot T^{T}\cdot Q^{T}=(\tilde{T}^{T})$

(2)不是。$\bar{T}\cdot A\cdot\bar{T}=Q\cdot T\cdot(Q^{T}\cdot A\cdot Q)\cdot T\cdot Q^{T}$,一般，$Q^{T}\cdot A\cdot Q\neq A$

3.5 已知：二阶张量 $\boldsymbol{T}$的张量函数$H=A\cdot T$（$\boldsymbol{A}$为二阶常张量）。

求：$\boldsymbol{A}$满足什么条件时，$\boldsymbol{H}$是 $\boldsymbol{T}$的各向同性函数。

解：当 $\boldsymbol{A}$是球形张量时，$H=A\cdot T$是 $\boldsymbol{T}$的各向同性函数。

$H=A\cdot T$是 $\boldsymbol{T}$的各向同性函数即$H=A\cdot T=(Q\cdot A\cdot Q^{T})\cdot T$,所以$Q\cdot A\cdot Q^{T}=A$

设二阶张量$\boldsymbol{A}$在在一组正交标准化基$\mathbf{e}_1$，$\mathbf{e}_2$，$\mathbf{e}_3$中的并矢展开式为

$A=A_{11}\mathbf{e}_1\mathbf{e}_1+A_{12}\mathbf{e}_1\mathbf{e}_2+A_{13}\mathbf{e}_1\mathbf{e}_3+A_{21}\mathbf{e}_2\mathbf{e}_1+A_{22}\mathbf{e}_2\mathbf{e}_2+A_{23}\mathbf{e}_2\mathbf{e}_3+A_{31}\mathbf{e}_3\mathbf{e}_1+A_{32}\mathbf{e}_3\mathbf{e}_2+A_{33}\mathbf{e}_3\mathbf{e}_3$

先证$\boldsymbol{A}$是对称张量。若取正交张量$Q=-\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2+\mathbf{e}_3\mathbf{e}_3$(为关于$x^2$，$x^3$平面的镜面反射)，则

$Q\cdot A\cdot Q^{T}=A_{11}\mathbf{e}_1\mathbf{e}_1-A_{12}\mathbf{e}_1\mathbf{e}_2-A_{13}\mathbf{e}_1\mathbf{e}_3-A_{21}\mathbf{e}_2\mathbf{e}_1+A_{22}\mathbf{e}_2\mathbf{e}_2+A_{23}\mathbf{e}_2\mathbf{e}_3-A_{31}\mathbf{e}_3\mathbf{e}_1+A_{32}\mathbf{e}_3\mathbf{e}_2+A_{33}\mathbf{e}_3\mathbf{e}_3$

由于$Q \cdot A \cdot Q^T = A$

故可证得，$A_{12} = -A_{12} = 0$，$A_{13} = -A_{13} = 0$，$A_{21} = -A_{21} = 0$，$A_{31} = -A_{31} = 0$

同理若设$Q = \mathrm{e}_1\mathrm{e}_1 - \mathrm{e}_2\mathrm{e}_2 + \mathrm{e}_3\mathrm{e}_3$

可证得 $A_{23} = A_{32} = 0$

故$A = A_{11}\mathrm{e}_1\mathrm{e}_1 + A_{22}\mathrm{e}_2\mathrm{e}_2 + A_{33}\mathrm{e}_3\mathrm{e}_3$是对称张量。

再证 A 是球形张量。即证 $A_{11} = A_{22} = A_{33}$

若取$Q = \mathrm{e}_2\mathrm{e}_1 - \mathrm{e}_1\mathrm{e}_2 + \mathrm{e}_3\mathrm{e}_3$（即绕 x^3 转动 90°）

$Q \cdot A \cdot Q^T = A_{11}\mathrm{e}_2\mathrm{e}_3 + A_{22}\mathrm{e}_1\mathrm{e}_1 + A_{33}\mathrm{e}_3\mathrm{e}_3$

由于$Q \cdot A \cdot Q^T = A$，故可证得，$A_{11} = A_{22}$

同理，若设$Q = \mathrm{e}_1\mathrm{e}_1 + \mathrm{e}_3\mathrm{e}_2 - \mathrm{e}_2\mathrm{e}_3$，可证得 $A_{22} = A_{33}$

故$A = A_{11}(\mathrm{e}_1\mathrm{e}_1 + \mathrm{e}_2\mathrm{e}_2 + \mathrm{e}_3\mathrm{e}_3) = A_{11}G$ 是球形张量。

3.15 设$\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j = T^{ij}\mathbf{g}_i\mathbf{g}_j$

则$\mathbf{H} = f'(\mathbf{T}) = \dfrac{\partial f}{\partial T_{ij}}\mathbf{g}_i\mathbf{g}_j$

$\mathbf{T}$的正交相似张量$\tilde{\mathbf{T}} = T_{ij}\tilde{\mathbf{g}}^i\tilde{\mathbf{g}}^j = T^{ij}\tilde{\mathbf{g}}_i\tilde{\mathbf{g}}_j$

其中$\tilde{\mathbf{g}}_i = \mathbf{Q}\cdot\mathbf{g}_i \quad \tilde{\mathbf{g}}^i = \mathbf{Q}\cdot\mathbf{g}^i$

由于$f(\mathbf{T})$是各向同性标量函数，$f\left(\tilde{\mathbf{T}}\right) = f(\mathbf{T})$

故$f'\left(\tilde{\mathbf{T}}\right) = \dfrac{\partial f}{\partial T_{ij}}\tilde{\mathbf{g}}_i\tilde{\mathbf{g}}_j = \dfrac{\partial f}{\partial T_{IJ}}\left(\mathbf{Q}\cdot\mathbf{g}_i\right)\left(\mathbf{Q}\cdot\mathbf{g}_j\right)$

$$= \frac{\partial f}{\partial T_{IJ}}\mathbf{Q}\cdot\mathbf{g}_i\mathbf{g}_j\cdot\mathbf{Q}^* = \mathbf{Q}\cdot\mathbf{H}\cdot\mathbf{Q}^* = \widetilde{\mathbf{H}}$$

因此，$\mathbf{H} = f'(\mathbf{T})$ 是各向同性张量函数。

3.16 设 $\boldsymbol{\upsilon} = \upsilon_i\mathbf{g}^i = \upsilon^i\mathbf{g}_i$

其旋转量 $\tilde{\boldsymbol{\upsilon}} = \upsilon_i\tilde{\mathbf{g}}^i = \upsilon^i\tilde{\mathbf{g}}_i$

其中 $\tilde{\mathbf{g}}_i = \mathbf{Q}\cdot\mathbf{g}_i$ ，$\tilde{\mathbf{g}}^i = \mathbf{Q}\cdot\mathbf{g}^i$

因为 $F(\boldsymbol{\upsilon})$ 是 $\boldsymbol{\upsilon}$ 的各向同性矢量函数，故：

$$F(\tilde{\boldsymbol{\upsilon}}) = \mathbf{Q}\cdot F(\boldsymbol{\upsilon})$$

设 $\mathbf{H} = F'(\boldsymbol{\upsilon}) = \dfrac{\partial F(\boldsymbol{\upsilon})}{\partial \upsilon_i}\mathbf{g}_i$

故 $F'(\tilde{\boldsymbol{\upsilon}}) = \dfrac{\partial F(\tilde{\boldsymbol{\upsilon}})}{\partial \upsilon_i}\tilde{\mathbf{g}}_i = \dfrac{\partial\left[\mathbf{Q}\cdot F(\boldsymbol{\upsilon})\right]}{\partial \upsilon_i}\mathbf{Q}\cdot\mathbf{g}_i$

$$= \mathbf{Q}\cdot\frac{\partial F(\boldsymbol{\upsilon})}{\partial \upsilon_i}\mathbf{g}_i\cdot\mathbf{Q}^* = \mathbf{Q}\cdot\mathbf{H}\cdot\mathbf{Q}^* = \widetilde{\mathbf{H}}$$

3.18 求 $\det\left(T^m\right)$ 的导数（T 为二阶张量）。

$$m\left(g_3^T\right)^m\left(T^{-1}\right)^T$$

3.19 求 $\dfrac{dT^T}{dT}$（T^T 为二阶张量 T 的转置张量）。

$$g^i g_j g^j g_i$$

3.20 求 $\dfrac{d\left[\left(T^T\right)^2\right]}{dT}$（$T^T$ 为二阶张量 T 的转置张量）

$$T^i{}_{\cdot j}\left(g^s g_i g^j g_s + g^j g_s g^s g_i\right)$$

3.21 求 $\det(\lambda\boldsymbol{G}-\boldsymbol{T})$ 对 λ 及对 $\boldsymbol{T}$ 的一阶、二阶导数（$\boldsymbol{T}$ 为二阶张量）。

解：$$\frac{d}{d\lambda}\left[\det(\lambda\boldsymbol{G}-\boldsymbol{T})\right] = 3\lambda^2 - 2\lambda\delta_1^T + \delta_2^T$$

$$\frac{d^2}{d\lambda^2}\left[\det(\lambda\boldsymbol{G}-\boldsymbol{T})\right] = 6\lambda - 2\delta_1^T$$

$$\frac{d}{d\boldsymbol{T}}\left[\det(\lambda\boldsymbol{G}-\boldsymbol{T})\right] = (-\lambda^2 + \lambda\delta_1^T - \delta_2^T)\boldsymbol{G} + (-\lambda + \delta_1^T)\boldsymbol{T}^* - (\boldsymbol{T}^2)^*$$

$$\frac{d^2}{d\boldsymbol{T}^2}\left[\det(\lambda\boldsymbol{G}-\boldsymbol{T})\right] = (\lambda - \delta_1^T)\boldsymbol{G}\boldsymbol{G} + \boldsymbol{G}\boldsymbol{T}^* + \boldsymbol{T}^*\boldsymbol{G} + (\delta_1^T - \lambda)\frac{d\boldsymbol{T}}{d\boldsymbol{T}^*} - \frac{d(\boldsymbol{T}^*)^2}{d\boldsymbol{T}}$$

3.22 已知：矢量 $\boldsymbol{v}$ 的标量函数 $\varphi = e^{\boldsymbol{v}^2}$，

求：（1） $\frac{d\varphi}{d\boldsymbol{\upsilon}}$

（2）是否为各向同性函数，并说明理由。

解：$2ve^{v^2}$ 。是各向同性矢量函数。

第四章

4.5 已知：φ 为标量场函数，$\boldsymbol{\upsilon}$ 为矢量场函数

求证：$\nabla(\varphi\boldsymbol{\upsilon})=\varphi(\nabla\boldsymbol{\upsilon})+(\nabla\varphi)\boldsymbol{\upsilon}$

证明：$\nabla(\varphi\boldsymbol{\upsilon})=\mathbf{g}^{\mathbf{i}}\frac{\partial}{\partial x^i}(\varphi\boldsymbol{\upsilon})=\mathbf{g}^{\mathbf{i}}\frac{\partial\varphi}{\partial x^i}\boldsymbol{\upsilon}+\mathbf{g}^{\mathbf{i}}\frac{\partial\boldsymbol{\upsilon}}{\partial x^i}\boldsymbol{\upsilon}=\varphi(\nabla\boldsymbol{\upsilon})+(\nabla\varphi)\boldsymbol{\upsilon}$

4.6 已知：$\boldsymbol{\upsilon},\boldsymbol{\omega}$ 均为矢量场函数。

求证：$\nabla(\boldsymbol{\upsilon}\bullet\boldsymbol{\omega})=(\nabla\boldsymbol{\omega})\bullet\boldsymbol{\upsilon}+(\nabla\boldsymbol{\upsilon})\boldsymbol{\omega}$

证明：

$$\nabla(\boldsymbol{\upsilon}\bullet\mathbf{w})=\mathbf{g}^{\mathbf{i}}\frac{\partial}{\partial x^i}(\boldsymbol{\upsilon}\bullet\mathbf{w})=\mathbf{g}^{\mathbf{i}}\frac{\partial\boldsymbol{\upsilon}}{\partial x^i}\mathbf{w}+\mathbf{g}^{\mathbf{i}}\boldsymbol{\upsilon}\frac{\partial w}{\partial x^i}=(\nabla\boldsymbol{\upsilon})\bullet\mathbf{w}+\mathbf{g}^{\mathbf{i}}\frac{\partial\mathbf{w}}{\partial x^i}\bullet\boldsymbol{\upsilon}=(\nabla\boldsymbol{\upsilon})\bullet\mathbf{w}+(\mathbf{g}\mathbf{w})\bullet\boldsymbol{\upsilon}$$

4.7 已知：$\boldsymbol{\upsilon}$ 为矢量场函数，$\mathbf{a}$ 为任意适量。

求证：$(curl\boldsymbol{\upsilon})\times\mathbf{a}=[\boldsymbol{\upsilon}\nabla-\nabla\boldsymbol{\upsilon}]\bullet\mathbf{a}$

证明：

$$(curl\boldsymbol{\upsilon})\times\mathbf{a}=\left(\mathbf{g}^{\mathbf{i}}\times\frac{\partial\boldsymbol{\upsilon}}{\partial x^i}\right)\times\mathbf{a}=\left(\mathbf{a}\bullet\mathbf{g}^{\mathbf{i}}\right)\frac{\partial\boldsymbol{\upsilon}}{\partial x^i}-\left(\mathbf{a}\bullet\frac{\partial\boldsymbol{\upsilon}}{\partial x^i}\right)\mathbf{g}^{\mathbf{i}}$$

$$=\mathbf{a}\bullet(\nabla\boldsymbol{\upsilon})-\mathbf{a}\bullet(\boldsymbol{\upsilon}\nabla)=\boldsymbol{\upsilon}\nabla\bullet\mathbf{a}-\nabla\boldsymbol{\upsilon}\bullet\mathbf{a}=[\boldsymbol{\upsilon}\nabla-\nabla\boldsymbol{\upsilon}]\bullet\mathbf{a}$$

4.8 已知：$\mathbf{u},\boldsymbol{\upsilon}$ 为矢量场函数。

求证：$\nabla(\mathbf{u}\bullet\mathbf{v})=\mathbf{u}\times(\nabla\times\mathbf{v})+\mathbf{v}\times(\nabla\times\mathbf{u})+\mathbf{u}\bullet(\nabla\mathbf{v})+\mathbf{v}\bullet(\nabla\mathbf{u})$

证明：

$$\mathbf{u}\times(\nabla\times\mathbf{v})+\mathbf{v}\times(\nabla\times\mathbf{u})+\mathbf{u}\bullet(\nabla\mathbf{v})+\mathbf{v}\bullet(\nabla\mathbf{u})$$

$$=\mathbf{u}\times\left(\mathbf{g}^{\mathbf{i}}\times\frac{\partial\mathbf{v}}{\partial x^{i}}\right)+\mathbf{v}\times\left(\mathbf{g}^{\mathbf{i}}\times\frac{\partial\mathbf{u}}{\partial x^{i}}\right)+\mathbf{u}\bullet(\nabla\mathbf{v})+\mathbf{v}\bullet(\nabla\mathbf{u})$$

$$=\left(\mathbf{u}\bullet\frac{\partial\mathbf{v}}{\partial x^{i}}\right)\mathbf{g}^{\mathbf{i}}-\left(\mathbf{u}\bullet\mathbf{g}^{\mathbf{i}}\right)\frac{\partial\mathbf{v}}{\partial x^{i}}+\left(\mathbf{v}\bullet\frac{\partial\mathbf{u}}{\partial x^{i}}\right)\mathbf{g}^{\mathbf{i}}-\left(\mathbf{u}\bullet\mathbf{g}^{\mathbf{i}}\right)\frac{\partial\mathbf{u}}{\partial x^{i}}+\mathbf{u}\bullet(\nabla\mathbf{v})+\mathbf{v}\bullet(\nabla\mathbf{u})$$

$$=\left(\mathbf{u}\bullet\frac{\partial\mathbf{v}}{\partial x^{i}}+\frac{\partial\mathbf{u}}{\partial x^{i}}\bullet\mathbf{v}\right)\mathbf{g}^{\mathbf{i}}=\mathbf{g}^{\mathbf{i}}\frac{\partial}{\partial x^{i}}(\mathbf{u}\bullet\mathbf{v})=\nabla(\mathbf{u}\bullet\mathbf{v})$$

4.9 已知：$\boldsymbol{u}$，$\boldsymbol{v}$是矢量场函数。

求证：$\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{v})=\boldsymbol{v}\cdot(\boldsymbol{\nabla}\boldsymbol{v})-\boldsymbol{v}\bullet(\boldsymbol{\nabla}\cdot\boldsymbol{u})+\boldsymbol{u}\bullet(\boldsymbol{\nabla}\cdot\boldsymbol{v})-\boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{v})$

证明 $\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{v})=\mathbf{g}^{i}\times\frac{\partial}{\partial x^{i}}(\boldsymbol{u}\times\boldsymbol{v})$

$$=\mathbf{g}^{i}\times\left(\frac{\partial\boldsymbol{u}}{\partial x^{i}}\times\boldsymbol{v}\right)+\mathbf{g}^{i}\times\left(\boldsymbol{u}\times\frac{\partial\boldsymbol{v}}{\partial x^{i}}\right)$$

$$=\left(\mathbf{g}^{i}\cdot\boldsymbol{v}\right)\frac{\partial\boldsymbol{u}}{\partial x^{i}}-\left(\mathbf{g}^{i}\cdot\frac{\partial\boldsymbol{u}}{\partial x^{i}}\right)\boldsymbol{v}+\left(\mathbf{g}^{i}\cdot\frac{\partial\boldsymbol{v}}{\partial x^{i}}\right)\boldsymbol{u}-\left(\mathbf{g}^{i}\cdot\boldsymbol{u}\right)\frac{\partial\boldsymbol{v}}{\partial x^{i}}$$

$$=\left(\boldsymbol{v}\cdot\mathbf{g}^{i}\right)\frac{\partial\boldsymbol{u}}{\partial x^{i}}-\boldsymbol{v}\left(\mathbf{g}^{i}\cdot\frac{\partial\boldsymbol{u}}{\partial x^{i}}\right)+\boldsymbol{u}\left(\mathbf{g}^{i}\cdot\frac{\partial\boldsymbol{v}}{\partial x^{i}}\right)-\left(\boldsymbol{u}\cdot\mathbf{g}^{i}\right)\frac{\partial\boldsymbol{v}}{\partial x^{i}}$$

$$=\boldsymbol{v}\cdot(\boldsymbol{\nabla}\boldsymbol{v})-\boldsymbol{v}\bullet(\boldsymbol{\nabla}\cdot\boldsymbol{u})+\boldsymbol{u}\bullet(\boldsymbol{\nabla}\cdot\boldsymbol{v})-\boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{v})$$

得证。

4.11 已知：某矢量场函数$\boldsymbol{u}$，$\mathrm{curl}\boldsymbol{u}=0,\mathrm{div}\boldsymbol{u}=0$

求证：$\boldsymbol{u}$是调和函数，即$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{u}=0$。

（提示：可先证 $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})-\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\boldsymbol{u})$）

本题中不对指标i求和

$$\Delta\times(\boldsymbol{\nabla}\times\boldsymbol{u})=\mathbf{g}^{i}\times\frac{\partial}{\partial x^{i}}\left(\mathbf{g}^{j}\times\frac{\partial\boldsymbol{u}}{\partial x^{j}}\right)$$

$$=\mathbf{g}^i\times\left(\frac{\partial\mathbf{g}^j}{\partial x^i}\times\frac{\partial\boldsymbol{u}}{\partial x^j}\right)+\mathbf{g}^i\times\left[\mathbf{g}^j\times\frac{\partial}{\partial x^i}\left(\frac{\partial\boldsymbol{u}}{\partial x^j}\right)\right]$$

$$=\frac{\partial\mathbf{g}^j}{\partial x^i}\left(\mathbf{g}^i\cdot\frac{\partial\boldsymbol{u}}{\partial x^j}\right)-\frac{\partial\boldsymbol{u}}{\partial x^j}\left(\mathbf{g}^i\cdot\frac{\partial\mathbf{g}^j}{\partial x^i}\right)+\mathbf{g}^j\left[\mathbf{g}^i\cdot\frac{\partial}{\partial x^i}\left(\frac{\partial\boldsymbol{u}}{\partial x^j}\right)\right]$$

$$-\frac{\partial}{\partial x^i}\left(\frac{\partial\boldsymbol{u}}{\partial x^j}\right)\left(\mathbf{g}^i\cdot\mathbf{g}^j\right)\nabla(\nabla\cdot\boldsymbol{u})-\nabla\cdot(\nabla\boldsymbol{u})$$

$$=\mathbf{g}^i\frac{\partial}{\partial x^i}\left(\mathbf{g}^j\cdot\frac{\partial\boldsymbol{u}}{\partial x^j}\right)-\mathbf{g}^i\cdot\frac{\partial}{\partial x^i}\left(\mathbf{g}^j\frac{\partial\boldsymbol{u}}{\partial x^j}\right)$$

$$=\mathbf{g}^i\left(\frac{\partial\mathbf{g}^j}{\partial x^i}\cdot\frac{\partial\boldsymbol{u}}{\partial x^j}\right)+\mathbf{g}^i\left(\mathbf{g}^j\cdot\frac{\partial^2\boldsymbol{u}}{\partial x^i\partial x^j}\right)-\left(\mathbf{g}^i\cdot\frac{\partial\mathbf{g}^j}{\partial x^i}\right)\frac{\partial\boldsymbol{u}}{\partial x^j}$$

$$-\left(\mathbf{g}^i\cdot\mathbf{g}^j\frac{\partial^2\boldsymbol{u}}{\partial x^i\partial x^j}\right)\nabla\times(\nabla\times\boldsymbol{u})-\left[\nabla(\nabla\cdot\boldsymbol{u})-\nabla\cdot(\nabla\boldsymbol{u})\right]$$

$$=\frac{\partial\mathbf{g}^j}{\partial x^i}\left(\mathbf{g}^i\cdot\frac{\partial\boldsymbol{u}}{\partial x^j}\right)-\mathbf{g}^i\left(\frac{\partial\mathbf{g}^j}{\partial x^i}\cdot\frac{\partial\boldsymbol{u}}{\partial x^j}\right)=\frac{\partial\boldsymbol{u}}{\partial x^j}\left(\frac{\partial\mathbf{g}^j}{\partial x^i}\times\mathbf{g}^i\right)$$

$$=-\frac{\partial\boldsymbol{u}}{\partial x^j}\times\left(\boldsymbol{\Gamma}^j_{im}\mathbf{g}^m\times\mathbf{g}^i\right)=-\frac{\partial\boldsymbol{u}}{\partial x^j}\times\left(\boldsymbol{\varepsilon}^{mil}\boldsymbol{\Gamma}^j_{im}\times\mathbf{g}_l\right)$$

因为$\boldsymbol{\varepsilon}^{mil}$关于指标$i$，$m$为对称，$\boldsymbol{\Gamma}^j_{im}$关于指标$i$，$m$为反对称。故

$\boldsymbol{\varepsilon}^{mil}\boldsymbol{\Gamma}^j_{im}\times\mathbf{g}_l=0$

则$\nabla\times(\nabla\times\boldsymbol{u})=\nabla(\nabla\cdot\boldsymbol{u})-\nabla\cdot(\nabla\boldsymbol{u})$

根据此式，当$\nabla\times\boldsymbol{u}=0,\nabla\cdot\boldsymbol{u}=0,$时，则

$\nabla\cdot(\nabla\boldsymbol{u})=0$

$\boldsymbol{u}$为调和函数。

4.12 已知：标量场函数ϕ，矢量场函数$\boldsymbol{F}=F^{(k)}\boldsymbol{e}_k$，

其中$\boldsymbol{e}_k=\dfrac{\boldsymbol{g}_k}{\sqrt{g_{kk}}}$（$k$不求和）

求：正交曲线坐标戏中 grandϕ，div$\boldsymbol{F}$，curl$\boldsymbol{F}$及$\nabla^2\phi=\nabla\cdot\nabla\phi$ =div grandϕ（要求按$\boldsymbol{e}_k$展开的表达式）。

解：$\mathrm{grand}\,\phi=\sum_{i=1}^{3}\frac{1}{\sqrt{g_{ii}}}\frac{\partial\phi}{\partial x^i}\boldsymbol{e}_i$

$$\mathrm{div}\boldsymbol{F}=\frac{1}{\sqrt{g}}\sum_{k=1}^{3}\frac{\partial}{x^k}\left(\sqrt{g}F^{(k)}\Big/\sqrt{g_{kk}}\right)$$

$$\mathrm{curl}\boldsymbol{F}=\left(g_{22}g_{33}\right)^{\frac{-1}{2}}\left[\frac{\partial}{\partial x^2}\left(\sqrt{g_{33}}F^{(3)}\right)-\frac{\partial}{\partial x^3}\left(\sqrt{g_{22}}F^{(2)}\right)\right]\boldsymbol{e}_1$$

$$+\left(g_{33}g_{11}\right)^{\frac{-1}{2}}\left[\frac{\partial}{\partial x^3}\left(\sqrt{g_{11}}F^{(1)}\right)-\frac{\partial}{\partial x^1}\left(\sqrt{g_{33}}F^{(3)}\right)\right]\boldsymbol{e}_2$$

$$+\left(g_{11}g_{22}\right)^{\frac{-1}{2}}\left[\frac{\partial}{\partial x^1}\left(\sqrt{g_{22}}F^{(2)}\right)-\frac{\partial}{\partial x^2}\left(\sqrt{g_{11}}F^{(1)}\right)\right]\boldsymbol{e}_3$$

$$\nabla^2\phi=\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\sqrt{\frac{g_{22}g_{33}}{g_{11}}}\frac{\partial\phi}{\partial x^1}\right)+\frac{\partial}{\partial x^2}\left(\sqrt{\frac{g_{33}g_{11}}{g_{22}}}\frac{\partial\phi}{\partial x^2}\right)+\frac{\partial}{\partial x^3}\left(\sqrt{\frac{g_{11}g_{22}}{g_{33}}}\frac{\partial\phi}{\partial x^3}\right)\right]$$

4.13 已知：圆柱坐标中矢量场函数 $\boldsymbol{F}$ 可表达为 $\boldsymbol{F}=F_r\boldsymbol{e}_r+F_\theta\boldsymbol{e}_\theta+F_z\boldsymbol{e}_z$（$\boldsymbol{e}_r$，$\boldsymbol{e}_\theta$，$\boldsymbol{e}_z$ 是方向的单位矢量）；标量场函数 ϕ。

求：Christoffel 符号与 $\boldsymbol{e}_r$，$\boldsymbol{e}_\theta$，$\boldsymbol{e}_z$ 对坐标的导数；求 $\mathrm{grand}\,\phi$，$\mathrm{div}\boldsymbol{F}$，$\mathrm{curl}\boldsymbol{F}$ 及 $\nabla^2\phi$。

解：$\Gamma^{\theta}_{r\theta}=\Gamma^{\theta}_{\theta r}=\frac{1}{r}$　　$\Gamma^{r}_{\theta\theta}=-r$　　其余为零

$$\mathrm{grand}\,\phi=\frac{\partial\phi}{\partial r}\boldsymbol{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta+\frac{\partial\phi}{\partial z}\boldsymbol{e}_z$$

$$\mathrm{div}\boldsymbol{F}=\frac{1}{r}\frac{\partial}{\partial r}\left(rF_r\right)+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial z}$$

$$\mathrm{curl}\boldsymbol{F}=\left(\frac{1}{r}\frac{\partial F_z}{\partial\theta}-\frac{\partial F_\theta}{\partial z}\right)\boldsymbol{e}_r+\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)\boldsymbol{e}_\theta+\frac{1}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)\boldsymbol{e}_z$$

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}$$

4.14 已知：圆柱坐标中矢量场函数 $\boldsymbol{F}$ 可表达为 $\boldsymbol{F}=F_r\boldsymbol{e}_r+F_\theta\boldsymbol{e}_\theta+F_\varphi\boldsymbol{e}_\varphi$（$\boldsymbol{e}_r$，$\boldsymbol{e}_\theta$，$\boldsymbol{e}_\varphi$ 是方向的单位矢量）；标量场函数 ϕ。

求：Christoffel 符号与 $\boldsymbol{e}_r$，$\boldsymbol{e}_\theta$，$\boldsymbol{e}_\varphi$ 对坐标的导数；求 $\mathrm{grand}\,\phi$，$\mathrm{div}\boldsymbol{F}$，$\mathrm{curl}\boldsymbol{F}$ 及 $\nabla^2\phi$。

解：$\Gamma^{r}_{\theta\theta}=-r$，　$\Gamma^{r}_{\varphi\varphi}=-r\sin^2\theta$，　$\Gamma^{\theta}_{r\theta}=\Gamma^{\theta}_{\theta r}=\dfrac{1}{r}$，　$\Gamma^{\theta}_{\varphi\varphi}=-\sin\theta\cos\theta$，

$\Gamma^{\varphi}_{\varphi r}=\Gamma^{\varphi}_{r\varphi}=\dfrac{1}{r}$，　$\Gamma^{\varphi}_{\varphi\theta}=\Gamma^{\varphi}_{\theta\varphi}=\mathrm{ctg}\,\theta$，　其余为零

$$\mathrm{grand}\phi=\frac{\partial\phi}{\partial r}\boldsymbol{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{e}_\varphi$$

$$\mathrm{div}\boldsymbol{F}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(F_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

$$\mathrm{curl}\boldsymbol{F}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\varphi\sin\theta\right)-\frac{\partial F_\theta}{\partial\varphi}\right)\boldsymbol{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{1}{r}\frac{\partial}{\partial r}\left(rF_\varphi\right)\right)\boldsymbol{e}_\theta+\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rF_\theta\right)-\frac{\partial F_r}{\partial\theta}\right)\boldsymbol{e}_\varphi$$

$$\nabla^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial\theta}\sin\theta\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

4.15　已知：二维空间中（n,s)坐标系如图 4.19.其中 s 是沿某一物体表面的曲线边界弧长（选择物体表面某一确定点为起始点），n 为沿物体表面外法线的长度（从物体表面起算),则物体外部域内每一点的坐标均可用 n，s 描述（n>>0）。物体表面每点处的曲率半径及其对 s 的各阶导数均为已知。求：用 R（s)及其导数，坐标 n,s 表示下列各项（$x^1=n,x^2=s$）：

（1）Lame 参数 A，B。

（2）用（n,s）坐标单位切向矢量 e_n,e_s 表示基矢量 g_α,g^β 。

（3）用矢量的物理量分量 $u<a>=u_n,u_s$ 表示其张量分量 u^a,u_a。

（4）Christoffel 符号 $\Gamma_{\alpha}{}^{\upsilon}{}_{\beta}$。

（5）若 f 为标量场，$u=u_ne_n+u_se_s$ 为矢量场，求 $\nabla f,\nabla\cdot u,\nabla\times u,\nabla^2 f$ 的表达式。（$\nabla^2 f=\nabla\cdot\nabla f$）。

解　（1）$\mathrm{A}=1$，　　$\mathrm{B}=1+\dfrac{n}{R(s)}$

（2）　$g_1=e_n$，　　$g_2=\left(1+\dfrac{n}{R(s)}\right)e$，

$$g^1 = e_n, \qquad g^2 = \left(\frac{R(s)}{R(s)+n}\right)e,$$

（3） $u_1 = u_n$, $\qquad u_2 = \left(1+\frac{n}{R(s)}\right)u$,

$$u^1 = u_n \ , \qquad u^2 = \left(\frac{R(s)}{R(s)+n}\right)u\ ;$$

（4） $\Gamma_{11}^1 = 0$, $\qquad \Gamma_{12}^1 = \Gamma_{21}^1 = 0$, $\qquad \Gamma_{11}^2 = 0$

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{\mathrm{R(s)}+n}, \qquad \Gamma_{22}^1 = -\frac{\mathrm{R(s)}+n}{\mathrm{R}^2(\mathrm{s})}$$

$$\Gamma_{22}^2 = -\frac{n}{\mathrm{R\ (R+n)}}R'(s)$$

(5) $\nabla f = \frac{\partial f}{\partial n}e_n + \frac{R}{R+n}.\frac{\partial f}{\partial s}e;$

$$\nabla\cdot u = \frac{\partial u_n}{\partial n} + \frac{u_n}{R+n} + \frac{R}{R+n}.\frac{\partial u_s}{\partial s}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial n^2} + \frac{R^2}{(R+n)^2}.\frac{\partial^2 f}{\partial s^2} + \frac{nRR'}{(R+n)^8}.\frac{\partial\ f}{\partial s} + \frac{1}{R+n}.\frac{\partial\ f}{\partial n}$$

$$\nabla\times u = [\frac{\partial u_n}{\partial n} - \frac{R}{R+n}\frac{\partial u_s}{\partial s} + \frac{u_s}{R+n}]g_3$$

(g_3为垂直于平面的单位矢量)

4.16 已知：z^k为直角坐标，x^k为抛物柱坐标，他们之间满足关系;

$$z^1 = a(x^1 - x^2)$$
$$z^2 = 2a\sqrt{x^1x^2}$$
$$z^3 = x^3$$

其中 a=常数>0

求：对于x^k坐标系（只研究上半平面）

（1）求基矢量，度量张量。

（2）用矢量的物理分量来表示矢量的张量分量。

（3）求 Christoffel 符号 Γ_{ij}^k。

（4）f 为标量场，$u = u^i g_i$为矢量场，求 $\nabla f, \nabla\cdot u, \nabla\times u, \nabla^2 f$ 的表达式

解

（1） $g_1 = ai + a\sqrt{\frac{x^2}{x^1}}j \qquad g^1 = \frac{1}{a(x^1+x^2)}(x^1 i + \sqrt{x^1 x^2}$

$g_2 = \text{-}ai + a\sqrt{\frac{x^1}{x^2}}j \qquad g^2 = \frac{1}{a(x^1+x^2)}(\text{-}x^2 i + \sqrt{x^1 x^2}j)$

$g_3 = k \qquad g^3 = k$

$g_{11} = \frac{1}{g_{11}} = \frac{(a)^2}{x^1}(x^1+x^2)$， $\qquad g_{22} = \frac{1}{g_{22}} = \frac{(a)^2}{x^2}(x^1+x^2)$

$g_{33} = 1$， 其余为零。

$$\sqrt{g} = \frac{(a)^2(x^1+x^2)}{\sqrt{x^1 x^2}}$$

（2） $u_1 = a\sqrt{1+\frac{x^2}{x^1}}u<1>$， $\qquad u_2 = a\sqrt{1+\frac{x^1}{x^2}}u<2>$，

$u_3 = u<3>$

$u^1 = \frac{u<1>}{a\sqrt{1+\frac{x^2}{x^1}}}$， $\quad u^2 = \frac{u<2>}{a\sqrt{1+\frac{x^1}{x^2}}}$， $\quad u^3 = u<3>$

（3） $\Gamma^1_{11} = \text{-}\frac{x^2}{2x^1(x^1+x^2)} \qquad \Gamma^1_{12} = \Gamma^1_{21} = \text{-}\frac{1}{2(x^1+x^2)}$

$\Gamma^2_{22} = \text{-}\frac{x^1}{2x^2(x^1+x^2)} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \text{-}\frac{1}{2(x^1+x^2)}$

$\Gamma^1_{22} = \text{-}\frac{x^1}{2x^2(x^1+x^2)} \qquad \Gamma^2_{11} = \text{-}\frac{x^2}{2x^1(x^1+x^2)}$

其余为零

（4）
$$\nabla f = \frac{1}{a(x^1+x^2)}[(x^1\frac{\partial f}{\partial x^1} - x^2\frac{\partial f}{\partial x^2})i + \sqrt{x^1x^2}(\frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^2})j] + \frac{\partial f}{\partial x^8}k$$

$$\nabla\cdot u=\frac{\partial u^1}{\partial x^1}+\frac{\partial u^2}{\partial x^2}+\frac{\partial u^3}{\partial x^3}+u^1\frac{(x^1-x^2)}{2x^1(x^1+x^2)}$$
$$+u^2\frac{(x^2-x^1)}{2x^2(x^1+x^2)}$$

$$\nabla\times u=\frac{\sqrt{x^1x^2}}{2(x^1+x^2)}\{[\frac{\partial u_3}{\partial x^2}-\frac{\partial u_2}{\partial x^3}-\frac{\partial u^1}{\partial x^3}+\frac{\partial u_3}{\partial x^1}]i$$
$$+[\sqrt{\frac{x^2}{x^1}}(\frac{\partial u_3}{\partial x^2}-\frac{\partial u_2}{\partial x^3})+\sqrt{\frac{x^1}{x^2}}(\frac{\partial u_1}{\partial x^3}-\frac{\partial u_3}{\partial x^1})j$$
$$+a[\frac{\partial u_2}{\partial x^1}-\frac{\partial u_1}{\partial x^2}]k$$

$$\nabla^2 f=\frac{x^1}{(a)^2(x^1+x^2)}\frac{\partial^2 f}{(\partial x^1)^2}+\frac{x^2}{(a)^2(x^1+x^2)}\frac{\partial^2 f}{(\partial x^2)^2}$$
$$+\frac{\partial^2 f}{(\partial x^3)^2}+\frac{1}{2(a)^2(x^1+x^2)}(\frac{\partial\ f}{\partial x^1}+\frac{\partial\ f}{\partial x^2})$$

4.20 求证 $\boldsymbol{g}_{(i)}$ 为完整系的必要条件（对于单连通域也是充分条件）为 $\beta^{(i)}_{j,k}=\beta^{(i)}_{k,j}$

证：（1）必要性：即已知 $\boldsymbol{g}_{(i)}$ 为完整系，求证 $\beta^{(i)}_{j,k}=\beta^{(i)}_{k,j}$。

$\boldsymbol{g}_{(i)}$ 为完整系则存在坐标 $x^{(i)}$，使 $\boldsymbol{g}_{(i)}=\dfrac{\partial \boldsymbol{r}}{\partial x^{(i)}}$

$\boldsymbol{g}_{(i)}$ 与 $\boldsymbol{g}_j$ 有转换关系（$\boldsymbol{g}_j$ 是另一完整系中基矢量）：$\boldsymbol{g}_j=\beta^{(i)}_j\boldsymbol{g}_{(i)}$

其中 $\beta^{(i)}_j=\dfrac{\partial x^{(i)}}{\partial x^j}$

则 $\dfrac{\partial\beta^{(i)}_j}{\partial x^k}=\dfrac{\partial^2 x^{(i)}}{\partial x^k\partial x^j}=\dfrac{\partial^2 x^{(i)}}{\partial x^j\partial x^k}=\dfrac{\partial\beta^{(i)}_k}{\partial x^k}$

（2）充分性：即已知 $\dfrac{\partial\beta^{(i)}_j}{\partial x^k}=\dfrac{\partial\beta^{(i)}_k}{\partial x^j}$，求证存在着曲线坐标系 $x^{(i)}$，使 $\beta^{(i)}_j=\dfrac{\partial x^{(i)}}{\partial x^j}$。

设 $a^{(i)}=\beta^{(i)}_j dx^j$，在单连通域，$a^{(i)}$ 为全微分，换言之，存在着 $x^{(i)}$，使 $a^{(i)}=dx^{(i)}$。

故：$dx^{(i)}=\beta^{(i)}_j dx^j=\dfrac{\partial x^{(i)}}{\partial x^j}dx^j$

从而 $\beta_j^{(i)}=\dfrac{\partial x^{(i)}}{\partial x^j}$。

4.21 试利用完整系与非完整系的转换关系，由完整系中任意正交曲线坐标的平衡方程导出圆柱坐标系$\left(r,\theta,z\right)$中用物理分量表示的平衡方程（应力的物理分量记为p_{rr}，$p_{\theta\theta}$ ，…）。

解：圆柱坐标系$\left(r,\theta,z\right)$中以物理分量表示的平衡方程：

$$
\begin{aligned}
&\frac{\partial \boldsymbol{p}_{\boldsymbol{rr}}}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{p}_{\boldsymbol{r}\theta}}{\boldsymbol{r}\partial\theta}+\frac{\partial \boldsymbol{p}_{rz}}{\partial z}+\frac{1}{r}\left(\boldsymbol{p}_{rr}-\boldsymbol{p}_{\theta\theta}\right)+\rho f_r=0\\
&\frac{\partial \boldsymbol{p}_{\theta\boldsymbol{r}}}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{p}_{\theta\theta}}{\boldsymbol{r}\partial\theta}+\frac{\partial \boldsymbol{p}_{\theta z}}{\partial z}+2\frac{\boldsymbol{p}_{\theta r}}{r}+\rho f_\theta=0\\
&\frac{\partial \boldsymbol{p}_{z\boldsymbol{r}}}{\partial \boldsymbol{r}}+\frac{\partial \boldsymbol{p}_{z\theta}}{\boldsymbol{r}\partial\theta}+\frac{\partial \boldsymbol{p}_{zz}}{\partial z}+\frac{\boldsymbol{p}_{zr}}{r}+\rho f_z=0
\end{aligned}
$$

4.22 同上题，试导出球坐标系$\left(r,\theta,\varphi\right)$中用物理分量表示的平衡方程（应力的物理分量记为$p_{rr}$，$p_{r\varphi}$ ，…）。

解：球坐标系$\left(r,\theta,\varphi\right)$中以物理分量表示的平衡方程：

$$
\frac{\partial \boldsymbol{p}_{\boldsymbol{rr}}}{\partial \boldsymbol{r}}+\frac{1}{\boldsymbol{r}}\frac{\partial \boldsymbol{p}_{\boldsymbol{r}\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial \boldsymbol{p}_{r\varphi}}{\partial\varphi}+\frac{1}{r}\left(2\boldsymbol{p}_{rr}-\boldsymbol{p}_{\theta\theta}-\boldsymbol{p}_{\theta z}+\boldsymbol{p}_{r\theta}\cot\theta\right)+\rho f_r=0
$$

$$
\frac{\partial \boldsymbol{p}_{\theta\boldsymbol{r}}}{\partial \boldsymbol{r}}+\frac{1}{\boldsymbol{r}}\frac{\partial \boldsymbol{p}_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial \boldsymbol{p}_{\varphi\theta}}{\partial\varphi}+\frac{1}{r}\left[3\boldsymbol{p}_{\theta r}+\left(\boldsymbol{p}_{\theta\theta}-\boldsymbol{p}_{\varphi\varphi}\right)\cot\theta\right]+\rho f_\theta=0
$$

$$
\frac{\partial \boldsymbol{p}_{\varphi\boldsymbol{r}}}{\partial \boldsymbol{r}}+\frac{1}{\boldsymbol{r}}\frac{\partial \boldsymbol{p}_{\varphi\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial \boldsymbol{p}_{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}\left[3\boldsymbol{p}_{\varphi r}+2\boldsymbol{p}_{\varphi\theta}\cot\theta\right]+\rho f_\varphi=0
$$

4.23 试导出任意正交曲线坐标系中用物理分量表示的平衡方程。设

$$A_1=\sqrt{g_{11}}\qquad A_2=\sqrt{g_{22}}\qquad A_3=\sqrt{g_{33}}$$

任意正交曲线坐标系中以物理分量表示的平衡方程：

$$\frac{1}{A_1A_2A_3}\left[\begin{array}{l}\frac{\partial}{\partial x^1}\left(A_2A_3p\langle 11\rangle\right)+\frac{\partial}{\partial x^2}\left(A_3A_1p\langle 12\rangle\right)+\frac{\partial}{\partial x^3}\left(A_1A_2p\langle 13\rangle\right)+\\ A_2\frac{\partial A_1}{\partial x^3}p\langle 31\rangle+A_3\frac{\partial A_1}{\partial x^2}p\langle 21\rangle-A_3\frac{\partial A_2}{\partial x^1}p\langle 22\rangle-A_2\frac{\partial A_3}{\partial x^1}p\langle 33\rangle\end{array}\right]+\rho f\langle 1\rangle=0$$

$$\frac{1}{A_1A_2A_3}\left[\begin{array}{l}\frac{\partial}{\partial x^1}\left(A_2A_3p\langle 21\rangle\right)+\frac{\partial}{\partial x^2}\left(A_3A_1p\langle 22\rangle\right)+\frac{\partial}{\partial x^3}\left(A_1A_2p\langle 23\rangle\right)+\\ A_3\frac{\partial A_2}{\partial x^1}p\langle 12\rangle+A_1\frac{\partial A_2}{\partial x^3}p\langle 32\rangle-A_3\frac{\partial A_3}{\partial x^2}p\langle 33\rangle-A_3\frac{\partial A_1}{\partial x^2}p\langle 11\rangle\end{array}\right]+\rho f\langle 2\rangle=0$$

$$\frac{1}{A_1A_2A_3}\left[\begin{array}{l}\frac{\partial}{\partial x^1}\left(A_2A_3p\langle 31\rangle\right)+\frac{\partial}{\partial x^2}\left(A_3A_1p\langle 32\rangle\right)+\frac{\partial}{\partial x^3}\left(A_1A_2p\langle 33\rangle\right)+\\ A_1\frac{\partial A_3}{\partial x^2}p\langle 23\rangle+A_2\frac{\partial A_3}{\partial x^1}p\langle 13\rangle-A_2\frac{\partial A_1}{\partial x^3}p\langle 11\rangle-A_1\frac{\partial A_2}{\partial x^3}p\langle 22\rangle\end{array}\right]+\rho f\langle 3\rangle=0$$

4.24 试导出小位移情况下圆柱坐标系中用物理分量表示的应变与位移的几何关系。（以 u_r， u_θ， u_z表示位移的物理分量， $\varepsilon_{rr},\dots\varepsilon_{r\theta},\dots$表示应变的物理分量。）

解：

$$\varepsilon_{rr}=\frac{\partial \boldsymbol{u}_r}{\partial \boldsymbol{r}}$$

$$\varepsilon_{\theta\theta}=\frac{\partial \boldsymbol{u}_\theta}{\boldsymbol{r}\partial\theta}+\frac{\boldsymbol{u}_r}{\boldsymbol{r}}$$

$$\varepsilon_{zz}=\frac{\partial \boldsymbol{u}_z}{\partial z}$$

$$\varepsilon_{r\theta}=\varepsilon_{\theta r}=\frac{1}{2}\left(\frac{\partial \boldsymbol{u}_r}{\boldsymbol{r}\partial\theta}+\frac{\partial \boldsymbol{u}_\theta}{\partial \boldsymbol{r}}-\frac{\boldsymbol{u}_\theta}{\boldsymbol{r}}\right)$$

$$\varepsilon_{\theta z}=\varepsilon_{z\theta}=\frac{1}{2}\left(\frac{\partial \boldsymbol{u}_\theta}{\partial z}+\frac{\partial \boldsymbol{u}_z}{\boldsymbol{r}\partial\theta}\right)$$

$$\varepsilon_{zr}=\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial \boldsymbol{u}_r}{\partial z}+\frac{\partial \boldsymbol{u}_z}{\partial \boldsymbol{r}}\right)$$

4.25 是导出小位移情况下球坐标中用物理分量表示的应变与位移的几何关系。（以 u_r,u_θ,u_φ 表示位移的物理分量， $\varepsilon_{rr},\dots\varepsilon_{\theta\varphi},\dots$ 表示应变的物理分量）。

小位移情况下求坐标系中用物理分量表示的几何关系；

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta}=\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r}$$

$$\varepsilon_{\varphi\varphi}=\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r}{r}+\frac{u_\theta}{r}ctg\theta$$

$$\varepsilon_{r\theta}=\varepsilon_{\theta r}=\frac{1}{2}(\frac{\partial u_r}{r\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r})$$

$$\varepsilon_{\theta\varphi}=\varepsilon_{\varphi\theta}=\frac{1}{2}[\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{1}{r}\frac{\partial u_\varphi}{\partial\theta}-\frac{u_\varphi}{r}ctg\theta]$$

$$\varepsilon_{\theta\varphi}=\varepsilon_{\varphi r}=\varepsilon_{\varphi r}=\frac{1}{2}[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}]$$

第五章

5.1 取圆柱面上的 Gauss 坐标为（ξ，φ），见图 5.18.求：$a_{\alpha\beta}$，$b_{\alpha\beta}$，主曲率 $\frac{1}{\mathrm{R}_1},\frac{1}{\mathrm{R}_2}$，平均曲率，Gauss 曲率。

解

由图得

$$\rho=\xi i+R\sin\varphi j+R\cos\varphi k$$

所以

$$\rho_1=\frac{\partial\rho}{\partial\xi}=i=(1,0,0)$$

$$\rho_2=\frac{\partial\rho}{\partial\varphi}=(0,R\cos\varphi,R\sin\varphi)$$

又因为 $a_{\alpha\beta}=\rho_\alpha\rho_\beta$

所以

$a_{11}=\rho_1\rho_1=1$，$a_{12}=a_{21}=\rho_1\rho_2=0$，$a_{22}=R^2$

$$a=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=R^2$$

$$n=\frac{\rho_1\times\rho_2}{\left|\rho_1\times\rho_2\right|}=\left(0,\sin\varphi,\cos\varphi\right)$$

$$b_{\alpha\beta}=-\frac{\partial n}{\partial\xi^{\beta}}\rho_{\alpha}$$

$$b_{11}=0, b_{12}=b_{21}=0, b_{22}=-R$$

$$b=\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}=0$$

所以$\dfrac{1}{R_1^{'}}=-\dfrac{b_{11}}{a_{11}}=0,\dfrac{1}{R_2^{'}}=-\dfrac{b_{22}}{a_{22}}=-\dfrac{-R}{R^2}=\dfrac{1}{R}$

$$H=\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{R_1^{'}}+\frac{1}{R_2^{'}}=\frac{1}{R} \qquad (1)$$

$$K=\frac{b}{a}=\frac{1}{R_1}\cdot\frac{1}{R_2}=0 \qquad (2)$$

联立（1）（2）式得$\dfrac{1}{R_1}=0,\dfrac{1}{R_2}=\dfrac{1}{R}$或$\dfrac{1}{R_1}=\dfrac{1}{R},\dfrac{1}{R_2}=0$

5.2 已知：旋转曲面上的 Gauss 坐标为（θ，z），见图 5.19，曲面上点的矢径 $\rho=f(z)\cos\theta i+f(z)\sin\theta j+zk$ 求：$a_{\alpha\beta}$，$b_{\alpha\beta}$，主曲率$\dfrac{1}{\mathrm{R}_1},\dfrac{1}{\mathrm{R}_2}$，平均曲率，Gauss 曲率。

解：

$$a_{11}=\left[f(z)\right]^2, a_{12}=a_{21}=0, a_{22}=1;$$

$$b_{11}=-\frac{\left[f(z)\right]^2}{\left|f(z)\right|}, b_{12}=b_{21}=b_{22}=0;$$

$$b_1^1=-\frac{1}{\left|f(z)\right|}, b_2^1=b_1^2=b_2^2=0;$$

$$\frac{1}{R_1}=b_1^1=-\frac{1}{\left|f(z)\right|},\frac{1}{R_2}=0, b_{\alpha}^{\alpha}=-\frac{1}{\left|f(z)\right|}, b=0$$

5.3 1.14 题中的斜圆锥面上，已求得（θ，z）坐标系中

$$[a_{\alpha\beta}]=\begin{bmatrix} \frac{R^2z^2}{H^2} & -\frac{RCz}{H^2}\sin\theta \\ -\frac{RCz}{H^2}\sin\theta & \frac{1}{H^2}(H^2+R^2+C^2+2RC\cos\theta) \end{bmatrix}$$

求：$a_{\alpha\beta}$，$b_{\alpha\beta}$，主曲率 $\frac{1}{\mathrm{R}_1},\frac{1}{\mathrm{R}_2}$，平均曲率，Gauss 曲率。

解

由题意得

$$a_{11}=\rho_1\cdot\rho_1=\frac{R^2z^2}{H^2}$$

$$a_{12}=a_{21}=\rho_1\cdot\rho_2=-\frac{RCz}{H^2}\sin\theta$$

$$a_{22}=\rho_2\cdot\rho_2=-\frac{1}{H^2}(H^2+R^2+C^2+2RC\cos\theta)$$

所以

$$\rho_1=\left(\frac{RZ}{H}\sin\theta,\frac{RZ}{H}\cos\theta,0\right)$$

$$\rho_2=\left(-\frac{C}{H},0,\frac{1}{H}\sqrt{H^2+R^2+2RC\cos\theta}\right)$$

$$n=\frac{\rho_1\times\rho_2}{|\rho_1\times\rho_2|}=(1,0,0)$$

$$b_{\alpha\beta}=-\frac{\partial n}{\partial\theta^\beta}\rho_\alpha$$

所以

$$b_{11}=b_{12}=b_{21}=b_{22}=0$$

$$\frac{1}{\mathrm{R}_1}=\frac{1}{\mathrm{R}_2}=0$$

$$\mathrm{H}=\frac{1}{\mathrm{R}_1}+\frac{1}{\mathrm{R}_2}=0$$

$$\mathrm{K}=\frac{1}{\mathrm{R}_1}\cdot\frac{1}{\mathrm{R}_2}=0$$

5.4 求证：$\dfrac{\partial a^{\alpha\beta}}{\partial \xi^{\lambda}}=-a^{\alpha\omega}\overset{\circ}{\Gamma}{}^{\beta}_{\lambda\omega}-a^{\omega\beta}\overset{\circ}{\Gamma}{}^{\alpha}_{\omega\lambda}$

证：因为 $a_{\omega\alpha}a^{\alpha\beta}=\delta^{\beta}_{\omega}$

所以 $\dfrac{\partial a_{\omega\alpha}a^{\alpha\beta}}{\partial \xi^{\lambda}}=0$

$$\frac{\partial a_{\omega\alpha}}{\partial \xi^{\lambda}}\cdot a^{\alpha\beta}+\frac{\partial a^{\alpha\beta}}{\partial \xi^{\lambda}}\cdot a_{\omega\alpha}=0$$

$$\frac{\partial a^{\alpha\beta}}{\partial \xi^{\lambda}}=-\frac{\partial a_{\omega\alpha}}{\partial \xi^{\lambda}}\cdot\frac{a^{\alpha\beta}}{a_{\omega\alpha}}$$

因为 $\dfrac{\partial a_{\omega\alpha}}{\partial \xi^{\lambda}}=\overset{\circ}{\Gamma}_{\lambda\omega\cdot\alpha}+\overset{\circ}{\Gamma}_{\lambda\alpha\cdot\omega}$

所以 $\dfrac{\partial a^{\alpha\beta}}{\partial \xi^{\lambda}}=-(\overset{\circ}{\Gamma}_{\lambda\omega\cdot\alpha}+\overset{\circ}{\Gamma}_{\lambda\alpha\cdot\omega})\cdot\dfrac{a^{\alpha\beta}}{a_{\omega\alpha}}$

$\dfrac{\partial a^{\alpha\beta}}{\partial \xi^{\lambda}}=-a^{\alpha\omega}\overset{\circ}{\Gamma}{}^{\beta}_{\lambda\omega}-a^{\omega\beta}\overset{\circ}{\Gamma}{}^{\alpha}_{\omega\lambda}$ 得证。

5.5 求证：$\overset{\circ}{\Gamma}{}^{\beta}_{\alpha\beta}=\dfrac{1}{\sqrt{a}}\dfrac{\partial\sqrt{a}}{\partial \xi^{\alpha}}$

证：因为

$$\frac{\partial\sqrt{a}}{\partial\xi^{\alpha}}=\frac{\partial(\rho_1\times\rho_2)\cdot n}{\partial\xi^{\alpha}}$$

$$=(\frac{\partial\rho_1}{\partial\xi^{\alpha}}\times\rho_2)\cdot n+(\frac{\partial\rho_2}{\partial\xi^{\alpha}}\times\rho_1)\cdot n+(\rho_2\times\rho_1)\cdot\frac{\partial n}{\partial\xi^{\alpha}}$$

$$=\left[\left(\overset{\circ}{\Gamma}{}^{1}_{1\alpha}\ \rho_1+b_{1\alpha}\right)\times\rho_2\right]\cdot n+\left[\left(\overset{\circ}{\Gamma}{}^{2}_{2\alpha}\ \rho_2+b_{2\alpha}\right)\times\rho_1\right]\cdot n$$

$$+\left(\rho_1\times\rho_2\right)\cdot\left(-\mathrm{b}^{\beta}_{\cdot\alpha}\rho_{\beta}\right)$$

$$=\overset{\circ}{\Gamma}{}^{1}_{1\alpha}\cdot\left(\rho_1\times\rho_2\cdot n\right)+\overset{\circ}{\Gamma}{}^{2}_{2\alpha}\cdot\left(\rho_1\times\rho_2\cdot n\right)$$

$$=(\overset{\circ}{\Gamma}{}^{1}_{1\alpha}+\overset{\circ}{\Gamma}{}^{2}_{2\alpha})\cdot\left(\rho_1\times\rho_2\cdot n\right)$$

$$=\overset{\circ}{\Gamma}{}^{\beta}_{\alpha\beta}\sqrt{a}$$

所以

$$\overset{\circ}{\Gamma}{}^{\beta}_{\alpha\beta}=\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}}{\partial\xi^{\alpha}}$$ 得证

5.6 求证：单位矢量的求导公式（5.2.15a,b）式，并进一步求证正交系中的单位矢量求导公式（5.2.17）式。

证：

$$\frac{\partial}{\partial\xi^{\alpha}}\left(\frac{\rho_1}{\sqrt{a_{11}}}\right)=\frac{\partial\rho_1}{\partial\xi^{\alpha}}\cdot\frac{1}{\sqrt{a_{11}}}+\frac{\partial}{\partial\xi^{\alpha}}\left(\frac{1}{\sqrt{a_{11}}}\right)\cdot\rho_1$$

$$=\frac{1}{\sqrt{a_{11}}}\left(\overset{\circ}{\Gamma}{}^{2}_{1\alpha}\ \rho_2+b_{1\alpha}n\right)-\rho_1\cdot\frac{\overset{\circ}{\Gamma}_{\alpha 1\cdot 1}}{a_{11}\sqrt{a_{11}}}$$

$$=\frac{1}{\sqrt{a_{11}}}\left(\overset{\circ}{\Gamma}{}^{2}_{1\alpha}\ \rho_2-\rho_1\cdot\frac{1}{a_{11}}\cdot\overset{\circ}{\Gamma}_{\alpha 1\cdot 1}+b_{1\alpha}n\right)$$

把

$$a^{22}=\frac{a_{11}}{a}$$ 代入得

$$\frac{\partial}{\partial \xi^{\alpha}}\left(\frac{\rho_1}{\sqrt{a_{11}}}\right)=\frac{1}{\sqrt{a_{11}}}\left(\frac{1}{a^{22}}\cdot \mathring{\Gamma}^{1}_{1\alpha}\,\rho^{2}+b_{1\alpha}n\right)$$

同理

$$\frac{\partial}{\partial \xi^{\alpha}}\left(\frac{\rho_2}{\sqrt{a_{22}}}\right)=\frac{1}{\sqrt{a_{22}}}\left(\frac{1}{a^{11}}\cdot \mathring{\Gamma}^{1}_{2\alpha}\,\rho^{1}+b_{2\alpha}n\right)$$

得证

进一步求证正交系中的单位矢量求导公式（5.2.17）式

证

$$\frac{\partial e_{\xi}}{\partial \xi}=\frac{\partial}{\partial \xi}\left(\frac{\rho_1}{A}\right)=\frac{\partial \rho_1}{\partial \xi}\cdot\frac{1}{A}+\frac{\partial}{\partial \xi}(\frac{1}{A})\cdot \rho_1$$

$$\frac{\partial e_{\xi}}{\partial \xi}=\frac{\partial}{\partial \xi}\left(\frac{\rho_1}{A}\right)=\frac{\mathrm{B}}{A}\cdot \mathring{\Gamma}^{1}_{11}-\frac{A}{R_1^{'}}n$$

$$=-\frac{1}{B}\frac{\partial A}{\partial \eta}e_{\eta}-\frac{A}{R_1^{'}}n$$

同理

$$\frac{\partial e_{\xi}}{\partial \eta}=\frac{1}{A}\frac{\partial B}{\partial \eta}e_{\eta}+\frac{\mathrm{B}}{R_{12}}n$$

$$\frac{\partial e_{\eta}}{\partial \xi}=\frac{1}{B}\frac{\partial A}{\partial \eta}e_{\xi}+\frac{\mathrm{A}}{R_{12}}n$$

$$\frac{\partial e_{\eta}}{\partial \eta}=-\frac{1}{A}\frac{\partial B}{\partial \xi}e_{\xi}-\frac{\mathrm{B}}{R_2^{'}}n$$

因为

$$\frac{\partial n}{\partial \xi}=\frac{\partial\left(e_{\xi}\times e_{\eta}\right)}{\partial \xi}=\frac{\partial e_{\xi}}{\partial \xi}e_{\eta}+\frac{\partial e_{\eta}}{\partial \xi}e_{\xi}$$

$$\frac{\partial e_{\xi}}{\partial \xi}=\frac{A}{R_{12}},\frac{\partial e_{\eta}}{\partial \xi}=-\frac{A}{R_1^{'}}$$

所以

$$\frac{\partial n}{\partial \xi}=-\frac{A}{R_{12}}e_{\eta}+\frac{A}{R_1^{'}}e_{\xi}$$

同理

$$\frac{\partial n}{\partial \eta}=-\frac{A}{R_{12}}e_{\xi}+\frac{A}{R_1^{'}}e_{\eta}$$

得证

5.7 求题 5.1 中圆柱面上（ξ，φ）坐标系中的 $\overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta}$。设 $e_1=\rho_1/A_1, e_2=\rho_2/A_2$，求：$\frac{\partial e_\alpha}{\partial \xi^\beta},\frac{\partial n}{\partial \xi^\beta}(\alpha,\beta,\gamma=1,2)$。

解

$$|\rho_1|=A_1=1$$
$$|\rho_2|=A_2=R$$

$$\overset{\circ}{\Gamma}{}^{1}_{11}=\frac{\partial A_1}{\partial \xi^1}=0 \quad \overset{\circ}{\Gamma}{}^{2}_{11}=-\frac{1}{R^2}\frac{\partial A_1}{\partial \xi^2}=0 \quad \overset{\circ}{\Gamma}{}^{1}_{12}=\frac{\partial A_1}{\partial \xi^2}=0$$

$$\overset{\circ}{\Gamma}{}^{2}_{12}=\frac{1}{R}\frac{\partial A_2}{\partial \xi^1}=\frac{1}{R} \quad \overset{\circ}{\Gamma}{}^{1}_{22}=-R\frac{\partial A_2}{\partial \xi^1}=-R \quad \overset{\circ}{\Gamma}{}^{2}_{22}=\frac{1}{R}\frac{\partial A_2}{\partial \xi^2}=\frac{1}{R}$$

$$\frac{\partial e_\alpha}{\partial \xi^\beta}=\frac{\partial\left(\frac{\rho_\alpha}{A_\alpha}\right)}{\partial \xi^\beta}=\frac{1}{A_\alpha}\frac{\partial \rho_\alpha}{\partial \xi^\beta}=\frac{1}{A_\alpha}\left(\overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta}\rho_\gamma+b_{\alpha\beta}n\right)$$

$$\frac{\partial e_1}{\partial \xi^1}=0 \quad \frac{\partial e_1}{\partial \xi^2}=\frac{\partial e_2}{\partial \xi^1}=0 \quad \frac{\partial e_2}{\partial \xi^2}=\frac{\cos\xi^2-R\sin\xi^2}{R}j-\frac{\sin\xi^2+R\cos\xi^2}{R}k$$

$$\frac{\partial n}{\partial \xi^\beta}=-b_{\alpha\beta}\rho^\alpha$$

$$b_{12}=b_{21}=0$$

$$\frac{\partial n}{\partial \xi^1}=\frac{\partial n}{\partial \xi^2}=0$$

5.8 求例 5.1 中圆环曲面上（θ，φ）坐标系中的$\overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta}$，$\overset{\circ}{R}_{1212}$。设 $e_1=\rho_1/A_1, e_2=\rho_2/A_2$，求：$\dfrac{\partial e_\alpha}{\partial\xi^\beta},\dfrac{\partial n}{\partial\xi^\beta}(\alpha,\beta,\gamma=1,2)$。

解

$$A_1=|\rho_1|=r_0$$
$$A_2=|\rho_2|=R+r_0\sin\theta$$

$$\overset{\circ}{\Gamma}{}^{1}_{11}=\frac{1}{A_1}\frac{\partial A_1}{\partial\theta}=0 \quad \overset{\circ}{\Gamma}{}^{2}_{11}=-\frac{1}{(R+r_0\sin\theta)^2}\frac{\partial A_1}{\partial\varphi}=0 \quad \overset{\circ}{\Gamma}{}^{1}_{12}=\frac{1}{A_1}\frac{\partial A_1}{\partial\varphi}=0$$

$$\overset{\circ}{\Gamma}{}^{2}_{12}=\frac{1}{R+r_0\sin\theta}\frac{\partial A_2}{\partial\theta}=0 \qquad \overset{\circ}{\Gamma}{}^{1}_{22}=-\frac{R+r_0\sin\theta}{r_0{}^2}\frac{\partial(R+r_0\sin\theta)}{\partial\theta}=\frac{(R+r_0\sin\theta)}{r_0}\cdot\cos\theta$$

$$\overset{\circ}{\Gamma}{}^{2}_{22}=\frac{1}{R+r_0\sin\theta}\frac{\partial(R+r_0\sin\theta)}{\partial\varphi}=0$$

$$\overset{\circ}{R}_{1212}=b_{11}b_{22}-b_{12}b_{21}=r_0(R+r_0\sin\theta)\sin\theta$$

$$\frac{\partial e_\alpha}{\partial\xi^\beta}=\frac{\partial\left(\dfrac{\rho_\alpha}{A_\alpha}\right)}{\partial\xi^\beta}=\frac{1}{A_\alpha}\frac{\partial\rho_\alpha}{\partial\xi^\beta}=\frac{1}{A_\alpha}\left(\overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta}\rho_\gamma+b_{\alpha\beta}n\right)$$

$$n=\sin\theta\cos\varphi i+\sin\theta\sin\varphi j+\cos\theta k$$

$$\frac{\partial e_1}{\partial\xi^1}=-\sin\xi_1\cos\xi_2 i-\sin\xi_1\sin\xi_2 j-\cos\xi_1 k \qquad \frac{\partial e_1}{\partial\xi^2}=\frac{\partial e_2}{\partial\xi^1}=0$$

$$\frac{\partial e_2}{\partial\xi^2}=-\sin^2\xi_1\cos\xi_2 i-\sin^2\xi_1\sin\xi_2 j-\sin\xi_1\cos\xi_1 k$$

$$\frac{\partial n}{\partial\xi^\beta}=-b_{\alpha\beta}\rho^\alpha$$

$$b_{12}=b_{21}=0$$

$$\frac{\partial n}{\partial\xi^1}=\frac{\partial n}{\partial\xi^2}=0$$

5.9 对于圆环曲面，验证 Codazzi 方程与 Gauss 方程。

解

Codazzi 方程

$$\overset{\circ}{\nabla}_{\gamma} b_{\alpha\beta} = \overset{\circ}{\nabla}_{\beta} b_{\alpha\gamma} \left(\alpha，\beta，\gamma = 1,2\right)$$

对于圆环曲面任一点的矢径为

$$\rho = \left(R + r_0 \sin \xi^1\right)\cos\xi^2 i + \left(R + r_0 \sin \xi^1\right)\sin \xi^2 j + r_0 \cos\xi^1 k$$

$$\rho_1 = \frac{\partial\rho}{\partial\xi^1} = r_0 \cos\xi^1 \cos\xi^2 i + r_0 \cos\xi^1 \sin \xi^2 j - r_0 \sin \xi^1 k$$

$$\rho_2 = \frac{\partial\rho}{\partial\xi^2} = -\left(R + r_0 \sin \xi^1\right)\sin \xi^2 i + \left(R + r_0 \sin \xi^1\right)\cos\xi^2 j$$

$$n = \sin \xi^1 \cos\xi^2 i + \sin \xi^1 \sin \xi^2 j + \cos\xi^1 k$$

$b_{\alpha\beta} = -\dfrac{\partial n}{\partial\xi^{\gamma}} \cdot \rho^{\gamma}$ 得到

$$\frac{\partial b_{\alpha\beta}}{\partial\xi^{\gamma}} = \frac{\partial b_{\alpha\gamma}}{\partial\xi^{\beta}} \left(\alpha,\beta,\gamma = 1,2\right)$$

$$b_{\mu\beta} \overset{\circ}{\Gamma}{}^{\mu}_{\alpha\gamma} = b_{\mu\gamma} \overset{\circ}{\Gamma}{}^{\mu}_{\alpha\beta} \left(\alpha,\beta,\gamma = 1,2\right)$$

所以

$$\frac{\partial b_{\alpha\beta}}{\partial\xi^{\gamma}} - b_{\mu\beta} \overset{\circ}{\Gamma}{}^{\mu}_{\alpha\gamma} - b_{\alpha\mu} \overset{\circ}{\Gamma}{}^{\mu}_{\beta\gamma} = \frac{\partial b_{\alpha\gamma}}{\partial\xi^{\beta}} - b_{\mu\gamma} \overset{\circ}{\Gamma}{}^{\mu}_{\alpha\beta} - b_{\alpha\mu} \overset{\circ}{\Gamma}{}^{\mu}_{\beta\gamma} \left(\alpha,\beta,\gamma = 1,2\right)$$

$\overset{\circ}{\nabla}_{\gamma} b_{\alpha\beta} = \overset{\circ}{\nabla}_{\beta} b_{\alpha\gamma}$ 成立

Gauss 方程

$$\overset{\circ}{R}{}^{\lambda}_{\cdot\alpha\gamma\beta} = b_{\alpha\beta} b^{\lambda}_{\cdot\gamma} - b_{\alpha\gamma} b^{\lambda}_{\cdot\beta} \quad \left(\alpha，\beta，\gamma，\lambda = 1,2\right)$$

对于圆环曲面

$$\frac{\partial \overset{\circ}{\Gamma}{}^{\lambda}_{\alpha\beta}}{\partial\xi^{\gamma}} + \overset{\circ}{\Gamma}{}^{\mu}_{\alpha\beta} \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\gamma} = b_{\alpha\beta} b^{\lambda}_{\cdot\gamma}$$

$$\frac{\partial \overset{\circ}{\Gamma}{}^{\lambda}_{\alpha\gamma}}{\partial\xi^{\beta}} + \overset{\circ}{\Gamma}{}^{\mu}_{\alpha\gamma} \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\beta} = b_{\alpha\gamma} b^{\lambda}_{\cdot\beta}$$

因为

$$\overset{\circ}{R}{}^{\lambda}_{\cdot\alpha\gamma\beta}=\frac{\partial\overset{\circ}{\Gamma}{}^{\lambda}_{\alpha\beta}}{\partial\xi^{\gamma}}-\frac{\partial\overset{\circ}{\Gamma}{}^{\lambda}_{\alpha\gamma}}{\partial\xi^{\beta}}+\overset{\circ}{\Gamma}{}^{\mu}_{\alpha\beta}\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\gamma}-\overset{\circ}{\Gamma}{}^{\mu}_{\alpha\gamma}\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\beta}$$

所以

$\overset{\circ}{R}{}^{\lambda}_{\cdot\alpha\gamma\beta}=b_{\alpha\beta}b^{\lambda}_{\cdot\gamma}-b_{\alpha\gamma}b^{\lambda}_{\cdot\beta}$ 成立

5.10 已知：旋转张量$c=c_{\alpha\beta}\rho^{\alpha}\rho^{\beta}$。求：$\overset{\circ}{\nabla}_{\lambda}c_{\alpha\beta}$，$\overset{\circ}{\nabla}c$

解

$$\overset{\circ}{\nabla}_{\lambda}c_{\alpha\beta}=\frac{\partial c_{\alpha\beta}}{\partial\xi^{\lambda}}-\overset{\circ}{\Gamma}_{\lambda\alpha,\beta}-\overset{\circ}{\Gamma}_{\lambda\beta,\alpha}$$

$$\overset{\circ}{\nabla}_{\lambda}c^{\alpha\beta}=\frac{\partial c^{\alpha\beta}}{\partial\xi^{\lambda}}+c^{\alpha\mu}\overset{\circ}{\Gamma}{}^{\beta}_{\lambda\mu}+c^{\mu\beta}\overset{\circ}{\Gamma}{}^{\alpha}_{\lambda\mu}$$

因为

$$\overset{\circ}{\nabla}_{\lambda}c_{\alpha\beta}=\overset{\circ}{\nabla}_{\lambda}c^{\alpha\beta}=\frac{\partial c^{\alpha\beta}}{\partial\xi^{\lambda}}+c^{\alpha\mu}\overset{\circ}{\Gamma}{}^{\beta}_{\lambda\mu}+c^{\mu\beta}\overset{\circ}{\Gamma}{}^{\alpha}_{\lambda\mu}=\frac{\partial c_{\alpha\beta}}{\partial\xi^{\lambda}}-\overset{\circ}{\Gamma}_{\lambda\alpha,\beta}-\overset{\circ}{\Gamma}_{\lambda\beta,\alpha}$$

所以$\overset{\circ}{\nabla}_{\lambda}c_{\alpha\beta}=\overset{\circ}{\nabla}_{\lambda}c^{\alpha\beta}=0$

$$\overset{\circ}{\nabla}c=\overset{\circ}{\nabla}_{\lambda}c_{\alpha\beta}\rho^{\lambda}\rho^{\alpha}\rho^{\beta}+b^{\cdot\omega}_{\lambda}c_{\omega\beta}\rho^{\lambda}n\rho^{\beta}+b^{\cdot\omega}_{\lambda}c_{\alpha\omega}\rho^{\lambda}\rho^{\alpha}n$$

$$=\overset{\circ}{\nabla}_{\lambda}c_{\alpha\beta}\rho^{\lambda}\rho^{\alpha}\rho^{\beta}+b_{\lambda\beta}\rho^{\lambda}n\rho^{\beta}+b_{\lambda\alpha}\rho^{\lambda}\rho^{\alpha}n$$

$$=b^{\cdot\omega}_{\lambda}c_{\alpha\omega}\left(\rho^{\lambda}\rho^{\alpha}n-\rho^{\lambda}n\rho^{\alpha}\right)$$

第六章

6.9 求 Almansi 应变张量 e 对时间 t 的率

证明：因为$\mathbf{e}=E_{ij}\hat{\mathbf{g}}^{i}\hat{\mathbf{g}}^{j}$　　$\hat{\mathbf{g}}^{i},\hat{\mathbf{g}}^{j}$为变形后（在$t$时刻）的逆变基，是随时间$t$变化的

所以　$\dot{\mathbf{e}}=\dfrac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(E_{ij}\hat{\mathbf{g}}^{i}\hat{\mathbf{g}}^{j}\right)=\dfrac{\mathrm{d}E_{ij}}{\mathrm{d}t}\hat{\mathbf{g}}^{i}\hat{\mathbf{g}}^{j}+E_{ij}\dfrac{\mathrm{d}\hat{\mathbf{g}}^{i}}{\mathrm{d}t}\hat{\mathbf{g}}^{j}+E_{ij}\hat{\mathbf{g}}^{i}\dfrac{\mathrm{d}\hat{\mathbf{g}}^{j}}{\mathrm{d}t}$　　（1）

又有　$\dot{E}_{ij}=\dfrac{\mathrm{d}E_{ij}}{\mathrm{d}t}=\hat{d}_{ij}=\dfrac{1}{2}\dfrac{\mathrm{d}\hat{g}_{ij}}{\mathrm{d}t}=\dfrac{1}{2}\left(\hat{\nabla}_{j}\hat{v}_{i}+\hat{\nabla}_{i}\hat{v}_{j}\right)=\dfrac{1}{2}\left(\hat{v}_{i;j}+\hat{v}_{j;i}\right)$

$$\dot{\hat{\mathbf{g}}}^{i}=-\nabla\mathbf{v}\cdot\hat{\mathbf{g}}^{i}\qquad\dot{\hat{\mathbf{g}}}^{j}=-\nabla\mathbf{v}\cdot\hat{\mathbf{g}}^{j}$$

代入（1）式得

$$\dot{\mathbf{e}} = \frac{1}{2}\left(\hat{\nu}_{i;j} + \hat{\nu}_{j;i}\right)\hat{\mathbf{g}}^i\hat{\mathbf{g}}^j - E_{ij}\left(\nabla\mathbf{v}\cdot\hat{\mathbf{g}}^i\right)\hat{\mathbf{g}}^j - E_{ij}\hat{\mathbf{g}}^i\left(\nabla\mathbf{v}\cdot\hat{\mathbf{g}}^j\right)$$