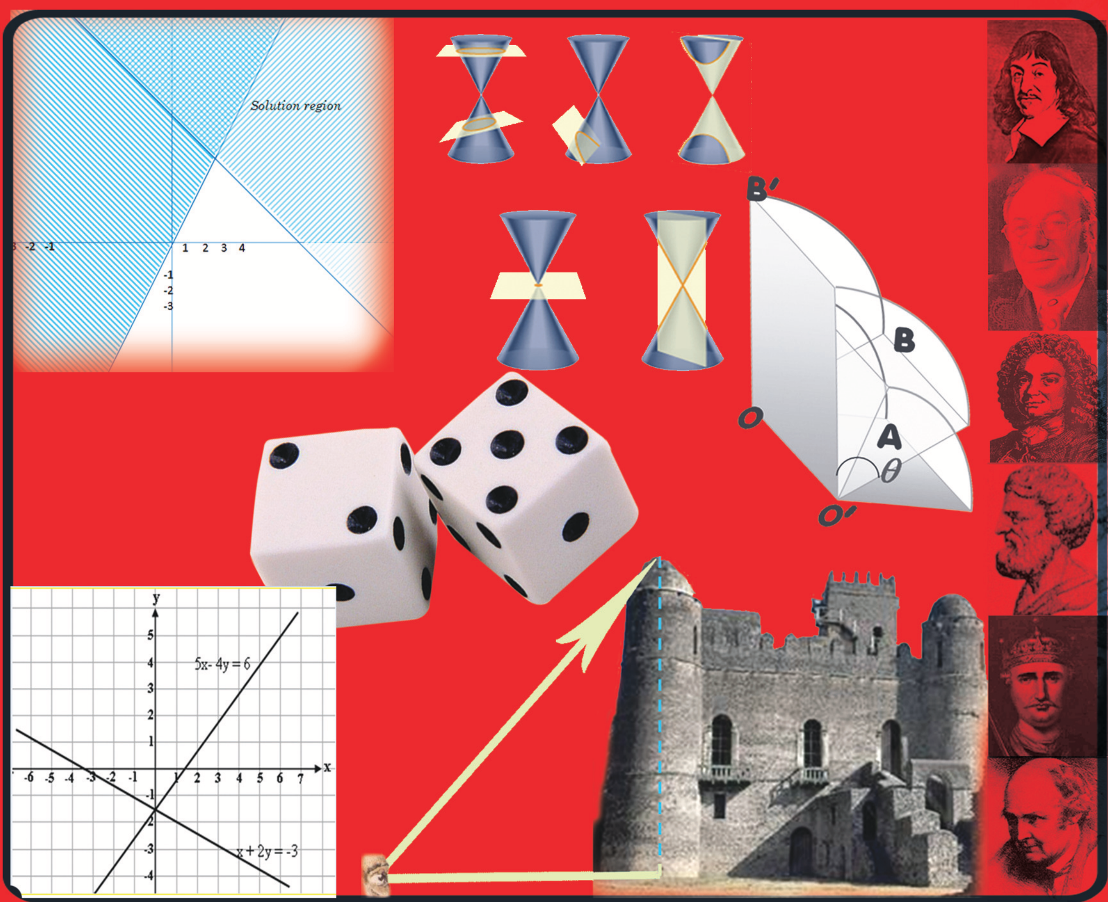
MATHEMATICS

TEACHER'S GUIDE GRADE 11



FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

MATHEMATICS

TEACHER'S GUIDE

GRADE 11

Authors, Editors and Reviewers:

Rachel Mary Z. (B.Sc.)

Kassa Michael (M.Sc.)

Kinfegabrail Dessalegn (M.Sc.)

Sebsibe Teferi (M.Sc.)

Tilahun Abebaw (Ph.D.)

Hunduma Legesse (M.Sc.)

Mulugeta Naizghi (M.Sc.)

Berhanu Bekele (Ph.D.)

Evaluators:

Tesfaye Ayele

Dagnachew Yalew

Tekeste Woldetensai

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA

MINISTRY OF EDUCATION

Published E.C. 2002 by the Federal Democratic Republic of Ethiopia, Ministry of Education, under the General Education Quality Improvement Project (GEQIP) supported by IDA Credit No. 4535-ET, the Fast Track Initiative Catalytic Fund and the Governments of Finland, Italy, Netherlands and the United Kingdom.

The Ministry of Education wishes to thank the many individuals, groups and other bodies involved-directly and indirectly-in publishing this textbook and the accompanying teacher guide.

Developed and Printed by

STAR EDUCATIONAL BOOKS DISTRIBUTORS Pvt. Ltd.

24/4800, Bharat Ram Road, Daryaganj,

New Delhi – 110002, INDIA

and

ASTER NEGA PUBLISHING ENTERPRISE

P.O. Box 21073

ADDIS ABABA, ETHIOPIA

Under GEQIP Contract No. ET-MoE/GEQIP/IDA/ICB/G01/09.

ISBN 978-99944-2-047-0

INTRODUCTION

The study of mathematics at this cycle, grades 11 -12, prepares our students for the future, both practically and philosophically. Studying mathematics provides them not only with specific skills in mathematics, but also with tools and attitudes for constructing the future of our society. As well as learning to think efficiently and effectively, our students come to understand how mathematics underlies daily life and, on a higher level, the dynamics of national and international activity. The students automatically begin to apply high-level reasoning and values to daily life and also to their understanding of the social, economic, political and cultural realities of the country. In turn, this will help them to actively and effectively participate in the ongoing process of developing the nation.

At this cycle, our students gain a solid knowledge of the fundamental mathematical theories, theorems, rules and procedures. They also develop reliable skills for using this knowledge to solve problems independently. To this end, the objectives of mathematics learning at this cycle are to enable students to

- apply the mathematical knowledge and capabilities gained to solve problems independently.
- develop mental abilities and high skills and competencies in calculations, especially, in the field of logical thinking, reasoning, proving, defining and use of mathematical language, terminologies and symbols correctly.
- develop an appreciation for the importance of mathematics as a field of study by learning its historical development, scope and its relationship with other disciplines.
- develop scientific outlook and personality characteristics such as working activities with algorithms, exactness, neatness, honesty and carefulness according to self-prepared plans for solving problems in line with the needs of the society.

Recent research gives strong arguments for changing the way in which mathematics has been taught. The rote-learning paradigm has been replaced by the student-centered model. A student-centered classroom stimulates student inquiry, and the teacher serves as a mentor who guides students as they construct their own knowledge base and skills. A primary goal when you teach a concept is for the students to discover the concept for themselves, particularly as they recognize threads and patterns in the data and theories that they encounter under your guidance.

One of our teaching goals is particularly fostered by the student-oriented approach. We want our students to develop personal qualities that will help them in real life.

For example, student-oriented teachers encourage students' self-confidence and their confidence in their knowledge, skills and general abilities. We motivate our students to express their ideas and observations with courage and confidence. Because we want them to feel comfortable addressing individuals and groups and to present themselves and their ideas well, we give them safe opportunities to stand before the class and present their work. Similarly, we help them learn to learn to answer questions posed directly to them by other members of the class.

Teamwork is also emphasized in a student-centered classroom. For example, the teacher creates favorable conditions for students to come together in groups and exchange ideas about what they have learned and about material they have read. In this process, the students are given many opportunities to openly discuss the knowledge they have acquired and to talk about issues raised in the course of the discussion.

This teacher's guide will help you teach well. For example, it is very helpful for budgeting your teaching time as you plan how to approach a topic. The guide suggests tested teaching-time periods for each subject you will teach. Also, the guide contains answers to the review questions at the end of each topic.

Each section of your teacher's guide includes student-assessment guidelines. Use them to evaluate your students' work. Based on your conclusions, you will give special attention to students who are working either above or below the standard level of achievement. Check each student's performance against the learning competencies presented by the guide. Be sure to consider both the standard competencies and the minimum competencies. Note that the *minimum requirement level* is not the *standard level of achievement*. To achieve the standard level, your students must fulfill all of their grade-level's competencies successfully.

When you identify students who are working either below the standard level or below the minimum level, give them extra help. For example, give them supplementary presentations and reviews of the material in the section, give them extra time to study, and develop extra activities to offer them. You can also encourage high-level students in this way. You can develop high-level activities and extra exercises for them and can offer high-level individual and group discussions. Some highlighting exercise problems are given in this teacher's guide. Be sure to show the high-level students that you appreciate their good performance, and encourage them to work hard. Also, be sure to discourage any tendencies toward complacency that you might observe.

Some helpful reference materials are listed at the end of this teacher's guide. For example, the internet is a rich resource for teachers, and searching for new web sites is well worth your time as you investigate your subject matter. Use one of the many search engines that exist – for example, Yahoo and Google.

Do not forget that, although this guide provides many ideas and guidelines, you are encouraged to be innovative and creative in the ways you put them into practice in your classroom. Use your own knowledge and insights in the same way as you encourage your students to use theirs.

Contents

ACTIVE LEARNING AND CONTINUOUS ASSESSMENT REQUIRED!

Dear mathematics teacher! For generations the technique of teaching mathematics at any level was dominated by what is commonly called the **direct instruction.** That is, students are given the exact tools and formulas they need to solve a certain mathematical problem, sometimes without a clear explanation as to why, and they are told to do certain steps in a certain order and in turn are expected to do them as such at all times. This leaves little room for solving varying types of problems. It can also lead to misconceptions and students may not gain the full understanding of the concepts that are being taught.

You just sit back for a while and try to think the most common activities that you, as a mathematics teacher, are doing in the class.

> *Either you explain (lecture) the new topic to them, and expect your students to remember and use the contents of this new topic or you demonstrate with examples how a particular kind of problem is solved and students routinely imitate these steps and procedures to find answers to a great number of similar mathematical problems.*

But this method of teaching revealed little or nothing of the meaning behind the mathematical process the students were imitating.

We may think that teaching is telling students something, and learning occurs if students remember it. But research reveals that teaching is not "pouring" information into students' brain and expecting them to process it and apply it correctly later.

Most educationalists agree that learning is an active meaning-making process and students will learn best by trying to make sense of something on their own with the teacher as a guide to help them along the way. This is the central idea of the concept Active Learning.

> **Active learning**, *as the name suggests, is a process whereby learners are actively engaged (involved) in the learning process, rather than "passively" absorbing lectures. Students are rather encouraged to think, solve problems, do activities carefully selected by the teacher, answer questions, formulate questions of their own, discuss, explain, debate, or brainstorm, explore and discover, work cooperatively in groups to solve problems and workout projects.*

The design of the course materials (student textbooks and teachers guides) for mathematics envisages active learning to be dominantly used. With this strategy, we feel that you should be in a position to help students understand the concepts through relevant, meaningful and concrete activities. The activities should be carried out by students to explore the world of mathematics, to learn, to discover and to develop interest in the subject. Though it is your role to exploit the opportunity of using active learning at an optimal level, for the sake of helping you get an insight, we recommend that you do the following as frequently as possible during your teaching:

- Engage your students in more relevant and meaningful activities than just listening.
- Include learning materials having examples that relate to students life, so that they can make sense of the information.
- Let students be involved in dialog, debate, writing, and problem solving, as well as higher-order thinking, e.g., analysis, synthesis, evaluation.
- Encourage students' critical thinking and inquiry by asking them thoughtful, open-ended questions, and encourage them to ask questions to each other.
- Have the habit of asking learners to apply the information in a practical situation. This facilitates personal interpretation and relevance.
- Guide them to arrive at an understanding of a new mathematical concept, formula, theorem, rule or any generalization, by themselves. You may realize this by giving them an activity in which students sequentially uncover layers of mathematical information one step at a time and discover new mathematics.
- Select assignments and projects that should allow learners to choose meaningful activities to help them apply and personalize the information. These need to help students undertake initiatives, discover mathematical results and even design new experiments to verify results.
- Let them frequently work in peers or groups. Working with other learners gives learners real-life experience of working in a group, and allows them to use their metacognitive skills. Learners will also be able to use the strengths of other learners, and to learn from others. When assigning learners for group work membership, it is advisable if it is based on the expertise level and learning style of individual group members, so that individual team members can benefit from one another's strengths.

In general, if mathematics is to develop creative and imaginative mathematical minds, you must overhaul your traditional methods of presentation to the more active and participatory strategies and provide learning opportunities that allow your students to be actively involved in the learning process. While students are engaged with activities, group discussions, projects, presentations and many others they need to be continuously assessed.

Continuous Assessment

You know that continuous assessment is an integral part of the teaching learning process. Continuous assessment is the periodic and systematic method of assessing and evaluating a person's attributes and performance. Information collected from continuous behavioral change of students will help teachers to better understand their strengths and weaknesses in addition to providing a comprehensive picture of each student over a period of time. Continuous assessment will afford student to readily see his/her development pattern through the data. It will also help to strengthen the parent teacher relationship and collaboration. It is an ongoing process more than giving a test or exam frequently and recording the marks.

Continuous assessment enables you to assess a wide range of learning competencies and behaviors using a variety of instruments some of which are:

- Tests/ quizzes (written, oral or practical)
- Class room discussions, exercises, assignments or group works.
- Projects
- Observations
- Interview
- group discussions
- questionnaires

Different competencies may require different assessment techniques and instruments. For example, oral questions and interviews may serve to assess listening and speaking abilities. They also help to assess whether or not students are paying attention, and whether they can correctly express ideas. You can use oral questions and interviews to ask students to restate a definition, note or theorem, etc. Questionnaires, observations and discussions can help to assess the interest, participation and attitudes of a student. Written tests/exams can also help to assess student's ability to read, to do and correctly write answers for questions.

When to Assess

Continuous assessment and instruction are integrated in three different time frames namely, Pre-instruction, During-instruction and Post-instruction. To highlight each briefly

1. Pre-instruction assessment

This is to assess what students luck to start a lesson. Hence you should start a lesson by using opportunities to fill any observed gap. If students do well in the pre-instruction assessment, then you can begin instructing the lesson. Otherwise, you may need to revise important concepts.

The following are some suggestions to perform or make use of pre-instruction assessment.

i. assess whether or not students have the prerequisite knowledge and skill to be successful, through different approaches.
ii. make your teaching strategies motivating.
iii. plan how you form groups and how to give marks.
iv. create interest on students to learn the lesson.

2. Assessment during instruction:

This is an assessment during the course of instruction rather than before it is started or after it is completed. The following are some of the strategies you may use to assess during instruction.

i. observe and monitor students' learning.

ii. check that students are understanding the lesson. You may use varying approaches such as oral questions, asking students to do their work on the board, stimulate discussion, etc.

iii. identify which students need extra help and which students should be left alone.

iv. ask a balanced type of exercise problems according to the students ability, help weaker students and give additional exercise for fast students.

v. monitor how class works and group discussions are conducted

3. Post Instruction Assessment:

This is an assessment after instruction is completed. It is conducted usually for the purpose of documenting the marks and checking whether competencies are achieved. Based on the results students scored, you can decide whether or not there is anything the class didn't understand because of which you may revise some of the lessons or there is something you need to adjust on the approach of teaching. This also help you analyze whether or not the results really reflect what students know and what they can do, and decide how to treat the next lesson.

Forming and managing groups

You can form groups through various approaches: mixed ability, similar ability, gender or other social factors such as socioeconomic factors. When you form groups, however, care need to be taken in that you should monitor their effort. For example, if students are grouped by mixed ability the following problems may happen.

1. Mixed ability grouping may hold back high-ability students. Here, you should give enrichment activities for high ability students.
2. High ability students and low ability students might form a teacher-student relationship and exclude the medium ability students from group discussion. In this case you should group medium ability students together.

When you assign group work, the work might be divided among the group members, who work individually. Then the members get together to integrate, summarize and present their finding as a group project. Your role is to facilitate investigation and maintain cooperative effort.

Highlights about assessing students

You may use different instruments to assess different competencies. For example, consider each of the following competencies and the corresponding assessment instruments.

Competency 1. Recall measures of central tendency.

Instrument: Oral question.

Question: What are the measures of central tendency?

Competency 2 - Students will calculate percentiles.

Instrument: class work/homework/ quiz /test

Question: Find the 30^{th} and 75^{th} percentiles for the following grouped data

Class	Frequency
1 – 7	2
8 – 14	4
15 – 21	7
22 – 28	3
29 – 35	4
	20

Competency 3 – Apply statistics in their daily life problems.

Instrument: Assignment/project.

Question: Go to your school's record office, collect data of students' scores in their EGSSCE, and then calculate each of the following. Mean, Median and Mode, 80^{th} and 60^{th} percentiles, and interpret the results.

How often to assess

Here are some suggestions which may help you how often to assess.

- Class activities / class works: Every day (when convenient).
- Homework/Group work: as required.
- Quizzes: at the end of every one (or two) sub topics.
- Tests: at the end of every unit.
- Exams: once or twice in every semester.

How to Mark

The following are some suggestions which may help you get well prepared before you start marking:

- use computers to reduce the burden for record keeping.
- although low marks may diminish the students motivation to learn, don't give inflated marks for inflated marks can also cause reluctance.

The following are some suggestions on how to mark a semester's achievement.

1. One final semester exam 30%.
2. Tests 25%
3. Quizzes 10%
4. Homework 10%
5. Class activities, class work, presentation demonstration skills 15%
6. Project work, in groups or individually 10%.

Moreover

In a group work allow students to evaluate themselves as follows using format of the following type.

	A	*B*	*C*	*D*	
The ability to communicate					
The ability to express written works					
Motivation					
Responsibility					
Leadership quality					
Concern for others					
Participation					
Over all					

You can shift the leadership position or regroup the students according to the result of the self evaluation. You can also consider your observation.

Reporting students' progress and marks to parents

Parents should be informed about their children's progress and performance in the class room. This can be done through different methods.

1. The report card: two to four times per year.
2. Written progress report: Per week/two weeks/per month/two months.
3. Parent – teacher conferences (as scheduled by the school).

The report should be about the student performance say, on tests, quizzes, projects, oral reports, etc that need to be reported. You can also include motivation or cooperation behavior. When presenting to parents your report can help them appraise fast learner, pay additional concern and care for low achieving student, and keep track of their child's education. In addition, this provides an opportunity for giving parents helpful information about how they can be partners with you in helping the student learn more effectively.

The following are some suggested strategies that may help you to communicate with parents concerning marks, assessment and student learning.

1. Review the student's performance before you meet with parents.
2. Discuss with parents the students good and poor performances.
3. Do not give false hopes. If a student has low ability, it should be clearly informed to his/her parents.
4. Give more opportunities for parents to contribute to the conversation.
5. Do not talk about other students. Don't compare the student with another student.
6. Focus on solutions

NB. ***All you need to do is thus plan what type of assessment and how many of each you are going to use beforehand (preferably during the beginning of the year/semester).***

UNIT 1 FURTHER ON RELATIONS AND FUNCTIONS

INTRODUCTION

This unit is devoted to the discussion of relations and functions. The concepts of relations and functions are very important for problem solving. They influence the study of mathematics from elementary level to advanced level. It is very important that students should have a firm grasp of these concepts.

Since students have studied this topic in grade 9, you need to make sure that the students have worked through the examples and exercises given in the unit.

This unit consists of five sections: revisions on relations, some additional types of functions, classification of functions, composition of functions, and inverse functions and their graphs.

Encouraging students to do the activities and exercises in pairs and small groups is advised because group work helps them to develop confidence in expressing their ideas. They may also develop team spirit in working in small groups.

In addition to the examples given in the textbook, help students to construct their own examples.

Unit Outcomes

After completing this unit, students will be able to:

- *know specific facts about relations.*
- *know additional concepts and facts about functions.*
- *understand methods and principles in composing functions.*

Suggested Teaching Aids in unit 1

In addition to the students textbooks and teachers guide, the following are recommended teaching aids for this unit.

- Chart showing sample graphs of relations, functions and their inverses
- Sample graphs showing the symmetry of R and R^{-1} on the line $y = x$
- Graph Board.
- Sample graph of invertible functions together with their inverses.
- Software such as Geometer's sketch pad, tinkerplot, Matlab, etc.

1.1 REVISION ON RELATIONS

Periods Allotted: 2 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *find out the inverse of a given relation.*
- *sketch the graph of a relation and its inverse.*

Vocabulary: Relation, Inverse, Domain, Range, Graph, Mirror image

Introduction

The main purpose of this sub-unit is to help the students find out inverses of given relations and sketch their graphs. But the students may need to revise what they learnt in Grades 9 and 10.

Since the students studied this topic in grades 9 and 10 you may first need to assess their background through oral questions and answers. Whenever you find it essential, you may start the sub-unit by revising the concepts of relation, domain, range and graph of a relation. In addition, give the graphs of simple relations and help students to write the relations that define them. However, before attempting to discuss these, it is advisable to give the opening problem to brainstorm the thinking of the students.

1.1.1 Inverse of a Relation

Teaching Notes

As a consequence of the opening problem, which the students might not be able to answer, you form groups and let them do Activity 1.1. The purpose of this activity is to help the students revise relations represented as a set of ordered pairs, and to determine their domains and ranges. They will also revise what happens if the ordered pairs in a relation are reversed. This will help them revise inverse of a relation too.

Answers to Activity 1.1

The purpose of this Activity is to help students recall the basic concepts of relations. Use it as an indicator of how much they recall of what they studied in the lower grades.

1. Domain of R = {1, 5, 6, 7, 8} and Range of R = {–1, 2, 4, 9}.

   Relation:

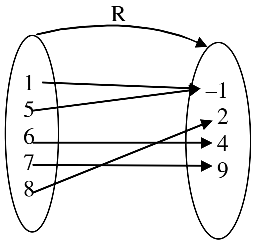


Figure 1.1

2. (–5, 6) and (π, 3.4) belong to R, while (–4, –6.234) does not belong to R.
3. When the order of each of the ordered pairs of the relation in 1 is reversed, we get the set {(–1, 5), (4, 6), (9, 7), (2, 8), (–1, 1)}.

Similarly, for the relation in 2, when the elements are reversed, we get the set

$\{(y, x): x < y\} = \{(x, y): y < x\}$.

After the revision through Activity 1.1, you can proceed with the new lesson by taking inverses of relations with finite elements and then proceed with more complex ones.

For Example: $R_1 = \{(1, 3), (b, a), (c, 1), (a, 2)\}$

$R_2 = \{(x, y): y = x + 1\}$

$R_3 = \{(x, y): y < x + 1\}$

$R_4 = \{(x, y): y < x + 1 \text{ and } y \geq 3x – 2\}$ and so on.

Students need to develop the ability of sketching graphs of relations. So, make sure that they identify basic components of the Cartesian plane and clearly indicate the difference of inequalities like ($y \geq 2x + 3$) and strict inequalities like ($y > 2x + 3$) in their sketches. (In sketching $y \geq 2x + 3$, the line $y = 2x + 3$ is expressed by unbroken line, while in sketching $y > 2x + 3$, the line $y = 2x + 3$ is given by a broken line).

At this stage, you can state the definition of a relation and its inverse given on page 3 and definition 1.1.

To let the students identify relations between two nonempty sets and determine their domain and range, you can form groups and let them do Group Work 1.1. This group work also guides the students to find inverse of a relation with its domain and range. Finally, it helps them relate the domain and range of a relation and its inverse. That is, it will help them to establish the fact that

Domain R^{-1} = Range R and Domain R = Range R^{-1}

For talented students you can give exercises to find maximum and minimum values of relations.

Example: Find the maximum and minimum values of the relation

$$R = \{(x, y): y \le x + 3,\ y \le -x + 1 \text{ and } y \le x - 4\}$$

Solution:

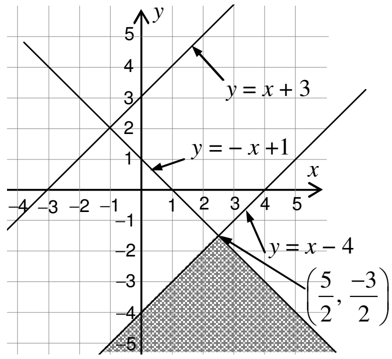


Figure 1.2

From the graph of the relation R, we have the maximum value at $\left(\frac{5}{2}, \frac{-3}{2}\right)$ and no minimum value.

Answers to Group Work 1.1

1. Only R_1 and R_4 are relations from A to B.
2. a. Domain of R = {–1, 0} and Range of R = {5, 6} = B

   b. Domain of R^{-1} = {5, 6}

   Range of R^{-1} = {–1, 0}

   c. Domain of R = Range of R^{-1} and Range of R = Domain of R^{-1}.
3. c and d
4. Domain of $R^{-1} = \mathbb{R}$ and Range of $R^{-1} = (1, \infty)$
5. a. $R^{-1} = \left\{(5,1), (-6,3), (3.5,4), \left(\frac{6}{5}, 1\right)\right\}$

   Domain of $R^{-1} = \{5, -6, 3.5, \frac{6}{5}\}$; Range of R^{-1} = {1, 3, 4}

   b. $R^{-1} = \{(y, x): y = 3x - 7\} = \{(x, y): x = 3y - 7\}$

   $= \left\{(x, y): y = \frac{1}{3}x + \frac{7}{3}\right\}$

   Domain of $R^{-1} = \mathbb{R}$ = Range of R^{-1}

   c. $R^{-1} = \{(y, x): y < -3x \text{ and } y \ge x - 4\} = \{(x, y): x < -3y \text{ and } x \ge y - 4\}$

   $= \left\{(x, y): y < -\frac{1}{3}x \text{ and } y \le x + 4\right\}$

   Domain of $R^{-1} = \mathbb{R}$; Range of $R^{-1} = (-\infty, 1)$

Assessment

You may use Exercise 1.1 for assessing the understanding of your students either as class work or home work.

Answers to Exercise 1.1

1. b, c, d and f are true. Check by inserting the points in the relation.

2. a.

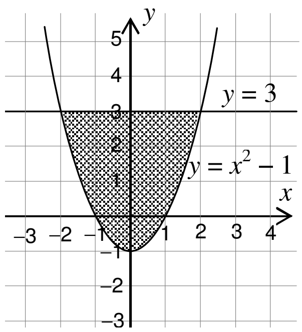


Figure 1.3

b. To find the domain, find the intersection points of $y = x^2 - 1$ and $y = 3$

$x^2 - 1 = 3 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Thus, the domain is $[-2, 2]$ and the range is $[-1, 3]$.

3. $R = \{(x, y): x \geq y^2 \text{ and } x^2 + y^2 \leq 9\}$

4. a. $R^{-1} = \{(x, y): y \text{ is a brother of } x\}$

b. $R^{-1} = \{(x, y): y^2 + 1 = x^2\}$

c. $R^{-1} = \{(x, y): x \geq y + 3 \text{ and } x < -3y - 1\}$

$$= \left\{(x, y): y \leq x - 3 \text{ and } y < -\frac{1}{3}x - \frac{1}{3}\right\}$$

5. a. $R^{-1} = \{(x, y): x \geq y^2 + 1\} = \{(x, y): x - 1 \geq y^2\}$

Now for each y, $y^2 \geq 0 \Rightarrow x - 1 \geq 0 \Rightarrow x \geq 1$

Thus Domain of $R^{-1} = [1, \infty)$ and Range of $R^{-1} = \mathbb{R}$.

b. $R^{-1} = \{(x, y): x \leq -y^2 \text{ and } x \geq -1\}$

$-1 \leq x \leq -y^2 \Rightarrow -1 \leq x \leq 0$, (since $-y^2 \leq 0$)

Thus Domain of $R^{-1} = [-1, 0]$

Besides $-1 \leq x \leq -y^2 \Rightarrow 1 \geq -x \geq y^2$.

So, $y^2 \leq 1 \Rightarrow |y| \leq 1 \Rightarrow$ Range of $R^{-1} = [-1, 1]$.

c. $R^{-1} = \{(x, y): -3 \leq y \leq 3, x \in \mathbb{R}\}$

Thus Domain of $R^{-1} = \mathbb{R}$ and Range of $R^{-1} = [-3, 3]$.

1.1.2 Graphs of Inverse Relations

By guiding the students to work in pairs through Activity 1.2 in the student textbook, and by giving additional examples, encourage them to arrive at the facts

Domain of R = Range of R^{-1} and Range of R = Domain of R^{-1}

Give additional examples, activities and group work to

- draw R^{-1} and R using the separate set of coordinate axes.
- draw R^{-1} and R using the same Cartesian coordinate axes and notice that they are mirror images on the line $y = x$.
- reflect R on the line $y = x$ to get R^{-1}.
- Find domain and Range of R^{-1} from the graph of R and that of R^{-1}.

Use large graph papers and display for the students. Note also that it is easier to find domain and range of a relation using its graph.

Answers to Activity 1.2

1. a. $R^{-1} = \{(-2, 1), (9, 3), (6, 4), (-7, 5), (2.5, 5)\}$

   b. Domain of $R^{-1} = \{-2, 9, 6, -7, 2.5\}$

   Range of $R = \{-2, 9, 6, -7, 2.5\}$

   $\Rightarrow$ Domain of R^{-1} = Range of R.

   c. Range of $R^{-1} = \{1, 3, 4, 5\}$ = Domain of R.

   d. $R^{-1} = \{(x, y): -3 \le y \le 3, x \in \mathbb{R}\}$

   Thus Domain of $R^{-1} = \mathbb{R}$ = Range of R and

   Range of $R^{-1} = [-3, 3]$ = Domain of R.

   e. For any relation R, Domain of R^{-1} = Range of R and

   Range of R^{-1} = Domain of R.

Once the students are able to relate domain and range of a relation and its inverse, you can now proceed to discuss the behavior of graphs of a relation and its inverse. To this end, you can use Group Work 1.2. The purpose of this Group Work is to show that, for any relation R, the graph of R^{-1} is the reflection of the graph of R with respect to the line $y = x$.

Answers to Group Work 1.2

1. a. $R^{-1} = \{(-1, 3), (2, 4), (3, 6), (1, -5)\}$

b.

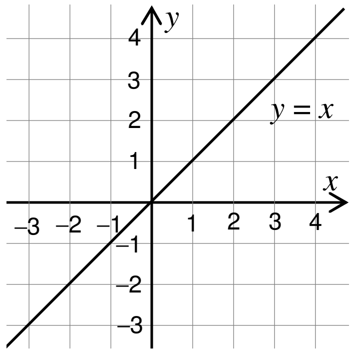


Figure 1.4

c.

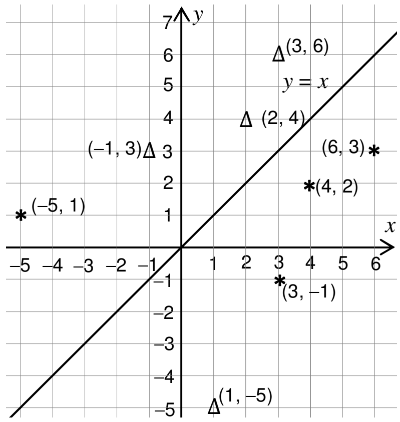


Figure 1.5

d. When you fold the graph in (c) above, along the line $y = x$, you can see that the points on the graph of R and the points on the graph of R^{-1} coincide.

2. As investigated in (1) above, the graph of R: $y= x^3$ and R^{-1}: $x = y^3$ is given below.

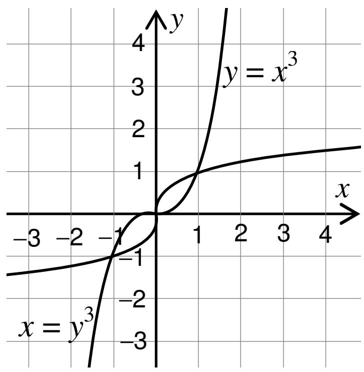


Figure 1.6

3.

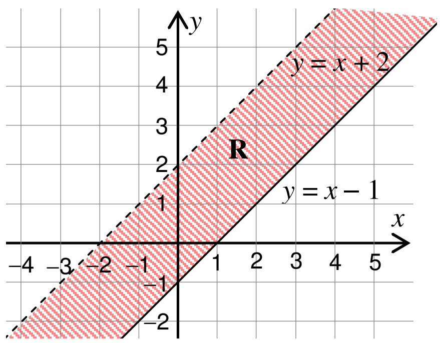


Figure 1.7

From the graph, if you turn the paper over and rotate the paper 90° clockwise, you will get the graph of R^{-1}.

Through the Group Work, the students will recall at the conclusion that the graph of a relation is mirror image of the graph of its inverse. After ensuring this, you can give exercise 1.2 as a class work or homework with which students can practice determining graphs of inverse relations.

Assessment

You can assess the understanding of the students either through homework or by giving them assignments consisting of problems related to the topic.

Answers to Exercise 1.2

1. a.

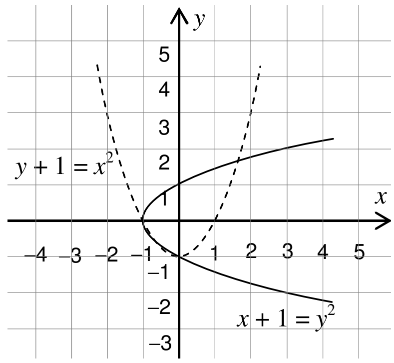


Figure 1.8

Note:- Curve in quadrant I is reflected on quadrant I

Curve in quadrant II is reflected on quadrant IV

Curve in quadrant III is reflected on quadrant III

Curve in quadrant IV is reflected on quadrant II

b. iii

2. a. b.

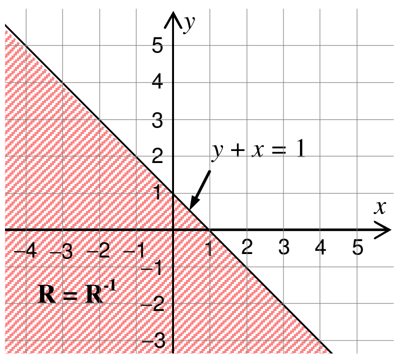


Figure 1.9

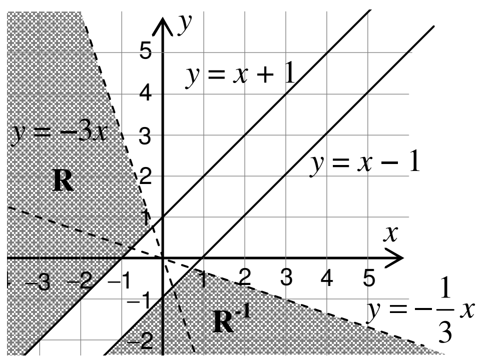


Figure 1.10

c.

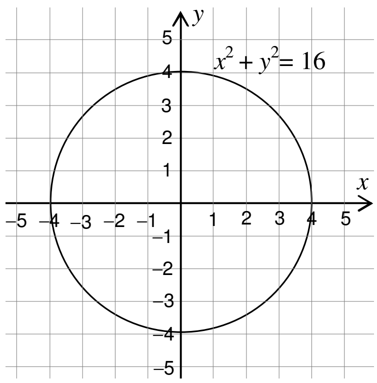


Figure 1.11

d.

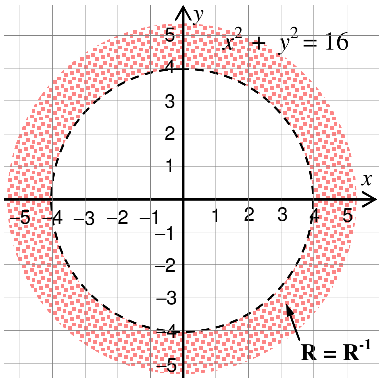


Figure 1.12

1.2. SOME ADDITIONAL TYPES OF FUNCTIONS

Periods Allotted: 4 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *define power functions.*
- *describe the properties of power functions in relation to their exponents.*
- *determine the domains and ranges of power functions.*
- *sketch the graphs of power functions.*
- *define the Modulus (Absolute value) function.*
- *determine the domain and the range of the modulus function.*
- *sketch the graph of the Modulus function.*
- *define the signum function.*
- *determine the domain and range of the signum function.*
- *sketch the graph of the signum function.*
- *define the (the greatest integer) function.*
- *determine the domain and range of the greatest integer function.*
- *sketch the graph of the greatest integer function.*

Vocabulary: Function, Parity, Power function, Modulus (absolute value) function, Signum function, Greatest integer (floor) function, Inflection point, Cusp

Introduction

In the same way as you did for revising relations, you can see Unit 4 of Grade 9 Student Textbook and Units 1 and 2 of Grade 10 Student Textbook to revise concepts so far covered on functions, polynomial, exponential and logarithmic functions.

The concept of a function is undoubtedly the most important concept in mathematics. Thus, enough attention must be given to revise this important concept. The purpose of this sub-unit is to familiarize the students with power functions and piece-wise defined functions. Piece-wise defined functions occur in many situations. For instance, the amount of money you pay for a parcel you send through the postal service versus the weight of the parcel gives a step-wise function.

1.2.1 Revision on Function

Teaching Notes

To begin the revision, you can guide the students to do Activity 1.3 which will be useful to discuss identifying functions from relations, determine domain of a function and finally evaluate functional value at a point. Following this activity, the students can also discuss different ways of representing functions. They will also be able to use vertical lines in diagnosing whether a certain graph represents a function.

Answers to Activity 1.3

1. b and c are functions
2. Domain of $f = \mathbb{R}$; domain of $g = \left[\frac{-7}{3}, \infty\right)$

   a. –20 b. –6.5 c. 4

Right after this, you can define different types of functions as even or odd, and introduce the term parity (being even or odd). Some other types of functions: exponential and logarithmic are also defined.

For talented students you can give exercises to determine the parity of rational functions such as the following.

Example: Determine the parity of $f(x) = \dfrac{x^5 - 5x^3 + x}{x^2 + 1}$.

Solution: $f(-x) = \dfrac{(-x)^5 - 5(-x)^3 + (-x)}{(-x)^2 + 1} = \dfrac{-x^5 + 5x^3 - x}{x^2 + 1} = -f(x)$.

Therefore, f is odd.

Assessment

To assess the understanding of the students, you can use Exercise 1.3 which you may give them as homework assignment.

Answers to Exercise 1.3

1. a.

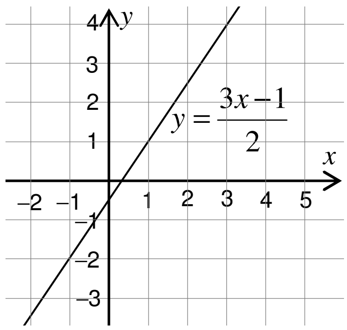


Figure 1.13

b.

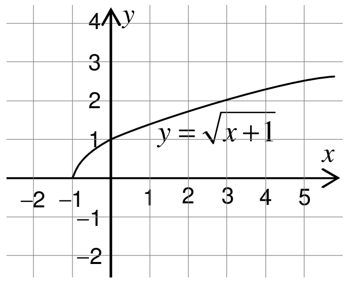


Figure 1.14

c.

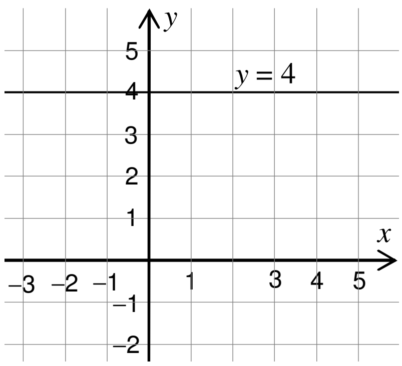


Figure 1.15

2. p(50) = 29%, p(100) = 26% , p(200) = 20% and p(400) = 8%

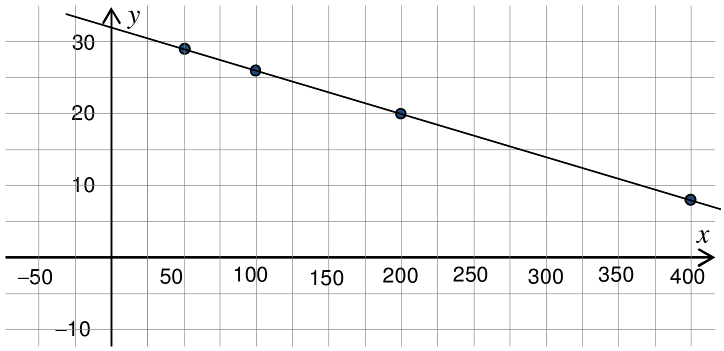


Figure 1.16

3. a. $g(-x) = \sqrt{8(-x)^4+1} = \sqrt{8x^4+1} = g(x) \Rightarrow g$ *is even*

b. $f(-x) = 4(-x)^3 + 5(-x) = -(4x^3 - 5x) = -f(x) \Rightarrow f$ is odd.

c. $f(-x) = (-x)^4 + 3(-x)^2 = x^4 + 3x^2 = f(x) \Rightarrow f$ is even

d. $h(-x) = \dfrac{1}{-x} = -\left(\dfrac{1}{x}\right) = -h(x) \Rightarrow h$ is odd.

4 a and b are functions; the others are not.

1.2.2 Power Functions

Power functions have many varieties. The definition and their general behaviors, when r is an integer or a fraction and when r is negative or positive, are given in the student textbook.

But, due to their variety, students may find them confusing. So, in order to clear their confusion they need to do as many examples as possible, through group work or home take exercises. To start discussing power functions, however, you can let the students do Activity 1.4. The purpose of this activity is to guide students to identify the functions as power and exponential functions.

Answers to Activity 1.4

a, b and d are power functions. But c and e are exponential functions.

Though students might differentiate power function and exponential function, for a function of type $f(x) = ax^r$ the behavior depends on the value of r. To identify the change in the behavior, you can group the students and let them do Group Work 1.3. The purpose of the Group Work is to help students identify the basic properties of power functions by themselves. The important properties to look for are domain, range, parity (whether they are odd, even or neither), symmetry (if the function has any) and also the regions where they are decreasing or increasing.

Here you need to note that any function of the form $y = a^x$ where a is fixed is exponential function and that of $y = ax^r$ where the base x is variable is power function.

Answers to Group Work 1.3

1. a. Domain of f = Range of f = $\mathbb{R}$

   b.

x	−2	−1	0	1	2
$f(x)$	−32	−4	0	4	32

   c.

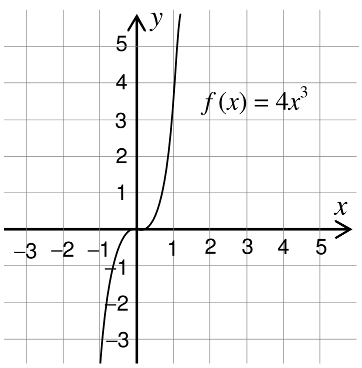


Figure 1.17

d. Since $f(-x) = 4(-x)^3 = -4x^3 = -f(x)$, then f is odd.

e. Since the function is odd, the graph of f is symmetric with respect to the origin.

2. a. Domain of $f = \mathbb{R}$

Range of $f = [0, \infty)$

b.

x	-2	-1	0	1	2
$f(x)$	16	4	0	4	16

c.

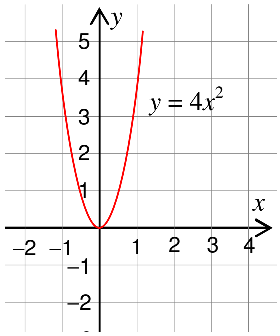


Figure 1.18

d. Since $f(-x) = 4(-x)^2 = 4x^2 = f(x)$, f is even.

e. Since f is even, the graph of f is symmetric with respect to y-axis

3. $f(x) = 2x^{-3}$

a. Domain of $f = \mathbb{R} \setminus \{0\}$

Range of $f = \mathbb{R} \setminus \{0\}$

b.

x	-2	-1	0	1	2
$f(x)$	$-\frac{1}{4}$	-2	$\nexists$	2	$\frac{1}{4}$

c.

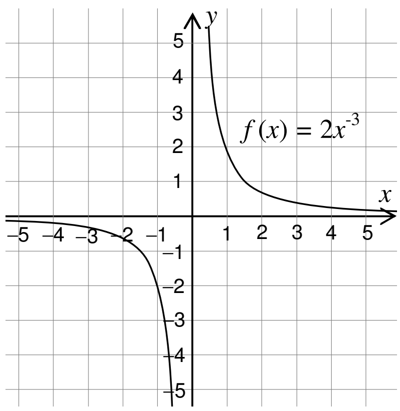


Figure 1.19

d. Since $f(-x) = 2(-x)^{-3} = -2x^{-3} = -f(x)$. $\Rightarrow f$ is odd

e. Since f is odd, its graph is symmetric with respect to the origin

4. $f(x) = 2x^{-2}$

a. Domain of $f = \mathbb{R} \setminus \{0\}$

Range of $f = (0, \infty)$

b.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{2}$	2	$\nexists$	2	$\frac{1}{2}$

c.

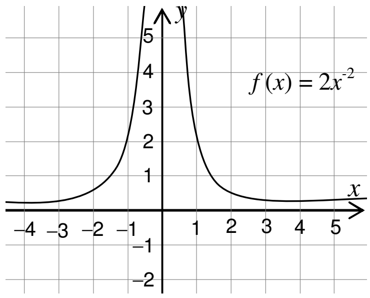


Figure 1.20

d. Since $f(-x) = 2(-x)^{-2} = 2x^{-2} = f(x)$, f is even.

e. Since f is even, the graph of f is symmetric with respect to the y-axis.

After the students do the Group Work and make some generalizations, it will be better to consider the behavior of power function where r is a rational number of the form $\frac{m}{n}$, where m and n are integers, with $n \neq 0$. Make charts of typical functions with their important behaviors listed and post them on the wall, for students to review as they go through the group work. Here, it is worth to note that students need to be reminded what it is and how they identify "inflection point". For further investigation of the domain, range, parities, symmetry and other related parameters, you can let the students discuss Activity 1.5.

Answers to Activity 1.5

a. $f(x) = x^{\frac{m}{n}}$, m even, n odd (Figure 1.9)

Specific example $f(x) = x^{\frac{2}{5}}$

1. Domain = $\mathbb{R}$, Range = $[0, \infty)$
2. f is even
3. symmetric with respect to the y-axis
4. f is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

b. $f(x) = x^{\frac{m}{n}}$, m odd, n even (Figure 1.10)

Specific example: $f(x) = x^{\frac{5}{2}}$

1. Domain = $[0, \infty)$; Range = $[0, \infty)$
2. f is neither odd nor even
3. it has no symmetry
4. f is increasing on $[0, \infty)$

c. $f(x) = x^{\frac{-1}{n}}$, n odd (Figure 1.11)

Specific example: $f(x) = x^{\frac{-1}{7}}$

1. Domain = $\mathbb{R}\setminus\{0\}$; Range = $\mathbb{R}\setminus\{0\}$
2. f is odd
3. symmetric with respect to the origin
4. f is decreasing on $(-\infty, 0) \cup (0, \infty)$

d. $f(x) = x^{\frac{-1}{n}}$, n even (Figure 1.12)

Specific example: $f(x) = x^{\frac{-1}{4}}$

1. Domain = $(0, \infty)$; Range = $(0, \infty)$
2. f is neither odd nor even
3. has no symmetry
4. f is decreasing on $(0, \infty)$

e. $f(x) = x^{-\frac{m}{n}}$, m even, n odd (Figure 1.13)

Specific example: $f(x) = x^{\frac{-2}{5}}$

1. Domain = $\mathbb{R}\setminus\{0\}$, Range = $(0, \infty)$
2. f is even
3. symmetric with respect to the y = axis
4. f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

Assessment

You can assess the students' understanding of power functions by using different approaches. You can give them homework and check their work.

Answers to Exercise 1.4

1. b and c are the only power functions because these are the only ones that can be written in the form of $f(x) = ax^r$.

2. a. $\mathbb{R}$ b. $[0, \infty)$ c. $\mathbb{R} \setminus \{0\}$ d. $(0, \infty)$

3.

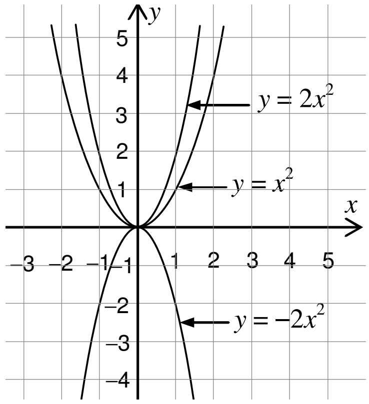


Figure 1.21

4. $a = 1$

5. a. $f(x) = \frac{a}{x}$; Domain = $\mathbb{R}\setminus\{0\}$; Range = $\mathbb{R}\setminus\{0\}$

   b. The graph is symmetrical w.r.t the origin.

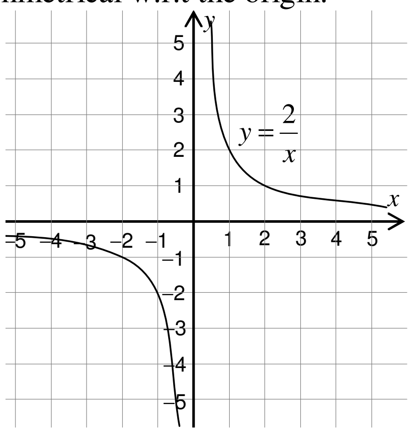


Figure 1.22

1.2.3 Absolute Value (Modulus) Function

Absolute values are essential in mathematics. The students have learned about absolute values in grade 9. Before giving the formal definition of absolute value function, it will be helpful to assess their prior knowledge and take any remedial measure if needed. After, ensuring the backlog of the students, you can guide them to do Activity 1.6 for the purpose of revising absolute values of numbers.

Answers to Activity 1.6

The purpose of this Activity is to determine the absolute value of a number.

a. 2   b. 3   c. 0   d. 6.014

It is also good to help the students recall some of the properties of absolute value and derive a few of them. The derivation of the basic properties are based on the fact that $|x| = \sqrt{x^2}$. Using this relation, discuss with the students how to find modulus of a number using calculators.

When the students see the tip that is helpful to derive properties of absolute values, you can give them Activity 1.7 to let them see how absolute values of a number and its opposite are equal.

Answers to Activity 1.7

1. a. $|-3.5| = |3.5| = 3.5$ b. $|4.213| = |-4.213| = 4.213$

   c. $|x| = |-x|$

2. a. i. $|xy| = |2.4 \times 3| = |7.2| = 7.2$

   $|x|\,|y| = |2.4| \times |3| = 2.4 \times 3 = 7.2$

   ii. $|xy| = |-6 \times 4| = |-24| = 24$

   $|x|\,|y| = |-6|\,|4| = 6 \times 4 = 24$

   b. $|xy| = |x|\,|y|$, for each $x, y \in \mathbb{R}$

By taking the presentations of Activity 1.7 and other numerous examples, help students to establish

i. $|x| = |-x|$ ii. $|x| \geq 0$ iii. $|xy| = |x|\,|y|$

iv. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, $y \neq 0$ v. $|x| = a \Leftrightarrow x = a$ or $x = -a$, if $a > 0$.

You can assign the proofs of the above facts to fast learners. (They can base their proofs on the fact $|x| = \sqrt{x^2}$)

Once they see the absolute values of numbers through examples and their properties, it will be better to give them the formal definition of absolute value. Afterwards, they can try to discuss absolute value function and its domain, range and graph.

Assessment

You can give various exercises of absolute values for the purpose of assessing your students.

Answers to Exercise 1.5

1. a. 13 b. 25 c. $\frac{4}{5}$

2. a. $\{-4, 4\}$ b. $\varnothing$ c. $\{-\frac{1}{3}\}$ d. $\left\{-2, \frac{4}{3}\right\}$

3. a. $\mathbb{R}$ b. $\mathbb{R}$ c. $\mathbb{R}\backslash\{0\}$ d. $\mathbb{R}$

4. a. c.

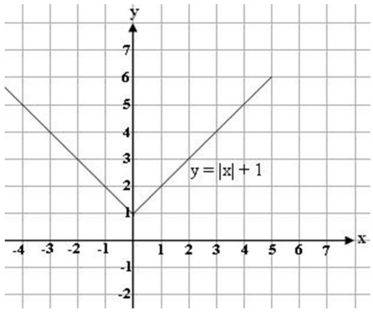


Figure 1.23

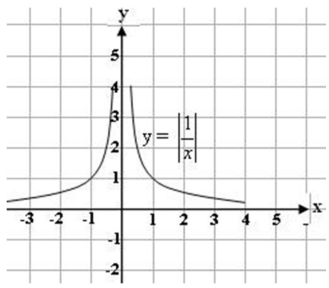


Figure 1.24

1.2.4 Signum Function

The students may not have been familiar with signum function. Before you start the lesson, you can let the students do Activity 1.8 so that they can revise functions defined differently at different intervals (subsets of the domain). This Activity will guide them to easily grasp the idea of signum function.

Answers to Activity 1.8

a. $\mathbb{R}$ b. $\{-2, 3\}$

c.

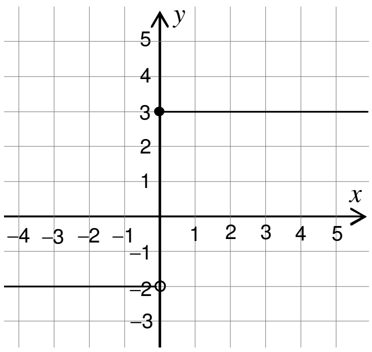


Figure 1.25

When the students discuss this activity, you can give instances where piece-wise defined functions that are similar to the function in the activity are treated. The following example dealing with the intervals and income tax in each interval can help you present the application of piece-wise defined function.

N.B. You may not let your students know each detail of this function. But, you give the highlights of the function since it is one of our practical applications.

Example: Income tax of a salary.

Income tax in Birr

$$f(x) = \begin{cases} 0 & 0 \le x \le 150 \\ (0.1)\,x & 151 \le x \le 650 \\ 0.15x & 651 \le x \le 1400 \\ 0.2x & 1401 \le x \le 2350 \\ 0.25x & 2351 \le x \le 3350 \\ 0.3x & 3551 \le x \le 5000 \\ 0.35x & x \ge 5001 \end{cases}$$

Then, discuss the signum function $f(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases}$

Discuss that f is related with the modulus (absolute) function. The graph is different from any graph the students have experienced so far. So, see to it that the students to pay attention to the jump discontinuity at $x = 0$. The break in the graph at $x = 0$ is called a jump (probably because anything that moves from left to right along the path given by $f(x)$ has to jump up to move from the part of the graph on the left of the y-axis to the part of the graph on the right of the y-axis). Pay attention to the facts that (0, –1) and (0, 1) do not belongs to the graph (shown by hollow circle) and that (0, 0) belongs to the graph (shown by filled-in small circle).

For talented students, you can give an exercise that applies such a function. First give the above income tax function where x is the gross salary. The pension tax function is given by $g(x) = 0.04x$, $x > 150$. If the gross salary of an employee is Birr 2000, ask them to find the amount of the net salary of the employee so that they can see the application of piece-wise defined functions.

Solution: First notice that the pension tax and income tax are calculated separately from the gross salary.

First calculate income tax as:

Income tax = $(650-150) \times 0.1 + (1400-650) \times 0.15 + (2000-1400) \times 0.2$

= Birr 282.50.

Second find pension tax as:

Pension tax = (0.04×2000) = Birr 80.00

Total deduction = Income tax + Pension tax = 282.50 + 80.00 = Birr 362.50

Therefore, the Net salary = Gross salary – Total deduction

= 2000 – 362.5 = Birr 1637.50.

Assessment

You can use exercise 1.6 for students to practice functions that consist of signum function and their graphs. You can let them do this exercise as homework and check their work.

Answers to Exercise 1.6

1. $y = x + \text{sgn}(x)$

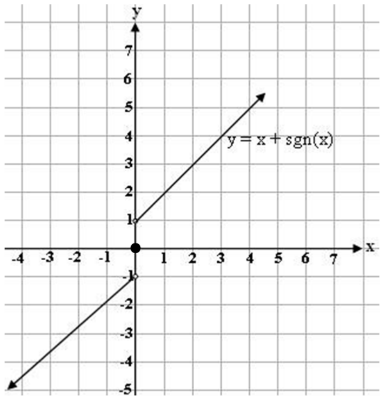


Figure 1.26

Domain = $\mathbb{R}$; Range = $(-\infty, -1) \cup \{0\} \cup (1, \infty)$

2. $f(x) = x\ \text{sgn}(x) = |x|$.

3. $y = x^2\ \text{sgn}(x)$

   Domain = $\mathbb{R}$;

   Range = $\mathbb{R}$

4. Domain = $\mathbb{R}$;

   Range = $[0, \infty)$

   Symmetry = y-axis

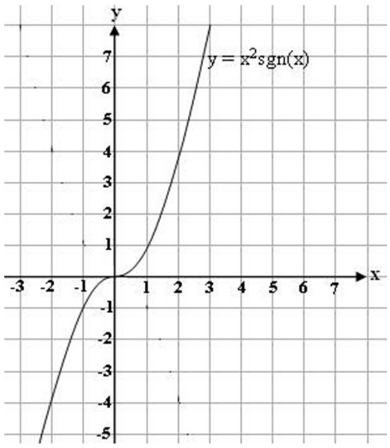


Figure 1.27

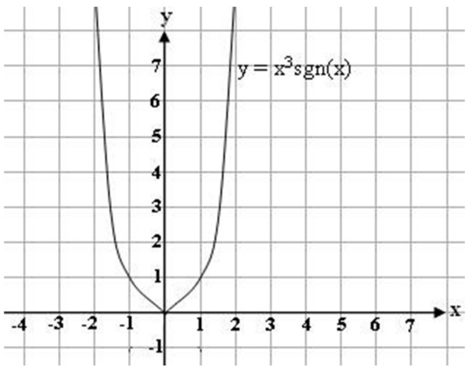


Figure 1.28

5. a. odd b. even

1.2.5 The Greatest Integer (floor) Function

Once the students have captured the ideas of piece-wise functions in general and signum function in particular, they can see some other functions that have similar nature with signum function but with a different level. To this end, the students need to discuss the floor function. But, before the students draw the graph of the floor function, let them do the examples on the textbook and give them as many examples as possible so that they can state the definition by themselves. Students need assistance in sketching the graph.

In order to help the students sketch the graph of floor function, you can let them do Activity 1.9.

Answers to Activity 1.9

1. a. $f(-3) = f(-2.7) = f(-2.5) = f(-2.1) = f(-2.01) = -3$

   b. $f(x) = -3$

   c.

x	$-3 \le x < -2$	$-2 \le x < -1$	$-1 \le x < 0$	$0 \le x < 1$	$1 \le x < 2$	$2 \le x < 3$
$f(x)$	-3	-2	-1	0	1	2

2.

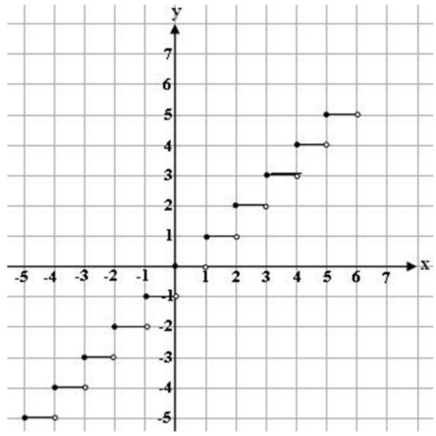


Figure 1.29

After doing the Activity, the students need to give an explanation about floor functions. Notice that for each $n \in \mathbb{Z}$, if $n \le x < n + 1$, then $f(x) = n$, and (n, n) is part of the graph (shown by a filled-in small circle), while $(n + 1, n)$ is not part of the graph (shown by a hollow small circle).

Domain = $\mathbb{R}$ while Range = $\mathbb{Z}$.

So $f : \mathbb{R} \to \mathbb{Z}$. The graph has infinite jumps.

Assessment

By giving exercises to sketch graphs and determine domain and range of different floor functions, you can assess your students.

Answers to Exercise 1.7

1. a. 3 b. –22 c. 21 d. 0

2. i. a. $f(4.25 + 6) = f(10.25) = 10$ and

$f(4.25) + 6 = 4 + 6 = 10$

$\Rightarrow f(4.25 + 6) = f(4.25) + 6$

b. $f(-3.21 + 7) = f(3.79) = 3$ and

$f(-3.21) + 7 = -4 + 7 = 3$

$\Rightarrow f(-3.21 + 7) = f(-3.21) + 7$

c. $f(8 + (-11)) = f(-3) = -3$

$f(8) + (-11) = 8 + (-11) = -3$

$\Rightarrow f(8+(-11)) = f(8) + (-11)$

ii. a. $f(4.25) + f(6.32) = 4+6 = 10$

$f(4.25+6.32) = f(10.57) = 10$

$4.25+ 6.32 =10.57$

$\Rightarrow f(4.25)+f(6.32) \le f(4.25+6.32) \le 4.25+6.32.$

b and c can be done similarly

iii. a. $f(2.5) = 2$, $x = 2.5$, $f(2.5) + 1 = 2+1=3$

$\Rightarrow f(2.5) \le 2.5 \le f(2.5) + 1$

b and c can be done similarly

3. a. $f(x) \le x < f(x) + 1 \Rightarrow \lfloor x \rfloor \le x < \lfloor x \rfloor + 1$

$\Rightarrow \lfloor x \rfloor - \lfloor x \rfloor \le x - \lfloor x \rfloor < \lfloor x \rfloor + 1 - \lfloor x \rfloor \Rightarrow 0 \le x - \lfloor x \rfloor < 1$

$\Rightarrow 0 \le a < 1$

b. $a = x - \lfloor x \rfloor \Rightarrow x = \lfloor x \rfloor + a$, $0 \le a < 1$.

c. $f(x + k) = f(\lfloor x \rfloor + a + k) = f(\lfloor x \rfloor + k + a)$

$= [\lfloor x \rfloor + k + a]$, where $\lfloor x \rfloor + k \in \mathbb{Z}$ and $0 \le a < 1$.

$= \lfloor x \rfloor + k = f(x) + k$

1.3 CLASSIFICATION OF FUNCTIONS

Periods Allotted: 2 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define one-to-one function.*
- *identify functions which are one-to-one.*
- *define onto function.*
- *identify functions which are onto.*
- *identify one-to-one correspondence.*

Vocabulary: One-to-one, Onto, One-to-one correspondence

Introduction

There are various classifications of functions in mathematical studies. But this unit is devoted to the study of one-to-one and onto functions.

Teaching Notes

Students are aware of a function. Yet functions can be either one-to-one or many-to-one. In this sub-unit, the students are required to study one-to-one functions so that they will be able to understand its essence in studying inverse of functions.

1.3.1 One-to-one Function

Before stating the definition of one-to-one function, it is advisable to enable students identify different types of functions and try to select those that are one-to-one. For this purpose, you can give them Activity 1.10.

Answers to Activity 1.10

f is not one-to-one because b and c are both mapped to 3 whereas g is one-to-one.

After that, you can give the definition of one-to-one function. Give examples in which the functions are defined by rules as well as by Venn diagrams. In addition, give examples of one-to-one functions which are not numeric.

Example: $f = \{(x, y)$: x is a student and y is his school identity number$\}$ or

$f = \{(x, y)$: x is a country and y is its president$\}$.

After discussing this, you may assess how the students try to identify whether a function is one-to-one. For this purpose, they may try to use the formal definition. However, relying on the formal definition alone makes it difficult to check the one-to-oneness of functions.

When the students are given a function f which is not one-to-one, they should be able to show the case by producing $x_1, x_2 \in$ Domain of f such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Example: If $f(x) = x^2$ then $-2, 2 \in$ Domain of f such that $-2 \neq 2$ and $f(-2) = f(2)$ implying $f(x) = x^2$ is not one-to-one.

It is also possible to use a horizontal line test to check whether a function is one-to-one. However, the horizontal line test works by using the graphs of real-valued functions. Therefore, students need not use this for all types of functions since it will require them to draw the graphs of the functions.

Assessment

By giving different types of functions, let the students try to identify those that are one-to-one functions.

Answers to Exercise 1.8

1. a, e, and f are one-to-one.
2. If $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2)$

$$\Rightarrow \frac{ax_1 + b}{cx_1 + d} = \frac{ax_2 + b}{cx_2 + d}$$

$$\Rightarrow (ax_1 + b)(cx_2 + d) = (ax_2 + b)(cx_1 + d)$$

$$\Rightarrow acx_1x_2 + adx_1 + bcx_2 + bd = acx_1x_2 + adx_2 + bcx_1 + bd$$

$$\Rightarrow (ad - bc)x_1 = (ad - bc)x_2$$

$$\Rightarrow x_1 = x_2 \text{, because } ad - bc \neq 0$$

Therefore, f is one-to-one

1.3.2 Onto Function

Once the students have discussed one-to-one functions, it will be better for the students to tell the condition in which a function will be onto. You can do this through question and answer. After the students try to explain these conditions, you can discuss their answers and give them the formal definition of onto functions.

Given a function f: A $\to$ B, f is onto if and only if Range f = B. This means, for any $y \in$ B, there exists $x \in$ A such that $y = f(x)$.

To enrich their understanding, you can give the students examples first by use of Venn diagrams and then other simple examples. At this moment, you can give students the chance to discuss how they check whether a function is onto. Finally, you can present the following to show f is onto. When f is given by a rule from A to B, to show that f is onto check the following. If for any $y \in$ B, there exists $x \in$ A such that $y = f(x)$, then f is onto.

Example: Let $f:\mathbb{R}\longrightarrow\mathbb{R}$, given by $f(x) = 9x + 5$.

Now $y = 9x + 5 \Rightarrow x = \dfrac{y-5}{9}$

From this we see that for any $y\in\mathbb{R}$, $x = \dfrac{y-5}{9} \in \mathbb{R}$ and $f\left(\dfrac{y-5}{9}\right) = y$. Thus f is onto.

But $f:\mathbb{R}\to\mathbb{R}$, given by $f(x) = x^4$ is not onto, because $y = x^4 \Rightarrow x = \sqrt[4]{y}$, and x is defined only when $y \geq 0$. This means that if we select negative $y\in\mathbb{R}$ then there is no $x\in\mathbb{R}$ such that $f(x) = y$ i.e., x is not defined for negative y.

Assessment

You can give various functions and ask the students to identify whether each is onto. They also need to give reasons for their answers. You can also use Exercise 1.9 for the purpose of assessment.

Answers to Exercise 1.9

1. a, c, d, are onto
2. a. $[2, \infty)$ b. $[5, \infty)$ c. $[0, \infty)$ d. $(-\infty, 1]$
3. a. $f(x_1) = f(x_2) \Rightarrow \dfrac{3x_1+1}{5} = \dfrac{3x_2+1}{5} \Rightarrow 3x_1 = 3x_2$

   $\Rightarrow x_1 = x_2$, for any $x_1, x_2 \in \mathbb{R} \Rightarrow f$ is one-to-one.

   For any $y\in\mathbb{R}$, $y = \dfrac{3x+1}{5}$ gives $x = \dfrac{5y-1}{3} \Rightarrow$, f is onto.

   Therefore, f is a one-to-one correspondence.

   b. Let $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

   $\Rightarrow \sqrt{x_1} = \sqrt{x_2} \Rightarrow x_1 = x_2$, by squaring.

   $\Rightarrow$, f is one-to-one

   For any $y\in[0, \infty)$, $y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow x \geq 0$, and

   $f(x) = f(y^2) = \sqrt{y^2} = |y| = y$ (because $y\in[0, \infty)$)

   $\Rightarrow f$ is onto

   Therefore, f is a one-to-one correspondence.

   c. For any $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2)$

   $\Rightarrow 5^{x_1} = 5^{x_2} \Rightarrow \log_5 x_1 = \log_5 x_2 \Rightarrow x_1 = x_2$

$\Rightarrow$ f is one-to-one and, for any $y \in [0, \infty)$, $y = 5^x$

$\Rightarrow \quad \log_5 y = \log_5 5^x \quad \Rightarrow \quad x = \log_5 y \in \mathbb{R}$. $\Rightarrow f$ is onto

Therefore, f is a one-to-one correspondence.

d. Let $x_1, x_2 \in [1, \infty)$, $f(x_1) = f(x_2)$

$\Rightarrow \quad (x_1 - 1)^2 + 1 = (x_2 - 1)^2 + 1 \quad \Rightarrow \quad (x_1 - 1)^2 = (x_2 - 1)^2$

$\Rightarrow \quad \sqrt{(x_1 - 1)^2} = \sqrt{(x_2 - 1)^2} \quad \Rightarrow |x_1 - 1| = |x_2 - 1|$

$\Rightarrow \quad x_1 - 1 = x_2 - 1$, $(x_1 \geq 1,\ x_2 \geq 1) \Rightarrow x_1 = x_2$

Therefore, f is one-to-one

Also $(x - 1)^2 \geq 0$, for any $x \in [1, \infty) \Rightarrow (x - 1)^2 + 1 \geq 1$

$\Rightarrow \quad f(x) \geq 1,\ \forall\ x \in [1, \infty) \qquad \Rightarrow$ Range $f = [1, \infty) \neq [0, \infty)$

$\Rightarrow$, f is not onto.

Therefore, f is not a one-to-one correspondence.

4. (Possible solutions)

a.

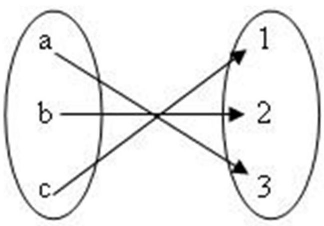


Figure 1.30

b. $f(x) = -x$

c. $f(x) = 3x + 2$

1.4 COMPOSITION OF FUNCTIONS

Periods Allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define the composition of functions.*
- *determine the composite function given the component functions.*
- *determine the domain and the range of a composite function of two given functions.*

Vocabulary: Function, Combination, Composition

Introduction

It is expected that the students have background about combinations of real valued functions. For this purpose it is advisable to revise a little about combinations and get into compositions of functions which is the main issue of this subtopic.

Teaching Notes

In discussing composition of functions, you can start by considering functions f and g such that

$f: A \to B$ and $g: B \to C$, with Range of $f \subseteq$ B.

Example 1

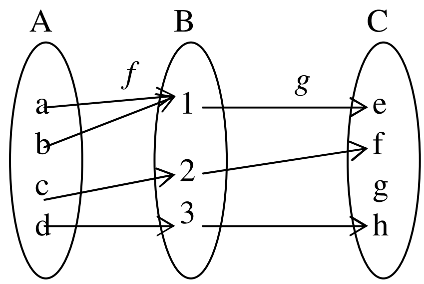


Figure 1.31

Example 2 Let $f: \mathbb{R}\setminus\{0\} \to (0,\infty)$ given by $f(x) = x^{-2}$ and

$g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \log x$

Then $gof: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is given by $\log\left(x^{-2}\right) = -2\log x$

Here Domain of gof = Domain of g.

After demonstrating the examples, you need to introduce the following.

1. When we think of composition of functions f : A $\to$ B and g: C $\to$D , it is required that $B \cap C \neq \varnothing$

   i.e., Range of $f \cap$ Domain of $g \neq \varnothing$.

   In case range of $f \cap$ domain of $g = \varnothing$, $gof(x)$ is not defined. i.e $gof = \varnothing$.
2. In all cases, Domain of $gof \subseteq$ Domain of f.

Right after the students have become able to determine the forward approach in finding composition of functions from their component ones, it may also be of interest to determine the component functions from another function that can be taken as a composition. This is where majority of students may find it difficult to determine how they can identify the component functions of a composite function.

Example: The composite function $f(x)=\log\sqrt{x^2+1}$ has three component functions in it.

Let $h(x) = x^2 + 1$, $g(x) = \sqrt{x}$, $k(x) = \log x$. If you ask your students the question "Which of the following is correct?"

$f = kogoh$ or $f = kohog$?

or ask them to express f as a composition of the functions g, h and k, you may get different replies.

When they try to determine the composition of these functions, they may not have a guiding principle. However, it is possible to assist them with the use of calculators in determining the order of component functions h, g and k whose composition will be f. For this purpose, ask the students to find the value of $f(x)$ at a given point a and narrate the steps they followed to calculate this value.

Students usually answer that they calculate $a^2 + 1$ first, then take the square root of $a^2 + 1$ and then take the logarithm of the result. With this step narration, you can relate that in finding the composition function the most inner function must be $a^2 + 1$, then square root of $a^2 + 1$,and finally the logarithm of the square root of $a^2 + 1$.

Hence, the order in the composition will be to get $f(x) = \log\sqrt{x^2+1} = (kogoh)(x)$. To help the students practice the ideas outlined above, you can give them Activity 1.11 as class work or homework, and some more examples on finding the component functions in a composition of functions.

Answers to Activity 1.11

In this activity, students will find composition of functions and then domain and range of composition of functions.

1. a. $(fog)(x) = \sqrt{x}+1$; $(gof)(x) = \sqrt{x+1}$

   b. i. $[0, \infty)$ ii. $[-1, \infty)$ iii. $[1, \infty)$ iv. $[0, \infty)$

2. a. $fog(x) = |x|^2 - 1$; $gof(x) = |x^2 - 1|$

   b. Domain $gof = \mathbb{R}$

   c.

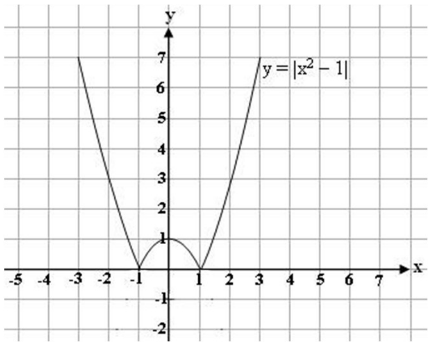
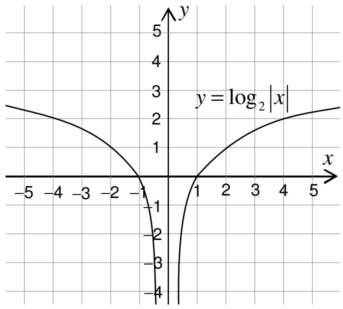


Figure 1.32

3. a. $fog(x) = \log_2|x|$; $gof(x) = |\log_2 x|$

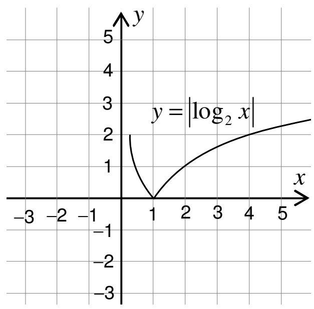
   b. Domain $fog = \mathbb{R}\setminus\{0\}$; Domain $gof = (0, \infty)$

   c. d.



$y = \log_2 |x|$

Figure 1.33



$y = |\log_2 x|$

Figure 1.34

For talented students, you may give functions which involve different types of functions and ask them either to find their composition or to find the component functions.

Example:

1. Let $f(x) = x^4 - 6x^2 + 6$, $g(x) = 2^{3x^2+1}$ and $h(x) = \sqrt{3x+1}$ then find $(fogoh)(x)$.

2. Let $f(x) = \sqrt{\log\left(\sin\frac{1}{x}\right) + \sqrt{2x+1}}$ then write f as a composition of at least three simpler (component) functions.

Solution:

1. $(fogoh)(x) = f(g(h(x))) = f(g(\sqrt{3x+1})) = f(2^{3\left(\sqrt{3x+1}\right)^2+1}) = f(2^{3(3x+1)+1}) = f(2^{9x+4})$

$$= \left(2^{9x+4}\right)^4 - 6\left(2^{9x+4}\right)^2 + 6 = 2^{36x+16} - 6\left(2^{18x+8}\right) + 6$$

2. Let $g(x) = \sqrt{x}$, $h(x) = \log x$, $k(x) = \sin x$, $l(x) = \frac{1}{x}$, $m(x) = \sqrt{x}$, $n(x) = 2x+1$,

then $f(x) = g\left(h(k(l(x))) + m(n(x)\right)$ or $f(x) = g\left(h(k(l(x))) + g(n(x)\right)$

Assessment

There are different ways that you can use to assess your students. However, approaches, such as giving a test/quiz or giving an assignment can be helpful.

Answers to Exercise 1.10

1. a. 3 b. 4 c. 9

2. a. –19 b. $\frac{61}{\sqrt{28}}$ c. $\left[-\frac{7}{3}, \infty\right)$ d. $\left[-\frac{7}{3}, \infty\right)$ e. $\left(-\frac{7}{3}, \infty\right)$

3. a. $(f+g)(x) = \frac{-x}{2x-1}$, $(f-g)(x) = \frac{11x}{2x-1}$

   $(fg)(x) = \frac{-30x^2}{(2x-1)^2}$, $\left(\frac{f}{g}\right)(x) = -\frac{5}{6}$

b. $(f + g)(x) = \dfrac{x+2}{\sqrt{x+1}}$ $(f - g)(x) = \dfrac{x}{\sqrt{x+1}}$,

$(fg)(x) = 1$, $\left(\dfrac{f}{g}\right)(x) = x+1,\ x > -1$

4. a. $(fog)(3) = 46$ b. $(fof)(0) = -8$ c. $(gof)(-5) = -84$
d. $(gog)(-7) = -169$ e. $(fogof)(2) = 61$

5. a. i. $(fog)(x) = 2(4x+2) - 1 = 8x + 3$;
ii. $(gof)(x) = 4(2x-1) + 2 = 8x - 2$
iii. $(fof)(x) = f(2x-1) = 2(2x-1) - 1 = 4x - 3$
iv. $(gog)(x) = g(4x+2) = 4(4x+2) + 2 = 16x + 10$

b. i. $(fog)(x) = x$ ii. $(gof)(x) = \sqrt{x^2} = |x|$
iii. $(fof)(x) = x^4$ iv. $(gog)(x) = \sqrt[4]{x}$

c. i. $(fog)(x) = 1 - 5|2x+3|$ ii. $(gof)(x) = |5 - 10x|$
iii. $(fof)(x) = 25x - 4$ iv. $(gog)(x) = |2|2x+3|+3|$

d. i. $(fog)(x) = 3(2^x)$ ii. $(gof)(x) = 2^{3x}$
iii. $(fof)(x) = 3(3x) = 9x$ iv. $(gog)(x) = 2^{2^x}$

6. a. $l(x) = \sqrt{3x} = hof(x)$ b. $k(x) = fog(x)$
c. $t(x) = hog(x)$

7. a. $g(x) = 3x + 1$, $h(x) = \sqrt{x}$, so, $f = (hog)(x)$
b. $g(x) = x^2$, $h(x) = 16x - 3$ and $f = (hog)(x)$ or
$g(x) = 4x$, $h(x) = x^2 - 3$ and $f = (hog)(x)$
c. $g(x) = 3x^2 + 1$, $h(x) = 2^x$ and $f = (hog)(x)$
d. $g(x) = 2^x$, $h(x) = 5x^2 + 3$ and $f = (hog(x)$
e. $g(x) = x^2$, $h(x) = x^2 - 6x + 6$ and $f = (hog)(x)$

8. $(fog)(x) = 4(3x + k) + 1 = 12x + 4k + 1$ and
$(gof)(x) = g(4x+1) = 3(4x+1) + k = 12x + 3 + k$
Now $(fog)(x) = (gof)(x) \Rightarrow 12x + 4k + 1 = 12x + 3 + k$

$$\Rightarrow 3k = 2 \Rightarrow k = \frac{2}{3}$$

9. $(gof)(x) = g(ax + b) = x$
Take $g(x) = cx + d$ (you get the same function if you consider
$g(x) = cx^2 + dx + e$)
Then $(gof)(x) = c(ax + b) + d = acx + bc + d$
Now $(gof)(x) = x \Rightarrow ac\,x + bc + d = x \Rightarrow$ $ac = 1$ and $bc + d = 0$

$$\Rightarrow c = \frac{1}{a}(a \neq 0) \text{ and } d = -bc = \frac{-b}{a}$$

Thus $g(x) = \dfrac{1}{a}x - \dfrac{b}{a}$.

10. $(fog)(x) = f(g(x)) = (2x+3)^4$ and $gof(x) = g(f(x)) = 2x^4 + 3$
Thus $(fog)(x) \neq (gof)(x)$. In general, this is the case.

1.5 INVERSE FUNCTIONS AND THEIR GRAPHS

Periods Allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define inverse function.*
- *describe the condition for the existence of the inverse function.*
- *determine the inverse function of an invertible function.*
- *determine whether or not two given functions are inverses of each other.*
- *sketch the graph of the inverse of a function.*
- *determine the domain and range of the inverse of a given function.*

Vocabulary: Inverse, Invertibility, Identity function

Introduction

At the beginning of this unit, the students have learned how to find inverses of relations. Since functions are relations, finding their inverses are not far from the ideas of finding inverses of relations. However, there are major questions that need to be addressed when dealing with inverses of functions. These will be addressed in this sub-topic.

Teaching Notes

In dealing with inverse functions, there are points that need focus. So as to relate these points and proceed to discuss inverses of functions, you can start the lesson by asking the following major questions.

a. Given a function f, is f^{-1} a function?

   f^{-1} can be a function only if f is one-to-one. In this case,

   $(f \circ f^{-1})(x) = x$, for all $x \in$ Domain f^{-1} and $(f^{-1} \circ f)(x) = x$, for all $x \in$ Domain f

b. If f is invertible, how can we find its inverse?

c. Can you recognize the graph of an invertible function?

Accordingly, revise what the students did about inverse of relations. Then lead them to try to find inverses of functions of type:

i. one-to-one

ii. Many-to-one and let them discuss if each inverse is a function.

For this purpose you can give Activity 1.12.

Answers to Activity 1.12

a. $f^{-1} = \left\{(x, y) : y = \dfrac{x+4}{3}\right\}$; f^{-1} is a function.

b. $R^{-1} = \left\{(x, y) : y \leq \dfrac{x+4}{3}\right\}$; f^{-1} is not a function.

c $f^{-1} = \{(x, y) : x = y^2\}$; f^{-1} is not a function.

d. $g^{-1} = \{(x, y) : y = 2^x\}$. g^{-1} is a function.

This leads them to conclude that f^{-1} is a function if and only if f is one-to-one. At this level, the students need to know that if a function f has an inverse, then it is called *invertible*.

When the students are aware of functions and their inverses, it will be recommended to let them know how they can find an inverse of a given function. For this purpose you can give the following steps and give them as many examples as possible to enable them practice the steps.

Algebraic method of finding f^{-1}:

a. Write the equation $y = f(x)$

b. Interchange x and y wherever they occur in the equation found in step a).

c. Solve the equation found in step b) for y in terms of x, to get $y = f^{-1}(x)$

Example Let $f(x) = 1 - 2x^3$

Step a. Put $y = 1 - 2x^3$

Step b. $x = 1 - 2y^3$

Step c. $x = 1 - 2y^3 \Leftrightarrow 2y^3 = 1 - x \Leftrightarrow y^3 = \dfrac{1-x}{2} \Leftrightarrow y = \sqrt[3]{\dfrac{1-x}{2}}$

Hence $f^{-1}(x) = \sqrt[3]{\dfrac{1-x}{2}}$

At this moment, you can define an identity function which is its own inverse.

Cognizant of the fact that the students are already capable of finding inverse of functions, it is recommended that they should be given the chance to identify by themselves, how the graphs of functions and their inverses look like.

As one of their observation assist the students to identify that

Domain f^{-1} = Range f and Range f^{-1} = Domain f.

It is also of interest to help the students determine graphs of inverse functions from the graph of the functions themselves. You can let the students note that graphs of functions and their inverses are mirror images along the line $y = x$. To detect whether a function has an inverse or not from its graph, you help them realize that a horizontal line intersects the graph of the function at only one point. That is, a function f is invertible if and only if no horizontal straight line intersects its graph more than once.

To help the students realize how to draw the graphs of inverse functions and to determine the properties of inverse functions, you can let them do Activity 1.13.

Answers to Activity 1.13

a.

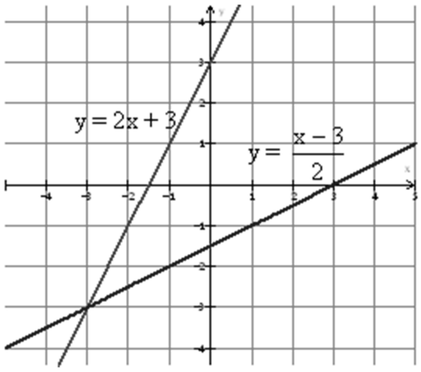


Figure 1.35

b.

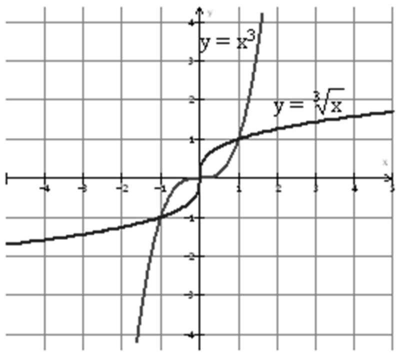


Figure 1.36

Assessment

Since this is an end of the unit, in addition to assessing students' understanding on inverses of functions, you can also give them a comprehensive evaluation for the unit. For this purpose, you may give them a test.

Answers to Exercise 1.11

1. a. $f^{-1}(x) = \frac{1}{2}3^x$; f^{-1} is a function

   b. $h^{-1}(x) = \dfrac{13-x}{5}$; h^{-1} is a function

   c. $g^{-1}(x) = (x - 1)^2$; g^{-1} is a function

   d. Take $k = \{(x, y): y = (x - 2)^2\}$

   Then $k^{-1} = \{(x, y): y = 2 \pm \sqrt{x}\}$; k^{-1} is not a function.

2. a. $\mathbb{R}$ b. $\mathbb{R}$ c. $[1, \infty)$ d. $[0, \infty)$

3. a. Let $f(x) = 3x + 2$, $g(x) = \dfrac{x-2}{3}$. Then $fog(x) = 3\left(\dfrac{x-2}{3}\right) + 2 = x$

   and $gof(x) = \dfrac{(3x + 2) - 2}{3} = x$. They are inverses of each other.

b. $fog\,(x) = \left(\sqrt[3]{x}\right)^3 = x$ and $gof\,(x) = \sqrt[3]{x^3} = x$

They are inverses of each other

c. They are not inverses of each other, because Range $f^{-1} \neq$ Domain g.

d. $fog\,(x) = \sqrt[3]{x^3 - 8 + 8} = x$ and $gof\,(x) = \left(\sqrt[3]{x + 8}\right)^3 - 8 = x$

They are inverses of each other

4. a. $f(x) = x^3$ is invertible.

   b. $g(x) = 4 - x^2$ is invertible if the domain is $[0, \infty)$ or $(-\infty, 0]$.

   c. $h\,(x) = \dfrac{-1}{3}x + 5$ is invertible.

   d. $f(x) = \log x^2$ is invertible when the domain is $(0, \infty)$ or $(-\infty, 0)$.

5. a and c are invertible

6. a.

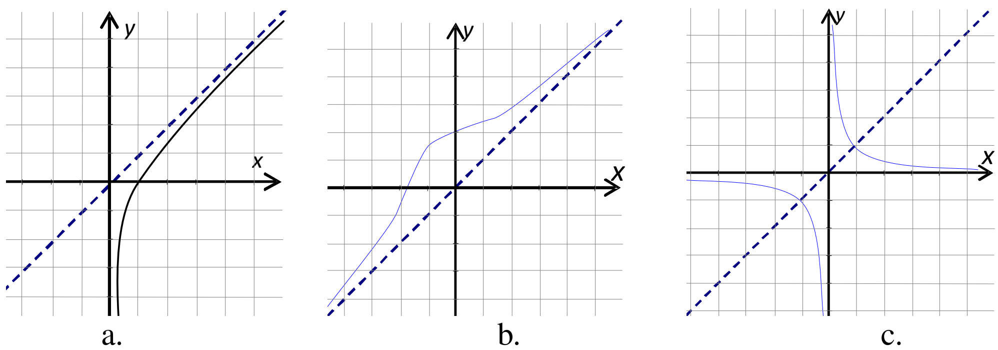


Figure 1.37

Answers to Review Exercises on Unit 1

1. a. $\{(-2, 2), (3, -3), (-4, -4)\}$; R^{-1} is a function.

   b. $\{(1, 2), (3, 2), (7, 2)\}$; R^{-1} is a function.

2. a. $f^{-1}(x) = \dfrac{x - 3}{2}$ (function)

   b. $f^{-1} = \{(x, y): x + 9 = y^2\}$(not a function)

   c. $f^{-1} = \left\{(x, y) : x = (y^2 - 9)^2\right\}$ (not a function)

   d. $f^{-1}(x) = 9x^2, x \geq 0$ (function)

3. $\mathbb{R}$

4. a. (0, 0), (–1, –1) and (1, 1) are all the intersection points of $y = x^5$ and $y = x^7$

   b. yes.

   c. i. $f(x) = 4x^3$ grows 4 times faster than $y = x^3$

      ii. The graph of f is shifted upwards by 4 units.

   d. $f(ab) = (ab)^3 = a^3b^3 = f(a)f(b)$. They are equal

   e. yes.

5. a. b.

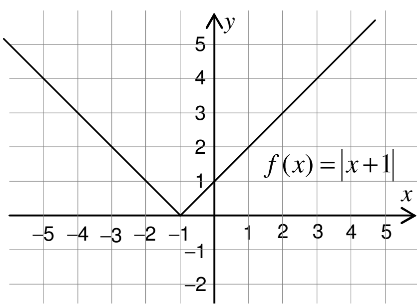


Figure 1.38

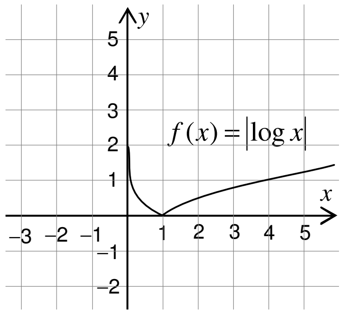


Figure 1.39

c.

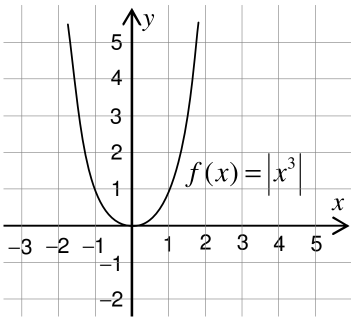


Figure 1.40

6. $\operatorname{sgn}(-x) = \begin{cases} -1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ 1 \text{ for } x < 0 \end{cases}$

$-\operatorname{sgn}(x) = -\begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases} = \begin{cases} -1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ 1 \text{ for } x < 0 \end{cases}$

$\Rightarrow \operatorname{sgn}(-x) = -\operatorname{sgn}(x)$

Therefore, $y = \operatorname{sgn} x$ is an odd function.

b. $h(-x) = \frac{1}{2}(\operatorname{sgn}(-x) + 1) = \frac{1}{2}(-\operatorname{sgn} x + 1)$

Thus $h(x) + h(-x) = \frac{1}{2}(\text{sgn } x + 1) + \frac{1}{2}(-\text{sgn } x + 1)$

$= \frac{1}{2}((\text{sgn } x + 1) + (-\text{sgn } x + 1)) = 1$

c. $h(x) = \frac{1}{2}(\text{sgn } x + 1) = \begin{cases} \frac{1}{2}(1+1) \text{ for } x > 0 \\ \frac{1}{2}(0+1) \text{ for } x = 0 \\ \frac{1}{2}(-1+1) \text{ for } x < 0 \end{cases} = \begin{cases} 1 \text{ for } x > 0 \\ \frac{1}{2} \text{ for } x = 0 \\ 0 \text{ for } x < 0 \end{cases}$

7. a. $f(-3.9 + (-16.4)) = f(-20.3) = -20$ and

$f(-3.9) + f(-16.4)) + 1 = -20$

$\Rightarrow f(-3.9 + (-16.4)) \le f(-3.9) + f(-16.4)) + 1$

b. $f(3.9 + (-16.4)) = f(-12.5) = -13$ and

$f(3.9) + f(-16.4)) + 1 = -13$

$\Rightarrow f(3.9 + (-16.4)) \le f(3.9) + f(-16.4)) + 1$

c and d can be done similarly.

8. Taking any value for x say $x = 3.75$.

$\Rightarrow \lfloor 2(3.75) \rfloor = \lfloor 3.75 \rfloor + \left\lfloor 3.75 + \frac{1}{2} \right\rfloor$

$\Rightarrow \lfloor 7.5 \rfloor = 3 + \lfloor 4.25 \rfloor$

$\Rightarrow 7 = 3 + 4$

9. a. $fog(x) = 1 + 2|x|$; $fof\ (x) = 1 + 2(1 + 2x) = 3 + 4x$

$gof\ (x) = |1 + 2x|$; $gog\ (x) = |x|$

b. $fog(x) = \log(3x + 1)$; $gof\ (x) = 3 \log x + 1$

$fof\ (x) = \log(\log x)$; $gog\ (x) = 3(3x + 1) + 1 = 9x + 4$

10. a. Domain $fog = \mathbb{R}$; Domain $gof = \mathbb{R}$

Domain $fof = \mathbb{R}$; Domain $gog = \mathbb{R}$

b. Domain $fog = \left(-\frac{1}{3}, \infty\right)$; Domain $gof = (0, \infty)$

Domain $fof = (1, \infty)$; Domain $gog = \mathbb{R}$

11. a. They are inverses of each other.

b. They are not inverses of each other.

UNIT 2 RATIONAL EXPRESSIONS AND RATIONAL FUNCTIONS

INTRODUCTION

In this unit, rational expressions and rational functions will be discussed. A rational expression is a quotient of two polynomials where the denominator is nonzero. Since students are familiar with polynomials, it may be useful to start the unit by revising the zeros of polynomials and domains of some familiar expressions and proceed to the domain of a rational expression by excluding the zeros of the denominator polynomials. Following this, the operations with rational expressions and decomposition into partial fractions will be discussed. Finally, students will be introduced to the concepts of rational equations, rational functions and their graphs.

Unit Outcomes

After completing this unit, students will be able to:

- *know methods and procedures in simplifying rational expressions.*
- *understand and develop efficient methods in solving rational equations and inequalities.*
- *know basic concepts and specific facts about rational functions.*

Suggested Teaching Aids in Unit 2

For this unit, several teaching aids can be used. Some of these include: coloured chalks (white board markers), chalk board, graphics software such as Geometers Sketchpad, Matlab, Mathematica, Tinkerplot, Fathom, etc (Free software may be found on the Internet), graphing calculators such as TI-83, or TI 84, etc, straight edged rulers and squared papers, graphed flip charts, etc.

2.1 SIMPLIFICATION OF RATIONAL EXPRESSIONS

Periods allotted: 4 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *define rational expression.*
- *identify the universal set of a given rational expression.*
- *show the simplified form and the necessary steps to simplify a given rational expression.*
- *perform the four fundamental operations on rational expressions.*
- *decompose rational expressions into sums of partial fractions.*

Vocabulary: Fractional expression, Rational expression, Universal set, Domain, Lowest term, Decomposition, Partial fraction, Proper rational fraction, Improper rational fraction

Introduction

This unit has three subunits: rational expression, rational equation and rational functions. To work on rational expressions, students need to realize that they will use the properties of addition, subtraction, multiplication and division of fractions. The subunit on simplification of rational expressions will be useful to express rational functions in simplest forms. For instance, knowing how to expand and simplify $\frac{x^2+4x}{5x^4+20x^3}$ will help students to observe later that, while the graph of $f(x)=\frac{x^2+4x}{5x^4+20x^3}$ has a hole at $x=-4$, it has vertical asymptote at $x=0$.

Teaching Notes

In this section, there are three topics. These are rational expressions, operations with rational expressions and decomposition of rational expressions into partial fractions. You may start the section by giving a list of different expressions which the students had seen in previous grades and making students discuss the domains and some properties of these expressions. For instance, ask students to identify the domains of polynomials, logarithmic and square root expressions. Then give some rational expressions like $\frac{3}{x}$, $\frac{2x}{x-7}$, . . . and ask students to give their domains.

For the purpose of revising and strengthening the ability of students in determining domain of an algebraic expression, you can let each student do Activity 2.1. A domain is the subset of $\mathbb{R}$ that makes a given expression meaningful. In the case of rational expressions, students have always to remember that the denominator of a rational expression cannot be zero. Before passing into discussing how students determine domain of rational functions, it will be useful to revise how to find domains of expressions and elementary functions which the students know. To help you discuss these, you can use Activity 2.1 which is meant to check how much the students are knowledgeable in finding domain of a function.

Answers to Activity 2.1

a. $\mathbb{R}$ b. $(-2, \infty)$ c. $\left(-\infty, \frac{1}{5}\right]$ d. $\mathbb{R} \setminus \{-5\}$

Let the students do the activity. In the process, clever students may find the question easier. You can ask these clever students questions like:
Find the domain of each of the following expressions.

a. $\log\sqrt{x-1}$ b. $\log_{x^2} 5$ c. $\frac{x^2+1}{\sqrt{x}}$ d. $\frac{x^3+x^2+1}{\left(x^2-1\right)(x+2)}$

whose answers are:
a. $x > 1$ b. $x \neq 0, \pm 1$ c. $x > 0$ d. $x \neq -2, \pm 1$

2.1.1 Rational Expressions

Introduce this lesson by giving the definition of a rational expression (together with its universal set) and providing additional examples. Ask students to evaluate rational expressions for given values of *x*. When simplifying rational expressions into lowest form, give emphasis to the universal set and cancel out common factors from the numerator and the denominator of a given rational expression.

Emphasize the fact that two expressions are equal only in their common domains (a common domain on which both are defined).

For Example, $\frac{5x^2-14x-3}{2x^2+x-21}$ has domain $\mathbb{R}\setminus\left\{-\frac{7}{2}, 3\right\}$, while $\frac{5x+1}{2x+7}$ has domain $\mathbb{R}\setminus\left\{-\frac{7}{2}\right\}$

This means, $\frac{5x^2-14x-3}{2x^2+x-21} = \frac{5x+1}{2x+7}$, only in $\mathbb{R}\setminus\left\{-\frac{7}{2}, 3\right\}$.

Therefore, when students simplify rational expressions, they should determine the universal set before they start simplifying. For this purpose, you can let the students do

Activity 2.2 in pairs. In this Activity, students will practice simplifying rational expressions to their lowest terms and finding domains of rational expressions.

Let the students answer the questions by discussing in pairs and ask some of the students to give their answers orally and loudly and write them on the board. Then guide the whole class to discuss the answers. Give corrections as necessary. When the students do the activity, you can round through the groups and identify the talents of the students so that you can regroup or prepare additional questions to support weaker students or engage gifted students in some more exercises.

Answers to Activity 2.2

a. $\{x : x \in \mathbb{R} \text{ and } x \neq 0\}$

b. $x^2+2x = x(x+2)$. Thus, $\dfrac{x^2+2x}{5x} = \dfrac{x(x+2)}{5x}$

c. $\dfrac{x^2+2x}{5x} = \dfrac{x(x+2)}{5x} = \dfrac{x+2}{5}$

d. When $x \neq 0$

e Given $\dfrac{9x^2-4}{9x^2+9x-10}$

i. $\left\{x : x \in \mathbb{R} \text{ and } x \neq \dfrac{-5}{2}, \dfrac{2}{3}\right\}$

ii. $\dfrac{(3x-2)(3x+2)}{(3x-2)(3x+5)}$

iii. $\dfrac{9x^2-4}{9x^2+9x-10} = \dfrac{3x+2}{3x+5}$

iv. Only when $x \neq -\dfrac{5}{3}, \dfrac{2}{3}$

You can give high achievers the following additional exercises.

1. Reduce to the lowest terms

   a. $\dfrac{cd-c^2}{c^2-d^2}$ b. $\dfrac{16x-32}{8y(x-2)-4(x-2)}$

2. Give the domain for each of the following

   a. $\dfrac{1+\frac{1}{x}}{x-\frac{1}{x}}$ b. $\dfrac{\frac{1}{x+y}-\frac{1}{x-y}}{\frac{2y}{x^2-y^2}}$

3. Simplify each of the following

   a. $\dfrac{1+\frac{1}{x}}{x-\frac{1}{x}}$ b. $\dfrac{\frac{1}{d}-\frac{1}{c}}{c^2-d^2}$ c. $\dfrac{\frac{1}{x+y}-\frac{1}{x-y}}{\frac{2y}{x^2-y^2}}$

Solution:

1. a. $\dfrac{c(d-c)}{(c-d)(c+d)} = \dfrac{-c}{c+d}; c \neq \pm d$ b. $\dfrac{16(x-2)}{4(x-2)(2y-1)} = \dfrac{4}{2y-1}; x \neq 2, y \neq \dfrac{1}{2}$

2. a. $\mathbb{R}\backslash\{0, 1\}$ b. $\{(x, y) : y \neq 0, x \neq \pm y\}$

3. a. $\frac{1}{x-1}; x \neq 0, 1$ b. $\frac{c-d}{cd(c-d)(c+d)} = \frac{1}{cd(c+d)}; c \neq \pm d, c, d \neq 0$

c. $$\frac{\frac{x-y-x-y}{(x+y)(x-y)}}{\frac{2y}{(x-y)(x+y)}} = -1; x \neq \pm y, y \neq 0$$

Assessment

You can give to the students various algebraic expressions and ask them to identify the rational expressions among them. You can also give them homework or class activities on determining the universal sets of rational expressions and on simplifying expressions.

Assessment should be on

1. Reducing to lowest terms.

   Example: Reduce to lowest terms

   a. $\frac{5x^2-14x-3}{2x^2+x-21}$ b. $\frac{x-3}{x^2+3x-18}$

2. Identifying rational expression

   Example: Choose the one that is a rational expression

   a. $\frac{x^2+\sqrt{x}+1}{x}$ b. $\frac{5x^2+5x}{\sqrt{x+1}}$ c. $\log\left(\frac{x+1}{x^3}\right)$ d. $\frac{x+4}{x^3+3x+2}$

   Solution:

1. a. $\frac{(5x+1)(x-3)}{(2x+7)(x-3)} = \frac{5x+1}{2x+7}; x \neq 3, -\frac{7}{2}$

   b. $\frac{(x-3)}{(x+6)(x-3)} = \frac{1}{x+6}; x \neq 3, -6$

2. d

Answers to Exercise 2.1

a. Since $4x = 0$ when $x = 0$, domain is $\{x \mid x \in \mathbb{R}, x \neq 0\} = \mathbb{R} \setminus \{0\}$. Thus

   $\frac{4x-12}{4x} = \frac{x-3}{x}$, for $x \neq 0$

b. The denominator $2x^2 + 5x - 12 = (2x - 3)(x + 4) = 0$ for $x = \frac{3}{2}$ and $x = -4$.

   Thus domain $= \mathbb{R} \setminus \left\{-4, \frac{3}{2}\right\}$.

   The numerator $6x^2 + 23x + 20 = (3x + 4)(2x + 5)$

   Thus $\frac{6x^2+23x+20}{2x^2+5x-12} = \frac{(3x+4)(2x+5)}{(2x-3)(x+4)}$

c. The domain is $\{x \mid x \in \mathbb{R}, x \neq -3\} = \mathbb{R} \setminus \{-3\}$ and $\frac{x^3+3x^2}{x+3} = \frac{x^2(x+3)}{x+3} = x^2$, for $x \neq -3$.

d. The denominator $x^4 + 3x^3 - 27x - 81 = (x-3)(x+3)(x^2+3x+9)$
Since $x^2 + 3x + 9 \neq 0$, for all $x \in \mathbb{R}$, the domain is $\mathbb{R} \setminus \{-3, 3\}$
Thus $\frac{x^3-27}{x^4+3x^3-27x-81} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)(x^2+3x+9)} = \frac{1}{x+3}$, for $x \neq 3, -3$

e. $\frac{x^2-5x+6}{3x^3-2x^2-8x} = \frac{(x-2)(x-3)}{x(x-2)(3x+4)} = \frac{x-3}{x(3x+4)}$, for $x \neq -\frac{4}{3}$, 0, 2

f. $\frac{x^4-8x}{3x^3-2x^2-8x} = \frac{x(x-2)(x^2+2x+4)}{x(x-2)(3x+4)} = \frac{x^2+2x+4}{3x+4}$, for $x \neq -\frac{4}{3}$, 0, 2.

2.1.2 Operations with Rational Expressions

You may start this lesson with a brief revision of operations on rational numbers. We use the rules of the operations of addition, subtraction, multiplication and division of rational numbers on rational expressions. In your discussion, emphasize on how to find the L.C.M. of the denominators of the expressions which should be factorized into prime polynomials. Let the students mention the rules for the operations alongside each step and mention the universal set at the beginning and the end of each example.

Since the students can simplify rational expressions, you can start this lesson by forming groups of students and letting them do Activity 2.3. The purpose of this activity is to help students revise addition and subtraction of rational numbers. Addition and subtraction of rational expressions are very closely related with addition and subtraction of rational numbers.

Answers to Activity 2.3

a. $\frac{12}{8} = \frac{3}{2}$ b. $\frac{19}{12}$ c. $\frac{7}{12}$ d. $\frac{3}{10}$

Assessment

For the purpose of assessing students understanding on using the operations and simplifying rational expressions, you can give them questions similar to those in Activity 2.3. You can also give them various algebraic expressions to simplify.

Answers to Exercise 2.2

a. $\frac{2x-3}{x^2+5x} - \frac{3x-5}{x^2+5x} = \frac{(2x-3)-(3x-5)}{x^2+5x} = \frac{-x+2}{x^2+5x}$, for $x \neq -5$, 0.

b. $\dfrac{x^2+3x-(2x+12)}{x^2+2x-15}=\dfrac{x^2+x-12}{x^2+2x-15}=\dfrac{(x\ +\ 4)(x-3)}{(x+5)(x-3)}=\dfrac{x+4}{x+5}$, for $x \neq -5, 3$

c. $\dfrac{12+x-2}{5x}=\dfrac{x\ +\ 10}{5x}$, for $x \neq 0$

d. $\dfrac{6y+11-(4y+4)}{4y^2+12y-7}=\dfrac{2y\ +\ 7}{(2y-1)\ (2y\ +\ 7)}=\dfrac{1}{2y-1}$, for $y \neq -\dfrac{7}{2}, \dfrac{1}{2}$

To simplify addition and subtraction of rational expressions, guide students to factorize the denominators of the expressions to be added and then find the LCM, using Activity 2.4 and example 6. Add some more examples if you find it necessary. It is good if the students practice good style of writing. Guide them to write the rule they have used for verifying each step. In Activity 2.4, the students will practice factoring the denominators of the rational expressions to be added and then finding the least common denominator. An additional purpose of this Activity is to lead the students to generalize the steps required in order to add and subtract rational expressions with unlike denominators. So, before you discuss the steps given in the student textbook, guide the students to reach the steps required.

Answers to Activity 2.4

a. $x^2 - 9 = (x - 3)\ (x + 3)$ and $x^2 + 11x + 24 = (x + 3)\ (x + 8)$.

Thus the domain of $\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}$ is $\mathbb{R} \setminus \{-8, -3, 3\}$

b The LCM of the denominators is $(x - 3)(x + 3)\ (x + 8)$

c $\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}=\dfrac{x-4}{(x-3)(x+3)}+\dfrac{x+2}{(x+3)(x+8)}$

$$=\frac{(x-4)(x+8)+(x+2)(x-3)}{(x-3)(x+3)(x+8)}$$

$$=\frac{2x^2+3x-38}{x^3+8x^2-9x-72}, \text{ for } x \neq -8, -3, 3$$

d Since there are no common factors, we leave the result as it is.

Therefore $\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}=\dfrac{2x^2+3x-38}{x^3+8x^2-9x-72}$, for $x \neq -8, -3, 3$

You can give high achievers problems on addition and subtraction of rational expressions with more than one variable.

Example: Find the LCD of the fractions, perform the operations and simplify each of the following.

a. $\dfrac{y}{y-z}-\dfrac{2y}{y+z}+\dfrac{3yz}{z^2-y^2}$

b. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(c-b)}-\dfrac{1}{(a-c)(b-c)}$

Solution:

a. $\dfrac{y^2+yz-2y^2+2yz+3yz}{(z-y)(z+y)}$

b. $\dfrac{b-c+(a-c)-(a-b)}{(a-b)(a-c)(b-c)}=\dfrac{2(b-c)}{(a-b)(a-c)(b-c)}=\dfrac{2}{(a-b)(a-c)};b\neq c$

Assessment

To evaluate students' understanding of the concepts discussed so far, give them problems on finding domain of an algebraic expression, and addition and subtraction of rational expressions. You can also use quiz, homework, etc to assess them.

Answers to Exercise 2.3

a. $\dfrac{25y^2}{5y-4}+\dfrac{16}{4-5y}=\dfrac{25y^2-16}{5y-4}=\dfrac{(5y-4)(5y+4)}{5y-4}=5y+4,\ \text{for } y\neq\dfrac{4}{5}$

b. $\dfrac{1}{x^2}+\dfrac{1}{x^2+x}=\dfrac{2x^2+x}{x^2(x^2+x)}=\dfrac{2x+1}{x^2(x+1)},\ \text{for } x\neq-1,\ 0$

c. $u+1+\dfrac{1}{u+1}=\dfrac{(u+1)^2+1}{u+1}=\dfrac{u^2+2u+2}{u+1}\ \text{for } u\neq-1$

d. $\dfrac{2}{a^2}-\dfrac{3}{ab}+\dfrac{4}{b^2}=\dfrac{2b^2-3ab+4a^2}{a^2b^2},\ \text{for } a\neq0,\ b\neq0$

e. $\dfrac{3}{x^2-x}-\dfrac{2}{x^2+x-2}=\dfrac{3(x+2)-2x}{x(x-1)(x+2)}=\dfrac{x+6}{x(x-1)(x+2)},\ \text{for } x\neq-2,\ 0,\ 1.$

f. $\dfrac{x}{(x+1)^2}+\dfrac{2}{x+1}=\dfrac{x+2(x+1)}{(x+1)^2}=\dfrac{3x+2}{(x+1)^2},\ \text{for } x\neq-1$

g. $\dfrac{-50x^2-55x+8-25x(3x-1)+(25x^2+15x)(5x+2)}{15x^2+x-2}=\dfrac{125x^3+8}{15x^2+x-2}\ \text{for } x\neq\dfrac{-2}{5},\dfrac{1}{3}$

$$=\dfrac{(5x+2)(25x^2-10x+4)}{(5x+2)(3x-1)}=\dfrac{25x^2-10x+4}{3x-1}$$

h. $\dfrac{16}{3(z+4)},\ z\neq-4$

Pursuant to their ability of applying the operations and simplifying rational expressions, you can group the students and proceed by letting each group do Activity 2.5.

The purpose of the group activity is to help the students revise multiplication and division of rational numbers. This in turn helps them to recognize the similarity of multiplication and division of rational numbers and rational expressions.

Answers to Activity 2.5

1. a. $\frac{9}{35}$ b. $\frac{2}{9}$ c. $\frac{8}{9}$ d. $\frac{3}{10}$

After the students have completed Activity 2.5, do the examples in class and let them do Exercise 2.4 in small groups. Let a student from each group do a problem from the exercise on the board.

You can give for high achievers the following types of problems:

Example: Give the domain, perform the operations and simplify.

a. $\frac{2u-v}{4u}\cdot\frac{2u-v}{4u^2-4uv+v^2}$ b. $\frac{\frac{6}{p^2+3p-10}-\frac{1}{p-2}}{\frac{1}{p-2}+1}$

c. $\frac{x^3-y^3}{x-y}\div\frac{x^2+xy+y^2}{x^2-2xy+y^2}$ d. $\frac{x^{2n+2}+x^{2n+1}+x^{2n}y^2}{x^{n+3}-x^n y^3}\div x^n$

Solution:

a. $\frac{2u-v}{4u}\cdot\frac{2u-v}{(2u-v)^2}=\frac{1}{4u}; v\neq 2u, u\neq 0$

b. $\frac{6}{(p+5)(p-2)}-\frac{1}{p-2}\cdot\left(\frac{p-2}{p-1}\right)=\frac{1-p}{(p+5)(p-2)}\cdot\frac{(p-2)}{(p-1)}$

$=-\frac{1}{5+p}; p\neq 2,1,-5$

c. $\frac{(x-y)(x^2+xy+y^2)}{x-y}\cdot\frac{(x-y)^2}{(x^2+xy+y^2)}=(x-y)^2; x\neq y$

d. $\frac{x^{2n}(x^2+x+y^2)}{x^n(x^3-y^3)}\times\frac{1}{x^n}=\frac{x^2+x+y^2}{x^3-y^3}; x\neq 0, x\neq y$

Assessment

To assess students' understanding of the concepts discussed so far, give them problems on multiplication and division of rational expressions and expressions that also involve addition and subtraction of rational expressions using quiz, homework, class group work etc. Ask them questions that check their understanding of the properties of the four operations.

Answers to Exercise 2.4

1. a. $\dfrac{x^2-x-12}{x^2-9}\times\dfrac{3+x}{4-x}=\dfrac{(x+3)(x-4)}{(x-3)(x+3)}\times\dfrac{(x+3)}{(4-x)}=\dfrac{-(x+3)}{x-3}$, $x\neq-3, 3, 4$

   b. $\dfrac{(x-3)(x^2+3x+9)(x+3)}{(x-3)(x+3)(x^2+3x+9)}=1$, for $x\neq-3, 3$

2. a. $\dfrac{(x-3)(x-4)}{4-x}\times\dfrac{5}{(x-3)(x+3)}=\dfrac{-5}{x+3}$, for $x\neq-3, 3, 4$

   b. $\dfrac{2x^2-3x-2}{x^2-1}\times\dfrac{x^2+x-2}{2x^2+5x+2}=\dfrac{(x-2)(2x+1)}{(x-1)(x+1)}\times\dfrac{(x-1)(x+2)}{(x+2)(2x+1)}=\dfrac{x-2}{x+1}$,

   for $x\neq -1,-2,-\dfrac{1}{2},1$

   c. $\dfrac{x^2-x-6}{3x^2-12}\times\dfrac{2-x}{x^2-3x}=\dfrac{(x-3)(x+2)}{3(x-2)(x+2)}\times\dfrac{2-x}{x(x-3)}=\dfrac{-1}{3x}$, for $x\neq-2, 0, 2, 3$

2.1.3 Decomposition of Rational Expressions into Partial Fractions

So far, students have been adding, subtracting, multiplying and dividing rational expressions. They also know how to simplify an expression by cancelling common factors from both the numerator and denominator. In this lesson, you guide them to use partial fraction decomposition. Decomposition of rational fractions into simpler partial fractions is a method your students will find useful in future (when for instance they learn integration in calculus). Do the examples given in the student textbook by considering the different cases addressed through each example. Explain the concepts and steps, and add some more examples. When they practice simplifying rational expressions, the students need to recognize that the simplified expression may have linear or quadratic factor that cannot be further simplified. To this end, guide them to understand the statement of Theorem 2.1. This theorem indicates that there may be quadratic factors that cannot be reduced further in the set of real numbers. These are those quadratic factors whose discriminants are negative. To help the students understand the theorem, encourage them to do the examples and Activity 2.6. The purpose of this Activity is to help students revise factorization of polynomials. They will also use the discriminant ($b^2 - 4ac$) to decide whether or not a quadratic expression is irreducible.

Answers to Activity 2.6

1. $x^3 - 3x^2 + 2x = x(x-1)(x-2)$
2. a. $x^2 - 6x + 9 = (x-3)^2$

b. $15x^2 + 14x - 8 = (3x + 4)(5x - 2)$

c. $x^2 - x + 2$ is not factorable.

3. a. $b^2 - 4ac = 0$

b. $b^2 - 4ac = 676 > 0$

c $b^2 - 4ac = -7 < 0$

Yes we can use $b^2 - 4ac$ to decide which quadratic polynomial can be factorized. When $b^2 - 4ac < 0$, we say the quadratic equation is not factorized.

4. $x^4 + 7x^3 + 12x^2 - 7x - 13 = (x - 1)(x + 1)(x^2 + 7x + 13)$

After the students performed the activity, you can proceed to discuss the different examples supplied in the students textbook.

Assessment

Give the students problems on partial fraction decomposition that evaluate each of the cases that are treated in the section. Give them questions like the following.

1. Factorize the following. Which of them have quadratic factors that cannot be reduced further? How can you check?

a. $x^2 - 5x + 6$ b. $x^3 - 2x^2 - 20x - 24$ c. $x^3 + x^2 + 4x + 4$

2. Based on what you did in 1), give the partial fraction decomposition of each of the following. Use the rules given in your text book.

a. $\dfrac{4}{x^2 - 5x + 6}$ b. $\dfrac{3x+4}{x^3 - 2x^2 - 20x - 24}$ c. $\dfrac{1}{x^2(x-1)}$

d. $\dfrac{8x^2 + 3x + 20}{x^3 + x^2 + 4x + 4}$ e. $\dfrac{x^3 - x^2}{(x^2+3)^2}$

Solution:

1. a. $x^2 - 5x + 6 = (x - 3)(x - 2)$

b. $x^3 - 2x^2 - 20x - 24 = (x + 2)^2(x - 6)$

c. $x^3 + x^2 + 4x + 4 = (x + 1)(x^2 + 4)$

2. a. $\dfrac{4}{x-3} - \dfrac{4}{x-2}$

b. $\dfrac{11}{32(x-6)} + \dfrac{1}{4(x+2)^2} - \dfrac{11}{32(x+2)}$

Notice that $x^3 - 2x^2 - 20x - 24 = (x-6)(x+2)^2$

c. $\dfrac{1}{x^2}-\dfrac{1}{x-1}-\dfrac{1}{x}$

d. $\dfrac{5}{x+1}+\dfrac{3x}{x^2+4}$

Notice that $x^3+x^2+4x+4=(x+1)(x^2+4)$

e. $\dfrac{x-1}{x^2+3}-\dfrac{3(x-1)}{(3+x^2)^2}$

Let the students do the problems in groups and submit the answers they have agreed upon. Then you might select a student from each group and let them do their answers on the board. Based on their performances, you may give them some additional problems.

Answers to Exercise 2.5

a. $\dfrac{7x+6}{(x-2)(x+3)}=\dfrac{A}{x-2}+\dfrac{B}{x+3}=\dfrac{4}{x-2}+\dfrac{3}{x+3}$

b. $\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+2}$ gives $A(x^2+x+2)+(Bx+C)(x-1)=5x+7$

$\Rightarrow \quad A+B=0$

$A-B+C=5 \Rightarrow A=3, B=-3$ and $C=-1$

$2A-C=7$

Thus $\dfrac{5x+7}{(x-1)(x^2+x+2)}=\dfrac{3}{x-1}-\dfrac{3x+1}{x^2+x+2}$

c. $\dfrac{3x+5}{(x-2)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}=\dfrac{3}{x-2}+\dfrac{11}{(x-2)^2}$

d. $\dfrac{(x+3)^2}{(x^2+1)(x+3)}=\dfrac{x+3}{x^2+1}$, for $x\neq-3$

e. $\dfrac{x^2+4x-3}{x-2}=x+6+\dfrac{9}{x-2}$ (by long division)

f. $\dfrac{7x^2-11x+6}{(x-1)(2x^2-3x+2)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{2x^2-3x+2}$ which gives

$2A+B=7$

$-3A-B+C=-11 \quad \Rightarrow \quad A=2, \; B=3$ and $C=-2$

$2A-C=6$

Thus $\dfrac{7x^2-11x+6}{(x-1)(2x^2-3x+2)}=\dfrac{2}{(x-1)}+\dfrac{3x-2}{2x^2-3x+2}$

2.2 RATIONAL EQUATIONS

Periods allotted: 3 Periods

Competency

At the end of this sub-unit, students will be able to:

- *solve rational equations.*

Vocabulary: Rational equation, Extraneous solution

Introduction

In this sub-unit, students will learn how to solve rational equations. Remind the students that an equation is an algebraic expression with the equality sign somewhere in between or simply equality of two expressions which they studied in grade 9. When one or more of the expressions involve a fractional one, it becomes a rational equation.

Teaching Notes

Before starting this lesson, it will be advisable to assess how much the students recall what they know about equations. This can be done in a question and answer form. Following this, you can write examples of equations and ask them to determine their domains. To this end, you can also enquire them about how they solve the equations and let them do Activity 2.7. This activity is meant for revising what the students have already learned on solving equations. Guide the students to do the activity individually and let some of them go to the board and explain the steps and the answers. Next, go through the examples. Here, you may write simple rational equations (together with their domain) and solve them together with active participation of the students. You may also give real life problems that can become rational equations. For solving rational equations, you need to consider different cases that refer to the domain. Your equations should include equations whose solution sets are empty and equations with extraneous solutions.

Example: Give the solution sets of each of the following.

a. $\frac{x+2}{x}=1$ (Please ask the students to give justifications for each step they take to answer the question). Here the solution is empty.

b. $\frac{1}{x(x-1)}-\frac{1}{x}=\frac{1}{x-1}$ c. $\frac{x^2-2x-15}{2x^2+3x-15}=0$

d. $\frac{(x-1)(x+2)(3-x)}{(x+3)(x^2-x-2)}=0$

Solution:

b. $\frac{1-(x-1)}{x(x-1)}-\frac{1}{x-1}=\frac{2-x}{x(x-1)}-\frac{1}{x-1}=\frac{2-2x}{x(x-1)}=-\frac{2}{x}=0 \Rightarrow S.S=\varnothing$

c. $\frac{(x-5)(x+3)}{2x^2+3x-15}=0 \Rightarrow x=5,-3 \Rightarrow S.S=\{-3,\ 5\}$

d. $\frac{(x-1)(x+2)(3-x)}{(x+3)(x^2-x-2)}=0 \Rightarrow (x-1)(x+2)(3-x)=0$

$\Rightarrow x = 1, -2, 3 \Rightarrow \text{S.S} = \{1, -2, 3\}$

Answers to Activity 2.7

This Activity is designed to help the students to recall ways of solving the equations.

a. The universal set can be any number system, and $x = 2$

b. $x + 2 - 3\ (x - 2) = 0 \Rightarrow -2x + 8 = 0 \Rightarrow x = 4$

c. $x = \frac{24}{13}$

d. $2\ (10x + 3) = 5x + 6 \Rightarrow x = 0$

e Let x be the number of staff members initially.

$200x = (x + 2)195 \Rightarrow 200x = 195x + 390$

$5x = 390 \Rightarrow x = 78$

Number of staff members now is 78 + 2 = 80.

Assessment

Give the students several rational equations to solve. Your equations should include equations whose solution sets are empty and equations with extraneous solutions. You can let the students do the problems in Exercise 2.6 in groups and submit the answers they have agreed upon. Then you might select a student from each group and send them to the board to do and state their answers. Based on their performances you may give them some additional problems.

Answers to Exercise 2.6

1. a. $\text{U}=\{x\in \mathbb{R} \mid x\neq 0 \wedge x\neq -2\}=\mathbb{R}\setminus\{0,-2\}$

$\frac{3}{x+2}-\frac{1}{x}=\frac{1}{5x}$ ⇔ $15x - 5(x+2) = x + 2$ ⇔ $9x = 12$ or $x = \frac{12}{9}=\frac{4}{3}$

or S.S = $\left\{\frac{4}{3}\right\}$

b. U = $\mathbb{R}\setminus\{0\}$

$\frac{x-6}{x} = \frac{x+4}{x} + 1$ ⇔ $x - 6 = x + 4 + x$, for $x \neq 0$ ⇔ $x = -10$

or S.S = $\{-10\}$

c. $U=\{a\in\mathbb{R}\,|\,a\neq 0 \wedge a\neq -4\}=\mathbb{R}\setminus\{0,-4\}$

$\frac{4}{a} = \frac{1}{a^2+4a}-\frac{a+3}{a^2+4a}$ ⇔ $4(a+4) = 1-(a+3)$ ⇔ $5a = -18$ or $a = \frac{-18}{5}$

or S.S = $\left\{-\frac{18}{5}\right\}$

d. U = $\mathbb{R}\setminus\{-1, 1, 4\}$

$\frac{2}{x-4}-\frac{3}{x+1} = \frac{6}{x-1}$ ⇔ $2(x+1)(x-1) - 3(x-4)(x-1) = 6(x-4)(x+1)$

⇔ $7x^2 - 33x - 10 = 0$

which gives the solution set $\left\{\frac{-2}{7},\ 5\right\}$

e. U = $\mathbb{R}$

$\frac{3x-2}{5} = \frac{4x}{7}$ ⇔ $7(3x-2) = 5(4x)$.

S.S = $\{14\}$

f. U = $\mathbb{R}\setminus\{-5, 5\}$

$\frac{x+4}{x-5}-\frac{1}{x+5} = \frac{10}{x^2-25}$ ⇔ $(x+4)(x+5)-(x-5) = 10$

⇔ $x^2 + 8x + 15 = 0$ ⇔ $x = -3$ or $x = -5$

But since $x = -5 \notin$ U, S.S = $\{-3\}$.

2. Suppose the first plane flew for x hrs, and the second plane flew for y hrs. Then $x = y + 1.5$ or $y = x - 1.5$

Since they flew at the same rate, we have rate = $\frac{\text{distance}}{\text{time}} = \frac{2700}{x} = \frac{2025}{y} = \frac{2025}{x-1.5}$

which gives $2700(x - 1.5) = 2025x \Rightarrow x = \frac{4050}{675} = 6$ hrs $\Rightarrow y = 4.5$ hrs.

Therefore, the first plane flies for 6 hours and the second plane flies for 4.5 hours.

3. Let the height of the tree be h feet. Then

$\frac{h}{3} = \frac{34}{1.7} = 20 \Rightarrow h = 3(20) = 60$ feet.

2.3 RATIONAL FUNCTIONS AND THEIR GRAPHS

Periods allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define rational function.*
- *determine the domain of a given rational function.*
- *determine the range of a given rational function.*
- *sketch the graph of a given rational function.*
- *determine the intercepts and symmetry of the graph of a given rational function.*
- *identify the types of asymptotes that the graph of a given function may have.*
- *tell the properties of a given rational function from its graph.*
- *use graphs of rational functions to solve rational inequalities.*

Vocabulary: Rational function, Vertical asymptote, Horizontal asymptote, Oblique asymptote, Zero of a rational function, x-intercept, y-intercept, Parity

Introduction

This sub-unit is devoted to rational functions and their graphs. Issues of asymptotes and holes, and the zero of functions will be addressed. By using these and other concepts such as parity, x and y-intercepts sketching graphs of rational functions will be discussed.

Teaching Notes

You may start the lesson by listing several functions familiar to the students and asking them to tell the type of the functions and their domains. Following this entry, you can let the students do Activity 2.8 individually. In this activity, students will summarize what they know about some of the commonest functions. This paves the way for stating rational functions and helps in finding their domains.

Answers to Activity 2.8

a. Polynomial function (linear function); dom $(f) = \mathbb{R}$

b. Polynomial function (quadratic function); dom $(g) = \mathbb{R}$

c. Logarithmic function ; dom $(f) = (-1, \infty)$

d. Exponential function ; dom $(g)=\mathbb{R}$

e. Trigonometric function; dom $(f) =\mathbb{R}$

f. Square root function ; dom $(g) = [-3, 3]$

2.3.1 RATIONAL FUNCTIONS

Once the students have become familiar with various functions and can identify rational functions, it may be sound to state the formal definition of rational function. To make it formal you can give the definition of a rational function as given in the student textbook and elaborate the fact that the denominator is non-zero. This will help the students in determining the domain of a rational function. At this moment, you may present rational functions and their domains with some examples. Once the students have the ability to identify a rational function and determine its domain, you can ask them to describe how they can determine functional values at a point. To help them practice, you can give activity 2.9 to the students to prepare table of values and plot the corresponding points with special attention to values near vertical asymptotes. In this activity, the students will practice calculating the values of functions for some specific *x* values in the domains of the functions.

Answers to Activity 2.9

1. a. $\frac{1}{2}$ b. $-\frac{1}{3}$ c. 2.5 d. $-\frac{2}{3}$
2. a. 0 b. 3 c. $\frac{1}{2}$ d. 5

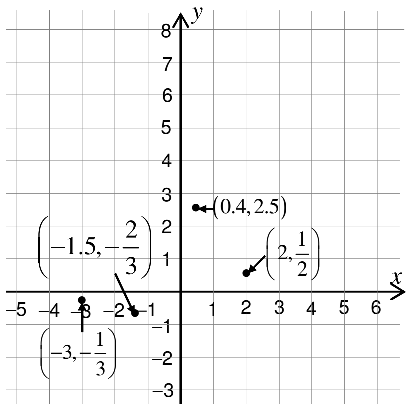


Figure 2.1

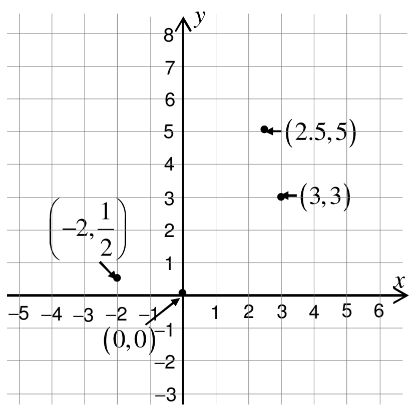


Figure 2.2

Now that the students have the ability to determine functional values, you can form students groups and let them do Group Work 2.1. Here inform the students that selection of points in the domain is essential. From this group activity, students will recognize the behaviour of the function $f(x) = \frac{1}{x}$. In so doing, they will be introduced to the concept of asymptotes of rational functions. The group work requires a lot of input from you. Students may not easily understand the meaning of “approaches from the left or from the right”, “grows without bound” and so on. Use graphs of familiar functions like $y = \log x$, or $y = 2^x$, if necessary. After filling in the table and observing the behaviour of the function from the table, let the students practice plotting these points on the coordinate plane and observe, if their graph looks like the one given in the group work.

Answers to Group Work 2.1

1. Domain = $\mathbb{R}\backslash\{0\}$

2.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$
x	1	0.5	0.1	0.01	0.001	$\to 0$
$f(x)$	1	2	10	100	1000	$\to \infty$

3. $x \to 0^-, f(x) \to -\infty$
   $x \to 0^+, f(x) \to \infty$
   $x \to \infty, f(x) \to 0^+$ (above)
   $x \to -\infty, f(x) \to 0^-$ (below)

4. Yes

With active participation of students, discuss how to determine the type of asymptotes a function may have. To this end, you can give Activity 2.10 for the students to do in pairs. In this activity, the students will find domain of functions and the asymptotes of the functions. However, guide the students to note the points where holes may occur and also realize the fact that oblique asymptotes occur when the degree of the numerator is larger than the degree of the denominator by only 1. In this case, they use long division to find the oblique asymptote.

Answers to Activity 2.10

a. dom (f) = $\mathbb{R} \setminus \{-4\}$ – Vertical asymptote , $x = -4$
   – horizontal asymptote , $y = 0$

b. dom (f) = $\mathbb{R} \setminus \{0\}$ – vertical asymptote, $x = 0$
   – horizontal asymptote, $y = 2$

c. dom (f) = $\mathbb{R} \setminus \{-3, 3\}$ – vertical asymptote ,$x = -3$
   $\left(\frac{x-3}{x^2-9} = \frac{1}{x+3}, x \neq 3\right)$ – a hole at $x = 3$, $y = \frac{1}{6}$
   – horizontal asymptote, $y = 0$

d. dom (f) = $\mathbb{R} \setminus \{-1\}$ – vertical asymptotes, $x = -1$

By long division, $\dfrac{x^2-x}{x+1} = x-2+\dfrac{2}{x-1}$

Thus oblique asymptote at $y = x - 2$.

e. dom (f) = $\mathbb{R} \setminus \left\{\dfrac{1}{3}\right\}$ – vertical asymptote $x = \dfrac{1}{3}$

– horizontal asymptote, $y = -\dfrac{4}{3}$

f. dom (f) = $\mathbb{R} \setminus \{1\}$ – vertical asymptote , $x = 1$

By long division, $\dfrac{x^2-x-2}{x-1} = x-\dfrac{2}{x-1}$

Thus oblique asymptote is, $y = x$.

2.3.2 GRAPHS OF RATIONAL FUNCTIONS

After ensuring the students understanding of rational function, the next point of discussion is on graphs of functions. But before dealing with graphs, it will be good to discuss the zero of a rational function. Knowing the zeros of a rational function helps the students to find the x- intercept of the function. Graphs of rational functions are difficult to construct. So, guide the students to note the domain, the intercepts, the asymptotes, and if there are points where the graph crosses its asymptotes. To help students practice, you can form groups and let them do Group Work 2.2. In this group work, the students will find domain of functions, x and y intercepts, the asymptotes of the functions and their parity (when existing). Doing this is necessary for sketching the graphs of rational functions.

Answers to Group Work 2.2

Domain	x-intercept	y-intercept	Asymptote	Parity
$\mathbb{R}\setminus\{-3, 2\}$	$x = -1$	$y = -\dfrac{1}{18}$	V.A, $x = 2, x = -3$ H.A, $y = 0$	Neither
$\mathbb{R}\setminus\{-1\}$	$x = -3, x = -2$	$y = 6$	V.A, $x = -1$ O.A, $y = x + 4$	Neither
$\mathbb{R}\setminus\{-2, 2\}$	---	$y = \dfrac{1}{2}$	V.A, $x = -2$ Hole, $x = 2$ H.A, $y = 0$	Neither

With these characterizations, guide the students how they can draw/sketch graphs of rational functions. You can use example 6 of the student textbook on Page 60 for the purpose of illustration. In addition to sketching graphs of rational functions, you may also guide students to determine behaviours of functions from their graphs. For example, you can encourage them to determine the range of a function from its graph. (Note that for graphs like in Example 6(d), it may be difficult at this stage, to tell the range, since you may need the concepts of calculus to find the turning point). Finally, you can give Exercise 2.7 as an assignment.

Answers to Activity 2.11

a. i. $f(x) = -\dfrac{1}{x^2} < 0$ on $(-\infty, 0) \cup (0, \infty)$

ii. $f(x) = -\dfrac{1}{x^2} > 0$, No solution.

b. i. $f(x) = \dfrac{3x^2}{(x-2)(x+1)} < 0$ on $(-1, 2)$

ii. $f(x) = \dfrac{3x^2}{(x-2)(x+1)} > 0$ on $(-\infty, -1) \cup (2, \infty)$

c. i. $f(x) = \dfrac{x+1}{(x-2)(x+3)^2} < 0$ on $(-1, 2) \cup (0, \infty)$

ii. $f(x) = \dfrac{x+1}{(x-2)(x+3)^2} > 0$ on $(-\infty, -3) \cup (-3, -1) \cup (2, \infty)$

d. i. $f(x) = \dfrac{x^2+5x+6}{x+1} < 0$ on $(-\infty, -3) \cup (-2, -1)$

ii. $f(x) = \dfrac{x^2+5x+6}{x+1} > 0$ on $(-3, -2) \cup (-1, \infty)$

d. i. $f(x) = \dfrac{x-2}{x^2-4} < 0$ on $(-\infty, -2)$

ii. $f(x) = \dfrac{x-2}{x^2-4} > 0$ on $(-2, \infty)$

Assessment

Graphing is a skill that improves with practice. So, guide the students to practice graphing in groups. Insist that they should use squared papers and rulers. Let them first prepare a table of values that will help them in sketching the graph. Your questions should check at least the following:

- Have they understood how to determine the significance of the domain, the intercepts, and the asymptotes?
- Can they explain the nature of the graph near its asymptote?
- Can they give the region in which the function is above (below) its oblique or horizontal asymptote, if any?

Answers to Exercise 2.7

1. a. $f(x)=\dfrac{x^2-x-12}{x^2-2x-8}=\dfrac{(x+3)(x-4)}{(x+2)(x-4)}=\dfrac{x+3}{x+2}$, *for* $x\neq -2,\ 4$

A hole at $x=4$, $y=\dfrac{7}{6}$

Vertical asymptote. $x=-2$

horizontal asymptote. $y=1$

y-intercept at $\left(0,\ \dfrac{3}{2}\right)$

x-intercept at $(-3, 0)$

Domain $\mathbb{R}\setminus\{-2, 4\}$

Range = $\mathbb{R}\setminus\{1\}$

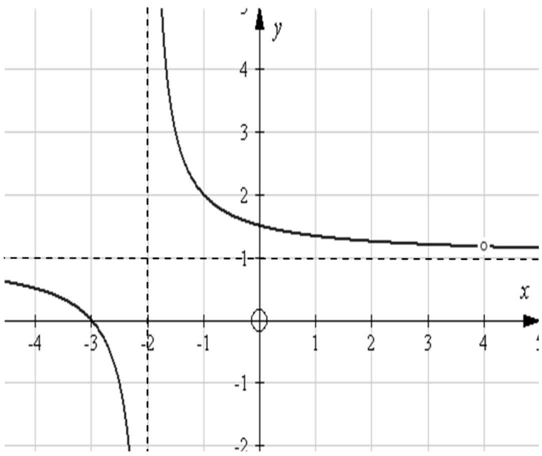


Figure 2.3

b. $f(x)=\dfrac{3x^2-5x-2}{x^2-1}$, for $x\neq -1, 1$

Vertical asymptote. $x=-1$ and $x=1$

Horizontal asymptote. $y=3$

y- intercept at $(0, 2)$

$3x^2-5x-2=0$

$\Rightarrow (3x+1)(x-2)=0$

$\Rightarrow$ x-intercepts at $\left(-\dfrac{1}{3},0\right)$ and $(2, 0)$

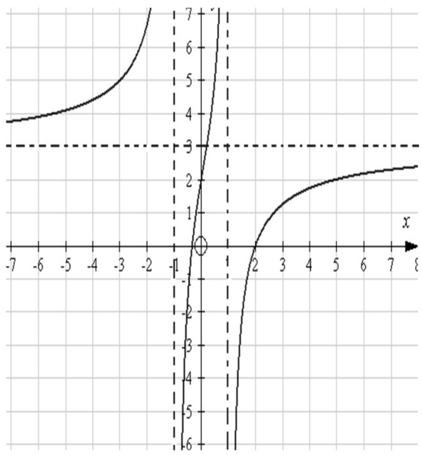


Figure 2.4

c. $f(x)=\dfrac{x^3+1}{x^2-1} = \dfrac{(x+1)(x^2-x+1)}{(x+1)(x-1)} = \dfrac{x^2-x+1}{x-1}$, for $x \neq -1, 1$

By long division,

$$\frac{x^2-x+1}{x-1} = x + \frac{1}{x-1}$$

Vertical asymptote. $x = 1$

A hole at $x = -1$, $y = -\dfrac{3}{2}$

y-intercept at $(0, -1)$

oblique asymptote ,$y = x$

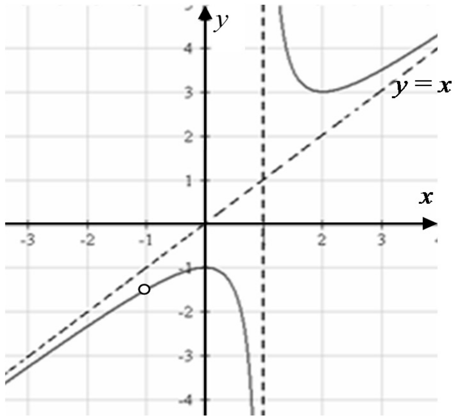


Figure 2.5

d. $f(x)=\dfrac{x^2+3x-4}{x-5}$, for $x \neq 5$

Vertical asymptote. $x = 5$

oblique asymptote. $y = x + 8$

y-intercept at $\left(0, \dfrac{4}{5}\right)$

x-intercepts at $(-4, 0)$ and $(1, 0)$

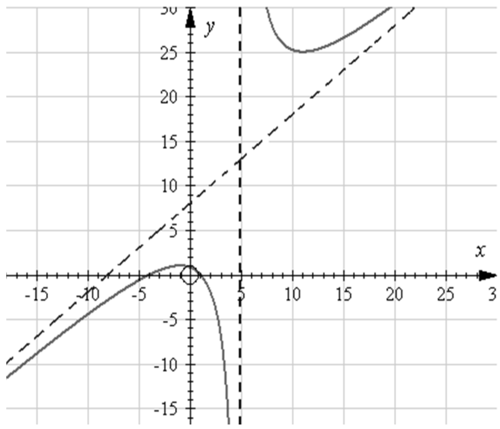


Figure 2.6

e. $f(x) = \dfrac{2x^2-3x+2}{x^2+1}$

No vertical asymptote

Horizontal asymptote $y = 2$

y-intercept at $(0, 2)$

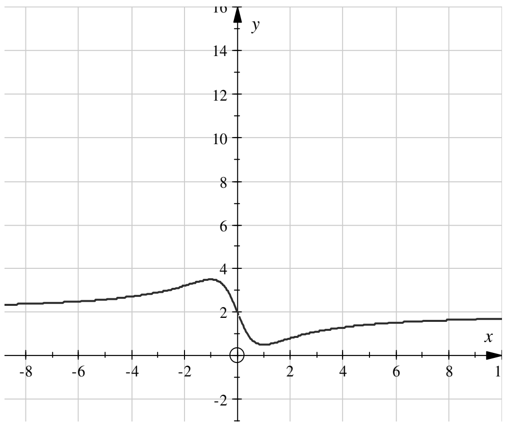


Figure 2.7

f. $f(x) = \dfrac{x + 2}{x^2 - 9}$, for $x \neq -3.3$

Vertical asymptote. $x = -3$ and $x = 3$

Horizontal asymptote. $y = 0$

x-intercept at $(-2, 0)$

y-intercept at $\left(0, -\dfrac{2}{9}\right)$

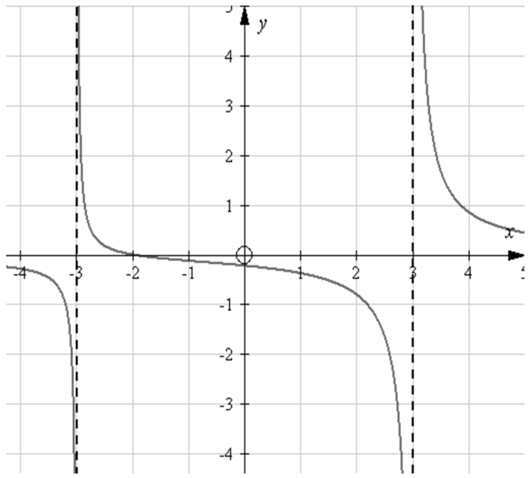


Figure 2.8

You can also plot some additional points to see where the graph lies, say $f(-4) = -\dfrac{2}{7}$ and $f(4) = \dfrac{6}{7}$. Note that the graph crosses its horizontal asymptote at $x = -2$.

g. $f(x) = \dfrac{-x}{x^2 + x - 2}$, for $x \neq -2, 1$

Intercept at (0, 0)

Vertical asymptote. $x = -2$, $x = 1$

Horizontal asymptote. $y = 0$

Plot additional points

x	-3	-1	2
$f(x)$	$\dfrac{3}{4}$	$-\dfrac{1}{2}$	$-\dfrac{1}{2}$

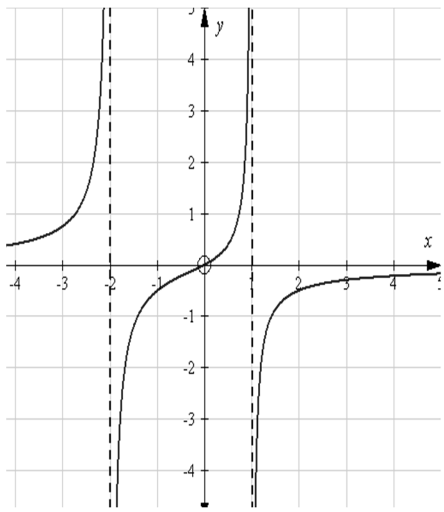


Figure 2.9

h. $f(x)=\dfrac{(x-1)\ (x+3)}{(x-2)}$

y- intercept $\left(0,\ \dfrac{3}{2}\right)$

x-intercept (1, 0) and (–3, 0)

Oblique asymptote $y = x + 4$

Vertical asymptote $x = 2$

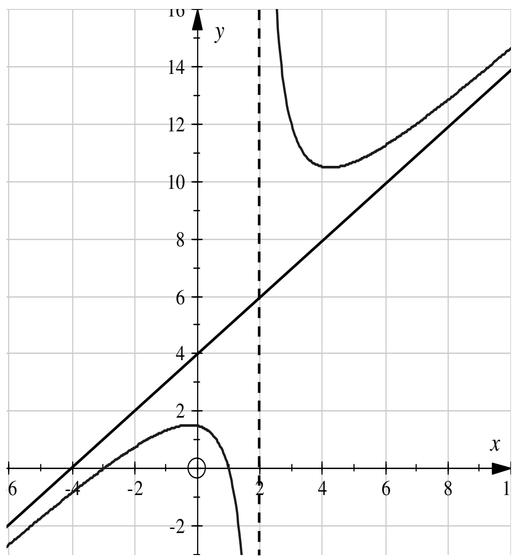


Figure 2.10

i. $f(x) = \dfrac{x^2-9}{x-3}$, for $x \neq 3$

For $x \neq 3, f(x) = x + 3$ except for a hole at $x = 3$, $y = 6$

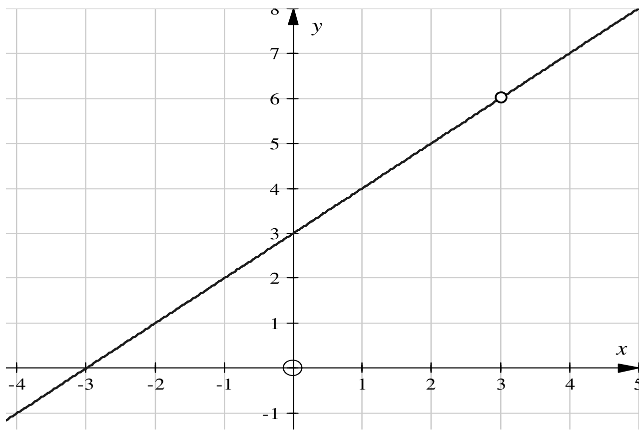


Figure 2.11

2. For 1b, $f(x) > 0$, for $x \in (-\infty, -1)$, $x \in \left(-\dfrac{1}{3}, 1\right)$, $x \in (2, \infty)$

$f(x) < 0$, for $x \in \left(-1, -\dfrac{1}{3}\right)$, $x \in (1, 2)$

For 1d, $f(x) > 0$ for $x \in (-4, 1) \cup (5, \infty)$ and $f(x) < 0$, for $x \in (-\infty, -4) \cup (1, 5)$

For 1h, $f(x) > 0$ for $x \in (-3, 1) \cup (2, \infty)$ and $f(x) < 0$, for $x \in (-\infty, -3) \cup (1, 2)$

Answers to Review Exercises on Unit 2

1. a. $\dfrac{2x-4}{x^2+x-6} = \dfrac{2(x-2)}{(x+3)\ (x-2)} = \dfrac{2}{x\ +\ 3}$, for $x \neq 2, -3$

b. $\dfrac{x^2-x-6}{x^2+3x+2} = \dfrac{(x-3)\ (x+2)}{(x+1)\ (x\ +\ 2)} = \dfrac{x-3}{x+1}$, for $x \neq -1, -2$

c. $\dfrac{x^2-5x}{x^2-25} = \dfrac{x(x-5)}{(x+5)\ (x-5)} = \dfrac{x}{x+5}$, for $x \neq -5, 5$

d. $\dfrac{x^3+8x^2+24x+45}{x^4+3x^3-27x-81} = \dfrac{(x+5)(x^2+3x+9)}{(x-3)(x+3)(x^2+3x+9)} = \dfrac{x+5}{(x-3)(x+3)}$, for $x \neq -3, 3$

2. a. x

b. $\dfrac{2x^2-162}{x+9} = \dfrac{2(x-9)\ (x+9)}{x+9} = 2x-18$

c. $\dfrac{2(x+1)+(x-1)\ (x-1)}{(x-1)\ (x+1)} \cdot \dfrac{x^2-1}{1} = 2\ (x + 1) + x^2 - 2x + 1 = x^2 + 3$

d. $\dfrac{(x-1)\ (x+1)}{(x-1)\ (x+4)} \cdot \dfrac{(x+4)\ (x-3)}{(x+1)\ (x+3)} = \dfrac{x-3}{x+3}$

e. $\dfrac{(x-5)\ (x+5)}{(x-5)^2} \cdot \dfrac{(x+4)^2}{x^2+16} = \dfrac{(x+5)(x+4)^2}{(x-5)\ (x^2+16)}$

f. $\dfrac{x}{x^3-1} \div \left[2-\dfrac{x-2}{x-1}\right] = \dfrac{x}{(x-1)\ (x^2+x+1)} \cdot \dfrac{x-1}{x} = \dfrac{1}{x^2+x+1}$

3. a. $\dfrac{3}{x^2-3x} = \dfrac{3}{x(x-3)} = \dfrac{A}{x} + \dfrac{B}{x-3} = \dfrac{A(x-3)+Bx}{x(x-3)}$

$\Rightarrow \quad (A + B)\ x - 3A = 3$

$\Rightarrow \quad A + B = 0 \ \Rightarrow\ A = -1, B = 1$

$-3A = 3$

Thus $\dfrac{3}{x^2-3x} = \dfrac{-1}{x} + \dfrac{1}{x-3}$

b. $\frac{x+1}{x^2+4x+3} = \frac{x+1}{(x+3)\ (x+1)} = \frac{1}{x+3}$

c. $\frac{2x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} = \frac{2}{x-1} - \frac{1}{(x-1)^2}$

d. $\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x\ (Bx+C)}{x\ (x^2+1)}$

$\Rightarrow$ A = 1; C = 1; A + B = 0 $\Rightarrow$ B = –1

Thus $\frac{x+1}{x^3+x} = \frac{1}{x} + \frac{-x+1}{x^2+1}$

e. $\frac{x-1}{x^3+x^2} = \frac{x-1}{x^2(x+1)} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} = \frac{Ax^2+Bx(x+1)+C(x+1)}{x^2(x+1)}$

$\Rightarrow$ A + B = 0 $\Rightarrow$ A = –2

B + C = 1 B = 2

C = –1 C = –1

Thus $\frac{x-1}{x^3+x^2} = \frac{-2}{x+1} + \frac{2}{x} - \frac{1}{x^2}$

f. $\frac{5x+1}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} = \frac{5}{4x} + \frac{1}{4x^2} - \frac{\frac{5}{4}x+\frac{1}{4}}{(x^2+4)}$

$= \frac{5}{4x} + \frac{1}{4x^2} - \frac{5x+1}{4(x^2+4)}$

4. a. U = $\mathbb{R} \setminus \{0\}$. Multiplying both sides by x^2, we get

$4 = 5x - 1 \Rightarrow$ S.S = {1}

b. U = $\mathbb{R} \setminus \{0\}$ S.S = {–10}

c. U = $\mathbb{R} \setminus \{-3\}$ Multiplying both sides by $y + 3$ gives

$3 + 3y = y + 3 \Rightarrow 2y = 0$ S.S = {0}

d. U = $\mathbb{R} \setminus \{0, 3\}$ Multiply by $y\ (y - 3)$ to get

$1 + y = 3 \Rightarrow$ S.S = {2}

5. a. $U = \mathbb{R} \setminus \{-2\}$

x-intercept: (3, 0)

y-intercept: $\left(0,\ -\dfrac{3}{2}\right)$

Vertical asymptote. $x = -2$

Horizontal asymptote. $y = 1$

$f(-3) = 6, f(-1) = -4$

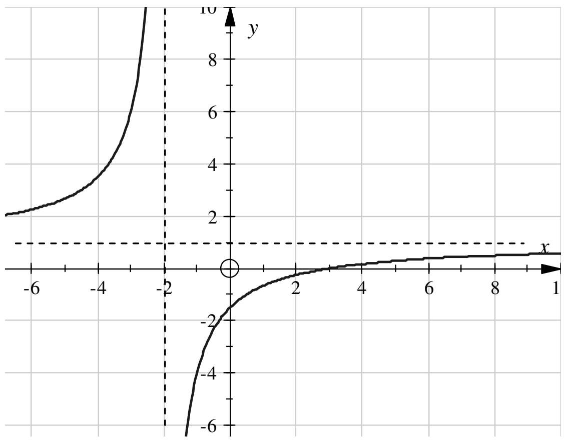


Figure 2.12

b. $U = \mathbb{R} \setminus \{\ 5\ \}$

No x-intercept

y-intercept $\left(0,\ \dfrac{3}{25}\right)$

Vertical asymptote $x = 5$

Horizontal asymptote $y = 0$

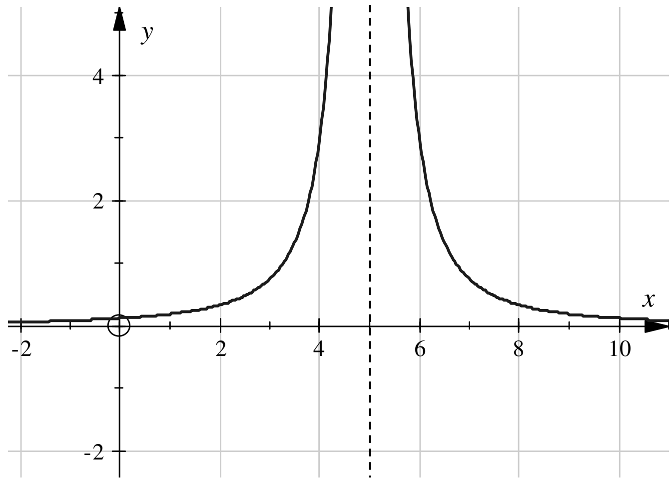


Figure 2.13

c. $U = \mathbb{R}$

Intercept : (0, 0)

No vertical asymptote.

Horizontal asymptote. $y = 1$

x	-2	-1	1	2
$f(x)$	$\frac{4}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{4}{5}$

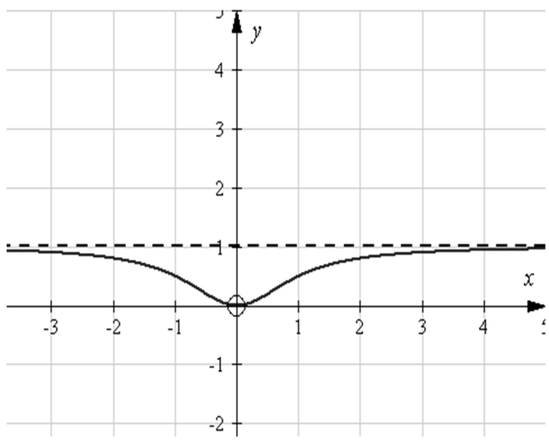


Figure 2.14

d. $U = \mathbb{R} \setminus \{-2, 2\}$

Intercept. (0, 0)

Vertical asymptote. $x = -2$ and $x = 2$

Horizontal asymptote. $y = 0$

x	-3	-1	1	3
$f(x)$	-3	$\frac{5}{3}$	$-\frac{5}{3}$	3

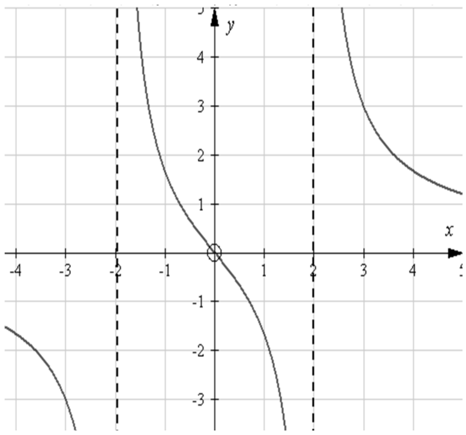


Figure 2.15

e. $f(x) = x + \dfrac{1}{x^2} = \dfrac{x^3+1}{x^2}$

$U = \mathbb{R} \setminus \{0\}$

x – intercept $(-1, 0)$

No y - intercept

Vertical asymptote $x = 0$

Oblique asymptote $y = x$

$f(1) = 2, f(2) = \dfrac{9}{4}$

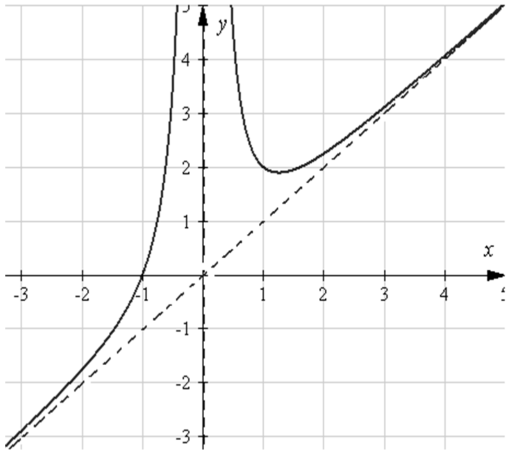


Figure 2.16

f. $g(x) = \dfrac{2x^3}{x^2+1} = 2x - \dfrac{2x}{x^2+1}$

$U = \mathbb{R}$

Intercept : (0, 0)

No vertical asymptote.

Oblique asymptote. $y = 2x$

x	-2	-1	1	2
$f(x)$	$-\frac{16}{5}$	-1	1	$\frac{16}{5}$

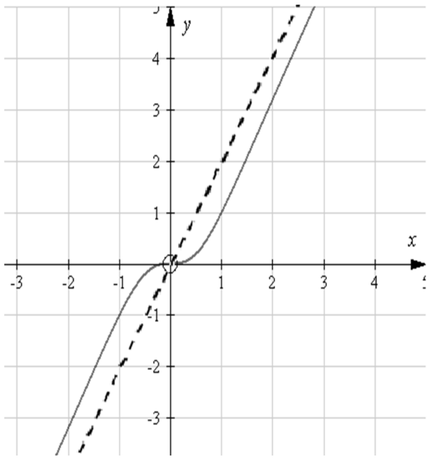


Figure 2.17

UNIT 3 COORDINATE GEOMETRY

INTRODUCTION

The plane figures that will be discussed in this unit namely lines, circles, parabolas, ellipses and hyperbolas are interesting curves that have wide applications. In this unit, we will deal with these plane figures. The unit is divided into two sections. Each section is divided into subsections. The first section is about straight lines and the second section is on conic sections: circles, parabolas, ellipses and hyperbolas. It is a known fact that the planets revolve around the sun in elliptical paths. Satellite receivers are made in parabolic shape because of a very interesting property which a parabola exhibits. Due to their stable property, smoke vents in nuclear reactors are made in hyperbolic shape.

Students may find this unit very interesting and practical. You can help the students to investigate their surroundings for objects that use these curves.

Unit Outcomes

After completing this unit, students will be able to:

- *understand specific facts and principles about lines and circles.*
- *know basic concepts about conic sections.*
- *know methods and procedures in solving problems on conic sections.*

Suggested Teaching Aids in Unit 3

In this unit, use of teaching aids is vital for the better understanding of the concepts. Most of the topics discussed in this unit are of practical nature. Therefore, apart from the student textbook and teachers' guide, the following teaching aids can be used for instruction purposes.

- Sample graphs of the conic sections.
- Cones made from cartons, wood or other materials.
- Ruler and Mathematical instruments.
- Thread of various lengths.
- Graphing softwares like Geometer's Sketch Pad, Fathom, Matlab, Mathematica, etc. (you can find free software from the internet).

3.1 STRAIGHT LINES

Periods Allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *write different forms of equation of a line.*
- *determine the slope, x-intercept and y-intercept of a line from its equation.*
- *determine the angle between two intersecting lines on the coordinate plane whose equations are given.*
- *determine the distance between a point and a line given on the coordinate plane.*

Vocabulary: Coordinate plane, Point, Line, Inclination, Angle, Slope, Intercept, Distance, Parallel, Perpendicular

Introduction

The purpose of the section is to revise lessons learnt about lines, slopes of lines and intercepts of lines, and to show the students how to relate the angle of inclination of a line and the slope of a line, and how to find distance from a point to a line.

Start the unit by revising the lessons the students have learned so far such as equation of straight lines, x-intercept and y-intercept. Encourage students to use straight edges to draw lines instead of doing free sketching. For delivery of this sub-unit, you need to tell the students to come to class with Mathematical Instruments.

3.1.1 Angle Between two Lines on the Coordinate Plane

Teaching Notes

Before starting the lesson, it is advisable to assess the background of the students about lines through oral questions and answers. Pursuant to this, you can start the lesson by revising about lines. To do so, form groups of students and let them do Activity 3.1. The purpose of this Activity is to revise what the students learned so far about lines, perpendicularity and parallelism. The important facts that students need to remember are pointed out in the student textbook.

Answers to Activity 3.1

1. Equation of the line through the points P (1, 4) and Q (3, –2) is $y = -3x + 7$.

   So, slope = –3 and y-intercept = 7.

2. a. neither b. horizontal c. vertical d. neither

3. a. intersecting but not perpendicular

   b. intersecting and perpendicular

   c. parallel

   d. Intersecting and perpendicular

After the students complete the activity, you can proceed to discuss angle between two lines on the coordinate plane. To do so, let them draw various lines on a coordinate plane and:

- measure their inclinations using compass.
- draw other lines with the same inclination and stress the fact that such lines are parallel.
- given a line, let students draw lines which are perpendicular to it by using compass.
- given a line, let students measure angle of inclination and find its slope using a trigonometric table or a calculator.
- by using a ruler, let students take a point on a line and measure the distance from the point to the x and y-axis and decide what the slope of the line is.

Activities like the above help the students to grasp the concepts of inclination, parallelism and perpendicularity. When you make sure that they can do the above activities, you can guide the students to do Group Work 3.1. The main purpose of this group work is to help the students relate the angle of inclination and slope of a line. In this group work, it is hoped that students will recognize that if α is the angle that a line makes with the positive x-axis, then the slope m of the line is $\tan \alpha$.

Answers to Group Work 3.1

a. $\tan\alpha = \dfrac{y}{x - x_0}$ b. Slope of $\ell_1 = \dfrac{y}{x - x_0}$ c. Slope of $\ell_1 = \tan\alpha$

d. $\alpha = 90^{\circ}$ e. 0° f. Yes

This time you can define angle of inclination of a line as presented in the student textbook. Here, what the students know is the angle of inclination that was considered with respect to the x-axis. But, it may sometimes be of interest to measure an angle of inclination of a line with respect to another line that we commonly call an angle between two lines. Before going into the definition of this angle, you can form groups and let them do activity 3.2. The purpose of this activity is to help students see the relationship between the angles of inclinations of the two lines and the angle between these lines.

Answers to Activity 3.2

a. the angle of inclination of ℓ_1 is α,

b. the angle of inclination of ℓ_2 is γ,

c. the relation between α, γ and β is,

$\beta = \gamma - \alpha$ and m_1 = slope of $\ell_1 = \tan\alpha$, m_2 = slope of $\ell_2 = \tan\gamma$

Following this activity, you need to define angle between two intersecting lines. To define angle between two intersecting lines, first let the students recall the relationship between the angles of inclinations of the lines and the angle between the lines from activity 3.2.

Since the relationship is $\beta = \gamma - \alpha$, it will be apparent to represent this as $\tan\beta = \tan(\gamma - \alpha)$.

This time guide the students to arrive at the relation

$$\tan\beta = \tan(\gamma - \alpha) = \frac{\tan\gamma - \tan\alpha}{1 + \tan\gamma\tan\alpha} = \frac{m_2 - m_1}{1 + m_1 m_2}, \text{ if } m_1 m_2 \neq -1.$$

Caution: Put emphasis on the fact that the above relation does not work if one of the two intersecting lines is vertical or when the two lines are perpendicular to each other.

Assessment

In order to assess the students' understanding, you can give them exercises as homework and check their work.

Answers to Exercise 3.1

1. a. $\dfrac{y-2}{x+6} = 4 \quad \Rightarrow \quad y - 2 = 4x + 24 \quad \Rightarrow \quad y = 4x + 26$

   b. $\dfrac{y-7}{x+1} = \dfrac{7-6}{-1-6} \quad \Rightarrow \quad \dfrac{y-7}{x+1} = \dfrac{-1}{7} \quad \Rightarrow \quad y - 7 = \dfrac{-1}{7}(x + 1)$

   $\Rightarrow y = \dfrac{-1}{7}x + \dfrac{48}{7}$

   c. $\dfrac{y+4}{x-2} = 7 \quad \Rightarrow \quad y + 4 = 7x - 14 \quad \Rightarrow y = 7x - 18$

   d. $\dfrac{y+4}{x-2} = \dfrac{-1}{2} \quad \Rightarrow \quad y + 4 = \dfrac{-1}{2}x + 1 \quad \Rightarrow y = \dfrac{-1}{2}x - 3$

   e. Let m be the slope of the line.

   Then $\tan 45^{\circ} = \dfrac{m-1}{1+m} \Rightarrow \ 1 = \dfrac{m-1}{m+1}$

   $\Rightarrow \quad m + 1 = m - 1$ which has no solution.

At this point, students may get confused because they have applied the formula properly, but cannot go further. What happens is that $y = x + 2$ makes 45° already and the required line makes another 45° with $y = x + 2$. In other words, the required line is vertical. Thus its equation is $x = 1$.

2. Let α be the angle from ℓ_1 to ℓ_2, then

   a. $\tan\alpha = \dfrac{-1-(-3)}{1+3} = \dfrac{2}{4} = \dfrac{1}{2}$ b. $\tan\alpha = \dfrac{4-3}{1+4\times 3} = \dfrac{1}{13}$

3. $5x + \mathrm{B}y - 6 = 0 \Rightarrow \mathrm{B}y = -5x + 6 \quad \Rightarrow \quad y = \dfrac{-5}{B}x + \dfrac{6}{B}$, provided that $B \neq 0$. Now

   a. to be parallel, $\dfrac{-5}{B} = \dfrac{4}{7} \Rightarrow B = -\dfrac{35}{4}$

   b. to be perpendicular, $\dfrac{-5}{B} = -\dfrac{7}{4} \quad \Rightarrow \quad B = \dfrac{20}{7}$

4. The fixed cost for 5 days is $5 \times 20 = 100$ Birr.
   Since 1 km costs 2 Birr, x km cost $2x$ Birr
   Thus $y = 2x + 100$

5. At 2001 ($x = 0$), it was 7gm per 1000l.
   At 2002 ($x = 1$), it is 0.75 less (Hence 6.25 g per 1000l)

   $\Rightarrow \quad \dfrac{y-6.25}{x-1} = \dfrac{7-6.25}{0-1} = -0.75$

   $\Rightarrow \quad y - 6.25 = -0.75x + 0.75 \Rightarrow y = -0.75x + 7$

3.1.2 Distance between a Point and a Line on the Coordinate Plane

To find distance between a point and a line, there are various approaches. However, that distance is the shortest one. To let the students arrive at this conclusion, you can let them do Activity 3.3.

Answers to Activity 3.3

In this activity, it is hoped that students will recognize that there are infinitely many lines through the point P and that the line segment perpendicular to the line has the shortest length. But, you can let them measure the distance and the angle to justify that the distance is the shortest one and the angle is a right angle.

With this understanding, you can proceed to deriving the formula for the distance from a point to a line. To do so, two steps are used. First, we find the formula for a distance from the origin to a line, i.e., the distance from (0, 0) to the line with equation

Ax + By + C = 0 which is $\dfrac{|\mathrm{C}|}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}$, A ≠ 0 or B ≠ 0 as discussed in the student textbook.

Then using the translation formulas

$x' = x - h$ and $y' = y - k$ which they will see after Group Work 3.2,

the general case for any point (x_o, y_o) is derived to be $\dfrac{|Ax_o + By_o + C|}{\sqrt{A^2+B^2}}$.

Although this approach seems to be easier to calculate the distance of a point from a line, there is also an alternative approach that applies distance between two points. To do so, first you need to find the line that passes through the given point and is perpendicular to the given line. Next, find the point of intersection of the two lines. Finally, compute the distance between the given point and the point of intersection.

Example: Find the distance of the point (3, 1) from the line $y = x + 1$.

To find this distance, we can use the above stated rule, and

$$d = \frac{|Ax_o + By_o + C|}{\sqrt{A^2+B^2}} = \frac{|-1(3)+1(1)+-1|}{\sqrt{1^2+1^2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

We also can determine the distance alternatively using another method. You can ask talented students to do the problem in a different approach as follows.

1. Since our line is $y = x + 1$, the line that passes through (3, 1) and perpendicular to $y = x + 1$ is $y = -x + 4$.
2. The point of intersection of these two lines $y = x + 1$ and $y = -x + 4$ is $\left(\frac{3}{2}, \frac{5}{2}\right)$.
3. The distance of the point (3, 1) from the line $y = x + 1$ is the same as the distance between the points (3, 1) and $\left(\frac{3}{2}, \frac{5}{2}\right)$ which is

$$d = \sqrt{\left(\frac{3}{2}-3\right)^2 + \left(\frac{5}{2}-1\right)^2} = \frac{3\sqrt{2}}{2}$$

To help them derive the translation formula, let them do Group Work 3.2 by themselves.

Answers to Group Work 3. 2

1. From the picture, guide the students to reach to the formulas

   $x' = x - h$ and $y' = y - k$

2. P (–3, 2) and C(h, k) = (3, 4)

   Thus $x' = x - h = -3 - 3 = -6$ and $y' = y - k = 2 - 4 = -2$

   Thus in the $x'y'$ – system P (–6, –2)

Assessment

You can assess the understanding of the students when they do group work. You can also give them homework a problem of the following type.

Given the points (1, 3), (-3, 1) and (0, -4).

a. Find equations of the lines that pass through any two of the points.

b. Angle of inclination of these lines.

c. Distance from one of the points to the line containing the other two points.

d. Angle between any two of the lines.

Answers to Exercise 3.2

1. a. $d = \dfrac{|C|}{\sqrt{A^2+B^2}} = \dfrac{10}{\sqrt{4^2+3^2}} = 2$ units

   b. $d = \dfrac{|C|}{\sqrt{A^2+B^2}} = \dfrac{2}{\sqrt{1^2+5^2}} = \dfrac{\sqrt{26}}{13}$ units

   c. $d = \dfrac{|C|}{\sqrt{A^2+B^2}} = \dfrac{7}{\sqrt{3^2+1^2}} = \dfrac{7\sqrt{10}}{10}$ units

2. a. $d = \dfrac{|5(-3)+4(2)-3|}{\sqrt{25+16}} = \dfrac{10}{\sqrt{41}} = \dfrac{10\sqrt{41}}{41}$ units

   b. $d = \dfrac{|2(4)-3(0)-2|}{\sqrt{2^2+(-3)^2}} = \dfrac{6}{\sqrt{13}} = \dfrac{6\sqrt{13}}{13}$ units

   c. $d = \dfrac{|2(-3)-3(-5)+11|}{\sqrt{2^2+(-3)^2}} = \dfrac{20}{\sqrt{13}} = \dfrac{20\sqrt{13}}{13}$ units

3.2 CONIC SECTIONS

Periods Allotted: 18 periods

Competencies

At the end of this sub-unit, students will be able to:

- *name the different types of conic sections.*
- *explain how the conic sections are generated (formed).*
- *define circle as a locus.*
- *write equation of a circle.*
- *find the radius and centre of a circle from its equation.*
- *determine whether a given line and circle have a point of intersection or not.*
- *determine the coordinates for the intersection points (s) of intersecting lines.*
- *write equation of a tangent line to a given circle, (where the point of tangency is given.)*
- *write the standard form of equation of a parabola.*
- *draw different types of parabolas.*
- *describe some properties of a parabola.*
- *define "ellipse" as a locus. (set of points on the plane which satisfy a certain given condition).*
- *write the standard form of equation of an ellipse.*
- *sketch ellipse.*
- *describe some related terms (latus rectum, eccentricity, major and minor axis)*
- *define a hyperbola as locus.*
- *write the standard form of equation of a hyperbola.*
- *describe related terms to hyperbola (foci, centre, transverse axis, asymptotes, conjugate axis . . .).*
- *sketch hyperbola based on its given equation.*
- *give eccentricity of a given hyperbola.*
- *solve problems on hyperbola.*

Vocabulary: Cone, Locus, Conic section, Circle, Center, Radius, Parabola, Focus, Directrix, Latus rectum, Axis, Ellipse, Major axis, Minor axis, Hyperbola, Conjugate axis, Transverse axis, Eccentricity

Introduction

The purpose of this section is to deal with the sections formed by the intersection of a cone and a plane; conic sections that include circles, parabolas, ellipses and hyperbolas.

To start the section, you need to introduce cone and sections of the cone using physical models. These will help students to understand why the plane curves discussed in the section got the name conic sections.

Teaching Notes

In this sub-unit, a conic section and each one of the sections formed by the intersection of a cone and a plane; conic sections that include circles, parabolas, ellipses and hyperbolas will be discussed.

3.2.1 Cone and sections of a cone

To present this topic, you may need to have models of a cone that can be made either by using cartons or any other material that can be accessed around the school. With these models, prepare one cone and the other sections of the cone as cut by a plane that represent circle, parabola, ellipse and hyperbola as explained in the student textbook. You may give activities for the students to prepare such prototypes and identify how the conic sections are formed. After doing this, you can discuss each conic section one by one.

3.2.2 Circles

To start this lesson, you may first make sure that the students can identify a circle from the above task. Pursuant to this effort, let students sketch loci of circles by using compass. The students can also use a piece of string to sketch loci of circles.

Tie one end of the string to a nail inserted at the centre of the circle. Attach the other end of the string to a pencil, and, holding the string tight, draw the circle. The radius will be the length of the string. To justify this, you can let them do Activity 3.4 and reach a similar conclusion.

Figure 3.1

Answers to Activity 3.4

In this activity, students will find circles that satisfy certain conditions by themselves. Guide them to take a point P(x, y) on the locus of the curves required and find the distance:

Let P(x, y) be a point on the locus. Then

a. $\sqrt{(x-0)^2+(y-0)^2}=5$ or $x^2+y^2=25$

b. $\sqrt{(x-1)^2+(y+2)^2}=4$ or $(x-1)^2+(y+2)^2=16$

When the students realize that the locus of a point on the circle is at equal distance from a fixed point, called the center, they can algebraically determine this distance between the two points to formulate the equation of a circle. For this purpose, you can use the example presented in the student textbook and encourage the students to determine equations of various circles. That is, you can define a circle using the idea of locus. By using different examples, help the students to find equations of circles and, from a given equation help them how to find radius and centre of a circle.

Once we have a circle on a plane, it is worth discussing possible presentations that the circle can have with a line on the same plane. To discuss this, you can let the students to Activity 3.5. The main purpose of this activity is to show the students all the possible cases of intersection of a circle and a line.

Answers to Activity 3.5

You can use the distance formula which we have seen in the previous section and guide students to compare the perpendicular distance (d) they find with the radius of the circle which is 4. Then using the solutions which are given below, you can define *tangent* line, *secant* line and *point of tangency*.

1. a. $d=\frac{18}{5}=3.6$ b. $d=4$ c. $d=\frac{21}{5}=4.2$

2 a.

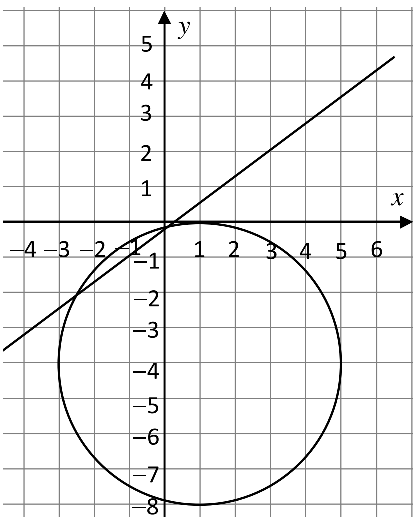


Figure 3.2

b.

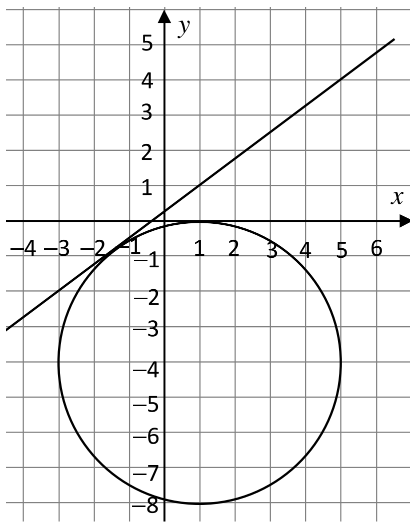


Figure 3.3

c.

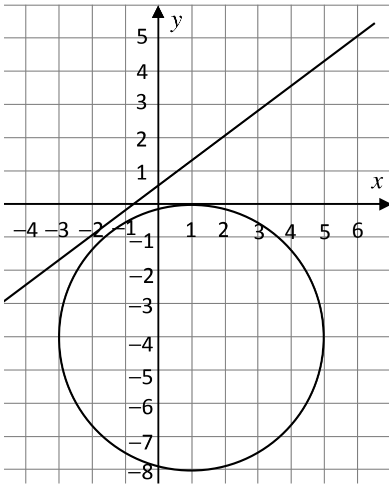


Figure 3.4

So, from the graphs we can observe that

ℓ_1: $3x - 4y - 1 = 0$ is secant line to the circle and $d = \frac{18}{5} < 4$.

ℓ_2: $3x - 4y + 1 = 0$ is tangent line to the circle and $d = 4$.

ℓ_3: $3x - 4y + 2 = 0$ and the circle are disjoint since the distance between the center of the circle and ℓ_3 is greater than the radius of the circle.

When we find the distance between the center of a circle and the line of interest, if:

- the distance is greater than the radius, then the line does not intersect the circle.
- the distance is equal to the radius, then the line intersects at only one point and hence it is tangent to the circle.
- the distance is less than the radius, then the line intersects the circle at two points and hence is secant to the circle.

Assessment

Using exercise 3.3, you can assess the students' understanding about circle and related ideas.

Answers to Exercise 3.3

1. a. $(x + 2)^2 + (y - 3)^2 = 25$ b. $(x - 8)^2 + (y - 2)^2 = 2$

   c. $(x + 2)^2 + (y + 1)^2 = 16$

2. a. C (2, 3) , $r = \sqrt{7}$ b. C (–7, –12), $r = 6$

c. C (–3, –2), $r = \dfrac{\sqrt{7}}{2}$ d. C (1, –3), $r = \sqrt{20} = 2\sqrt{5}$

e $x^2 + y^2 - 8x + 12y - 12 = 0$

$\Leftrightarrow x^2 - 8x + 16 + y^2 + 12y + 36 = 12 + 16 + 36$

$\Leftrightarrow (x - 4)^2 + (y + 6)^2 = 64$ and hence C (4, –6), $r = 8$

f $(x - 1)^2 + (y + 2)^2 = -3$. The equation does not represent a circle (the circle is imaginary).

3. a. C (5, 2). To find r, $r = \sqrt{(0-5)^2 + (0-2)^2} = \sqrt{29}$

Thus, the circle is $(x - 5)^2 + (y - 2)^2 = 29$

b. $(x - 3)^2 + (y + 4)^2 = 9$

c. $d = \sqrt{(4+2)^2 + (5+3)^2} = \sqrt{36 + 64} = 10$

$\Rightarrow r = \dfrac{d}{2} = 5$ and C $(h, k) = \left(\dfrac{4+(-2)}{2}, \dfrac{5+(-3)}{2}\right) = (1, 1)$

Thus $(x - 1)^2 + (y - 1)^2 = 25$ is equation of the circle.

4. C (5, 12). To find r, find the distance from the centre to $2x - y + 3 = 0$

$\Rightarrow r = \dfrac{|2(5) - 12 + 3|}{\sqrt{4+1}} = \dfrac{1}{\sqrt{5}}$ is radius of the circle.

$\Rightarrow (x - 5)^2 + (y - 12)^2 = \dfrac{1}{5}$ is equation of the circle.

5. a. $x^2 + y^2 = 145$; p (9, –8)

Centre C $(h, k) = (0, 0)$ and point of tangency is (9, –8), then,

$\dfrac{y - y_o}{x - x_o} = \dfrac{0-(-8)}{0-9} = \dfrac{-8}{9}$ is the slope of the radius.

The equation of the tangent line is given by: $\dfrac{y - y_o}{x - x_o} = \dfrac{9}{8}$

$\Rightarrow \dfrac{y-(-8)}{x-9} = \dfrac{9}{8} \quad \Rightarrow \dfrac{y+8}{x-9} = \dfrac{9}{8}$

$\Rightarrow y = \dfrac{9}{8}x - \dfrac{145}{8}$ is equation of the tangent line.

b. C (2, 3), P = (–1, 2)

$\Rightarrow m = \dfrac{2-3}{-1-2} = \dfrac{-1}{-3} = \dfrac{1}{3}$ is the slope of the radius.

$\Rightarrow$ The slope of the tangent line is –3.

$\dfrac{y-2}{x+1} = -3$ is equation of the tangent line. Or $y = -3x - 1$

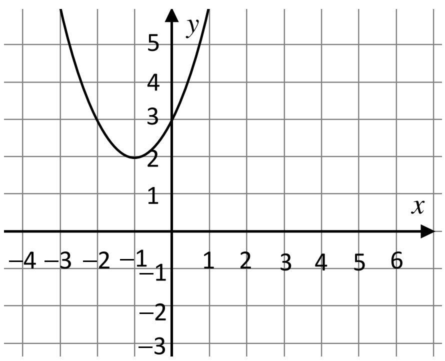
3.2.3 Parabolas

You can start the section by giving the students physical examples of parabolas, such as arcs in bridges, satellite dishes, car head lamps, etc. You can also use local materials that represent parabolas. Then, draw the graphs of different quadratic functions which are parabolas. You can use Activity 3.6 for this purpose. Put emphasis on axis, vertex of a parabola and whether the parabola opens upward or downward. Then, you can define a parabola and different parts of a parabola. Guide the students to derive the standard form of equation of parabolas with vertical and horizontal axes. You can also give different examples on these points.

Answers to Activity 3.6

The activity helps students to observe that quadratic functions have parabolic graphs and that the line of symmetry of the quadratic function is the axis of a parabola.

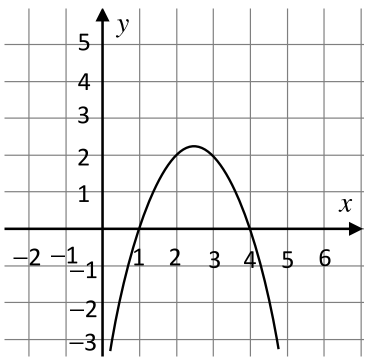
1. a.



$y = x^2 + 2x + 3$

Figure 3.5

b.



$y = -x^2 + 5x - 4$

Figure 3.6

2 a. Axis: $x = -1$ b. Axis: $x = \dfrac{5}{2}$

Although there are various types of parabola say, parabolas whose axes are parallel to the x or y axes, there are also others whose axes might not be parallel to any of these axes. Such parabolas are not discussed in good detail in this grade level. However, if there are talented students who might want to investigate more, you can give them as an assignment of the following type.

Example: Find the equation of the parabola having focus at (1, 1) and directrix y = -x for the talented students.

Solution: Let P(x, y) be any point on the parabola. Then by definition of the parabola, the distance from P to the directrix $y = -x$ is equal to the distance from p to F(1, 1).

From this, $\sqrt{(x-1)^2+(y-1)^2} = \dfrac{|x+y|}{\sqrt{2}}$

Squaring both sides,

$$(x-1)^2+(y-1)^2 = \frac{(x+y)^2}{2} \Leftrightarrow x^2+y^2-2xy-4x-4y+4=0$$

Assessment

For assessing students' understanding, you can use the questions in exercise 3.4. You can also give an assignment for the students to identify a parabola from their surrounding and report the parts in the parabola.

Example: If we consider a satellite dish in their surrounding, the dish represents the parabola and the decoder is located at the focus, etc.

Answers to Exercise 3.4

1. a. $h = -2$, $k = 5$ and $k + p = -8 \Rightarrow 5 + p = -8 \Rightarrow p = -13$

   $\Rightarrow$ The parabola opens downwards. Thus, the equation is

   $(x + 2)^2 = -52\ (y - 5)$

   b. $h = -3$, $k = 4$ and $k + p = 12 \Rightarrow 4 + p = 12 \Rightarrow p = 8$

   $\Rightarrow$ The parabola opens upwards, thus the equation is

   $(x + 3)^2 = 32\ (y - 4)$

   c. $h = 4$, $k = 6$ and $h + p = -8 \Rightarrow 4 + p = -8 \Rightarrow p = -12$

   $\Rightarrow$ The parabola opens to the left, thus the equation is

   $(y - 6)^2 = -48\ (x - 4)$

   d. $h = -1$, $k = 8$ and $h + p = 6 \Rightarrow -1 + p = 6 \Rightarrow p = 7$

   $\Rightarrow$ The parabola opens to the right

   Thus, the equation is $(y - 8)^2 = 28\ (x + 1)$

2. a. $x^2 = 2y \Leftrightarrow (x - 0)^2 = 2\ (y - 0)$

   $\Rightarrow$ Vertex V (0, 0), and $4p = 2 \Rightarrow p = \dfrac{1}{2}$.

The parabola opens upwards, with focus $F\left(0, 0+\frac{1}{2}\right) = \left(0, \frac{1}{2}\right)$ and the directrix is $y = k - p = 0 - \frac{1}{2} = \frac{-1}{2}$, i.e., $y = \frac{-1}{2}$

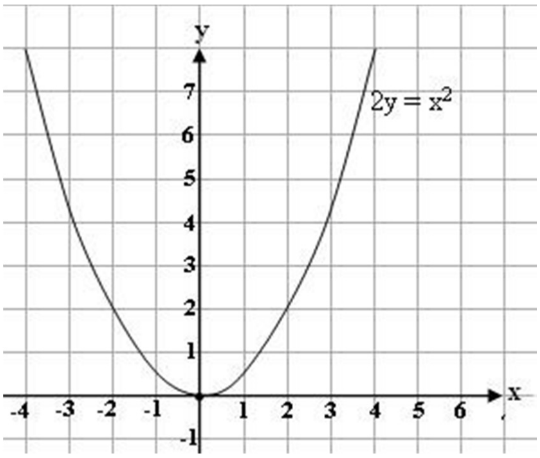


Figure 3.7

b. $(x + 2)^2 = 4(y - 6)$

$\Rightarrow$ V (–2, 6) and $4p = 4 \Rightarrow p = 1$. The parabola opens upwards, with F (–2, 7) and directrix $y = 5$

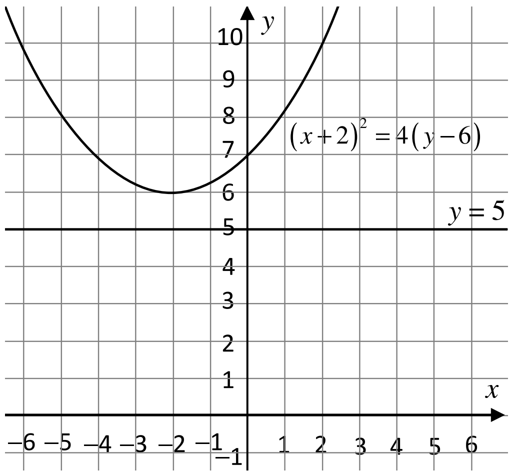


Figure 3.8

c. $(y + 2)^2 = -16(x - 3)$

$\Rightarrow$ V = (3, –2) and $4p = 16 \Rightarrow p = 4$

The parabola opens to the left.

$\Rightarrow$ F = (3+ (–4), –2) = (–1, –2) and directrix $x = h + p = 3 + 4 = 7$

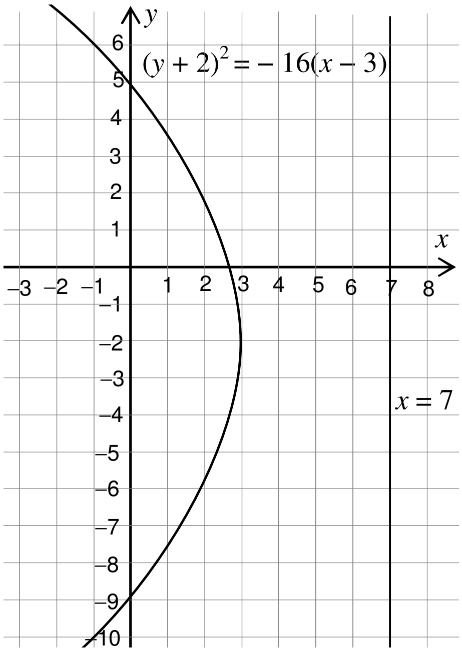


Figure 3.9

d. $(x - 3)^2 = 4y = 4(y - 0)$

$V = (3, 0)$ and $4p = 4 \Rightarrow p = 1$

$\Rightarrow F = (3, 0 + p) = (3, 1)$ and directrix: $y = -1$

The parabola opens upwards

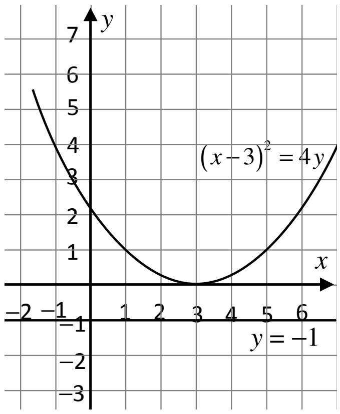


Figure 3.10

3. a. F (3, 5); directrix : $y = 3$

The focus is above the directrix $y = 3$. The distance from F to $y = 3$ is $2p$.

$\Rightarrow p = 1$ and the vertex is V(3, 4).Thus the equation is $(x - 3)^2 = 4(y - 4)$

b. V (–2, 1) ; axis $y = 1$ and $p = 1$,hence the parabola opens to the right

$\Rightarrow F(-2 + 1, 1) \Rightarrow F(-1, 1))$

So $(y - 1)^2 = 4(x + 2)$

c. V (4, 3) ; passes through (5, 2); vertical axis

$\Rightarrow$ Axis $x = 4$

$\Rightarrow$ $(x - 4)^2 = 4p\ (y - 3)$

Since it passes through (5, 2), $(5 - 4)^2 = 4p\ (2 - 3)$

$\Rightarrow$ $1 = 4p\ (-1) \Rightarrow -4p = 1 = p = -\frac{1}{4}$

Hence, we have $(x - 4)^2 = -(y - 3)$

d. F (5, 0); $p = 4$; vertical axis

$\Rightarrow$ $k + p = 0 \Rightarrow k + 4 = 0 \Rightarrow k = -4$

V = (5, –4)

Now the equation is:

$$(x - 5)^2 = 16\ (y + 4)$$

4. a. V (0, 0); Axis $y = 0$

Passes through A (3, 6). Hence the parabola is

$y^2 = 4px$, since it passes through A(3,6)

So $36 = 4p \times 3 \Rightarrow p = \frac{36}{12} = 3$ $\Rightarrow$ the parabola opens to the right

Thus the equation is $y^2 = 12x$

b. V (4, 2) ; Axis $y = 2$; passing through A (8, 7)

$\Rightarrow$ The equation is $(y - 2)^2 = 4p\ (x - 4)$

From this,

$$(7 - 2)^2 = 4p\ (8 - 4) \Rightarrow 25 = 4p\ (4) \Rightarrow 4p = \frac{25}{4}$$

Thus $(y - 2)^2 = \frac{25}{4}\ (x - 4)$

c. V (5, –3); Axis $x = 5$; B (1, 2) is to the left of Axis and above the vertex.

$\Rightarrow$ $(x - 5)^2 = 4p\ (y + 3) \Rightarrow (1 - 5)^2 = 4p\ (2 + 3)$

$\Rightarrow$ $16 = 4p\ (5) \Rightarrow 4P = \frac{16}{5}$

$\Rightarrow$ $(x - 5)^2 = \frac{16}{5}\ (y + 3)$

5. a. Consider the following representation of the reflector.

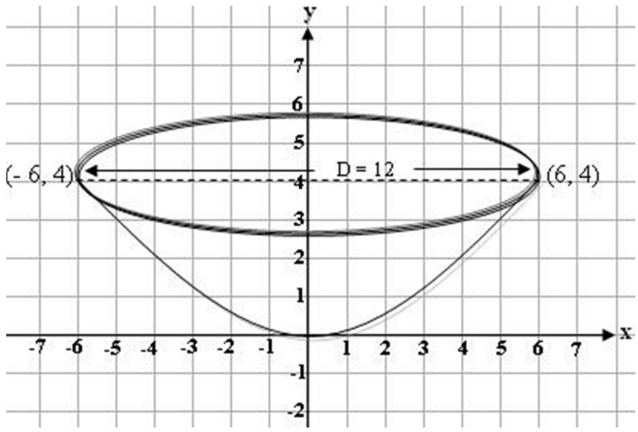


Figure 3.11

We get $x^2 = 4py$

$$\Rightarrow\ 6^2 = 4p4 \Rightarrow 4p = \frac{36}{4} = 9$$

Thus the parabola has the equation $x^2 = 9y$

$\Rightarrow\ p = \frac{9}{4}$ and $F\left(0,\ \frac{9}{4}\right)$. Thus the focus must be at 2.25m.

b. Consider the figure below.

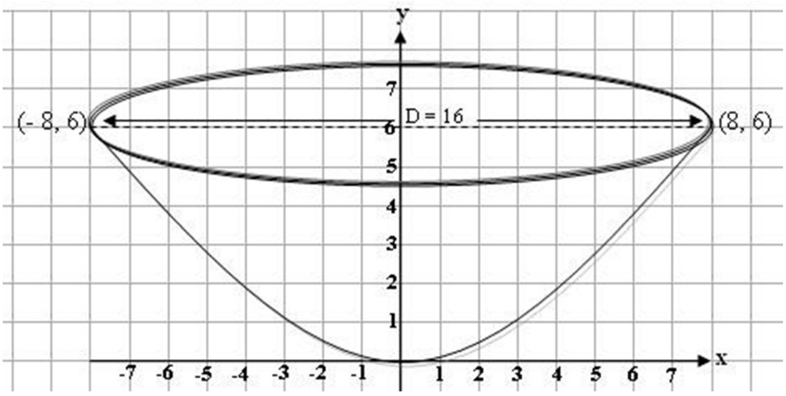


Figure 3.12

The equation is $x^2 = 4py$ and $8^2 = 4p\ (6)\ \Rightarrow\ \frac{64}{6} = \frac{32}{3} = 4p$

$\Rightarrow$ The equation is $x^2 = \frac{32}{3}y$

Now the focus is $F\left(0,\ \frac{8}{3}\right)$.

To find how wide the lamp is at the focus (at the depth of $\frac{8}{3}$ cm) insert

$y = \frac{8}{3}$ in the equation.

$\Rightarrow x^2 = \frac{32}{3}\left(\frac{8}{3}\right) \Rightarrow x^2 = \frac{256}{9}$

$\Rightarrow x = \pm\frac{16}{3}$. The points $\left(\frac{-16}{3}, \frac{8}{3}\right)$ and $\left(\frac{16}{3}, \frac{8}{3}\right)$ are on the parabola.

Now the width is the distance between these points, which is $\frac{32}{3}$.

Notice that this equals to $4p$, and the length is called the **latus rectum**.

Thus for any parabola, latus rectum = the width of the parabola at the focus = $4p$.

6. a. V (1, 2) ; Axis $y = 2$; passes through (6, 3)

$\Rightarrow (y - 2)^2 = 4p\,(x - 1)$

$\Rightarrow (3 - 2)^2 = 4p\,(6 - 1)$ (since the parabola passes through (6, 3))

$\Rightarrow \frac{1}{5} = 4p$ and hence $(y - 2)^2 = \frac{1}{5}(x - 1)$ is equation of the parabola.

b. F (3, 4) ; directrix: $x = 8$

$\Rightarrow$ Axis $y = 4$ and directrix $x = 8$ both meet at (8, 4)

Thus $V = \left(\frac{3+8}{2}, 4\right) = \left(\frac{11}{2}, 4\right)$ and $P = FV = \frac{5}{2}$

Besides, the directrix is to the right of the focus F. Thus the parabola opens to the left and hence $(y - 4)^2 = -10\left(x - \frac{11}{2}\right)$.

3.2.4 Ellipses

Before starting this lesson, it would be good to give the students a reading assignment about the path of the planets around the sun, and some applications of ellipses.

Let the students practice drawing ellipses; to draw an ellipse, put pins on the plane at foci F_1 and F_2, take a piece of string with length ℓ, $\ell > F_1F_2$ (the distance between F_1 and F_2), hold the string tight and draw a curve around F_1 and F_2 as shown in the figure below. The resulting figure will be an ellipse with equation $PF_1 + PF_2 = \ell$, for every point P on the curve.

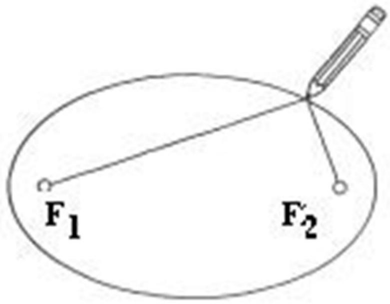


Figure 3.13

Before we proceed to discuss ellipse, it will be advisable for the students to differentiate a circle and an ellipse. For this purpose, you can let them do Group Work 3.3.

Answers to Group Work 3.3

Though there could be different observation notes, the following are recognized differences between a circle and an ellipse.

1. A circle has only one reference point (the center) and the distance from any point of the circle to this reference point is the same.
2. An ellipse has two reference points called foci. Though any point on the ellipse might not be at constant distance from any of the reference points, the length of the string is always constant.
3. They have different shapes.

Construct an ellipse on a gridded plane with centre mid way between F_1 and F_2 and ask students if they can give its equation.

Give the definition of an ellipse and the parts on the ellipse. Then using the definition, derive the standard form of equation of the ellipse by considering horizontal and vertical axes. Referring the drawing on the student textbook, derive the relation between *a*, *b* and *c*. To make them relate these, it will be good to discuss the examples in the students textbook. However, ellipses that have only vertical or horizontal axes are treated in this unit. Those ellipses with oblique axes are not treated. But for talented students, they may raise the questions. So, you can give exercises of the following type.

Example: Find the equation of an ellipse whose center is the origin, one focus at (2, 2) and length of the semi minor axis is $\sqrt{8}$.

Solution: The position of the ellipse is as shown in the following figure.

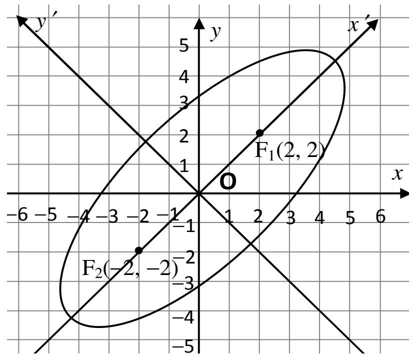


Figure 3.14

If we choose the axes x' and y' so that the x'-axis coincides with the major axis of the ellipse then $c = OF_1 = \sqrt{8}$ and $b = \sqrt{8}$, and from the relation

$a^2 = b^2 + c^2 = 8 + 8 = 16$ we find that $a = 4$.

Let P(x, y) be any point on the ellipse then $PF_1 + PF_2 = 2a$

$$\Rightarrow \sqrt{(x-2)^2 + (y-2)^2} + \sqrt{(x+2)^2 + (y+2)^2} = 8$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-2)^2} = 8 - \sqrt{(x+2)^2 + (y+2)^2}$$

Squaring both sides gives us

$$(x-2)^2 + (y-2)^2 = 64 - 16\sqrt{(x+2)^2 + (y+2)^2} + (x+2)^2 + (y+2)^2$$

$$\Rightarrow x^2 - 4x + 4 + y^2 - 4y + 4 = 64 - 16\sqrt{(x+2)^2 + (y+2)^2} + x^2 + 4x + 4 + y^2 + 4y + 4$$

$$\Rightarrow -8x - 8y = 64 - 16\sqrt{(x+2)^2 + (y+2)^2}$$

$$\Rightarrow x + y + 8 = 2\sqrt{(x+2)^2 + (y+2)^2}$$

Squaring both sides, gives us:

$$\Rightarrow x^2 + y^2 + 2xy + 16x + 16y + 64 = 4x^2 + 16x + 16 + 4y^2 + 16y + 16$$

$$\Rightarrow 3x^2 + 3y^2 - 2xy - 32 = 0$$

The equation of the ellipse is $3x^2 + 3y^2 - 2xy - 32 = 0$

Assessment

It is possible to use the questions in Exercise 3.5 for the purpose of assessing students understanding. You can assess the students by giving home work and check their work.

You can also give them assignments. For example, the planets revolve around the sun in an elliptical orbit. So you can ask the students to find one of the foci of the orbit and other parts of the ellipse in which Mars revolve around by considering sun as the other focus. You can also give Exercise 3.5 question 4 for this purpose.

Answers to Exercise 3.5

1. a. $\frac{x^2}{36} + \frac{y^2}{16} = 1$

   b. Foci (–3, 0), (3, 0), $a = 8$

   $\Rightarrow$ C (0, 0) and $c = 3 \Rightarrow b^2 = a^2 - c^2 = 64 - 9 = 55$

   $\Rightarrow \frac{x^2}{64} + \frac{y^2}{55} = 1$ is equation of an ellipse.

   c. $\frac{y^2}{64} + \frac{x^2}{36} = 1$ d. $\frac{(x-5)^2}{25} + \frac{y^2}{4} = 1$

2. a. C (3, 4), $a^2 = 25$, $b^2 = 16$

   $\Rightarrow c^2 = 9 \Rightarrow c = 3$

   Hence $F_1(0, 4)$ and F_2 (6, 4), V_1 (–2, 4) and V_2 (8, 4)

   B_1 and B_2 (end points of the minor axis) are sometimes called co-vertices.

   They are B_1 (3, 0) and B_2 (3, 8)

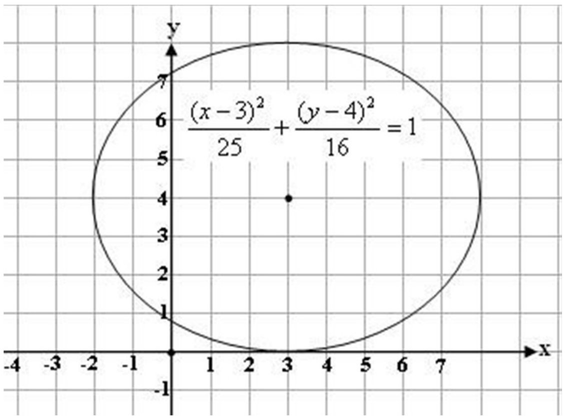


Figure 3.15

   b. C (1, – 2)

   $a = 5, b = 2 \Rightarrow c = \sqrt{25-4} = \sqrt{21}$

   Thus F_1 $(1, -2 - \sqrt{21})$ and F_2 $(1, -2 + \sqrt{21})$

   V_1 (1, –7), V_2 (1, 3), B_1 (–1, –2), B_2 (3, –2)

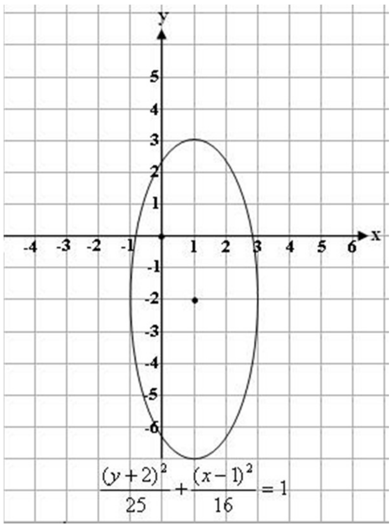


Figure 3.16

c. C (3, 2) $a = 5$, $b = \sqrt{5}$

$\Rightarrow$ $c = \sqrt{20} = 2\sqrt{5}$ $\Rightarrow$ $F_1 (3, 2 - 2\sqrt{5})$, $F_2 (3, 2 + 2\sqrt{5})$

$V_1 (3, -3)$, $V_2 (3, 7)$, $B_1 (3 - \sqrt{5}, 2)$, $B_2 \left(3 + \sqrt{5}, 2\right)$

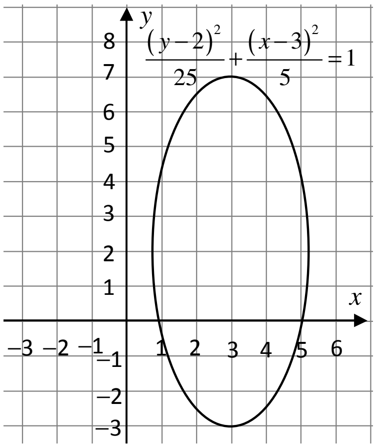


Figure 3.17

3. a. $a = CV_2 = 9$, $b = B_1C = 2$

$$\Rightarrow \frac{(x-1)^2}{81} + \frac{(y-4)^2}{4} = 1$$

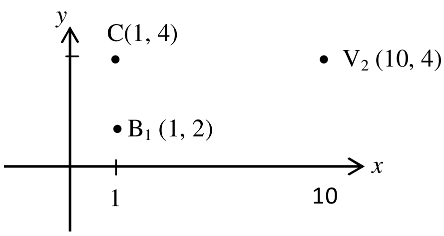


Figure 3.18

b. $F_1 = (-1, 0)$, F_2 $(1, 0)$

$\Rightarrow$ C $(0, 0)$ and $c = 1$

$a = 3 \Rightarrow b = \sqrt{a^2 - c^2} = 2\sqrt{2}$

$\Rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

c. $\Rightarrow c = 1, a = 6 \Rightarrow b^2 = 36 - 1 = 35$

$\dfrac{x^2}{36} + \dfrac{y^2}{35} = 1$

d. C $\left(0, -\dfrac{1}{2}\right)$ F_2 is above C $\left(0, -\dfrac{1}{2}\right)$

$$\Rightarrow \frac{\left(y + \frac{1}{2}\right)^2}{a^2} + \frac{x^2}{b^2} = 1$$

Now $c = \dfrac{3}{2} \Rightarrow a^2 = b^2 + c^2 = b^2 + \dfrac{9}{4}$

$$\Rightarrow \frac{\left(y + \frac{1}{2}\right)^2}{b^2 + \frac{9}{4}} + \frac{x^2}{b^2} = 1$$

Since it passes through (2, 2),

$$\frac{4}{b^2} + \frac{\left(2 + \frac{1}{2}\right)^2}{b^2 + \frac{9}{4}} = 1$$

$$\Rightarrow \frac{4}{b^2} + \frac{\frac{25}{4}}{\frac{4b^2 + 9}{4}} = 1 \Rightarrow \frac{16b^2 + 36 + 25b^2}{b^2(4b^2 + 9)} = 1$$

$\Rightarrow 41b^2 + 36 = 4b^4 + 9b^2 \Rightarrow 4b^4 - 32b^2 - 36 = 0$

$\Rightarrow b^4 - 8b^2 - 9 = 0 \Rightarrow (b^2)^2 - 8\,(b^2) - 9 = 0$

$\Rightarrow b^2 = 9 \Rightarrow 3b = 3 \Rightarrow a^2 = 9 + \dfrac{9}{4} = \dfrac{45}{4}$

Hence, $4\dfrac{\left(y + \frac{1}{2}\right)^2}{45} + \dfrac{x^2}{9} = 1$

e. C $(0, 0)$; V_1 $(0, -5)$; $b = 4 \Rightarrow a = 5$ and $\dfrac{y^2}{25} + \dfrac{x^2}{4} = 1$

4. a. $a = 288$, $b = 227$

$$\Rightarrow \quad c = \sqrt{(228)^2 - (227)^2} = \sqrt{455} \approx 21.33$$

Since the sun is at one of the foci, the distance from the sun to the other focus is the distance between the two foci, i.e. $2c$.

Hence, the distance from the sun to the other foci is 42.66 million km.

b. The shortest distance between Mars and the sun is the distance between the adjacent focus and vertex.

$\Rightarrow$ 228 – 21.33 = 206.67 million km

c. The longest distance is $2c$ + shortest distance (or equivalently length of major axis – shortest distance)

i.e., 2 (21.33) + 206.67 (in millions) = 42.66 + 206.67 = 249.33 millions

or

$2a$ – shortest distance (in millions) = 2 (228) – 206.67

= 456.00 – 206.67 = 249.33 millions

3.2.5 Hyperbolas

Since students have already discussed parabolas, you can brainstorm to begin discussing hyperbolas by asking oral questions such as the following.

What will happen if two parabolas are put on one axis of symmetry but opposite in direction?

Students need to relate, if they can, the foci of each part of the parabola, the distance between the vertices of each parabola, etc. But there might not be definite solution.

Following their inquiry, you can give them the formal definition of hyperbola and related terminologies as stated in the student textbook. Students may have problems with the relationship between a, b and c, since these fixed numbers are used in both equations of ellipses and hyperbolas. Help them to understand that

$a^2 = b^2 + c^2$ for ellipses

while $a^2 + b^2 = c^2$ for hyperbolas.

Besides, while $a > b$ for ellipses, and a corresponds with the major axis, for hyperbolas,

$a < b$, $a = b$ or $a > b$ are possible.

In this case, a corresponds with the transverse axis which also lies on $\overline{F_1F_2}$.

Based on this definition you can derive the formula for a hyperbola. Try to guide the students to take part in the derivation. When you finish the derivation, it will be important to discuss some applications of a hyperbola.

Here are some applications of hyperbolas

1. Most comets that do not move in elliptical orbits around the sun move in hyperbolic orbits.
2. If a sound is heard at three different locations, the sound source can be found using hyperbolas.
3. Notice that the graph of $y = \frac{k}{x}$ is a hyperbola

   $\Rightarrow$ i.e., $xy = k$ is a hyperbola.

   Thus, Boyle's law, PV = C, (V-volume, P- pressure) is a hyperbola.

Cognizant of the fact that the students have now got the ideas of a hyperbola and its formula, they need to practice sketching hyperbolas and related asymptotes. To do so, you can let them do activity 3.7.

Answers to Activity 3.7

This activity helps students to draw asymptotes of hyperbolas using squared papers. Encourage them to find the equations of the asymptotes from the graph.

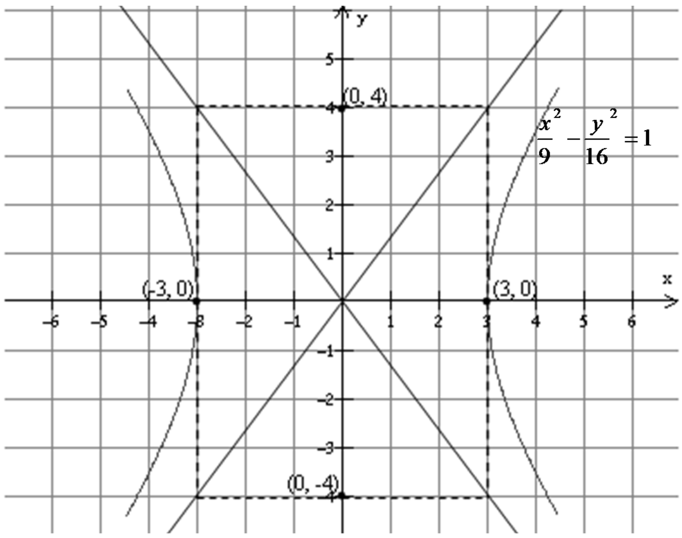


Figure 3.19

When they practice sketching and tried to relate hyperbolas with practical meaning, you may ask students to consider Boyle's Law (for natural science students) and supply and demand curves (for social science students).

Assessment

To assess the understanding of students, you can use several approaches. One approach could be giving them exercise 3.6 as homework. The other approach could be collectively giving a test/quiz that represents all conic sections.

Answers to Exercise 3.6

1. a. $\frac{x^2}{64}-\frac{y^2}{25}=1$

   b. $a = 8$; C (0, 0) ; transverse axis: horizontal and $c = 10$

   $\Rightarrow b^2 = c^2 - a^2 = 100 - 64 = 36$

   $\Rightarrow \frac{x^2}{64}-\frac{y^2}{36}=1$

   c. C (–1, 4), $a = 2$, $b = 3$, vertical transverse axis

   $\Rightarrow \frac{(y-4)^2}{4}-\frac{(x+1)^2}{9}=1$

   d. V_1 (–2, 1), V_2 (2, 1) ; $b = 2$

   $\Rightarrow$ C = (0, 1) and $2a = 4 \Rightarrow a = 2$

   $\Rightarrow \frac{x^2}{4}-\frac{(y-1)^2}{4}=1$

2. a. $\frac{x^2}{36}-\frac{y^2}{81}=1$

   C (0, 0),

   $a = 6, b = 9 \Rightarrow c = \sqrt{117}$

   $\Rightarrow$ Foci $(\pm 3\sqrt{13}, 0)$, V (± 6, 0), B (0, ± 9)

   The asymptotes are $y = \pm\frac{3}{2}x$

   b. C (–3, –6),

   $a = 3, b = 6 \Rightarrow c = \sqrt{45}$

   $\Rightarrow F_1 = (-3-3\sqrt{5}, -6), F_2 = (-3+3\sqrt{5}, -6)$

   V_1 (–6, –6), V_2 (0, –6)

   B_1 (–3, –12), B_2 (–3, 0)

   Asymptotes $y = \pm 2(x + 3) + -6$

   $\Rightarrow y = 2x$ and $y = -2x - 12$

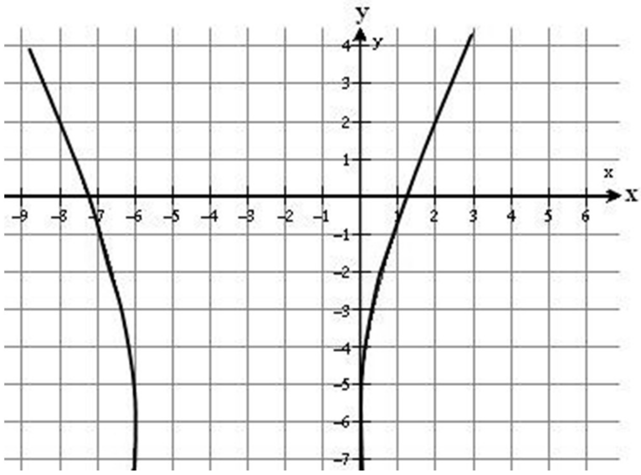


Figure 3.20

c. $\frac{y^2}{25}-\frac{x^2}{16}=1$

C (0, 0), $a = 5$, $b = 4 \Rightarrow c = \sqrt{41}$

$\Rightarrow$ F$\left(0, \pm\sqrt{41}\right)$ and V $(0, \pm 5)$, B $(\pm 4, 0)$

Asymptotes: $y = \pm\frac{5}{4}x$

d. $\frac{(y-3)^2}{25}-\frac{(x-2)^2}{25}=1$

C (2, 3), $a = b = 5 \Rightarrow c = 5\sqrt{2}$

$\Rightarrow$ $F_1\left(2, 3-5\sqrt{2}\right)$, $F_2\left(2, 3+5\sqrt{2}\right)$

V_1 (2, –2), V_2 (2, 8), B_1 (–3, 3), B_2 (7, 3)

Asymptotes: $y = \pm(x-2)+3$

$y = -x+5$ and $y = x+1$

3. a. $\Rightarrow a = 2, c = 3 \Rightarrow b^2 = c^2 - a^2 = 9 - 4 = 5$

$\Rightarrow \frac{(x-4)^2}{4}-\frac{(y+2)^2}{5}=1$

b. $a = 3$ and it is y-hyperbola (transverse axis: vertical)

$\Rightarrow \frac{(y-2)^2}{9}-\frac{(x-4)^2}{16}=1$

Now $4y - 3x = -4 \Rightarrow 4y = 3x - 4 \Rightarrow y = \frac{3}{4}x-1$

$\Rightarrow \frac{a}{b}=\frac{3}{4} \Rightarrow b = 4$

$\Rightarrow \frac{(y-2)^2}{9}-\frac{(x-4)^2}{16}=1$ is the equation of the hyperbola.

c. V_1 (0, –4), V_2 (0, 4), F_1 (0, –5) and F_2 (0, 5)

$\Rightarrow$ C = $\left(0,\ \frac{-4+4}{2}\right) = (0, 0)$

It is a y-hyperbola with $a = 4$, $c = 5$ and $b^2 = c^2 - a^2 = 9$

$\Rightarrow \frac{y^2}{16} - \frac{x^2}{9} = 1$

d. V_1 (–2, 3), V_2 (6, 3), F_1 (–4, 3)

C $\left(\frac{-2+6}{2},\ 3\right) = (2, 3)$

$a = 4$, and $c = 6 \Rightarrow b^2 = 36 - 16 = 20$

$\Rightarrow \frac{(x-2)^2}{16} - \frac{(y-3)^2}{20} = 1$

e. C (2, 0), a = 4 , b = 3

$\frac{(x-2)^2}{16} - \frac{y^2}{9} = 1$

f. $a = 4$, B_1 (5, –5), B_2 (5, 3) $\Rightarrow$ C (5, –1) and $b = 4$

$\Rightarrow \frac{(x-5)^2}{16} - \frac{(y+1)^2}{16} = 1$

4. Suppose $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is equilateral

$\Rightarrow a = b$

Now the asymptotes are

$y = \pm \frac{b}{a}(x-h) + k \Rightarrow$ their slopes are $\frac{b}{a}$ and $\frac{-b}{a}$

Now $\frac{b}{a} . \frac{-b}{a} = \frac{-b^2}{a^2} = \frac{-b^2}{b^2} = -1$, because $a = b$

$\Rightarrow y = \frac{-b}{a}(x-h) + k$ and $y = \frac{b}{a}(x-h) + k$ are perpendicular

Conversely suppose

$y = \frac{-b}{a}(x-h) + k$ and $y = \frac{b}{a}(x-h) + k$

are perpendicular. Then

$\frac{-b}{a} . \frac{b}{a} = -1 \Rightarrow \frac{b^2}{a^2} = 1 \Rightarrow a^2 = b^2$

$\Rightarrow a = b$

We could equally have used $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ and its asymptotes

$y = \pm \frac{a}{b}(x-h) + k$. Thus the proof is done

Answers to Review Exercises on Unit 3

1. a. $0.x + 1.\ y + 3 = 0$ b. $1.x + 0.y - 9 = 0$ c. $x - 2y + 8 = 0$

   d. $x + y - 7 = 0$ e. $3x + 4y - 7 = 0$

2. a. $\frac{y-3}{x+2} = -1 \Rightarrow y - 3 = -x - 2$ or $y = -x + 1$

   b. $\frac{y-7}{x-3} = \frac{-10-7}{6-3} \Rightarrow \frac{y-7}{x-3} = \frac{-17}{3}$

   $\Rightarrow y - 7 = \frac{-17}{3}(x-3) \Rightarrow y - 7 = \frac{-17}{3}x + 17$

   $\Rightarrow y = \frac{-17}{3}x + 24$

   c. $\frac{y+2}{x-3} = 3 \Rightarrow y + 2 = 3x - 9 \Rightarrow y = 3x - 11$

   d. $6x = 2y - 4 \Rightarrow 2y = 6x + 4 \Rightarrow y = 3x + 2$

   Now a line perpendicular to $y = 3x + 2$, has slope $\frac{-1}{3}$.

   Thus, the required line is $y = \frac{-1}{3}x + 4$

3. a. $\begin{cases} y = -2x + 2 \\ y = -3x - 1 \end{cases} \Rightarrow \tan\alpha = \frac{-2-(-3)}{1+6} = \frac{1}{7}$

   b. $x - 6y + 5 = 0 \Rightarrow 6y = x + 5 \Rightarrow y = \frac{1}{6}x + \frac{5}{6}$

   $2y - x - 1 = 0 \Rightarrow 2y = x + 1 \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$

   $\Rightarrow \tan\alpha = \frac{\frac{1}{2}-\frac{1}{6}}{1+\frac{1}{2}.\frac{1}{6}} = \frac{\frac{2}{6}}{1+\frac{1}{12}} = \frac{1}{3} \times \frac{12}{13} = \frac{4}{13}$

   c. $-x - 5y - 2 = 0 \Rightarrow y = \frac{-1}{5}x - \frac{2}{5}$

   $y - 4x + 7 = 0 \Rightarrow y = 4x - 7 \Rightarrow \tan\alpha = \frac{4-\left(\frac{-1}{5}\right)}{1-\frac{4}{5}} = 21$

d. $x - 6y + 5 = 0 \Rightarrow 6y = x + 5 \Rightarrow y = \frac{1}{6}x + \frac{5}{6}$

$2y - x - 1 = 0 \Rightarrow 2y = x + 1 \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$

$$\Rightarrow \tan \alpha = \frac{\frac{1}{2} - \frac{1}{6}}{1 + \frac{1}{2}\cdot\frac{1}{6}} = \frac{4}{13}$$

4. a. $\frac{|2(4) - 3(3) + 2|}{\sqrt{4+9}} = \frac{1}{\sqrt{13}}$ b. $\frac{2}{\sqrt{4+9}} = \frac{2}{\sqrt{13}}$

c. $\frac{|2(-1) + 2|}{\sqrt{4+9}} = 0$ d. $\frac{|3(-2) - 4(4) - 1|}{\sqrt{9+16}} = \frac{23}{5}$

5. a. $d = \frac{|2-6|}{\sqrt{4+9}} = \frac{4}{\sqrt{13}}$

b. $4y = 3x - 1 \Rightarrow 3x - 4y - 1 = 0$

$8y = 6x - 7 \Rightarrow 6x - 8y - 7 = 0 \Rightarrow 3x - 4y - \frac{7}{2} = 0$

Hence $d = \frac{\left|-1 - \left(\frac{-7}{2}\right)\right|}{\sqrt{9+16}} = \frac{\frac{5}{2}}{5} = \frac{1}{2}$

6. a. $(x - 3)^2 + (y + 7)^2 = 9$

b. $r = \frac{|2(3) + 3(-7) - 4|}{\sqrt{4+9}} = \frac{19}{\sqrt{13}}$

The circle is

$(x - 3)^2 + (y + 7)^2 = \frac{361}{13}$

c. $C\left(\frac{3+4}{2}, \frac{-7+3}{2}\right) = \left(\frac{7}{2}, -2\right)$

$d = \sqrt{(4-3)^2 + (3-(-7))^2} = \sqrt{1 + 100} = \sqrt{101}$

$\Rightarrow r = \frac{d}{2} = \frac{\sqrt{101}}{2}$

$\Rightarrow \left(x - \frac{7}{2}\right)^2 + (y + 2)^2 = \frac{101}{4}$

7. $(h, k) = (3, 4)$; $(x_o, y_o) = (1, 0)$

The tangent line is $\dfrac{y - y_o}{x - x_o} = \dfrac{-(x_o - h)}{(y_o - k)}$

$$\Rightarrow \frac{y}{x-1} = \frac{-(1-3)}{-4} \Rightarrow y = \frac{-1}{2}(x-1)$$

8. a. $y^2 = -8x$ b. $(x - 3)^2 = 4(y - 2)$

c. C (0, 0)

$x^2 = 4py \Rightarrow 1 = 4p(1) \Rightarrow 4p = 1 \Rightarrow x^2 = y$

9. a. $(x - 1)^2 = y + 2$

$(x - 1)^2 = 1(y + 2) \Rightarrow 4p = 1 \Rightarrow p = \dfrac{1}{4}$

and V $(1, -2) \Rightarrow$ Focus F $\left(1, -2 + \dfrac{1}{4}\right) = \left(1, -\dfrac{7}{4}\right)$

Axis: $x = 1$

Directrix: $y = \dfrac{-9}{4}$

b. $x^2 = -6y$

V(0, 0)

$4P = 6 \Rightarrow P = \dfrac{6}{4} = \dfrac{3}{2}$ and F $\left(0, -\dfrac{3}{2}\right)$

Directrix: $y = \dfrac{3}{2}$; Axis : $x = 0$

c. $4(x + 1) = 2(y + 2)^2 \Rightarrow 2(x + 1) = (y + 2)^2$

V $(-1, -2)$, $4p = 2 \Rightarrow p = \dfrac{1}{2}$.

Now, the parabola opens to the right

$$\Rightarrow F\left(-1+\frac{1}{2}, -2\right) = \left(-\frac{1}{2}, -2\right)$$

Axis: $y = -2$

Directrix: $x = -1 - \dfrac{1}{2}$ or $x = \dfrac{-3}{2}$

10. a. C (0, 0); $c = 3$; $a = 5 \Rightarrow b^2 = a^2 - c^2 = 25 - 9 = 16$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

b. C (3, 0) ; $a = 4$ and $c = 2$

$\Rightarrow b^2 = 16 - 4 = 12$

Now $\dfrac{y^2}{16} + \dfrac{(x-3)^2}{12} = 1$

c. $b = \dfrac{9}{2}$; C (0, 7) and $c = 4$

$$\Rightarrow a^2 = b^2 + c^2 = \frac{81}{4} + 16 = \frac{81 + 64}{4} = \frac{145}{4}$$

Hence $\dfrac{4x^2}{145} + \dfrac{4(y-7)^2}{81} = 1$

d. $a = 4$, $c = 3$

$\Rightarrow b^2 = a^2 - c^2 = 16 - 9 = 7$

Thus $\dfrac{(x-6)^2}{16} + \dfrac{(y+2)^2}{7} = 1$

11. a. $4x^2 + y^2 = 8$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{8} = 1$$

C (0, 0) $a = 2\sqrt{2}$ $b = \sqrt{2}$

$\Rightarrow$ F $(0, \pm\sqrt{6})$, V $(0, \pm 2\sqrt{2})$, B = $(\pm\sqrt{2}, 0)$

b. $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$

C (1, –2) $a = 3, b = 2$

$\Rightarrow c^2 = a^2 - b^2 = 9 - 4 = 5 \Rightarrow c = \sqrt{5}$

Now $F_1(1, -2 - \sqrt{5})$, $F_2(1, -2 + \sqrt{5})$

V_1 (1, –5), V_2 (1, 1), B_1 (–1, –2), B_2 (3, –2)

12. a. C (0, 0); $c = 9$, $a = 4$

$\Rightarrow b^2 = 81 - 16 = 65 \Rightarrow \dfrac{x^2}{16} - \dfrac{y^2}{65} = 1$

b. $a = 3$, $c = 6 \Rightarrow b^2 = c^2 - a^2 = 36 - 9 = 27$

C (0, 0) $\Rightarrow \dfrac{y^2}{9} - \dfrac{x^2}{27} = 1$

c. $c = 10$ C(0, 0)

Asymptote of the hyperbola is

$$y - 10 = \pm\frac{b}{a}x$$

$$\Rightarrow \frac{b}{a} = 3$$

$b = 3a$

Now $a^2 + b^2 = 10^2$

$(3a)^2 + a^2 = 10^2 \Rightarrow 10a^2 = 100$

$a^2 = 10 \Rightarrow a = \sqrt{10}$

$\Rightarrow \quad b^2 = (3a)^2 = \left(3\sqrt{10}\right)^2 = 90$

$\therefore$ Equation of the hyperbola is $\dfrac{y^2}{90} - \dfrac{x^2}{10} = 1$

13. a. $9x^2 - 16y^2 = 144$

$\Rightarrow \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

$\Rightarrow$ C (0, 0) $\quad a = 4$, $\quad b = 3$

$\Rightarrow c^2 = 4^2 + 3^2 = 16 + 9 = 25 \quad \Rightarrow c = 5$

$\Rightarrow$ F ($\pm$ 5, 0), V ($\pm$ 4, 0) , B (0, $\pm$ 3)

Asymptotes: $y = \pm\dfrac{3}{4}x$.

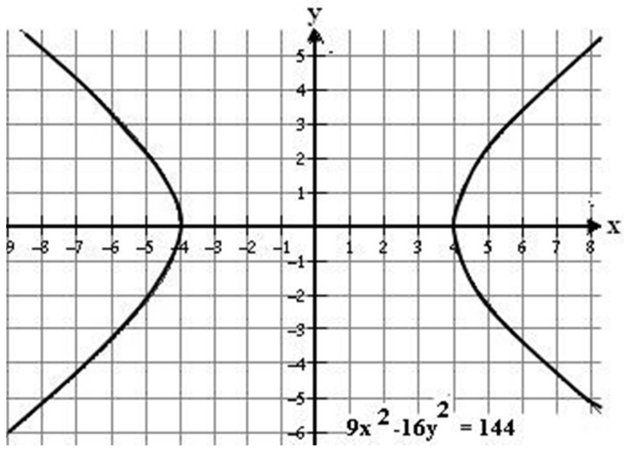


Figure 3.21

b. $\dfrac{(x+3)^2}{25} - \dfrac{(y+1)^2}{144} = 1$

$a = 5$, $b = 12 \Rightarrow c^2 = 25 + 144 = 169 \Rightarrow c = 13$

C $(-3, -1)$;

F_1 $(-16, -1)$, $F_2(10, -1)$;

V_1 $(-8, -1)$, V_2 $(2, -1)$;

B_1 $(-3, -13)$, B_2 $(-3, 12)$.

Asymptotes: $y = \pm\dfrac{12}{5}\left(x + 3\right) + -1$

$\Rightarrow y = \dfrac{12}{5}x + \dfrac{31}{5}$ or $y = \dfrac{-12}{5}x - \dfrac{41}{5}$

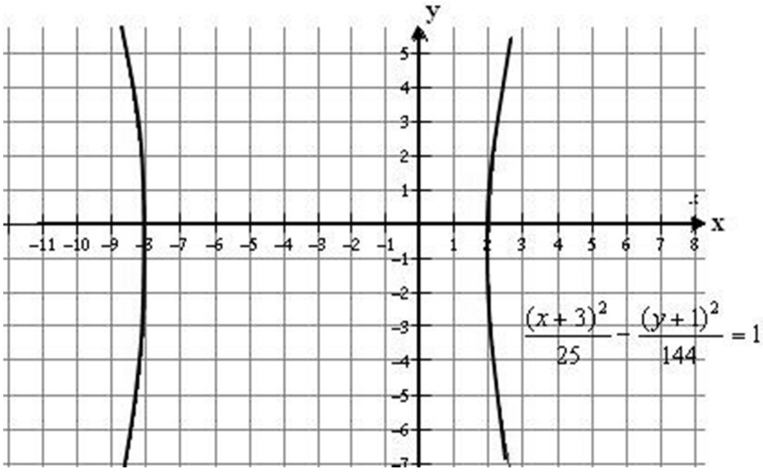


Figure 3.22

14. $a = 25$ and $b = 20$ with center $(0, 0)$

The equation of the ellipse is $\dfrac{x^2}{625} + \dfrac{y^2}{400} = 1$

For $x = 10$, the equation becomes $\dfrac{x^2}{625} + \dfrac{100}{400} = 1$

$$\Rightarrow \frac{x^2}{625} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow x = \frac{25}{2}\sqrt{3}$$

Therefore, the width of the arc at a height of ten meter is $25\sqrt{3}$.

15.

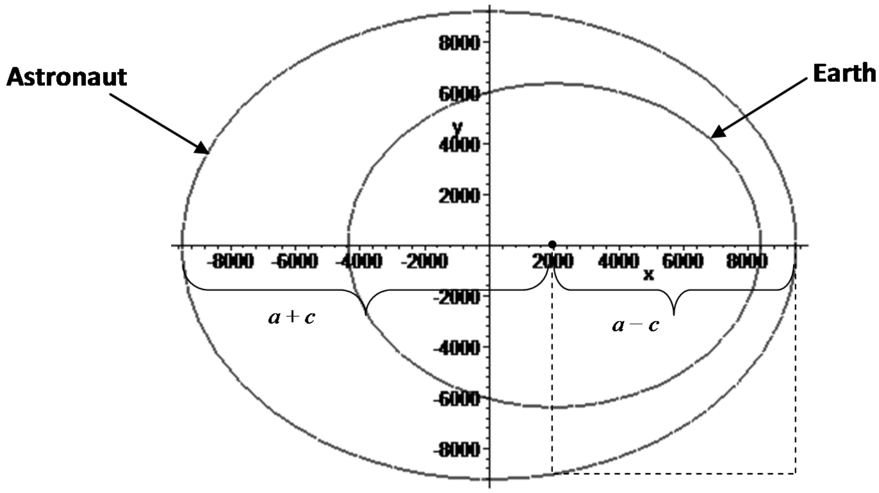


Figure 3.23

From Figure 3.23, since the orbit is elliptical, we have $b^2 + c^2 = a^2$

Where $a - c$ = (6400 + 800)(radius of the earth + shortest distance)

$a + c$ = (6400 + 5400)(radius of the earth + longest distance)

$$\Rightarrow \begin{cases} a - c = 7200 \\ a + c = 11800 \end{cases}$$

$\Rightarrow a = 9500$ and $c = 2300$.

From $b^2 = a^2 - c^2$

Therefore, the path of the astronaut vehicle is $\dfrac{x^2}{90250000} + \dfrac{y^2}{84,960000} = 1$

In our discussion so far, we approached conic sections by using geometric approach where we used string and pin (or nail) to construct each. However, it is possible to get in some books information that conic sections are approached through the following approach. This may not be directly applicable to all students. But, you can give this option to clever students so that they can have another outlook to conic sections.

Optional Topic (Alternative characterization of a conic section)

Define a conic section as a locus of a point which moves so that the ratio of its distance from a fixed point F (Focus) and its distance from a fixed line *d* (Directrix) is a constant *e* (called eccentricity)

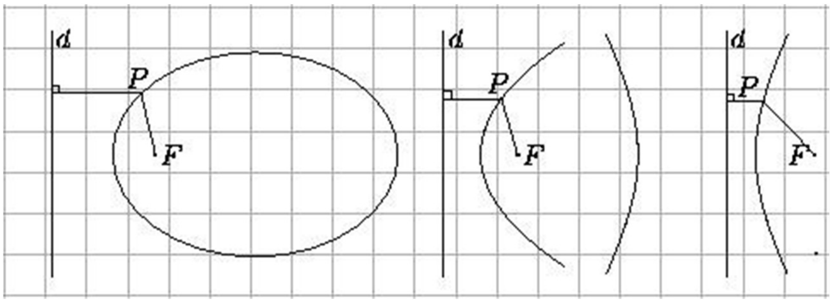


Figure 3.24

For the parabola, the above definition coincides with the definition given earlier so that $e = 1$.

For the ellipse, we can find the range for e as follows:

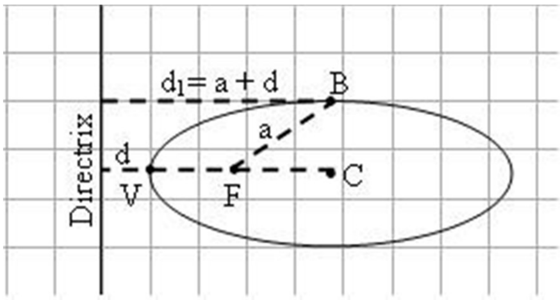


Figure 3.25

Let d be the distance from V to the directrix. Then $\frac{a-c}{d} = e$ and $\frac{a}{a+d} = e$

$\Rightarrow$ $a - c = de$ and $a = ae + de$

$\Rightarrow$ $a - c = a - ae$

$\Rightarrow$ $c = ae$

$\Rightarrow$ $\boxed{e = \frac{c}{a}}$

$\Rightarrow$ $e = \frac{\sqrt{a^2 - b^2}}{a}$

Now $e = 0 \Leftrightarrow a^2 - b^2 = 0 \Leftrightarrow a = b$ i.e $c = 0$

$\Leftrightarrow$ the locus is a circle

On the other hand, $0 < e < 1 \Leftrightarrow 0 < c < a \Leftrightarrow$ the locus is an ellipse $e = \dfrac{c}{a}$ happens to hold for a hyperbola as well.

But $c = \sqrt{a^2 + b^2}$ in this case and $c > a$.

Thus $e > 1 \Leftrightarrow$ the locus is a hyperbola.

Now $e = \dfrac{c}{a} = \dfrac{\sqrt{a^2+b^2}}{a} = \dfrac{\sqrt{a^2\left(1+\dfrac{b^2}{a^2}\right)}}{a} = \sqrt{1+\dfrac{b^2}{a^2}}$

$\Rightarrow$ e infinite corresponds to b infinite

$\Leftrightarrow$ $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ give lines

Indeed, eccentricity is the extent to which a conic section deviates from being circular.

i.e.,

$e = 0$	gives circle
$0 < e < 1$	gives ellipse
$e = 1$	gives parabola
$e > 1$	gives hyperbola
e infinity	gives line

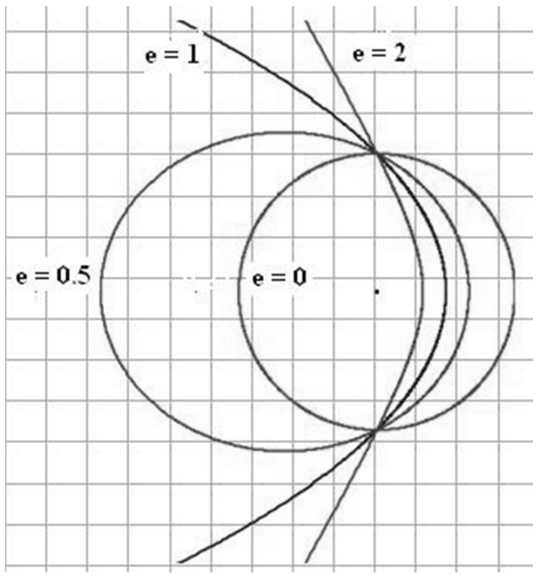


Figure 3.26

UNIT 4 MATHEMATICAL REASONING

INTRODUCTION

The main task of this unit is to introduce mathematical reasoning. *Mathematical Reasoning* helps the students devise strategies to solve a wide variety of mathematical problems. Mathematical Logic is very important in that it enhances the power of reasoning. Mathematical logic has a wide range of applications, particularly in judging the correctness of a certain chain of reasoning as in mathematical proofs. So it is very important to familiarize the students with propositional logic, logical connectives (or logical operators) and make the students determine the truth values of compound propositions given the truth values of component propositions. Finally this unit presents a criterion for evaluating whether a certain argument form is valid or invalid.

Unit Outcomes

After completing this unit, students will be able to:

- *know basic concepts about mathematical logic.*
- *know methods and procedures in combining and determining the validity of statements.*
- *know basic facts about argument and validity.*

Suggested Teaching Aids in Unit 4

You know that students learn in a variety of ways. Some are visually oriented and more inclined to acquire information from photographs or videos. Other students do much better when they hear instructions rather than read them. Teachers use teaching aids to provide students with these different ways of learning. Therefore, it is recommended that you may use charts presenting truth tables for this unit. You may also prepare charts of proofs and let the students argue on the logical flow and validity.

4.1 LOGIC

Periods Allotted: 13 periods

Competencies

At the end of this sub-unit, students will be able to;

- *explain the difference between "statement" and "open statement"*
- *determine the truth value of a statement.*
- *describe the rules for each of the five logical connectives.*
- *use the symbols* $\neg, \wedge, \vee, \Rightarrow$, *and* $\Leftrightarrow$ *to make compound statements.*
- *determine the truth values of compound statements connected by each of the logical connectives.*
- *determine truth values of two or three statements connected by two or three connectives.*
- *describe the properties and laws of logical connectives.*
- *determine the equivalence of two statements.*
- *Define the terms "contradiction" and "tautology".*
- *determine that a given compound statement is either a contradiction or tautology or neither of them.*
- *find the "converse" of a given compound statement.*
- *determine the truth value of the converse of a given compound statement.*
- *find the "contra-positive" of a given statement.*
- *determine the truth value of the contra-positive of a given statement.*
- *describe the two types of quantifiers.*
- *determine the truth value of statements involving quantifiers.*

Vocabulary: Logic, Proposition, Open proposition, Logical connectives, Truth value, Compound proposition, Contradiction, Tautology, Converse, Contrapositive, Quantifiers

Introduction

Under this sub-unit about seven sub-topics are treated separately. Therefore, you can start this topic by raising questions like what logic is. After that, discuss with students as given in the introduction part of the students' textbook and immediately pass to propositional logic.

Teaching Notes

You may start this section by simply asking students the opening problem as given in the students' textbook. The purpose of this opening problem is not to let students discuss it but to inspire their thinking. Hence, you just introduce the opening problem for triggering their thinking. Don't try to show whether the argument is valid or not at this moment because this part will be discussed at the end of this unit. The purpose of the opening problem here is just to show that logic has a power of reasoning.

You can use the opening problem to assess the students' background. Engage them in debate on the opening problem as to whether such a reasoning is accepted or rejected. Only listen to their opinions. Do not go into the detail at this moment.

4.1.1 Statement and Open Statement

The main task of this sub-topic is to make students identify whether a given sentence is a statement (proposition), open statement or neither a proposition nor an open proposition. For this purpose, you may start the lesson by forming groups of students and asking them what a sentence is as given in Group Work 4.1. Make all students construct their own sentences and discuss with their peers in each group whether he/she has constructed a correct sentence.

After discussing the issue of sentence with the students, you may ask them what is meant by assertive or declarative sentence. Or you may proceed as in Group Work 4.1 in identifying a sentence which can be said is True, False or neither True or False. The purpose of this group work is to guide the students identify an assertive or declarative sentence by telling whether each sentence in the group work is true, false or neither.

Answers to Group work 4.1

1. Just give a precise definition of English sentence. Encourage students to construct their own sentence.
2. a. It can be said true.

   b. Cannot be said true or false.

   c. It can be said false.

   d. It can be said true.

e. Cannot be said true or false.

f. Cannot be said true or false.

g. Cannot be said true or false.

h. Cannot be said true or false.

i. It can be said true or false. However, at present, due to the absence of records, we do not know whether it is true or false. But we are certain that it can't both be True and False.

j. Cannot be said True or False.

k. Cannot be said True or False.

l. Cannot be said True or False.

After discussing Group Work 4.1 with participation of students, you systematically define a statement (proposition) and an open proposition. Then give examples of a statement and an open statement and ask them to identify from among various sentences those that are propositions, open propositions, and those that are neither propositions nor open propositions. You can as well use example 1 on page 116 of the student textbook.

Assessment

Engage students in:

- debate on Group Work 4.1 to assess the background of students through discussion. Make the students individually or in group construct a sentence and assess them.
- presentation of assignments and home-works individually or in groups. (For this purpose, you can use Exercise 4.1.) The students might be asked to hand in their home-works and assignments.

Answers to Exercise 4.1

(a), (b), (c), (e), (f), (g), (k), (m) and (n) are all propositions; (i) and (j) are open propositions; but (d), (h) and (l) are neither propositions nor open propositions.

4.1.2 Fundamental Logical Connectives (Or Logical Operators)

The main purpose of this subtopic is to make students use logical connectives (or logical operators) and know the rules that govern these connectives when communicating through logic.

You may start the lesson with statements that are taken from real life situations and connected by the words “and”, “or”, “if…, then…”, “if and only if”; and “not” which is applied only to one or more statements as a whole. You may form groups of students and let them do Activity 4.1. The purpose of this activity is to give chance for each student to answer each of the questions and then discuss them in their group. All of the sentences under Activity 4.1 are formed by using logical operators. First, ask the students to determine (or guess) the truth value of each of the sentences. After collecting information from the students, use the symbols for the logical operators and show the students how to write and read as given in the table in the student textbook. You can then proceed to the rule which is agreed upon systematically and give the rules with examples as given in the truth table in the student textbook.

Answers to Activity 4.1

a.	F	b.	T	c.	T	d.	T	e.	T
f.	T	g.	T	h.	F	i.	T	j.	T

Assessment

Engage students in:

- debate on Activity 4.1 to assess the background of students through discussion. Determining individually or in group the truth values for each statement under Activity 4.1.
- presentation of assignments and home-works individually or in groups. For this purpose, you can use Exercise 4.2.

Answers to Exercise 4.2

Given that:

p: Man is mortal. (True)

q: Botany is the study of plants. (True)

r: 6 is a prime number. (False)

Thus, by using the rules for logical operators we have the following:

a.	T	b.	F	c.	T	d.	T	e.	F
f.	F	g.	F	h.	F	i.	T		

4.1.3 Compound Propositions

So far, students know how to use logical connectives and determine the truth values of statements which are connected by logical connectives. Now, the purpose of this sub-topic is just to give a meaning to statements formed by joining two or more statements by using logical connectives.

Actually, you have several options or possibilities to begin this section. You may use the previous lesson which involves logical connectives. All such statements which involve logical connectives are called compound statements. After having discussed the issue, you can now give the definition of compound proposition as given in the students' textbook. Under this sub-topic, students may also discover compound propositions having the same truth values for each component proposition. Such pairs of compound propositions which have the same truth values are called equivalent propositions.

Assessment

Engage students in:

- discussion on what the students know about compound sentence in the usual English sentence to assess the background of students. Make the students individually or in group to construct truth tables determine the truth values of any given compound statements.
- presentation of assignments and home-works individually or in groups. For this purpose, you can use Exercise 4.3.

Answers to Exercise 4.3

1. The truth values of *p*, *q* and *r* are given in Example 9 of the student textbook as T, F and T respectively. Then applying the rules for logical operators, we have the truth values of each statement as follows:

   a. F   b. T   c. T   d. F   e. F

2. Given that:

   p: The sun rises due East. (T)

   q: 5 is less than 2. (F)

   r: Pigeons are birds. (T)

   s: Laws and orders are dynamic. (T)

   t: Lake Tana is found in Ethiopia. (T)

   a. The sun rises due East and pigeons are birds. (T)

   b. The sun rises due East or pigeons are birds. (T)

   Alternatively one can write as:

   Either the sun rises due East or pigeons are birds. (T)

   c. If the sun rises due East and pigeons are birds, then 5 is less than 2. (F)

d. The sun rises due East and pigeons are not birds if and only if 5 is not less than 2. (F)

e. If the sun rises due East, then either 5 is less than 2 or pigeons are birds. (T)

f. The sun rises due East if and only if 5 is less than 2 and pigeons are birds. (F)

g. If Laws and orders are dynamic, then Lake Tana is found in Ethiopia. (T)

h. Laws and orders are dynamic if and only if Lake Tana is found in Ethiopia. (T)

i. Laws and orders are dynamic and Lake Tana is found in Ethiopia. (T)

3. This problem requires constructing truth tables.

a.

p	**q**	**$p \Rightarrow q$**	**$p \Rightarrow (p \Rightarrow q)$**
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

b.

P	**q**	**r**	**$\neg(p \wedge r)$**	**$p \Rightarrow \neg(p \wedge r)$**
T	T	T	F	F
T	T	F	T	T
T	F	T	F	F
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

c.

p	**q**	**$\neg p$**	**$p \Rightarrow q$**	**$\neg p \vee q$**	**$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$**
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

d.

p	**q**	**$p \wedge q$**	**$p \vee q$**	**$(p \wedge q) \Leftrightarrow (p \vee q)$**
T	T	T	T	T
T	F	F	T	F
F	T	F	T	F
F	F	F	F	T

4. Given that: $p \Rightarrow q$ has truth value T; this is possible only in three cases as far as the rule of implication is concerned.

   These are: a. both p and q have truth value T

   b. both p and q have truth value F

   c. p has truth value F and q has truth value T

   Therefore, in the first two cases (a) and (b) $(p \wedge q) \Leftrightarrow (p \vee q)$ has truth value T, but it has truth value F in the third case (c).

5. $p \Leftrightarrow q$ has truth value T whenever p and q have the same truth value, that is, if both of them are true or both of them are false. From this one can see that p and $\neg q$ have different truth values. Likewise, $\neg p$ and q have different truth values.

   Therefore, both problems in (a) and (b) have truth value F. But (c) is T.

4.1.4 Properties and Laws of Logical Connectives

The main task in this sub-topic is to make the students construct truth tables and tell some properties just by looking at the table. It focuses on familiarizing students with commutative property, associative property, distributive property and De Morgan's laws.

You may start the section by engaging students in Activity 4.2 which requires constructing truth tables. The purpose of this activity is to help the students practice identifying the truth values of each component proposition and the compound proposition. They will also see the properties of logical connectives (operators). Here, the participation of the students is very important. The students need to compare the truth values of each table and come to the conclusion of those properties such as commutative, associative, distributive properties and De Morgan's law as given in the student textbook. Finally, you can let them write some of the properties in their own words and, after a thorough discussion, guide the students to summarize the main properties.

Answers to Activity 4.2

a. Equivalent (show by constructing truth table)

b. Equivalent (show by constructing truth table)

c. Equivalent (show by constructing truth table)

d. Not Equivalent (show by constructing truth table)

e. Not Equivalent (show by constructing truth table)

f. Equivalent (show by constructing truth table)

g. Equivalent (shown in the textbook)

h. Equivalent (shown in the textbook)

Assessment

You can ask the students to tell or write the properties of the logical connectives.

4.1.5 Contradiction and Tautology

This sub-topic focuses on determining whether a certain compound proposition has a truth value T or F regardless of the truth values of the component propositions. Therefore, you discuss Tautology and Contradiction.

You may start this section by forming groups of students and letting them do Group Work 4.2 as given in the students textbook. The purpose of this group work is to let the students complete the tables in the group work which would help them identify relations between component propositions and their connections. Finally, they will find out whether the compound proposition has truth values all T or all F. In this discussion, you will come across compound propositions having truth values all T, all F or none regardless of the truth values of each component proposition. You may discuss with students those group works which are given at the beginning of the section and make them observe the truth tables. Having done this, come to the definition or give a name for such type of compound proposition. Following this discussion, give the definitions, of Tautology and Contradiction as given in the students' textbook.

Answers to group work 4.2

a.

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$	$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

From the above truth table, observe that $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ always has truth value T regardless of the truth values of the component propositions p and q.

b.

p	q	$\neg q$	$p \Rightarrow q$	$p \wedge \neg q$	$(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	T	F	F

From the above truth table, observe that $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$ always has truth value F regardless of the truth values of the component propositions p and q.

c.

p	q	$\neg$ q	p $\vee$ q	p $\vee \neg$ q	(p $\vee$ q) $\Leftrightarrow$ (p $\vee \neg$ q)
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

From the above truth table, observe that $(p \vee q) \Leftrightarrow (p \vee \neg q)$ always has truth value T and F depending on the truth values of p and q.

Now you are going to give a meaning for compound propositions of the type as in group work 4.2 (a) a tautology and 4.2 (b) a contradiction as given in the student textbook.

Assessment

You can make students:

- debate on Group work 4.2 to assess their background through discussion.
- use Exercise 4.4 for home-works and assignments.

Answers to Exercise 4.4

By constructing truth tables, one can reach at the following conclusions:

a. Tautology b. Tautology c. Tautology

d. Neither e. Neither f. Neither

g. Contradiction h. Tautology

4.1.6 Converse and Contrapositive

Under this sub-topic, you make clear to the students, the concepts converse and contrapositive of a given conditional statements. To do this, you may follow the approaches presented under teaching notes.

It is advisable to check at the outset whether the students have properly understood equivalences of compound propositions that they have studied previously. When you make sure that there is solid background, you may start this section by revising conditional statements and trying to interchange the statements under *if clause* and under the *then clause*. Use Activity 4.3 to motivate students and then give the definitions of converse and contrapositive as given in the students textbook.

You may take the following examples to elaborate the relations.

Example: If he is an Ethiopian, then he knows the rules of the country.

(Given conditional sentences)

If he knows the rules of the country, then he is an Ethiopian.

(The converse of the given conditional sentences)

If he doesn't know the rules of the country, then he is not an Ethiopian.

(The contrapositive of the given conditional sentences)

As already shown in previous exercises or by constructing truth tables, we can conclude that:

The given conditional statement is not equivalent to its converse in general.

But the given conditional statement is equivalent to its contrapositive.

Answers to Activity 4.3

a. If a child has the right to be free of corporal punishment, then the sun rise due North.

b. If the sun rise due North, then a child has the right to be free of corporal punishment.

c. If the sun does not rise due North, then a child does not have the right to be free of corporal punishment.

Assessment

Engage students in:

- Debate on Activity 4.3 to assess the background of students through discussion.
- Make the students individually writing each compound sentence under this activity in good English.
- Comparing their works and discussing them.
- Presentation of assignments and home-works individually or in groups.
- For this purpose, you can use Exercise 4.5.

Answers to Exercise 4.5

1. By constructing truth tables of $\neg(p \Rightarrow q), \neg p \Rightarrow \neg q$ and $p \wedge \neg q$, you can show them $\neg(p \Rightarrow q) \equiv p \wedge \neg q$ and $\neg(p \Rightarrow q) \not\equiv \neg p \Rightarrow \neg q$

2. a. Converse: If 6 is prime, then $2 > 3$.

   Contrapositive: If 6 is not prime, then $2 \le 3$.

   b. Converse: If Sudan is in Africa, then Ethiopia is in Asia.

   Contrapositive: If Sudan is not in Africa, then Ethiopia is not in Asia.

   c. Converse: If life would have been simpler, then Ethiopia is in Europe.

   Contrapositive: If life would not have been simpler, then Ethiopia is not in Europe.

3. Use proof by contradiction as in the case of Example 12(c).

   You can do as follows:

   Assumption: k^2 is odd.

   We need to show k is odd.

   Suppose that k is even,

   From which $k = 2n \Rightarrow k^2 = 4n^2 = 2(2n^2)$ which is also even.

   This contradicts, however, the assumption that k^2 is odd. Therefore, our supposition k is even was false. Hence, k is odd is true.

4.1.7 Quantifiers

In this section, you are required to make students aware of the fact that it is possible to convert open statements into statements without explicitly replacing variable(s) by individual entity. This can be done by using quantifiers.

You may start this section by revising open propositions and then showing how to change open propositions to proposition by replacing the blank spaces or the variable(s) by certain entities. Now, systematically, use the phrases "there is" or "for all" in conjunction with the open proposition given and ask students whether the open propositions are converted to propositions or not. You may use Activity 4.4 as given in students' textbook for the purpose of checking whether the open propositions (open statement) become a proposition (statement) or not.

Answers to Activity 4.4

You can do this activity as follows:

Given: $P(x): x+5=7, x$ is a natural number, and

$Q(x): x^2 \ge 0, x$ is a natural number as open propositions, now form the following sentences using the phrase "there is" and the phrase "for all" as follows:

a. There is a natural number x such that $x+5=7$

b. For all natural number x, $x+5=7$

c. There is a natural number x such that $x^2 \geq 0$.

d. For every natural number x, $x^2 \geq 0$

First, make the students analyze each of the above sentences and come to the conclusion that the above sentences (a)-(d) are no more open propositions, since we can determine their truth values as; for example, $x=2$ satisfies (a) and hence it is true, (b) is false since for example $x=3$ does not satisfy it. In a similar manner, you can give arguments for (c) which is true and for (d) which is also true.

Therefore, you can generalize that we can convert open proposition to proposition using the phrases "there is" and "for all" without explicitly substituting the variables by entities.

After having done this, use the symbols as the notation instead of the phrases as given in the students' textbook just below Activity 4.4. Use alternative phrases such as "there is at least one", "there exists one *x*", "for some *x*" for the phrase "there is an *x*" denoted by $\exists x$ which is called existential quantifier; and "for every *x*", "for each *x*" for the phrase "for all" denoted by $\forall x$ which is called universal quantifier.

Thus $(\exists x)P(x)$, $(\exists x)Q(x)$, $(\forall x)P(x)$ and $(\forall x)Q(x)$ are all no more open propositions and hence they are all propositions.

Assessment

Engage students in:

- Debate on Activity 4.4 to assess the background of students through discussion.
- Determining individually and in group the truth values for each statement under Activity 4.4.
- Presentation of assignments and home-works individually or in groups.
- For this purpose you can use Exercise 4.6.

Answers to Exercise 4.6

a. $(\exists x)$ $(4x - 3= -2x + 1)$;its truth value is T

b. $(\exists x)$ $(x^2 + x + 1=0)$;its truth value is F in $\mathbb{R}$.

c. $(\exists x)\,(x^2 + x + 1 > 0)$;its truth value is T

d. $(\exists x)\,(x^2 + x + 1 < 0)$;its truth value is F

e. $(\forall x)\,(x^2 > 0)$; its truth value is F

f. $(\forall x)\,(x^2 + x + 1 \neq 0)$; its truth value is T

g. $(\forall x)\,(4x - 3 = -2x + 1)$; its truth value is F

Relations Between Quantifiers

The main task of this sub-topic is to examine the relation between existential quantifier and universal quantifier. Therefore, to make things clear about this issue, you may follow the methods proposed under the teaching notes.

Having a proposition, it is obvious that the negation is also a proposition. Hence, one can ask what is the form of the negation of $(\exists x)P(x)$ and the form of the negation of $(\forall x)P(x)$. To this effect, you may use Group Work 4.3.

Answers to Group Work 4.3

1. If there is at least one value of x, that makes the statement true.
2. If the statement cannot be false for any value of x. Or the statement is always true for any arbitrary selection of x.
3. If there is at least one value of x, that makes the statement false.
4. If the statement is always false.

Based on the results of Group Work 4.3, you can guide the students to summarize the following.

The proposition $(\forall x)P(x)$ will be false only if we can find an individual "a" such that $P(a)$ is false. If we succeed in getting such an individual "a", then $P(a)$ is false, that means, $(\exists x)\neg P(x)$ is true and hence $(\forall x)P(x)$ is false. Therefore, the negation of $(\forall x)P(x)$ is $(\exists x)\neg P(x)$. Thus, we can describe it as follows using symbols:

$$\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$$

To search the symbolic form of the negation of $(\exists x)P(x)$, proceed as follows: $(\exists x)P(x)$ is false if there is no an individual "a" for which $P(a)$ is true.

Thus, for every x, $P(x)$ is false means for every x, the negation of $P(x)$ is true. Therefore, the negation of $(\exists x)P(x)$ is $(\forall x)\neg P(x)$. Thus, we can describe it as follows using symbols:

$$\neg(\exists x)P(x) \equiv (\forall x)\neg P(x)$$

Use examples to elaborate the concepts discussed here. You may use the examples that are given in the students' textbook.

Assessment

Engage students in:

- Debate on Group Work 4.3 to assess the background of students through discussion.
- Presentation of assignments and home-works individually or in groups.
- For this purpose, you can use Exercise 4.7.

Answers to Exercise 4.7

1. a. $(\forall x)(4x-3 \neq -2x+1)$, its truth value is F.
   b. $(\forall x)(x^2+1 \neq 0)$, its truth value is T.
   c. $(\exists x)(x^2+1 \leq 0)$, its truth value is F.
   d. $(\exists x)(x^2 \geq 0)$, its truth value is T.
   e. $(\forall x)(x^2+x+1 \neq 0)$, its truth value is T.
2. a. T b. T c. T d. F
   e. F f. T g. T

Quantifiers Occurring in Combinations

This sub-topic is concerned with converting open statements involving two variables into a statement. It involves the use of two quantifiers.

Since students are aware of quantifiers, you may begin this section by letting each student do Activity 4.5. The purpose of this activity is to help students discuss the joint consideration of two quantifiers and observe how quantifiers occur in combination. While students do the activity you may round and assist them how to determine the truth values in Activity 4.5.

Answers to Activity 4.5

1. Yes, given a natural it is possible to find a natural number greater than it.
2. No, because, if we take a natural number 1, there is no natural number less than 1.

3. Yes, given an integer n, it is always possible to have an integer $n - 1$ so that $n - 1 < n$.
4. Yes, given x we can find $y = -x$ such that $x + y = 0$
5. a. Yes, $x = 0$

   b. No, since this is true only for $y = 0$ but not for other values of y.

Continue this part by introducing open proposition involving two variables like

$$P(x, y): \ x + y = 5, \text{ where } x \text{ and } y \text{ are natural numbers.}$$

This open proposition can be changed to a proposition either by replacing both variables by certain numbers explicitly or by using quantifiers. To use quantifiers, we have to use either one of the quantifiers twice or both quantifiers in combinations. So, it is important to make some vocabularies on how to read and write such quantifiers. You are expected to make the students practice such writing systems. They can have the following possibilities.

$(\exists x)(\exists y)P(x, y) \equiv$ There is some x and some y so that property P is satisfied.

This statement is true if one can succeed in finding one individual x and one individual y which satisfy property P.

$(\exists x)(\forall y)P(x, y) \equiv$ There is some x for every y so that property P is satisfied.

$\equiv$ There is some x which stands for all y so that property P is satisfied.

This statement is true if one can succeed in finding one individual x which stands for every y so that property P is satisfied.

$(\forall x)(\exists y)P(x, y) \equiv$ For every x there is some y so that property P is satisfied.

$\equiv$ Given x we can find y so that property P is satisfied.

This statement is true if one can succeed in finding one individual y corresponding to a given x so that property P is satisfied.

$(\forall x)(\forall y)P(x, y) \equiv$ For every x and every y property P is satisfied.

This statement is false if one succeeds in finding an individual x and an individual y which does not satisfy property P.

Thus, if we apply this for the open statement:

$P(x, y): \ x + y = 5,$ where x and y are natural numbers,

$(\exists x)(\exists y)P(x, y)$ has truth value T.

$(\exists x)(\forall y)P(x, y)$ has truth value F

$(\forall x)(\exists y)P(x, y)$ has truth value F, since, if x is given to be 6, we cannot find a natural number y so that $6 + y = 5$.

$(\forall x)(\forall y)P(x, y)$ has truth value F.

But, if we change the universe from natural numbers to integers as:

$P(x, y)$: $x + y = 5$ where x and y are integers, then

$(\exists x)(\exists y)P(x, y)$ has truth value T.

$(\exists x)(\forall y)P(x, y)$ has truth value F.

$(\forall x)(\exists y)P(x, y)$ has truth value T, since, given x, we can take $y = 5 - x$ which is also an integer.

$(\forall x)(\forall y)P(x, y)$ has truth value F.

You can also discuss those examples and exercises which are given in the students' textbook in a similar way.

Assessment

Engage students in:

- Debate on Activity 4.5 to assess the background of students through discussion.
- Presentation of assignments and home-works individually or in groups.
- For this purpose you can use Exercise 4.8.

Answers to Exercise 4.8

1.	a.	F	b.	F	c.	F	d.	F		
	e.	F	f.	T	g.	T				
2.	a.	T	b.	F	c.	F	d.	T	e.	F
	f.	F	g.	F	h.	F	i.	F	j.	T

4.2 ARGUMENTS AND VALIDITY

Periods Allotted: 3 periods

Competencies

At the end of this sub unit, students will be able to:

- *describe what is meant by "argument".*
- *check the validity of a given argument.*
- *use rules of inference to demonstrate the validity of a given argument.*

Vocabulary: Argument, Validity, Premise, Conclusion

Introduction

This section is concerned with problems of decision making. We decide whether a certain chain of reasoning is accepted to be correct or rejected to be incorrect on the basis of its forms. It provides the rules of inferences which play a central role in the principle of reasoning. The theory of inference may be applied to test the validity of an argument in everyday life.

Teaching Notes

You may start this topic by considering simple examples from daily life like:

> If I were a bird, I could fly.
>
> I am not a bird.
>
> Therefore, I can't fly.

Then, ask the students whether the conclusion "I can't fly" is the direct consequence of the sentences given. Or you can begin by considering Activity 4.6 as given in the students' textbook. This activity is meant to help the students check the truth value of a proposition by considering the truth values of other propositions. This will guide them to be able to decide the validity of certain arguments.

Answers to Activity 4.6

1. q must be T.
2. q must be T.
3. q may be T or F**.** Since it is given that p is T and p or q is T, it is possible for q to be either T or F.

After having discussed Activity 4.6 with students, define terms like premises, conclusion, argument forms, the way we write an argument form, valid arguments and invalid arguments as given in the students textbook. Then, by using tables and formal proofs show clearly how to check the validity of argument forms as in Examples 1 and 2 of the students textbook.

Assessment

Use Activity 4.6 to assess the background of the students. In addition, give class works and home-works in order to assess the performance of students. For this purpose you can use Exercise 4.9.

Answers to Exercise 4.9

1. You are required to help the students decide whether the given argument forms are valid or invalid. Therefore, by using truth table method as illustrated in the students' textbook, help the students to evaluate the arguments given. But, for illustration purpose, let's do (a) here.

   a.

p	**q**	**¬p**	**¬p ⇒ q**	**(¬p ⇒ q)∧q**	**(¬p ⇒ q)∧q ⇒ p**
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	T	F
F	F	T	F	F	T

Since $(\neg p \Rightarrow q) \wedge q \Rightarrow p$ is not a tautology, the given argument form is not valid.

Or equivalently you can judge whether the argument form is valid or not, by looking at columns 2, 4 and 6 of the table, there the truth values of the premises are both true in the 3rd row but the truth value of the conclusion is false. Therefore you can conclude that the argument form is invalid.

b. Valid c. Valid d. Invalid e. Invalid

2. (I) a. **Premises:**

   If the rain does not come, then the crops will be ruined and as a result, the people will starve.

   The crops will not be ruined or the people will not starve.

   Conclusion:

   The rain comes.

   b. You can use any appropriate symbol to designate each component sentences

   p: The rain comes.

   q: The crops will be ruined.

   r: The People will starve.

   c. The argument forms can be written as follows using symbols:

$$\neg p \Rightarrow (q \wedge r)$$

$$\frac{\neg q \vee \neg r}{p}$$

d. You are required to help the students decide whether the given argument forms are valid or invalid. Therefore, by using both methods as illustrated in the students' textbook, you may guide the students to evaluate the arguments given.

The workout for checking the validity of d(I) using formal proof is the following.

1. $\neg p \Rightarrow (q \wedge r)$ is true . . . . . .premise
2. $\neg q \vee \neg r$ is true . . . . .premise
3. $\neg(q \wedge r)$ *is true* . . . . 2 and De Morgan's law
4. $q \wedge r$ *is false* . . . . 3 and rule of negation
5. $\neg$p is false . . . . 1 and rule of implication
6. P is true . . . . 5 and rule of negation

Therefore the argument is valid. You can also use truth table.

The workout for checking the validity of d(II):

a. Using truth table

p	**q**	**r**	**¬q**	**¬r**	**p⇒¬q**	**r⇒q**	**(p⇒¬q)∧(r⇒q)**	**(p⇒¬q)∧ (r⇒q) ∧p**	**(p⇒¬q)∧(r⇒q) ∧p ⇒¬r**
T	T	T	F	F	F	T	F	F	T
T	T	F	F	T	F	T	F	F	T
T	F	T	T	F	T	F	F	F	T
T	F	F	T	T	T	T	T	T	T
F	T	T	F	F	T	T	T	F	T
F	T	F	F	T	T	T	T	F	T
F	F	T	T	F	T	F	F	F	T
F	F	F	T	T	T	T	T	F	T

Therefore, the argument is valid.

(II) a. **Premises:**

If the team is late, then it cannot play the game.

If the referee is here, then the team can play the game.

The team is late.

Conclusion:

The referee is not here.

b. You can use any appropriate symbol to designate each component sentences

p: The team is late.

q: It can play the game.

r: The referee is here.

c. The argument forms can be written as follows using symbols:

$$\begin{array}{l} p \Rightarrow \neg q \\ \underline{r \Rightarrow q} \\ \neg r \end{array}$$

d. using formal proof:

1. $p \Rightarrow \neg q$ is true . . . . . .premise
2. p is true . . . . premise
3. $r \Rightarrow q$ is true . . . . premise
4. $\neg q$ is true . . . . 1 and rule of implication
5. r is false . . . . .3 and rule of implication
6. $\neg$ r is true . . . . 5 and rule of negation

Hence, the argument is valid.

Rules of Inferences

The purpose of this section is to show students a method of proof called formal proof which does not involve constructing truth table.

In this section, you are expected to make students be aware of the fact that as the number of component sentences are increasing, using truth tables to decide whether a given argument form is valid or invalid becomes very tedious. It may require 16 rows if the number of component sentence is 4, or 32 rows, if the number of component sentence is 5, and so on. Therefore, it is recommended to make students use the formal proofs and use (as given) some of the rules of inferences listed in the students' textbooks.

Assessment

Engage students in:

- presentation of assignments and home-works on Exercise 4.10 individually or in groups.

Answers to Exercise 4.10

1. a.

$$\begin{array}{c} P \Rightarrow Q \\ R \Rightarrow P \\ R \\ \hline Q \end{array}$$

Proof: 1. R (True) ……………. Premise

2. $R \Rightarrow P$ (True)………..Premise

3. P (True) …………..Modes Ponens

4. $P \Rightarrow Q$ (True)………...Premise

5. Q (True)……...from step (3), (4) again we apply Modes Ponens.

Therefore, the argument form given is valid.

b.

Proof: 1. $\neg Q$ (True) …………...Premise

2. $P \Rightarrow Q$ (True)………...Premise

3. $\neg P$ (True) ……....Modes Tollens

4. $\neg P \Rightarrow \neg R$ (True)……..Premise

5. $\neg R$ (True)…………….from step (3), (4) Modes Ponens.

6. R (False) ……………...from step (5) and rule of negation.

Therefore, the argument form given is not valid.

c.

$$\begin{array}{c} P \Rightarrow \neg Q \\ R \Rightarrow Q \\ \hline \neg R \end{array}$$

Proof: 1. $P \Rightarrow \neg Q$ is True ……...Premise

2. P is True……………....Premise

3. $\neg Q$ is True ……......Modes Tollens

4. Q is False……………...from steps1 and 2

5. $R \Rightarrow Q$ is True………...Promise.

6. R is False ……………...from steps 24 and 5.

6. $\neg R$ is True …………….from step 6 and rule of negation.

Therefore, the argument form is valid.

d.

$$\begin{array}{c} \neg P \wedge \neg Q \\ (\neg Q \Rightarrow R) \Rightarrow P \\ \hline \neg R \end{array}$$

Proof: 1. $\neg P \wedge \neg Q$ is True ....Premise

2. $(\neg P \vee Q)$ is True.............from 1 and De'Morgans law

3. $P \vee Q$ is False ....from 2 and rule of negation

4. Both P and Q are False ......from step 3.

5. $(\neg Q \Rightarrow R) \Rightarrow P$ is True .....Premise.

6. $\neg Q \Rightarrow R$ is False ....from 4 and 5.

7. R is False ....from 4 and 6.

8. $\neg R$ is True.....................from 7 and rule of negation.

Therefore, the argument form is valid.

2. a. The premise is "if a person stays up late tonight, then he/she will be dull tomorrow and if he/she does not stay up late tonight, then he/she will feel that life is not worth living".

The conclusion is "either the person will be dull tomorrow or will feel that life is not worth living".

b. Let P: a person stays up late to night

Q: a person will be dull tomorrow

R: a person feels that life is worth living

c. $p \Rightarrow q, \neg p \Rightarrow \neg r \vdash \neg q \vee \neg r$

d. The argument is not valid.

Answers to Review Exercises on Unit 4

1.	a.	Neither	b.	Neither	c.	Tautology	d.	Contradiction		
2.	a.	T	b.	T	c.	T	d.	F	e.	F
	f.	T	g.	F	h.	T	i.	F		

3. a. 1. $\neg p \wedge q$ is true . . . . premise

2. $(q \vee r) \Rightarrow p$ is true . . . . premise

3. $\neg p$ and q are true . . . . . (1) and rule of "$\wedge$"

4. p is false . . . . (3) and rule of "$\neg$"

5. $q \vee r$ is false . . . . (4) and rule of "$\Rightarrow$"

6. $\neg r$ is true . . . . (3), (5) and rule of "$\vee$"

Therefore, the argument is valid.

b. 1. $p \Rightarrow (q \vee r)$ is true . . . .premise

2. $\neg r$ is true . . . . ……….premise

3. p is true . . . . ………..premises

4. r is false . . . . (2) and rule of "$\neg$"

5. $q \vee r$ is true . . . . (1), (3) and rule of "$\Rightarrow$"

6. q is true . . . . (5) and rule of "$\vee$"

Therefore, the argument is valid.

c. Let: P: Mathematics is a good subject.

q: It is worth learning.

r: The grading system is fair

The argument form is:

$$\begin{array}{l} p \Rightarrow q \\ \neg r \vee \neg p \\ \underline{\quad r \quad} \\ \neg p \end{array}$$

Proof: 1. $p \Rightarrow q$ is true . . . .premise.

2. $\neg r \vee \neg q$ is true . . . . . . .premise.

3. r is true . . . . ……premise.

4. $\neg r$ is false ……………from (3) and rule of $\neg$.

5. $\neg q$ is true……………...from (2) and rule of $\vee$.

6. q is false……………….from (5) and rule of $\neg$.

7. p is false……………….from (1) and rule of $\Rightarrow$.

8. $\neg p$ is true……………..from (7) and rule of $\neg$.

Therefore, the argument is valid.

UNIT 5 STATISTICS AND PROBABILITY

INTRODUCTION

The word statistics has different senses. When we use it in the plural sense it has a meaning equivalent to refilling numerical facts, figures or data. When it is used in singular form, statistics is concerned with the development and application of methods and techniques for the collection, organization, analysis and interpretation of quantitative data. In this unit, we will confine the concepts of statistics to the second meaning. However, the teaching-learning need to be organized with the anticipation that students will have the chance to practice the statistical concepts in real life problems.

Unit Outcomes

After completing this unit, students will be able to:

- *know specific facts about types of data.*
- *know basic concepts about grouped data.*
- *know principles of counting.*
- *apply facts and principle in computation of probability.*

Suggested Teaching Aids in Unit 5

Since statistics is one of the fields that is practically applied in our day-to-day experience, there might be a lot to apply depending on where the discussion is held. Yet, because of supply and access, some of the teaching aids that can be used for teaching statistics and particularly this unit are: Coloured chalks (white board markers), chalk board, different coloured objects (marbles), playing cards, dice, coins, paper for drawing graphs, colour pencils, and straight edged ruler for drawing charts, graphs and various data from different sources.

5.1 STATISTICS

Periods allotted: 14 periods

Competencies

At the end of this sub-unit, students will be able to:

- *identify qualitative and quantitative data.*
- *describe the difference between discrete and continuous variables (data).*
- *identify ungrouped and grouped data.*
- *determine class interval (class size) as required to form grouped data from a given ungrouped data.*
- *make cumulative frequency table for grouped data (for both discrete and continuous).*
- *describe terms related to grouped continuous data, i.e, class limit, class boundary, class interval and class midpoint.*
- *determine class limit, class boundary, class interval and class midpoint for grouped continuous data.*
- *find the mean of a given grouped data.*
- *find median for grouped discrete data.*
- *find median for grouped data (continuous variable).*
- *determine the mode of a given grouped data.*
- *identify data that are unimodal, bimodal and multimodal.*
- *determine the quartiles for a given grouped data.*
- *determine the required deciles of a given frequency distribution.*
- *determine the required percentile of a given frequency distribution.*
- *describe the dispersion of data values.*
- *find the range of a given data.*
- *compute variance for ungrouped data.*
- *calculate variance for grouped data.*
- *solve problems on variance.*

Vocabulary: Statistics, Data, Qualitative data, Quantitative data, Variable, Continuous variable, Frequency distribution, Discrete frequency distribution, Grouped frequency distribution, Class interval, Class limit, Class boundary, Mean, Median, Mode, Quartiles, Deciles, Percentiles, Range, Standard deviation, Variance, Discrete

Introduction

In this unit the students will continue studying methods of statistics that they started in grade 9. The students will learn about types of data, and techniques for presentation of results from any statistical investigation. Measures of location and measures of dispersion for grouped data will be given treatment.

Teaching Notes

So far as students are expected to have some background from grade 9, you can introduce the sub-unit by asking students to define the basic statistical terms and check if they can state the meaning of statistics. After this, it may be good to rehearse the students' understanding about grouped data. For this purpose, you may group the students and ask them to do the opening problem given in the student textbook on page 146. To enrich the students understanding, organize students in different groups and assign them to do tasks like:

a. Collecting data of ages, heights, weights of the students in their class.
b. Group the data into classes.
c. Present the data in any form of a diagram.
d. Compute some measures of central tendency.
e. Calculate the variation of the data they collected.

5.1.1 Types of Data

You may start this lesson by revising the major concepts of statistics that students had studied in their grade 9. Through the use of sufficient and appropriate examples, discuss with students the types of data, i.e qualitative and quantitative data and let the students explain the differences between these types of data. At this juncture, you may also discuss what is meant by "variable" in statistics i.e. the characteristic which can be measured and expressed in quantitative or numerical terms. You may as well introduce the ideas of discrete and continuous variables and lead the students to come up to the conclusion that discrete variable can only have observed values at isolated points along a scale of values. Similarly, let the students conclude that a continuous variable assumes a value at any fractional point along a specified interval of values and generated by the process of measuring. To strengthen the discussion, you can group the students and let them do Activity 5.1. In order to do so, let the students answer the two questions by discussing in pairs and ask some of them to orally give their answers loudly and also write them on the board. Next guide the whole class to discuss the answers which may help them reach the meanings of the statistical terms outlined above. Give corrections whenever necessary.

Answer to Activity 5.1

1. a. qualitative b. quantitative c. qualitative
   d. quantitative e. qualitative f. qualitative
2. a. continuous (if we refer the tag on a shirt it can be discrete)
   b. discrete c. continuous d. discrete
   e. continuous f. continuous

To help you assess the understanding of the students, ask them to give their own examples of qualitative and quantitative data. Let them also give examples of continuous and discrete data. To help them organize their understanding, let them do Group Work 5.1 in groups and discuss their answers.

Answer to Group Work 5.1

1. a. qualitative b. qualitative c. qualitative
   d. discrete quantitative e. continuous quantitative
   f. continuous quantitative g. continuous quantitative
   h. continuous quantitative i. qualitative
2. a. the suitable scale to measure height is centimetre or meter.
   b. the suitable scale to measure speed of a car is km/hour.
   c. the suitable scale to measure monthly income can be in Birr or Dollars or any other unit.

5.1.2 Introduction to Grouped Data

You may start this lesson with a brief description of frequency distribution, which is a table in which possible values for a variable are grouped into classes and the number of observed values which fall into each class is recorded. Following this, introduce grouped data as those data which are organized in a frequency distribution, and explain to the students that we use grouped frequency distribution for the purpose of summarizing a large sample of data. With this brief introduction, you can state the definition in the student text and, by forming groups, give chance for the students to do Example 3. During the discussion, encourage some of the students to give explanations to the whole class and jointly examine the correct results. Help the students to have a clear understanding of the way that is useful in determining class size for a grouped frequency distribution. To check whether the students are able to identify classes and their respective frequencies, let them do Activity 5.2.

Answers to Activity 5.2

1. 66 2. 4

Pursuant to the discussion on grouped frequency distribution, ask the students to determine a frequency of less than some value so that they will be directed to the use of cumulative frequency.

Example: From the following grouped frequency distribution representing scores of students in an exam that counts out of 30%.

Class	Frequency
1 – 6	2
7 – 12	4
13 – 18	7
19 – 24	5
25 – 30	2
	20

Ask the students the following questions:

a. How many students have scored less than 19?

b. How many students have scored less than 15?

Solution:

a. 13 students have scored less than 19 which is represented by the first three classes.

b. With this representation, it is not possible to determine the number of students who scored less than 15. This is so because we cannot determine how many of the 7 students in the third class have scored 13 and 14.

After they see this, guide them to note the idea of cumulative frequency distribution which they will see in subsequent section.

At the end of this subsection, explain terms like the lower and upper class limits, lower and upper class boundaries or exact limits, and class interval and class midpoint. You can give exercise 5.1 as homework.

Assessment

You can assess students through different ways that may include giving them revision exercises from grade 9 or giving them raw data and asking them to form groups. You can also give them variates and ask them to identify whether they represent quantitative or qualitative, and whether they are discrete or continuous.

Answers to Exercise 5.1

1. (a) and (d) are qualitative, and (b) and (c) are quantitative.
2. (a) and (c) are continuous and (b) and (d) are discrete.
3. Maximum value = 76; Minimum value = 22; Number of Classes = 7

Class interval = $\frac{76-22}{7} \approx 7.714 \approx 8$

Score (x)	**Number of Students (x)**
22 – 29	1
30 – 37	8
38 – 45	8
46 – 53	12
54 – 61	6
62 – 69	3
70 – 77	2

a. 8 b. 30 c. 37 d. 1

4.

Score (x)	**Number of Students (x)**
12 – 17	2
18 – 23	3
24 – 29	7
30 – 35	2
36 – 41	5
42 – 47	6
48 – 53	5
54 – 59	9
60 – 65	8
66 – 71	1
72 – 78	2

a. 11

b.

Score (x)	Number of Students (x)	Cumulative Frequency
12 – 17	2	2
18 – 23	3	5
24 – 29	7	12
30 – 35	2	14
36 – 41	5	19
42 – 47	6	25
48 – 53	5	30
54 – 59	9	39
60 – 65	8	47
66 – 71	1	48
72 – 78	2	50

c. 25 d. 2 e. 30

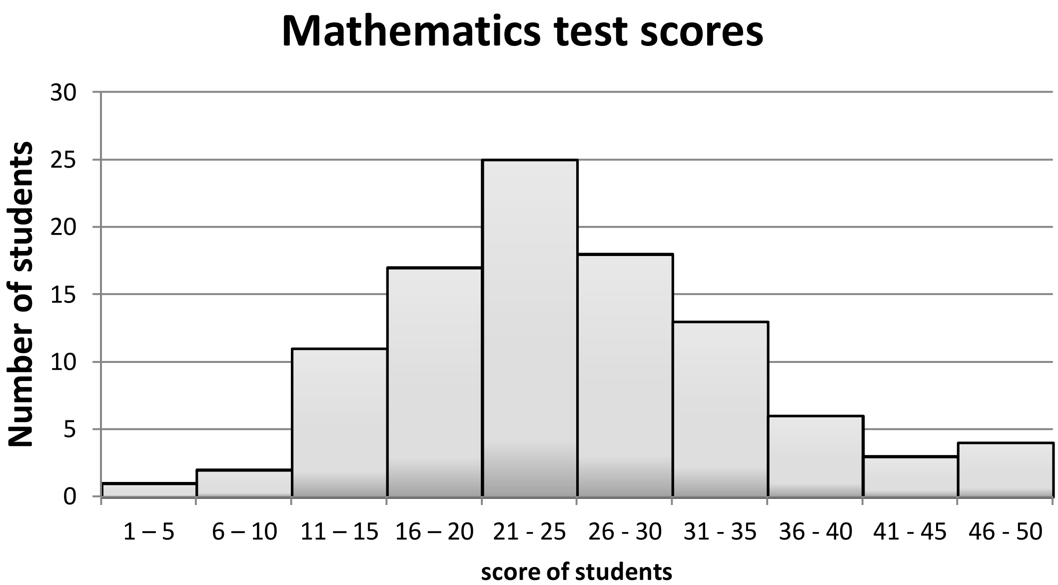
In the previous session, students discussed statistical terms and some ideas about frequency distribution. Here, you need to consolidate the discussion and enrich the use of cumulative frequency. For this purpose, group your students and let them discuss Group Work 5.2. By rounding, assist the groups and identify the group that did better and encourage some of these groups to present their work to the whole class. Summarize their discussion and finally make sure that the students have understood the steps for constructing a frequency distribution stated on pages 153 and 154 of the student textbook. You can give Exercise 5.2 as an assignment so that students can practice it.

Answers to Group Work 5.2

1.

Score (class limit)	Number of students	Class boundaries	Class midpoint	Cumulative frequency
1 – 25	5	0.5 – 25 .5	13	5
26 – 50	10	25. 5 – 50 . 5	38	15
51 – 75	30	50. 5 – 75. 5	63	45
76 – 100	15	75.5 – 100.5	88	60

2.



You can ask clever students to justify why we use class boundaries (true limits) to draw histograms. You can also ask them what will it be if they redraw by using class limits. (this will lead them to think of bar charts).

You can as well ask them the following: Why is the use odd class interval (class width) recommended when constructing grouped frequency distribution? Let them know that it is useful to have odd class width in order to have integer class marks which will make calculations easier.

Assessment

You can assess the understanding of the students by checking their activity in the group work and by keeping records. You can also give them an assignment to collect some data (say number of students by sex and section in grade 11, their age and their weight) and draw a histogram.

Answers to Exercise 5.2

1. a. 20 b. 21 c. 40

 d. Tree type A = 5

 Tree type B = 4

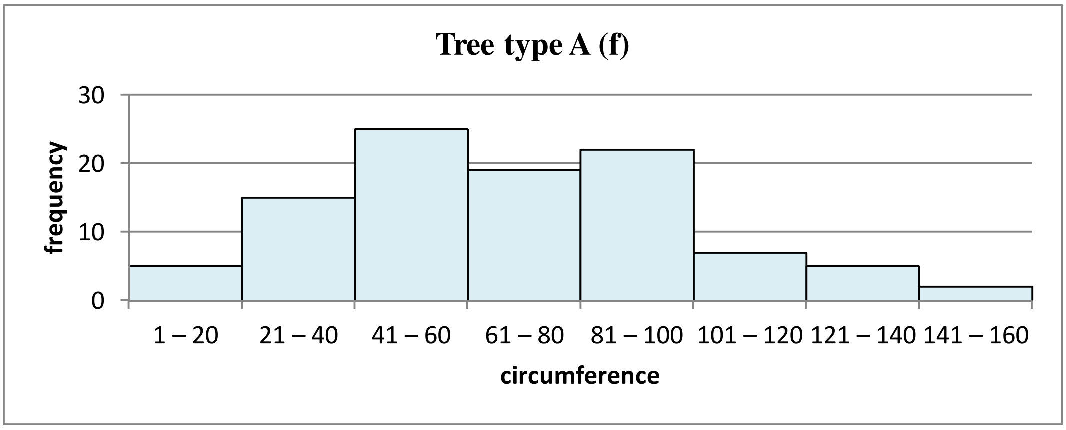
e.

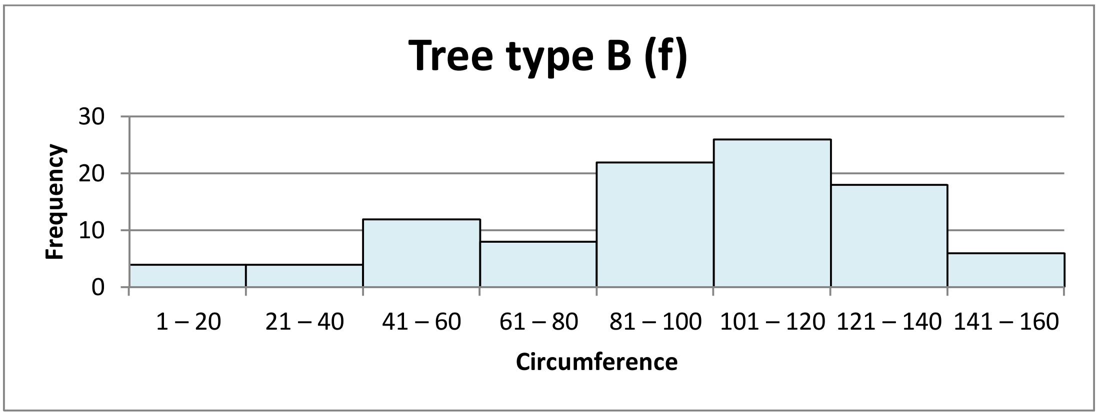
Circumference (cm)	Class boundaries	Class midpoint	Tree type A (f)
1 – 20	0.5 – 20 .5	10.5	5
21 – 40	20. 5 – 40 . 5	30.5	15
41 – 60	40. 5 – 60. 5	50.5	25
61 – 80	60.5 – 80.5	70.5	19
81 – 100	80.5 – 100.5	90.5	22
101 – 120	100.5 – 120.5	110.5	7
121 – 140	120.5 – 140.5	130.5	5
141 – 160	140.5 – 160. 5	150.5	2

f.

Circumference (cm)	Class boundaries	Class midpoint	Tree type B (f)
1 – 20	0.5 – 20.5	10.5	4
21 – 40	20. 5 – 40.5	30.5	4
41 – 60	40. 5 – 60.5	50.5	12
61 – 80	60.5 – 80.5	70.5	8
81 – 100	80.5 – 100.5	90.5	22
101 – 120	100.5 – 120.5	110.5	26
121 – 140	120.5 – 140.5	130.5	18
141 – 160	140.5 – 160.5	150.5	6

g.





2.

Score (x)	Number of Students (x)	Cumulative Frequency
12 – 17	5	2
18 – 23	4	5
24 – 29	6	12
30 – 35	5	14
36 – 41	3	19
42 – 47	1	25
48 – 53	2	30
54 – 59	2	39
60 – 65	1	47
66 – 71	0	48
72 – 78	1	50

a. 24 b. 47.5 c. 0.5 d. 20.5 e. 6

Assessment

To evaluate students' understanding of the concepts discussed so far and to also involve them in the learning actively, give them projects to collect and classify quantitative data based on issues taken from real life. Here are some possible projects which you can assign to your students:

1. Using clinical thermometers (sterilized after each person) take temperatures of the students in the class. Take the readings to the nearest 0.1°c. Ask questions like:

- Does everybody in the class have the same body temperature?
- Are females warmer than males? Or is it the other way round?
- Are males' temperatures more variable?
- Construct grouped frequency distribution.
- Draw histogram for the frequency distribution.

2. Go to the nearest post office and get the data for the number of letters that arrived each day for, say, a month. Which days have more letters? Which days of the week have fewer letters? Construct grouped frequency distribution and Draw histogram for the frequency distribution.
3. You can also consider the distance your students travel from home to school.
4. You can also consider some other examples.

5.1.3. Measure of Location for Grouped Data

a. Mean

You may start this lesson by asking the students to state the definition of the mean. Based on their answers, help them to revise, in brief, the measures of location for ungrouped data and define the concepts of mean or arithmetic mean and clarify it with the help of several examples, and discuss with students how to find mean for ungrouped data. To help them realize how to compute the mean and application for ungrouped data you can give them Activity 5.3. In this activity students will revise what they have learned in grade 9 concerning the mean. Let each student answer the questions and check what they have done. Based on their answers, give them additional problems if necessary.

Answers to Activity 5.3

1. 4.402 meters
2. 76.52
3. To calculate the mean for two graphs of data, first we need to calculate the number of values of each group and also the total number of values.

   Generally, to find the mean of two groups of data,

$$\text{Mean} = \frac{\text{Mean}_1 \times \text{Number of values of } 1^{st} \text{ group} + \text{Mean}_2 \times \text{Number of values of the } 2^{nd} \text{ group}}{\text{Total Number of values obtained from the two groups}}$$

Following this, encourage the students to discuss calculating mean for grouped data. Pursuant to their discussion, let the students clearly understand that, when data have been grouped in a frequency distribution, the midpoint of each class is used as an approximation of all values contained in the class. In addition to this, based on the formal definition of "mean" you already stated and with the help of examples, encourage students to come to the formula for mean of a grouped data is given by:

$\overline{x} = \dfrac{\sum(fx_c)}{\sum f}$ or $\overline{x} = \dfrac{\sum(fx_c)}{n}$ where x_c is the class mark (midpoint) of a class. You can then give exercise 5.3 as a class work or homework to assess the students' understanding. Ask high achievers to calculate the class marks by using $\dfrac{\text{lower class boundary} + \text{upper class boundary}}{2}$ and compare the results with the class mark they calculated by using class limits. Let them conclude that both are always possible and are always equal.

Assessment

You can assess the students by giving them raw data and then asking them (a) to find the mean, (b) to group the data, (c) to find the mean of the grouped data, and finally asking them to compare the results. You can do these either as homework or assignment.

Answers to Exercise 5.3

1. a.

Marks of Students	**Class Mid-point (x_c)**	**Number of Students (f)**	fx_c
10 – 12	11	4	44
13 – 15	14	7	98
16 – 18	17	10	170
19 – 21	20	13	260
23 – 25	24	16	384

$\sum f = 50$ $\qquad$ $\sum fx_c = 956$

Then, $\overline{x} = \dfrac{\sum fx_c}{\sum f} = \dfrac{956}{50} = 19.12$

b.

Age	Class mid-point (x_c)	Number of students (f)	fx_c
13 – 15	14	6	84
16 – 18	17	6	102
19 – 21	20	3	60
22 – 23	22.5	2	45

$\sum f = 17$ $\sum fx_c = 291$

Hence, $\bar{x} = \dfrac{\sum fx_c}{\sum f} = \dfrac{291}{17} = 17.12$

2.

Age	Class mid-point (x_c)	Number of students (f)	fx_c
54 – 58	56	2	112
59 – 63	61	5	305
64 – 68	66	10	660
69 – 73	71	14	994
74 – 78	76	10	760
79 – 83	81	5	405

$\sum f = 46$ $\sum fx_c = 3236$

Hence, $\bar{x} = \dfrac{\sum fx_c}{\sum f} = \dfrac{3236}{46} = 70.35$

3. a. 24.44

   b. The mean depends on the frequency distribution you make.

4. You can let the students form frequency distributions. You can consider the following classes.

With 6 classes

6 – 11	2
12 –17`	6
18 – 23	12
24– 29	16
30 – 35	11
36 – 42	3
	50

Mean = 24.97

With 9 classes

6 – 9	2
10 –13`	2
14 – 17	4
18– 21	8
22 – 25	12
26 – 29	8
30– 33	8
34– 37	4
38– 42	2
	50

Mean = 24.64

By doing so, let them calculate the means.

ii. All the four means are not exactly equal but are close to one another.

iii. The generalization is that the mean depends on the constructed frequency distribution. However, each mean will be closer to one another.

b. Median

After stating the way students can use to determine median of ungrouped data, give a few examples and the questions in exercise 5.4 so that students can discuss and revise their previous discussions from grade 9. Then guide the students to actively participate and discuss the methods and procedures that are used to find the median of a grouped data.

During the discussion, guide the students to use the following procedures and find the median for grouped data.

a. The data should be given in a cumulative frequency distribution.

b. The class which contains the median value has to be determined first.

c. Once this class is identified, the specific value of median is determined by the formula

$$Median = B_L + \left[\frac{\frac{n}{2} - cf_B}{f_c}\right] i$$

Where: B_L = lower boundary of the class containing the median.

n = total number of observations in the frequency distribution

cf_B = the cumulative frequency in the class preceding the class containing the median.

f_c = the number of observations in the class containing the median.

i = the size of the class interval.

You can then give some more examples, such as the questions in Exercise 5.5.

Assessment

You can assess the students by giving them raw data and then asking them (a) to find the median, (b) to group the data, (c) to find the median of the grouped data, and finally asking them to compare the results. You can also use Exercise 5.4 and Exercise 5.5. For the purpose of assessment. You can do these either as homework or assignment.

Answers to Exercise 5.4

1. a. First, we have to arrange in an increasing order and arranging gives:

4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 10, 10, 10

Since the number of observations is 30, which is an even number,

$$m_d = \frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2} \text{ item} = \frac{\left(\frac{30}{2}\right)^{th} \text{ item} + \left(\frac{30}{2}+1\right) \text{ item}}{2}$$

$$= \frac{15^{th} \text{ item} + 16^{th} \text{ item}}{2} = \frac{7+7}{2} = 7$$

b.

x	4	5	6	7	8	10
f	2	5	7	6	7	3

2. From the distribution table we have n = 25 number of observations, which is an odd number.

Thus, $m_d = \dfrac{\left(\dfrac{n+1}{2}\right)^{th} \text{item}}{2} = \dfrac{26}{2} = 13^{th}$

The 13^{th} item = 7

3. a. The median bill is 65. b. The mean bill is 66.

The mean is greater than the median which means that the data is skewed to the right.

4. a. The median is 15. b. The median is 15.

Answers to Exercise 5.5

a. $m_d = 16$

b.

Age	Number of students
12 – 13	6
14 – 15	9
16 – 17	15
18 – 19	8
20 – 21	2

c. $m_d = 16.17$

2. a.

Daily income (in Birr)	Number of Students (frequency)	Commutative frequency (cf)
10 – 14	4	4
15 – 19	11	15 = 4 + 11
20 – 24	17	32 = 15 + 17
25 – 29	16	48 = 32 + 16
30 – 34	8	56 = 48 + 8
35 – 39	4	60 = 56 + 4

The median class is the class containing $\left(\frac{60}{2}\right)^{th}$ item = 30^{th} item and found in the third class.

The median class is 20 – 24

Thus, $B_L = 19.5$, $\frac{n}{2} = 30, f_c = 17$, $cf_b = 15$, i = 5

$$M_d = B_L + \left(\frac{\frac{n}{2} - cf_b}{cf}\right)i = \ 19.5 + \left(\frac{30-15}{17}\right)5 = 19.5 + 4.41 = 23.91$$

Therefore, the median daily income is 23.91. Birr

b.

Marks	Class boundaries	Frequency	Cumulative frequency
20 – 29	19.5 – 29.5	2	2
30 – 39	29.5 – 39.5	12	14 = 2 + 12
40 – 49	39.5 – 49.5	15	29 = 14 + 15
50 – 59	49.5 – 59.5	10	39 = 29 + 10
60 – 69	59.5 – 69.5	4	43 = 39 + 4
70 – 79	69.5 – 79.5	4	47 = 43 + 4
80 – 89	79.5 – 89.5	3	50 = 47 + 3

The median class is the class containing $\left(\frac{50}{2}\right)^{th}$ item$=25^{th}$ item which is found in the third class and its class is 39.5 – 49.5.

Thus, $M_d = 39.5 + \left(\frac{25-14}{15}\right)10 = 395 + 7.33 = 46.83$

Therefore, the median mark is 46.83

3. a. Median = 89.5

b.

Amount of drops	Number of samples
63 – 68	8
69 – 74	2
75 – 80	14
81 – 86	8
87 – 92	13
93 – 98	12
99 – 104	14
105 – 110	3
111 – 116	5
117 – 122	1

$$Median = B_L + \left[\frac{\frac{n}{2} - cf_B}{f_c}\right] i = 86.5 + \left[\frac{40-32}{13}\right] 6 = 90.19$$

4. a. Mean = 25 and Median = 25 b. They are equal.

As you did for mean and median, start the lesson by giving the formal definition of mode and then give examples that explain how to determine the mode of a given ungrouped data. In addition to these, by using sufficient examples, introduce the unimodal, bimodal and multimodal distributions. At this time, you can use exercise 5.6 to assess students understanding.

Assessment

In order to assess whether students understand the concept mode or not, you can use different approaches. You can ask them to list the ages of students in their section and identify the modal age. You can also give them exercise problems such as those given in Exercise 5.6 and ask them to find the mode(s).

Answers to Exercise 5.6

1. a. 7 b. 7 and 12 c. no mode d. 7, 10 and 12

2. a. Mode is 27 b. At 27^{th}
3. a. 8 b. Mahder, because she is the one with mode vote.
4. a. 39 b. Variety of models have shoe size of 39

In the discussion, emphasize that, for a data grouped in frequency distribution with equal class interval, the class containing the mode is determined first by identifying the class with the greatest number of observations known as the modal class. Then, with the modal class, the mode can be determined by using the formula,

$$\text{Mode} = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i$$

Where; B_L = lower boundary of the modal class

d_1 = the difference between the frequency in the modal class and the frequency in the preceding class.

d_2 = the difference between the frequency in the modal class and the frequency in the next class.

i = the size of the class interval.

Let the students discuss the concepts in groups and give them exercise 5.7 as class work or homework to help you assess their understanding. You can also give them as many problems as possible.

Assessment

You can assess the students by giving them raw data and then asking them (a) to group the data, (b) to find the mode(s) of the grouped data, and finally asking them to interpret what it means. You can also use Exercise 5.7 for this purpose. You can do these either as homework or assignment. You can also give them a test/quiz that encompasses mean, median and mode.

Answers to Exercise 5.7

1. a. 8 b. No mode c. 8 and 9 d. 8 and 10 e. 25.125
2. Mode = 256.75
3. Mode = 45.21
4. a. Mode = 87
   b. This has two modes where, mode 1 = 78.5 and mode 2 = 99.42
5. The mode prize is Medal.

You can give high achievers the following questions which will help them think deeper and refer more.

1. Compare and contrast mean, median and mode. This will guide them into the issue of symmetry and skewness.
2. When is it better to use mean as a representative measure of data? This will be better when the effect of extreme value is very small which will lead them into use of standard deviation that they are going to discuss later for selecting either mean, median or mode.

c. Quartiles, Deciles and Percentiles

From the previous discussions, encourage students to explain the measures of central tendency. Let them also explain what makes median different from mean and mode. One simple explanation could be that median is simply measure of location which tells the value at which the distribution is divided into two equal parts. This time, you may give them a clue by asking, say, the question "If we need to classify a distribution into four equal parts, into ten equal parts and into hundred equal parts, then what will the measures be?" After some inquiry, relating with median (which divides a given distribution into two halves) introduce the other measures of locations, i.e. *quartiles* which divide the data into four quarters, the *deciles* which divides the data in to 10 tenths and the *percentiles* which divides the data into 100 parts and guide students to come to the formulas for ungrouped. The formulas for quartiles, deciles and percentiles of ungrouped data are given as follows.

$$Q_k = \left(\frac{kn}{4}\right)^{th} \text{item, or } Q_k = \left(\frac{k(n+1)}{4}\right)^{th} \text{ for quartiles}$$

$$D_j = \left(\frac{in}{10}\right)^{th} \text{item, or } D_j = \left(\frac{i(n+1)}{10}\right)^{th} \text{ for deciles}$$

$$P_t = \left(\frac{tn}{100}\right)^{th} \text{item, or } P_t = \left(\frac{t(n+1)}{100}\right)^{th} \text{ for percentiles}$$

Each formula for grouped data is also given as follows.

$$Q_k = B_L + \left[\frac{\frac{kn}{4} - cf}{f_k}\right] i, \text{ for quartiles}$$

$$D_j = B_L + \left[\frac{in}{10} - cf\right] i, \text{ for deciles}$$

$$P_t = B_L + \left(\frac{\frac{tn}{100} - cf}{f}\right) i, \text{ for percentiles}$$

Here you need to take note that percentiles are different from percentages. Some explanations and illustrative examples are given in the student textbook from pages 166 to 168. After they realize quartiles, deciles and percentiles for ungrouped data, you can group the students and guide them to formulate the formulas for determining quartiles, deciles and percentiles of grouped data. Here, make a note that these measures are more or less derivatives of the formula for median except for determining the class that consists of the specific measure. The formulas and some examples are given in the textbook from pages 168 to 173. You can use some of the questions in Exercise 5.8 for the purpose of enriching students understanding of the way quartiles are calculated and how Q_2 is related with the median.

From the above discussion, you may raise a question on how different formulas can work for the same thing. It is because quartiles have different understanding and representations. The approach given in the student textbook is only one of the different representations and understandings of quartiles. Because of this, quartiles seem to be simple in concept but can be complicated in execution.

Here's where it starts to get confusing. The terms 'quartile', 'upper quartile' and 'lower quartile' each has two meanings. One definition refers to the *subset of all data values* in each of those parts. For example, if I say "my score was in the upper quartile on a math test", I mean that my score was one of the values in the upper quartile subset (i.e. the top 25% of all scores on that test).

But the terms can also refer to *cut-off values* between the subsets. The 'upper quartile' (labelled Q_3) can refer to a cut-off value between the upper quartile subset and the upper middle quartile subset. Similarly, the 'lower quartile' (labelled Q_1) can refer to a cut-off value between the lower quartile subset and the lower middle quartile subset.

The term 'quartiles' is sometimes used to collectively refer to these values plus the median (which is the cut-off value between the upper middle quartile subset and the lower middle quartile subset). John Tukey, the statistician who invented the box-and-whisker plot, referred to these cut-off values as 'hinges' to avoid confusion. Unfortunately, not everyone followed his lead on that.

It gets worse. Statisticians don't agree on whether the quartile values ('hinges') should be points from the data set itself, or whether they can fall between the points (as the median can when there are an even number of data points). Furthermore, if the quartile value is not required to be a point in the data set itself, most data sets don't have a unique set of values $\{Q_1, Q_2, Q_3\}$ that divides the data into four "roughly equal" portions. The SAS statistical software package, for example, allows you to choose from among five different methods for calculating the quartile values. How then do we choose the "best" value for the quartiles?

The answer to that question depends in part on the statisticians' objective in finding quartile values. Tukey wanted a method that was simple to use, "without the aid of calculating machinery." Others seek to minimize the bias in selecting the quartile values. Thus, *different methods have been developed for calculating the quartile values.*

Tukey's method for finding the quartile values is to find the median of the data set, then find the median of the upper and lower halves of the data set. If there are an odd number of values in the data set, include the median value in both halves when finding the quartile values. For example, if we have the data set: {1, 4, 9, 16, 25, 36, 49, 64, 81} we first find the median value, which is 25. Since there are an odd number of values in the data set (9), we include the median in both halves. To find the quartile values, we must find the medians of: {1, 4, 9, 16, 25} and {25, 36, 49, 64, 81}.

Since each of these subsets has an odd number of elements (5), we use the middle value. Thus the lower quartile value is 9 and the upper quartile value is 49.

The TI-83 uses a method described by Moore and McCabe (sometimes referred to as "M-and-M") to find quartile values. Their method is similar to Tukey's, but you do not include the median in either half when finding the quartile values. Using M-and-M on the data set above: {1, 4, 9, 16, 25, 36, 49, 64, 81} we first find that the median value is 25. This time we will exclude the median from each half. To find the quartile values, we must find the medians of: {1, 4, 9, 16} and {36, 49, 64, 81}.

Since each of these data sets has an even number of elements (4), we average the middle two values. Thus the lower quartile value is $\frac{(4+9)}{2}=6.5$ and the upper quartile value is $\frac{(49+64)}{2}=56.5$.

With each of the above methods, the quartile values are always either one of the data points, or exactly half way between two data points.

Those methods involve only simple arithmetic and are easily extendable to octiles (eighths), hexadeciles (sixteenths), etc. They are not, however, extendable to quintiles (fifths) or percentiles (hundredths), etc. Furthermore, they tend to have a high bias. (That is, the quartile values calculated on subsets of the data set tend to vary more, and are not good predictors of the quartile values of the entire data set.)

Mendenhall and Sincich, in their text-Statistics for Engineering and the Sciences, define a different method of finding quartile values. To apply their method on a data set with n elements, first calculate: $Q_1=\frac{1(n+1)}{4}$ and round to the nearest integer. If Q_1 falls halfway between two integers, round up. The Q_1^{th} element is the lower quartile value.

Next calculate: $Q_3 = \frac{3(n+1)}{4}$ and round to the nearest integer. If Q_3 falls halfway between two integers, round down. The Q_3th element is the upper quartile value. So for our example data set: {1, 4, 9, 16, 25, 36, 49, 64, 81}.

n = 9, so $Q_1 = \frac{1(n+1)}{4} = \frac{1(9+1)}{4} = 2.5$ which becomes 3 after rounding up. The lower quartile value is the 3rd data point, 9. Similarly: $Q_3 = \frac{3(n+1)}{4} = \frac{3(9+1)}{4} = 7.5$ which becomes 7 after rounding down. The upper quartile value is the 7th data point, 49.

Using this method, the upper and lower quartile values are always two of the data points.

Minitab uses the same method, except it doesn't round the values of Q_1 and Q_3. Instead, it uses linear interpolation between the two closest data points. For our example above, instead of rounding Q_1 to 3, Minitab would let Q_1 = 2.5 and find the value half way between the 2nd and 3rd data points. In our example, that would be $\frac{(4+9)}{2} = 6.5$. Similarly, the upper quartile value would be half way between the 7th and 8th data points, which would be $\frac{(49+64)}{2} = 56.5$. If Q_1 were 2.25, Minitab would find the value one fourth of the way between the 2nd and 3rd data points and if Q_1 were 2.75, Minitab would find the value three fourths of the way between the 2nd and 3rd data points.

Excel uses a method described by Freund and Perles, which almost no one else uses. To apply this method on a data set with n elements, Excel first calculates $Q_1 = \frac{1(n+3)}{4}$. The Q_1th element is the lower quartile value. If Q_1 is not an integer, Excel uses linear interpolation. Next it calculates $Q_3 = \frac{1(3n+1)}{4}$. The Q_3th element is the upper quartile value. If Q_3 is not an integer, Excel again uses linear interpolation. So for our example data set:{1, 4, 9, 16, 25, 36, 49, 64, 81},

n = 9, so $Q_1 = \frac{1(n+3)}{4} = \frac{1(9+3)}{4} = 3$

The lower quartile value is the 3rd data point, 9.

$$Q_3 = \frac{1(3n+1)}{4} = \frac{1(3\times 9+1)}{4} = 7$$

The upper quartile value is the 7th data point, 49.

As we can see, these methods sometimes (but not always) produce the same results. To further illustrate, consider the following data sets:

A = {1, 2, 3, 4, 5, 6, 7, 8}

B = {1, 2, 3, 4, 5, 6, 7, 8, 9}

C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

D = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

Here are the upper and lower quartile values, as calculated by each method described above:

		Tukey	M and amp	M	M and amp;S	Excel
Set A	Q_1:	2.5	2.5	2	2.25	2.75
	Q_3:	6.5	6.5	7	6.75	6.25
Set B	Q_1:	3.0	2.5	3	2.50	3.00
	Q_3:	7.0	7.5	7	7.50	7.00
Set C	Q_1:	3.0	3.0	3	2.75	3.25
	Q_3:	8.0	8.0	8	8.25	7.75
Set D	Q_1:	3.5	3.0	3	3.00	3.50
	Q_3:	8.5	9.0	9	9.00	8.50

After making sure that the students have captured what is required of them, you can use Exercise 5.8 for the purpose of assessment of calculating quartiles.

Assessment

You can assess students' learning by giving them raw and grouped data and asking them to find the quartiles and interpret their meaning. You can use Exercise 5.8 for this purpose.

Answers to Exercise 5.8

1. a. $Q_1 = 28.75$ $Q_2 = 35$ $Q_3 = 55.5$

   b. $Q_1 = 2.75$ $Q_2 = 5$ $Q_3 = 8$

   c. $Q_1 = 14$ $Q_2 = 15$ $Q_3 = 15$

2. a. $Q_1 = 18$ $Q_2 = 24$ $Q_3 = 31$

   b. $Q_2 - Q_1 = 6$ $Q_3 - Q_2 = 7$ $Q_3 - Q_1 = 13$

Therefore, $Q_3 - Q_1 = Q_3 - Q_2 + Q_2 - Q_1$

3. By forming the following frequency distribution

Fertilizer	**Quintals**	***cf***
6 – 9	2	2
10 – 13	2	4
14 – 17	4	8
18 – 21	8	16
22 – 25	12	28
26 – 29	8	36
30 – 33	8	44
34 – 37	4	48
38 – 41	1	49
42 – 45	1	50

a. $Q_1 = 19.75$; $Q_2 = 24.5$; $Q_3 = 30.25$

b. Median = 24.5 and Median = Q_2

4.

Marks	***F***	***Cf***
9.5 – 14.5	7	7
14.5 – 19.5	12	19
19.5 – 29.5	8	27
29.5 – 39.5	9	36

Q_1 is found $\left(\frac{36}{4}\right)^{th}$ item = 9^{th} item, 2^{nd} class

$$Q_1 = 14.5 + \left(\frac{9-7}{12}\right)5 = 14.5 + \frac{2\times 5}{12} = 14.5 + \frac{10}{12} = 15.333$$

$$Q_2 \text{ is } \left[\ 2\left(\frac{36}{4}\right)\right]^{\text{th}} \text{ item} = 18^{\text{th}} \text{ item, } 2^{\text{nd}} \text{ class}$$

$$Q_2 = 14.5 + \left(\frac{18-7}{12}\right)^5 = 14.5 + 4.583 = 19.083$$

$$Q_3 = \left[3\left(\frac{36}{4}\right)^{\text{th}}\right] \text{item } = 27^{\text{th}} \text{ item, } 3^{\text{rd}} \text{ class}$$

$$Q_3 = 19.5 + \left(\frac{27-19}{8}\right)5 = 19.5 + 5 = 24.5$$

a. $24.5 = Q_3$ b. $15.333 = Q_1$

In the discussion conducted so far, the students were able to determine quartiles. Now, it is possible to continue the discussion on deciles and percentiles. You can let your students realize the similarity of the approaches except the partitioning references of the respective deciles and percentiles.

After they discuss deciles and percentiles, they need to generalize the relationship between $Q_2 = D_5 = P_{50.}$ For this purpose, you can give them Activity 5.4 and guide them to come to the conclude that $Q_2 = D_5 = P_{50}$.

Answers to Activity 5.4

Median = 61.17, Q_2 = 61.17, D_5 = 61.17, and P_{50} = 61.17

They need to identify that median = $Q_2 = D_5 = P_{50}$

You can ask high achievers, the following additional question.

1. Given a data set, when will it be convenient to use only the data between Q_1 and Q_3, instead of considering all the data? The answer to this question is when the data is clustered around the centre.

Assessment

To assess students' understanding, you can use Exercise 5.9 which will give students the chance to practice calculating deciles and percentiles. Here you can give them an assignment to ask, compile answers and report how the percentiles in grade eight are calculated, and what makes percentile different from percentages.

Answers to Exercise 5.9

1. a. arranging the data,

10, 19, 28, 31, 35, 35, 35, 46, 48, 58, 68, 78

$Q_2 = 35$; $P_{12} = 18.46$

$Q_3 = 55.5$; $P_{24} = 29.14$

$D_4 = 35$; $P_{87} = 67.4$

$D_8 = 59$

b. $Q_2 = 15$; $P_{12} = 10$

$Q_3 = 15$; $P_{24} = 14$

$D_4 = 14$; $P_{87} = 19$

$D_8 = 15$

c. $Q_2 = 26$; $P_{12} = 14.67$

$Q_3 = 35.214$; $P_{24} = 18.17$

$D_4 = 22.83$; $P_{87} = 40.85$

$D_8 = 37.35$

2.

Profit	Frequency	Commulative frequency
1 – 100	12	12
101 – 200	18	30
201 – 300	27	57
301 – 400	20	77
401– 500	17	94
501 – 600	6	700

$n = 100$

Q_1 is $\left(\frac{100}{4}\right)^{th}$ item i.e 25^{th} item which is in the 2^{nd} class.

$C_f = 12$, $f_1 = 18$, $i = 100$, $B_L = 100.5$

$$Q_1 = 100.5 + \left(\frac{\frac{1 \times 100}{4} - 12}{18} \right) 100 = 100.5 + \left(\frac{13}{18} \right) 100 = 172.72.$$

Q_3 is $\left(\frac{3 \times 100}{4} \right)^{th}$ item, i.e 75^{th} item which is in the fourth class.

$C_f = 27, f_1 = 20, i = 100, B_L = 300.5$

$$Q_3 = 300.5 + \left(\frac{\frac{3 \times 100}{4} - 27}{20} \right) 100 = 540.5$$

$D4$ is $\left(\frac{3 \times 100}{10} \right)^{th}$ item = 30^{th} item. It is found in the 2^{nd} class.

$$\text{So, } D_3 = 100.5 + \left(\frac{\frac{3 \times 100}{10} - 12}{18} \right) 100$$

$= 100.5 + 100 = 200.5$

$P_{70} \left(\frac{70 \times 100}{100} \right)^{th}$ item = 70^{th} item, which is found in the fourth class.

$$\text{So, } P_{70} = 300.5 + \left(\frac{\frac{70 \times 100}{100} - 27}{100} \right) 100 = 343.5$$

3. a. $D_2 = 18$; $D_7 = 28.5$; $P_{20} = 18$; $P_{60} = 26$
   b. $P_{50} - P_{25} = 6$; $P_{75} - P_{50} = 7$; $P_{75} - P_{25} = 13$
4. a. $Q_1 = 19.75$ $D_3 = 21$; $P_{70} = 29$
   b. 68%
   c. 31 Quintals of fertilizer

Assessment

Give to the students a frequency distribution table and ask them to:

a. find lower and upper class limits and class boundaries

b. find class interval, class midpoint

c. give the mean

d. give the median

e. give the mode

f. give the quartiles, deciles and percentiles

Let your questions be on both discrete and continuous variables. You can give them these types of questions as homework or assignment. You can also give them test/quiz.

5.1.4 Measures of Dispersion

Before starting this session, it will be helpful to give students data sets of different type that have equal mean but different values. From this, help the students to enquire what difference might be addressed on each data set. With this beginning, you may start this topic with introductory discussion on what is meant by dispersion among values of a given data. Right after that, you may group the students, let them do Activity 5.5 and discuss each question. The purpose of this activity is to remind the students that they should not rely only on the mean to arrive to a conclusion. A very large or a very small value in the items might make the mean biased. They also understand the need for measuring variation of data from some reference point, usually the average (mean, median or mode).

Answers to Activity 5.5

a. $\overline{x} = 11.35$

b. No

c. determine the range between the highest and the lowest value.

Once the students realize the need for measures of dispersion, you can continue the discussion by defining range, variance for ungrouped and grouped data, and standard deviation for ungrouped and grouped data, and illustrate each measure with several examples.

In addition, discuss with the students calculations of standard deviation for grouped data and let them practice using the formula;

$$sd = \sigma = \sqrt{\frac{\sum\left[f_i(x_i - \overline{x})^2\right]}{\sum f_i}}$$

Where; x_i = the value of the i^{th} item

$\overline{x}$ = the mean of the data

f_i = the frequency of the i^{th} class interval.

The steps to calculate the variance and standard deviation of grouped frequency distribution are outlined in the student textbook. They also need to understand why standard deviation is the one measure of dispersion that is most commonly used.

You can give additional questions to high achievers to lead them into other measures of dispersion such as quartile deviation and mean deviation. For this purpose, you can ask them the following questions:

1. Find $\frac{Q_3 - Q_1}{2}$ for some data. This is called quartile deviation.
2. Giving them as a hint that $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$, ask them to find $\frac{\sum_{i=1}^{n}|x_i - \bar{x}|}{n}$ for the same data, and check out what it means. This is called mean deviation.

Assessment

For the purpose of assessing the students' understanding, you can give some of the questions from exercise 5.10 as a home work and some others as an assignment and record the scores of the students.

Answers to Exercise 5.10

1. a. R = 18 – 2 = 16

   Variance = 21.23

   Standard deviation = 4.61

   b. R = 7 – 3 = 4

   Variance = 2

   Standard deviation = 1.41

   c. R = 50 – 31 = 19

   Variance = 26.25

   Standard deviation = 5.12

   d. R = 89 – 30 = 59, variance = 251.92, standard deviation = 15.87
2. To study how the data are dispersed from each other.
3. 6
4. Constant temperature
5. a. $S_A = 2.58$, $S_B = 2.61$

   b. Player A is more consistent because he/she has less standard deviation.
6. a.

	Range	Variance	Standard Deviation
Farmer 1	7	5.58	2.36
Farmer 2	2	0.67	0.82
Farmer 3	14	21.73	4.66

b. Farmer 3; amount of yield largely varies from year to year.

c. Farmer 2 d. Farmer 2

After the students have proved successful in capturing the concepts of measures of central tendency and dispersion, they need to relate and apply the concepts in real life problems. For this purpose, you may group the students and encourage them to do Group Work 5.3. When you give them such a group work, you can give similar but different questions to different groups so that they can have wider applications problems.

Assessment

Give the students sample data and ask them to find the following:

a. Range of the grouped data

b. What information the range gives about the data

c. Variance and what information it gives

d. Standard deviation and what information it gives

Let your questions be on both discrete and continuous data.

5.2 PROBABILITY

Periods allotted: 17 periods

Competencies

At the end of this sub-unit, students will be able to;

- *determine the number of different types of possible selection from a given set of objects (by using the multiplication principle).*
- *determine the factorial of a given non-negative integer.*
- *find the possible ways of arranging objects by using the principles of permutation.*
- *compute the possible arrangement of objects around the circle (using the principle of circular permutation).*

- *describe the difference between arrangement of objects and selection of objects.*
- *describe what is meant by "combination of objects".*
- *determine the number of different combinations of n objects taken r at a time.*
- *explain the computational relationship between permutation and combination of objects.*
- *prove simple facts about combination.*
- *solve practical problems on combination of objects.*
- *write up to* 6^{th} *power of a binomial expression* $(x + y)^n$ *(i.e. when* $n = 0, 1, 2, 3, 4, 5$*) in its expanded form by using direct multiplication.*
- *describe what they observe in the expansion of* $(x + y)^n$ *where* $n = 1, 2, 3, 4, 5$.
- *describe "Pascal's triangle" and its use.*
- *apply the "binomial theorem" in expanding the* n^{th} *power of binomial terms i.e.* $((x + y)^n$ *, where* $n \in \mathbb{Z}^+$*).*
- *determine any term in the expanded form of* $(x + y)^n$ *where* $n \in \mathbb{Z}^+$.
- *solve problems on binomial expansion.*
- *describe what is meant by "random experiment".*
- *explain what is meant by an outcome of a random experiment.*
- *describe what is meant by sample space of a given random experiment.*
- *list some of the sample points of sample space for a given experiment.*
- *define "equally likely outcomes" of a given trial in his/her own words.*
- *define "favourable outcomes/cases".*
- *determine events of a given random experiment.*
- *identify sample (elementary) events and compound events.*
- *determine the number of events of a given sample space.*
- *describe the occurrence or non-occurrence of an event.*
- *explain an event denoted by "not E" where "E" is a given event.*
- *explain events connected by "or" and "and".*
- *describe the simplified forms of events by using the properties of operations on sets.*
- *identify exhaustive events.*
- *identify mutually exclusive events.*
- *describe events that are both exhaustive and mutually exclusive.*
- *identify independent events.*
- *identify dependent events.*
- *describe the axiomatic approach of probability.*
- *interpret basic facts in the theory of probability.*
- *find probabilities of events based on axiomatic approach.*
- *describe the odds in favour of an event or the odds against an event.*

- *find the probability of $E_1 \cup E_2$ where E_1 and E_2 are events in a random experiment.*
- *determine the probability of mutually exclusive events.*
- *find probability of the joint occurrence of independent events (by using rule of multiplication).*
- *describe the outcomes of events using a tree diagram.*
- *determine the probability of the joint occurrence of dependent events (using multiplication rule).*
- *describe the outcomes of events using tree diagram to determine their probability.*
- *identify whether given events are independent or dependent (by comparing the equation for probability of joint occurrence of independent event).*

Introduction

Probability theory, as the students may recall, is a mathematical model to describe the likelihood of an event's occurrence. The likelihood of an event's occurrence is assigned a number between 0 and 1, in which 0 is assigned to an impossible event, and 1 is assigned to an event whose occurrence is certain (a must).

The students were introduced to some important ideas about probability in grade 9. Issues like experimental and theoretical approaches of probability and determining probability of sample events were considered there. In this subunit, the students first consider permutation and combination. The study of permutation and combination is concerned with finding the number of logical or acceptable possibilities of an event without necessarily considering every individual case. They are very basic in probability considerations. They will also study the basic assumptions and rules for calculating probabilities.

Teaching Notes

To begin this subunit, you may start by asking the students to give explanations on the meaning/definition of the terms such as experiment, outcome, sample space, event, and equally likely.

After their deliberation, you can direct them to practice these terminologies with examples of calculating probabilities. You may also group the students in pairs and let them do Activity 5.6. The purpose of this activity is on the one hand to let the students calculate probability of different events and on the other, to practically guide to the terminologies of certain event and impossible event by way of revising what the students have learned in Grade 9. You may also guide the students through questions and answers to recall the concepts they covered in grade 9.

Answers to Activity 5.6

a. $\frac{1}{6}$ b. $\frac{1}{2}$ c. 0 d. 1 e. $\frac{5}{6}$

5.2.1 Permutation and Combination

With the help of simple day-to-day activities, introduce the idea of fundamental principle of counting which is used to find the number of ways of occurrence of events in a given order. For clarity and simplicity, you can use simple examples that represent places in the surrounding of students and look like the one given as example 4 on page 187 of the student textbook. Here, it will be essential to help the students to differentiate between addition principle and multiplication principle. They also need to note that the counting principle works for any number of events. You can group the students and give them exercise 5.11 as a class work so that they can have a better understanding of the counting principle.

Assessment

You can assess students' understanding during class instruction by using oral questions. You can also give them some exercises of counting, let them discuss with their peers, and assess their understanding by letting them present report.

Answers to Exercise 5.11

1. All are events. (d is also an event as empty set)
2. a. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

   b. $\frac{3}{8}$
3. a. $S = \{R_1R_2, R_1B_1, R_1B_2, R_1B_3, R_2B_1, R_2B_2, R_2B_3, B_1B_2, B_1B_3, B_2B_3\}$

   b. $\frac{1}{10}$

   c. $\frac{6}{10}$
4. The first book has 6 choices, the 2nd book has 5 choices, and the third, fourth, fifth and the sixth book has 4, 3, 2, 1, choices respectively. Thus, the total number of arrangement is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways
5. $2 \times 3 = 6$

Following the discussion on counting principle, discuss with the students the idea of factorial and present them with several examples. By defining the concept factorial, encourage the students to develop a meaning to permutation. Right after their trial give them the definition of permutation, explain its usefulness for finding number of arrangements of objects and introduce its notation, P(n, r). To make things easier and give the students the chance to apply permutation, you can group the students and let them do Group work 5.4. After they finish their group work, select some of the groups and let them present their work to the whole class. Make corrections when necessary.

Answers to Group Work 5.4

1. a. $\frac{6!}{4!} = 30$ b. $\frac{8!}{3!} = 6720$ c. $\frac{1000!}{1!} = 1000!$

2. 5!

3. ${}_5P_3 = 60$

4. CALL, ACLL, CLAL, ALCL, CLLA, ALLC, LLAC, LACL, LALC, LLCA, LCAL, LCLA

When the students finish the group work, you can give them a question of type 4 in the group work 5.4 where the two Ls in the word CALL are identical, and let them discuss it for a few minutes. This will lead them into permutation of duplicate terms which is presented on page 191 of the student textbook. You can also explain to them the circular permutation.

For the purpose of assessing your students, you can give them the questions in exercise 5.12 as a home work.

Assessment

You can assess your students by giving them questions on permutation and asking them to calculate. You can also ask them to derive the formula for permutation whenever you find it convenient. You can use Exercise 5.12 for assessment purpose.

Answers to Exercise 5.12

1. a. 6! = 720 b. 8! = 40,320 c. 12! = 479,001,600

2. a.

6	6	6	1

$6 \times 6 \times 6 \times 1 = 216$ (if repetition is allowed)

b.

1	6	6	6

$1 \times 6 \times 6 \times 6 = 216$ (if repetition is allowed)

3. a.

W	M	M

1 2 1

$1 \times 2 \times 1 = 2$ ways

That is $WM_1\ M_2$ or $WM_2\ M_1$ are favourable cases.

b.

M	W	M

2 1 1

$2 \times 1 \times 1 = 2$ ways, that is $M_1\ WM_2$ or M_2WM_1

4. a. $\dfrac{10!}{2!2!2!}$ b. $\dfrac{4!}{2!}\dfrac{7!}{2!2!}$ c. $\dfrac{9!}{2!2!2!}$

5. $\dfrac{10!}{3!4!3!} = 4{,}200$

6. $_nP_{n-1} = \dfrac{n!}{[n-(n-1)]!} = \dfrac{n!}{1!} = \dfrac{n!}{0!} = \dfrac{n!}{(n-n)!} = {}_nP_n$

The next point of discussion will be combination. Before defining combination of objects, it will be better to start from permutation which they have discussed previously, by giving the following as a guide;

If we have three volunteer students to serve as members of a committee of two, how many such combinations are there?

Let the students discuss and see how combination differs from permutation. Following this enquiry question, you can state the formal definition of combination and its notation $C(n, r)$ or ^nC_r or $\begin{pmatrix} n \\ r \end{pmatrix}$ for combination of n objects taken r at a time, $0 < r \le n$.

You can then let the students do as many examples as possible to understand the idea of combination. For further developing and understanding of the properties of combination, you can group the students in pairs and let them do Activity 5.7. In addition you can assist students to do practical problems on combination and give them some exercises.

Answers to Activity 5.7

a. $C\,(n\,,0) = \dfrac{n!}{(n-0)!\,.\,0!} = \dfrac{n!}{n!.1} = 1$

b. $C\,(n\,,n-r\,) = \dfrac{n!}{(n-(n-r))!\,(n-r)!} = \dfrac{n!}{(n-n+r)!\,.\,(n-r)!}$

$$= \frac{n!}{r!.\,(n-r)!} = \frac{n!}{(n-r)!\,.\,r!} = C(n,\,r)$$

c. $\binom{n}{r}+\binom{n}{r-1} = \dfrac{n!}{(n-r)!r!}+\dfrac{n!}{(n-r+1)!(r-1)!} = \dfrac{n!}{(n-r)!r!}+\dfrac{n!}{(n+1-r)!(r-1)!}$

$$= \frac{n!}{(n-r)!r!}+\frac{rn!}{(n+1-r)!r!} = \frac{n!}{(n-r)!r!}+\frac{rn!}{(n+1-r)(n-r)!r!}$$

$$= \frac{n!(n+1-r)+rn!}{(n+1-r)!r!} = \frac{n!n+n!-rn!+rn!}{(n+1-r)!r!}$$

$$= \frac{n!(n+1)}{(n+1-r)!r!} = \frac{(n+1)!}{(n+1-r)!r!} = \binom{n+1}{r}$$

Assessment

So as to assess students' understanding of combinations, you can ask them to do several exercises on permutations and combinations. You can also ask them to derive the formula for combination and some properties of combinations similar to the questions in Activity 5.7. You can also use the questions in Exercise 5.13 for the purpose of assessment.

Answers to Exercise 5.13

1. a. $C(8,\,0) = \dfrac{8!}{(8-0)!\,0!} = \dfrac{8!}{8!\,0!} = 1$

   b. $C(n,\,n) = \dfrac{n!}{(n-n)!\,n!} = \dfrac{n!}{0!\,n!} = 1$

   c. $C\,(8,\,6) = \dfrac{8!}{(8-6)!\,6!} = \dfrac{8!}{2!\,6!} = \dfrac{8\times 7\times 6!}{2!\times 6!} = 4\times 7 = 28$

2. $C(n, 6) = C(n, 4)$

$$\frac{n!}{(n-6)!\ 6!} = \frac{n!}{(n-4)!\,4!} \Rightarrow \frac{n!}{(n-6)!\,6!} = \frac{n!}{(n-4)\ (n-5)\ (n-6)!\,4!}$$

$$\Rightarrow \frac{n!}{(n-6)!\,6\times 5\times 4!} = \frac{n!}{(n-4)(n-5)\ (n-6)!\,4!}$$

$$\Rightarrow \frac{1}{30} = \frac{1}{(n-4)\ (n-5)}$$

$$\Rightarrow n^2 - 9n + 20 - 30 = 0$$

$$\Rightarrow n^2 - 9n - 10 = 0$$

Solving the quadratic equation gives $n = 10$.

3. (10, 5) = 252

4. a. $C(9, 2)\ .\ C(9, 3) = 36 \times 42 = 1512$ ways

   b. $C(9, 0)\ .\ C(9, 5) = 1 \times 126 = 126$ ways

   c. $C(9, 5).\ C(9, 0) = 126 \times 1 = 126$ ways

   d. $C(9, 3).C(9, 2) + C(9, 4).C(9, 1) + C(9, 5).C(9, 0)$

   $= 42 \times 36 + 126 \times 9 + 126 \times 1 = 2772$ ways.

   e. $C(9, 0)\ .\ C(9, 5) + C(9, 1)\ .\ C(9, 4) + C(9, 2)\ C(9, 3) + C(9, 3).\ C(9, 2)$

   $= 1 \times 126 + 9 \times 126 + 36 \times 42 + 42 \times 36 = 4284$ ways.

5. a. 190 games

   b. 120 games

6. a. 112 b. 10 c. 219

5.2.2 Binomial Theorem

You may start the lesson by revising how to expand the square and cube of a given binomial expression using the distributive property of multiplication over addition.

Example:

$$(x + y)(x + y) = x^2 + 2xy + y^2$$

After this, in order to give students the chance to practice by themselves and reach some generalization on Binomial theorem, you can group them and let them do Group Work 5.5. In this group work, students will use inductive approach to see the relationship between the coefficients of the terms of the binomial theorem and the elements of the Pascal Triangle.

Answers to Group Work 5.5

1. For $n = 1$, $(a + b)^1 = (a + b)$

   For $n = 2$, $(a + b)^2 = a^2 + 2ab + b^2$

   For $n = 3$, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

   For $n = 4$, $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

   For $n = 5$, $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

2. $(a + b)^n = C(n, 0)a^n + C(n, 1)a^{n-1}b + C(n, 2)a^{n-2}b^2 + \ldots + C(n, n-2)a^2b^{n-2}$

   $+ C(n, n-1)ab^{n-1} + b^n$

3. a. In $(a + b)^n$ there are $(n + 1)$ terms

   b. The exponent of "a" decreases while the exponent of "b" increases.

   c. The sum of exponents of a and b is always n.

   d. The coefficient of the first and the last terms is always 1.

   e. Yes. The coefficients are as given in 2.

   f. The completed PASCAL Triangle is given below.

   g. It is their sum.

   h. The coefficients given by $C(n, r) = C(n, n-r)$

To help them use Pascal's Triangle in expanding binomial powers, you can group them in pairs and let them do Activity 5.8.

Answers to Activity 5.8

1 . . . . $(a + b)^0$

1 1 . . . . $(a + b)^1$

1 2 1 . . . . $(a + b)^2$

1 3 3 1 . . . . $(a + b)^3$

1 4 6 4 1 . . . . $(a + b)^4$

1 5 10 10 5 1 . . . . $(a + b)^5$

1 6 15 20 15 6 1 . . . . $(a + b)^6$

1 7 21 35 35 21 7 1 . . . . . $(a + b)^7$

1 8 28 56 70 56 28 8 1 . . . $(a + b)^8$

From the above Pascal's triangle we have;

$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

$(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$ and

$(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$.

Assessment

As binomial expansion and Binomial theorem are useful in calculating probability, students would benefit from understanding these. To assess students learning, you can give them questions as given in Exercise 5.14 and ask them to do each question. You can do these as class work and homework.

Answers to Exercise 5.14

1. a. $(a+b)^5 = C(5, 0)\,a^5 + C(5, 1)\,a^4\,b + C(5, 2)\,a^3\,b^2 + C(5, 3)\,a^2b^3$
$+ C(5, 4)\,ab^4 + C(5, 5)b^5$
$= a^5 + 5a^4b + 10a^3b^2 + 10a^2\,b^3 + 5ab^4 + b^5$

   b. $(a+b)^7 = C(7, 0)a^7 + C(7, 1)\,a^6\,b + C(7, 2)\,a^5\,b^2 + C(7, 3)\,a^4\,b^3$
$+ C(7, 4)a^3\,b^4 + C(7, 5)a^2b^5 + C(7, 6)\,ab^6 + C(7, 7)b^7.$
$= a^7 + 7a^6\,b + 21a^5b^2 + 35a^4b^3 + 35a^3\,b^4 + 21a^2b^5 + 7ab^6 + b^7.$

   c. $(3x-4y)^6 = 729x^6 - 5832x^5y + 19440x^4y^2 - 34560x^3y^3 + 34560x^2y^4$
$- 18432xy^5 + 4096y^6$

2. a. $C(8, 5) = 56$ b. $C(6, 2) = 15$ c. $C(6, 4) = 15$
3. The terms that have equal coefficients are x^3 and y^3; and x^2y and xy^2.
4. a. 11 b. a^8b^2 and a^2b^8.
5. a. 5000 b. x^4y
6. 12

5.2.3 Random Experiments and its Outcomes

This topic has been discussed in grade 9. You can start the session by asking the students to tell what a random experiment and outcomes are. Following their responses, you can introduce a “Random experiment” as an experiment, which when repeated under identical conditions does not produce the same result or outcomes. You may also explain it as a trial whose outcome cannot be determined in advance. To help them have a better understanding, you can let them try some experiments such as tossing a coin, throwing a die, etc. From the experiments they try, you can also let them write down the set of possible outcomes. Some examples are given in the student textbook.

5.2.4 Events

You may start the lesson by revising the concept of sample space of a given random experiment. And then using simple examples consider situations which ensure the happening of a particular condition among the members of the sample space of an experiment. Based on this, define an event that is any subset of a sample space and usually denoted by "E". For the purpose of practice in identifying events from some sample space, you can let the student individually do Activity 5.9.

Answers to Activity 5.9

As each of the following are sample spaces, any subset of each will be an event.

a. S = { HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}

b. S = { Defective, non-defective}

c. S = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

d. S = {Red, white}

e. S = { Boy, Girl}

When the students understand determining events, you can guide them through examples how they can identify the types of events as simple or compound. You can also guide them to see the occurrence and non-occurrence of an event. Following this, they need to understand a complement of an event and practice how they determine it. After they understand this, you can let the students discuss in pairs the questions in Activity 5.10. The purpose of this activity is to ensure that the students have understood the basic definitions on algebra of events and to state the results.

Answers to Activity 5.10

a. $E_1 \cup E_2$ is either E_1 or E_2 or both and $E_1 \cap E_2$ is both E_1 and E_2

b. See student textbook.

c. If $E_1, E_2, E_3, \ldots, E_n$ are random experiments in a sample space

   $S = E_1 \cup E_2 \cup \ldots \cup E_n$, then the collection of the events

   $E_1, E_2, \ldots, E_n$ forms a mutually exclusive events if $E_i \cap E_j = \varnothing, i \neq j$ and an exhaustive set of events where

   $S = E_1 \cup E_2 \cup \ldots \cup E_n$.

d. Two events are said to be independent if the occurrence or non- occurrence of one event doesn't affect the occurrence or non-occurrence of the other event.

Following the activity, it will be essential for the students to discuss and understand some of the terminologies in the study of probability namely: exhaustive events, mutually exclusive events, and dependent and independent events which are explained through examples in the student textbook.

5.2.5 Probability of an Event

Once the students are capable of identifying the counting principles and determining permutations and combinations, it will be essential to discuss probability. You may start the lesson by introducing the common experiences students had in determining probability of an event through classical approach or empirical approach. You may need to give them some examples of determining probability of some events. The students might realize how they determine probability of an event. At this point it may be possible to ask them if they can determine the probability of an event without using the counting principles or the relative frequency approach. This will lead them into the modern theory of probability known as Axiomatic approach of probability. You need to explain to the students that the Axiomatic approach includes both the empirical and classical definitions of probability that they have learned in grade 9. In addition to these, you can discuss the probability of mutually exclusive events and probability of independent and dependent events.

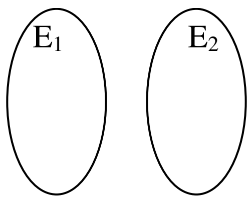
After the discussion outlined above, it will be essential to let the students realize the rules of probability which is one of the useful ways of computing probability of an event mathematically. For this purpose, you can give Activity 5.11 to the students to do it in group.

Answers to Activity 5.11

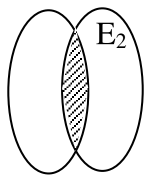
a. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
The condition applied here is when the two events E_1 and E_2 are not mutually exclusive.

b. $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, the condition is
E_1 and E_2 are mutually exclusive events.
i.e., $E_1 \cap E_2 = \varnothing$

c. i. For (b)



ii. For (a)



Pursuant to the activity, guide the students to identify the addition and multiplication rules of probability. Illustrate each with the help of examples. Some guiding examples are given in the student textbook. Although it may seem easier to apply the rules stated earlier, there may be time where occurrence of one event may affect the occurrence of another event. In this case, there will be a need to discuss the concept of conditional probability. Guide the students to understand the issues of conditional probability with the help of the examples given in the student textbook.

Assessment

Here you can assess students by giving them a test/quiz focusing on the terminologies in probability and giving example for each. You can also ask questions on rules and properties of probability. It is also possible to ask them questions to find probability of some events. You can also give them the questions in exercise 5.15 as an assignment or as a home work.

Answers to Exercise 5.15

1. a. $\frac{1}{6}$ b. $\frac{1}{3}$

2. a. $\{5\}$ b. 1 c. 0 d. $\frac{1}{6}$ e. $\frac{5}{6}$ f. $\frac{5}{6}$

3. a. $\frac{8}{52}$ b. $\frac{28}{52}$

4. $\frac{1}{36}$

5. a. $\frac{1}{5}$ b. $\frac{3}{5}$

6. a. $\frac{16}{2704}=\frac{1}{169}$ b. $\frac{16}{2652}=\frac{4}{663}$

7. n (S) = 24, n (Red) = 10

   n (S) = n (not Red) + n (Red)

   n (not Red) = 14

   Thus, P(not Red) = $\frac{14}{24}=\frac{7}{12}$

8. a. a, c, d and e are invalid assignments because in (a), the sum of the probabilities is less than 1, in (c), the sum is greater than 1, in (d), some of the probabilities are negative and in (e), the sum is greater than 1.

   b. (b) is valid for two reasons. Each probability is between 0 and 1 and their sum is 1.

9. $\frac{5}{6}$

10. $\frac{22}{49}$

When the students attempt to do the exercises, you can help them use tree diagram to determine the possible outcomes with which they can determine probability of an event. Such a tree diagram is also useful for determining probability of joint events. Some examples are outlined in the student text and you can also give them some more other examples. Finally, you can give them exercise 5.16 as a class work or home work with which they can practice applying tree diagrams.

For high achievers, you can give them questions of the following type that are useful in calculating probabilities.

1. Justify that; if a box contains n_1 white and n_2 red balls, and if someone picked up r balls at random, then the probability that k of them will be white is

$$P(x=k)=\frac{\binom{n_1}{k}\binom{n_2}{r-k}}{\binom{n_1+n_2}{r}}$$

Assessment

You can assess students learning by giving them questions similar to those listed in Exercise 5.16 and ask them to find the probability. You can also let them determine sample spaces and events by using tree diagrams, and calculate probabilities.

As this is the end of the unit, you can also give them test encompassing all the subtopics in this unit.

Answers to Exercise 5.16

1. a. $\frac{5}{11}+\frac{6}{11}=\frac{11}{11}=1$ b. $1-\frac{5}{11}=\frac{6}{11}$ c. $\frac{0}{11}=0$

2. a. $\frac{4}{52}\times\frac{4}{52}\times\frac{4}{52}=\frac{64}{140608}=\frac{1}{2197}$

   b. $\frac{4}{52}\times\frac{3}{51}\times\frac{2}{50}=\frac{24}{132600}=\frac{1}{5525}$

3. a. $\frac{1}{36}$ b. 0 c. $\frac{3}{36}=\frac{1}{12}$

4. a. $\frac{1}{30}$ b. $\frac{5}{6}$ c. $\frac{4}{15}$

5. a. $\frac{1}{11}$ b. $\frac{16}{33}$ c. $\frac{10}{11}$

6. a. $\frac{1}{30}$

Answers to Review Exercises on Unit 5

1.

Classes	Frequencies
1 – 3	7
4 – 6	6
7 – 9	3
10 – 12	5
13 – 15	8
16 – 18	7
19 – 21	2
22 – 24	2

2. a. 4 b. 5 and 7 c. 110.4
3. a. 15 b. 3.5 c. 340.23
4. a. 9.1 b. 4.7 c. 15 d. 74.5
5. $Q_2 = 12.5$ $D_3 = 12.5$ $P_{20} = 7.5$
6. a. $\sigma = 2.85$ b. σ s = 1.12 c. $\sigma = 3.7229$

   $\sigma^2 = 8.1225$ b. $\sigma^2 = 1.2544$ c. $\sigma^2 = 13.86$
7. a. $\frac{1}{64}$ b. $\frac{15}{64}$
8. $\frac{(n+1)!}{n!} = 5 \Rightarrow \frac{(n+1)n!}{n!} = 5$

   $\Rightarrow n = 4$

9. a.

4	3	2

$4 \times 3 \times 2 = 24$ ways

b.

4	4	4

$4 \times 4 \times 4 = 64$ ways

10. a. 15 b. 28 c. 3

11. a. $\dfrac{\binom{3}{2}}{\binom{12}{2}} = \dfrac{3}{66}$ b. $\dfrac{\binom{9}{2}}{\binom{12}{2}} = \dfrac{36}{66}$ c. $\dfrac{\binom{9}{1}\binom{3}{1}}{\binom{12}{2}} = \dfrac{27}{66}$

12. $7! = 5040$

13. a. 120 b. 120

14. a. $\binom{8}{2}\binom{7}{3} = 28 \times 35 = 980$

b. $\binom{7}{2}\binom{8}{3} + \binom{7}{3}\binom{8}{2} + \binom{7}{4}\binom{8}{1} + \binom{7}{5}\binom{8}{0} = 2457$

c. $\binom{8}{0}\binom{7}{5} + \binom{8}{1}\binom{7}{4} + \binom{8}{2}\binom{7}{3} + \binom{8}{3}\binom{7}{2} + \binom{8}{4}\binom{7}{1}$

$$= \frac{21 + 280 + 980 + 588 + 490}{3003} = \frac{2359}{3003}$$

15. $\dfrac{3}{11}$

16. $\dfrac{4}{52} \times \dfrac{4}{51} \times \dfrac{4}{50} = \dfrac{64}{132600} = \dfrac{8}{16575}$

17. $\dfrac{4}{36} = \dfrac{1}{9}$

UNIT 6 MATRICES AND DETERMINANTS

INTRODUCTION

This unit deals with matrices and determinants. Matrices and determinants deal with an array of numbers with their own rules for addition and multiplication. It is expected that the students have some idea of systems of linear equations and how to solve them by using substitution and elimination methods.

In this unit, they will learn how to solve systems of linear equations by systematically using Gauss' method. Expressing linear systems of equations using matrices makes them easier to solve, using programmable calculating machines.

Unit Outcomes

After completing this unit, students will be able to:

- *know basic concepts about matrices.*
- *know specific ideas, methods and principles concerning matrices.*
- *perform operations on matrices.*
- *apply principles of matrices to solve problems.*

Suggested Teaching Aids in Unit 6

Although teaching aids may not be excessively exploited for this unit, you can present different charts that represent different types of matrices. Apart from use of the student textbook, you need to elaborate more application problems from your surrounding so that students can best appreciate and see how useful matrices are.

You can group students, give them hints on a problem and let them assess such a problem from the perspective of their daily life and develop their matrix form. The following materials, if available, are very helpful:

- Calculators or computers.
- Colored chalks

6.1 MATRICES

Periods Allotted: 4 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define matrix.*
- *determine the sum and difference of two given matrices of the same order.*
- *multiply a matrix by a scalar.*
- *describe the properties of multiplication of matrices by scalars.*
- *determine the product of two matrices.*
- *describe the properties of the product of two matrices.*
- *determine the transpose of a matrix.*

Vocabulary: Matrix, Column, Row, Vector, Square matrix, Zero matrix, Diagonal matrix, Unit matrix, Scalar matrix, Principal diagonal, Lower triangular matrix, Upper triangular matrix, Symmetric matrix, Skew-symmetric matrix.

Introduction

In this sub-unit, the definition of a matrix is presented. Operations of addition, subtraction, multiplication of a matrix by a scalar, product of matrices, transposition of matrix and properties of these operations are addressed. To deal with these, active participation of students is a requirement.

Matrices have many scientific applications. They are useful to organize a vast amount of data. One can represent a system of linear equations by using matrices. Matrix representation is better suited for solving systems of linear equations by using computing devices.

For instance, to solve the opening problem, you can formulate the problem as a system of linear equations. You can write the system of linear equations by letting:

x be the number of eggs.

y be the number of cups of milk.

z be the number of cups of juice.

Then the system of linear equations becomes:

$$\begin{cases} 80x + 160y + 110z = 540 \\ 6x + 9y + 2z = 25 \end{cases}$$

The system is equivalent to:

$$\begin{cases} 8x + 16y + 11z = 54 \\ 6x + 9y + 2z = 25 \end{cases}$$

Using matrix notation, this system of linear equations can be written as:

$$\begin{pmatrix} 8 & 16 & 11 & 54 \\ 6 & 9 & 2 & 25 \end{pmatrix}.$$

After discussing 6.3, students will be able to solve this system, using matrices. But for now, guide them to solve the system by elimination method which they studied in grade 9.

The system has infinite solution with:

$$\left.\begin{array}{l} x = \dfrac{67t - 86}{24} \\ y = \dfrac{124 - 50t}{24} \\ z = t \end{array}\right\}, \ t \in \mathbb{Z} \geq 0$$

Teaching Notes

Start the discussion of the subunit by considering tables containing rows and columns of items so that the students will easily grasp the concept of a matrix. Examples are given in the student textbook. Here are additional examples. You may also construct examples of your own.

Example1: Suppose the age(in years), weight(in kg) and height (in cm) of a student in your class is given by (16, 55, 169). (16, 55, 169) is called a 1 by 3 (1×3) matrix (or a row vector). Notice that the order is important.

Similarly, for any real numbers $a_1, a_2, \ldots, a_n$, $(a_1, a_2, \ldots, a_n)$ is called a $1 \times n$ matrix or a row vector.

If you write them in columns, as

$$\begin{pmatrix} 16 \\ 55 \\ 169 \end{pmatrix} \text{ and } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

you will call them column vectors (or a 3×1 and $n \times 1$ matrices respectively).

Remark: Sometimes you will see the elements of a row vector separated by commas. Commas are not necessary unless confusion from not using them.

Example 2: Suppose there are four soccer clubs A, B, C and D. Suppose in a competition, A won seven, lost six, and tied one, B won five, lost eight, and tied one, C won two, lost twelve, and had no ties, D won nine, lost five, and had no ties. We can represent this data using the four vectors (7 6 1), (5 8 1), (2 12 0), and (9 5 0). You can combine this information in a matrix as follows:

$$\begin{pmatrix} 7 & 6 & 1 \\ 5 & 8 & 1 \\ 2 & 12 & 0 \\ 9 & 5 & 0 \end{pmatrix}$$

Example 3: The information in the following table is made up of names of students against their marks in three tests.

Name	Test 1	Test 2	Test 3
Siraj	9	10	7
Bizunesh	9	10	6
Welela	8	10	8
Ojulu	9	10	8

This now can be written in a matrix form like

$$\begin{pmatrix} 9 & 10 & 7 \\ 9 & 10 & 6 \\ 8 & 10 & 8 \\ 9 & 10 & 8 \end{pmatrix}$$

In general, a matrix is an arrangement of numbers into rows and columns.

To help students practice, you can give them Activity 6.1. The purpose of this Activity is to prepare students to define some important types of matrices. Ask voluntary students to give the sizes of the matrices in the activity.

Answers to Activity 6.1

- For matrix A, the number of rows is 2 and the number of columns is 2.
- For matrix B, the number of rows is 3 and the number of column is 1.
- For matrix C, the number of rows is 3 and the number of columns is 2.
- For matrix D, the number of rows is 1 and the number of columns is 3.

Using active learning methods like drill partners, please make sure that

- students do not confuse row and column
- size(order) is given as number of rows × number of columns. Thus, a matrix with, say, 5 rows and 8 columns has size 5×8.
- for the entry a_{ij}, i represents the row and j represents the column in which a_{ij} is found in the matrix.
- students identify the various kinds of matrices listed in their textbook and give examples of each.

To assess their understanding of the definition and terms give students exercise problems that ask the size of matrices, identifying entries, and types of matrices. Also ask students to construct various matrices (include real life examples).

The next lesson deals on addition and subtraction of matrices. But, before starting to discuss these, it will be helpful to discuss equality of matrices. Right after identifying equality of matrices, the students can discuss addition and subtraction of matrices. To begin this discussion, you can let the students do Activity 6.2 which is supposed to prepare them for the definition of addition of matrices.

Let the students do the activity in small groups. Let the students add the number of books component wise (Biology books with Biology books and so on). Then, let them represent the number of books in stock and the newly arrived books in matrix form

$$\begin{pmatrix} 101 & 89 & 72 & 75 \\ 62 & 58 & 70 & 43 \\ 57 & 65 & 71 & 94 \\ 81 & 87 & 91 & 93 \end{pmatrix} \text{ and } \begin{pmatrix} 60 & 65 & 54 & 45 \\ 27 & 35 & 50 & 27 \\ 55 & 66 & 65 & 44 \\ 75 & 68 & 70 & 51 \end{pmatrix} \text{ respectively.}$$

Let the students give the sum table they have found in matrix form. You then explain to them that the sum table is defined to be equal to:

$$\begin{pmatrix} 101 & 89 & 72 & 75 \\ 62 & 58 & 70 & 43 \\ 57 & 65 & 71 & 94 \\ 81 & 87 & 91 & 93 \end{pmatrix} + \begin{pmatrix} 60 & 65 & 54 & 45 \\ 27 & 35 & 50 & 27 \\ 55 & 66 & 65 & 44 \\ 75 & 68 & 70 & 51 \end{pmatrix}$$

Answers to Activity 6.2

Subject	Grade Level			
	7	8	9	10
Biology	101 + 60 = 161	89 + 65 = 154	72 + 54 = 126	75 + 45 = 120
Physics	62+27 = 89	58 + 35 = 93	70 + 50 = 120	43 + 27 = 70
Chemistry	57+55 = 112	65 + 66 = 131	71 + 65 = 136	94 + 44 = 138
Mathematics	81 + 75 = 156	87 + 68 = 155	91 + 70 = 161	93 + 51 = 144

Therefore, $\begin{pmatrix} 101 & 89 & 72 & 75 \\ 62 & 58 & 70 & 43 \\ 57 & 65 & 71 & 94 \\ 81 & 87 & 91 & 93 \end{pmatrix} + \begin{pmatrix} 60 & 65 & 54 & 45 \\ 27 & 35 & 50 & 27 \\ 55 & 66 & 65 & 44 \\ 75 & 68 & 70 & 51 \end{pmatrix} = \begin{pmatrix} 161 & 154 & 126 & 120 \\ 89 & 93 & 120 & 70 \\ 112 & 131 & 136 & 138 \\ 156 & 155 & 161 & 144 \end{pmatrix}$

Matrices of the same dimensions are added by adding corresponding elements. For instance, a_{ij} corresponds to b_{ij} because they both lie in the i^{th} row and j^{th} column of their respective matrices. Therefore, you would add, $a_{ij} + b_{ij}$ to obtain the $(i, j)^{th}$ element of A + B. Similarly, matrices of the same dimensions (sizes) are **subtracted** by subtracting corresponding elements. Therefore, if

$$A = \begin{pmatrix} 4 & -6 \\ 0 & 12 \\ -53 & 27 \end{pmatrix},\ B = \begin{pmatrix} 7 & 6 & 1 \\ 5 & 8 & 1 \\ 2 & 12 & 0 \\ 9 & 5 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 23 & -6 \\ 11 & 17 \\ 5 & 28 \end{pmatrix}$$

A and B cannot be added or subtracted, because their sizes are not the same. But A and C can be added, and the sum equals to

$$\mathrm{A} + \mathrm{C} = \begin{pmatrix} 27 & -12 \\ 11 & 29 \\ -48 & 55 \end{pmatrix} \text{ and } \mathrm{A} - \mathrm{C} = \begin{pmatrix} -19 & 0 \\ -11 & -5 \\ -18 & -1 \end{pmatrix}.$$

Once the students can add matrices, the next important point is identifying and verifying properties of matrix addition and subtraction. To this end, you can let them do Activity 6.3.

Answers to Activity 6.3

a. $(A+B)+C = \begin{pmatrix} 14 & -4 \\ 3 & 10 \end{pmatrix}$, b. $(A+B)+C = \begin{pmatrix} 14 & -4 \\ 3 & 10 \end{pmatrix}$

Hence, $(A + B) + C = A + (B + C)$.

c. $A - A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, d. $A + 0 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$

e. $A + B = \begin{pmatrix} 7 & -1 \\ 1 & 5 \end{pmatrix}$, f. $B + A = \begin{pmatrix} 7 & -1 \\ 1 & 5 \end{pmatrix}$

Hence, $A + B = B + A$

The purpose of this Activity is to verify some properties of matrix addition. They are:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A - A = 0$
4. $A + 0 = A$

Note:- 0 denotes the zero matrix having the same size as A.

Let the students verify these properties by using the matrices given in the activity. Give them additional group activities, if necessary. But, explain to them that they can prove that, say, $A + B = B + A$ in general by looking at the general $(i,j)^{th}$ element of each side of the equation. The $(i,j)^{th}$ element of $A + B$ is $a_{ij} + b_{ij}$ and the $(i,j)^{th}$ element of $B + A$ is $b_{ij} + a_{ij}$. Therefore, using the commutative law of addition for real numbers,

$A + B = B + A$.

You can also guide the students to see if these properties also hold true for subtraction where the commutative property fails to hold. That is, $A - B \neq B - A$.

To assess their understanding of addition and its properties, give to the students matrices of different sizes and ask them to check if they can be added or subtracted together. Let them add or subtract those that can be added or subtracted.

Following their practices on adding and subtracting matrices, you can proceed to multiplying a matrix by a scalar by letting the students do Activity 6.4. The purpose of this activity is to prepare students for the definition of scalar multiplication of matrices.

To multiply a matrix by a scalar r, the students must know that they have to multiply each entry or element of the matrix by r.

Thus, if $A = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix}$, and $r = -5$,

$$-5A = \begin{pmatrix} -5\times 2 & -5\times 5 \\ -5\times -2 & -5\times 0 \end{pmatrix} = \begin{pmatrix} -10 & -25 \\ 10 & 0 \end{pmatrix}.$$

In the activity, guide the students to write the matrix of the marks. Let them denote it by M. Let them give the resulting matrix after the marks are converted. Explain to them that the resulting matrix is denoted by 2M.

Answers to Activity 6.4

The marks of Nigist and Hagos (out of 100) in their examinations are given below.

	Nigist	**Hagos**
English	$37\times 2 = 74$	$31\times 2 = 62$
Mathematics	$46\times 2 = 92$	$44\times 2 = 88$
Biology	$28\times 2 = 56$	$25\times 2 = 50$

The resulting matrix is $2M = \begin{pmatrix} 74 & 62 \\ 92 & 88 \\ 56 & 50 \end{pmatrix}$

After discussing the definition and doing examples given in the student textbook, let the students work Activity 6.5 in groups. This activity will help the students to verify the properties of scalar multiplication, i.e. distributive properties, associative property and multiplication by 0 and 1.

The properties the students are expected to verify and learn are the following:

If A and B are matrices of the same order (size) and r and s are any scalars (i.e. real numbers), then:

a. $r(A + B) = rA + rB$ b. $(r + s)A = rA + sA$

c. $(rs)A = r(sA)$ d. $1\cdot A = A$ and $0\cdot A = 0$

Answers to Activity 6.5

a. $-7(A+B)=\begin{pmatrix} 0 & -7 & 0 \\ -56 & 21 & -14 \end{pmatrix}$

b. $(-7A)+(-7B)=\begin{pmatrix} 0 & -7 & 0 \\ -56 & 21 & -14 \end{pmatrix}$

Hence, $-7(A + B) = (-7A)+(-7B)$.

c. $(-7\times 4)A=-28A=\begin{pmatrix} 28 & -28 & 28 \\ -168 & 56 & 28 \end{pmatrix}$,

d. $(-7)\times(4A)=\begin{pmatrix} 28 & -28 & 28 \\ -168 & 56 & 28 \end{pmatrix}$

Therefore, $(-7\times 4)A=-7(4A)$.

e. $(-7+4)A=-3A=\begin{pmatrix} 3 & -3 & 3 \\ -18 & 6 & 3 \end{pmatrix}$,

f. $-7A+4A=\begin{pmatrix} 3 & -3 & 3 \\ -18 & 6 & 3 \end{pmatrix}$

Therefore, $(-7+4)A=(-7A)+4A$.

g. $1A=\begin{pmatrix} -1 & 1 & -1 \\ 6 & -2 & -1 \end{pmatrix}=A$,

h. $0A=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=0$

Assessment

You can assess the understanding of the students by giving them Exercise 6.1 as homework. You can assign problem 7 in Exercise 6.1 to high achievers. The rest of the questions can be given to all the students. Let the students complete the exercise and check their solutions. At this stage, you can give students a test or a quiz on the concepts covered so far. The questions should focus on definition of a matrix, size or order of a matrix, equality of matrices, and addition and subtraction of matrices and their properties.

Answers to Exercise 6.1

1. a. 9 b. 2 c. 9 d. 51
2. a. 2×2 b. 2×3 c. 3×2 d. 1×3 e. 1×1
3. a. 1, –4, 1 b. 0, 1, 71, 4

4. $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 & -5 \\ 4 & 2 & 0 & -2 \\ 7 & 5 & 3 & 1 \end{pmatrix}$

5. a. $\begin{pmatrix} -3 & 2 & -2 \\ 0 & 3 & 6 \end{pmatrix}$ b. $\begin{pmatrix} 5 & -2 & -2 \\ 2 & 1 & 0 \end{pmatrix}$ c. $\begin{pmatrix} -10 & 6 & -4 \\ -1 & 7 & 15 \end{pmatrix}$

d. $\begin{pmatrix} -3 & 2 & -2 \\ 0 & 3 & 6 \end{pmatrix}$ e. $\begin{pmatrix} -10 & 6 & -4 \\ -1 & 7 & 15 \end{pmatrix}$

6. a. $C = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ -1 & 1 & 2 \end{pmatrix}$ b. $\begin{pmatrix} 4 & \frac{-5}{2} & \frac{9}{2} \\ \frac{7}{2} & 3 & \frac{13}{2} \\ \frac{3}{2} & \frac{1}{2} & 4 \end{pmatrix}$

7. a. $\begin{pmatrix} 295 & 371 \\ 398 & 331 \end{pmatrix}$ b. The matrix of sell is $\begin{pmatrix} 797.5 & 1002.5 \\ 1094.5 & 918 \end{pmatrix}$

Thus, from the boys they raised 1800 Birr. From the girls they raised 2012.50 Birr. In Kebele 1, they raised 1892 Birr. And the total amount raised is 3812.50 Birr.

When you make sure that students have well understood addition and subtraction of matrices, the next point of interest will be multiplying matrices.

Start matrix multiplication by considering column vectors and row vectors. Thus,

a. $(3 \quad 5 \quad 1)\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = (17)$ or 17, while

b. $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}(3 \quad 5 \quad 11) = \begin{pmatrix} 3 & 5 & 11 \\ 6 & 10 & 22 \\ 12 & 20 & 44 \end{pmatrix}$

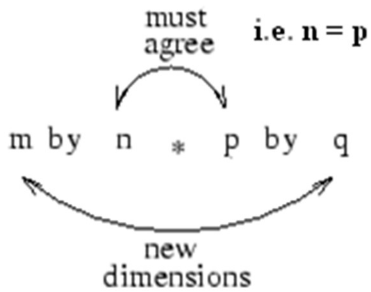
In a) a 1×3 is multiplied by 3×1 matrix resulting in a 1×1 matrix (a number).

In b) a 3×1 matrix is multiplied by a 1×3 matrix. The result is a 3×3 matrix.

Students can decide whether two matrices can be multiplied with each other by observing their sizes and then can find the product when possible.

Students might find matrix multiplication difficult to understand. The definition is not easy to grasp. So, give the students as many examples as needed. They also easily make arithmetic mistakes. Since the purpose is not to develop the calculating power of the students, it is not necessary to give them large matrices to multiply; rather drill them by giving them many examples of small sized matrices.

When you multiply a 1 by 3 matrix by a 3 by 4 matrix, you get a 1 by 4 matrix. This pattern will always hold when you multiply. When we multiply matrices, the middle numbers must be the same. The following picture expresses the requirements on the dimensions:



To make sure that they have understood the definition, give them matrices such that

1. Multiplication is impossible, like, $\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$ with $\begin{pmatrix} 4 \\ -2 \\ 23 \end{pmatrix}$

2. Multiplication is possible like:

   a. $\begin{pmatrix} 4 \\ -2 \\ 23 \end{pmatrix}$ with $\begin{pmatrix} 7 & -8 \end{pmatrix}$

   b. $\begin{pmatrix} 2 & 11 & -5 \end{pmatrix}$ with $\begin{pmatrix} 4 \\ -2 \\ 23 \end{pmatrix}$

   c. $\begin{pmatrix} 4 \\ -2 \\ 23 \end{pmatrix}$ with $\begin{pmatrix} 2 & 11 & -5 \end{pmatrix}$

   d. $\begin{pmatrix} 4 & 7 \\ 1 & -2 \\ 1 & 3 \end{pmatrix}$ with $\begin{pmatrix} 4 & 3 & -5 & 7 \\ 3 & 2 & 8 & 9 \end{pmatrix}$

Ask the whole class questions like:

- Tell which of the given pairs can be multiplied with each other by seeing their size.
- If a pair can be multiplied, what will be the size of a product?
- Give the product of each pair, if possible.

Let the students think by themselves and then do group discussion and finally share what they did to the whole class. When they share, let a group representative go to the board and do it. You can use their performance as a formative assessment and give them some more problems to work on, if necessary. To help you do so, you can give them Activity 6.6. The purpose of this activity is to prepare the students for the definition of multiplication of matrices.

Answers to Activity 6.6

a. $A_1B^1 = 3\times5 + 2\times2 + 0\times2 = 15 + 4 + 0 = 19$

b. $A_1B^2 = 3\times3 + 2\times4 + 0\times1 = 9 + 8 + 0 = 17$

c. $A_1B^3 = 3\times3 + 2\times2 + 0\times2 = 9 + 4 + 0 = 13$

d. $A_2B^1 = 2\times5 + 1\times2 + 1\times2 = 10 + 2 + 2 = 14$

e. $A_2B^2 = 2\times3 + 1\times4 + 1\times1 = 6 + 4 + 1 = 11$

f. $A_2B^3 = 2\times3+1\times2+1\times2 = 6 + 2 + 2 = 10$

Therefore, $AB = \begin{pmatrix} 19 & 17 & 13 \\ 14 & 11 & 10 \end{pmatrix}$.

Right after they perform Activity 6.6, you can form groups of students and let them do Activity 6.7. This is designed to help the students realize the properties of matrix multiplication. Multiplication of matrices is associative, left as well as right distributive over addition of matrices. In this activity, these properties are illustrated.

Guide the students to check that AB may not be the same as BA, in general, by taking examples like:

$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}$. Notice that $AB = \begin{pmatrix} 6 & 10 \\ 14 & 15 \end{pmatrix}$ and $BA = \begin{pmatrix} -2 & -4 \\ -1 & 23 \end{pmatrix}$.

Thus, $AB \neq BA$.

Answers to Activity 6.7

a. $A(BC) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\left[\begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -6 & 8 \\ 12 & -11 \end{pmatrix} = \begin{pmatrix} 18 & -14 \\ 42 & -41 \end{pmatrix}$

b. $(AB)C = \left[\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}\right]\begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 14 & 15 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 18 & -14 \\ 42 & -41 \end{pmatrix}$

Therefore, $A(BC) = (AB)C$.

c. $A(B+C) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\left(\begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 11 & 22 \end{pmatrix}$

d. $AB + AC = \begin{pmatrix} 9 & 8 \\ 11 & 22 \end{pmatrix}$

Therefore, $A(B+C) = AB + AC$.

e. $(B+C)A=\begin{pmatrix}1 & -4\\ 4 & 6\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}=\begin{pmatrix}5 & -10\\ -2 & 26\end{pmatrix}$

f. $BA+CA=\begin{pmatrix}5 & -10\\ -2 & 26\end{pmatrix}$

Therefore, $(B+C)A=BA+CA.$

Another important idea in the study of matrices is transposition. The transpose of a matrix can be simply considered as a change of roles of row and column. But, to let students practice by themselves, you can give them Activity 6.8. The purpose of this activity is to verify some properties of transpose of a matrix. Before reading the properties, guide the students to work the activity in groups.

Answers to Activity 6.8

a. $A^T=\begin{pmatrix}1 & 0\\ 2 & 1\\ 3 & 4\end{pmatrix}$

b. $\left(A^T\right)^T=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{pmatrix}=A$

c. $3A^T=3\begin{pmatrix}1 & 0\\ 2 & 1\\ 3 & 4\end{pmatrix}=\begin{pmatrix}3 & 0\\ 6 & 3\\ 9 & 12\end{pmatrix}$

d. $(3A)^T=\begin{pmatrix}3 & 6 & 9\\ 0 & 3 & 12\end{pmatrix}^T=\begin{pmatrix}3 & 0\\ 6 & 3\\ 9 & 12\end{pmatrix}$

Therefore, $3A^T=(3A)^T$.

e. $(AB)^T=\begin{pmatrix}9 & 9\\ 6 & 3\end{pmatrix}$

f. $B^TA^T=\begin{pmatrix}9 & 9\\ 6 & 3\end{pmatrix}$

Hence $(AB)^T=B^TA^T$

To help the students practice more, you can give them the following questions as an assignment.

Question 1 Based on the given matrices, perform (when they are possible) the operations listed below.

$$A=\begin{pmatrix}5 & 5 & 2\\ 1 & 0 & -6\\ 3 & 3 & 8\end{pmatrix},\ B=\begin{pmatrix}7 & 1 & -6\\ 9 & 4 & 4\\ 2 & 10 & 9\end{pmatrix},\ C=\begin{pmatrix}3 & 1 & 2\end{pmatrix},\text{ and } D=\begin{pmatrix}8\\ 1\\ 5\end{pmatrix}$$

a. 4C	b. AD	c. DA	d. BC
e. 3CB	f. C (A + B)	g. AB	h. BA
i. CAD	j. DBC	k. AD + $(CB)^T$	l. DC
m. CD			

Solution:

a. 4C = 4 (3 1 2) = (12 4 8)

b. $AD = \begin{bmatrix} 5 & 5 & 2 \\ 1 & 0 & -6 \\ 3 & 3 & 8 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 55 \\ -22 \\ 67 \end{bmatrix}$

c. DA is impossible to compute.

d. BC is impossible to compute.

e. $3CB = 3\,(3 \;\; 1 \;\; 2) \begin{bmatrix} 7 & 1 & -6 \\ 9 & 4 & 4 \\ 2 & 10 & 9 \end{bmatrix} = 3\,(34 \;\; 27 \;\; 4) = (102 \;\; 81 \;\; 12)$

f. $C(AB) = (3\ 1\ 2) \left[\begin{bmatrix} 5 & 5 & 2 \\ 1 & 0 & -6 \\ 3 & 3 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 1 & -6 \\ 9 & 4 & 4 \\ 2 & 10 & 9 \end{bmatrix} \right]$

$= (3 \;\; 1 \;\; 2) \begin{bmatrix} 12 & 6 & -4 \\ 10 & 4 & -2 \\ 5 & 13 & 17 \end{bmatrix} = (56 \;\; 48 \;\; 20)$

g. $AB = \begin{bmatrix} 5 & 5 & 2 \\ 1 & 0 & -6 \\ 3 & 3 & 8 \end{bmatrix} \begin{bmatrix} 7 & 1 & -6 \\ 9 & 4 & 4 \\ 2 & 10 & 9 \end{bmatrix} = \begin{bmatrix} 84 & 45 & 8 \\ -5 & -59 & -60 \\ 64 & 95 & 66 \end{bmatrix}$

h. $BA = \begin{bmatrix} 7 & 1 & -6 \\ 9 & 4 & 4 \\ 2 & 10 & 9 \end{bmatrix} \begin{bmatrix} 5 & 5 & 2 \\ 1 & 0 & -6 \\ 3 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 18 & 17 & -40 \\ 61 & 57 & 26 \\ 47 & 37 & 16 \end{bmatrix}$

i. CAD = (CA) . D = 277. But it is impossible that CAD = C.(AD).

j. It is impossible to compute.

k. $AD + (CB)^T = \begin{bmatrix} 55 \\ -22 \\ 67 \end{bmatrix} + \begin{bmatrix} 34 \\ 27 \\ 4 \end{bmatrix} = \begin{bmatrix} 89 \\ 5 \\ 71 \end{bmatrix}$

l. $DC = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix} (3 \;\; 1 \;\; 2) = \begin{bmatrix} 24 & 8 & 16 \\ 3 & 1 & 2 \\ 15 & 5 & 10 \end{bmatrix}$

m. $CD = (3 \;\; 1 \;\; 2) \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix} = (35).$

Question 2 The matrix below expresses the approximate distance, in miles, between any two of the following four cities: A, B, C, and D.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 150 & 111 & 137 \\ 150 & 0 & 217 & 265 \\ 111 & 217 & 0 & 241 \\ 137 & 265 & 241 & 0 \end{pmatrix} \end{array}$$

a. What a special kind of matrix is this (other than square and 4 by 4)?

b. If we want to know the same information in kilometers, what should we do? Take one mile is equal to 1.6 kilometers.

c. What is the resulting matrix when you perform the operation that you suggested in part (b)?

You may give high achievers problems of the following type (as an extra work, not for summative assessment purpose.)

Question 3 Does $r(AB) = A(rB)$ where r is a scalar, A is a 2 by 2 matrix, and B is a 1 by 2 column vector? Explain your answer.

Question 4 Where would you place a parenthesis to minimize the number of multiplications performed to do this problem? How many simple multiplications did it take to find T? What is T?

$$T = \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \begin{pmatrix} 8 & 6 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ -4 \end{pmatrix} \begin{pmatrix} 1 & -7 & 8 \end{pmatrix}$$

Assessment

After the students go through the previous activities and assignments, assess the students' understanding of matrix multiplication and transpose, by asking them to do additional problems. You can also give the students a test or a quiz as a summative assessment.

Answers to Exercise 6.2

1. a. $AB = \begin{pmatrix} 9 & 4 & -3 \\ 3 & 2 & -15 \end{pmatrix}$, BA does not exist.

b. Both AB and BA do not exist

c. $AB = \begin{pmatrix} 5 & 10 & 5 \\ 13 & 14 & -5 \\ -3 & 6 & 15 \end{pmatrix}$, $BA = \begin{pmatrix} 12 & -6 \\ 1 & 22 \end{pmatrix}$

d. $\begin{pmatrix} 29 \\ -10 \\ -15 \end{pmatrix}$

2. a. 3×2 b. $C=\begin{pmatrix}12 & -11\\ -1 & -7\\ 12 & 0\end{pmatrix}$ Thus, $C_{32}=0$; $C_{11}=12$; $C_{21}=-1$

3. $-4AB=\begin{pmatrix}-48 & 44\\ 4 & 28\\ -48 & 0\end{pmatrix}$; $AA=\begin{pmatrix}3 & -1 & 15\\ 1 & 0 & 3\\ 0 & 0 & 9\end{pmatrix}$; $A(AB)=\begin{pmatrix}61 & -15\\ 13 & -4\\ 36 & 0\end{pmatrix}$

4. a. $TP=\begin{pmatrix}5 & 2 & 2\\ 3 & 6 & 0\\ 4 & 4 & 1\\ 6 & 0 & 3\end{pmatrix}\begin{pmatrix}2 & 3\\ 1 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}12 & 17\\ 12 & 15\\ 12 & 16\\ 12 & 18\end{pmatrix}$

   The new system is better to rank the teams

   b. D stood first, while B stood last

5. $A+A^T=\begin{pmatrix}3 & -1\\ 0 & \frac{4}{3}\end{pmatrix}+\begin{pmatrix}3 & 0\\ -1 & \frac{4}{3}\end{pmatrix}=\begin{pmatrix}6 & -1\\ -1 & \frac{8}{3}\end{pmatrix}$

   $B+B^T=\begin{pmatrix}3 & 2 & 2\\ 2 & 4 & -2\\ -1 & 0 & 1\end{pmatrix}+\begin{pmatrix}3 & 2 & -1\\ 2 & 4 & 0\\ 2 & -2 & 1\end{pmatrix}=\begin{pmatrix}6 & 4 & 1\\ 4 & 8 & -2\\ 1 & -2 & 2\end{pmatrix}$

   Therefore, both $A+A^T$ and $B+B^T$ are symmetric.

6. $AA^T=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

   $=\begin{pmatrix}\cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta\\ \sin\theta\cos\theta-\sin\theta\cos\theta & \sin^2\theta+\cos^2\theta\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

   Similarly $A^TA=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

7. Suppose A is a square matrix of order n.
   Then $(A+A^T)^T=A^T+(A^T)^T=A^T+A=A+A^T$
   Hence, $A+A^T$ is a symmetric matrix.

8. a. $A+A^T=\begin{pmatrix}0 & -1 & 4\\ 1 & 0 & 7\\ -4 & -7 & 0\end{pmatrix}+\begin{pmatrix}0 & 1 & -4\\ -1 & 0 & -7\\ 4 & 7 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

   $\Rightarrow A$ is skew – symmetric

b. $B + B^T = \begin{pmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{pmatrix} + \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow B$ is skew-symmetric

9. Let A be a square matrix. Then,

$$(A - A^T) + (A - A^T)^T = A - A^T + A^T - A$$
$$= A - A + A^T - A^T$$
$$= 0$$

10. The diagonal elements of A and A^T are the same.

If A is a skew – symmetric then

$A + A^T = 0$

$\Rightarrow 2a_{ii} = 0$, for each diagonal element a_{ii}, $i = 1, 2, \ldots, n$

$\Rightarrow a_{ii} = 0$

Thus, the diagonal elements are zero.

6.2 DETERMINANTS AND THEIR PROPERTIES

Periods Allotted: 6 Periods

Competencies

At the end of this sub-unit, students will be able to;

- *determine the determinant of a square matrix of order two.*
- *determine the minor and cofactor of a given element of a matrix.*
- *calculate the determinant of square matrix of order 3.*
- *describe the properties of determinants.*

Vocabulary: Determinant, Minor, Cofactor

Introduction

Determinant is defined for a square matrix. The **determinant** of a square matrix is a real number that is assigned to the matrix according to certain rules. The determinant gives you valuable information about the matrix. In this sub-unit, determinants of only 1 by 1, 2 by 2 and 3 by 3 matrices are treated.

Teaching Notes

The symbols det(A) and $|A|$ represent the determinant of A. Remind the students that the straight bars do NOT mean absolute value; they represent the determinant of the matrix. The determinant of a 1 by 1 matrix is simply the element of the matrix. When the size of the matrix is more than 3 or 4, the calculation of the determinant can be involved. In such cases, it is better to use the properties of determinants rather than the definition, as much as possible.

While defining determinants of 2 by 2 matrices, remind students that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ are different. The first one is a matrix (not a number) and the second one is a determinant (which is a real number).

Guide students to note that every 2 by 2 matrix has a determinant (This is also true for any square matrix). First, let the students try to calculate the determinant of some square matrices (2 by 2) and understand the definition. Then guide the students to work Activity 6.9. The purpose of this activity is to let the students identify and verify some of the properties of determinants. Some of the properties of determinants are:

- determinant of a matrix and its transpose are equal.
- determinant of a product is equal to product of the determinants.

A more complete list of properties of determinants is given after Group Work 6.1.

Answers to Activity 6.9

1. a. $|A| = 3 - 2 = 1,$ b. $|B| = 10 - 3 = 7,$ c. $|A^T| = 1.$

2. $AB = \begin{pmatrix} -9 & 1 \\ 2 & -1 \end{pmatrix}.$

   Then $|AB| = 7$ and also $|A|\,|B| = 1 \times 7 = 7$

   Hence, $|AB| = |A||B|$

3. $A + B = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}.$

   Then $|A + B| = -10$ and $|A| + |B| = 1 + 7 = 8$

   $\Rightarrow \ |A + B| \neq |A| + |B|$

Once the students become capable of calculating determinant of a 2 by 2 matrix and verify its properties, you can as well proceed to discuss the determinant of a 3 by 3 matrix. To define determinant of a 3 by 3 matrix, first define minor and cofactor. Then give them some 3 by 3 matrices and ask them to find the minors and cofactors of their entries. Then define the determinant of a 3 by 3 matrix. Give them sufficient examples to work through. In the students' textbook, the determinant of a 3 by 3 matrix is calculated by use of minors and cofactors which is a general rule for calculating determinants of any squared order. But, for only 3 by 3 matrix, the following approach can also be used to calculate determinants.

Example: Calculate the determinant of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5 \end{pmatrix}$$

To calculate such a determinant:

1. Write the first two columns beside the third column.
2. Multiply and add the triple numbers presented left to right.
3. Multiply and add triple numbers presented right to left.
4. Subtract result of (2) from result of (3).
5. The resulting number is the determinant.

Solution:

1. $A = \begin{pmatrix} 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ -3 & 2 & 5 & -3 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ -3 & 2 & 5 & -3 & 2 \end{pmatrix}$ Multiply these and add

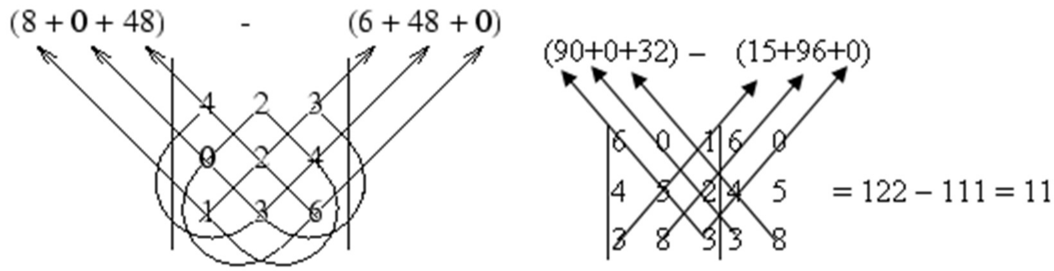
   $(-3)(1)(0) + (2)(4)(2) + (5)(1)(1) = 21$

3. $A = \begin{pmatrix} 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ -3 & 2 & 5 & -3 & 2 \end{pmatrix}$ Multiply and add.

   $(2)(1)(5) + (1)(4)(-3) + (0)(1)(2) = -2$

4. $|A| = -2 - 21 = -23$

Example2: The figures below give two ways of finding determinants of 3 by 3 matrices. Encourage your students to explain how they work.



You can give various exercises for the students to practice calculating determinant of a matrix. Pursuant to their ability of calculating matrices, it will be essential to study properties of determinants. Before telling them anything, guide the students to work Group Work 6.1 by forming groups. The group work will help them to study properties of determinants. Once they go through the group work, give them some examples on the properties.

Answers to Group Work 6.1

1. a. $|A| = 17$, $|A^T| = 17$ b. $|A| = |A^T|$
2. a. $|B| = -17$ b. $|B| = -|A|$
3. a. $|C| = 85$ b. $|C| = 5|A|$
4. a. $|D| = 17$ b. $|D| = |A|$

The Group Work activity verifies further properties of determinants. You need to give students some more similar examples so that they can become comfortable in using the properties. You can give them examples of the following kind.

Example1. Let a, b, c, $d \in \mathbb{R}$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 4$. Determine the value of each of the following.

a. $\begin{vmatrix} 3a & b \\ 3c & d \end{vmatrix}$ b. $\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix}$

2. Let A be a 3×3 matrix. If $|A| = 21$, what is $|5A|$?
3. Evaluate each of the following.

a. $\begin{vmatrix} a & d & a \\ b & e & b \\ c & f & c \end{vmatrix}$ b. $\begin{vmatrix} a & b & e \\ c & d & f \\ a & b & e \end{vmatrix}$

4. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

5. Find t if a. $\begin{vmatrix} 1 & -2 & 3 \\ t & 3 & 2 \\ 6 & 1 & t \end{vmatrix} = 0$ b. $\begin{vmatrix} 1 & 1 & 1 \\ t & 1 & t \\ 1+t & 2 & 3 \end{vmatrix} = 0$

6. Find the following determinant using both methods:

$$\begin{vmatrix} 5 & 7 & 3 \\ -2 & 8 & 5 \\ 7 & 9 & 2 \end{vmatrix}$$

Solution:

1. a. $\begin{vmatrix} 3a & b \\ 3c & d \end{vmatrix} = 3(ad - bc) = 3(4) = 12$

   b. $\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} = 9(ad - bc) = 9(4) = 36$

2. $|5\text{A}| = 5|\text{A}| = 5(21) = 105$

3. a. $\begin{vmatrix} a & d & a \\ b & e & b \\ c & f & c \end{vmatrix} = a\begin{vmatrix} e & b \\ f & c \end{vmatrix} - d\begin{vmatrix} b & b \\ c & c \end{vmatrix} + a\begin{vmatrix} b & e \\ c & f \end{vmatrix}$

   $= a\,(ec - bf) - d(bc - bc) + a(bf - ce) = 0$

   b. $\begin{vmatrix} a & b & e \\ c & d & f \\ a & b & e \end{vmatrix} = a\begin{vmatrix} d & f \\ b & e \end{vmatrix} - b\begin{vmatrix} c & f \\ a & e \end{vmatrix} + e\begin{vmatrix} b & d \\ a & b \end{vmatrix}$

   $= a\,(de - bf) - b(ce - af) + e(bc - ad) = 0.$

4. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 1\begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - 1\begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + \begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$

   $= bc^2 - b^2c - ac^2 + a^2c + ab^2 - ba^2$

   $= (a^2c + bc^2 - abc - ac^2 - a^2b + b^2c - b^2a + abc)$

   $= c\,(a^2 + bc - ab - ac) - b(a^2 - ab + bc - ac)$

   $= (c - b)\,(a^2 + bc - ab - ac)$

   $=(c - b)\,(b - a)\,(c - a).$

5. a. $\begin{vmatrix} 1 & -2 & 3 \\ t & 3 & 2 \\ b & 1 & t \end{vmatrix} = 1\begin{vmatrix} 3 & 2 \\ 1 & t \end{vmatrix} + 2\begin{vmatrix} t & 2 \\ 6 & t \end{vmatrix} + 3\begin{vmatrix} t & 3 \\ 6 & 1 \end{vmatrix} = 0$

   $= 3t - 2 + 2\,(t^2 - 12) + 3\,(t - 18) = 0$

   $\Rightarrow t^2 + 3t - 40 = 0$

   Solving the quadratic equation yields $t = -8$ or $t = 5$.

b. $$\begin{vmatrix} 1 & 1 & 1 \\ t & 1 & t \\ 1+t & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & t \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} t & t \\ 1+t & 3 \end{vmatrix} + 1\begin{vmatrix} t & 1 \\ 1+t & 2 \end{vmatrix} = 0$$

$$= 3 - 2t - (3t - t(1 + t)) + 2t - (1 + t) = 0$$

$$\Rightarrow t^2 - 3t + 2 = 0$$

Solving the quadratic equation gives

$t = 1$ or $t = 2$.

Now the students are ready to do Exercise 6.3. You can give them the Exercise as homework or as a group work and they can present their work in class.

Here are some additional examples

1. Evaluate the determinant of A.

$$A = \begin{pmatrix} 3 & 1 & 4 \\ -6 & -2 & -8 \\ 1 & 5 & -7 \end{pmatrix}$$

2. Give the minors and cofactors of the entries –2, 5 and –7 of A.
3. Use the properties to find the following determinants.

a. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 1 & -1 \end{vmatrix}$ b. $\begin{vmatrix} 1 & -5 & 2 \\ -4 & -1 & 5 \\ 3 & -4 & 3 \end{vmatrix}$

Solution:

1. $$\begin{vmatrix} 3 & 1 & 4 \\ -6 & -2 & -8 \\ 1 & 5 & -7 \end{vmatrix} = 3\begin{vmatrix} -2 & -8 \\ 5 & -7 \end{vmatrix} - 1\begin{vmatrix} -6 & -8 \\ 1 & -7 \end{vmatrix} + 4\begin{vmatrix} -6 & -2 \\ 1 & 5 \end{vmatrix}$$

$$= 3(14 + 40) - (42 + 8) + 4(-30 + 2)$$
$$= 162 - 50 + 112 = 0.$$

2. Minor of (–2) = $\begin{vmatrix} 3 & 4 \\ 1 & -7 \end{vmatrix} = -21 - 4 = -25$

Cofactor of (–2) = $(-1)^{2+2}(-25) = -25$

Minor of (5) = $\begin{vmatrix} 3 & 4 \\ -6 & -8 \end{vmatrix} = -32 + 24 = -8$

Cofactor of (5) = $(-1)^{3+2}(-8) = 8$

Minor of (–7) = $\begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} = -6 + 6 = 0$

Cofactor of (–7) = $(-1)^{3+3}(0) = 0$.

3. a. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$

$= -3 + 2 + 2\,(-2 + 6) + 3(2 - 9) = 14$

b. $\begin{vmatrix} 1 & -5 & 2 \\ -4 & -1 & 5 \\ 3 & -4 & 3 \end{vmatrix} = 1\begin{vmatrix} -1 & 5 \\ -4 & 3 \end{vmatrix} + 5\begin{vmatrix} -4 & 5 \\ 3 & 3 \end{vmatrix} + 2\begin{vmatrix} -4 & -1 \\ 3 & -4 \end{vmatrix}$

$= -3 + 20 + 5\,(-12 - 15) + 2(16 + 3) = -\,80$

You can challenge the high achievers by giving them matrices with orders 4 by 4 and ask them to find their determinants by using the properties, like:

Example

a. Show that each of the following holds:

1. $\begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 5 \\ 7 & 6 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{vmatrix} = 60$

2. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = (1-r)^3.$

3. $\begin{vmatrix} 1+a & a & a & a \\ b & 1+b & b & b \\ c & c & 1+c & c \\ d & d & d & 1+d \end{vmatrix} = 1 + a + b + c + d$

b. Evaluate $\begin{vmatrix} \log 100 & 2 \\ \log 10 & 3 \end{vmatrix}$

c. Show that in the xy plane an equation of the line that contains the two points (a, b) and (c, d) is given by $\begin{vmatrix} x & y & 1 \\ a & b & 1 \\ c & d & 1 \end{vmatrix} = 0$

Solution:

a. 1. $\begin{vmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 5 \\ 7 & 6 & 1 & 2 \\ 1 & 1 & 3 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 & 5 \\ 6 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 & 5 \\ 7 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 & 5 \\ 7 & 6 & 2 \\ 1 & 1 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 1 & 4 \\ 7 & 6 & 1 \\ 1 & 1 & 3 \end{vmatrix}$

$= 1(-5) - (-10) + 2\,(45) - 35 = 60$

2. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 & 1 \\ r & 1 & 1 \\ r & r & 1 \end{vmatrix} - 1\begin{vmatrix} r & 1 & 1 \\ r & 1 & 1 \\ r & r & 1 \end{vmatrix} + 1\begin{vmatrix} r & 1 & 1 \\ r & r & 1 \\ r & r & 1 \end{vmatrix} - 1\begin{vmatrix} r & 1 & 1 \\ r & r & 1 \\ r & r & r \end{vmatrix}.$$

$= (1 - 2r + r^2) - (0) + 0 - (r^3 - 2r^2 + r)$
$= 1 - 2r + r^2 - r^3 + 2r^2 + r$
$= -r^3 + 3r^2 - 3r + 1$
$= (r^2 - 2r + 1)\ (1 - r) = (1 - r)^3.$

3. $$\begin{vmatrix} 1+a & a & a & a \\ b & 1+b & b & b \\ c & c & 1+c & c \\ d & d & d & 1+d \end{vmatrix} = (1+a)\begin{vmatrix} 1+b & b & b \\ c & 1+c & c \\ d & d & 1+d \end{vmatrix}$$

$$-a\begin{vmatrix} b & b & b \\ c & 1+c & c \\ d & d & 1+d \end{vmatrix} + a\begin{vmatrix} b & 1+b & b \\ c & c & c \\ d & d & 1+d \end{vmatrix} - a\begin{vmatrix} b & 1+b & b \\ c & c & 1+c \\ d & d & d \end{vmatrix}$$

$= (1 + a)\ (1 + d + c + b) - a(b) + a(-c) - a(d)$
$= 1 + a + b + c + d$

b. $$\begin{vmatrix} \log_{100} & 2 \\ \log_{10} & 3 \end{vmatrix} = 3\log_{100} - 2\log_{10} = 4$$

c. Given the points (a, b) and (c, d)

To find equation of the line,

$$\frac{y-b}{x-a} = \frac{d-b}{c-a}.$$

$\Rightarrow (y - b)\ (c - a) - (x - a)\ (d - b) = 0.$

$\Rightarrow x(b - d) - y(a - c) + ad - bc = 0$

Thus, the equation of the line is given by $\begin{vmatrix} x & y & 1 \\ a & b & 1 \\ c & d & 1 \end{vmatrix} = 0.$

Assessment

You can assess their understanding of the definition and properties of determinants, by giving them 2 by 2 and 3 by 3 matrices and asking them to calculate their determinants individually or in groups. You can also let them give the minors and cofactors of matrices. Give students matrices, in which they are asked to apply properties of determinants, as given in Exercise 6.3.

Answers to Exercise 6.3

1. a. -32

   b. -13

   c. $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix} = a^2 - b^2 - a^2 = -b^2$

2. a. $\begin{vmatrix} 2x & x \\ 4 & x \end{vmatrix} = 0 \Leftrightarrow 2x^2 - 4x = 0$

   $\Rightarrow 2x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$

   b. $\begin{vmatrix} 2 & -2 & 1 \\ x & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 1 \Leftrightarrow 2(2) - x(-4-1) + 3(-1) = 1$

   $\Leftrightarrow 4 + 5x - 3 = 1 \Leftrightarrow 5x + 1 = 1 \Rightarrow x = 0$

   c. $\begin{vmatrix} x+1 & 2 & 1 \\ 1 & 1 & 2 \\ x-1 & 1 & x \end{vmatrix} = 0 \Leftrightarrow (x+1)\begin{vmatrix} 1 & 2 \\ 1 & x \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ x-1 & x \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ x-1 & 1 \end{vmatrix} = 0$

   $\Leftrightarrow (x+1)(x-2) - 2(x-2x+2) + (1-x+1) = 0$

   $\Leftrightarrow x^2 - x - 2 + 2x - 4 + 2 - x = 0$

   $\Leftrightarrow x^2 - 4 = 0 \Rightarrow x = 2$ or $x = -2$

3. a. $C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 9 & 3 \end{vmatrix} = -(3-27) = 24$

   b. $C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1$

   c. $C_{23} = (-1)^{2+3} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3$

4. a. $\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & a-x & b-y \\ 0 & c-x & d-y \end{vmatrix} = \begin{vmatrix} a-x & b-y \\ c-x & d-y \end{vmatrix}$

   $= (a-x)(d-y) - (c-x)(b-y)$
   $= ad - ay - dx + xy - (bc - cy - bx + xy)$
   $= ad - ay - dx + xy - bc + cy + bx - xy$
   $= (ad - bc) - dx + cy + bx - ay$

b. Let the straight line pass through (a, b) and (c, d)

$$\frac{y-b}{x-a}=\frac{d-b}{c-a} \Rightarrow (y-b)(c-a)-(d-b)(x-a)=0$$

$$\Rightarrow yc - ya - bc + ab + ad - dx + bx - ab = 0$$

$$\Rightarrow (ad - bc) - (dx - cy) + (bx - ay) = 0$$

$$\Rightarrow 1\begin{vmatrix} a & b \\ c & d \end{vmatrix} - 1\begin{vmatrix} x & y \\ c & d \end{vmatrix} + 1\begin{vmatrix} x & y \\ a & b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = 0$$

5\. a. $\begin{vmatrix} x & t+w \\ y & s+u \end{vmatrix} = xs+xu-(yt+yw) = (xs-yt)+(xu-yw) = \begin{vmatrix} x & t \\ y & s \end{vmatrix} + \begin{vmatrix} x & w \\ y & u \end{vmatrix}$

b. $\begin{vmatrix} a+rb & b \\ c+rd & d \end{vmatrix} = (a+rb)d - b(c+rd) = ad + rbd - bc - brd = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

c. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-c & c+a-b-c \\ 0 & c-a & a+b-b-c \end{vmatrix}$

$$= \begin{vmatrix} b-a & a-b \\ c-a & a-c \end{vmatrix} (b-a)(a-c)-(c-a)(a-b)$$

$$= (ab - bc - a^2 + ac - (ac - cb - a^2 + ab)$$

$$= (ab - bc - ac - a^2 - ac + bc + a^2 - ab) = 0$$

6.3 INVERSE OF A SQUARE MATRIX

Periods allotted: 4 periods

Competency

At the end of this sub-unit, students will be able to;

- *determine inverse of a square matrix.*

Vocabulary: Inverse, Invertible, Singular, Non-singular, Adjoint

Introduction

In this subunit, the students will learn how to find inverses of some square matrices. Finding inverses of square matrices, whenever they exist, is important in solving systems of linear equations. When students work to find inverses of non – singular matrices, they do lots of arithmetic. In the process, they may make arithmetic mistakes. Guide them to check whether or not they have found the correct inverse matrix by multiplying the given matrix and the matrix they have found. If they have found the correct matrix, the product will be the identity.

Teaching Notes

In this sub-unit, students will learn how to find inverses of non-singular matrices.

The important facts you need to stress here are that non - square matrices have no inverses. This means, if a matrix is not a square, students should conclude that it has no inverse. Moreover, A is non-singular $\Leftrightarrow$ $|A| \neq 0$. Thus, if $|A| = 0$, then since A is singular, they need not try to find the inverse.

To practice these and reach a conclusion (generalization), you can let the students form pairing and do Activity 6.10. When they do the activity, you may go round the groups and assist them to reach some generalization. Remind them to check their answers.

In the activity, if they have calculated B to be the inverse of A, then they need to check that $AB = BA =$ I. If this is not the case, then they have to recheck their calculations.

As the students do Activity 6.10.d, ask them to find the determinant of the given matrix B. They will get $|B| = 0$. Give them some more 2 by 2 matrices whose determinant is zero and ask them to check if they were invertible. Let them generalize.

Answers to Activity 6.10

a. $AI_2 = A$ b. $I_2A = A$

c. Let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $\begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -3a+2c & -3\mathrm{b}+2\mathrm{d} \\ a-c & \mathrm{b}-\mathrm{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solving this gives you, $\mathrm{C} = \begin{pmatrix} -1 & -2 \\ -1 & -3 \end{pmatrix}$

d. If there is a $D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that $BD = I_2$, then

$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 6e+2g & 6f+2h \\ 3e+g & 3f+h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 6e+2g=1 \\ 3e+g=0 \end{cases} \text{ and } \begin{cases} 6f+2h=0 \\ 3f+h=1 \end{cases}$$

But, the system has no solution. Hence, there is no such a matrix D.

In previous lessons, students have known about cofactors and transpose. Here they will discuss adjoint which is the ***transpose*** of the matrix of cofactors. Class Activity 6.11 helps to impress these points and use each in determining inverse of a matrix. The purpose of this activity is to prepare the students for the definition of inverse of a square matrix. The definition gives meaning to inverse of a matrix and a student can use it for checking whether or not a matrix, he or she has found by using adjoint and determinant, is indeed the inverse. However, you need to be cautious that usually students tend to take the matrix of cofactors to represent the adjoint instead of the transpose of the matrix of cofactors.

The activity also helps to show that the inverse of a matrix, if it exists, can be found using the adjoint and determinant of the matrix. Showing that this is always the case for any non-singular matrix is beyond the scope. But, in the activity, the students will do it for the case of 2 by 2 matrix.

Once students find that adj$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & ad-bc \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= ad-bc\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, $A^{-1} = \dfrac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Answers to Activity 6.11

1. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $c_{11} = (-1)^{1+1}|d| = d$, $c_{12} = (-1)^{1+2}|c| = -c$,

   $c_{21} = (-1)^{2+1}|b| = -b$ and $c_{22} = (-1)^{2+2}|a| = a$.

   Therefore, adj $A = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d-b \\ -c \quad a \end{pmatrix}$.

2. $(A)\,(\text{adj } A)=\begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix}=\ (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. a. Let $A^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $AA^{-1}=I_2$

$$\Rightarrow\begin{pmatrix} 5a-3c & 5b-3d \\ 4a+2c & 4b+2d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Rightarrow\begin{cases} 5a-3c=1 \\ 4a+2c=0 \end{cases}\text{ and }\begin{cases} 5b-3d=0 \\ 4b+2d=1 \end{cases}$$

Solving this gives you: $A^{-1}=\begin{pmatrix} \frac{1}{11} & \frac{3}{22} \\ \frac{-2}{11} & \frac{5}{22} \end{pmatrix}$

b. $\text{adj } A=\begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix}$ c. $|\mathrm{A}| = 22$ d. $\frac{1}{\det A}(\text{adj } A)=A^{-1}$

After the students have performed the activity and have done some more examples, it will be useful to see one strong property of inverse presented as theorem 6.2 in the student textbook. To help you assess the students' understanding, you can let them work on Exercise 6.4. Encourage them to do it in groups and check their answers.

You may give high achievers the following kinds of problems:

a. Find k such that $\begin{pmatrix} 1 & 2 & k \\ 3 & -1 & 1 \\ 5 & 3 & -5 \end{pmatrix}$ is singular.

b. Show that if A is non-singular then A^{T} is also non-singular. They might answer it by comparing $|\mathrm{A}|$ and $|\mathrm{A}^{\mathrm{T}}|$.

c. Verify that $(\mathrm{AB})^{\mathrm{T}}=\mathrm{B}^{\mathrm{T}}\,\mathrm{A}^{\mathrm{T}}$, for $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} e & g \\ f & h \end{pmatrix}$.

d. Using c) above show that if A is invertible then $(A^T)^{-1}=(A^{-1})^T$.

Assessment

In addition to what has been mentioned above, assessment of the students' understanding of the concept discussed in this sub-unit should be done on the following points:

- Given a square matrix, is it invertible or not? (They can check this by calculating the determinant. If the determinant is 0, then it is not invertible.)
- Given two matrices, can they check if they are inverses of each other or not?
- Can they state and prove properties of inverses?
- Can they find inverse of a matrix by finding minor and cofactor?

Answers to Exercise 6.4

1. $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -3 & 3 \\ 4 & 1 & 8 \end{pmatrix}\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{I}$

   Hence the two matrices are inverses of each other.

2. a. $\begin{pmatrix} \frac{3}{2} & \frac{-5}{2} \\ -1 & 2 \end{pmatrix}$ b. $\begin{pmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & \frac{-4}{5} & \frac{-14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{pmatrix}$ c. $\begin{pmatrix} \frac{3}{14} & \frac{-1}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{3}{14} & \frac{-1}{14} \\ \frac{-1}{14} & \frac{5}{14} & \frac{3}{14} \end{pmatrix}$

3. $\text{A} = \begin{pmatrix} 3-k & 6 \\ 2 & 4-k \end{pmatrix}$

   A is singular if $|A| = 0$.

   Now $\begin{vmatrix} 3-k & 6 \\ 2 & 4-k \end{vmatrix} = 0 \Rightarrow (3-k)(4-k) - 12 = 0$

   $\Rightarrow 12 - 3k - 4k + k^2 - 12 = 0 \Rightarrow k^2 - 7k = 0$

   $\Rightarrow k = 7$ or $k = 0$

   When $k = 1$,

   $A = \begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix} \quad \Rightarrow A^{-1} = \begin{pmatrix} \frac{-1}{2} & 1 \\ \frac{1}{3} & \frac{-1}{3} \end{pmatrix}$

4. $c_{11} = (-1)^{1+1}\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$, $\quad c_{12} = (-1)^3\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta$

   $c_{13} = (-1)\begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0$, $\quad c_{21} = (-1)^3\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta$

   $c_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = cos\theta$, $\quad c_{23} = (-1)^{-5}\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0$

   $c_{31} = 0$, $\quad c_{32} = 0$, $\quad c_{33} = \cos^2\theta + \sin^2\theta = 1$

   $|A| = \cos\theta\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} -\sin & 0 \\ 0 & 1 \end{vmatrix} + 1$

   $= \cos^2\theta + \sin^2\theta = 1$

   Hence $\text{adj}(A) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\ \text{adj}\ (A) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Also, } A^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore A^{-1} = A^T$

5. $(AB) = \begin{pmatrix} -1 & 7 \\ -3 & 6 \end{pmatrix} \Rightarrow (AB)^{-1} = \dfrac{1}{-6+21}\begin{pmatrix} 6 & 7 \\ 3 & -1 \end{pmatrix}$

$$\Rightarrow (AB)^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{-7}{15} \\ \frac{1}{5} & \frac{-1}{15} \end{pmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{15} \end{pmatrix}$$

$$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\Rightarrow B^{-1}A^{-1} = \frac{1}{15}\begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{15} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{15}\begin{pmatrix} \frac{6}{15} & \frac{-7}{15} \\ \frac{3}{15} & \frac{-1}{15} \end{pmatrix}\begin{pmatrix} \frac{2}{5} & \frac{-7}{15} \\ \frac{1}{5} & \frac{-1}{15} \end{pmatrix}$$

$\therefore \quad (AB)^{-1} = B^{-1}A^{-1}$

6. $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1}A)\, B = (A^{-1}A)\, C$

$\Rightarrow B = C$

If A is singular, this may fail. Take $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ and C $= \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

An obvious choice is $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow AB = AC$, but $B \neq C$

6.4 SYSTEMS OF EQUATIONS WITH TWO OR THREE VARIABLES

Periods allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to;

- *find associated augmented matrix of equations.*
- *perform elementary operations on matrices.*
- *solve systems of equations in two or three variables using the elementary operations.*

Vocabulary: Linear equation, System of linear equation, Homogeneous, Non-homogeneous, Coefficient matrix, Constant vector, Variable vector, Augmented matrix, Elementary row operations

Introduction

Systems of linear equations have many applications. For instance, they are used to give mathematical models of traffic flows and electrical networks. In grade 9, students have discussed solving systems of linear equations in two variables. The students already know how to find solutions of linear systems with two equations and two variables, using substitution or elimination methods. Gauss' method is a systematic application of the elimination method. In this subunit, the students will learn how to find solutions of systems of linear equations by using elementary row operations (Gauss method).

What is important here is that they have to use matrix representation of coefficients and the constants (the augmented matrix), and also use the properties of operations on matrices to apply the row operations.

Teaching Notes

Start the section by revising the substitution and elimination methods. Gauss' method which will be discussed in this section is the elimination method which is done by using matrices. Gauss' method, as we discuss it here, is just changing the augmented matrix of a linear system to an upper triangular matrix and then solving the system using back-substitution.

Changing the system to upper triangular matrix is done by expressing the system into echelon form (This is Gauss' method). On the other hand, if we change the system into reduced echelon form (a modification by Jordan and hence Gauss – Jordan method), we do not need to use back substitution.

The basic rules for the students to follow are:

1. Change the augmented matrix (A: *B*) into echelon form. To change the system into echelon form, you use elementary row operations. There are three of them: swapping, rescaling, and pivoting. We use them to transform the system into an upper triangular matrix and, starting from the bottom, we substitute back into the variables and find the solution, if they exist.
2. In echelon form or reduced echelon form, if a row of the form (0, 0,…0, *b*,) with $b \neq 0$ appears, then the system does not have a solution.
3. If there is a row (or rows) of all zeros at the bottom and there are more variables than non-zero rows, the system has infinite solutions.
4. Otherwise, the system has a unique solution.

As an entry to the discussion, you can let each student individually do Activity 6.12. In solving a system of linear equations, we may need to transform the system to another system of linear equations that have the same solution with the original system of linear equations. The purpose of this activity is to illustrate this idea and help the students revise what they studied in grade 9.

Answers to Activity 6.12

a. $\text{S.S} = \{(3,2)\}$ b. $\text{S.S} = \{(3,2)\}$ c. $\text{S.S} = \{(0,1)\}$

When the students revise these ways of solving systems of equations, help them to systematically understand forming equivalent systems that are helpful in getting solutions. Application of raw operations is the way used to form equivalent systems of equations. They also need to know how useful it is to present a system of equations in a form of augmented matrix. Let the students do more examples similar to the ones presented in the student textbook to help them practice augmented matrix representation and use of elementary raw operations. By doing so, they will also gain more experience in solving systems of equations.

You can give high achievers problems of the following kind.

a. Solve the following systems of linear equations

$$\begin{cases} \dfrac{1}{s} + \dfrac{4}{t} - \dfrac{3}{u} = 4 \\ \dfrac{2}{s} - \dfrac{3}{t} + \dfrac{1}{u} = 1 \\ \dfrac{-3}{s} + \dfrac{2}{t} + \dfrac{2}{u} = -3 \end{cases}$$

b. Give the restrictions on the real numbers *a* and *b* such that the following system has

i. no solutions ii. a unique solution

iii. infinitely many solutions

$$\begin{cases} x - 2y + 3z = 4 \\ 2x - 3y + az = 5 \\ 3x - 4y + 5z = b \end{cases}$$

c. Find the value of *a* for which the following system is consistent and solve the system for this particular *a*:

$$\begin{cases} x + y = 1 \\ ax + y = a \\ (1 + a)x + 2y = 3 \end{cases}$$

For which a does the homogeneous system $\begin{cases} x + (a - 3)y = 0 \\ (a - 3)x + y = 0 \end{cases}$ has a non-zero solution?

Assessment

To assess students' progress and ability of solving systems of linear equations, give them systems of linear equations and guide them to

a. determine the augmented matrix of the system of linear equations.
b. find solutions of the system using the method they have now learned.
c. identify the type of solutions they have found.

Let students do the problems in groups, check their answers and give them feedback.

Answers to Exercises 6.5

1. a. $R_1 \longrightarrow -3R_2 + R_1$ or $R_1 \longrightarrow -3R_3 + R_1$

   b. $R_1 \longrightarrow \frac{1}{8}R_2 + R_1$

2. a. $\begin{pmatrix} 5 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{5}R_1} \begin{pmatrix} 1 & 0 & \frac{-1}{5} \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

$$\xrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & \frac{-1}{5} \\ 0 & 1 & \frac{-1}{5} \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 0 & \frac{-1}{5} \\ 0 & 1 & \frac{-1}{5} \\ 0 & 0 & \frac{21}{5} \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{5}{21} R_3} \begin{pmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

b. $$\begin{pmatrix} 1 & -1 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{pmatrix} \xrightarrow{R_2 \to -4R_1 + R_2} \begin{pmatrix} 1 & -1 & 1 & 5 \\ 0 & 12 & -3 & -14 \end{pmatrix}$$

c. $$\begin{pmatrix} 1 & -1 & 3 & -6 \\ 5 & 3 & -2 & 4 \\ 1 & 3 & 4 & 11 \end{pmatrix} \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -5R_1 + R_2} \begin{pmatrix} 1 & -1 & 3 & -6 \\ 0 & 8 & -17 & 34 \\ 0 & 4 & 1 & 17 \end{pmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{8} R_2} \begin{pmatrix} 1 & -1 & 3 & -6 \\ 0 & 1 & \frac{-17}{8} & \frac{17}{4} \\ 0 & 4 & 1 & 17 \end{pmatrix}$$

$$\xrightarrow{R_3 \to -4R_2 + R_3} \begin{pmatrix} 1 & -1 & 3 & -6 \\ 0 & 1 & \frac{-17}{8} & \frac{17}{4} \\ 0 & 0 & \frac{19}{2} & 0 \end{pmatrix}$$

3. a. $$\begin{pmatrix} 3 & 5 & -1 & -4 \\ 2 & 5 & 4 & 9 \\ -1 & 1 & -2 & 11 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{3} R_1} \begin{pmatrix} 1 & \frac{5}{3} & \frac{-1}{3} & \frac{-4}{3} \\ 2 & 5 & 4 & 9 \\ -1 & 1 & -2 & 11 \end{pmatrix}$$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to -2R_1 + R_2} \begin{pmatrix} 1 & \frac{5}{3} & \frac{-1}{3} & \frac{-4}{3} \\ \frac{5}{3} & 0 & \frac{14}{3} & \frac{-19}{3} \\ 0 & \frac{8}{3} & -\frac{7}{3} & \frac{29}{3} \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & -5 & 5 \\ 0 & \frac{5}{3} & \frac{14}{3} & \frac{-19}{3} \\ 0 & \frac{8}{3} & \frac{-7}{3} & \frac{29}{3} \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{-8}{3}R_2 + R_3} \begin{pmatrix} 1 & 0 & -5 & 5 \\ 0 & \frac{5}{3} & \frac{14}{3} & \frac{-19}{3} \\ 0 & 0 & \frac{-49}{5} & \frac{99}{5} \end{pmatrix} \xrightarrow{R_3 \to \frac{-5}{49}R_3} \begin{pmatrix} 1 & 0 & -5 & 5 \\ 0 & 1 & \frac{14}{3} & \frac{-19}{3} \\ 0 & 0 & 1 & \frac{-99}{49} \end{pmatrix}$$

$$\xrightarrow[R_3 \to \frac{-14}{5}R_3 + R_2]{R_1 \to 5R_3 + R_1} \begin{pmatrix} 1 & 0 & 0 & \frac{-250}{49} \\ 0 & 1 & 0 & \frac{13}{7} \\ 0 & 0 & 1 & \frac{99}{49} \end{pmatrix}$$

b. $$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{pmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to R_1 + R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{pmatrix}$$

$$\xrightarrow[R_2 \to \frac{1}{2}R_2]{R_1 \to R_1 - R_2} \xrightarrow{R_3 \to \frac{3}{2}R_2 + R_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \xrightarrow{R_3 \to -2R_3}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \to -\frac{3}{2}R_3 + R_2]{R_1 \to 2R_3 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. a. $AX = B \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

b. Suppose A is non singular. Then A^{-1} exists

$\Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow X = A^{-1}B$

c. $A = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 4 \\ 17 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{-7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix}$

$$\Rightarrow X = A^{-1}B = \begin{pmatrix} \frac{-4}{7} & \frac{3}{7} \\ \frac{5}{7} & \frac{-2}{7} \end{pmatrix}\begin{pmatrix} 4 \\ 17 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

5. a. $\begin{pmatrix} 2 & -2 & 12 \\ -2 & 3 & 10 \end{pmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 2 & -2 & 12 \\ 0 & 1 & 22 \end{pmatrix}$

Thus, $y = 22$,

$2x - 2(22) = 12 \Rightarrow 2x = 12 + 44 = 56$

$\Rightarrow x = 28$

Therefore, $x = 28$, $y=12$

b. $\begin{pmatrix} 2 & -5 & 8 \\ 6 & 15 & 18 \end{pmatrix} \xrightarrow{R_2 \to -3R_1 + R_2} \begin{pmatrix} 2 & -5 & 8 \\ 0 & 30 & -6 \end{pmatrix}$

Thus, $30y = -6 \Rightarrow y = \frac{-1}{5} \Rightarrow 2x - 5\left(\frac{-1}{5}\right) = 8 = x = \frac{7}{2}$

c. $\begin{cases} \frac{x}{3} + \frac{3y}{5} = 4 \\ \frac{x}{6} - \frac{y}{2} = -3 \end{cases}$

$$\begin{pmatrix} \frac{1}{3} & \frac{3}{5} & 4 \\ \frac{1}{6} & \frac{-1}{2} & -3 \end{pmatrix} \xrightarrow[R_2 \to 6R_2]{R_1 \to 3R_1} \begin{pmatrix} 1 & \frac{9}{5} & 12 \\ 1 & -3 & -18 \end{pmatrix} \xrightarrow{R_2 \to -R_1+R_2} \begin{pmatrix} 1 & \frac{9}{5} & 12 \\ 0 & \frac{-24}{5} & -30 \end{pmatrix}$$

$$\frac{-24}{5}y = 30 = y = \frac{-30\times 5}{-24} = \frac{25}{4} \Rightarrow y = \frac{25}{4} \Rightarrow x + \frac{9}{5}\left(\frac{25}{4}\right) = 12$$

$$\Rightarrow x = 12 - \frac{45}{4} = \frac{48-45}{4} = \frac{3}{4} \Rightarrow \frac{3}{4} \Rightarrow x = \frac{3}{4}$$

d. $\begin{pmatrix} 1 & -3 & 1 & -1 \\ 2 & 1 & -4 & -1 \\ 6 & -7 & 8 & 7 \end{pmatrix} \xrightarrow[R_3 \to -6R_1 + R_3]{} \begin{pmatrix} 1 & -3 & 1 & -1 \\ 0 & 7 & -6 & 1 \\ 0 & 11 & 2 & 13 \end{pmatrix}$

$$\xrightarrow{R_2 \to \frac{1}{7}} \begin{pmatrix} 1 & -3 & 1 & -1 \\ 0 & 1 & \frac{-6}{7} & \frac{1}{7} \\ 0 & 11 & 2 & 13 \end{pmatrix} \xrightarrow[R_3 \to -11R_2 + R_3]{R_1 \to 3R_2 + R_1} \begin{pmatrix} 1 & 0 & \frac{-11}{7} & \frac{-4}{7} \\ 0 & 1 & \frac{-6}{7} & \frac{1}{7} \\ 0 & 0 & \frac{80}{7} & \frac{80}{7} \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{7}{80}R_3} \begin{pmatrix} 1 & 0 & \frac{-11}{7} & \frac{-4}{7} \\ 0 & 1 & \frac{-6}{7} & \frac{1}{7} \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_1 \to \frac{11}{7}R_3 + R_1]{} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Hence, $x = 1$, $y = 1$ and $z = 1$

e. Using $\begin{pmatrix} 4 & 2 & 3 & 6 \\ 2 & 7 & -3 & 0 \\ -3 & -9 & 2 & -13 \end{pmatrix}$ we get

$$x = \frac{-91}{11}, y = \frac{68}{11}, z = \frac{98}{11}$$

6. Construct the augmented matrix

$\begin{pmatrix} 2 & -4 & 6 \\ 0 & 0 & 9+c \end{pmatrix}$ (Use $R_2 \to \frac{3}{2}R_1 + R_2$)

Now, for the system to have infinite solution

$9 + c = 0 \Rightarrow c = -9$

There will be only one equation $2x - 4y = 6$

Solving for $x = 3+2y$, the solution set will be

$\{(3 + 2y, y): y \in \mathbb{R}\}$

This is an infinite set.

7. The augmented matrix is $\begin{pmatrix} 1 & 2 & -3 & 5 \\ 2 & -1 & -1 & 8 \\ k & 1 & 2 & 14 \end{pmatrix}$

$$\xrightarrow[R_3 \to kR_1 + R_3]{R_2 \to -2R_1 + R_2} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & -5 & 5 & -2 \\ 0 & 1-2k & 2+3k & 14-5k \end{pmatrix}$$

$$R_2 \to \frac{-1}{5}R_2 \;\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & \frac{2}{5} \\ 0 & 1-2k & 2+3k & 14-5k \end{pmatrix}$$

$$R_3 \to (2k-1)R_2+R_3 \;\longrightarrow \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & \frac{2}{5} \\ 0 & 0 & 3+k & \frac{68-21k}{5} \end{pmatrix}$$

Hence, the system has a unique solution if $k + 3 \neq 0$.

An alternative way is to use the fact that the system has a unique solution, if $|A| \neq 0$

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ k & 2 \end{vmatrix} + -3\begin{vmatrix} 2 & -1 \\ k & 1 \end{vmatrix}$$

$$= (-2+1) - 2(4+k) - 3(2+k)$$

$$= -1 - 8 - 2k - 6 - 3k$$

$$= -5k - 15$$

Now $|A| \neq 0 = -5k - 15 \neq 0$

$$\Rightarrow k \neq -3$$

8. Inserting the points (0, 4) and (2, 16) into the equation $cx + dy = 2$,

$$\begin{cases} 4d = 2 \\ 2c + 16d = 2 \end{cases}$$

Gives $d = \frac{1}{2}$ $= 2c = 2 - 16d = 2 - 8 = -6$

$\Rightarrow c = -3$

Hence, the straight line is

$$-3x + \frac{y}{2} = 2$$

9. At the point (1, 9) we get $a + b + c = 9$

At the point (4, 6), we get $16a + 4b + c = 6$

At the point (6, 14), we get $36a + 6b + c = 14$

Thus, we get a liner system $\begin{cases} a + b + c = 9 \\ 16a + 4b + c = 6 \\ 36a + 6b + c = 14 \end{cases}$

$a = 1$, $b = -6$ and $c = 14$

Hence, the function is $y = x^2 - 6x + 14$

6.5 CRAMER'S RULE

Periods allotted: 3 Periods

Competency

At the end of this sub-unit, students will be able to;

- *apply Cramer's rule to solve systems of linear equations.*

Vocabulary: Cramer's rule

Introduction

In this sub-unit, another alternative way of solving system of linear equations, called Cramer's rule will be discussed.

Teaching Notes

When the coefficient matrix of a system of linear equations A is such that $|A| \neq 0$, and A is a 2×2 or a 3 × 3 matrix, Cramer's Rule can effectively be used. To solve $AX = B$ where matrix A *is* of order more than 3 use of Cramer's rule is impractical due to the large number of calculations that are required. Gauss' method is preferable in these cases. Besides, Cramer's Rule does not work when $|A| = 0$.

For 2 by 2 case the rule can also be derived as follows:

Suppose you want to solve $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$, and $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

Then the system is $AX = B$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} e \\ f \end{pmatrix}$.

In this system A^{-1} exists since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

By multiplying both sides by A^{-1}, we get $X = A^{-1}B$, but,

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A) \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}B = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} de-bf \\ af-ce \end{pmatrix} = \begin{pmatrix} \dfrac{de-bf}{ad-bc} \\ \dfrac{af-ce}{ad-bc} \end{pmatrix}$$

$$\Rightarrow x = \frac{de-bf}{ad-bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & c \\ c & d \end{vmatrix}} \text{ and } y = \frac{af-ce}{ad-bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & c \\ c & d \end{vmatrix}}$$

which gives a solution to the system of linear equations. This rule is called Cramer's rule.

This rule is valid for any system of linear equations in so far as the coefficient matrix is square. But, the repeated calculation of determinants may be tiresome.

Let the students do more examples as homework or assignment.

Before winding up of the unit, it is advisable to let the students know that there is an alternative way to find inverses of non-singular matrices. We will show it by an example. But, before that, let us state two facts:

1. If matrix A is non-singular then, when we change it to reduced echelon form, we get the identity matrix.
2. If A is non-singular and A is augmented with an identity matrix of the same order i.e., (A : I), then when the augmented matrix is changed into reduced row echelon form we get (I : A^{-1}) i.e.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is non-singular, and when the matrix

$\begin{pmatrix} a_{11} & a_{12} & a_{13}\,. & 1 & 0 & 0 \\ a_{21} & a_{22} & {}_{23}\,. & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33}\,. & 0 & 0 & 1 \end{pmatrix}$ is transformed into reduced echelon form to find

$\begin{pmatrix} 1 & 0 & 0 & . & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & . & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & . & b_{31} & b_{32} & b_{33} \end{pmatrix}$ then, $A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$.

You may give exercises on these methods, like the following ones:

1. Check if the matrix $\begin{pmatrix} 3 & 2 & 1 \\ 6 & -4 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is non-singular.
2. Find the inverse of the following matrix by using elementary row operations.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -3 \end{pmatrix}$$

so that the students can use $X = A^{-1}B$ as solution to a system of linear equations. This way of finding inverse follows the application of raw operations.

Assessment

Guide students to discuss the rule in groups and give them problems to solve using Cramer's rules. The review exercises below provide some problems to do. You can assess their understanding of the rule by giving them home works and tests as well.

Answers to Exercise 6.6

1. a. $x=\dfrac{\begin{vmatrix}4 & 5\\ 6 & 2\end{vmatrix}}{\begin{vmatrix}-3 & 5\\ 7 & 2\end{vmatrix}}=\dfrac{8-30}{-6-35}=\dfrac{22}{41}$, $y=\dfrac{\begin{vmatrix}-3 & 4\\ 7 & 6\end{vmatrix}}{\begin{vmatrix}-3 & 5\\ 7 & 2\end{vmatrix}}=\dfrac{-18-28}{-6-35}=\dfrac{46}{41}$

   b. $x=\dfrac{\begin{vmatrix}0 & 1\\ 7 & -6\end{vmatrix}}{\begin{vmatrix}4 & 1\\ 1 & -6\end{vmatrix}}=\dfrac{7}{25}$, $y=\dfrac{\begin{vmatrix}4 & 0\\ 1 & 7\end{vmatrix}}{\begin{vmatrix}4 & 1\\ 1 & -6\end{vmatrix}}=-\dfrac{28}{25}$

   c. $x=\dfrac{\begin{vmatrix}5 & 2 & -1\\ -15 & -1 & 3\\ -28 & 1 & 7\end{vmatrix}}{\begin{vmatrix}3 & 2 & -1\\ 1 & -1 & 3\\ 2 & 1 & 7\end{vmatrix}}=-1$, $y=\dfrac{\begin{vmatrix}3 & 5 & -1\\ 1 & -15 & 3\\ 2 & -28 & 7\end{vmatrix}}{\begin{vmatrix}3 & 2 & -1\\ 1 & -1 & 3\\ 2 & 1 & 7\end{vmatrix}}=2$, $z=\dfrac{\begin{vmatrix}3 & 2 & 5\\ 1 & -1 & -15\\ 2 & 1 & -28\end{vmatrix}}{\begin{vmatrix}3 & 2 & -1\\ 1 & -1 & 3\\ 2 & 1 & 7\end{vmatrix}}=-4$

   d. $x=\dfrac{\begin{vmatrix}5 & 3 & 0\\ 6 & 0 & 3\\ 11 & 5 & -1\end{vmatrix}}{\begin{vmatrix}2 & 3 & 0\\ 1 & 0 & 3\\ 0 & 5 & -1\end{vmatrix}}=\dfrac{-14}{9}$, $y=\dfrac{\begin{vmatrix}2 & 5 & 0\\ 1 & 6 & 3\\ 0 & 11 & -1\end{vmatrix}}{\begin{vmatrix}2 & 3 & 0\\ 1 & 0 & 3\\ 0 & 5 & -1\end{vmatrix}}=\dfrac{73}{27}$, $z=\dfrac{\begin{vmatrix}2 & 3 & 5\\ 1 & 0 & 6\\ 0 & 5 & 11\end{vmatrix}}{\begin{vmatrix}2 & 3 & 0\\ 1 & 0 & 3\\ 0 & 5 & -1\end{vmatrix}}=\dfrac{68}{27}$

2. a. $\begin{vmatrix}-3 & 5\\ 7 & 2\end{vmatrix} = -6-35 = -41 \neq 0.$

   Hence, the system has a solution

   $x=\dfrac{\begin{vmatrix}0 & 5\\ 0 & 2\end{vmatrix}}{\begin{vmatrix}-3 & 5\\ 7 & 2\end{vmatrix}}=\dfrac{0}{-41}=0$ and $y=\dfrac{\begin{vmatrix}-3 & 0\\ 7 & 0\end{vmatrix}}{\begin{vmatrix}-3 & 5\\ 7 & 2\end{vmatrix}}=\dfrac{0}{-41}=0$

b. $\begin{vmatrix} 3 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 26 \neq 0$

$$x = \frac{\begin{vmatrix} 0 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{vmatrix}}{26} = 0, \quad y = \frac{\begin{vmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 5 & 0 & -1 \end{vmatrix}}{26} = 0, \quad z = \frac{\begin{vmatrix} 3 & 2 & 0 \\ 2 & 1 & 0 \\ 5 & -2 & 0 \end{vmatrix}}{26} = 0$$

Answers to Review Exercises on Unit 6

1. $a = 2, \quad d = -1, \quad e = 3$

2. $5A - 2B = \begin{pmatrix} 10 & 15 & 20 \\ 0 & 20 & 30 \\ 25 & 40 & 45 \end{pmatrix} - \begin{pmatrix} 6 & 0 & 10 \\ 10 & 6 & 4 \\ 0 & 8 & 14 \end{pmatrix} = \begin{pmatrix} 4 & 15 & 10 \\ -10 & 14 & 26 \\ 25 & 32 & 31 \end{pmatrix}$

3. a. $AB = \begin{pmatrix} 15 & 16 \\ -2 & 7 \\ 16 & 32 \end{pmatrix}$ b. Does not exist because $B_{3\times2}$ and $A_{3\times3}$.

c. $BC = \begin{pmatrix} 2 & 15 \\ 14 & 10 \\ -4 & 0 \end{pmatrix}$ d. CB does not exist.

e. $\begin{pmatrix} 68 \\ -14 \end{pmatrix}$ f. $X^T CC^T = (223 \quad -24)$

g. Since B^TA is 2×3 and $-2B$ is 3×2, they do not have the same order.
Hence, $B^TA - 2B$ does not exist.

h. $X^TX = (113)$ i. $B^TB + 4C = \begin{pmatrix} 29 & 29 \\ 1 & 38 \end{pmatrix}$

4. a. $25\% = \frac{1}{4}$. To find the answer, multiply the matrix

$\begin{pmatrix} 300 & 400 & 500 & 600 \\ 500 & 400 & 700 & 750 \\ 400 & 400 & 600 & 500 \end{pmatrix}$ by 1.25. Thus, we get

	A	B	C	D
Beef meat	375	500	625	750
Tomato	625	500	875	937.5
Soya Beans	500	500	750	625

b. $$0.15\begin{pmatrix}300 & 400 & 500 & 600\\ 500 & 400 & 700 & 750\\ 400 & 400 & 600 & 500\end{pmatrix}=\begin{pmatrix}45 & 60 & 75 & 90\\ 75 & 60 & 105 & 112.5\\ 60 & 60 & 90 & 75\end{pmatrix}$$

Thus, the new order is 85% of the original. Thus, she ordered

	A	B	C	D
Beef meat	255	340	425	510
Tomato	425	340	595	637.5
Soya beans	340	340	510	425

5. a. $$\begin{array}{l}\\ \\ \\ \end{array}\begin{array}{c} \quad\ \text{Kelecha} \quad \text{Alemu} \\ \begin{pmatrix}\text{Hammer} & 1 & 1\\ \text{Saw} & 1 & 2\\ \text{Nails} & 2 & 3\end{pmatrix}\end{array} \text{ Hence, } I=\begin{pmatrix}1 & 1\\ 1 & 2\\ 2 & 3\end{pmatrix}$$

b. $$P=\begin{pmatrix}30 & 35 & 7\\ 28 & 37 & 6\end{pmatrix}$$

c. $$PI=\begin{pmatrix}30 & 35 & 7\\ 28 & 37 & 6\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 2\\ 2 & 3\end{pmatrix}=\begin{pmatrix}79 & 121\\ 77 & 120\end{pmatrix}$$

This means

	Kelecha	**Alemu**
Shop 1	79	121
Shop 2	77	120

d. Kelecha's cost at shop 1 is 79 Birr, while Alemu's cost at shop 2 is 120 Birr

e. They should buy from shop 2.

6. $$A+A^T=\begin{pmatrix}0 & -3 & -4\\ m & 0 & 8\\ 4 & -8 & 0\end{pmatrix}+\begin{pmatrix}0 & m & 4\\ -3 & 0 & -8\\ -4 & -8 & 0\end{pmatrix}=\begin{pmatrix}0 & -3+m & 0\\ m-3 & 0 & 0\\ 8 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Now, $A+A^T=0=m-3=0\Rightarrow m=3$

7. a. $\left(\dfrac{A+A^T}{2}\right)^T = \dfrac{1}{2}(A^T + (A^T)^T) = \dfrac{1}{2}(A^T + A) = \dfrac{A+A^T}{2}$ is symmetric.

$$\frac{A-A^T}{2} + \left(\frac{A-A^T}{2}\right)^T = \frac{A-A^T}{2} + \frac{A^T-(A^T)^T}{2}$$

$$= \frac{A-A^T}{2} + \frac{A^T-A}{2} = 0$$

$\Rightarrow \dfrac{A+A^T}{2}$ is symmetric, while $\dfrac{A-A^T}{2}$ is skew– symmetric.

8. a. $\begin{vmatrix} 4 & 3.5 \\ -7 & -20 \end{vmatrix} = -55.5$ b. -94

9. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$

Thus $|rA| = \begin{vmatrix} ra & rb \\ rc & rd \end{vmatrix} = r\begin{vmatrix} a & rb \\ c & rd \end{vmatrix} = r^2\begin{vmatrix} a & b \\ c & d \end{vmatrix} = r^2|A|$

10. $(a+b)\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} - a\begin{vmatrix} c & c \\ b & c+a \end{vmatrix} + b\begin{vmatrix} c & c \\ b+c & a \end{vmatrix}$

$= (a+b)\ \left[bc+ba+c^2+ca-ab\right] -ac\ (a+c-b)+bc(a-b-c)$

$= (a+b)(bc+ac+c^2) - a^2c - ac^2 + abc + abc - b^2c - bc^2$

$= abc + a^2c + ac^2 + b^2c + abc + bc^2 - a^2c - ac^2 + abc + abc - b^2c - bc^2$

$= 4abc$

11. a. $\begin{vmatrix} 3x & -1 \\ x & -3 \end{vmatrix} = \dfrac{3}{2} \Rightarrow -9x + x = \dfrac{3}{2}$

$\Rightarrow -8x = \dfrac{3}{2}$

$\Rightarrow x = \dfrac{-3}{16}$

b. $\begin{vmatrix} -3 & -x \\ 3x & 4 \end{vmatrix} = 15 \Rightarrow -12 + 3x^2 = 15$

$\Rightarrow 3x^2 = 27 \Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

12. $\begin{pmatrix} -\frac{1}{8} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{2} \\ \frac{1}{8} & -\frac{1}{2} & \frac{5}{4} \end{pmatrix}$

13. $\begin{pmatrix} 0 & -1 & 5 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{pmatrix}$

$\xrightarrow{R_3 \to -2R_1 + R_3} \begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 5 \\ 0 & -5 & 8 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -5 \\ 0 & -5 & 8 \end{pmatrix}$

$\xrightarrow[R_1 \to -3R_2 + R_1]{R_3 \to 5R_2 + R_3} \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & -17 \end{pmatrix} \xrightarrow{R_3 \to -\frac{1}{17} R_3} \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow[R_2 \to 5R_3 + R_2]{R_1 \to -13R_3 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

14. $\begin{pmatrix} 3 & -a & 1 \\ b & 4 & 6 \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{3} R_1} \begin{pmatrix} 1 & -\frac{a}{3} & \frac{1}{3} \\ b & 4 & 6 \end{pmatrix}$

$\xrightarrow{R_2 \to -bR_1 + R_2} \begin{pmatrix} 1 & -\frac{a}{3} & \frac{1}{3} \\ 0 & 4+\frac{ab}{3} & 6-\frac{b}{3} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{a}{3} & \frac{1}{3} \\ 0 & \frac{12+ab}{3} & \frac{18-b}{3} \end{pmatrix}$

a. The system has only one solution if $\frac{12+ab}{3} \neq 0 \Rightarrow ab \neq -12$

b. The system has no solution if $\frac{12+ab}{3} = 0$ and $\frac{18-b}{3} \neq 0$

$\Rightarrow ab \neq -12$ and $b = 18 \Rightarrow a \neq -\frac{2}{3}$

c. The system has infinite solution if $ab = -12$ and $b = 18$

$\Rightarrow a = -\frac{2}{3}$

15. $\begin{pmatrix} 3 & -2 & 1 & b \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{pmatrix} \xrightarrow[R_3 \to \frac{-2}{3}R_1 + R_3]{R_2 \to \frac{-5}{3}R_1 + R_2} \begin{pmatrix} 3 & -2 & 1 & b \\ 0 & -\frac{14}{3} & \frac{22}{3} & \frac{9-5b}{3} \\ 0 & \frac{7}{3} & a\frac{-2}{3} & \frac{-2b-3}{3} \end{pmatrix}$

$\xrightarrow{R_2 \to \frac{-3}{14}R_2} \begin{pmatrix} 3 & -2 & 1 & b \\ 0 & 1 & \frac{-11}{7} & \frac{5b-9}{14} \\ 0 & \frac{7}{3} & \frac{3a-2}{3} & \frac{-2b-3}{3} \end{pmatrix}$

$\xrightarrow{R_3 \to \frac{-7}{3}R_2 + R_3} \begin{pmatrix} 3 & -2 & 1 & b \\ 0 & 1 & \frac{-11}{7} & \frac{5b-9}{14} \\ 0 & 0 & a+3 & \frac{1-3b}{2} \end{pmatrix}$

a. $a \neq -3$ b. $a = -3, \quad b = \frac{1}{3}$ c. $a = -3, b \neq \frac{1}{3}$

16. $\begin{pmatrix} 1 & 2 & -1 & 12 \\ 2 & -1 & -2 & 2 \\ 1 & -3 & k & 11 \end{pmatrix} \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -2R_1 + R_2} \begin{pmatrix} 1 & 2 & -1 & 12 \\ 0 & -5 & 0 & -22 \\ 0 & -5 & k+1 & -1 \end{pmatrix}$

$\xrightarrow{R_3 \to -R_2 + R_3} \begin{pmatrix} 1 & 2 & -1 & 12 \\ 0 & -5 & 0 & -22 \\ 0 & 0 & k+1 & 21 \end{pmatrix}$

When $k = -1$, $1 + k = 0$ and the system has no solution.

17. a. $x = 1, \quad y = 1, \quad z = 1$

b. $\begin{pmatrix} 2+\beta & -\beta & 5 \\ -\beta & 1+\beta & 0 \end{pmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{pmatrix} 2+\beta & -\beta & 5 \\ 2 & 1 & 5 \end{pmatrix}$

$\xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 2+\beta & -\beta & 5 \\ 1 & \frac{1}{2} & \frac{5}{2} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 2+\beta & -\beta & 5 \end{pmatrix}$

$$\xrightarrow{R_2 \to -(2+\beta)R_1 + R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & \frac{-2-3\beta}{2} & -\frac{5}{2}\beta \end{pmatrix}$$

$$\Rightarrow y = \frac{5\beta}{2+3\beta} \quad \text{and} \quad x + \frac{5\beta}{4+6\beta} = \frac{5}{2}$$

$$\Rightarrow x = \frac{5}{2} - \frac{5\beta}{4+6\beta} = \frac{10+15\beta-5\beta}{4+6\beta} = x$$

$$\Rightarrow \frac{10+10\beta}{4+6\beta} = \frac{5+5\beta}{2+3\beta}$$

18. a. $x = \dfrac{\begin{vmatrix} 7 & 1 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}} = 2$ and $y = \dfrac{\begin{vmatrix} 2 & 7 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}} = 3$

b. $x = \dfrac{\begin{vmatrix} 1 & 4 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 4 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \dfrac{-1}{5}$, $y = \dfrac{\begin{vmatrix} -1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 4 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \dfrac{2}{5}$, and $z = \dfrac{\begin{vmatrix} -1 & 4 & -1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 4 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}} = \dfrac{4}{5}$

19. $A^{-1} = \begin{pmatrix} \frac{2}{5} & 1 & -\frac{3}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} \\ -\frac{3}{5} & -1 & \frac{7}{5} \end{pmatrix}$. Hence $x = A^{-1}B = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{pmatrix}$

UNIT 7

THE SET OF COMPLEX NUMBERS

INTRODUCTION

The main task of this unit is to extend the real number system to another new number system called complex number system. In this introduction part, you are required to see to it that the students discuss the development of number systems. For example, the extension of: whole numbers to integers; integers to rational; and rational to real number system. In this way, the students will realize the need for the new number system. The students also learn how to represent the new numbers in the system and calculate in it. Thus, you are expected to create experiences that engage students and support their own explanation, evaluation, communication and application of the mathematical models needed to make sense of these experiences. Thus after the students conduct the discussion thoroughly, familiarize them with the notation of complex numbers and enable them to represent and calculate with complex numbers.

Unit Outcomes

After completing this unit, students will be able to;

- *know basic concepts about complex numbers.*
- *know the general principle of performing operation on complex numbers.*
- *understand facts and procedures in simplifying complex numbers.*
- *show the geometric representation of complex numbers on the Argand plane.*

Suggested teaching aids in Unit 7

You know that students learn in a variety of different ways. Some are visually oriented and more inclined to acquire information from photographs or videos. Other students do very well when they hear instructions rather than read them. Teachers use teaching aids to provide these different ways of learning. It is, therefore, recommended that you may use models of planes and charts for this unit. You can also use instruments for drawing graphs like squared paper, ruler and compass.

7.1 THE CONCEPT OF COMPLEX NUMBERS

Periods Allotted: 2 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define a complex number*
- *identify the real and imaginary parts of a given complex number.*
- *determine the equality of two complex numbers.*
- *describe the set of complex numbers and its relation to the set of real numbers.*

Vocabulary: Imaginary number, Complex number, Real part, Imaginary part

Introduction

When you introduce the notion of complex numbers, you should make clear to the students that an equation has a solution depending on the domain of the variable. They should note that the need for extending a number system is mainly to be able to solve an equation that cannot be solved in the present system.

Teaching Notes

Complex numbers originate from a desire to extract square roots of negative numbers. If we want square roots of negative numbers, it is enough to introduce i = $\sqrt{-1}$. Once you introduce this assumption to the students, for instance using examples like,

$\sqrt{-4} = \sqrt{-1}\sqrt{4}$ =2i, then you can define a complex number $x + yi$, and state the set of complex numbers:

$$\mathbb{C}=\{x + yi : x, y \in \mathbb{R}, i = \sqrt{-1}\}.$$

Since any real number b can be written as $b + 0i$, therefore, $\mathbb{R} \subseteq \mathbb{C}$. Hence $\mathbb{C}$ is an extension of $\mathbb{R}$.

Equality in $\mathbb{C}$ is defined by $x + yi = u + vi$ if and only if $x = u$ and $y = v$.

Students might wonder why they need this extension to another number system, while they have many of them already. Give the students some quadratic equations to solve. Give them the necessary orientation and guidance to enable them observe that the quadratic equations with negative discriminants have no solution in the set of real numbers, while they have solutions in the set of complex numbers. In $\mathbb{C}$**,** one can solve any quadratic equation. It is also true that every number has a square root in $\mathbb{C}$ (while, as you know, some numbers have no square roots in $\mathbb{R}$).

To illustrate what we mean here, suppose

$$\sqrt{1+2i} = x + yi$$

$$\Rightarrow 1+2i = x^2 + 2xyi - y^2 \text{ (squaring both sides)}$$

$$\Rightarrow x^2 - y^2 = 1 \text{ and } 2xy = 2$$

$$\Rightarrow x^2 - \frac{1}{x^2} = 1, \text{ inserting the second equation in the first.}$$

Solving this quadratic equation, we get $x^2 = \dfrac{1 \pm \sqrt{5}}{2}$.

For real x, we have to take, $x^2 = \dfrac{1+\sqrt{5}}{2}$

$$\Rightarrow \mathrm{y}^2 = \frac{1}{\mathrm{x}^2} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$$

This gives you that

$$\sqrt{1+2\mathrm{i}} = \sqrt{\frac{1+\sqrt{5}}{2}} + \mathrm{i}\sqrt{\frac{\sqrt{5}-1}{2}}$$

Every complex number has a square root, and existence of square roots means that quadratic equations can always be solved.

Once you discuss the definition, give students Exercise 7.1 as homework, and check their work. Encourage them to go to the chalk board and do some of the problems. You can assign question 2 to high achievers, while all can do the rest of the questions.

With this background, you may start this section by opening problem as given in the student textbook. After having discussed the openingproblem, you introduce the new number called “imaginary number” denoted by i and read as “iota” which stands for $\sqrt{-1}$. Having this new notation now, you can define what a complex number is, its form, set of complex number; and characterize equality of complex numbers as given in the student textbook.

Assessment

Use the opening problem to assess the background of students. In this opening problem, you can ask them whether a quadratic equation has always a real root. You can give homework and class-works and assess students by checking their exercise books on this issue.You can also you use Exercise 7.1.

Answers to Exercise 7.1

1. a. $-i$ b. 1 c. $-i$

   d. 1 e. 1 f. i

   g. 1 h. $-i$

2. $i^{2n} = \begin{cases} 1 \text{ if } n \text{ is even} \\ -1 \text{ if } n \text{ is odd} \end{cases}$

   $i^{2n+1} = \begin{cases} i \text{ if } n \text{ is even} \\ -i \text{ if } n \text{ is odd} \end{cases}$

3.

	Complex number	Real part	Imaginary part
a	$\frac{3-5i}{7}$	$\frac{3}{7}$	$\frac{-5}{7}$
b	$\sqrt{5}+2i\sqrt{2}$	$\sqrt{5}$	$2\sqrt{2}$
c	7+0 i	7	0
d	$0+5i$	0	5

4. a. $x = 2$ and $y = \frac{-1}{4}$ b. $t = 7$ and $y = -5$

5. a. $3 = 3 + 0i$ b. $-7 = -7 + 0i$

   c. $0 = 0 + 0i$ d. $\sqrt{13} = \sqrt{13} + 0i$

6. From Exercise 7.1 (5), we observed that every real number r can be written as $r = r + 0i$ and hence every real number can be expressed in the form of $a + bi$.

7. Yes. From Exercise 7.1 (6), we can conclude that the set of real number is a subset of the set of complex numbers.

7.2 OPERATIONS ON COMPLEX NUMBERS

Periods Allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *add complex numbers.*
- *subtract complex numbers.*
- *describe the closure property of complex numbers.*
- *describe the commutative and associative properties of complex numbers.*
- *identify the additive identity of a complex number.*
- *identify the additive inverse of a complex number.*
- *determine the product of complex numbers.*
- *describe the basic properties of multiplication of complex numbers.*
- *identify the multiplicative identity of a complex number.*

Vocabulary: Commutative, Associative, Distributive, Identity

Introduction

The purpose of this topic is to systematically extend the four operations on the set of real numbers to the set of complex numbers. Before defining addition and subtraction on the set of complex numbers, try to check the experiences of the students on adding and subtracting terms involving variables by asking them questions.

Teaching Notes

You may use Activity 7.1 to introduce the notion of addition and subtraction of complex numbers. First, give the students chances to do activity 7.1 and see the experiences of students on how to add terms in the expressions. After that, you may summarize experiences in adding terms like $(3-5x)+(6+7x)$. You may explain how they do it by combining similar terms in the expressions. For example, if we were to simplify the expression $(3-5x)+(6+7x)$ by combining similar terms, then the constant 3 and 6 would be combined to yield 9, and the terms $(-5x)$ and $(7x)$ would be combined to yield $2x$; and the simplified form is

$$(3-5x)+(6+7x)=(3+6)+(-5x+7x)=9+2x$$

Answers to Activity 7.1

This Activity is intended to make students aware of the properties of the set of complex numbers with respect to the operation addition.

a. $7x-4y$ b. $-3x+6y$ c. $8-2k$ d. $-8+2h$

In a similar fashion as in the case of the above activities, we combine like terms (the real part to the real part and the imaginary part to the imaginary part) in complex numbers to add or subtract. For instance, given two complex numbers $z_1 = 3 + 4i$ and $z_2 = 5 + 2i$ to find $z_1 + z_2$ we add 3 and 5 together (the real parts) and add 4 and 2 (the imaginary parts) to get $8 + 6i$; and to find $z_1 - z_2$: we subtract 5 from 3 (the real parts) and 2 from 4(the imaginary parts) to get $-2 + 2i$. After having done this,let the students discuss how these operations are done and what their properties look like.You can group the students and let them do Group Work 7.1. As a prelude to this discussion, we can consider the following:

The set of complex numbers given by

$\mathbb{C}=\{x + y\mathrm{i} : x, y \in \mathbb{R}, \mathrm{i} = \sqrt{-1}\}$ is closed under addition.

This means, if we add two complex numbers, the result is a complex number.

Besides, + is commutative and associative and $0 = 0 + 0i$ is the identity for +. The inverse of $z = a + b\mathrm{i}$ with respect to addition is $-z = -a + -b\mathrm{i}$.

To verify one of them, $\forall a + b\mathrm{i}, c + d\,\mathrm{i}, e + f\mathrm{i}$ in $\mathbb{C}$,

$$\begin{aligned} a + b\mathrm{i} + ((c + d\mathrm{i}) + (e + f\,\mathrm{i})) &= a + b\mathrm{i} + ((c + e) + (d + f)\mathrm{i}) \\ &= (a + (c + e)) + (b + (d + f))\mathrm{i} \\ &= ((a + c) + e) + ((b + d) + f)\mathrm{i} \\ &= (a + c) + (b + d)\mathrm{i} + (e + f\,\mathrm{i}) \\ &= ((a + b\mathrm{i}) + (c + d\mathrm{i})) + (e + f\,\mathrm{i}). \end{aligned}$$

Thus, addition is associative on the set of complex numbers.

Answer to Group Work 7.1

You are required to guide students discuss the group work. After the group discussion, you can summarize important points of the discussion as follows:

a. The set of complex number is closed under addition.

b. Addition is commutative on the set of complex numbers.

c. Addition is associative on the set of complex numbers.

d. The number 0 is the additive identity in the set of complex numbers.

e. For any complex number z, its additive inverse is $-z$.

Assessment

Use Activity 7.1and Group Work 7.1 to assess the background of students. You can give homework, class-works and assess students by checking their exercise books.

Answers to Exercise 7.2

1. a. $0+11i$ b. $6+2i$ c. $2+8i$ d. $4-23i$

   e. $1-i$ f. $-1+i$ g. $0+i$ h. $-3+2i$

2. a. $x=7$ and $y=3$ b. $x=8$ and $y=10$

   c. $x=\frac{-8}{7}$ and $y=-4$ d. $x=\frac{-19}{2}$ and $y=\frac{-5}{4}$

After you make sure that students have understood addition and subtraction of complex numbers, you can proceed to discuss multiplication and division. For this purpose,first you let each student do Activity 7.2 for discussing multiplication since it is assumed that the students have the background from previous grades and sessions.

The activities are very important to the necessityof the definition of multiplication on the set of complex numbers. Thus, you may help the students to come to the definition of the product of two complex numbers. Make the students aware that the students' that they need not memorize the definition,but that they just get the result simply by multiplying like terms containing variables. But, in this case, the variable is i which is simplified by using the notation $i^2=-1$.

Answers to Activity 7.2

1. a. $a^2+2ab+b^2$ b. a^2-b^2

   c. $2x^2+xy-15y^2$ d. x^3+3x^2+x+3

2. a. $3-i$ b. $-19+61i$ c. 25 d. $-7+24i$

After having done this, let the students discuss how these operations are done and what their properties look like.You can group the students and let them do Group Work 7.2.

This group work is intended to make students aware of the properties of the set of complex numbers with respect to the operation multiplication. For illustration,

The set of complex numbers given by

$\mathbb{C}=\{x+y\mathrm{i} : x, y\in\mathbb{R}, i=\sqrt{-1}\}$ is closed under multiplication.

This means, if we multiply two complex numbers, the result is a complex number. The operation of multiplication, "×" is also commutative and associative, and "×" is distributive over "+". $1 = 1 + 0i$ is the multiplicative identity in $\mathbb{C}$.

Let us check that × is distributive over + (Since multiplication is commutative we only check left distributivity):

Let $z = a + bi$, $u = c + di$ and $v = e + f\,i$ be complex numbers. Now

$$\begin{aligned} z\,(u + v) &= (a + bi)(c + di + e + f\,i) \\ &= (a + bi)((c + e) + (d + f\,)i) \\ &= a(c + e) + a(d + f)i + (c + e)bi - b(d + f) \\ &= ac + ae + adi + afi + cbi + ebi - bd - bf \\ &= ((ac - bd) + (ad + bc)i\,) + ((ae - bf) + (af + be)i) \\ &= zu + zv \end{aligned}$$

By using this group work, help the students to come to the conclusion given as answers to the Group Work.

Answers to Group Work 7.2

a. Yes. The set of complex number is closed under multiplication. This is called product property.

b. Yes. Multiplication is commutative on the set of complex numbers.

c. Yes. Multiplication is associative on the set of complex numbers.

d. Yes. Multiplication is left distributive over addition.

e. Yes. Multiplication is right distributive over addition

f. $z_1 \cdot 1 = 1 \cdot z_1 = z_1$. 1 is the multiplicative identity in the set of complex numbers.

When the students discuss multiplication and generate some of its properties, you can continue to discuss division. To get into discussing division of complex numbers, you can form groups of students and let them do Group Work 7.3. This group work makes the students aware of finding multiplicative inverse of a non-zero complex number and, on the way, enable them to come to the definition of division of a complex number. At this stage, conjugate of a complex number is not introduced yet.Following the task in Group Work 7.3, introduce the concept of conjugates and tell students that the concept of conjugate further simplifies division of complex numbers.

Answers to Group Work 7.3

1. $$\left(\frac{1}{2+3i}\right)\left(\frac{1}{2-3i}\right) = \frac{1}{(2+3i)(2-3i)} = \frac{1}{4+6i-6i+9} = \frac{1}{13}$$

$$\frac{1}{2+3i} = \left(\frac{1}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right) \quad \text{Multiplication by 1}$$

$\frac{1}{2+3i}$ is multiplicative inverse of 2 +3*i*, because $\left(\frac{1}{2+3i}\right)(2+3i)=1$

$$\left(\frac{2}{13}-\frac{3i}{13}\right)(2+3i)=\left(\frac{2-3i}{13}\right)(2+3i)=\frac{13}{13}=1$$

2. You can do the same as in (1).

At the end of Group Work 7.3, you may help students to conclude that: for any non-zero complex number $z = x + yi$,its multiplicative inverse is given by $\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i$.

At this stage, the four operations are defined and their properties are given. But, you may need to give students additional exercises to work in small groups. Let them discuss the solutions of the problems until all agree on the solutions. Then, let them submit their common answers to you. You may correct the answers and then return the papers with comments. You can give the comments to each group or bring the difficult questions to the whole class for discussion. An alternative way is to ask a representative from each group to go to the board and do some of the problems.

Assessment

To assess the understanding of students, you can give homework or assignments or quiz that can be done individually or in groups, that ask students to give examples of complex numbers. Let them identify the real and imaginary parts of a complex number.

Ask them to solve equations of the form $x^2+ 6x + 25 = 0$.

Ask them questions based on equality of complex numbers like:

If $x + 2yi = -x^2 -2x – + 6$i, then give the real and imaginary parts of $x + yi$.

You can also ask them to simplify powers of*i*, like, $2i^{30} – 5i^{25} + i^2$.

Give them problems on addition, subtraction, multiplication and division of complex numbers.Check if they can expand $(z + w)^2$, $(z – w)^2$, $(z + w)^3$ and $(z – w)^3$.

Answers to Exercise 7.3

1. $2+14i$	2. $12+6i$	3. $34-13i$
4. $5-i$	5. $0-2i$	6. $\frac{12}{13}-\frac{5}{13}i$
7. $6+8i$	8. $1+i$	9. $0-i$
10. $\frac{2}{13}-\frac{3}{13}i$	11. $\frac{13}{29}+\frac{11}{29}i$	12. $3+i$

7.3 COMPLEX CONJUGATE AND MODULUS

Periods Allotted: 2 periods

Competencies

At the end of this sub-unit, students will be able to:

- *determine the conjugate of a given complex number.*
- *find the modulus of any given complex number*
- *use conjugates in the division process.*
- *describe basic properties of conjugates and modulus.*

Vocabulary: Conjugate, Modulus

Introduction

In this subunit, the students are expected to learn about conjugates and modulus of complex numbers. Since they are already aware of the operations and their properties, you can make them reach at conjugates and modulus by doing activities presented in the student textbook.

Teaching Notes

In order to present this topic, it would be advisable to let the students do Activity 7.3 which may help you to assess their pre-instruction background. Let them observe that the product of such complex numbers is always real. Based on this, give the meaning of complex conjugates. After that, go through the examples which are given in the textbook and in Example 2, help the students fill in the two columns in the table which is left as an exercise.

Help the students to do Activity 7.4 by themselves and guide them to discover those properties which are given in Theorem 7.1 of their textbook. The purpose of this activity is to verify various properties of the complex conjugate. Let the students go through this activity, before they try the proofs of Theorem 7.1 and Theorem 7.2 (Understanding that a proof is very important for their latter learning). Therefore, let the students present the proofs of the two theorems on the board. Do not let them directly copy the proofs from the textbook, but let them express the proofs in their own words.

Answers to Activity 7.4

a.	$3-4i$	b.	$5+2i$	c.	$8-2i$	d.	$8+2i$
e.	$8-2i$	f.	$23-14i$	g.	$23-14i$	h.	$\frac{7}{29}-\frac{26}{29}i$
i.	$\frac{7}{29}-\frac{26}{29}i$	j.	3+4i	k.	5–2i		

Activity 7.4 verifies the theorem; but now let them take any generic complex number $z_1 = a + bi$ and $z_2 = c + di$ and follow the method they used in Activity 7.4 to prove the theorem. We will show two of them here:

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

Then $\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1+x_2) - i(y_1+y_2)$

$$= (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 + \bar{z}_2$$

Similarly,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)} = \overline{\left(\frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}\right)}$$

$$= \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} - i\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

On the other hand,

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{\overline{(x_1 + iy_1)}}{\overline{x_2 + iy_2}} = \frac{(x_1 - iy_1)}{x_2 - iy_2} = \frac{(x_1 - iy_1)(x_2 + iy_2)}{(x_2 - iy_2)(x_2 + iy_2)}$$

$$= \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} - i\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}.$$

Thus, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, provided, $z_2 \neq 0$.

You can ask the students to multiply $\frac{z_1}{z_2}$ by its conjugate $\frac{\overline{z_1}}{\overline{z_2}}$ and see the importance of conjugates.

For any $z = a + bi \neq 0$, $\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = z^{-1}$.

Thus, for any z_1 and for any $z_2 \neq 0$, $\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z_2}}{\overline{z_2}}$

One of the important features of complex conjugate is to facilitate division.Like rationalization in simplification of expressions of real numbers, we multiply the denominator and numerator of any given complex expression by the conjugate of the denominator to get the complex expression simplified. You can illustrate this by taking several examples like example 3 given in student textbook under this topic.

Having discussed this, define modulus (or absolute value) of complex number.And then, there are properties of modulus and conjugate given as Theorem 7.2.Guide the students to verify the theorem and understand the proofs which are given in the student textbook. Afterwards, you can form groups of students and let them do Exercise 7.4. Give them question number 3 as an assignment to help them develop their own generalization.

Assessment

In addition to the formative assessments you used during instruction, you can give assignments, test/quiz to assess students understanding. You can also ask students to give the conjugate and modulus of complex numbers and ask them to prove some of the properties of conjugate and modulus. Give them problems to find inverses of non – zero complex numbers and also to divide a complex number by a non – zero complex number. They can do the problems in groups or individually

Answers to Exercise 7.4

1. a. $\frac{2}{13}-\frac{3}{13}i$ b. $\frac{22}{13}-\frac{7}{13}i$ c. $\frac{2}{29}+\frac{19}{58}i$

   d. $\frac{2}{25}+\frac{11}{25}i$ e. $\frac{-7}{41}+\frac{22}{41}i$ f. $\frac{1}{17}-\frac{13}{17}i$

   g. $\frac{-5}{2}+7i$ h. $8-6i$

2. a. 5 b. 10 c. 50 d. 50

   e. They are equal

   f. $|z_1+z_2|=\sqrt{97}$, $|z_1|=5$, $|z_2|=10$ implying $|z_1|+|z_2|=15$.

   Therefore, $|z_1+z_2|<|z_1|+|z_2|$.

   g. $|z_1-z_2|=\sqrt{153}$, $|z_1|=5$, $|z_2|=10$ implying $|z_1|-|z_2|=-5$.

   Therefore, $|z_1-z_2|>|z_1|-|z_2|$.

   h. $|z_1|-|z_2|=-5$, $\left||z_1|-|z_2|\right|=|-5|=5$.

   Therefore, $|z_1|-|z_2|<\left||z_1|-|z_2|\right|$

3. The answer is yes, because

$$|z_1||z_2|=\sqrt{x^2+y^2}\sqrt{a^2+b^2}$$

$$|z_1z_2|=|(x+yi)(a+bi)|=|xa-yb+(ay+bx)i|$$

$$=\sqrt{(xa-yb)^2+(ay+bx)^2}$$

$$=\sqrt{x^2a^2+y^2b^2+a^2y^2+b^2x^2}$$

$$=\sqrt{x^2(a^2+b^2)+y^2(b^2+a^2)}=\sqrt{(x^2+y^2)(b^2+a^2)}$$

7.4 SIMPLIFICATION OF COMPLEX NUMBERS

Periods Allotted: 3 periods

Competency

At the end of this sub-unit, the students will be able to:

- *write the simplified form of expressions involving complex numbers.*

Vocabulary: Simplification

Introduction

The students are expected to know the four operations and their properties, the conjugate of a complex number and its properties and the modulus of a complex number and its properties. In this subunit, they will apply their knowledge on the above concepts to simplify complex numbers.

Teaching Notes

The concept of simplification is not new to the students because they know how to simplify expressions involving variables. But what makes this simplification different from other simplifications is the use of the imaginary number i alongside the variables. In this case, exponents of the imaginary number can be simplified further. At this point, make the students aware that every equation of the form $z^2 = w$ has a solution in $\mathbb{C}$.

In order to start this lesson, you may need to check whether the students can recall and describe the operations and properties. Afterwards, you can start the subunit by giving them a revision exercise on the concepts covered so far. Let them do in groups the revision exercises presented as examples in the student textbook and let some of the groups present their agreed upon answers to the whole class. Next, discuss some more examples and the proof of Theorem 7.3. You can then give them Exercise 7.5 as homework. In Exercise 7.5, question number 3, we have considered equations upto fourth roots only. But, you may encourage able students to find the 5^{th}, 6^{th}, 7^{th} and 8^{th} roots of 1,and make the students generalize for the n^{th} roots of 1.

Assessment

Give students exercise problems on simplification of expressions, and let them present some of their work on the board. If you decide to give them as group work, let one of the students present their answer to the whole class.

Answers to Exercise 7.5

1. a. $$\frac{13}{3-2i}-\frac{i^3}{1+i}=\frac{13(3+2i)}{(3-2i)(3+2i)}-\frac{(-i)(1-i)}{(1+i)(1-i)}$$
$$=\frac{13(3+2i)}{13}+\frac{(i+1)}{2}$$
$$=3+2i+\frac{i+1}{2}=\frac{7}{2}+\frac{5}{2}i$$

b. $$\frac{5}{(i-1)(2-i)(3-i)}=\frac{5}{10i}=-\frac{1}{2}i$$

c. $i^{120}-4i^{94}+3i^{31}=1-4(-1)+3(-i)=5-3i$

d. $$\left(2+\sqrt{-25}-\left(3-\sqrt{-216}\right)+\left(1+\sqrt{-9}\right)\right)=(2+5i-3+6\sqrt{6}\,i)+1+3i$$
$$=\left(8+6\sqrt{6}\right)i$$

e. $$\frac{1+2i}{3-4i}+\frac{2-i}{5i}=\frac{(1+2i)(3+4i)}{(3-4i)(3+4i)}+\frac{2-i}{5i}\times\frac{i}{i}$$
$$=\frac{-5+10i}{25}+\frac{2i+1}{-5}=-\frac{1}{5}+\frac{2}{5}i-\frac{2}{5}i-\frac{1}{5}=\frac{-2}{5}$$

f. $i^{29}+i^{42}+i=i-1+i=2i-1$

g. $i^{400}+3i^{200}+5i-3=1+3(1)+5i-3=1+5i$

h. $$\frac{12i}{11i}=\frac{12}{11}$$

i. $$\left(\sqrt{-12}\right)^3=\left(2i\sqrt{3}\right)^3=2^3(i^3)\left(\sqrt{3}\right)^3=-24i\sqrt{3}$$

j. $$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}=\frac{12+5i}{4+3i}$$
$$=\frac{(12+5i)\ (4-3i)}{(4+3i)\ (4-3i)}$$
$$=\frac{63-16i}{25}=\frac{63}{25}-\frac{16i}{25}$$

2. a. $z_1^3=(2+i)^3=2+11i$

$-3z_2^2=-3(5-12i)=-15+36i$

$$\Rightarrow z_1^3-3z_2^2+4z_3=2+11i-5+36i+4\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$
$$=-15+\left(47+2\sqrt{3}\right)i$$

b. $\overline{z_3^4} = \overline{\left(z_3^2\right)^2} = \overline{\left(-\frac{1}{2}-\frac{1}{2}i\sqrt{3}\right)^2} = \overline{\left(-\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)} = -\frac{1}{2}+\frac{1}{2}i\sqrt{3}$

c. $\left|3\overline{z}_1 - 4\overline{z}_2 + z_3\right| = \left|6-3i-12-8i-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right|$

$$= \left|-\frac{13}{2}+\left(-11+\frac{\sqrt{3}}{2}\right)i\right|$$

$$= \sqrt{\left(-\frac{13}{2}\right)^2+\left(-11+\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{169}{4}+\frac{487}{4}-11\sqrt{3}} = \sqrt{164-11\sqrt{3}}$$

d. $\frac{z_1 z_2}{z_3} = \frac{8-i}{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}$

$$=(8-i)\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right) = -\frac{(8+\sqrt{3})}{2}+\frac{(1-8\sqrt{3})i}{2}$$

3. a. $z^2 = -4 \Rightarrow z = \pm 2i$

b. $z^2+12=0 \Rightarrow z^2 = -12$

$\Rightarrow z = \sqrt{-12} = \pm 2\sqrt{3}$

c. $z^2+z+1=0 \Rightarrow z = \frac{-1\pm\sqrt{3}}{2}i$

d. $z^3 = -1 \Rightarrow z^3+1=0$

$\Rightarrow (z+1)(z^2-z+1)=0$

$\Rightarrow z = -1$ or $z^2-z+1=0$

But $z^2-z+1=0 \Rightarrow z = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm i\sqrt{3}}{2}$

$\Rightarrow$ S.S $= \left\{-1, \frac{1\pm i\sqrt{3}}{2}\right\}$

f. $z^4 = 1 \Rightarrow z^4-1=0$

$\Rightarrow (z^2-1)(z^2+1)=0$

$\Rightarrow z = \pm 1$ or $z = \pm 2$

$\Rightarrow$ S.S $= \{\pm i, \pm 1\}$

4. a. 6 b. -6 c. $6i$ d. $6i$

The values in (a) and (b) are different; and the values in (c) and (d) are the same.

5. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ if and only if a and b are nonnegative real numbers.

7.5 ARGAND DIAGRAM AND POLAR REPRESENTATION OF COMPLEX NUMBERS

Periods Allotted: 3 periods

Competencies

At the end of this sub-unit, the students will be able to:

- *describe how to set up the Argand plane.*
- *plot the point corresponding to a given complex number.*
- *identify the complex number that corresponds to a given point in the Argand plane.*
- *represent a complex number in the polar form.*
- *determine the modulus and argument of a given complex number.*

Vocabulary: Argand diagram (Complex plane), Polar form, Real axis, Imaginary axis, Argument (amplitude)

Introduction

The main issue in this topic is to associate the Cartesian coordinate plane to complex numbers. You may begin this lesson by discussing the Cartesian coordinate plane and particularly how to represent a point in the plane.

Teaching Notes

You can let each student do Activity 7.5 which will help them revise the coordinate plane and identify points on the coordinate plane.

Answers to Activity 7.5

1. a. They represent different points on the plane and hence $(2,3) \neq (3,2)$.

   b. $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$

   c. (7, 5)

   d. Yes; $(a, b) + (c, d) = (a+c, b+d)$

2. All points of the form: $(x, 0)$ lie on the x-axis; and $(0, y)$ lie on the y-axis.

After ensuring the revision of the coordinate plane and points on the coordinate plane, you can form groups of students and let them do Group Work 7.4. The purpose of the group work is to establish the required one-to-one correspondence between a point on the coordinate plane and a complex number represented as $f: \mathbb{R}^2 \to \mathbb{C}$, given by

$$f(x, y) = x + yi.$$

In the Group Work, they are expected to show that f is one-to-one (question 1) and f is also onto(question 2). For this purpose, you are required to guide the students to discover by themselves that there is a one-to-one correspondence between the set of points in the plane and the set of complex numbers. After having discussed all these, discuss the geometric representation, the Argand plane or complex plane and polar representation of complex numbers. Meanwhile, introduce the concept of real axis, imaginary axis and the complex plane or the z-plane.

Answers to Group Work 7.4

1. It is impossible, since

$$f(x, y) = f(a, b) \Leftrightarrow x + iy = a + ib$$

$$\Leftrightarrow x = a \text{ and } y = b$$

Therefore $(x, y) \neq (a, b) \Rightarrow f(x, y) \neq f(a, b)$

2. Yes, $x = a$ and $y = b$

Since $f(a, b) = x + iy$

$$\Rightarrow a + ib = x + iy$$

$$\Rightarrow a = x \text{ and } b = y$$

The students will have a better understanding of the modulus – argument form of a complex number, if you guide them to revise the angle sum rule and the relationship of a trigonometric function of an angle and the quadrant in which its terminal side falls.

So discuss some of the following:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

Terminal side of angle	Sign of cos	sin	tan	cot	cosec	sec
Quadrant I	+	+	+	+	+	+
Quadrant II	–	+	–	–	+	–
Quadrant III	–	–	+	+	–	–
Quadrant IV	+	–	–	–	–	+

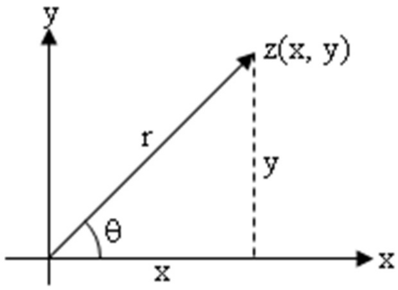


Figure 7.1

$z = x + iy$ is identified with $z(x, y)$, as shown in the figure. This gives us,

$\tan\theta = \dfrac{y}{x}$, and $r = \sqrt{x^2 + y^2}$.

r is called the modulus of z and θ is called the amplitude or argument of z. Remind the students that $\tan(\theta + k\pi) = \tan(\theta)$, for any integer k. In other words, argument of z denoted by arg z, is not unique. The students may notice that tanθ is not defined when

$x = 0$. In this case, help them to notice that, arg $z = \dfrac{(2n+1)}{2}\pi$, for some integer n.

To avoid duplication, we can use the angle θ such that $-\pi < \theta \le \pi$. This angle is called the principal argument and is denoted by Arg z.

Arg z and Arg z are related by the relationship:

arg z = Arg z + 2kπ, for some integer k.

The students need help in finding Arg$z$; so, you can guide them to calculate it as follows:

For $x \neq 0$, if $\tan\alpha = \left|\dfrac{y}{x}\right|$, then $\text{Arg}\, z = \begin{cases} \alpha, & \text{if } x > 0 \\ \pi - \alpha, & \text{if } x < 0, y > 0 \\ \alpha - \pi, & \text{if } x < 0, y < 0 \end{cases}$

For $x = 0$, Arg $z = \dfrac{\pi}{2}$.

Remark: For angles in 1st and 2nd quadrant, Arg z is measured in anticlockwise direction.

Example 1 Express z = $-\sqrt{3} + i$ in modulus argument form by using Arg z.

Solution: The terminal side is in the second quadrant. i.e. $x < 0$ and $y > 0$.

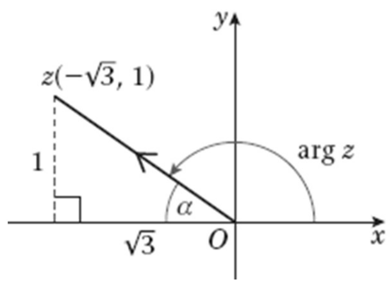


Figure 7.2

$$\text{Arg}z = \pi - \alpha = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}, \text{ and}$$

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

Therefore, $-\sqrt{3} + i = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right)$

Remark: For angles in 3rd and 4th quadrant, Arg$z$ is measured in clockwise direction.

Example 2 Express $z = -1 + i$ in modulus argument form by using Arg$z$.

Solution: The terminal side is in 3rd quadrant i.e. $x < 0$ and $y < 0$.

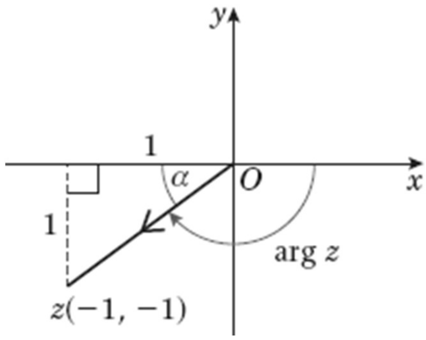


Figure 7.3

$$\text{Arg } z = \alpha - \pi = \tan^{-1}\left(\frac{1}{1}\right) - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}, \text{ and}$$

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Therefore, $-1 - i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)\right)$

Example 3 Express $z = 1 - i$ in modulus argument form by using Arg z.

Solution:

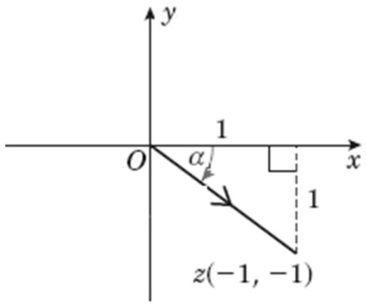


Figure 7.4

$$\text{Arg}z = \alpha = -\tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4} = -\frac{\pi}{4}, \text{ and}$$

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Therefore, $1 - i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right)$

Assessment

You can ask the students to plot complex numbers as points in the Argand Plane, or conversely, you can ask them to give a point in the plane in the form $x + y$i. You can also ask them to give the quadrant to which a given complex number belongs. Let students find the argument – modulus form of a complex number and express products, quotients, and powers of complex numbers using the polar form. Here you can give problems of various degrees that can be done by low achievers as well as high achievers that can be done in various active learning methods.

Answers to Exercise 7.6

1. a. (1, 1) first quadrant b. (2, –3) fourth quadrant
   c. (3, 4) first quadrant d. (–1, –2) third quadrant

2. a. $3(\cos 0 + i\sin 0)$ b. $3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

   c. $3(\cos\pi + i\sin\pi)$ d. $3(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})$

   e. $4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ f. $4(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$

   g. $2\sqrt{2}(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6})$ h. $\sqrt{3}(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$

3. If (r, θ) is the polar form, then the corresponding complex number is of the form

$z = r(\cos\theta + i\sin\theta)$, thus you can substitute and evaluate the values which are given in

a. $\frac{5}{2}+\frac{5\sqrt{3}i}{2}$ b. $3\sqrt{3}+3i$

c. $\frac{5\sqrt{2}}{2}+\frac{5\sqrt{2}i}{2}$ d. $\frac{-5}{2} - \frac{5\sqrt{3}i}{2}$

4. a. $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ b. $\theta = -\tan^{-1}\left(\frac{3}{4}\right)$

c. $\theta = \frac{3\pi}{4}$ d. $\theta = -\frac{3}{4}\pi$

Answers to Review Exercises on Unit 7

1. a. $x= 3, y = 4$

b. $\frac{1+2i}{x+yi} = 1-\sqrt{-4}$

$\Rightarrow \frac{1+2i}{1-2i} = x+yi \Rightarrow \frac{-3}{5}+\frac{4}{5}i = x+yi$

$\Rightarrow x = \frac{-3}{5}, y = \frac{4}{5}$

c. $(3+i)(x+yi)(3+4i) = 3+9i$

$\Rightarrow (3+i)(3+4i)(x+yi) = 3+9i$

$\Rightarrow (5+15i)(x+yi) = 3(+3i)$

$\Rightarrow 5(1+3i)(x+yi) = 3(+3i) \Rightarrow x+yi = \frac{3}{5} \Rightarrow x = \frac{3}{5}, y = 0$

d. $(2x+yi)(i+4) = \frac{1}{3+5i}$

$(2x+yi) = \frac{1}{(3+5i)(i+4)} \Rightarrow \frac{7}{578}-\frac{23}{578}i$

$\Rightarrow 2x = \frac{7}{578} \Rightarrow x = \frac{7}{1156}, y = -\frac{23}{578}$

e. $x = 3, y = 11$

2. a. $\overline{z} = 3-4i$ b. 5   c. 5

d. $5(\cos\theta + i\sin\theta)$, where $\theta = \tan^{-1}(\frac{4}{3})$

3. a. conjugate: $\overline{\left(\dfrac{3+i}{5-4i}\right)} = \dfrac{\overline{3+i}}{\overline{5-4i}} = \dfrac{3-i}{5+4i}$

Note that: $\dfrac{3-i}{5+4i} = \dfrac{(3-i)(5-4i)}{5^2+4^2} = \dfrac{15-12i-5i+4i^2}{41}$

$$= \frac{15-4-17i}{41} = \frac{11}{41} - \frac{17i}{41}$$

Argument:

$$\arg\left(\frac{3+i}{5-4i}\right) = \arg(3+i) - \arg(5-4i) = \arc\tan\left(\frac{1}{3}\right) - \arc\tan\left(\frac{-4}{5}\right)$$

$$= -57^{\circ}.$$

Modulus:

$$\frac{3+i}{5-4i} = \left|\frac{3+i}{5-4i}\right| = \frac{\sqrt{3^2+1^2}}{\sqrt{5^2+(-4)^2}} = \frac{\sqrt{10}}{\sqrt{41}} = \frac{\sqrt{410}}{41}$$

b. Conjugate:

$$\overline{\left(\frac{(2-3i)(4+i)}{\left(i\sqrt{3}+1\right)\left(\frac{1}{2}i+5\right)}\right)} = \frac{\overline{(2-3i)(4+i)}}{\overline{\left(i\sqrt{3}+1\right)\left(\frac{1}{2}i+5\right)}} = \frac{(2+3i)(4-i)}{\left(1-i\sqrt{3}\right)\left(5-\frac{1}{2}i\right)}$$

$$= \frac{11+10i}{\left(5-\frac{\sqrt{3}}{2}\right) - \left(5\sqrt{3}+\frac{1}{2}\right)i}$$

$$= \left(\frac{50}{101} - \frac{111\sqrt{3}}{202}\right) + \left(\frac{50\sqrt{3}}{101} + \frac{111}{202}\right)i$$

Argument:

$$\arc\tan\left(\frac{-\left(\frac{50\sqrt{3}}{101}+\frac{111}{202}\right)}{\left(\frac{50}{101}-\frac{111\sqrt{3}}{202}\right)}\right) = \arc\tan(1.885) = -108^{\circ}.$$

Modulus:

$$\frac{|2-i||4+i|}{\left|i\sqrt{3}+1\right|\left|\frac{1}{2}i+5\right|} = \frac{\sqrt{13}\times\sqrt{17}}{\sqrt{101}} = \frac{\sqrt{221}}{\sqrt{101}} = \frac{\sqrt{22321}}{101}$$

c. $\dfrac{(i+2)(3-4i)(5+3i)}{(2i+1)(4i+3)(5i-3)}=\dfrac{-3}{5}+\dfrac{4}{5}i$

Conjugate: $\dfrac{-3}{5}-\dfrac{4}{5}i$

Argument: arc tan (1.33) = 127°.

Modulus: $\sqrt{\left(\dfrac{3}{5}\right)^2+\left(\dfrac{4}{5}\right)^2}=1$

4. a. $i^{320}-5i^{121}+3i^{45}=1-5i+3i=1-2i$

b. $\dfrac{-3}{10}+\dfrac{-4i}{10}$

c. $-5+29i$

d. $\left(\dfrac{36x^2+20x+1}{180x^2+5}\right)+\left(\dfrac{12x^2-3}{180x^2+5}\right)i$

e. $\left(\dfrac{1-i}{\sqrt{2}}\right)^{40}=\left(\left(\dfrac{1-i}{\sqrt{2}}\right)^2\right)^{20}=1=1+0i$

f. $2^{40}=2^{40}+0i$

g. $\left(\dfrac{i-\sqrt{3}}{1-i}\right)^{30}=\dfrac{\left(\left(i-\sqrt{3}\right)^3\right)^{10}}{\left((1-i)^2\right)^{15}}=\left(\dfrac{(8i)^{10}}{(-2i)^{15}}\right)=\left(\dfrac{8^{10}(-1)}{-2^{15}(-i)}\right)=2^{15}i$

5. Proof:

$$\begin{aligned}|z_1+z_2|^2+|z_1-z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)}+(z_1-z_2)\overline{(z_1-z_2)}\\ &= (z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2})\\ &= z_1\overline{z_1}+z_1\overline{z_2}+z_2\overline{z_1}+z_2\overline{z_2}+z_1\overline{z_1}-z_1\overline{z_2}-z_2\overline{z_1}+z_2\overline{z_2}\\ &= 2(|z_1|^2+|z_2|^2)\end{aligned}$$

6. a. Using the rational root test, $1^3+2(1)^2+1-4=0$

$\Rightarrow x-1$ is a factor of x^3+2x^2+x-4

Long division gives:

$x^3+2x^2+x-4=(x-1)(x^2+3x+4)$

The roots of $x^2+3x+4=0$ are

$$x=\frac{-3\pm\sqrt{3^2-4\times1\times4}}{2(1)}=\frac{-3\pm\sqrt{-7}}{2}=\frac{-3\pm\sqrt{7}\,i}{2}$$

$$\Rightarrow \text{S.S}=\left\{\frac{-3}{2}-\frac{\sqrt{7}i}{2},\frac{-3}{2}+\frac{\sqrt{7}i}{2},1\right\}$$

b. S.S = $\{-1-\sqrt{2}i, -1+\sqrt{2}i\}$

c. S.S = $\{-2, 2-i, 2+i\}$

d. In $x^4+2x^2+2=0$,

Let $y = x^2$ so that $x^4 = y^2$

$\Rightarrow y^2 + 2y + 2 = 0$

$$y = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2} = \frac{-2\pm 2i}{2} = -1\pm i$$

$\Rightarrow x^2 = y = -1 \pm i$

$\Rightarrow x = \pm\sqrt{-1\pm \mathrm{i}}$

S.S=$\left\{\sqrt{-1+i}, -\sqrt{-1+i}, \sqrt{-1-i}, -\sqrt{-1-i}\right\}$

7. If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$, for example substituting $n = 10$ in each of the following, you get

a. $z = 8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $z^{10} = 8^{10}\left(\cos\frac{10}{3}\pi + i\sin\frac{10}{3}\pi\right)$

b. $z = 6\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right)$, $z^{10} = 6^{10}\left(\cos\frac{70}{4}\pi + i\sin\frac{70}{4}\pi\right)$

c. $z = 4\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{7}{6}\pi\right)$, $z^{10} = \left(4\sqrt{2}\right)^{10}\left(\cos\frac{70}{6}\pi + i\sin\frac{70}{6}\pi\right)$

d. $z = \frac{2\sqrt{3}}{5}\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right)$, $z^{10} = \left(\frac{2\sqrt{3}}{5}\right)^{10}\left(\cos\frac{110}{6}\pi + i\sin\frac{110}{6}\pi\right)$.

8. a. $\frac{5}{13} + -\frac{i}{13}$ b. $\frac{-30}{37} + \frac{32i}{37}$ c. $\frac{21-2\sqrt{5}}{13} + -i\frac{(14+3\sqrt{5})}{13}$

9. a. A circle centered at (1,0) with radius 1.

b. Circular region of the circle in (a).

c. Regions outside the circle in (a).

10. a. $(-1, -1)$ b. $(-1, 1)$ c. $(-1, 0)$ d. $\left(\frac{-5}{2}, \frac{-5\sqrt{3}}{2}\right)$

UNIT 8 VECTORS AND TRANSFORMATION OF THE PLANE

INTRODUCTION

This unit deals with vectors and transformation of the plane. This chapter follows the concepts which students discussed vectors in grade 9. Here, more discussions on vectors that include representation of vectors, scalar (dot) product of vectors, applications of vectors and transformation of the plane will be treated. Narrations and illustrative examples are offered for each sub-unit. As students are expected to have some background on vectors, the first sub-unit is devoted to revision of vectors and scalars that were discussed in grade 9. Thus, you need to focus more on the subsequent sections. You are expected to go for at most four periods for sub-units 8.1 and 8.2.

In this unit, students need also to develop as many applications of the concept of a vector and vector algebra in their daily life. Although details of applications of a vector are discussed, some of them include determining work done, angle between two vectors, writing equations in vector form, and formulating equations such as tangent and circle.

It is also expected to extend this prior discussion to deliberating about transformation of the plane which includes translation, reflection and rotation.

Involvement of students in various aspects of this chapter is also expected to help them have a better understanding of the concepts. Cognizant of this fact, efforts should be made to explore local issues that can best describe a vector. Representation of a vector, dot product of vectors and applications and transformation of the plane should also be dealt with along discussions of the ideas and examples delivered in the student textbook.

Unit Outcomes

After completing this unit, students will be able to:

- *know basic concepts and procedures about vectors and operation on vectors.*
- *know specific facts about vectors.*
- *apply principles and theorem about vectors in solving problems involving vectors.*
- *know basic concepts about transforming the plane.*
- *apply methods and procedures in transforming plane figures.*

Suggested Teaching Aids in Unit 8

Setting of models, organizing flip charts that describe translations, reflections and rotations are essential.

As an additional input, you can also constitute different groups of students so that they can develop local examples which will help as teaching aid for a better understanding of the notions of vectors and their applications.

8.1 REVISION ON VECTORS AND SCALARS

Period allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define a scalar quantity.*
- *identify the everyday application of scalars.*
- *define a vector quantity.*
- *identify the everyday application of vectors.*
- *describe the difference between a vector and a scalar quantity.*
- *represent vector by different notations.*
- *determine the sum of two or more vectors.*
- *determine the difference of two vectors.*
- *multiply a vector by a scalar.*

Vocabulary: Scalar, Vector, Representation of a vector, Vector addition, Scalar multiplication, Properties of vector operations.

Introduction

Students already have some background on vectors which they have studied in grade 9. This subunit is devoted to strengthening their background by revising vectors and scalars, addition of vectors and multiplication of vectors by a scalar, and representation of vectors.

Teaching Notes

You may start the lesson by revising important points which the students had learnt about scalars and vectors in grade 9. You may review grade 9 student text and organize the revision accordingly. While revising, you may proceed with an activity which deals with the concepts of "scalar quantity" so that students can define scalars as a quantity with size or magnitude only. You can assist the students to realize every day examples of scalars like mass (10kg), time (5sec), distance (5km), money (100Birr). You can then consolidate the concept of scalar quantity as having only magnitude with some unit. After consolidating students' understanding of scalars, you may proceed to vector starting from their discussion in grade 9. To do so, you may proceed with an activity which deals with the concept of "vector quantity" so that students can define and understand vector as a quantity with size or magnitude and direction included. In this regard, you may need to let them understand how direction of a vector is treated as an angle with respect to horizontal line or with respect to a vertical line in case of compass. Examples for this may include weight (whose direction is towards the center of the earth and whose magnitude is given in Newton (N)). You may also add more examples such as velocity, acceleration, etc.

For the purpose of simplifying revision, you may organize students in pairs and encourage them to do Activity 8.1.

Answers to Activity 8.1

1. a, c, and f are scalars whereas b, d, and e are vectors.

2.

Scalars	**Vectors**
Distance	Displacement
Speed	Velocity
Work	Acceleration
Area	Force
Time	Momentum
Volume	Weight
Density	
Temperature	
Mass	

After recognizing the understanding of the students about the concepts of scalars and vectors, you need to revise the way vectors are represented. At this stage, students need to recognize that there are different ways to describe vectors. Some of these ways may include representation as a directed line segment, as an ordered pair, as a standard vector etc. In order to further consolidate their understanding of scalars and concepts, you may assist students to exercise the different ways of representing vectors. To enrich these ideas, you may form groups of students and give them Group work 8.1 to discuss and present to the class. When they do each work in groups, you can round in the class and see if there are students who are deficient somehow and whom you need to assist. And if you have gifted students you can give them some more questions which you can prepare before coming to class.

When they discuss the group work, it will be helpful if you can encourage the students to state in their own words, the definitions of scalar and vector quantities, and representation of vectors. Finally, you can give them exercise 8.1 as a homework which may help you to assess the understanding of the students as well.

Additional exercise problems for high ability students

1. Ask some of the high ability students to demonstrate the sum **a** + **b** + **c** + **d** + **e** + **f** on the black board.

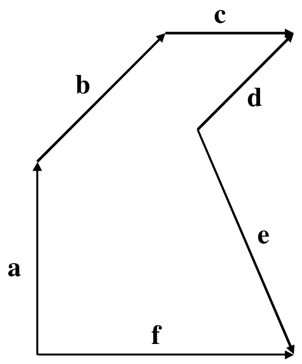


Figure 8.1

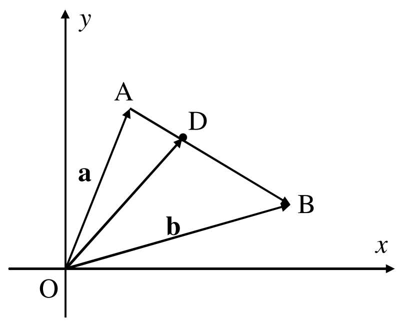


Figure 8.2

2. **a** and **b** are the position vectors of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as shown in Figure 8.2. If D divides $\overline{AB}$ in the ratio $m{:}n$, express the position vector of D in terms of **a** and **b**.

Solution: $\dfrac{AD}{BD} = \dfrac{m}{n} \Rightarrow \dfrac{AD}{AB} = \dfrac{m}{n+m}$

$$\overrightarrow{AD} = \left(\frac{m}{n+m}\right)\overrightarrow{AB} = \frac{m}{n+m}(b-a)$$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = a + \frac{m}{n+m}(b-a) = \frac{na+mb}{n+m}.$$

Answers to Group Work 8.1

1. A vector **v** as coordinates point can be represented by $\mathbf{v} = (x, y)$ where the initial point is the origin and terminal point is (x, y).

   $\mathbf{v} = (x, y)$ can be represented as a column vector in the form of $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

2. Two vectors are equal if their corresponding components are equal.
   Example: $\mathbf{v} = (2,3)$ and $\mathbf{v} = (1, 1) + (1, 2)$

3. The standard form of a vector $\mathbf{v}$ is $\mathbf{v} = (x, y)$
4. a. The components are 3 and 4 b. $|\mathbf{v}| = 5$

   c. $\theta = \tan^{-1}\left(\frac{4}{3}\right)$
5. It depends

Assessment

For this sub-unit, you may use the following assessments:

- Ask students to list out many examples of scalar quantities and vector quantities
- Ask students to identify the difference between a vector and a scalar quantity through examples
- Ask students to determine the different ways of representation of vectors and explain their answers through examples

You can do any of these through activity, assignments or homework, group work, or quizzes/tests.

Answers to Exercise 8.1

1. Initial point and terminal point $|\overrightarrow{PQ}|$, length.

2. The representation of the vector, coordinate components

3. $\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2)$

   $|\overrightarrow{PQ}| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$

4. $(3, -4) = \mathbf{v}$. $|\mathbf{v}| = 5$.

Now you may proceed to operations on vectors (addition, subtraction and multiplication of a vector by a scalar). You may start this discussion by revising the addition of vectors using the "triangle law of addition" and then proceed to the "parallelogram law of addition" of vectors. For the purpose of revising these rules, it may be of help to let each student do Activity 8.2. Select some students and encourage them to do each activity on the board. Then discuss their solutions by giving corrections when necessary. Here, you may need to discuss the properties of vector addition such as commutative and associative properties with the active involvement of students. You can also approach the difference of vectors as a consequence of vector addition and let the students practice computing difference of vectors with several exercises.

Answers to Activity 8.2

1.

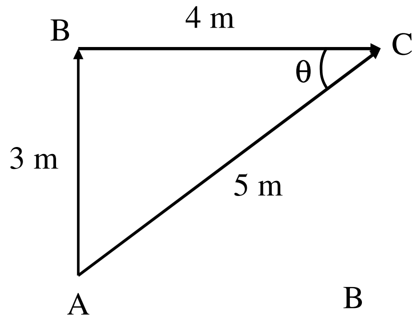


Figure 8.3

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$\left|\overrightarrow{AC}\right| = 5$ m at an angle $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

2.

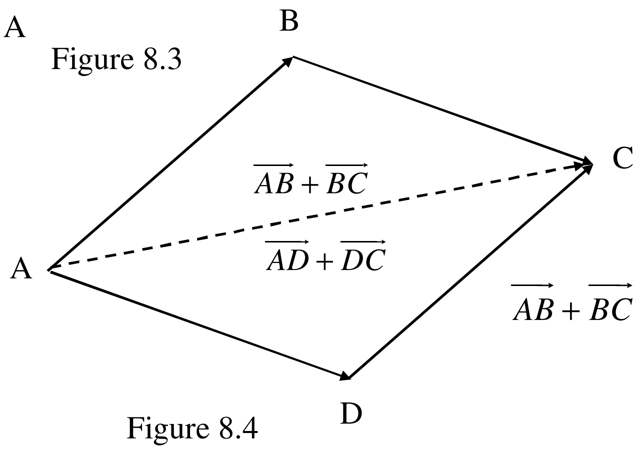


Figure 8.4

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AD}+\overrightarrow{DC}=\overrightarrow{AC}$$

Once the students have practiced discussing addition and subtraction of vectors, there is a need to proceed into multiplication of vectors by a scalar. To revise multiplication of vectors by a scalar, you may let students form groups and do Group Work 8.2. The purpose of this group work is to help students to recognize the relationship between the resulting vector when multiplied by a scalar and the original vector as parallel. If **v** is a vector then, k**v** is a vector which is parallel to vector **v** and the direction is the same if $k > 0$ and opposite if $k < 0$. You may need to give assignments or group work for practicing further concepts and for consolidating the revision of this unit.

Answers to Group Work 8.2

1. If $k > 0$, then $k\overrightarrow{PQ}$ and $\overrightarrow{PQ}$ have the same direction.
   If $k < 0$, then $k\overrightarrow{PQ}$ and $\overrightarrow{PQ}$ have opposite directions.
2. $\left|\overrightarrow{PQ}\right| = \left|-\overrightarrow{PQ}\right|$ and $\overrightarrow{PQ}+\left(-\overrightarrow{PQ}\right)=0$
3. $\overrightarrow{PQ}+\left(-\overrightarrow{PQ}\right)=\overrightarrow{PQ}-\overrightarrow{PQ}=0$

4.

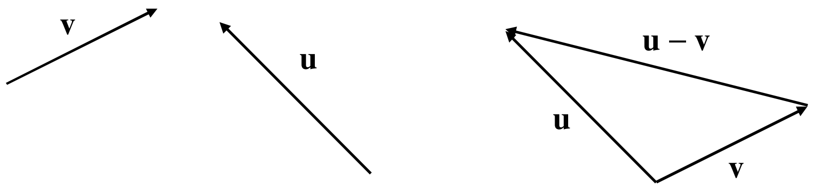


Figure 8.5

At last, you can use the questions in exercise 8.2 for the purpose of assessing students' understanding by giving them the questions as group work.

Assessment

For this sub-unit you may use the following assessments:

- Ask students to determine the sum and difference of some pair of vectors.
- Ask them to describe some of the properties of operations of vectors.
- Give them exercise problems on scalar multiplication of vectors.

You can do any of the above through activity, assignments or homework, group work, or quizzes/tests.

Answers to Exercises 8.2

1. 5km
2. The displacement is $\sqrt{15^2+25^2}$ km $=\sqrt{850}$ km
3. $m\mathbf{v} = n\mathbf{v} \quad \Rightarrow (m-n)\,\mathbf{v} = 0$

   $\Rightarrow m - n = 0$ since $\mathbf{v} \neq 0$

   $\Rightarrow m = n.$
4. a. (3, 18) b. (–13, 26) c. (3, 5)
5. $\sqrt{200-20\sqrt{2}}$
6.

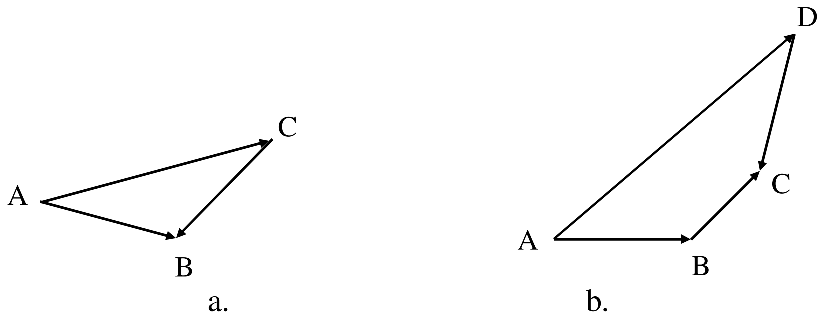


Figure 8.6

7. Consider Figure 8.7.

$\overrightarrow{CD} = \mathbf{w} - \mathbf{v}$

$\overrightarrow{DE} = -\mathbf{v}$

$\overrightarrow{EF} = -\mathbf{w}$

$\overrightarrow{AF} = \mathbf{w} - \mathbf{v}$

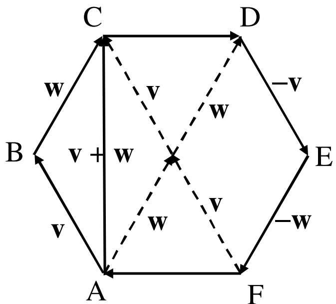


Figure 8.7

8.

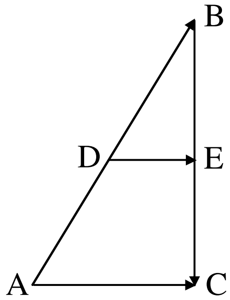


Figure 8.8

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$

$\Rightarrow \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$

$\Rightarrow \overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$

8.2 REPRESENTATION OF VECTORS

Periods allotted: 1 Period

Competencies

At the end of this sub-unit, students will be able to:

- *resolve a given vector into two components.*
- *use unit vectors to determine the column representation of a given vector.*
- *determine the magnitude of a vector.*

Vocabulary: Components of a vector, Magnitude, Unit Vector, Standard unit vectors

Introduction

This subunit is meant to revise the previous knowledge of students about representation of vectors. Some practical examples on vector representation and on the use of components of a vector will be discussed in this subunit.

Teaching Notes

You may start this lesson with an activity on the different ways of representing vectors with more emphasis on resolving vectors or with oral question on the possibility of representing a vector as a sum of two other vectors. For this purpose, you may need to give them Activity 8.3 which will help them think of the properties of vector addition and subtraction by way of resolving some force vectors given as a position vector using its x and y components on the coordinate plane. While students try to do the activity, you need to help them to practice component representation of vectors. You may then proceed introducing the unit vectors **i** and **j** on the coordinate plane and explain how a given vector is expressed as a sum and scalar multiplication of the vectors. With this move, you need to assist the students by showing them how a vector **P** = x**i** + y**j** can be resolved into its horizontal and vertical components.

This can be shown as follows,

If **P** = x**i** + y**j**, then we may consider the two vectors **h** = x**i** and **v** = y**j** where **h** = $(x, 0)$ and **v** = $(0, y)$ and the unit vectors are $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that **P** = x**i** + y**j** becomes

$$\mathbf{P} = \mathbf{h} + \mathbf{v} = \begin{pmatrix} x & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & y \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathbf{i} + y\mathbf{j}.$$

Classifying vectors into components will help students to easily understand computing the magnitude of a vector in a way if **P** = x**i** + y**j** then its magnitude (length) is given by $|\mathbf{P}| = \sqrt{x^2 + y^2}$. Here you may give chance for students to determine magnitude of a vector as a distance between the initial and terminal points of the vector. You also need to help students to identify unit vectors. You may also need to let students practice more exercises. When you finish your lesson, you may give students a reading assignment about scalar (dot) product which will be discussed in the subsequent lesson.

Answers to Activity 8.3

1. Given a vector **w**, you may find two vectors **u** and **v** such that **u** + **v** = **w** where **u** and **v** are components of vector **w**.

2. i. $(a\mathbf{i} + b\mathbf{j}) + (c\mathbf{i} + d\mathbf{j})$
   $= a(1, 0) + b(0, 1) + c(1, 0) + d(0, 1)$
   $= (a + c)(1, 0) + (b + d)(0, 1)$
   $= (a + c)\mathbf{i} + (b + d)\mathbf{j}$

   ii. $(a\mathbf{i} + b\mathbf{j}) - (c\mathbf{i} + d\mathbf{j})$
   $= a(1, 0) + b(0, 1) - c(1, 0) - d(0, 1)$
   $= (a - c)(1, 0) + (b - d)(0, 1)$
   $= (a - c)\mathbf{i} + (b - d)\mathbf{j}$

   iii. $t(a\mathbf{i} + b\mathbf{j}) = t(a(1, 0) + b(0, 1))$
   $= ta(1, 0) + tb(0, 1) = (ta)(1, 0) + tb(0, 1)$
   $= (ta)\mathbf{i} + (tb)\mathbf{j}$

You can give for high achievers additional exercises of the following type:

A car of mass 1800 kg climbs a hill at a constant speed of 60 km/hr. If the slope of the hill is $\frac{1}{2}$, find the work done by the car against gravity in one minute.

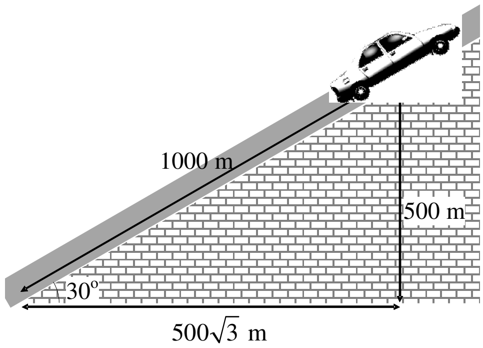


Figure 8.9

Solution:

In one minute, the car travelled 1000 m up the hill.

$\Rightarrow$ The vertical distance raised in one minute is 5000 m.

$\Rightarrow$ The work done against gravity in one minute is 1800 $\times$500 J = 900,000 J.

Assessment

For this sub-unit, you may use the following assessments:

- Ask students to resolve some vectors into their components and check their work.
- Ask students to determine the length of some vectors.

You can do any of these through activity, assignments or homework, or group work.

Answers to Exercise 8.3

1. a. $\mathbf{u} + \mathbf{v} = (7, 6)$ b. $\mathbf{u} + \mathbf{v} = (6, -2)$
   c. $\mathbf{u} + \mathbf{v} = (0, 1)$ d. $\mathbf{u} + \mathbf{v} = (1, 2)$

2. a. $|\mathbf{u}| = \sqrt{1^2 + 1^2} = \sqrt{2}$ b. $|\mathbf{u}| = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2} = \frac{3}{2}$
   c. $|\mathbf{u}| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$ d. $|\mathbf{u}| = \sqrt{x^2 + y^2}$

3. a. $\mathbf{u}+\mathbf{v} = \left(3+\frac{7}{2}\right)\mathbf{i} + \left(\frac{5}{2}-\frac{1}{4}\right)\mathbf{j} = \frac{13}{2}\mathbf{i} + \frac{9}{4}\mathbf{j}$
   b. $\mathbf{u}-\mathbf{v} = \left(3-\frac{7}{2}\right)\mathbf{i} + \left(\frac{5}{2}+\frac{1}{4}\right)\mathbf{j} = \frac{-1}{2}\mathbf{i} + \frac{11}{4}\mathbf{j}$
   c. $t\mathbf{u} = 3t\mathbf{i} + \frac{5}{2}t\mathbf{j},\ t \in \mathbb{R}.$
   d. $2\mathbf{u}-\mathbf{v} = \left(6-\frac{7}{2}\right)\mathbf{i} + \left(5+\frac{1}{4}\right)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{21}{4}\mathbf{j}$

4. a. $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ or $\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

   b. $\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$ or $\left(\frac{-\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}\right)$

   c. $\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$ is a unit vector in the same direction and

   $\left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right)$ is a unit vector in the opposite direction to the vector (x, y)

5. The coordinate form of the zero vector is (0, 0).
   Let $\mathbf{u} = (x, y)$ for arbitrary x and y, and $\mathbf{0} = (0, 0)$
   $\mathbf{u} + \mathbf{0} = (x, y) + (0, 0) = (x + 0, y + 0) = (x, y) = \mathbf{u}$

8.3 SCALAR (DOT) PRODUCT OF VECTORS

Period allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to:

- *find the scalar product (dot product) of two vectors.*
- *describe some properties of scalar product of vectors.*

Vocabulary: Scalar (dot) product

Introduction

Cognizant of the fact that students have revised and understood the previous lessons, in this subunit, they will discuss scalar product of vectors, which is sometimes called **inner product** or **dot product**. They will also discuss some applications of the dot product in determining angles between two vectors, or length of a vector given length of the other vector and the included angle.

Teaching Notes

In order to start this subunit, you may group the students and let them discuss Group Work 8.3. Following their discussion, you can select one group to present its work so that the whole class will discuss the ideas behind the group work.

Answers to Group Work 8.3

1. Discussion

2. $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) \text{ where } 0 \le \theta \le \pi$$

Then discuss.

Following their discussion, you may proceed by stating the definition of scalar product as:

i. $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ for vectors **a** and **b** and angle θ between them and

ii. $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$

After discussing these definitions, you may need to discuss some of the properties of scalar product of vectors as listed in the student text.

You also need to explain to the students that, for purposes of computation, it is desirable to have a formula that expresses the dot product of two vectors in terms of the components of the vectors that lead to what is called cosine law, which is helpful in application.

You should stress at this stage on the use of dot product to determine the angle between two vectors and, through this, on orthogonality of vectors. You may also assist students to observe the application of the concept of vector algebra in calculating work done, the angle between two vectors and its application to real situation. At this point, you may form group of students so that they can refer to physics texts and consolidate more applications of scalar products. They can also find other real life problems that require application of scalar product of vectors. Some examples are outlined in the student textbook.

Finally, you may group students and let them do each of the questions in Exercise 8.4 as a group work. But, for high achievers you can give questions of type:

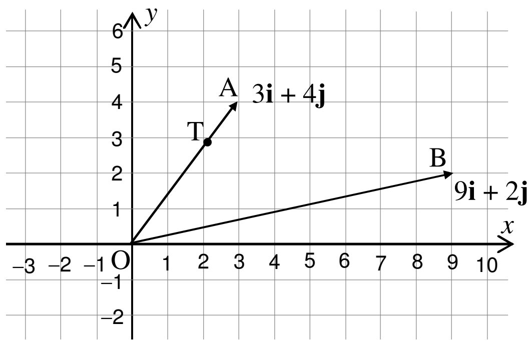


Figure 8.10

If T is a point on the line OA that is closest to B, find the position vector of OT.

Solution:

$$\overrightarrow{OA}.\overrightarrow{TB} = 0$$

Let $\overrightarrow{OT} = k\overrightarrow{AB}$. Then BT = $(9 - 3k)\mathbf{i} + (2 - 4k)\mathbf{j}$

$$\overrightarrow{OA}.\overrightarrow{TB} = 0 \Rightarrow 27 - 9k + 8 - 16k = 0 \Rightarrow k = \frac{35}{25} = \frac{7}{5}$$

$$\Rightarrow \overrightarrow{OT} = \frac{21}{5}\mathbf{i} + \frac{28}{5}\mathbf{j}$$

Assessment

For this sub-unit, you may use the following assessments:

- Ask students to state the possible scalar products of vectors,
- Give exercise problems on scalar product of vectors
- Give exercise problems on application of scalar products

You may perform these assessments through activity, or you may give the students assignments or homework, or group work on the basis of which they can practice more to relate scalar products with their daily life.

Answers to Exercises 8.4

1. a. $\mathbf{z} = (8, 7)$ and $\mathbf{z}' = (-2, 8)$

   b. $\mathbf{z} = \left(\frac{35}{3}, \frac{31}{10}\right)$ and $\mathbf{z}' = \left(\frac{58}{15}, \frac{29}{15}\right)$

2. a. -6 b. -7 c. 13 d. $2\sqrt{7}$

3. a. $(\mathbf{u}+\mathbf{v})^2 = (\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v}) = \mathbf{u}.(\mathbf{u}+\mathbf{v}) + \mathbf{v}.(\mathbf{u}+\mathbf{v}) = \mathbf{u}.\mathbf{u} + \mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{u} + \mathbf{v}.\mathbf{v}$

   $= \mathbf{u}^2 + \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{v} + \mathbf{v}^2$

   $= \mathbf{u}^2 + 2\mathbf{u}.\mathbf{v} + \mathbf{v}^2$

   b. $(\mathbf{u}-\mathbf{v})^2 = (\mathbf{u}-\mathbf{v}).(\mathbf{u}-\mathbf{v}) = \mathbf{u}.(\mathbf{u}-\mathbf{v}) - \mathbf{v}.(\mathbf{u}-\mathbf{v}) = \mathbf{u}.\mathbf{u} - \mathbf{u}.\mathbf{v} - \mathbf{v}.\mathbf{u} + \mathbf{v}.\mathbf{v}$

   $= \mathbf{u}^2 - \mathbf{u}.\mathbf{v} - \mathbf{u}.\mathbf{v} + \mathbf{v}^2$

   $= \mathbf{u}^2 - 2\mathbf{u}.\mathbf{v} + \mathbf{v}^2$

   c. $(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v}) = \mathbf{u}.(\mathbf{u}-\mathbf{v}) + \mathbf{v}.(\mathbf{u}-\mathbf{v}) = \mathbf{u}.\mathbf{u} - \mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{u} - \mathbf{v}.\mathbf{v}$

   $= \mathbf{u}.\mathbf{u} - \mathbf{v}.\mathbf{v}$

   $= \mathbf{u}^2 - \mathbf{v}^2$

   d. $(\mathbf{u}+\mathbf{v}).(\mathbf{w}+\mathbf{z}) = \mathbf{u}.(\mathbf{w}+\mathbf{z}) + \mathbf{v}.(\mathbf{w}+\mathbf{z}) = \mathbf{u}.\mathbf{w} + \mathbf{u}.\mathbf{z} + \mathbf{v}.\mathbf{w} + \mathbf{v}.\mathbf{z}$

4. The angle θ between **u** and **v** is calculated as follows.

From scalar product, $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$ which is the same as $\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}=\cos\theta$

a. $\cos\theta=\frac{(1,-1).(1,1)}{\sqrt{2}\times\sqrt{2}}=0$

b. $\frac{(1,1).(-2,3)}{\sqrt{2}\sqrt{13}}=\frac{1}{\sqrt{26}}$

c. $\cos\theta=\frac{(1,-1).(-2,3)}{\sqrt{2}\sqrt{13}}=\frac{-5}{\sqrt{26}}$

5. Since **u** . **v** = 0 for all vectors **v**, it is true for **v** = **u.** That is, **u**. **u** = **0** $\Rightarrow$ **u** = **0**
6. Here we have to show both directions i.e, the "if part" and the "only if" part.

Suppose **u** + **v** and **u** – **v** are perpendicular. We then need to show that $|\mathbf{u}|=|\mathbf{v}|$.

Since **u** + **v** and **u** – **v** are perpendicular, the angle between them is 90°.

Hence $0=\cos 90=\frac{(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v})}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}$. From this,

$\frac{\mathbf{u}.\mathbf{u}-\mathbf{v}.\mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=0\Rightarrow\frac{|\mathbf{u}|^2-|\mathbf{v}|^2}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=0$ this further implies

$|\mathbf{u}|^2-|\mathbf{v}|^2=0\Rightarrow|\mathbf{u}|^2=|\mathbf{v}|^2\Rightarrow|\mathbf{u}|=|\mathbf{v}|$. To show the backward direction, suppose $|\mathbf{u}|=|\mathbf{v}|$.

Then, the angle between **u** + **v**, and **u** – **v** is

$\cos\theta=\frac{(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v})}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=\frac{\mathbf{u}.\mathbf{u}-\mathbf{v}.\mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=\frac{|\mathbf{u}|^2-|\mathbf{v}|^2}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}$ but $|\mathbf{u}|=|\mathbf{v}|$ by assumption.

Thus, $\cos\theta=\frac{(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v})}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=\frac{\mathbf{u}.\mathbf{u}-\mathbf{v}.\mathbf{v}}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=\frac{|\mathbf{u}|^2-|\mathbf{v}|^2}{|\mathbf{u}+\mathbf{v}||\mathbf{u}-\mathbf{v}|}=0$

Therefore, the vectors **u** + **v** and **u** – **v** are perpendicular.

7. If **u** = **0** and **v** = **0** then the inequality reduces to $0\geq 0$ which is true.

Suppose $\mathbf{v}\neq\mathbf{0}$. For any real numbers s and t, we have

$0\leq|s\mathbf{u}+t\mathbf{v}|^2=(s\mathbf{u}+t\mathbf{v}).(s\mathbf{u}+t\mathbf{v})$

$= s^2\mathbf{u}.\mathbf{u}+2st\mathbf{u}.\mathbf{v}+t^2\mathbf{v}.\mathbf{v}$

Now, take $s=\mathbf{v}.\mathbf{v}=|\mathbf{v}|^2$, and $t=-\mathbf{u}.\mathbf{v}$. Thus, the above inequality gives

$$|\mathbf{v}|^4|\mathbf{u}|^2-2|\mathbf{v}|^2(\mathbf{u}.\mathbf{v})^2+(\mathbf{u}.\mathbf{v})^2|\mathbf{v}|^2=|\mathbf{v}|^2\left[|\mathbf{v}|^2|\mathbf{u}|^2-2(\mathbf{u}.\mathbf{v})^2+(\mathbf{u}.\mathbf{v})^2\right]$$

$$=|\mathbf{v}|^2\left[|\mathbf{v}|^2|\mathbf{u}|^2-(\mathbf{u}.\mathbf{v})^2\right]$$. Since $\mathbf{v}\neq 0$, $|\mathbf{v}|^2>0$.

Thus $0 \le |\mathbf{v}|^2 |\mathbf{u}|^2 - (\mathbf{u}.\mathbf{v})^2 \Rightarrow (\mathbf{u}.\mathbf{v})^2 \le |\mathbf{v}|^2 |\mathbf{u}|^2$

Hence $(\mathbf{u}.\mathbf{v})^2 \le (\mathbf{u}.\mathbf{u})(\mathbf{v}.\mathbf{v})$ because $\mathbf{u}.\mathbf{u} = |\mathbf{u}|^2$ and $\mathbf{v}.\mathbf{v} = |\mathbf{v}|^2$

Equality holds if **v** is a scalar multiple of **u** or **u** is a scalar multiple of **v**. (That is, **u** and **v** are linearly dependent)

8. a. Suppose that $\mathbf{u}.\mathbf{v} = 0$

Then $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v}).(\mathbf{u}+\mathbf{v}) = \mathbf{u}.\mathbf{u} + 2\mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{v} = |\mathbf{u}|^2 + |\mathbf{v}|^2$ since $\mathbf{u}\cdot\mathbf{v} = 0$

Conversely, suppose $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$

Then, $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v}) = \mathbf{u}.\mathbf{u} + 2\mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{v} = |\mathbf{u}|^2 + 2\mathbf{u}.\mathbf{v} + |\mathbf{v}|^2$

$$= |\mathbf{u}+\mathbf{v}|^2 + 2\mathbf{u}.\mathbf{v}$$

Thus, $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}+\mathbf{v}|^2 + 2\mathbf{u}.\mathbf{v} \Rightarrow 2\mathbf{u}.\mathbf{v} = 0 \Rightarrow \mathbf{u}.\mathbf{v} = 0$

b. The generalized Pythagoras theorem

9. a $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v})(\mathbf{u}+\mathbf{v}) = \mathbf{u}.\mathbf{u} + 2\mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{v} = |\mathbf{u}|^2 + 2\mathbf{u}.\mathbf{v} + |\mathbf{v}|^2$

$$= 3+2(|\mathbf{u}||\mathbf{v}|\cos\tfrac{\pi}{6})+1 = 3+2(\sqrt{3}\cdot 1\cdot\frac{\sqrt{3}}{2})+1 = 4 + 3 = 7$$

Therefore, $|\mathbf{u}+\mathbf{v}|^2 = 7 \Rightarrow |\mathbf{u}+\mathbf{v}| = \sqrt{7}$

b. $|\mathbf{u}-\mathbf{v}|^2 = (\mathbf{u}-\mathbf{v})(\mathbf{u}-\mathbf{v}) = \mathbf{u}.\mathbf{u} - 2\mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{v} = |\mathbf{u}|^2 - 2\mathbf{u}.\mathbf{v} + |\mathbf{v}|^2$

$$= 3-2(|\mathbf{u}||\mathbf{v}|\cos\frac{\pi}{6}) + 1$$

$$= 3-2(\sqrt{3}\cdot 1\cdot\frac{\sqrt{3}}{2}) + 1 = 4 - 3 = 1$$

Therefore, $|\mathbf{u}-\mathbf{v}|^2 = 1 \Rightarrow |\mathbf{u}-\mathbf{v}| = 1$

10. a $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u}.\mathbf{v} + |\mathbf{v}|^2 \Rightarrow 2\mathbf{u}.\mathbf{v} = |\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2$

$$\Rightarrow \mathbf{u}.\mathbf{v} = \frac{1}{2}\left(|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2\right) = \frac{1}{2}(576-169-361) = \frac{1}{2}(46) = 23$$

b. $|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 - 2\mathbf{u}.\mathbf{v} + |\mathbf{v}|^2 \Rightarrow |\mathbf{u}-\mathbf{v}|^2 = 169–46+361 = 484$

Thus, $|\mathbf{u}-\mathbf{v}| = \sqrt{484} = 22$

c. $|3\mathbf{u}+4\mathbf{v}|^2 = (3\mathbf{u}+4\mathbf{v})(3\mathbf{u}+4\mathbf{v}) = 9|\mathbf{u}|^2 + 24\mathbf{u}.\mathbf{v} + 16|\mathbf{v}|^2$

$$= 9(169) + 24(23) + 16(361) = 7849$$

Thus, $|3\mathbf{u}+4\mathbf{v}| = \sqrt{7849} \approx 88.6$.

8.4 APPLICATION OF VECTORS

Period allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *apply vectors to solve problems on geometry, algebra, mechanics and other. related problems.*
- *write the parametric equation of a line.*
- *write equation of a circle by applying vectors.*
- *determine the equation of the tangent line to a circle using vectors.*

Vocabulary: Application of vectors, Concurrence, Tangent,

Introduction

This subunit is devoted to the application of vectors that include verifying Pythagoras theorem and some of the properties related to concurrence in triangles of the perpendicular bisectors of the sides, altitudes, and angle bisectors.

Teaching Notes

In order to start this lesson, you may need to help students recall the discussions on scalar product of vectors which is useful for discussing their application. After reviewing scalar products, you may proceed to discussing the proof of some theorems from geometry using vector algebra with active involvement of the students.

When you ensure that students have captured some of the important applications of scalar product, you may need to proceed to relating vectors and lines. Here, with students' active participation, you can verify the geometric properties of concurrence of bisectors of angles and sides of a triangle and that of Pythagoras theorem. You then discuss parametric vector equation of a line through different examples and exercises.

When the students become capable of expressing parametric vector equation of a line, you may need to proceed to further application of vectors among which is discussing the use of vectors in writing equations of circles. You may assist students in writing equations of different circles through different examples.

Finally, you may also need to help students to identify and characterize a tangent line to a circle and to write the equation of a tangent line to a given circle through examples and exercises. You may group the students to do some more practical application of vectors from problems related to, say, physics some examples of which are given in the student textbook. This will help you assess their understanding of vectors. You can also give them Exercise 8.5 as an assignment. You can also give for high achievers questions of type:

Three forces, $\mathbf{F_1} = \mathbf{i} + \mathbf{j}$, $\mathbf{F_2} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{F}_3 = 2\mathbf{i} + 3\mathbf{j}$ act on a particle which moves from point A to point B and then to point C. A, B and C have position vectors $\mathbf{i} + 2\mathbf{j}$, $-\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + \mathbf{j}$ respectively.

Find the work done by

a. each force in moving from A to B.

b. each force in moving from B to C.

c. the resultant force in moving from A to B.

d. the resultant force in moving from B to C.

Solution:

$\overrightarrow{AB} = -\mathbf{i} - \mathbf{j} - (\mathbf{i} + 2\mathbf{j}) = -2\mathbf{i} - 3\mathbf{j}$ $\quad \overrightarrow{BC} = (2\mathbf{i} + \mathbf{j}) - (-\mathbf{i} - \mathbf{j}) = 3\mathbf{i} + 2\mathbf{j}$

a. $\mathbf{F}_1 . \overrightarrow{AB} = (\mathbf{i} + \mathbf{j}).(2\mathbf{i} - 3\mathbf{j}) = -5\text{J}$

$\mathbf{F}_2 . \overrightarrow{AB} = (3\mathbf{i} - 4\mathbf{j}).(-2\mathbf{i} - 3\mathbf{j}) = 6 \text{ J}$

$\mathbf{F}_3 . \overrightarrow{AB} = (2\mathbf{i} + 3\mathbf{j}).(-2\mathbf{i} - 3\mathbf{j}) = -13 \text{ J}$

b. $\mathbf{F}_1 . \overrightarrow{BC} = (\mathbf{i} + \mathbf{j}).(3\mathbf{i} + 2\mathbf{j}) = 5 \text{ J}$

$\mathbf{F}_2 . \overrightarrow{BC} = (3\mathbf{i} - 4\mathbf{j}).(3\mathbf{i} + 2\mathbf{j}) = 1 \text{ J}$

$\text{F}_3 . \overrightarrow{BC} = (2\mathbf{i} + 3\mathbf{j}).(3\mathbf{i} + 2\mathbf{j}) = 12 \text{ J}$

c. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 6\mathbf{i}$

$(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3). \overrightarrow{AB} = 6\mathbf{i}\ (-4 - 3\mathbf{j}) = -12 \text{ J}$

d. $6\mathbf{i}.(3\mathbf{i} + 2\mathbf{j}) = 18 \text{ J}$

Assessment

You can assess the understanding of students about the points discussed in this sub-unit by using different mechanisms some of which may be:

- giving exercise problems on the application of vector algebra.
- asking students to write the parametric equation of a line.
- giving exercise problems on writing equations of tangent to a given circle and check their work.

You may do these assessments through activity, or you may give to the students' assignments or homework, or group work on the basis of which they can practice more on applications of vectors that include problems from their daily life.

Answers to Exercises 8.5

1. a. The vector equation of the line is

$P = (-2, 1) + \lambda(-1, 1), \lambda \in \mathbb{R}$

$= (-2-\lambda, 1+\lambda)$

The parametric equations are

$x = -2 - \lambda$

$y = 1 + \lambda, \lambda \in \mathbb{R}$

and the standard equations are

$$\frac{x+2}{-1} = \frac{y-1}{1}$$

b. The vector equation of the line is:

$(x, y) = (1, 1) + \lambda(2, 2), \lambda \in \mathbb{R}$

The parametric equations are

$x = 1 + 2\lambda$

$y = 1 + 2\lambda, \lambda \in \mathbb{R}$ and the standard equations are

$$\frac{x-1}{2} = \frac{y-1}{2}$$

2. Let $P(x, y)$ be a point on the circle

Let $\overline{P}$ and $\overline{C}$ be the position vectors of P and C, respectively

Thus, equation of the circle is

$$(x, y)\cdot(x, y) - 2(x, y)\cdot(1, -2) + (1, -2)\cdot(1, -2) = \left(\frac{3}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 - 2(x - 2y) + 1 + 4 = \frac{9}{4}$$

$$\Rightarrow x^2 + y^2 - 2x + 4y + \frac{11}{4} = 0$$

3. a. The center of the circle is (1, 2). Thus, the equation of the tangent line is:

$(x-1)(1-1) + (y-2)(4-2) = 4$

$\Rightarrow 2y = 8$

$\Rightarrow y = 4$

b. The center of the circle is: (–2, 0).

The equation of the tangent line at $\left(-1, \sqrt{2}\right)$ is:

$(x+2)(-1+2) + (y-0)\left(\sqrt{2}-0\right) = 3$

$\Rightarrow (x+2) + \sqrt{2}\,y = 3 \Rightarrow x + \sqrt{2}\,y - 1 = 0$

4. a. they are not collinear

b. they are collinear

5. A line that passes through (3, 5) and (–2, 3) is the same as the line that passes through one of these points and parallel to the vector (–2–3, 3–5) = (–5, –2). The vector equation of the line passing through (3, 5) and parallel to (–5,–2) is

$$P = (3, 5) + \lambda(-5, -2),\ \lambda \in \mathbb{R}$$
$$= (3 - 5\lambda, 5 - 2\lambda)$$

The parametric equation is

$$x = 3 - 5\lambda$$
$$y = 5 - 2\lambda,\ \lambda \in \mathbb{R}$$

and the standard equation is

$$\frac{x-3}{-5} = \frac{y-5}{-2} \Rightarrow 5y - 2x = 19$$

6. a. The point is said to lie on the circle if it satisfies the equation $x^2 + y^2 - 2x - 4y - 9 = 0$. Here $1^2 + 4^2 - 2(1) - 4(4) - 9 = -10$. This is not 0.
Hence, we say the point does not lie on the circle and therefore, there is no tangent line at the point (1, 4)

b. $(x+2)^2 + y^2 = 3$ at $P_1(-1, \sqrt{2})$ gives $(-1+2)^2 + \sqrt{2}^2 = 3$. Thus, the point satisfies the equation and hence it lies on the circle. The circle has its center at (–2, 0) and radius $r = \sqrt{3}$. Thus, the equation of the tangent line is:

$$(x + 1)(-1-(-2)) + (y - \sqrt{2})(\sqrt{2} - 0) = 0$$
$$\Rightarrow x + 1 + \sqrt{2}y - 2 = 0$$
$$\Rightarrow x + \sqrt{2}y - 1 = 0$$
$$\Rightarrow y = \frac{1}{\sqrt{2}}(-x+1)$$ is the equation of the tangent line to the circle at $P_1(-1, \sqrt{2})$

7. Let O be the fixed origin of the given vectors since $\mathbf{v} - \mathbf{u} = \overrightarrow{AB}$ and $\mathbf{w} - \mathbf{z} = \overrightarrow{DC}$ are parallel and equal. Again, since $\mathbf{v} - \mathbf{u} = \mathbf{w} - \mathbf{z}$, i.e.
$\mathbf{w} - \mathbf{v} = \mathbf{z} - \mathbf{u}$
$\Rightarrow \overrightarrow{BC} = \overrightarrow{AD}$.
Thus, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel and equal.
Hence, ABCD is a parallelogram.

8. Using the definition of the dot product, the solutions for the problems are given below.

a. $\mathbf{F}_1 . \mathbf{F}_2 = |\mathbf{F}_1||\mathbf{F}_2|\cos\theta$
$= 4 \times 10 \times \cos 60 = 20$

b. $\mathbf{F}_5 . \mathbf{F}_6 = |\mathbf{F}_5||\mathbf{F}_6|\cos\theta$
$= 8 \times 4 \cos 45^\circ = 16\sqrt{2}$

c. $(\mathbf{F}_1 + \mathbf{F}_2 - \mathbf{F}_3).(\mathbf{F}_4 + \mathbf{F}_5 - \mathbf{F}_6) = (\mathbf{F}_1.\mathbf{F}_4 + \mathbf{F}_1.\mathbf{F}_5 - \mathbf{F}_1.\mathbf{F}_6) + (\mathbf{F}_2.\mathbf{F}_4 + \mathbf{F}_2.\mathbf{F}_5 - \mathbf{F}_2.\mathbf{F}_6)$
$+ (-\mathbf{F}_3.F_4 - \mathbf{F}_3.F_5 + \mathbf{F}_3.\mathbf{F}_6)$

$\Rightarrow$ $\mathbf{F}_1.\mathbf{F}_4 + \mathbf{F}_1.\mathbf{F}_5 - \mathbf{F}_1.\mathbf{F}_6 = 4 \times 6 \cos 150^\circ + 4 \times 8 \times \cos 90^\circ - 4 \times 4 \times \cos 45^\circ$

$$= 24\left(-\frac{\sqrt{3}}{2}\right) + 0 - 8\sqrt{2} = -12\sqrt{3} - 8\sqrt{2}$$

$\mathbf{F}_2.\mathbf{F}_4 + \mathbf{F}_2.\mathbf{F}_5 - \mathbf{F}_2.\mathbf{F}_6 = 10 \times 6 \cos 150^\circ + 10 \times 8 \times \cos 150^\circ - 10 \times 4 \cos 105^\circ$

$$= 60\left(\frac{-\sqrt{3}}{2}\right) + 80\left(\frac{-\sqrt{3}}{2}\right) - 49\left(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right)$$

$$= -70\sqrt{3} - 10\sqrt{2} + 10\sqrt{6}$$

$-\mathbf{F}_3.\mathbf{F}_4 - \mathbf{F}_3.\mathbf{F}_5 + \mathbf{F}_3.\mathbf{F}_6 = -10 \times 6 \cos 120^\circ - 10 \times 8 \cos 180^\circ + 10 \times 4 \cos 135^\circ$

$$= -60\left(-\frac{1}{2}\right) - 80(-1) + 40\left(\frac{-\sqrt{2}}{2}\right)$$

$$= 110 - 20\sqrt{2}$$

$$\Rightarrow (\mathbf{F}_1 + \mathbf{F}_2 - \mathbf{F}_3).(\mathbf{F}_4 + \mathbf{F}_5 - \mathbf{F}_6) = (-12\sqrt{3} - 8\sqrt{2}) + (-70\sqrt{3} - 10\sqrt{2} + 10\sqrt{6}) + (110 - 20\sqrt{2})$$

$$= 110 - 82\sqrt{3} - 38\sqrt{2} + 40\sqrt{6}$$

9. a. The unit vector in the direction of **a** + **b** is

$$\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|} = \frac{(3\mathbf{i}+\mathbf{j})+(2\mathbf{i}-2\mathbf{j})}{|(3\mathbf{i}+\mathbf{j})+(2\mathbf{i}-2\mathbf{j})|}$$

$$= \frac{5\mathbf{i}-\mathbf{j}}{\sqrt{26}}$$

$$= \frac{5}{26}\mathbf{i} - \frac{1}{\sqrt{26}}\mathbf{j}$$

b. $2\mathbf{a}+\mathbf{b}-\frac{3}{2}\mathbf{c} = 2(3\mathbf{i}+\mathbf{j}) + (2\mathbf{i}-2\mathbf{j}) - \frac{3}{2}(\mathbf{i}+3\mathbf{j})$

$$= 6.5\mathbf{i} - 4.5\mathbf{j}$$

$$\Rightarrow \left|2\mathbf{a}+\mathbf{b}-\frac{3}{2}\mathbf{c}\right| = \sqrt{(6.5)^2 + (-4.5)^2}$$

$$= 2.5\sqrt{10}$$

Therefore, the unit vector in the direction of $2\mathbf{a}+\mathbf{b}-\frac{3}{2}\mathbf{c}$ is

$$\frac{6.5}{2.5\sqrt{10}}\mathbf{i} - \frac{4.5}{2.5\sqrt{10}}\mathbf{j} = \frac{13\sqrt{10}}{50}\mathbf{i} - \frac{9\sqrt{10}}{50}\mathbf{j}$$

10. The displacement AB = (3**i** + 4**j**) – (**i** – 2**j**) = 2**i** + 6**j**.
The total work done,

w = ($\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$). **AB**
= ((2**i** + 3**j**) + (**i** + 2**j**) + (3**i** – **j**)).(2**i** + 6**j**)
= (6**i** + 4**j**).(2**i** + 6**j**) = 36J

8.5 TRANSFORMATION OF THE PLANE

Period allotted: 8 periods

Competencies

At the end of this sub-unit, students will be able to:

- *explain what is meant by transformation of the plane.*
- *describe the main properties of rigid motion.*
- *translate points, lines and circles using vectors.*
- *reflect points, lines, circles and some other plane figures.*
- *determine the images of a given plane figure when rotated through a certain angle.*

Vocabulary: Transformation, Translation, Reflection and Rotation

Introduction

Transformations are of practical importance in daily life problems. These can be managed in different forms: those that maintain direction and those that change direction. In this subunit, attention will be focused on discussing the most common types of transformations, namely, translations, reflections and rotations. It will be important to facilitate learning of the students by encouraging them to deal with problems from their surrounding which will help them realize the ideas of the concepts of transformations.

Teaching Notes

In order to start the lesson, you can group the students and let them do Group Work 8.4 which will help them distinguish shape and size changes, change of positions with or without changing shape and size. By selecting some group members, encourage them to present their work to the whole class. Finally, you can organize and guide them into the discussions of transformations and rigid motions which will be helpful to discuss translations, reflections and rotations.

Answers to Group Work 8.4

1. a. Shape and size changes b. No change
   c. No change d. No change
   e. Only size changes
2. A' = (1, –1) B' = (–2, –2) C' = (3, 0)

   The image of ΔABC is $\Delta A'B'C'$ and $\Delta ABC \cong \Delta A'B'C'$ by SSS

3. $AB = \sqrt{58}$

   a. A' = (2, 0); B' = (5, 0); AB ≠ A' B'

   b. A' =(2, 3); B' = (5, –4); AB = A' B'

   c. A' = (3, –6); B' = (6, 1); A' B' = AB

   d. A' = (1, –6); $B' = \left(\frac{5}{2}, 8\right)$; AB ≠A' B'

4. Discuss some rigid and some non-rigid motions among those listed by students.

After students discuss the group work, you can assign them in pairs and ask them if they can react to the terminologies of transformations, translation, reflection and rotation and what each roughly means. Following discussions, you may proceed to defining transformation of the plane and rigid motion as a special type of transformation. And then, you may proceed with the help of several examples to discussing the main properties of rigid motion, (a motion which preserves distance).

You may state the types of transformation, translation, reflection and rotation and proceed to discussing them one by one. When starting with translation, you may discuss what it means and its effects on the coordinate system. You may need to assist students to translate points, lines and circles with sufficient examples. At this level, for the purpose of assessing students' understanding of the concept of translation, you can give them Exercise 8.6 as an assignment.

Answers to Exercises 8.6

1. The image of the points A (3, 1), B (5, 1), C (5, 4) and D (3, 4) are

   T(3, 1) = (3 + (–3), 1 + 2) = (0, 3)

   T(5, 1) = (5 + (–3), 1 + 2) = (2, 3)

   T(5, 4) = (5 + (–3), 4 + 2) = (2, 6)

   T(3, 4) = (3 + (–3), 4 + 2) = (0, 6)

   Thus, the image of the rectangle is the rectangle whose vertices are A'(0, 3), B'(2, 3), C'(2, 6) and D'(0, 6).

2. We have A' = T(2, 1) = (2+4, 1+3) = (6, 4)

   B' = T(3, 5) = (3+4, 5+3) = (7, 8)

   C' = T(–1, –2) = (–1+4, –2+3) = (3, 1)

3. We have A' = T(1, 2) = (1+3, 2+(–2)) = (4, 0)

   B' = T(3, 4) = (3+3, 4+(–2)) = (6, 2)

   C' = T(7, 4) = (7+3, 4+(–2)) = (10, 2)

   D' = T(2, 5) = (2+3, 5+(–2)) = (5, 3)

4. Circle

5. The translation vector is (–4 – 1, 3 + 1) = (–5, 4). Thus, the point (x, y) is translated to the point $(x – 5, y + 4)$. This implies, the image of the circle is

   $$((\boldsymbol{x}+5)+1)^2+(\boldsymbol{y}-4-3)^2=5 \Rightarrow (x+6)^2+(y-7)^2=5$$

6. i. S(T(0, 0)) = S(3, –2) = (3+(–2), –2+(–1)) = (1, –3)

   ii. T(S(0, 0)) = T(–2, –1) = (–2+3, –1+(–2)) = (1, –3)

7. ℓ': $2(x + 4) – 3(y – 6) = 7 \Rightarrow \ell'$: $2x – 3y = –19$.

   Both ℓ and ℓ' have the same slope $\frac{2}{3}$.

1. The transition vector is **a** = (4, –5).

   a. ℓ' : $y = 3x + b$, from slope

   Take A = (0, 7) on ℓ: $y = 3x + 7$, then

   A' = (4, 2) lies on ℓ'

   $\Rightarrow 2 = 3\ (4) + b$

   $\Rightarrow b = –10$

   $\Rightarrow \ell'$: $y = 3x – 10$

   b. The transition vector is parallel to the line $4y + 5x = 10$

   $\Rightarrow \ell = \ell'$

9. The transition vector is

   a = (7, 10) – (3, –2)

   = (4, 12)

   a. The image of the ellipse is

   $$4(x-4)^2+3(y-12)^2-2(x-4)+6(y-12)=0$$

   $$\Rightarrow 4x^2+3y^2-34x-66y+432=0$$

   b. The image of the parabola is

   $(y – 12)^2 = 4\ (x – 4)$

c. The image of the hyperbola is $(x - 4)\ (y - 12) = 1$
$\Rightarrow xy - 12x - 4y + 47 = 0$

d. The translation shifts the graph of the function f 4 units in the positive x direction and 12 units in the positive y direction.$\Rightarrow$The image of the function is $f\,(x - 4) -12 = x^3 - 15x^2 + 72x - 96$.

When you make sure that your students have captured translation and its properties and its effect on the coordinate system, you can proceed to discussing reflection. You may start the discussion by asking students to express their ideas about reflection, for example, while they use plane mirrors. Following their expressions of ideas about reflection, you may proceed to stating what reflection is and discuss its effect on the coordinate plane. Before discussing the main points, however, you may group your students and let them do Activity 8.4 so that they can determine reflection points and characterize reflected images. Here, essential questions on reflection need to be raised and discussed, questions such as size of the reflected image, distance between an image and its reflected image from the reflection line, etc. You then need to assist students in reflecting points, lines, circles and some other plane figures along a given line through examples and exercises.

You may also need to help students to relate the concept of reflection with problems from daily life through examples and exercises.

Answers to Activity 8.4

1. Ask some students to demonstrate the images on the black board.
2. The perpendicular bisection of $\overline{AA'}$ is the axis of symmetry.
3. $x = 3$

Reflections about different lines, say the x and y axes, the line $y = mx$, and $y = mx + b$ need to be discussed. For the purpose of starting this discussion you can form pairs of students and let them do Activity 8.5 whose purpose is to conduct discussions, share ideas and reach their own understanding. Afterwards, select some pair of students to forward their work in a presentation form and resume discussion on their presentation. You then guide and assist students discuss their answers and present to them as many examples as possible for consolidating their understanding. You also help them arrive at the mathematical rule for determining reflection by use of trigonometric identities as outlined in the student textbook. They also need to identify some particular properties of reflection.

Answers to Activity 8.5

1. The image of $f(x) = e^{x}$ after refection
   a. in the y-axis is $f(x) = e^{-x} = f(-x)$
   b. in the x-axis is $f(x) = -e^{x} = -f(x)$
   c. in the line $y = x$ is $f(x) = \ln x = f^{-1}(x)$

2. a. $(a, b) \rightarrow (a, -b)$ b. $(a, b) \rightarrow (-a, b)$
   c. $(a, b) \rightarrow (b, a)$ d. $(a, b) \rightarrow (-b, -a)$

   Therefore, the image of the line is

   a. $y = -mx - b$ b. $y = mx + b$
   c. $my = x - b$ d. $my = x + b$

   The image of the circle is

   a. $(x-h)^2 + (y+k)^2 = r^2$ b. $(x+h)^2 + (y+k)^2 = r^2$
   b. $(x-k)^2 + (y-h)^2 = r^2$ c. $(x+k)^2 + (y+h)^2 = r^2$

When the students have realized the use of reflection formula derived with the help of trigonometric identities, they need to realize other analytical approach to finding reflection whose steps are described on page 329. Guide them do each step one by one so that they can internalize the meaning and achievements through several examples and exercises.

For the purpose of assessing students' understanding, you can group the students and give them questions from Exercise 8.7 as a home work.

Answers to Exercise 8.7

1. a. The images of A (2, 1), B (3, –2) and C (5, –3) after a reflection in the x-axis are A'(2, –1), B'(3, 2) and C'(5, 3), respectively.

   b. The images of A, B and C after a reflection in the y – axis are A'(–2, 1), B'(–3,–2) and C'(–5, –3).

   c. The images of A, B and C after a reflection in the line $y = -x$ are A'(–1, –2), B'(2, –3) and C'(3, –5), respectively.

   d. The images of A, B and C after a reflection in the line $y = x$ are A'(1, 2), B'(–2, 3) and C'(–3, 5), respectively.

2. The slope of ℓ is 1. Thus, the slope of the line through (–4, 3) and perpendicular to ℓ is –1. Thus, its equation is given by $\frac{y-3}{x+4}=-1 \Rightarrow y=-x-1$

   The intersection of $y = x - 2$ and $y = -x - 1$ is M $\left(\frac{1}{2}, -\frac{3}{2}\right)$. Taking M$\left(\frac{1}{2}, -\frac{3}{2}\right)$ as the midpoint of $\overline{PP}$,

   where P'(x', y') is the image of $P(-4, 3)$, we obtain

   $$\frac{-4+x'}{2}=\frac{1}{2} \text{ and } \frac{3+y'}{2}=-\frac{3}{2}$$

   $\Rightarrow x' = 5$ and $y' = -6$

   Hence, the image of P(–4, 3) is P'(5, –6)

3. The slope of the line through the points P(–1, 2) and P'(1, 0) is –1. The mid-point of $\overline{PP'}$ is $\left(\frac{-1+1}{2}, \frac{2+0}{2}\right) = (0, 1)$. Thus, the line of reflection is the line through (0, 1) with slope 1. It's equation is given by $\frac{y-1}{x-0} = 1 \Rightarrow y = x + 1$

4. Every point on $y = x$ is fixed and also every point that is symmetric with respect to the $y = x$ is fixed.

   For example, every line perpendicular to $y = x$, every circle with centre on $y = x$, every isosceles triangle with the vertex angle on $y = x$ are some of those fixed figures.

5. Clearly, point P(0, 4) is on ℓ.

   Let the image of P(0, 4) be P′(a, b) when it is reflected by the line L: $y = x - 3$. Then,

   i. Using slope, $\frac{b-4}{a} = -1$, you have $b = 4 - a$.

   ii. Using the mid-point, $\left(\frac{a}{2}, \frac{b+4}{2}\right)$, $b = a - 10$.

   Solving the equations in i and ii gives $a = 7$ and $b = -3$.

   Therefore, the equation of the image of ℓ is computed as:

$$\ell' : \frac{y+3}{x-7} = 1 \Rightarrow \ell' : y = x - 10.$$

6. Solving the system $\begin{cases} y = 2x+1 \\ y = 3x+2 \end{cases}$

gives $x = -1$

$y = -1$

Therefore M (– 1, – 1) = (–1, –1)

Consider, (0, 1) a point on $y = 2x + 1$.

Let M (0, 1) = (1, b)

$\Rightarrow \frac{b-1}{a} = -\frac{1}{3}$ using the slope

$\Rightarrow \; 3b - 3 = -a$

$\Rightarrow \; a = 3 - 3b$

$\frac{b+1}{2} = 3\left(\frac{a}{2}\right) + 2$ using the midpoint

$b + 1 = 3a + 4$

$b = 3a + 3$

Solving $\begin{cases} a = 3 - 3b \\ b = 3a + 3 \end{cases}$ gives b = 1. 2 and a = –0.6

The image line passes through (–0.6, 1.2)

$2y - 11x = 9$

7. The center of the circle is O (2, 3) and the radius is 5. The equation of the line through (2, 3) and perpendicular to $y = x + 3$ is $y = -x + 5$. The point of intersection of the two lines is M(1, 4). Taking M(1, 4) as a midpoint of $\overline{OO'}$, where O' is the image of O(2, 3) . When reflected in the line $y = x + 3$, we compute the coordinates of O' (x', y') as follows:

$\frac{x'+2}{2} = 1$ and $\frac{y'+3}{2} = 4$

$\Rightarrow x' = 0$ and $y' = 5$

Thus, the new center is (0, 5) and the equation of the image circle is:

$$(x-0)^2 + (y-5)^2 = 5^2$$

$$\Rightarrow x^2 + (y-5)^2 = 25$$

8. The axis of reflection is the perpendicular bisector of the line segment with end points the center of the circles.

$$x^2 + y^2 - x + 2y = 0 \Rightarrow \left(x-\frac{1}{2}\right)^2 + (y+1)^2 = \frac{5}{4}$$

$$x^2 + y^2 - 2x + y = 0 \Rightarrow (x-1)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\text{M}\left(\frac{1}{2},-1\right) = \left(1,\frac{-1}{2}\right).$$

The slope of the line of the centers is $m = \dfrac{-\frac{1}{2}-(-1)}{1-\frac{1}{2}} = 1$, hence ℓ has slope -1.

The midpoint of the segment is $\left(\frac{3}{4},-\frac{3}{4}\right)$. Hence ℓ has an equation $y = -x$.

9. a. The translation vector is $a = (3, -2)$

Let ℓ be the axis of reflection, then,

$$\ell\colon y-2 = \frac{-1}{2}(x-1) \Rightarrow \ell : y = \frac{-1}{2}x+\frac{5}{2}$$

$\Rightarrow$ The axis of reflection is $\ell : y = -\frac{1}{2}x+\frac{5}{2}$

Let M (1, 3) = (a, b), then

Using slope, $\dfrac{b-3}{a-1} = 2 \quad \Rightarrow b = 20+1$

Using the mid-point, $\dfrac{b+3}{2} = \dfrac{-1}{2}\left(\dfrac{a+1}{2}\right)+\dfrac{5}{2} \Rightarrow b = a+\dfrac{5}{2}$

From slope, $\dfrac{b-4}{a-1} = 2 \Rightarrow b = 2a+2$.

From midpoint. $\left(\dfrac{a+4}{21}, \dfrac{b+1}{2}\right)$ lies on the axis of reflection

$$\Rightarrow \frac{b+1}{2} = \frac{-1}{2}\left(\frac{a+4}{2}\right) + \frac{5}{2}$$

$$\Rightarrow \quad b = \frac{-a}{2} + 2$$

Solving $\begin{cases} b = 2a+2 \\ b = \dfrac{-a}{2} + 2 \end{cases}$ gives

$$(a,b) = \left(\frac{18}{5}, \frac{1}{5}\right)$$

$$\therefore M\,(T\,(1,\ 3)) \ = \ (4,1) = \left(\frac{18}{5}, \frac{1}{5}\right)$$

Solving $\begin{cases} b = 2a+1 \\ b = a + \dfrac{5}{2} \end{cases}$ gives (a, b) = $\left(\dfrac{3}{2}, 4\right)$

$\therefore$ T(M(1, 3)) = T $\left(\dfrac{3}{2}, 4\right) = \left(\dfrac{3}{2} + 3, 4 - 2\right) = (4.5, 2)$

b. M (T(1, 3)) = M (4, 1)

10. The line ℓ: $y - 2x = 3$ and its image ℓ': $2y - x = 9$ meet at (1, 5).

Hence, the axis of reflection L passes through (1, 5).

Clearly, L is the bisector of the angle formed by ℓ and ℓ'. So you have two reflecting lines, say L_1 and L_2.

See figure 8.11

Choose A = (0, 3) on ℓ, then M (A) = B or M(A) = D

$\Rightarrow \Delta$AOB and ΔAOD are isosceles triangles.

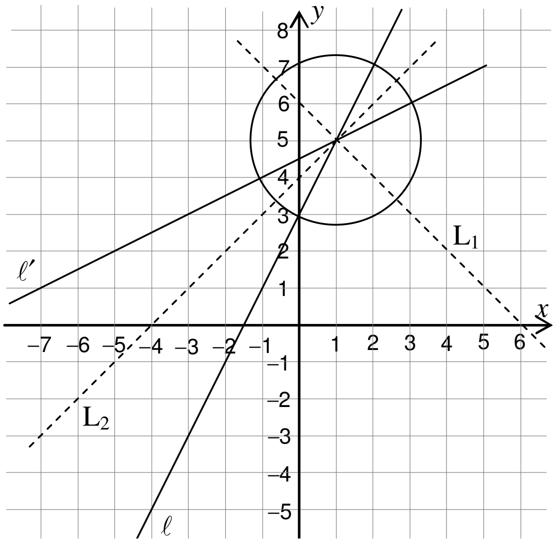


Figure 8.11

The points A, B, C and D lie on the same circle with circle with centre O (1, 5) and radius

OA = $\sqrt{(1-0)^2+(5-3)^2} = \sqrt{5}$

It suffices to determine the intersection points of the circle and the lines ℓ and ℓ'.

The circle has an equation $(x - 1)^2 + (y - 5)^2 = 5$.

After the students have mastered the concepts of translation and reflection, you can proceed to the third type of transformation called rotation. You may start this by forming groups and letting them do Group Work 8.5 which is more of a practical type. This group work will help them realize the concepts of rotations in general and rotation on the coordinate plane in particular.

Answers to Group Work 8.5

1. a. B = (–3, 2), C = (–2, –3), D = (3, –2)

   b. B = $(-y, x)$, C = $(-x, -y)$, D = $(y, -x)$

2. a. A' = (–1, 3) B' = (–7, 3) C' = (0, 10) D' = (–4, 6) E' = (–4, 14) and F' = (–3, 10)

   b. Yes, the origin.

   c. It is a rigid motion because $\mathrm{AB} \cong \mathrm{A'B'}$ and $ABCDEF \cong A'B'C'D'E'F'$.

   d. $-y$ and x

3. To define rotation you need the centre of rotation and the amount of rotation.

When each group tries the work, let some selected group present and demonstrate its work to the whole class. When defining rotation, let the students enquire what might happen on the object with changing value of the angle θ. By way of giving some more activity on conducting rotational movements with active involvement of students and defining the concept of rotation, and with the help of examples, discuss rotation of points through 90°, 180° and through any angle θ about the origin. At this juncture, encourage the students to perform the derivation of the rule for rotation by use of trigonometric identities, which is outlined on page 336 of the student text. While you define rotation, you may need to guide a discussion on the effect of rotation of some plane figures through 90°, 180° clockwise and anti-clockwise directions about the origin and then proceed with rotation through a given angle about the origin. After discussing these, with active participation of students, set up the relation between the coordinates of a point and that of its image.

When you ensure their understanding, you may proceed to assisting students to determine the images of plane figures after rotating through a given angle θ about a given point (a, b), which will enable them to generalize rotation at any angle.

Following their generalization, you may require them to determine rotation when the center of rotation is arbitrary point (x_0, y_0). For this purpose, you can form groups of students and let them do Activity 8.6. Assist them by rounding when they do the activity and finally facilitate discussion on the activity. The purpose of this activity is to generalize the rule they used to determine rotation when the center was the origin (0, 0).

Answers to Activity 8.6

1. The centre of rotation is the intersection point of the perpendicular bisectors of $\overline{AA}$ and $\overline{BB}'$.

2. First, translate the centre of rotation from (3, 2) to (0, 0). Hence the P (2, 0) will be translated to P' (1, –2). The image of P' (1, –2) in rotation about (0, 0) through $\frac{\pi}{4}$ radian is $\left(\frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Finally translate the image of P' by the translation vector (–3, –2) which is $\left(\frac{6+3\sqrt{3}}{2}, \frac{4-\sqrt{2}}{2}\right)$

You can give for high achievers questions of the following type.

1. Find the image of the point P(a, b) when it is reflected in the line $y = mx$.

Answer $\left(\dfrac{a+2mb-am^2}{1+m^2}, \dfrac{bm^2+2am-b}{1+m^2}\right)$

2. M(p, q) in the line $y = mx + b$.

Answer $\left(\dfrac{-pm^2+2mq-mb+p}{1+m^2}, \dfrac{m^2q+2mp+2b-q}{1+m^2}\right)$

3. Find the image of the point P(5, 1) if it is first rotated $\dfrac{\pi}{3}$ radian about the origin and then reflected by the line $y = \sqrt{3}\,x$,

Answer $\left(\dfrac{5}{2}+\dfrac{\sqrt{3}}{2}, \dfrac{5\sqrt{3}}{2}-\dfrac{1}{2}\right)$

Assessment

For assessing students' understanding you may:

- Give exercise problems on translating some points, lines, circles with given translation
- Ask students to reflect points, lines, and some plane figures along given lines
- Give exercise problems on rotating points, lines, and some plane figures through different angles in either direction about a given point.

In order to help you assess understanding of your students, you may use activity, or you may give them assignments or homework, group work or project work on the basis of which they can practice more on transformations (Translation, Reflection and Rotation) and applications of transformations that include problems from their daily life.

Answers to Exercise 8.8

1. The image of the vertices of the rectangle after a rotation about the origin through an angle $\theta = \pi$ are A'(–1, –2), B'(–4, –2), C'(–4, 1), D'(–1, 1)

2. a. If the image of (–3, 4) is P'(x', y'), then

$$x' = 3\cos 90^\circ - 4\sin 90^\circ = -4$$

$$y' = -3\sin 90^\circ - 4\cos 90^\circ = -3$$

Thus, the image of (–3, 4) is (–4, –3)

b. If the image of (–2, 0) is P'(x', y'), then

$$x' = -2\cos(60^o) - 0\times \sin 60^o = -1$$

$$y' = -2\sin 60^o + 0\times\cos 60^o = -\sqrt{3}$$

Hence, the image of (–2, 0) is $\left(-1,\ -\sqrt{3}\right)$

c. We have,

$$x' = 0\times\cos\frac{\pi}{4} + 1\times\sin\ \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$y' = 0\times\sin\frac{\pi}{4}\ - 1\times\cos\ \frac{\pi}{4} = \frac{-\sqrt{2}}{2}$$

Hence, the image of (0, –1) is $\left(\dfrac{\sqrt{2}}{2}, \dfrac{-\sqrt{2}}{2}\right)$

d. We have

$$x' = -1\times\cos\ (30^o) - 2\times\sin(30^o) = \frac{-\sqrt{3}}{2} - 1 = \frac{-2-\sqrt{3}}{2}$$

$$y' = -1\times\sin(30^o)\ + 2\cos(30^o) = \frac{-1}{2} + \sqrt{3} = \frac{-1+2\sqrt{3}}{2}$$

Hence, the image of (–1, 2) is $\left(\dfrac{-2-\sqrt{3}}{2}, \dfrac{-1+2\sqrt{3}}{2}\right)$

3. a. The angle θ satisfies tanθ = $\dfrac{3}{4}$. Thus, $\sin\theta = \dfrac{3}{5}$ and $\cos\theta = \dfrac{4}{5}$.

The points P(1, –1) and Q(5, 2) are on the given line. If image of P(1, –1) under the given rotation is P' (x', y')

$$x' = 1\left(\frac{4}{5}\right) - (-1)\left(\frac{3}{5}\right) = \frac{7}{5}$$

$$y' = 1\left(\frac{3}{5}\right) - 1\left(\frac{4}{5}\right) = \frac{-1}{5}$$

That is, $P'\left(\frac{7}{5}, -\frac{1}{5}\right)$ is the image of P(1, –1)

Similarly, the image of Q(5, 2) is $Q'\left(\frac{14}{5}, \frac{23}{5}\right)$. Thus, the equation of the image line is given passing through P′ and Q′ is given by $7y - 24x + 35 = 0$

b. Using the points P (0, 3) and Q (–1, 5) on the line $2x + y = 3$

You have $P' = \left(\frac{-3\sqrt{2}}{2}, \frac{3}{2}\right)$ and $Q'\left(-\frac{1}{2}, \frac{-5\sqrt{3}}{2}\right)$

Therefore, the equation image line is eh equation of the line through P' and Q', which is $y = \left(1 - \frac{3}{4} \times \sqrt{3}\right)x - \frac{3}{4} + \frac{3}{2}\sqrt{3}$

4. a. The center of the given circle is C(0, 0). If P'(a, b) is the image of C(0, 0), we have, $a = 0$ and $b = 0$.

Thus, the equation of the image circle is $(x')^2 + (y')^2 = 1$, which is the same as the given circle.

b. The center of the circle is (–1, 2). Let (a, b) be the image of (–1, 2). Then,

$$a = -1\cos\left(\frac{\pi}{4}\right) - 2\sin\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2} - 2\left(\frac{\sqrt{2}}{2}\right) = \frac{-2\sqrt{2} - \sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$$

$$b = 1\ \sin - 1\sin\left(\frac{\pi}{4}\right) + 2\cos\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2} + 2\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

Thus, the equation of the image circle is $\left(x + \frac{3\sqrt{2}}{2}\right)^2 + \left(y - \frac{\sqrt{2}}{2}\right)^2 = 9.$

5. Substitute $(x_o, y_o) = (3, 2)$, $\theta = -60^o$ and $(x, y) = (1, 0)$ in the formula in corollary 8.4.

$$x' = x_o + (x - x_o)\cos\theta - (y - y_o)\sin\theta$$

$$= 3 + (1 - 3)\cos(-60^o) - (0 - 2)\sin(-60^o)$$

$$= 3 + (-2)\left(\frac{1}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right) = 2 - \sqrt{3}$$

$y' = y_o + (x - x_o) \sin \theta + (y - y_o) \cos \theta$

$= 2 + (1 - 3) \sin (-60^o) + (0 - 2) \cos (-60^o)$

$= 2 - 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{1}{2}\right) = 1 - \sqrt{3}$

$\Rightarrow$ R $(1, 0) = \left(2-\sqrt{3}, 1-\sqrt{3}\right)$

6. a. In this problem, a rotation R is followed by a reflection M.

   You know that $R_{90^o}(3, 0) = (0, 3)$ and reflection about the line $y = -x$ sends (0, 3) to (–3, 0)

   b. Thus, M (R (3, 0)) = M (0, 3) = (–3, 0).

   Here a reflection M is followed by a rotation R.

   R (M (3, 0)) = R (0, –3) = (3, 0).

   In general,

   M (R (x, y)) = M $(-y, x) = (-x, y)$ and

   R(M (x, y)) = R $(-y, -x) = (x, -y)$

7. The lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are not parallel.

   The intersection point of the perpendicular bisectors of $\overline{AA'}$ and $\overline{BB'}$ is the centre of rotation.

   The equation of the perpendiculars of $\overline{AA'}$ and $\overline{BB'}$ is the centre of rotation.

   The equation of the perpendiculars of $\overline{AA'}$ and $\overline{BB'}$ are $y = x - 1$ and $y = \frac{5}{3}x - 3$ respectively.

   Let O be the center of the rotation.

   Solving the system of equations

$$\begin{cases} y = x - 1 \\ y = \frac{5}{3}x - 3 \end{cases}$$

   gives O = (3, 2).

   This shows the amount of rotation is 90^o. Therefore, the image of (0, 0) under this rotation is found by using the formula in **corollary 8.4**.

$$x' = 3 + (0-3)\cos 90^{\circ} - (0-2)\sin 90^{\circ} = 5$$

$$y' = 2 + (0-3)\sin 90^{\circ} + (0-2)\cos 90^{\circ} = -1$$

R(0, 0) = (5, −1)

8.

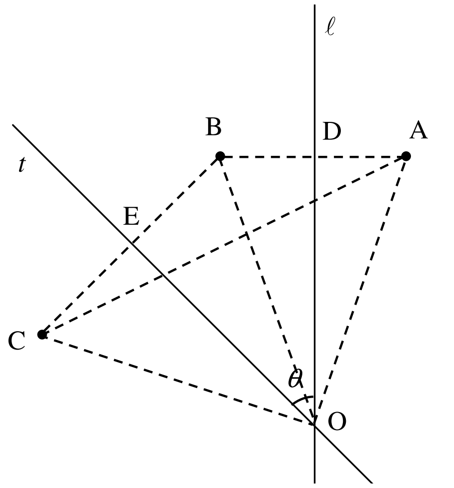


Figure 8.12

The perpendicular bisectors of ΔABC are concurrent at O.

Hence, there is a rotation about 0 that maps A to C through an angle equal to m(∠AOC).

But, ∠ COE ≅∠ BOE and ∠BOD ≅∠ AOD

⇒ m (∠AOC) = m (∠COB) + m(∠AOB)

= 2m (∠BOE) + 2m (∠BOD)

= 2[m(∠BOE) + m (∠BOD)]

= 2θ

Answers to Review Exercises on Unit 8

1. a. $\mathbf{u} - \mathbf{v} + 2\mathbf{w} = (15, 8) \Rightarrow |\mathbf{u} - \mathbf{v} + 2\mathbf{w}| = 17$

   b. $2\mathbf{u} + 3\mathbf{u} - \mathbf{w} = (-10, 16) \Rightarrow |2\mathbf{u} + 3\mathbf{v} - \mathbf{w}| = 2\sqrt{89}$

   c. $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \left(\dfrac{2}{\sqrt{29}}, \dfrac{5}{29}\right)$

   d. $\mathbf{z} = \mathbf{v} - \mathbf{w} - \mathbf{u} = (-10, -5)$

   e. $\mathbf{z} = \dfrac{1}{2}(3\mathbf{v} - \mathbf{u}) = \left(-\dfrac{11}{2}, 2\right)$

2.

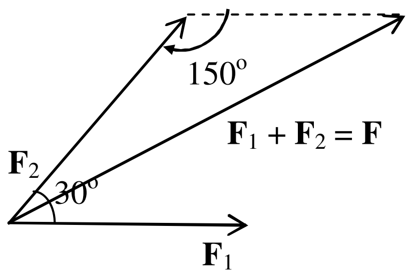


Figure 8.13

$$|\mathbf{F}| = \sqrt{40^2 + 30^2 - 2 \times 40 \times 30 \times \cos 150^{\circ}}\ \text{N} = 10\sqrt{25 + 12\sqrt{3}}\ \text{N}$$

3. The intersection point of the perpendicular bisectors of AA' and BB' is the center of rotation.

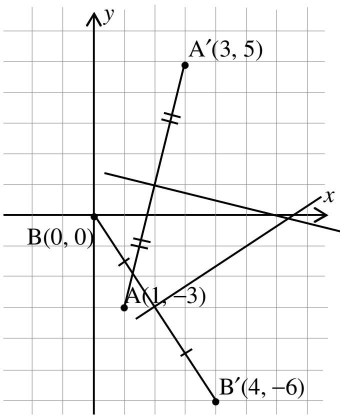


Figure 8.14

The perpendicular bisector of AA' has an equation: $4y + x = 6$. The perpendicular bisector of BB' has an equation of $3y - 2x = -13$.

Solving the system

$$4y + x = 6$$
$$3y - 2x = -13, \text{ gives}$$
$$x = \frac{70}{11} \text{ and } y = -\frac{1}{11}$$

$\Rightarrow$ The center of rotation is $\left(\frac{70}{11}, -\frac{1}{11}\right)$.

4. $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0$

5. $W = 10 \times 50 \times \cos 30^\circ = 250\sqrt{3}$ J

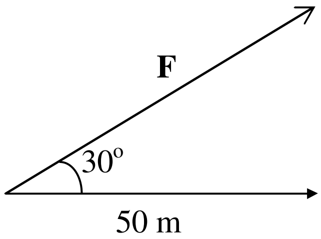


Figure 8.15

6. $x^2 + y^2 - x + y = 6 \Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{13}{2}$

$\Rightarrow$ The center of the circle is $\left(\frac{1}{2}, -\frac{1}{2}\right)$

Let ℓ be the line tangent to the circle at $(1, -3)$.

$$r_o = (1, -3) - \left(\frac{1}{2}, \frac{-1}{2}\right) = \left(\frac{1}{2}, \frac{-5}{2}\right)$$

a.

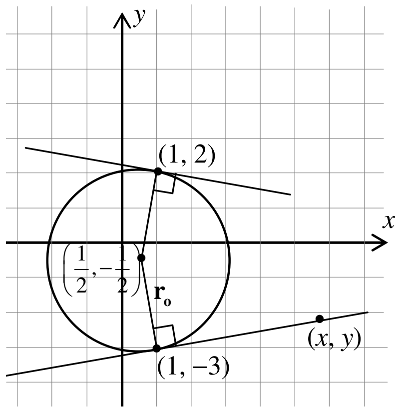


Figure 8.16

$$\Rightarrow (x-1,\ y+3).\left(\frac{1}{2}, \frac{-5}{2}\right)=0 \ \Rightarrow \ x-5y=16$$

b. Similarly, $r_o = (1,\ 2)-\left(\frac{1}{2}, \frac{-1}{2}\right)=\left(\frac{1}{2}, \frac{5}{2}\right)$

$$(x-1,\ y-2).\left(\frac{1}{2}, \frac{5}{2}\right) = 0 \Rightarrow \ x+5y-11=0$$

7. The translation vector, (9, – 10) – (7, –12) = (2, 2).

a. $\ell : y = 2x - 5 \ \Rightarrow \ell' : (y-2) = 2\,(x-2) - 5 \Rightarrow \ell' : y = 2x - 7.$

b. $\ell'\ \ 2\,(y-2) - 5\,(x-2) = 4 \Rightarrow 2y - 5x = -2$

c. ℓ': $(x-2) + (y-2) = 10 \Rightarrow x + y = 14$

d. $(x-2)^2 + (y-2)^2 = 3 \Rightarrow x^2 + y^2 - 4x - 4y = -5$

e. $(x-2)^2 + (y-2)^2 - 2(x-2) + 5\,(y-2) = 0$ which gives

$$(x-3)^2+\left(y+\frac{1}{2}\right)^2=\frac{29}{4}.$$

8. The line of reflection is the perpendicular bisector of $\overline{PP'}$.

Mid-point of $\overline{PP'}$ is (5, 6).

Slope of $\overline{PP'} = \frac{10-2}{3-7} = -2$.

Therefore, the slope of the perpendicular bisector is $\frac{1}{2}$ and it passes through (5, 6).

$$\frac{y-6}{x-5}=\frac{1}{2} \Rightarrow 2y-x=7$$ is equation of the line.

9. Using the formula

$$x' = a + (x - a)\cos\theta + (y - b)\sin\theta$$

$$y' = b - (x - a)\sin\theta + (y - b)\cos\theta$$

Together with the given $(a, b) = (1, 4)$ and $\theta = 30^{o}$ we have,

$$x' = 1 + (x - 1)\cos 30^{o} + (y - 4)\sin 30^{o}$$

$$= 1 + (x - 1) \times \frac{\sqrt{3}}{2} + (y - 4) \times \frac{1}{2}$$

and $y' = 4 + (x - 1)\sin 30^{o} + (y - 4)\cos 30^{o}$

$$= 4 - (x - 1) \times \frac{1}{2} + (y - 4) \times \frac{\sqrt{3}}{2}$$

Hence, substituting the corresponding values of x and y gives:

a. $R\left((-3, 2)\right) = \left(-2\sqrt{3},\ 2 - \sqrt{3}\right)$

b. $x^2 + y^2 - 2x - 8y = 10$

$\Rightarrow\ (x - 1)^2 + (y - 4)^2 = 27$

$\Rightarrow$ The center of the circle is $(1, 4)$ and radius $3\sqrt{3}$.

$R\left((1, 4)\right) = (1, 4)$ which is the same as the center of rotation.

$\Rightarrow$ The circle is fixed in this rotation.

c. $x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$

$$R\left(\left(0, \frac{3}{2}\right)\right) = \left(\frac{-1}{4}, \frac{\sqrt{3}}{2}, \frac{9}{2}\ \frac{-5\sqrt{3}}{4}\right)$$

$\Rightarrow$ The image of the circle is

$$\left(x + \frac{1}{4} - \frac{\sqrt{3}}{2}\right)^2 + \left(y - \frac{9}{2} + \frac{5\sqrt{3}}{4}\right)^2 = \frac{9}{4}$$

d. Consider the points $(0, 4)$ and $(1, 5)$ on ℓ: $y = x + 4$.

$$R\left((0, 4)\right) = \left(1 - \frac{\sqrt{3}}{2}, \frac{9}{2}\right) \text{ and } R\left((1, 5)\right) = \left(\frac{1}{2}, 4 + \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \ell' : \ y = x + \frac{7 + \sqrt{3}}{2}$$

10. Consider a regular hexagon

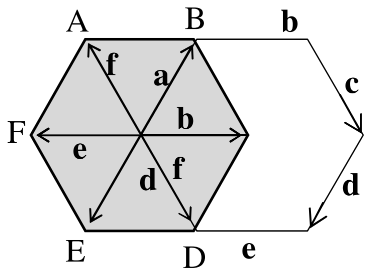


Figure 8.17

If the vectors are joined end to head, a regular hexagon will be obtained.

So, the sum of the vectors is **0**.

Students may practice this by considering any regular polygon.

11. Given

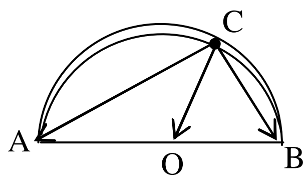


Figure 8.18

$$\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$$

$$\overrightarrow{OB} = \overrightarrow{OA}$$

From the figure $\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$ and

$$\overrightarrow{OB} = -\overrightarrow{OA}$$

Then $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB} = \overrightarrow{CO} - \overrightarrow{OA}$ $\left(\text{since } -\overrightarrow{OA} = \overrightarrow{OB}\right)$

$$\therefore \overrightarrow{CA} \,.\, \overrightarrow{CB} = \left(\overrightarrow{CO} + \overrightarrow{OA}\right) . \left(\overrightarrow{CO} - \overrightarrow{OA}\right).$$

$$\overrightarrow{CO} \,.\, \overrightarrow{CO} - \overrightarrow{CO} \,.\, \overrightarrow{OA} + \overrightarrow{OA} \,.\, \overrightarrow{CO} - \overrightarrow{OA} \,.\, \overrightarrow{OA}.$$

$$\left|\overrightarrow{CO}\right|^2 - \left|\overrightarrow{OA}\right|^2 = r^2 - r^2 = 0$$

$$\Rightarrow \overrightarrow{CA} \perp \overrightarrow{CB}$$

$\Rightarrow$<ACB = 90°.

12. $|a+b| = \sqrt{|a|^2 + |b|^2 + |a||b|\cos\theta}$

$$= \sqrt{10^2 + 6^2 + 2\times 10\times 6\cos\theta} = \sqrt{136 + 120\cos\theta}$$

a. $|a+b| = \sqrt{136 + 120\cos 30^\circ} = \sqrt{136 + 60\sqrt{3}} \approx 15.5$

b. $|a+b| = \sqrt{136 + 120\cos 120^\circ} = 2\sqrt{19} \approx 8.7$

c. $|a+b| = \sqrt{136 + 120\cos 150^\circ} = 2\sqrt{34 - 15\sqrt{3}} \approx 5.7$

13. $F = (3i + 4j) + (3i - 5j) + (5i + 4j) + (2i + j) = 13i + 4j$

14.

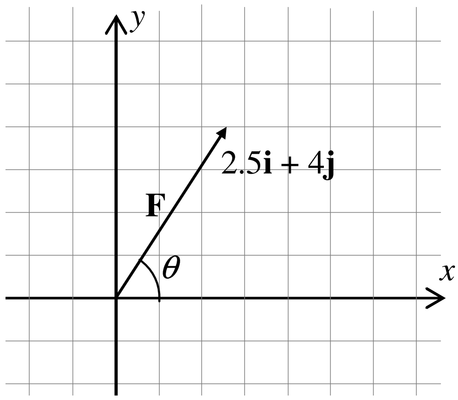


Figure 8.19

15. a. B = (600, 1200) = 600i + 1200j; C = (1400, 300) = 1400i + 300j

b. $\overrightarrow{AT} = \frac{2}{3}(600i + 1200j) = 400i + 800j$

$\overrightarrow{TC} = (1000i - 500j)$ $\overrightarrow{TC} \times \overrightarrow{AB} = 0$

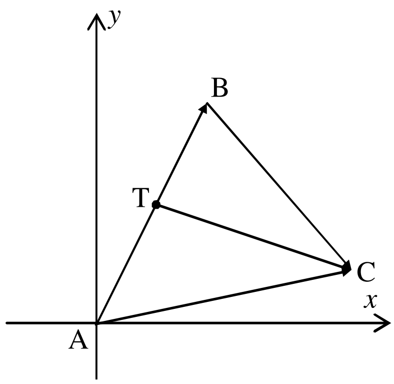


Figure 8.20

16. In this problem, students are expected to apply the rule of reflection about the line DC.

    First allow students to reflect the points A and B about $\overleftrightarrow{DC}$.

    Next, let them draw A'B and AB' each crossing $\overline{DC}$ at point E. *See Figure* 8.21.

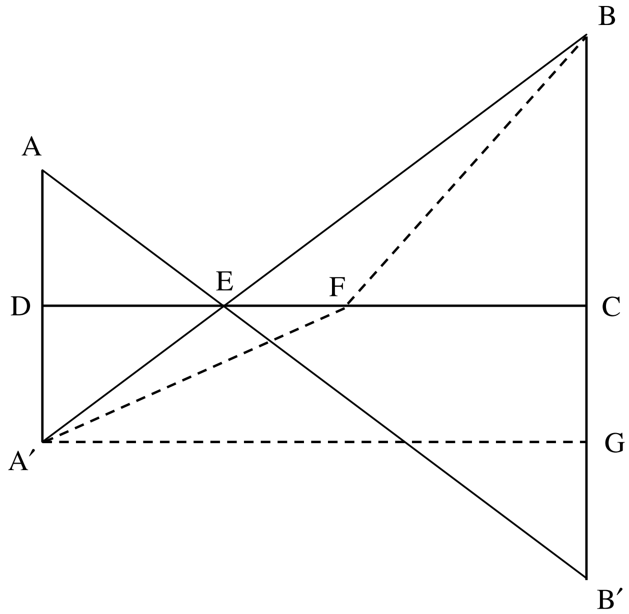


Figure 8.21

Let F be an arbitrary point on CD. Prove that AF + FB > AE + EB

This is proved as follows.

In Δ A'BF, A'F + BF > A'B by triangle inequality.

$\Rightarrow$ AF + BF > A'E + BE

$\Rightarrow$ AF + BF > AE + BE

Also, $A'B^2 = A'G^2 + BG^2$

$\Rightarrow$ AF + BF > A'E + BE

$\Rightarrow$ AF + BF > AE + BE

Also, $A'B^2 = A'G^2 + BG^2$

$\Rightarrow$ A'B = 10

This shows that the sum of the minimum distance from the power supplier to both villages is 10 km.

UNIT 9 FURTHER ON TRIGONOMETRIC FUNCTIONS

INTRODUCTION

This unit has two main objectives. The first objective is to enable students understand trigonometric functions in detail as presented in the student textbook. Special emphasis is given to the reciprocal trigonometric functions, inverse trigonometric functions, and graphs of some trigonometric functions of the form $y = a \sin (kx + b) + c$ and $y = a \cos(kx + b) + c$.

The second objective is to enable students solve practical real world problems, such as navigation, lengths related to angle of elevation and depression, and solving a triangle.

The unit also enables students to solve problems in other subjects such as geometry and physics. It presents each topic in detail by producing illustrative model examples.

Therefore, in this unit, students are expected to show mastery of the subject with encompassing various applications. Activities that are designed to revise the previous topics and introduce the current topics are also formulated. They also guide to arriving at some generalization of the concepts through ways of group discussions. This helps students not to waste time in learning the unit. They also help you as a starting board to teach each and every topic of the unit.

Unit outcomes

After completing this unit, the students will be able to:

- *know basic concepts about reciprocal functions.*
- *sketch graphs of some trigonometric functions.*
- *apply trigonometric functions to solve related problems.*

Suggested Teaching Aids

As teaching aids, there are various possibilities to select from. But, as a guide, the following can be considered as suggested teaching aids for this unit. These include:

- Charts containing graphs of $y = \sec x$, $y = \csc x$ and $y = \cot x$ that are showing the symmetric and periodic properties.
- Charts showing graphs of some inverse trigonometric functions.
- Calculators
- Tables
- Charts of some trigonometric functions
- Graphing calculators
- Softwares such as Geometers sketchpad, Matlab, Minitab, Mathematica, Tinkerplots, etc
- Pictures or photos containing model and solved problems.
- Additional materials: Calculators and software graphing.

9.1 THE FUNCTIONS $Y = \text{SEC } X$, $Y = \text{CSC } X$ AND $Y = \text{COT } X$

Periods Allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define and describe the functions* sec *x,* csc *x and* cot *x.*
- *sketch graphs of* sec *x,* csc *x and* cot *x.*

Vocabulary: Sin *x,* Cos *x*, Tan *x*. Sec *x,* Csc *x,* Cot *x,* Domain, Range, Period, Symmetry

Introduction

This subunit is devoted to strengthening the previous concepts of trigonometric functions the students have studied in grade 10 and to discuss the reciprocals of the ratios that define the sine, cosine and tangent. The main theme of this subunit will then be discussing the three trigonometric functions: cosecant, secant and cotangent, and their graphs.

Teaching Notes

This topic can be started by considering the reciprocals of the trigonometric functions of an acute angle and then extending it to a general angle. In the graphing part, you may start by presenting some simple reciprocal functions and constructing their graphs. Here, you are expected to help the students use the techniques of graphing the reciprocal trigonometric functions. For this purpose, you can form groups of students and let them do Activity 9.1. This activity is designed to revise the trigonometric functions sin x, cos x and tan x, and evaluate their reciprocals. After discussing the activity, you can define sec x, csc x and cot x using right angled triangles.

Answers to Activity 9.1

1. a. $\sin A = \frac{4}{5}$ b. $\sin B = \frac{3}{5}$

   c. $\cos B = \frac{4}{5}$ d. $\tan B = \frac{3}{4}$

2. a. $\frac{1}{\sin\theta} = \frac{13}{5}$ b. $\frac{1}{\cos\theta} = \frac{13}{12}$ c. $\frac{1}{\tan\theta} = \frac{12}{5}$

When the students are capable of evaluating the reciprocals of the ratios of trigonometric function, they are expected to practice sketching graphs of y = sec x, y = csc x, and y = cot x for different intervals. For this purpose, you can form groups of students and let them do Group Work 9.1 from which students will be able to revise graphs of y = sin x, y = cos x and y = tan x that students studied in grade 10, and determine domain, range and period of these three trigonometric functions. They will also discuss how to determine domain and range of y = sec x, y = csc x, and y = cot x, and issues of period and symmetry and symmetric properties of secant, cosecant and cotangent functions. When they do that, you are expected to assist students to correctly determine the domain and range of these functions and understand issues of period.

Answers to Group Work 9.1

1. a.

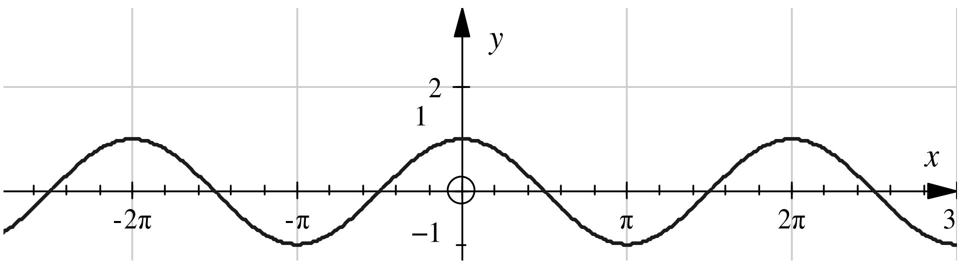


Figure 9.1 Graph of $y = \cos x$

Domain $= \mathbb{R}$

Range = [–1, 1];

Period = 2π

b.

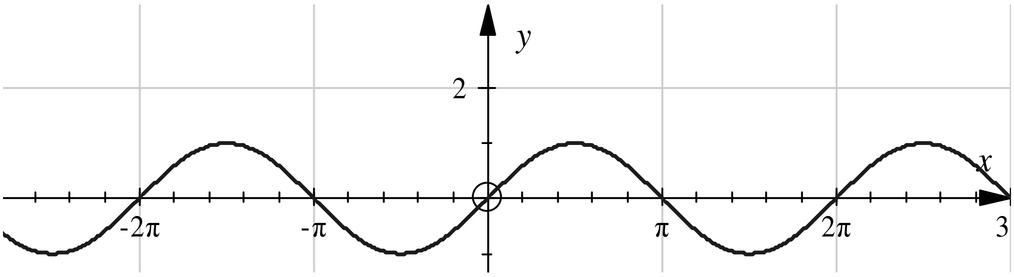


Figure 9.2 Graph of $y = \sin x$

Domain = $\mathbb{R}$

Range = $[-1, 1]$,

Period = 2π

c. D = $\left\{x \in \mathbb{R} : x \neq (2k+1)\dfrac{\pi}{2}; k \in \mathbb{Z}\right\}$

Range = $\mathbb{R}$,

Period = π

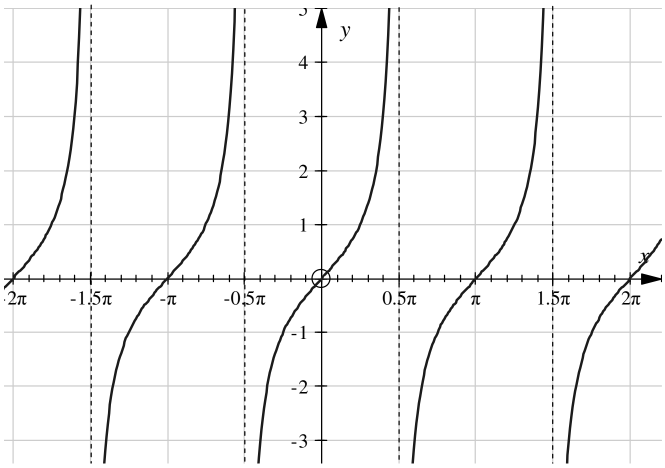


Figure 9.3 Graph of $y = \tan x$

2. Give this problem as oral question.

3. a. $y = \csc x = \dfrac{1}{\sin x}$

$\sin x = 0 \Rightarrow x = k\pi\ ;\ k \in \mathbb{Z}$

$\Rightarrow$ The domain of cosecant function is $\{x \in \mathbb{R} : x \neq k\pi;\ k \in \mathbb{Z}\}$

b. The domain of secant function is $\mathbb{R}$

$$\left\{x \in \mathbb{R} : x \neq (2k+1)\frac{\pi}{2};\ k \in \mathbb{Z}\right\}$$

c. The domain of cotangent function is $\{x \in \mathbb{R} : x \neq k\pi; k \in \mathbb{Z}\}$

4. $\dfrac{1}{|\sin x|} = |\csc x| \geq 1$

Thus, the range of cosecant and secant functions is $\{x \in \mathbb{R}\ x \leq -1 \text{ or } x \geq 1\}$

But the range of cotangent is $\mathbb{R}$

5. $\sec\ (x + 2\pi) = \dfrac{1}{\cos(x + 2\pi)} = \dfrac{1}{\cos x} = \sec x$

Also, $\cot\ (x + \pi) = \dfrac{1}{\tan(x + \pi)} = \dfrac{1}{\tan x} = \cot x$

$\Rightarrow$ secant and cosecant have period 2π. The period of cotangent is π.

6. $\csc\ (-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin x} = -\csc x$

$\sec\ (-x) = \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos x} = \sec\ x$

$\cot\ (-x) = \dfrac{1}{\tan(-x)} = \dfrac{1}{-\tan x} = -\cot x$

$\Rightarrow$ cosecant and cotangent are odd functions but secant is an even function.

Once the students have become capable of determining the domain and range, period, and other properties such as symmetry, the next discussion will be graphing these trigonometric functions. As an example, the graph of cosecant function is deliberated on page 351 of the student textbook. The graphs of the other trigonometric functions can be given as an assignment by forming groups of students. In addition, you can give them Exercise 9.1 which may help you to assess their progress.

You can give for high achievers additional exercises of the following type.

Draw the graphs of each of the following trigonometric functions for one cycle and demonstrate to the class.

a. $f(x) = 2 \sec x$

b. $f(x) = 2 - 3 \csc x$

c. $f(x) = -\dfrac{1}{4} \cot\left(x - \dfrac{\pi}{2}\right)$.

Solution:

1.

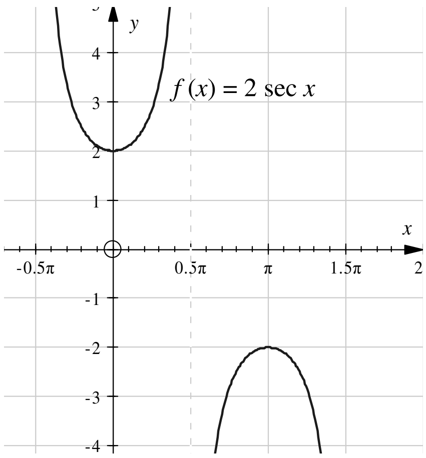


Figure 9.4

2.

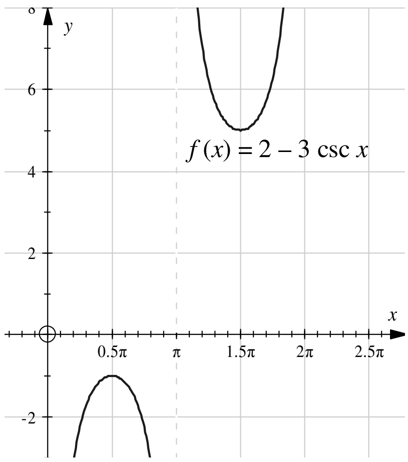


Figure 9.5

3.

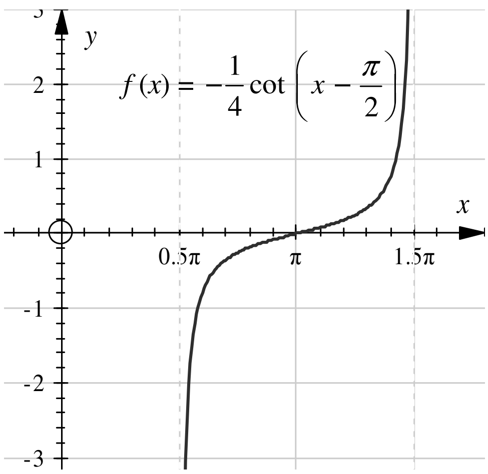


Figure 9.6

Assessment

In addition to the formative assessments, you conducted previously, you can also perform the following for the purpose of assessing understanding of your students.

- ask oral questions so that the students restate the definition of sec x, csc x and cot x.
- use the examples in the textbook to assess students during instruction.
- give the exercises from the textbook as class work or homework. You may give your own additional exercise problems on sketching the graphs of those functions.

Answers to Exercise 9.1

1. a. $\sqrt{2}$ b. -1 c. 1 d. 2

   e. -2 f. $-\sqrt{3}$ g. -2 h. $\dfrac{2\sqrt{3}}{3}$

   i. $\sqrt{3}$ j. undefined k. undefined l. undefined

2. a. $\left[\dfrac{\pi}{2}, \pi\right) \cup \left(\pi, \dfrac{3}{2}\pi\right]$ b. $\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$ c. $\varnothing$

3 a. $\sec x \sin x = \dfrac{\sin x}{\cos x} = \tan x$.

   b. $\tan x \csc x = \dfrac{\sin x}{\cos x} \times \dfrac{1}{\sin x} = \sec x$

   c. $1 + \dfrac{\tan x}{\cos x} = \dfrac{\cos^2 x + \sin x}{\cos^2 x}$

   d. $\csc\left(x + \dfrac{\pi}{2}\right) = \dfrac{1}{\sin\left(x + \dfrac{\pi}{2}\right)} = \dfrac{1}{\cos x} = \sec x$

   e. $\sec\left(x - \dfrac{\pi}{2}\right) = \dfrac{1}{\cos\left(x - \dfrac{\pi}{2}\right)} = \dfrac{1}{\sin x} = \csc x$

   f. $\tan\left(x + \dfrac{\pi}{2}\right) = \dfrac{\sin\left(x + \dfrac{\pi}{2}\right)}{\cos\left(x + \dfrac{\pi}{2}\right)} = \dfrac{\cos x}{-\sin x} = -\cot x$

4. It is known that $|\sec x| \geq 1$

   $|\cos x| \leq 1 \Rightarrow |-3\cos x| \leq 3 \Rightarrow |-3\sec x| \geq 3$

   Hence, the range of $y = -3 \sec x$ is $(-\infty, -3] \cup [3, \infty)$

5. Using the Pythagoras', identity, $\sin^2 x + \cos^2 x = 1$, dividing both sides by $\cos^2 x$ gives $\tan^2 x + 1 = \sec^2 x$ and dividing both sides by $\sin^2 x$ gives $1 + \cot^2 x = \csc^2 x$. Hence,

   a. $\sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1$

   b. $\csc^2 x - \cot^2 x = (1 + \cot^2 x) - \cot^2 x = 1$

9.2 INVERSE OF TRIGONOMETRIC FUNCTIONS

Periods Allotted: 4 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define the inverse trigonometric functions.*
- *sketch the graph of the inverse trigonometric functions.*

Vocabulary: Inverse sin x, Inverse cos x, Inverse tan x, Inverse sec x, Inverse csc x, Inverse cot x, Invertible, One-to-one, Restricted domain, and Graph.

Introduction

In the preceding subunit, students have exhaustively discussed the six trigonometric functions. In this unit, they will discuss the inverse of these trigonometric functions. They will also discuss some facts about inverse functions. The need for restricting the domain when trying to find the inverse trigonometric function will be discussed in this subunit. Following these, they will draw graphs of the inverse trigonometric functions.

Teaching Notes

The topic of inverse trigonometric functions involves an arrangement of the domain of the trigonometric functions. This is so because the functions whose inverse will be determined need to be one-to-one in the selected domain. For this reason, you may start the lesson by asking questions such as:

i. Is the inverse of a function f a function?

ii. If f^{-1} is the inverse of f, what is the rule for $f^{-1}(x)$?

Finally, you may give opportunities to students to determine the domain of the newly defined inverse functions.

After deliberating on these points, you can ask if whether any trigonometric function has an inverse. If yes, ask them where the inverse is.

This will lead the students to discuss the fact that trigonometric functions are not one-to-one everywhere but they can be one-to-one in some restricted domain in which inverse of the trigonometric functions can be discussed.

To enrich their understanding, you can assist the students to revise the inverse of functions through examples and then introduce and define the inverse trigonometric function.

Following the explanation through examples, you can form groups of students and let them do Activity 9.2 which will give chance for the students to determine an interval on which the sine function will be one-to-one. It will also help them try to draw the graph of $f(x) = \sin x$ in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and see its reflection along the line $y = x$.

Answers to Activity 9.2

1. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2. The graph of sine function, $y = \sin x$, is given on the student text (Figure 9.11 (a)) whose reflection is also in Figure 9.11 (b).

Pursuant to the discussion of the students on the activity, you can define inverse sine function and assist students to point out the remark stated in the student text. You can also let them do some exercises of computing the value of the inverse function of a number. Here, you may help students to practice how they can use calculators to determine values of inverse trigonometric functions. You may also use, if available, some computer software to demonstrate how the graphs of trigonometric function and the graphs of inverse trigonometric functions behave by simulating or animating these graphs.

In the same way, you can proceed with discussing the other trigonometric functions and their inverses that are presented with illustrative examples.

When you discuss these ideas,

- allow students to distinguish inverse trigonometric functions from reciprocal functions. $(\sin x)^{-1}$ from $\sin^{-1}x$, $(\cos x)^{-1}$ from $\cos^{-1}x$ and $(\tan x)^{-1}$ from $\tan^{-1}x$ by producing examples.
- help students to practice sketching graphs of inverse trigonometric functions through reflection in the line $y=x$.
- help students to determine the domain and range for the inverse of trigonometric functions.
- motivate students to use a calculator and tables to solve problems in inverse trigonometric functions.

You can give clever students additional exercises of the following type:

1. Draw the graph of each of the following inverse trigonometric functions.

   a. $y = \text{arc}\sin\left(\frac{1}{2}x\right)$ b. $y = \text{arc}\cos(2x)$.

2. Simplify each of the following expressions.

   a. $\tan(2\,\text{arc}\sin x)$ b. $\cos(\text{arc}(\sin 2x))$ c. $\sin(\text{arc}\cos(2x))$

Solution:

1. a.

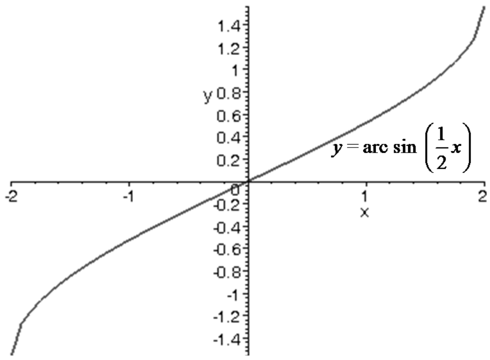


Figure 9.7

b.

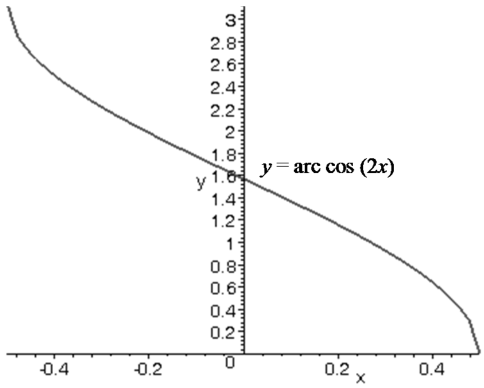


Figure 9.8

2. a. Let θ = arcsin x, then sin $\theta = x$ and tan $\theta = \dfrac{x}{\sqrt{1-x^2}}$

$\Rightarrow$ tan (2 arcsin x) = tan $(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{\dfrac{2x}{\sqrt{1-x^2}}}{1-\left(\dfrac{x}{\sqrt{1-x^2}}\right)^2} = \frac{2x}{\sqrt{1-x^2}} \times \frac{1-x^2}{1-2x^2} = \frac{2x\sqrt{1-x^2}}{1-2x^2}$$

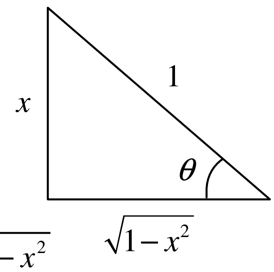


Figure 9.9

b. Let θ= arcsin (2x).

Then sin $\theta = 2x$

$\Rightarrow$ cos(arcsin(2x)) = cos $\theta = \sqrt{1-4x^2}$

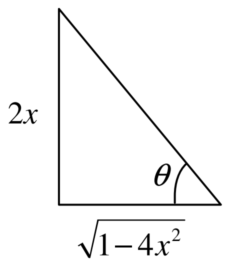


Figure 9.10

c. Let θ = arccos(2x).

$\Rightarrow \cos\theta = 2x$

$\Rightarrow$ sin (arccos(2x)) = sin θ = $\sqrt{1-4x^2}$

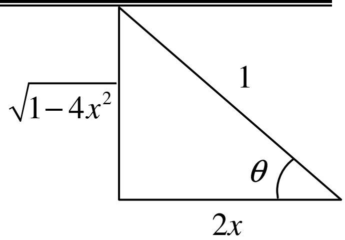


Figure 9.11

Assessment

To assess the understanding of the students you can

- ask oral questions so that the students restate the definition of inverse trigonometric functions
- give exercise problems on sketching graphs of inverse trigonometric functions
- give the exercise from the textbook as class work and homework.

Or forming groups of students, you can give them exercise 9.2 as an assignment.

Answer to Exercise 9.2

1. a. $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$ b. $\cos^{-1}(3)$ doesn't exist

c. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ d. $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) = -\csc^{-1}\left(\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$

e. $\sec^{-1}\left(\sqrt{2}\right) = \frac{\pi}{4}$ f. $\cot^{-1}(-1) = \frac{3\pi}{4}$

g. Consider a right angle triangle.

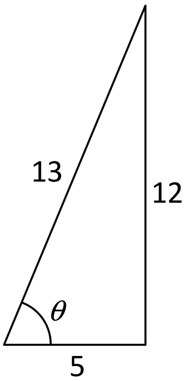


Figure 9.12

$$\sin^{-1}\left(\frac{12}{13}\right) = \theta \Rightarrow \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow \cos\theta = \sqrt{1-\left(\frac{12}{13}\right)^2}, \text{since } \cos\theta \geq 0 \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow \cos\theta = \frac{5}{13}$$

$$\therefore \cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right) = \frac{5}{13}$$

h. $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \sin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$

i. $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

j. $\text{arc cos}\left(\cos\frac{5}{6}\pi\right) = \text{arc cos}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \text{arc cos}\frac{\sqrt{3}}{2} = \pi - \frac{\pi}{6} = \frac{5}{6}\pi$

k. $\frac{\sqrt{3}}{2}$ l. $\sqrt{3}$ m. $\tan\left(\text{arc sin}\left(\frac{\sqrt{3}}{2}\right)\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

n. $\cos^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right) = \cos^{-1}(-1) = \pi - \cos^{-1}(1) = \pi - 0 = \pi$

2. Consider the following right angle triangle.

Let $\text{arc tan } x = \theta, \theta \in \left(\frac{-\pi}{2}, \frac{-\pi}{2}\right) \Rightarrow \tan\theta = \frac{1}{x} = x$

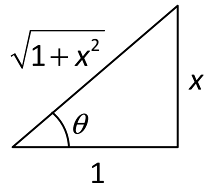


Figure 9.13

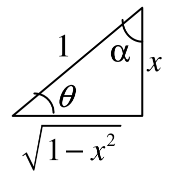
a. $\sin(\text{arc tan } x) = \sin\theta = \frac{x}{\sqrt{1+x^2}}$

If $x < 0$, $\sin(\text{arc tan } x) = -\sin(\text{arc tan }(-x)) = -\frac{-x}{\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$

Hence the value does not depend on the sign of x.

b. Let $\text{arc sin } x = \theta$

$\Rightarrow \cos(\arcsin x) = \cos\theta = \sqrt{1-x^2}$



c. Let $\arccos x = \alpha \Rightarrow \tan(\arccos x) = \tan\alpha = \frac{\sqrt{1-x^2}}{x}$

Figure 9.14

3. a. Let $y = \tan^{-1}(-x) \Rightarrow \tan y = -x \Rightarrow -\tan y = x$

$\Rightarrow \tan(-y) = x \Rightarrow -y = \tan^{-1} x$

$\Rightarrow y = -\tan^{-1} x \Rightarrow \tan^{-1}(-x) = -\tan^{-1} x$

b. let $y = \arccos\left(\frac{1}{x}\right) \Rightarrow \cos y = \frac{1}{x}$

$\Rightarrow \frac{1}{\cos y} = x$

$\Rightarrow \sec y = x \Rightarrow y = \sec^{-1} x$

Hence, $\sec^{-1}x = \arccos\frac{1}{x}$

c. let $y = \sin^{-1}\left(\frac{1}{x}\right) \Rightarrow \sin y = \frac{1}{x}$

$\Rightarrow \frac{1}{\sin y} = x$

$\Rightarrow \csc y = x \Rightarrow y = \csc^{-1} x$

Hence, $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

4. a. $y = \text{arccsc } x$ b. $y = \text{arcsec } x$

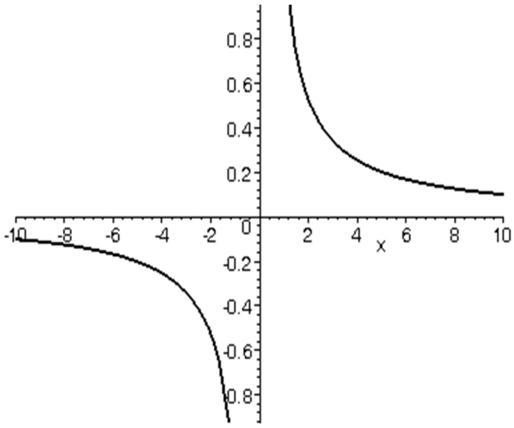


Figure 9.15

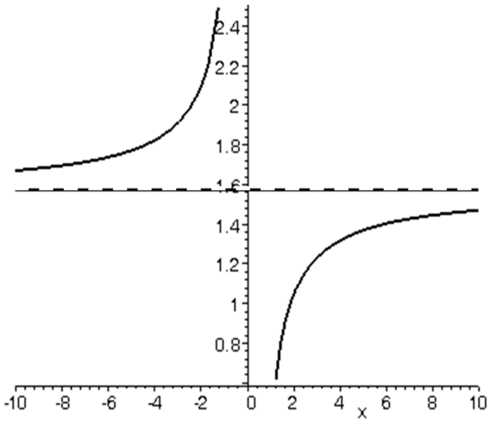


Figure 9.16

c. $y = \text{arccot } x$

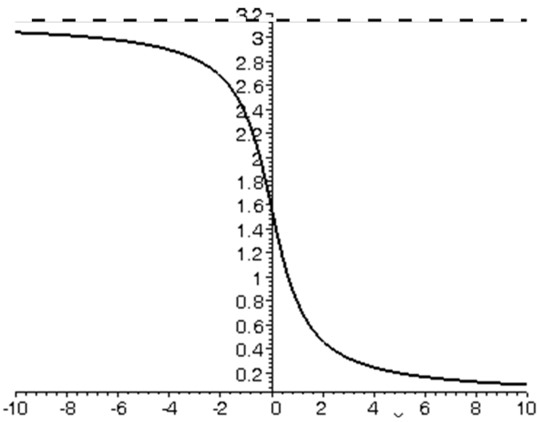


Figure 9.17

5. $y = 3 + 2 \arcsin (5x - 1)$

$$\Rightarrow |5x-1| \le 1 \Rightarrow -1 \le 5x-1 \le 1 \Rightarrow 0 \le x \le \frac{2}{5}$$

Also, $\frac{y-3}{2} = \arcsin\ (5x-1)$

$$\Rightarrow \ \sin\left(\frac{y-3}{2}\right) = 5x-1$$

$$\Rightarrow \ x = \frac{1}{5}\left(1+\sin\left(\frac{y-3}{2}\right)\right)$$

Also, $\left|\frac{y-3}{2}\right| \le \frac{\pi}{2} \ \Rightarrow \ -\pi \le y - 3 \le \pi$

$$\Rightarrow \ 3 - \pi \le y \le 3 + \pi$$

9.3 GRAPHS OF SOME TRIGONOMETRIC FUNCTIONS

Periods Allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *sketch the graphs of*
  $y = a \sin x$
  $y = a \sin (kx)$
  $y = a \sin (kx + b)$ *and*
  $y = a \sin (kx + b) + c$
- *list the properties of these graphs.*
- *sketch the graphs of*
  $y = a \cos x$
  $y = a \cos (kx)$
  $y = a \cos (kx + b)$
  $y = a \cos (kx + b) + c$
- *list the properties of these graphs.*

Vocabulary: Graph, Amplitude, Period, Symmetry, General form of trigonometric function.

Introduction

This sub-unit is devoted to generalizing the discussions the students have conducted in the preceding subunit. Here the general forms of trigonometric functions will be presented and properties related to changes in any one of the parameters of the general form of the functions will be addressed. With this concept, the students are expected to be able to graph any trigonometric function.

For the purpose of demonstrating the graphs in relation to changes in amplitude, period or constant term, either some organized charts or computer softwares are useful.

Teaching Notes

Students already know how to draw the graphs of trigonometric functions in their standard forms. Here, they are expected to discuss general forms of trigonometric functions of types

$$y = a \sin (kx + b) + c \text{ and } y = a \cos (kx + b) + c$$

and deal with graphing of these functions. These equations are important in analysis of sound waves, x-rays, electric circuits, vibrations, spring-mass systems, etc. They need to expand the technique on how to sketch graphs of trigonometric functions. The graphing technique emphasizes that students manipulate the functional values; determine the amplitude, period, phase shift, and the horizontal or vertical expansion or contraction.

To get into the lesson, you can form groups of students and let them do Group Work 9.2 which will help students to find functional values and sketch graphs. It also helps them to determine ranges, periods and amplitude. Here, when the students do the group work, you can round and identify students who face difficulties and those who are gifted to do these easily. For those who need assistance, you can guide them or group them with the talented students so that they can cope. For those who are far ahead, you can give them additional problems from the exercises.

Answers to Group Work 9.2

1.

θ	$\sin\theta$	$2\sin\theta$	$\frac{1}{2}\sin\theta$	$\cos\theta$	$-3\cos\theta$	$\frac{2}{3}\cos\theta$
0	0	0	0	1	–3	$\frac{2}{3}$
$\frac{\pi}{6}$	$\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{\sqrt{3}}{2}$	$-3\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{2}$	$-3\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{3}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{3}$
$\frac{\pi}{2}$	1	2	$\frac{1}{2}$	0	0	0
$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{3}$
$\frac{3}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{2}$	$3\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{3}$
$\frac{5}{6}\pi$	$\frac{1}{2}$	1	$\frac{1}{4}$	$-\frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
π	0	0	0	–1	3	$-\frac{2}{3}$

$\frac{7}{6}\pi$	$-\frac{1}{2}$	-1	$-\frac{1}{4}$	$-\frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$\frac{5}{4}\pi$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$-\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{2}}{2}$	$3\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{3}$
$\frac{4}{3}\pi$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{3}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{2}$	$-3\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{3}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	-1	$-\frac{1}{4}$	$\frac{\sqrt{3}}{2}$	$-3\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
2π	0	0	0	1	-3	$\frac{2}{3}$

2. a

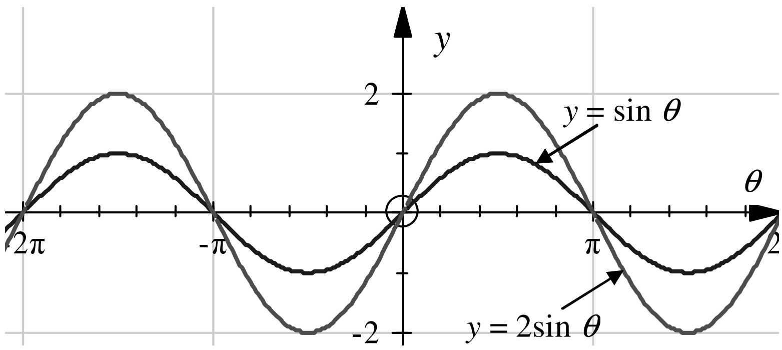


Figure 9.18

b.

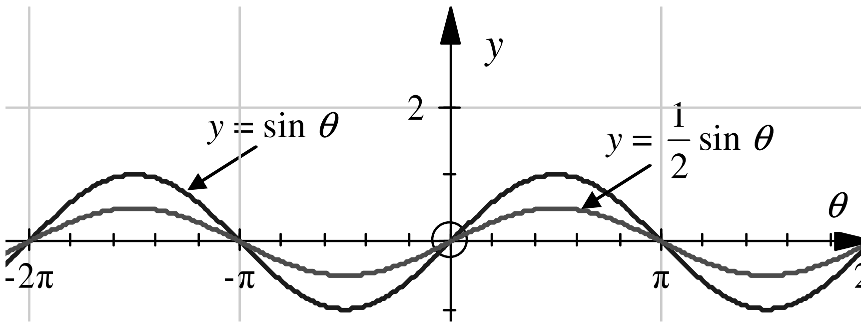


Figure 9.19

c.

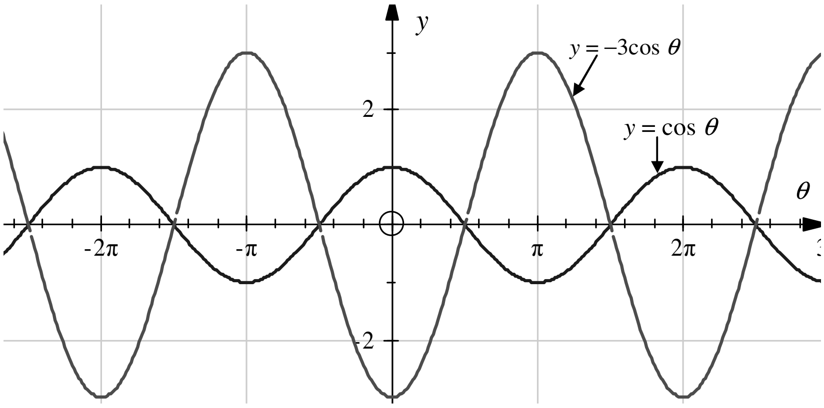


Figure 9.20

d.

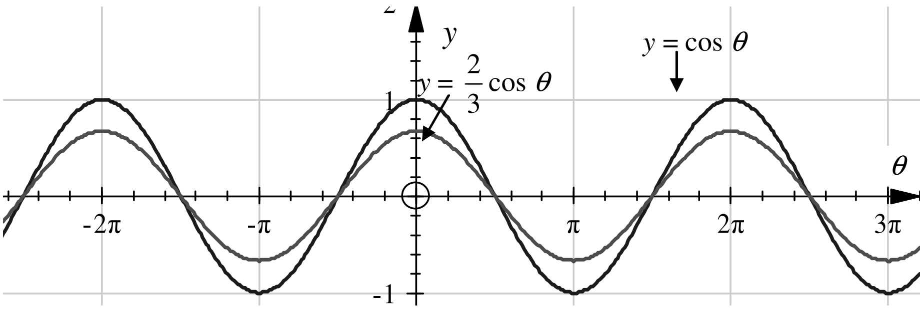


Figure 9.21

3. a. Range =[–2, 2], period= 2π b. Range= $\left[-\frac{1}{2}, \frac{1}{2}\right]$, period= 2π

c. Range=[–3, 3], period= 2π d. Range= $\left[-\frac{2}{3}, \frac{2}{3}\right]$, period= 2π

4. a. 1 b. 1 c. 0.25 d. no e. 6 f. 0.5

Following the group work, encourage students to explain how it is possible to determine graph of $y = a\sin x$ from the graph of $y = \sin x$. Let them also discuss amplitude and period. Once they discuss how they can draw graphs of $y = a\sin x$ from the graph of $y = \sin x$, assist them to draw the graphs of $y = 2\sin x$ and $y = \frac{1}{2}\sin x$ as a transformation of the graph of $y = \sin x$. When they draw the graphs, you can assist them to use the graphing procedures. Cognizant of their understanding to relate $y = a\sin x$ from the graph of $y = \sin x$, you can proceed to discussing the graph of $f(x) = \sin kx$, $k > 0$. To continue with discussion, you can form groups of students and let them do Group Work 9.3. This group work will guide students to determine functional values, their minimum and maximum that can be used to draw their graphs. This will also help them determine the period of the function. When the students try to do the group work, you need to assist them and guide them with hints and directives.

Answers to Group Work 9.3

1. A calculator or tables can be used

x	$2x$	$\frac{1}{2}x$	$\sin(2x)$	$\sin\left(\frac{1}{2}x\right)$
0	0	0	0	0
$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{8}$	1	0.3825
$\frac{\pi}{2}$	π	$\frac{\pi}{4}$	0	$\frac{\sqrt{2}}{2}$
$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{3\pi}{8}$	–1	0.9237
π	2π	$\frac{\pi}{2}$	0	1
$\frac{5\pi}{4}$	$\frac{5\pi}{2}$	$\frac{5\pi}{8}$	1	0.9239
$\frac{3\pi}{2}$	3π	$\frac{3}{4}\pi$	0	$\frac{\sqrt{2}}{2}$
$\frac{7\pi}{4}$	$\frac{7\pi}{2}$	$\frac{7\pi}{8}$	–1	0.3827
2π	4π	π	0	0

2. a. The maximum value is 1 and the minimum is –1.

   b. Maximum value = 1 minimum value = – 1

3. a. $f(x) = \sin(2x)$

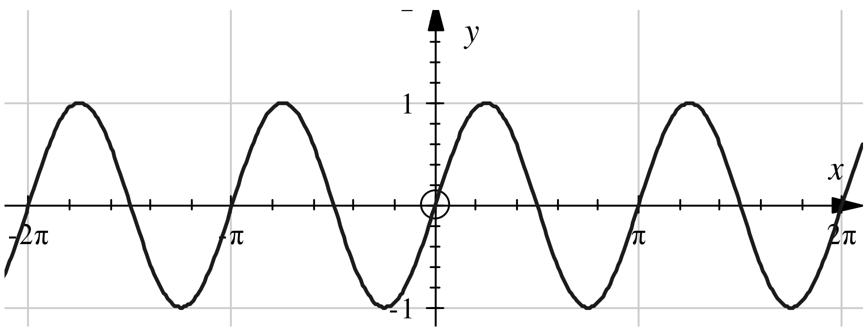


Figure 9.22

b. $g(x) = \sin\left(\frac{1}{2}x\right)$

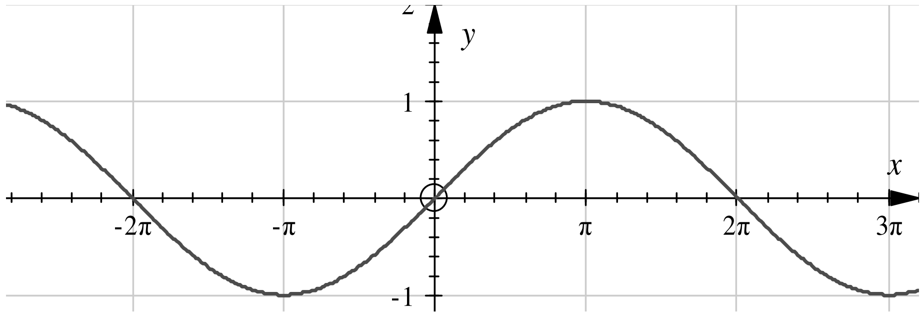


Figure 9.23

To enrich their trial, you can consolidate the discussion with examples and proceed to jointly consider trigonometric function of type $y = a \sin kx$ and $y = a \cos kx$. To get them right, you can assist them to discuss and explain the graphing procedures, and help them draw graphs of different function of the above forms. To check their level of understanding, you can pair the students and give them exercise 9.3 as an assignment.

Answer to Exercise 9.3

1. a. $f(x) = 4 \sin x$

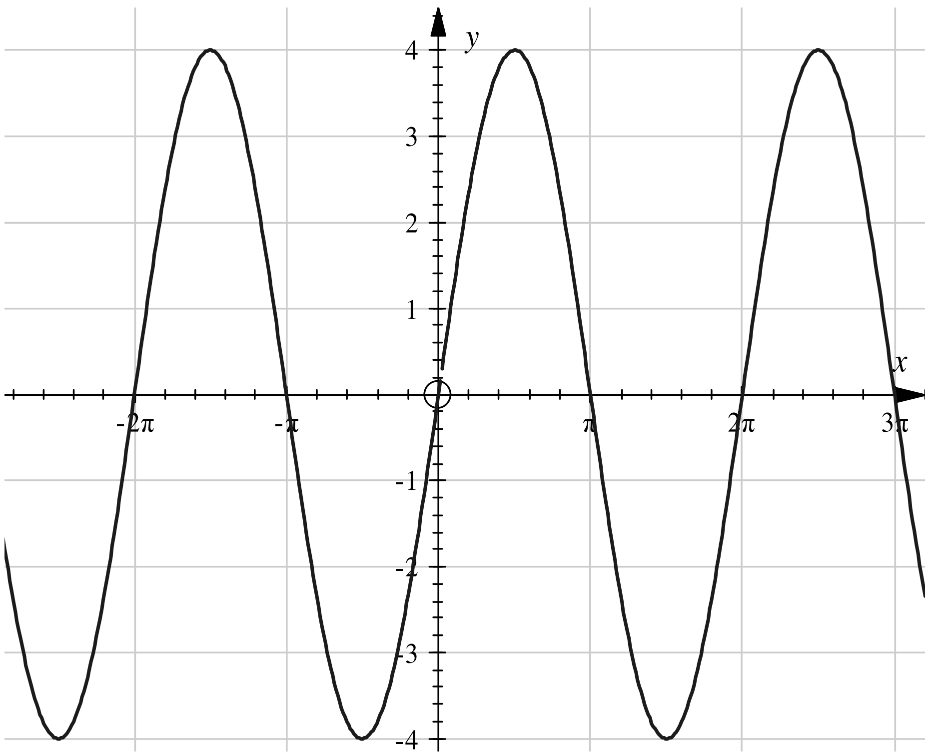


Figure 9.24

b. $f(x) = -2\cos x$

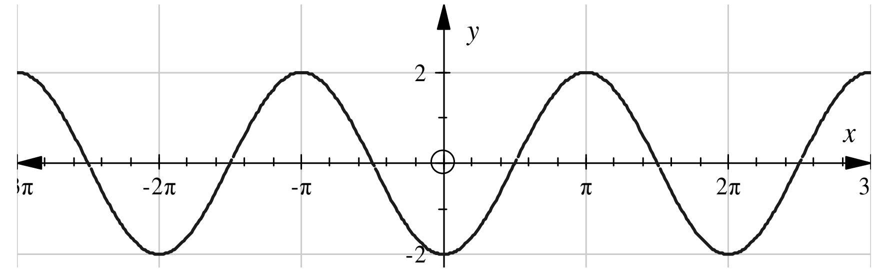


Figure 9.25

c. $f(x) = \frac{2}{3}\sin x$

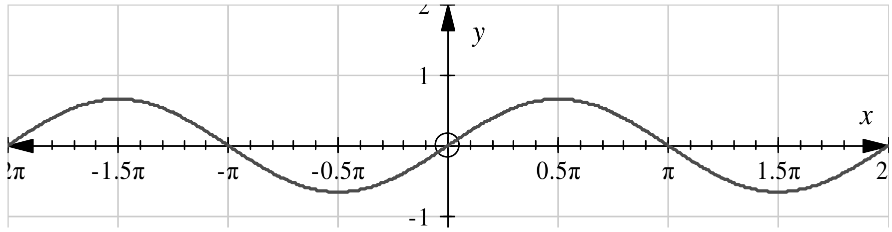


Figure 9.26

d. $f(x) = \frac{1}{4}\cos x$

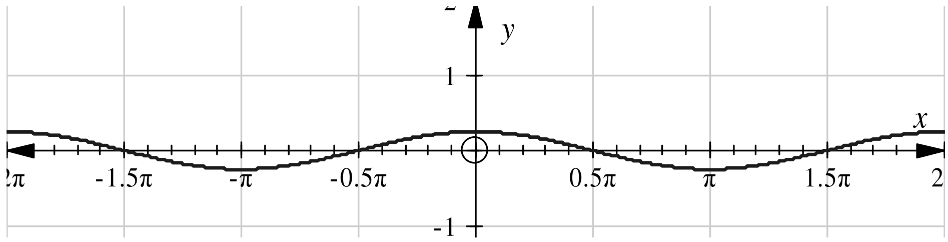
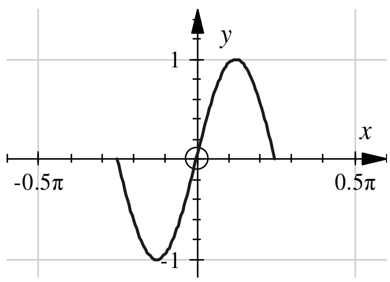


Figure 9.27

2. Follow the graphing procedure. Another graphing technique can be used if necessary.

a. $f(x) = \sin(4x)$

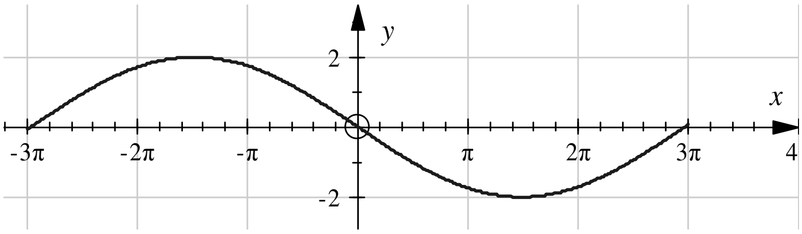


Period = $\frac{\pi}{2}$

$a = 1$

Figure 9.28

b. $f(x) = -2\sin\left(\frac{1}{3}x\right)$

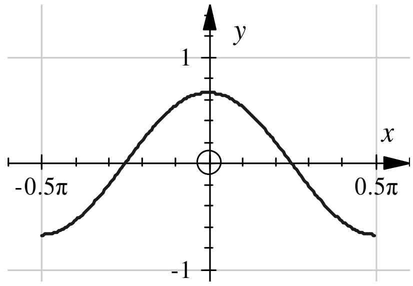


P = 6π

$a = 2$

Figure 9.29

c. $f(x) = \frac{2}{3}\cos(2x)$

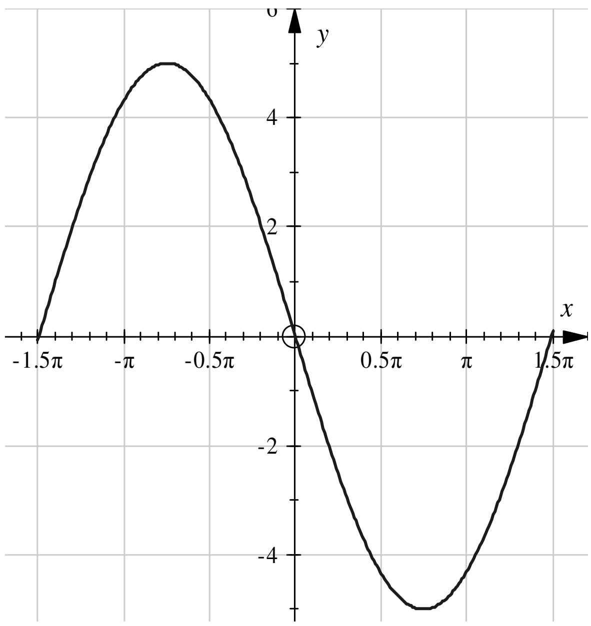


P = π

$a = \frac{2}{3}$

Figure 9.30

d. $f(x) = 5\sin\left(\frac{-2}{3}x\right)$

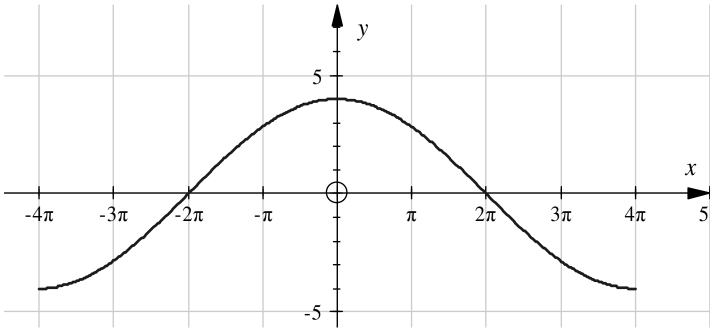


P = 3π

$a = 5$

Figure 9.31

e. $f(x) = 4\cos\left(\frac{1}{4}x\right)$

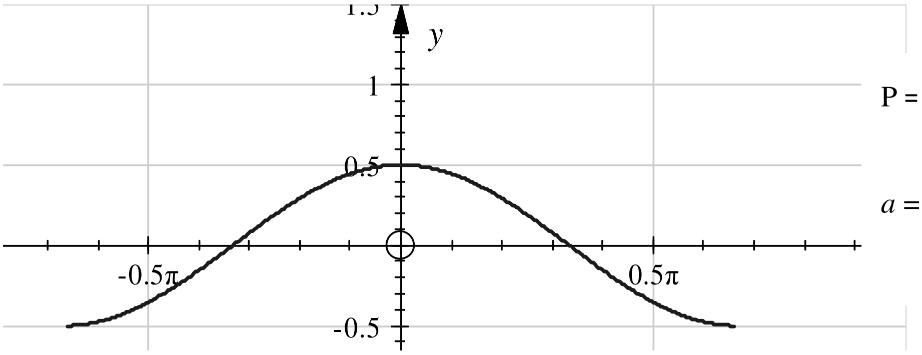


P = 8π

$a = 4$

Figure 9.32

f. $f(x) = \frac{1}{2}\cos\left(\frac{-3}{2}x\right)$



P = $\frac{4\pi}{3}$

$a = \frac{1}{2}$

Figure 9.33

Finally, you can proceed to discussing sketching the graphs of $y = a \sin (kx + b) + c$ and $y = a \cos (kx + b) + c$. Here what is most important is investigating the geometric effect of the constants b and c in drawing the graph of the function. For simplifying procedures of drawing graphs, you can form groups of students and let them do Activity 9.3. When they do the activity you need to assist them.

Answers to Activity 9.3

1. a. $0 \le 2x + 1 \le 2\pi \Rightarrow -\frac{1}{2} \le x \le \pi - \frac{1}{2}$

   b. $0 \le 3x - 1 \le 2\pi \Rightarrow \frac{1}{3} \le x \le \frac{2\pi}{3} + \frac{1}{3}$

   c. $0 \le 2x - \frac{\pi}{3} \le 2\pi \Rightarrow \frac{\pi}{6} \le x \le \frac{7}{6}\pi$

   d. $0 \le \pi x + \frac{\pi}{2} \le 2\pi \Rightarrow -\frac{1}{2} \le x \le \frac{3}{2}$

2. a. $x = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi$ b. $x = \frac{1}{4}, \frac{1}{4} + \frac{\pi}{8}, \frac{1}{4} + \frac{\pi}{4}, \frac{1}{4} + \frac{3\pi}{8}, \frac{1}{4} + \frac{\pi}{2}$

3.

x	$\frac{1}{4}$	$\frac{1}{4}+\frac{\pi}{8}$	$\frac{1}{4}+\frac{\pi}{4}$	$\frac{1}{4}+\frac{3\pi}{8}$	$\frac{1}{4}+\frac{\pi}{2}$
$3 \sin (4x - 1)$	0	3	0	–3	0
$3 \cos (4x - 1)$	3	0	–3	0	3

When they finish the activity, select some group members and encourage them to demonstrate their work to the whole class. You also need to organize the ideas discussed in this regard and encourage students to generalize the properties of the functions of type $y = a \sin x$, $y = a \sin (kx)$ and $y = a \sin (kx + b) + c$. You also need to ensure that the procedures of drawing graphs are well taken by the students. For this purpose, you can let the students do as many examples as possible and, finally, you can give them exercise 9.4 as an assignment with which you can check the level of understanding of your students.

You can also give for high achievers additional exercise problems of the following type:

1. Given below is the graph of $y = a \cos (\omega x - \phi) + k$.

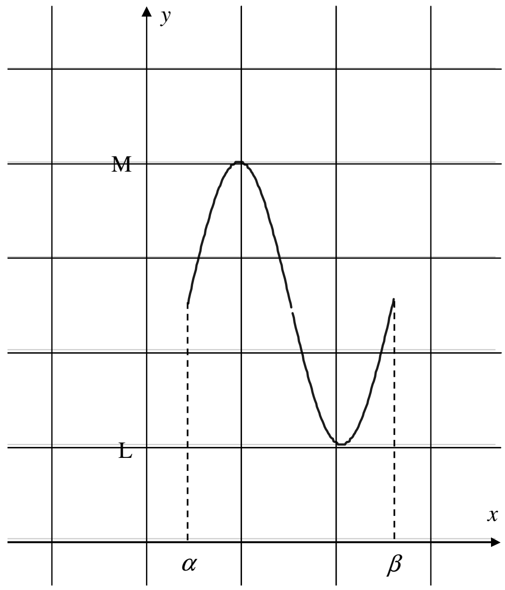


Figure 9.34

1. Express each of the following letters in terms of the others.

a. a b. ω c. ϕ d. k

Answer a. $\frac{M-L}{2}$ b. $\omega = \frac{2\pi}{\beta - \alpha}$ c. $\phi = \frac{(3\alpha + \beta)\pi}{2(\beta - \alpha)}$ d. $k = \frac{L+M}{2}$

2. Solve $2\sin^2 x + \cos x - 1 = 0$

Solution: $2\sin^2 x + \cos x - 1 = 0$

$\Rightarrow 2(1 - \cos^2 x) + \cos x - 1 = 0$

$\Rightarrow 2\cos^2 x - \cos x - 1 = 0$

$\Rightarrow \cos x = -\frac{1}{2}$ or $\cos x = 1$

$\Rightarrow S.S = \left\{\frac{3}{2}\pi \pm 2n\pi\right\} \cup \{2n\pi\}$; Where $n \in \mathbb{Z}$

$\Rightarrow S.S = \left\{\frac{3}{2}\pi \pm 2n\pi, 2n\pi;\ n \in \mathbb{Z}\right\}$

Assessment

- Give exercise problems on sketching graphs of
  $y = 2\sin x$, $y = \frac{1}{2}\sin x$, $y = -3\sin x$, etc. by using the graph of $y = \sin x$.
- Give exercise problems on sketching the graphs of
  $y = a\sin(kx)$, $y = a\cos(kx)$, $y = a\sin(kx + b)$, $y = a\cos(kx + b)$ for different values of a, b and k.
- Use the exercise problems in the textbook as class work / homework.

Answers to Exercise 9.4

1. $f(x) = -\frac{1}{2}\sin(2x-1)$	1. $\lvert a\rvert = \frac{1}{2}$. P = π; phase angle = $\frac{1}{2}$
2. $f(x) = \frac{1}{2}\cos(3x+2)$	2. $\lvert a\rvert = \frac{1}{2}$, P = $\frac{2}{3}\pi$ phase angle = $\frac{-2}{3}$
3. $f(x) = 3\sin\left(\frac{1}{2}x+3\right)-2$	3. $\lvert a\rvert = 3$, P = 4π phase angle = -6
4. $f(x) = \sin(\pi x)+3$	4. $\lvert a\rvert = 1$, P = 2 phase angle = 0
5. $f(x) = 2\cos(2x-\pi)$	5. $\lvert a\rvert = 2$, P = π phase angle = $\frac{\pi}{2}$
6. $f(x) = 3-2\cos\left(\frac{x}{2}\right)$	6. $\lvert a\rvert = 2$, P = 4π phase angle = 0
7. $f(x) = \frac{-3}{2}\sin\left(3x+\frac{3}{4}\pi\right)$	7. $\lvert a\rvert = \frac{3}{2}$, P = $\frac{2}{3}\pi$ phase angle = $-\frac{\pi}{4}$
8. $f(x) = 2-\frac{1}{2}\cos\left(\frac{3\pi}{2}x+\frac{\pi}{4}\right)$	8. $\lvert a\rvert = \frac{1}{2}$, P = $\frac{4}{3}\pi$ phase angle = $-\frac{1}{6}$

1.

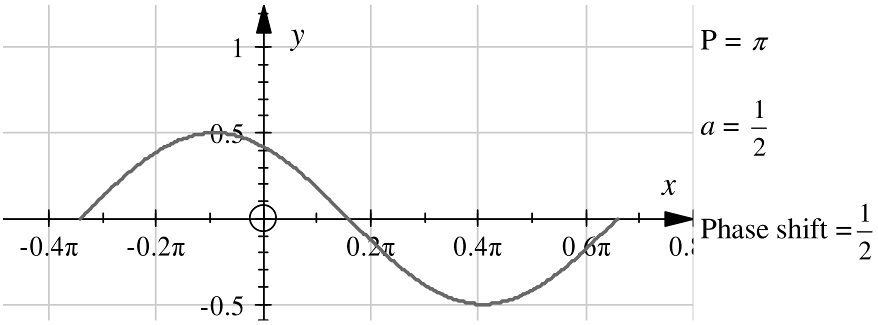


Figure 9.35

2.

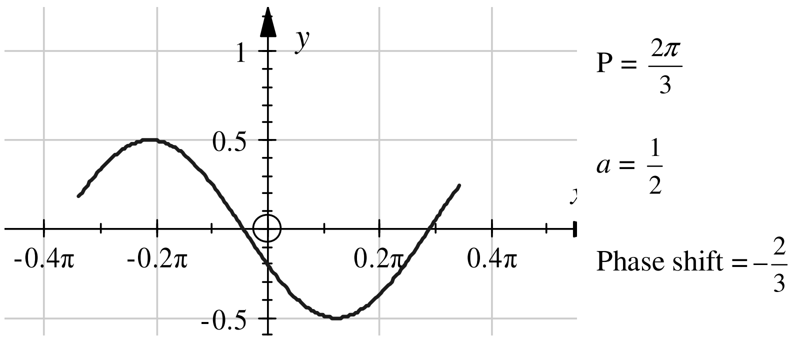


Figure 9.36

3.

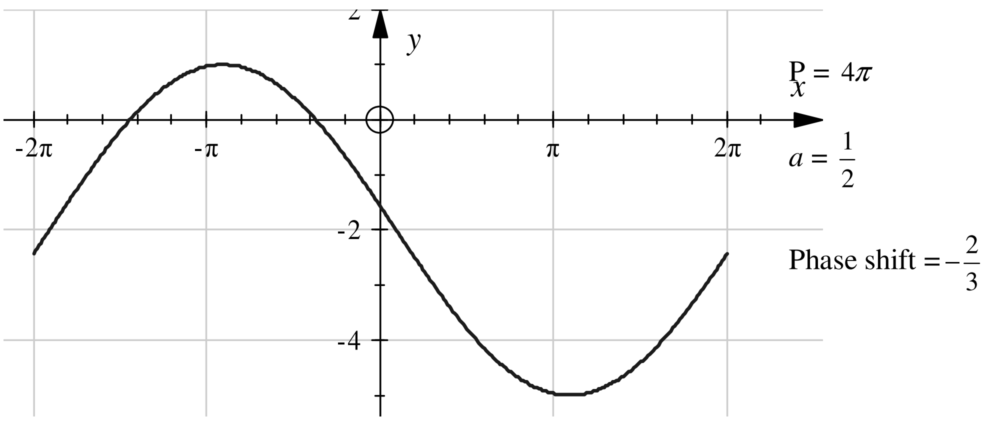
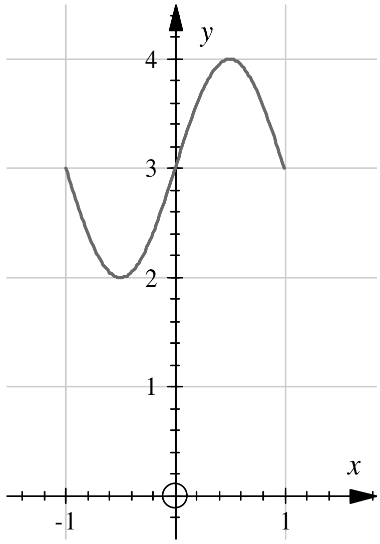


Figure 9.37

4.



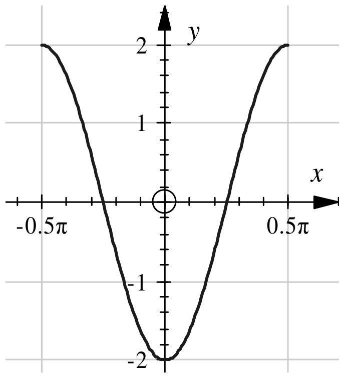
P = 2

$a = 1$

Phase shift =0

Figure 9.38

5.



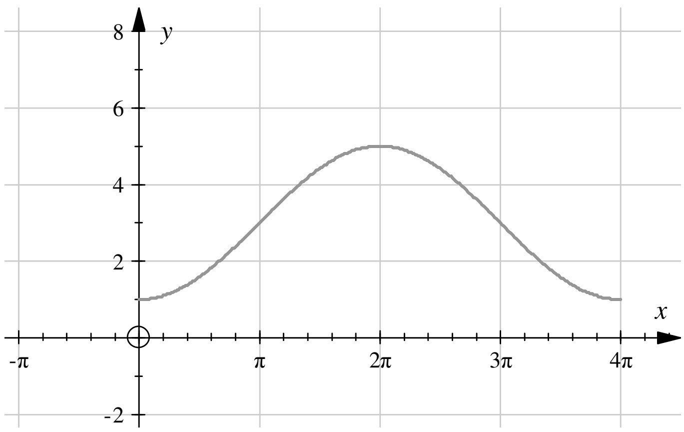
P = π

$a = 2$

Phase shift = $\frac{\pi}{2}$

Figure 9.39

6.



P = 4π

$a = 2$

Phase shift = 0

Figure 9.40

7.

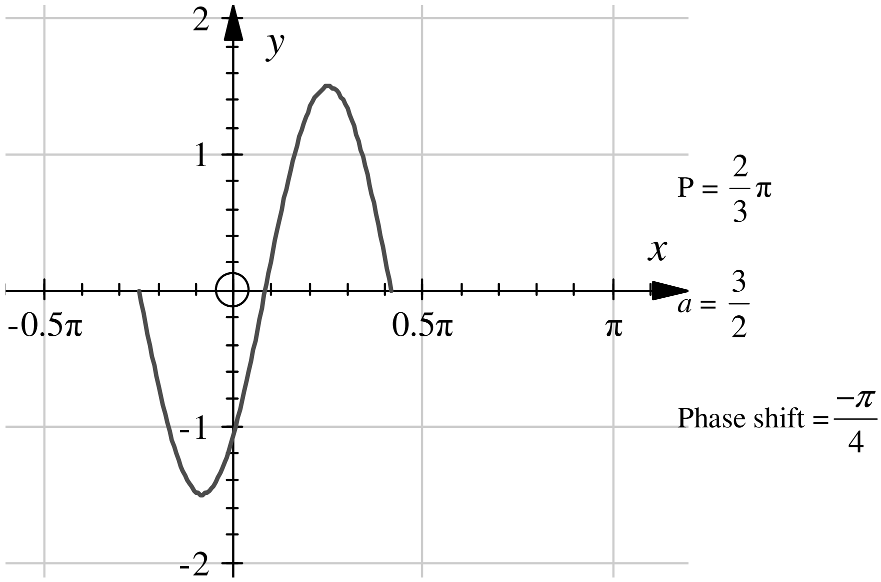


Figure 9.41

8.

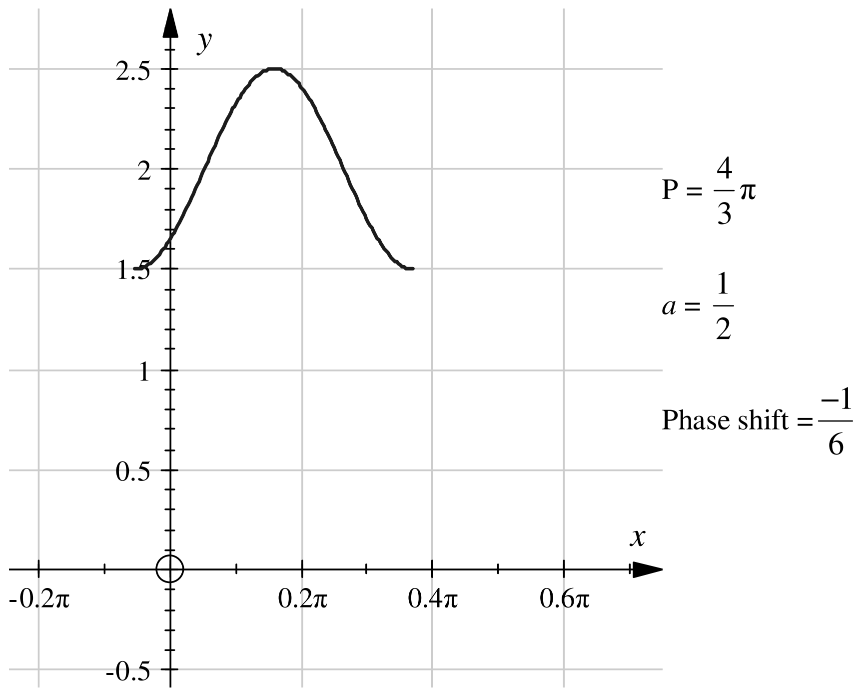


Figure 9.42

So far, students have discussed the various trigonometric functions and the ways the graphs of such functions are drawn. They have also discussed relationships between each type of trigonometric functions. In this subunit, they are supposed to discuss the applications of graphs particularly in solving trigonometric equations. To proceed with this discussion, you can form groups of students and let them do Activity 9.4 which will help them realize particular solutions among the infinitely many ones relating it with the concept of period, domain or range. While letting them do this activity it is recommended if students themselves discover the rules for the general solutions of simple trigonometric equations such as: $\sin x = \frac{1}{2}$. It is also good if you encourage students to draw the graphs of the trigonometric functions and the constant functions together when they do the activity.

Answers to Activity 9.4

1. a. Let students draw the graph $f(x) = \tan x$ and $y = t$, for an arbitrary real number t and see that they intersect at exactly one point on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. From this, they will conclude that the equation $\tan x = t$ has exactly one solution on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

   The solution for $\tan x = 1$ is $\left\{\frac{\pi}{4}\right\}$ on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

   b. Let students draw the graphs of $y = \sin x$ and $y = b$; $0 < |b| < 1$ together and see that the equation $\sin x = b$ has exactly two distinct solutions on $[0, 2\pi]$.

   The general solution set for $\tan x = 1$ is $\left\{\frac{\pi}{4} \pm n\pi; n \in \mathbb{Z}\right\}$

2. Apply the same technique for $\cos x = b, |b| < 1$ on $[-\pi, \pi]$.

   The solution set for the equation $\cos x = \frac{1}{2}$ in the rage $[-\pi, \pi]$ is $\left\{\frac{-\pi}{3}, \frac{\pi}{3}\right\}$

3. As given in the student text book, $\cos x = b$; $|b| \leq 1$.

   $\Rightarrow S.S = \left\{\frac{\pi}{3} \pm 2n\pi; n \in \mathbb{Z}\right\}$

When they perform the activity, it is advisable to give them as many examples as possible for consolidating the concept discussed. Finally, to help them share ideas and work together, you can group your students and give them Exercise 9.5 as an assignment. You can also use their result for the purpose of assessing their understanding.

Answers to Exercise 9.5

1. a. $\sin x = -\frac{1}{2} \Rightarrow \sin(-y) = \frac{1}{2} \Rightarrow -y_1 = \frac{\pi}{6}$ and $-y_2 = \frac{5\pi}{6}$

$$\Rightarrow y_1 = \frac{-\pi}{6} \text{ and } y_2 = \frac{-5}{6}\pi$$

$$\Rightarrow x_1 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}, x_2 = 2\pi - \frac{5}{6}\pi = \frac{7}{6}\pi$$

$$\Rightarrow S.S = \left\{\frac{11\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi\right\}$$

b. $\cos x = \frac{\sqrt{3}}{2} \Rightarrow x_1 = \frac{\pi}{6}$ and $x_2 = \frac{-\pi}{6} \Rightarrow S.S = \left\{2n\pi \pm \frac{\pi}{6}\right\}$

c. $\tan x = \sqrt{3} \Rightarrow x_1 = \frac{\pi}{3} \Rightarrow S.S = \left\{\frac{\pi}{3} + n\pi\right\}$

d. $2\cos^2 x + 3\sin x = 0 \Rightarrow 2(1 - \sin^2 x) + 3\sin x = 0$

$$\Rightarrow -2\sin^2 x + 3\sin x + 2 = 0$$

$$\Rightarrow 2\sin^2 x - 3\sin x - 2 = 0$$

$$\Rightarrow \sin x = \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm 5}{4} = 2, \frac{-1}{2}$$

$$\Rightarrow \sin x = 2 \quad \text{or} \quad \sin x = \frac{-1}{2}$$

But $|\sin x| \le 1 \Rightarrow \sin x = \frac{-1}{2}$

$$\Rightarrow \text{from (1), S.S} = \left\{\frac{11\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi\right\}$$

e. The identity $\cos(2x) = \cos^2 x - \sin^2 x$ should be given to students. In fact, this is introduced in section 9.4.2.

$$\cos(2x) + \sin^2 x = 0$$

$$\cos^2 x - \sin^2 x + \sin^2 x = 0 \Rightarrow \cos^2 x = 0$$

$$\Rightarrow x_1 = \frac{\pi}{2} \text{ and } x_2 = -\frac{\pi}{2}$$

$$\Rightarrow \text{S.S} = \left\{2n\pi \pm \frac{\pi}{2}\right\}$$

f. $\sin(6x) = \frac{\sqrt{3}}{2} \Rightarrow 6x_1 = \frac{\pi}{3}$ and $6x_2 = \frac{2\pi}{3}$

$\Rightarrow x_1 = \frac{\pi}{18}$ and $x_2 = \frac{\pi}{9}$

$\Rightarrow S.S. = \left\{\frac{\pi}{18} + \frac{n\pi}{3}, \frac{\pi}{9} + \frac{n\pi}{3}\right\}$

2. $\sin^2 x - \sin x \cos x = 0 \Rightarrow \sin x\ (\sin x - \cos x) = 0$

$\Rightarrow \sin x = 0$ or $\sin x = \cos x$

$\Rightarrow x = 0, \pi, 2\pi$ or $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$\Rightarrow$ S.S $= \left\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi\right\}$

3. a. For $0 \le x \le 2\pi$

$$\left.\begin{cases} \cos x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{6}, \frac{11}{6}\pi \\ \tan x = -\frac{\sqrt{3}}{3} \Rightarrow x = \frac{11}{6}\pi \end{cases}\right\} \Rightarrow S.S = \left\{\frac{11}{6}\pi\right\}$$

b. The period for $y = \cos\left(\frac{\pi}{3}x - 2\right)$ is $\frac{2\pi}{\left(\frac{\pi}{3}\right)} = 6$.

$$\cos\left(\frac{\pi}{3}x - 2\right) = \frac{1}{2} \Rightarrow \frac{\pi}{3}x_1 - 2 = \pm\frac{\pi}{3}$$

$$x_1 = \left(2 \pm \frac{\pi}{3}\right)\left(\frac{3}{\pi}\right) = \frac{6}{\pi} \pm 1$$

$$\Rightarrow \text{S.S} = \left\{6n + \frac{6}{\pi} \pm 1, n = -3, -2, -1, 0, 1, 2\right\}$$

c. $\sec\left(\frac{3}{2}x - \frac{\pi}{3}\right) = 2$ and $\cot x < 0$ on $[0, 2\pi]$

$\Rightarrow \frac{3\pi}{2} < x < 2\pi$ and $\cos x = \frac{1}{2}$

$\Rightarrow$ S.S $= \left\{\frac{5\pi}{3}\right\}$

d. $2\sin^2 x + \cos^2 x - 1 = 0 \Rightarrow \sin^2 x + \sin^2 x + \cos^2 x - 1 = 0 \Rightarrow \sin^2 x = 0$

$\Rightarrow x = 0, \pi, 2\pi$

$\Rightarrow$ S.S $= \{0, \pi, 2\pi\}$

9.4 APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

Periods Allotted: 6 periods

Competency

At the end of this sub-unit, students will be able to:

- *apply trigonometric functions to solve problems from fields of science, navigation, engineering, etc.*

Vocabulary: Application

Introduction

So far, students were discussing trigonometric functions, their inverses, properties and some applications in solving trigonometric equations. However, trigonometric functions have applications in science, navigation, wave motions and optics among many others. This subunit is, thus, devoted to discussing applications of trigonometric functions. Some of the applications discussed in this subunit include solving triangles, trigonometric formula for the sums and differences, navigation, optics problems and simple harmonic motion.

You can also expand the discussion to other applications when available.

Teaching Notes

You know that there are tremendous applications of trigonometric functions. In this topic, some of the applications such as solving triangles, trigonometric formula for the sums and differences, navigation, optics problems and the simple harmonic model are discussed. To do this, the laws of sines and cosines and some trigonometric identities are included. You may give additional exercise problems as project work. For discussing these practical applications, it is advisable if the students are grouped and each application is discussed in the form of group works. While the students do the problems in group, you can assist some of the groups who need help. You may also organize additional application problems that will be useful for gifted students. Finally, you assign each group to search for some application problems in other fields such as physics and do each problem as an assignment. This will help them try every problem by themselves.

You can also give additional exercise problems of the following type for high achievers:

1. In Figure 9.43, O is the centre of the circle and θ is in radian measure. Show that the area of the segment is $\frac{1}{2}r^2(\theta - \sin\theta)$.

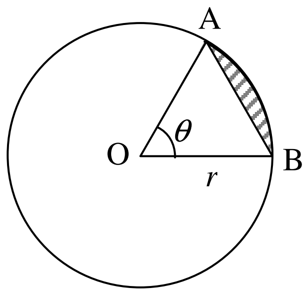


Figure 9.43

2. In Figure 9.44a, $\overline{AB}$ and $\overline{CD}$ are parallel chords that are equidistant from the centre O. If the distance between $\overline{AB}$ and $\overline{CD}$ is $2x$ units, find the area of the shaded part in terms of the radius r.

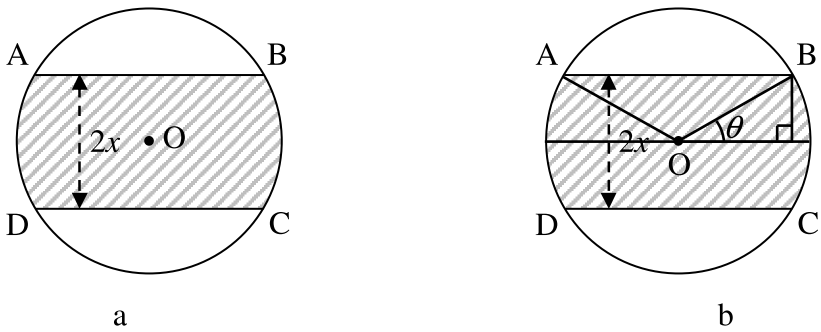


Figure 9.44

Solution:

$$\sin\,\theta = \frac{x}{r} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{r}\right)$$

$$A_\theta = \frac{\pi r^2\theta}{2\pi} = \frac{r^2 \sin^{-1}\left(\frac{x}{r}\right)}{2} \Rightarrow 4\,A_\theta = 2r^2\,\sin^{-1}\left(\frac{x}{r}\right).$$

$$a(\Delta\,\text{AOB}) = \frac{1}{2}r^2\sin(\beta) = \frac{1}{2}r^2\sin(\pi - 2\theta)$$

$$= \frac{1}{2}r^2\sin\left(\pi - 2\sin^{-1}\left(\frac{x}{r}\right)\right) = \frac{1}{2}r^2\sin\left(2\sin^{-1}\left(\frac{x}{r}\right)\right)$$

$$= \frac{1}{2}r^2\sin 2\theta = r^2\sin\theta\cos\theta$$

$$= r^2 \times \frac{x}{r} \times \frac{\sqrt{r^2 - x^2}}{r} = x\sqrt{r^2 - x^2}$$

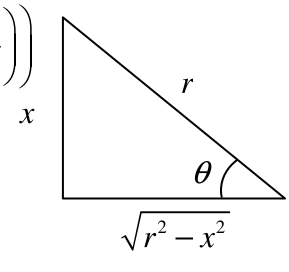


Figure 9.45

$2\,a(\Delta\text{AOB}) = 2x\sqrt{r^2 - x^2}$

The area of the shaded part = $4A_\theta + 2a(\Delta\text{AOB})$

$$= 2r^2 \sin^{-1}\left(\frac{x}{r}\right) + 2\,x\sqrt{r^2 - x^2}$$

3. In Figure 9.46, prove that $\dfrac{\cos\alpha}{a} + \dfrac{\cos\beta}{b} + \dfrac{\cos\gamma}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$.

Proof: $a^2+b^2+c^2 = 2a^2+2b^2+2c^2 - 2ab\cos\gamma - 2\,bc\cos\alpha - 2ac\cos\beta$

Dividing both sides by $2abc$ completes the proof.

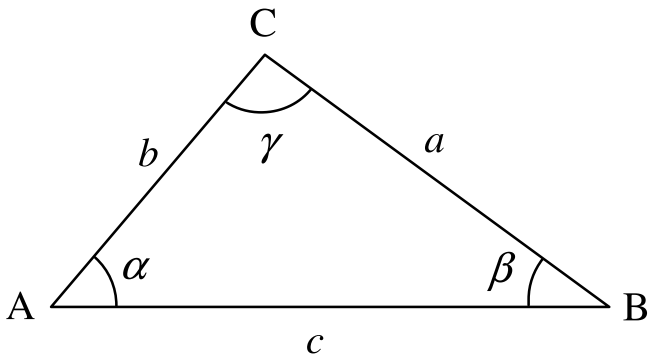


Figure 9.46

Assessment

Give exercise problems on application of trigonometric functions.

You may also give project work on measuring

- The height of a tree
- The width of a river/pond
- The heights of historical buildings.
- The height of a cliff
- Waterfall

Answer to Exercise 9.6

1.

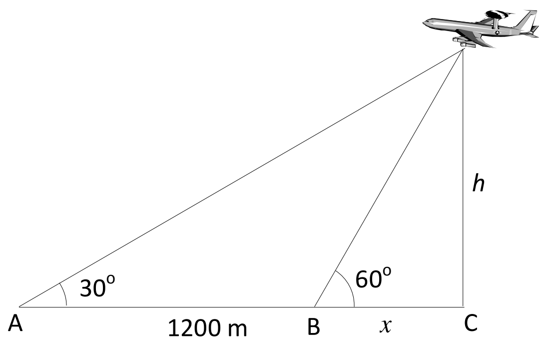


Figure 9.47

$$\tan 60^\circ = \frac{h}{x} \Rightarrow x = \frac{h}{\tan 60^\circ} \Rightarrow x = \frac{h}{\sqrt{3}}$$

Also, $\tan 30^\circ = \dfrac{h}{x+1200} \Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{h}{x+1200}$

$\Rightarrow \quad x+1200 = h\sqrt{3}$. But $x = \dfrac{h}{\sqrt{3}}$

$\Rightarrow \quad \dfrac{h}{\sqrt{3}} + 1200 = h\sqrt{3}$

$\Rightarrow \quad h = 600\sqrt{3} \approx 1039$

$\Rightarrow$ The altitude of the airplane is about 1039m

2. a. $\sin 30^\circ = \dfrac{x}{8} \Rightarrow x = 4$

$\cos 30^\circ = \dfrac{y}{8} \Rightarrow y = 4\sqrt{3}$

b. $\theta = 180^\circ - 120^\circ - 20^\circ = 40^\circ$

$$\frac{\sin 20^\circ}{3} = \frac{\sin 120^\circ}{x} \Rightarrow x = \frac{3\sin 120^\circ}{\sin 20^\circ} \approx 7.6$$

$$\frac{\sin 20^\circ}{3} = \frac{\sin 40^\circ}{y} \Rightarrow y = \frac{3\sin 40^\circ}{\sin 20^\circ} \approx 5.6$$

c. $x^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 150^{\circ} = 113 + 56\sqrt{3}$

$$\Rightarrow x = \sqrt{113 + 56\sqrt{3}}$$

By law of sines, $\dfrac{7}{\sin\theta} = \dfrac{\sqrt{113+56\sqrt{3}}}{\sin 150^{\circ}} \Rightarrow \dfrac{7}{\sin\theta} = 2\sqrt{113+56\sqrt{3}}$

$$\Rightarrow \sin\theta = \frac{7}{2\sqrt{113+56\sqrt{3}}}$$

$$\Rightarrow \sin\theta \approx 0.2415 \Rightarrow \theta \approx 14^{\circ}$$

d. By law of sines

$$\frac{10}{\sin 30^{\circ}} = \frac{x}{\sin 20} \Rightarrow x = \frac{10 \times \sin 20^{\circ}}{\sin 30^{\circ}} \approx 20(0.342) = 6.84$$

e. $(1.5)^2 = 2^2 + 1^2 - 2(2)(1)\cos\theta$

$2.25 = 4 + 1 - 4\cos\theta \Rightarrow 0.6875 = \cos\theta$

$$\Rightarrow \theta = 46.57^{\circ}$$

3. $\tan 60^{\circ} = \dfrac{y}{x} \Rightarrow x = \dfrac{y}{\sqrt{3}}$

$\tan 70^{\circ} = \dfrac{3+y}{x} \Rightarrow \tan 70^{\circ} = \dfrac{3+y}{\left(\dfrac{y}{\sqrt{3}}\right)} = \dfrac{(3+y)\sqrt{3}}{y}$

$$\Rightarrow 2.7475 = \frac{(3+y)\sqrt{3}}{y}$$

$$\Rightarrow 2.7475y = (3+y)\sqrt{3}$$

$$\Rightarrow y \approx 5.12$$

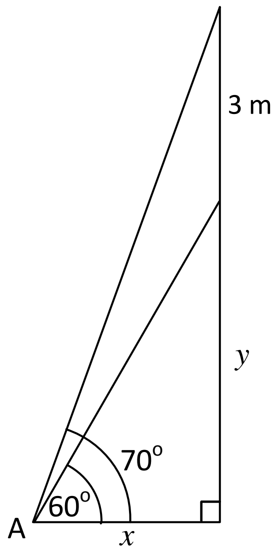


Figure 9.48

$\Rightarrow$ The height of the building is (3 + 5.12) m = 8.12 m.

4.

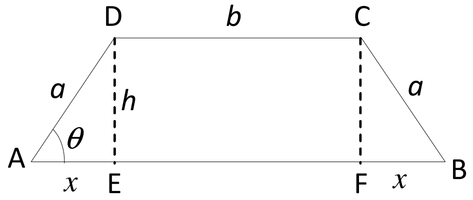


Figure 9.49

$$\sin\theta = \frac{h}{a} \Rightarrow h = a\sin\theta$$

$$\cos\theta = \frac{x}{a} \Rightarrow x = a\cos\theta$$

The area of the trapezium ABCD = $\frac{1}{2}xh + bh + \frac{1}{2}xh$

$= xh + bh$

$= a\cos\theta\, a\sin\theta + ba\sin\theta$

$= a\sin\theta(a\cos\theta + b)$.

5.

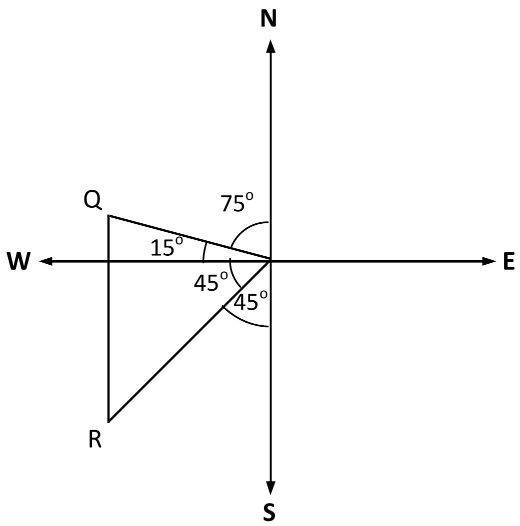


Figure 9.50

$RQ^2 = 60^2 + 80^2 - 2\times 60\times 80\times\cos 60^\circ = 5200$

$\Rightarrow$ PQ = $\sqrt{5200} \approx 72\,km$

6. $\dfrac{\sin\ \alpha}{\sin\ \beta} = \mu \quad \Rightarrow \quad \dfrac{\sin 45^{\circ}}{\sin \beta} = 1.33$

$\Rightarrow \quad \sin\ \beta = \dfrac{1}{1.33\sqrt{2}} \approx 0.5317$

$\Rightarrow \quad \beta = \sin^{-1}(0.5317)$

$\Rightarrow \quad \beta = 32^{\circ}$

7. a $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ b. $\dfrac{\sqrt{2}-\sqrt{6}}{4}$ c. $2+\sqrt{3}$

d. $\sqrt{2}-\sqrt{6}$ e. $\sqrt{3}-2$ f. $-\sqrt{2}-\sqrt{6}$

8. a. $\tan(175^{\circ} - 130^{\circ}) = \tan 45^{\circ} = 1$

b. $\sin x$

c. Note that (i) $\sin(2x) + \sin(4x) = \sin(2x) + 2\sin(2x)\cos(2x)$

$= \sin(2x)(1 + 2\cos(2x))$

(ii) $\cos(2x) - \cos(4x) = \cos(2x) - (\cos^2(2x) - \sin^2(2x))$

$= \cos(2x) - (2\cos^2(2x) - 1))$

$= \cos(2x)(1 - \cos(2x)) + (1 + \cos(2x))$

$= (1 - \cos(2x))(1 + 2\cos(2x))$

Then the quotient will be simplified to $\dfrac{\sin(2x)}{1-\cos(2x)}$.

d. 1; by direct substitution of $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$.

e. Applying the addition identity for sin gives,

$$\sin\left(\sin^{-1}\left(\frac{12}{13}\right)\right)\cos\left(\cos^{-1}\left(\frac{5}{13}\right)\right)+\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$$

$$=\frac{12}{13}\times\frac{5}{13}+\frac{5}{13}\times\frac{12}{13}=\frac{120}{169}$$

9. a. Amplitude = 20 period = $\dfrac{2\pi}{40\pi} = \dfrac{1}{20}$.

b. $f = \dfrac{w}{2\pi} = \dfrac{40\pi}{2\pi} = 20$

10.

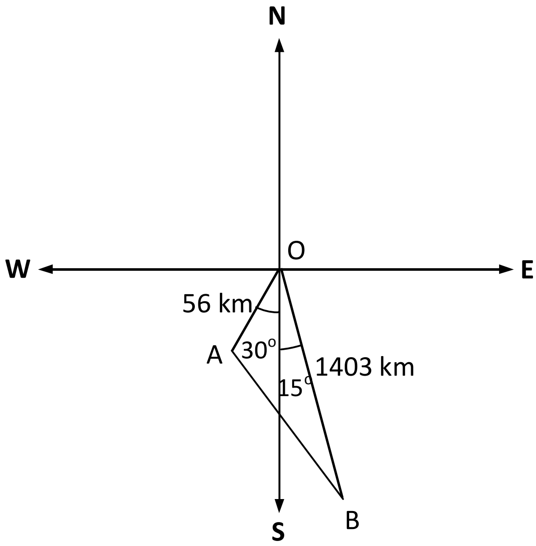


Figure 9.51

$$|AB| = \sqrt{56^2 + 1403^2 - 2\times 56\times 1403\times \frac{\sqrt{2}}{2}} \approx 1364$$

Also, $\dfrac{\sin(\mathrm{A})}{1403} = \dfrac{\sin 45^\circ}{1364} \Rightarrow \sin(\mathrm{A}) \approx 0.7272$

$$\Rightarrow m(\angle \mathrm{A}) \approx 47^\circ$$

$\Rightarrow$ The velocity of the airplane relative to the ground is 1364 km/hr S73°E

11. 20 km/hr N 75°E is the resultant velocity.

$$|\mathrm{BA}| = \sqrt{20^2 + 6^2 - 2\times 20\times 6\ \cos 75^\circ} \approx 19.34$$

$$\frac{\sin A}{6} = \frac{\sin 75^\circ}{19.34} \Rightarrow \sin \mathrm{A} \approx 0.3$$

$$\Rightarrow \mathrm{m}(\angle \mathrm{A}) \approx 17.5^\circ$$

$\Rightarrow$ The velocity of the boat relative to the ground is 18 km/hr N 57.5°E.

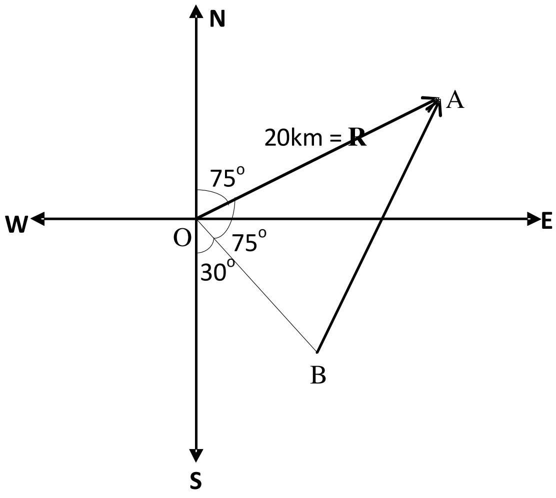


Figure 9.52

12. In ΔXYZ, $x = 23.5$, $y = 9.8$, $\angle X = 39.7^{o}$.

First a drawΔXYZ and organize the given information

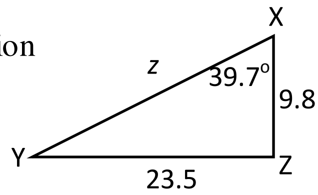


Figure 9.53

$\angle X = 39.7^{o}$ $\quad x = 23.5$

$\angle Y = ?$ $\quad y = 9.8$

$\angle Z = ?$ $\quad z = ?$

Using the SSA situation and beginning by finding y with the law of sines:

$$\frac{x}{\sin X} = \frac{y}{\sin Y} \Rightarrow \frac{23.5}{\sin 39.7^{o}} = \frac{9.8}{\sin Y}$$

$$\Rightarrow \sin Y = 0.2664$$

There are two angles less than 180^{o} with a sine of 0.2664. They are 15.4^{o} and 164.6^{o}, to the nearest of a degree. An angle of 164.6^{o} cannot be an angle of this triangle because it already has an angle of 39.7^{o} and these two angles would total more than 180^{o}. Thus 15.4^{o} is the only possibility for Y.

Therefore, $\angle Z = 180^{o} - (39.7^{o} + 15.4^{o}) \approx 124.98$

We now find z.

$$\frac{z}{\sin Z} = \frac{x}{\sin X} \Rightarrow \frac{z}{\sin 124.9^{o}} = \frac{23.5}{\sin 39.7^{o}}$$

$$\Rightarrow z \cong 30.2.$$

13. In ΔABC, $b = 15$, $c = 20$, and $\angle B = 29^o$.

Listing the known measures shows that you have the SSA situation.

$\angle A = ?$ $a = ?$

$\angle B = 29^o$ $b = 15$

$\angle C = ?$ $c = 20$

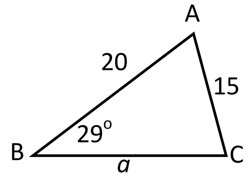


Figure 9.54

We first find C:

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{15}{\sin 29^o} = \frac{20}{\sin C}$$

$$\Rightarrow \sin C = \frac{20 \sin 29^o}{15} \approx 0.6464$$

There are two angles less than 180^o with a sine of 0.6464. They are 40^o and 140^o, to the nearest degree. This gives you two possible solutions.

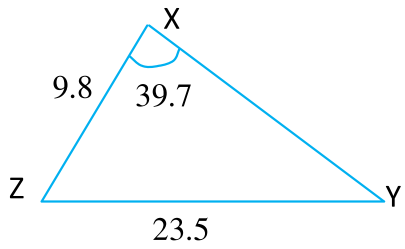


Figure 9.55

Possible solution I

If $C = 40^o$

$A = 180^o - (29^o + 40^o) = 111^o$

Then, we find a

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 111^o} = \frac{15}{\sin 29^o}$$

$$\Rightarrow a = \frac{15.\sin 111^o}{\sin 29^o} \approx 29$$

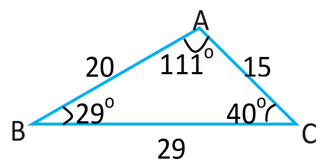


Figure 9.56

These measures make two triangles as shown in figure 9.56 and 9.57.

Possible solution II

If $C = 140^\circ$, then

$A = 180^\circ - (29^\circ + 140^\circ) = 11^\circ$

Then, we find a:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{\sin 11^\circ} = \frac{15}{\sin 29^\circ}$$

$$a = \frac{15 \sin 11^\circ}{\sin 29^\circ} \approx 6$$

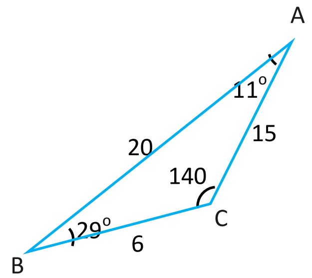


Figure 9.57

14. $x^2 = a^2\cos^2\theta - 2ab\cos\theta\sin\theta + b^2\sin^2\theta$
$y^2 = a^2\sin^2\theta + 2ab\cos\theta\sin\theta + b^2\cos^2\theta$

15. a. $T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{1.2}{9.8}} \approx 2.2\text{seconds}$

b. $\omega = \frac{2\pi}{T} = \frac{2\pi}{2.2} \approx 2.9/\text{sec}.$

c. $y = a\cos(\omega t) = 0.06\cos(2.9t).$

Answers to Review Exercises on Unit 9

1. The statements can be proved using the symmetric properties of sine, cosine and tangent and periodic property of cotangent in question 1(a).

2. a. $\sqrt{2}$ b. 2 c. 0

3. Shifting the graph of $y = \csc x$ by $\frac{\pi}{2}$ units to the left gives the graph of $y = \sec x$.

4. $f(x) = a\sin(kx)$

a. $|a| = 3,\ \frac{2\pi}{k} = \frac{2}{5}\pi \Rightarrow k = 5$

$\Rightarrow f(x) = \pm 3\sin(5x)$

b. $|a| = \frac{2}{5},\ f(3) = 0 \Rightarrow \sin(3k) = 0$

$\Rightarrow 3k = 0, \pi, 2\pi$

$\Rightarrow k = 0, \frac{\pi}{3}, \frac{2}{3}\pi$, but $k = 0$ cannot be a solution.

$$\Rightarrow f(x) = \pm\frac{2}{5}\sin\left(\frac{\pi}{3}x\right) \text{ or } \pm\frac{2}{5}\sin\left(\frac{2}{3}\pi x\right)$$

c. $\sin\left(k\left(\frac{\pi}{3}\right)\right)=1 \Rightarrow \frac{k\pi}{3}=\frac{\pi}{2} \Rightarrow k=\frac{3}{2}$

$\Rightarrow f(x)=5\sin\left(\frac{3}{2}x\right)$

d. $f(x)=\pm 2\sin(kx)$ *with* $\sin\left(k\left(\frac{\pi}{3}\right)\right)=0$

$\Rightarrow \frac{k\pi}{3}=0, \pi, 2\pi$

$\Rightarrow k=0, 3\pi, 6\pi$

$\Rightarrow f(x)=\pm 2\sin(3x) \text{ or } \pm 2\sin(6x)$

5. $f(x)=a\cos(kx)$

a. $f(x)=\pm 3\cos(5x) \Rightarrow f(3)=0$

b. $\Rightarrow 3k=\frac{\pi}{2}, \frac{3}{2}\pi \Rightarrow k=\frac{\pi}{6}, \frac{\pi}{2}$

$\Rightarrow f(x)=\pm\frac{2}{5}\cos\left(\frac{\pi}{6}x\right) \text{ or } \pm\frac{2}{5}\cos\left(\frac{\pi}{2}x\right)$

c. $\cos\left(\frac{\pi}{3}k\right)=1 \Rightarrow \frac{\pi}{3}k=0, 2\pi$

$\Rightarrow k=0, 6$ but $k=0$ is not possible.

$\Rightarrow f(x)=5\cos 6x.$

d. $\cos\left(\frac{\pi}{3}k\right)=0 \Rightarrow \frac{\pi}{3}k=\frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow k=\frac{3}{2}, \frac{9}{2}$

$\Rightarrow f(x)=\pm 2\cos\left(\frac{3}{2}x\right) \text{ or } \pm 2\cos\left(\frac{9}{2}x\right)$

6. a. $-\frac{\pi}{4}$ b. $\frac{\pi}{4}$ c. $-\frac{\pi}{3}$

7. a. 5.4^{o} b. 56.56^{o} c. -67.5^{o}

8. a. $\frac{3}{5}$ b. 0.025 c. $\frac{-3}{4}$ d. $\frac{3\sqrt{7}}{8}$

e. $\sqrt{1-x^2}$ f. $\sqrt{1-x^2}$ g. $\dfrac{\sqrt{65}}{4}$

h. Using the fact that sine and arctangent are odd function, you have:

$$\sin\left(2\tan^{-1}\left(-\frac{4}{5}\right)\right) = -\sin\left(2\tan^{-1}\left(\frac{4}{5}\right)\right)$$

Let $\theta = \tan^{-1}\left(\dfrac{4}{5}\right)$ *See Figure 9.57*

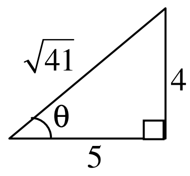


Figure 9.58

Then, $-\sin\left(2\tan^{-1}\left(\dfrac{4}{5}\right)\right) = -\sin(2\theta) = -2\sin\theta\cos\theta$

$$= 2\times\frac{4}{\sqrt{41}}\times\frac{5}{\sqrt{41}} = \frac{40}{41}$$

9. $\sin\alpha = \sin(\alpha+\theta-\theta) = \sin(\alpha+\theta)\cos\theta - \cos(\alpha+\theta)\sin\theta$

But $\cos\theta = \dfrac{4}{5}$ and $\cos(\alpha+\theta) = \sqrt{1-\left(\dfrac{55}{73}\right)^2} = \dfrac{48}{73}$

Hence, $\sin\alpha = \dfrac{55}{73}\times\dfrac{4}{5} - \dfrac{48}{73}\times\dfrac{3}{5} = \dfrac{76}{365}$

10. Clearly

$$\cos x = \frac{-35}{37} \text{ and } \cos\frac{x}{2} < 0$$

From the identify

$$\cos^2\frac{x}{2} = \pm\sqrt{\frac{\cos x+1}{2}} \Rightarrow \cos\frac{x}{2} = \pm\sqrt{\frac{\frac{-35}{37}+1}{2}} = \pm\sqrt{\frac{1}{37}}$$

$$\Rightarrow \cos\frac{x}{2} = -\frac{1}{\sqrt{37}}$$

11. a. $y = 2\sin\left(x - \frac{\pi}{2}\right)$

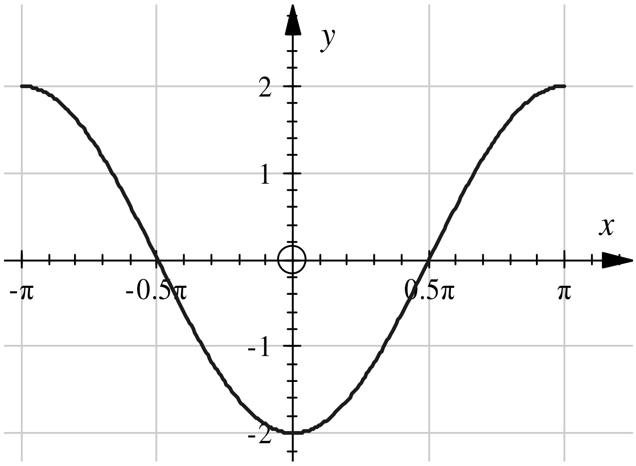


Figure 9.59

b. $y = \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)$

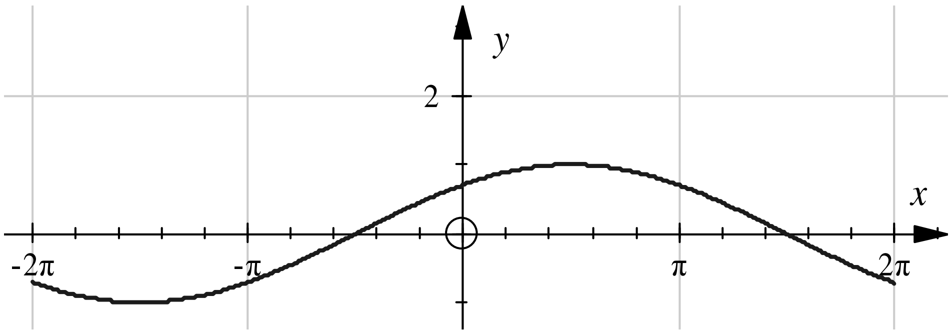


Figure 9.60

c. $y = 3 - \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

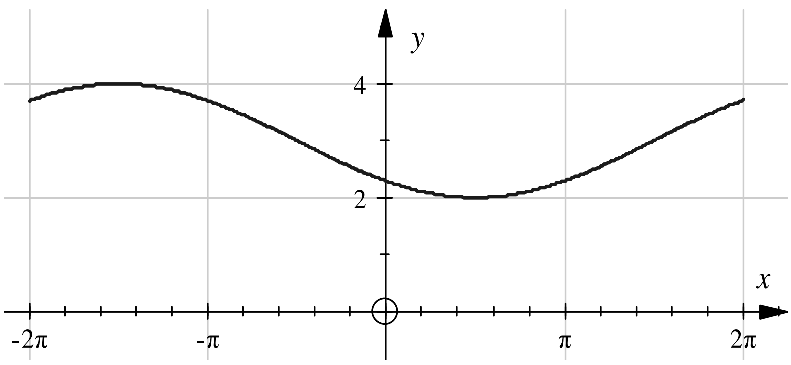


Figure 9.61

d. $y = 2\cos\left(\frac{\pi}{4}x\right) + 3$

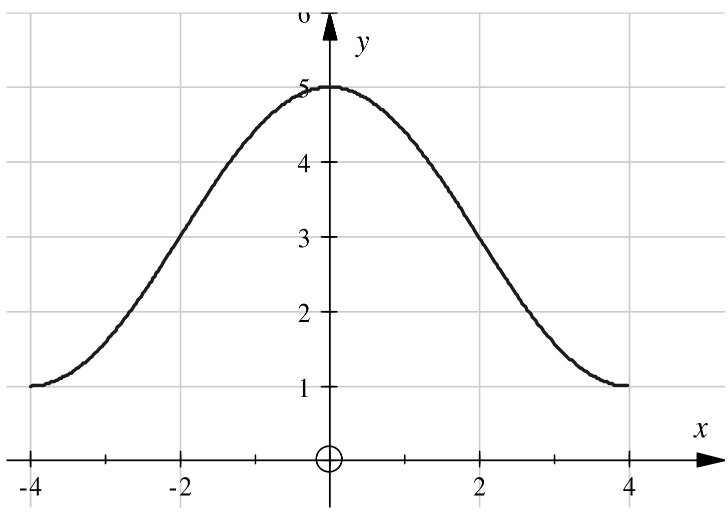


Figure 9.62

12. a. $\frac{5}{\sin\alpha} = \frac{c}{\sin\gamma} \Rightarrow \frac{5}{\sin 60^{\circ}} = \frac{c}{\sin 70^{\circ}}$

$$\Rightarrow c = \frac{5}{\left(\frac{\sqrt{3}}{2}\right)} \times \sin 70^{\circ}$$

$$\Rightarrow c \approx \frac{10\sqrt{3}}{3}(0.9397)$$

$$\Rightarrow c \approx 5.4254$$

and $\frac{5}{\sin 60^{\circ}} = \frac{b}{\sin 50^{\circ}} \Rightarrow b = \frac{10\sqrt{3}}{3} \times \sin 50^{\circ}$

$$\Rightarrow b \approx \frac{10\sqrt{3}}{3}(0.76602)$$

$$\Rightarrow b \approx 4.4225$$

b. $\frac{5}{\sin 45^{\circ}} = \frac{3}{\sin\beta} \Rightarrow \sin\beta = \frac{3}{5}\sin 45^{\circ} = 0.6\left(\frac{\sqrt{2}}{2}\right)$

$$\Rightarrow \sin\beta \approx 0.4243$$

$$\Rightarrow \beta \approx 25^{\circ}$$

c. $\dfrac{11}{\sin 59.5^{\circ}} = \dfrac{24}{\sin\beta} \Rightarrow \sin\beta = \dfrac{24}{11} \times \sin 59.5^{\circ}$

$$= \frac{24}{11}(0.8616) \approx 1.8799$$

This is impossible sine $|\sin\beta| \le 1.$ There is no such triangle.

13. a. $c^2 = a^2 + b^2 - 2ab\cos\gamma = 5^2 + 6^2 - 2\times 5\times 6\times\cos 60$

$= 31$

$\Rightarrow c = \sqrt{31}$

b. $a^2 = b^2 + c^2 - 2ab\cos\alpha$

$= 8^2 + 7^2 - 2\times 8\times 7\cos 30^{\circ}$

$= 113 - 56\sqrt{3}$

$\Rightarrow a = \sqrt{113 - 56\sqrt{3}}$

c. $b^2 = a^2 + c^2 - 2ac\cos\beta$

$= 20^2 + 30^2 - 2\times 20\times 30\times\cos 110^{\circ}$

$= 400 + 900 - 1200(-0.3420)$

≈ 1710.4

$\Rightarrow b = 41.357$

14. a. $\sin(2x) = \sqrt{3}\sin x \Rightarrow 2\sin x\cos x = \sqrt{3}\sin x$

$\Rightarrow 2\sin x\cos x - \sqrt{3}\sin x = 0$

$\Rightarrow \sin x\left(2\cos x - \sqrt{3}\right) = 0$

$\Rightarrow \sin x = 0 \ or \ 2\cos x - \sqrt{3} = 0$

$\Rightarrow x = n\pi \ or \ x = 2n\pi \pm \dfrac{\pi}{6};\ n \in \mathbb{Z}$

$\Rightarrow S.S = \left\{n\pi, 2n\pi \pm \dfrac{\pi}{6};\ n \in \quad\right\}$

b. $\sin(2x) = -\dfrac{1}{\sqrt{2}}$

The particular solutions are $2x_1 = \dfrac{5\pi}{4}, 2x_2 = \dfrac{7\pi}{4}$

$$\Rightarrow \; x_1 = \frac{5\pi}{8}, x_2 = \frac{7\pi}{8}$$

$$\Rightarrow \text{ S.S } = \left\{\frac{5\pi}{8} + n\pi, \frac{7\pi}{8} + n\pi;\ n \in \mathbb{Z}\right\}$$

Also, $S.S = \left\{\dfrac{5\pi}{8}(-1)^n + \dfrac{n\pi}{2}; n \in \quad \right\}$

c. $\tan\left(3x - \dfrac{\pi}{4}\right) = \sqrt{3} \Rightarrow 3x_1 - \dfrac{\pi}{4} = \dfrac{\pi}{3} \Rightarrow x_1 = \dfrac{7\pi}{36}$

$$\Rightarrow \quad S.S = \left\{\frac{7\pi}{36} + n\pi; n \in \quad \right\}$$

d. $2 \sin x = \sin(2x) \Rightarrow 2 \sin x = 2 \sin x \cos x$

$$\Rightarrow \; \sin x = 0 \text{ or } \cos x = 1$$

$$\Rightarrow \text{ S.S} = \{n\pi, 2n\pi; n \in \quad \} = \{n\pi;\ n \in \quad \}$$

e. $\tan\left(\dfrac{x}{2}\right) - 2\sin x = 0 \Rightarrow \tan\left(\dfrac{x}{2}\right) - 4\sin\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{2}\right) = 0$

$$\sin\left(\frac{x}{2}\right)\left(\frac{1}{\cos\left(\frac{x}{2}\right)} - 4\cos\left(\frac{x}{2}\right) = 0\right) \Rightarrow \; \sin\left(\frac{x}{2}\right) = 0 \text{ or } 1 - 4\cos^2\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \; \frac{x}{2} = n\pi \quad \text{or} \quad \cos\left(\frac{x}{2}\right) = \pm \frac{1}{2}$$

$$\Rightarrow x = 2n\pi, 4n\pi + \frac{2\pi}{3} \; or \; x = 4n\pi \pm \frac{4}{3}\pi$$

$$\Rightarrow \; S.S = \left\{2n\pi,\ 4n\pi \pm \frac{2\pi}{3},\ 4n\pi \pm \frac{4}{3}\pi\right\} = \left\{2n\pi,\ 2n\pi \pm \frac{2\pi}{3}\right\}$$

15. If A drives 80 km/hr, then it drives 120 km in $1\frac{1}{2}$ hours. Similarly if B drives 90 km/hr, it drives 135 km in $1\frac{1}{2}$ hours. *See figure 9.62*.

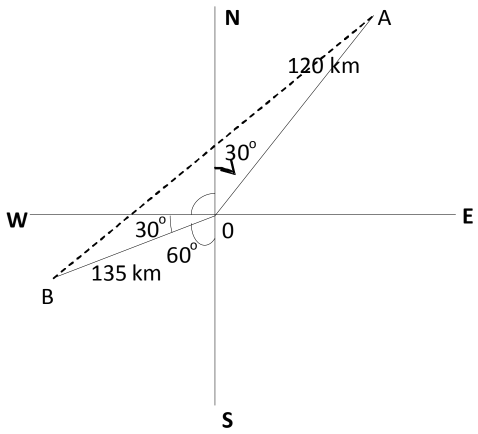


Figure 9.63

$AB^2 = (135)^2 + (120)^2 - 2 \times 135 \times 120 \cos(150^\circ)$

$= 32625 + 16200\sqrt{3}$

AB = 246.34

They are 246.34 km far apart.

16. Let x be the width of the river, then

$\tan 15^\circ = \frac{h}{x} \Rightarrow h = x \tan 15^\circ \approx 0.2679\,x$

$\tan 30^\circ = \frac{h+15}{x}$

$\Rightarrow h + 15 = x \tan 30^\circ = \frac{x\sqrt{3}}{3}$

But $h = 0.2679x$

$\Rightarrow 0.2679x + 15 = \frac{x\sqrt{3}}{3}$

$\Rightarrow x = 48.4731$

$\Rightarrow$ The width of the river is about 48.5 m.

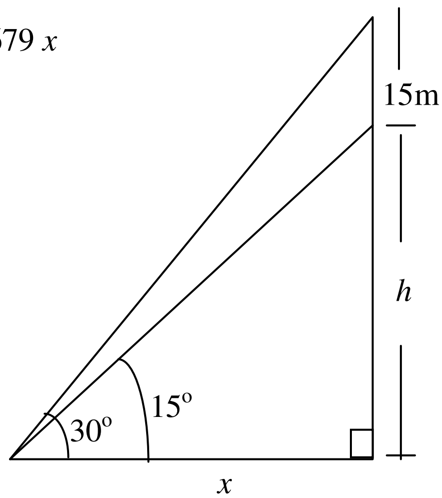


Figure 9.64

17. $\dfrac{\sin 40^\circ}{\sin \beta} = 1.309 \Rightarrow \dfrac{0.6428}{\sin \beta} = 1.309$

$$\Rightarrow \sin \beta = \frac{0.6428}{1.309} \approx 0.4912$$

$$\Rightarrow \beta \approx 29.42^\circ$$

18. a. $\cos^4 x - \sin^4 = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$

$$= \cos^2 x - \sin^2 x$$

$$= \cos(2x)$$

b. $\dfrac{\cot x - 1}{\cot x + 1} = \dfrac{\dfrac{\cos x}{\sin x} - 1}{\dfrac{\cos x}{\sin x} + 1} = \dfrac{\cos x - \sin x}{\cos x + \sin x}$

$$= \frac{\cos x - \sin x}{\cos x + \sin x} \cdot \frac{\cos x + \sin x}{\cos x + \sin x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x + 2\sin x \cos x}$$

$$= \frac{\cos(2x)}{1 + \sin(2x)}$$

19. Let $\sin^{-1} x = \theta$, then $\tan(2\sin^{-1} x)) = \tan 2\theta$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2x\sqrt{1 - x^2}}{1 - 2x^2}$$

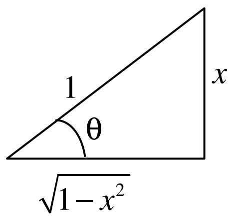


Figure 9.65

20. $P(t) = 5 + 3\sin\left(\dfrac{2\pi t}{5}\right); \; 0 \le t \le 12$

a. The initial population is when $t = 0$

$$P(0) = 5 + 3\sin\left(\frac{2\pi \times 0}{5}\right) = 5$$

$\Rightarrow$ The initial population is 500 birds.

b. The population is the largest if $\sin\left(\frac{2\pi t}{5}\right)=1$ and it is the smallest if

$\sin\left(\frac{2\pi t}{5}\right)=-1$.

Hence, the maximum population is $(5 + 3 \times 1) \times 100 = 800$ birds and the minimum population is $(5+3(-1))\times 100 = 200$ birds

c. $P(t)\times 100=350 \Rightarrow 5+3\sin\left(\frac{2\pi t}{5}\right)=3.5 \Rightarrow \sin\left(\frac{2\pi t}{5}\right)=-0.5$

$\Rightarrow \frac{2\pi t}{5}=\frac{7\pi}{6} \Rightarrow t=2\frac{11}{12}$.

The population for the first time reaches 350 birds, almost at the third month.

d. $P(12)=5+3\sin\left(\frac{2n(12)}{5}\right) = 5+3\sin\left(4\pi+\frac{4}{5}\pi\right)$

$= 5+3\sin\left(\frac{4}{5}\pi\right) = 5+3\sin\left(\frac{\pi}{5}\right)$

$= 5 + 3\,(0.5878)$

$= 6.7634$

$\Rightarrow$ After one year, the population reaches 676 birds.

UNIT 10 INTRODUCTION TO LINEAR PROGRAMMING

INTRODUCTION

Many real life problems involve finding the maximum or minimum value of a function. For this purpose, there are different approaches of finding the maximum or minimum value of a function that may represent a real life problem under certain conditions. In particular, linear programming is a field of mathematics that deals with the problem of finding the maximum or minimum value of a given linear function subject to certain conditions expressed as linear equalities or inequalities. In this unit, the graphical method of solving linear programming problems involving two variables is discussed. For problems with more than two variables and many constraints, the simplex method is used which is not included in this text.

Unit Outcomes

After completing this unit, students will be able to:

- *identify regions of inequality graphs.*
- *create real life examples of linear programming problems using inequalities and solve them.*

Suggested Teaching Aids in Unit 10

This unit starts with a review of linear graphs and you may use a big ruler to draw lines on the board. When you find graphical solutions of systems of linear inequalities you can use different colored chalks to sketch solution regions. For real life linear programming problems, you can encourage students to identify several points by asking them whether a point is in the feasible region or not. In order to do this, you may prepare graphed flip charts. You can also use various softwares than can freely be found in the internet through which you can describe the ideas, feasible region, critical points and solution to a linear programming problem.

10.1 REVISION ON LINEAR GRAPHS

Periods Allotted: 4 periods

Competency

At the end of this sub-unit, students will be able to:

- *draw graph of linear inequalities $y \leq mx + c$, $y \geq mx + c$, $ax + by \geq c$, and $ax + by \leq c$.*

Introduction

In unit 3, students have studied the concepts of coordinate geometry where the first section was the straight line. This subunit is intended to give students a quick review of graphing straight lines which they have discussed previously.

Teaching Notes

It is expected that students have a good background on the different forms of equations of straight lines. Both Activity 10.1 and Exercise 10.1 are intended to see if students recall the concepts discussed in unit 3. You can encourage students to identify parallel and perpendicular lines and indicate the form of the equation of the given lines. To help them recall some of the key ideas from what they have learned about lines in unit 3, you can let them do Activity 10.1.

Answers to Activity 10.1

a. Slope $m = \frac{3-(-2)}{k-1} = \frac{5}{k-1} = 5 \quad \Rightarrow \quad k = 2$

b. Slope of ℓ_1 is $m_1 = \frac{3-1}{-2-1} = -\frac{2}{3}$

Slope of ℓ_2 is $m_2 = \frac{6-2}{-3-3} = -\frac{4}{6} = \frac{-2}{3} = m_1$

Therefore, ℓ_1 is parallel to ℓ_2.

You can also let them do the examples given in the student textbook and some more other exercises.

Assessment

To make sure that the students have recalled what they have studied in unit 3 and can proceed further, you can give them exercise 10.1 (1) as class work (2) and (3) as homework and check their results.

Answers to Exercise 10.1

1. a. $y - 3 = 4\,(x + 1)$ or $y = 4x + 7$

   b. $m = \dfrac{1-2}{-4-1} = \dfrac{-1}{-5} = \dfrac{1}{5}$

   Equation: $y - 2 = \dfrac{1}{5}(x - 1) \Rightarrow y = \dfrac{1}{5}x + \dfrac{9}{5}$

   c. $y = -2x + 5$

2. Slope of the line $4x + ky = 8$ is $m_1 = \dfrac{-4}{k}$

   Slope of the line $x + 2y = 0$ is $m_2 = \dfrac{-1}{2}$

   Parallel means $m_1 = m_2 \Rightarrow \dfrac{-4}{k} = \dfrac{-1}{2} \Rightarrow k = 8$

3. As we can observe from the graph, the two lines $y = 2x - 1$ and $y = 2x + 3$ do not cross each other because they have the same slope (they are parallel). But lines $y = 2x - 1$ and $3x - 2y = 4$ meet at (–2, –5). Similarly, $y = 2x + 3$ and $3x - 2y = 4$ meet (crosses) at (–10, –17).

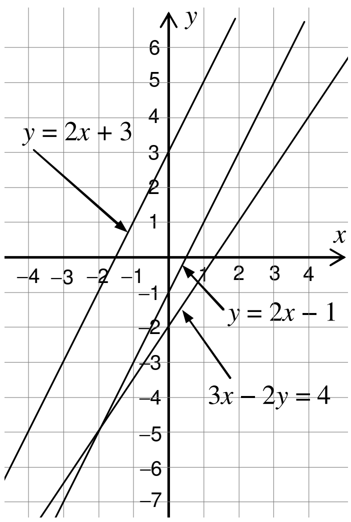


Figure 10.1

10.2 GRAPHICAL SOLUTIONS OF SYSTEMS OF LINEAR INEQUALITIES

Periods Allotted: 2 periods

Competency

At the end of this sub-unit, students will be able to:

- *draw graphs of linear inequalities.*

Introduction

In this sub-unit, students are expected to identify regions satisfied by a system of linear inequalities. Such regions are called solution (feasible) regions. In addition to half planes and possibly their intersections, finding feasible regions will involve solving systems of two equations with two variables simultaneously.

Teaching Notes

Once students are familiar with drawing lines, discuss the fact that a line divides the plane into two half-planes. A linear inequality is satisfied by one of the two half-planes. To check which half is satisfied by the linear inequality, you can test a point not on the line. An inequality of the form $ax + by < c$ does not contain the boundary line, (use broken line); whereas $ax + by \leq c$ contains the boundary line (use unbroken or solid line).

A graphical solution of a system of linear inequalities is then the intersection of the regions satisfied by all the given inequalities. Finding the corner points or vertices of such regions needs solving equations simultaneously. You can check if students can recall this method by asking some of them to stand up and do it on the board.

To make the students practice shading feasible regions, you can let them do Activity 10.2. This activity has dual purpose. One is drawing lines and the other is identifying the feasible region.

Answers to Activity 10.2

This activity will enable students to shade regions of single linear inequalities.

a.

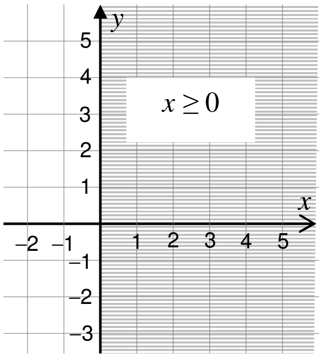


Figure 10.2

b.

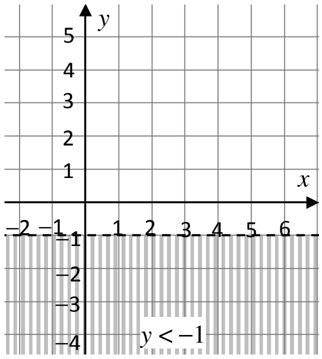


Figure 10.3

c.

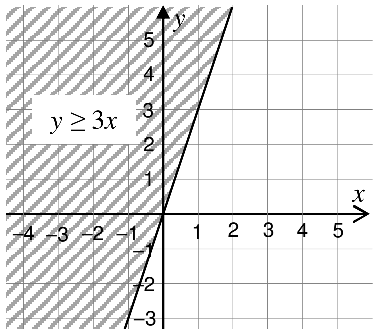


Figure 10.4

d.

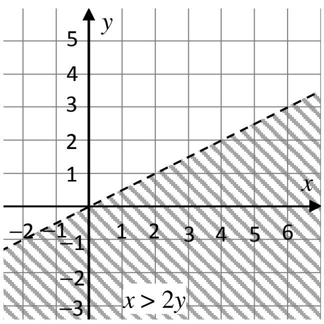


Figure 10.5

e.

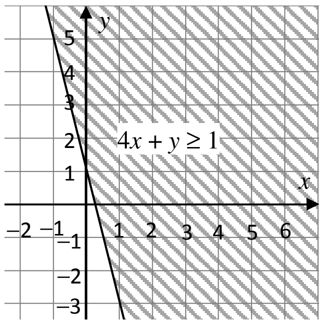


Figure 10.6

f.

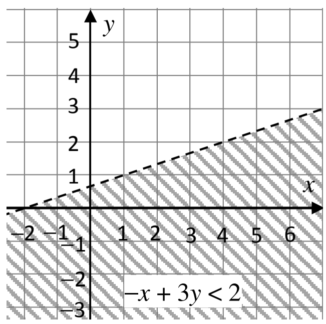


Figure 10.7

Once you make sure that the students can shade a feasible region, you can give them some ordered pairs and ask them whether they are feasible or not feasible, i.e. whether they lie in the feasible region or not. For instance, in Activity 10.2 (c) you can take the point (2, 8) which is feasible and the point (4, 3) which is not feasible. It is also possible to let the students practice checking whether points lie in the feasible region from the questions in Exercise 10.2.

After they have done these, you can proceed to discussing two inequalities and determine feasible regions. You can use example 3 and 4 of the student textbook. Continuing in the same way, you can let the students practice to find feasible (solution) region from more than two inequalities. To do this, you can discuss example 5 of the student text book and give them more questions. Let them also determine the corner points and practice if some other points lie in the feasible region.

You can give Exercise 10.2 as homework. On this question, you can also ask them to determine the corner points for each feasible region.

Assessment

To make sure that the students have understood feasible regions and corner points, you can give them several problems which they each need to do individually. You can also use Exercise 10.2 and ask the students whether or not the points (2, 2), (−1, 3) and (3, 5) are feasible in (b).

Answers to Exercise 10.2

a.

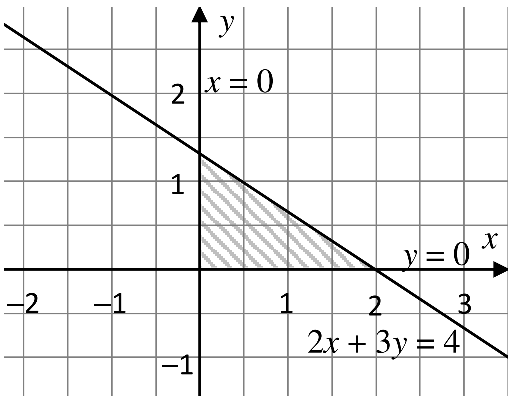


Figure 10.8

b.

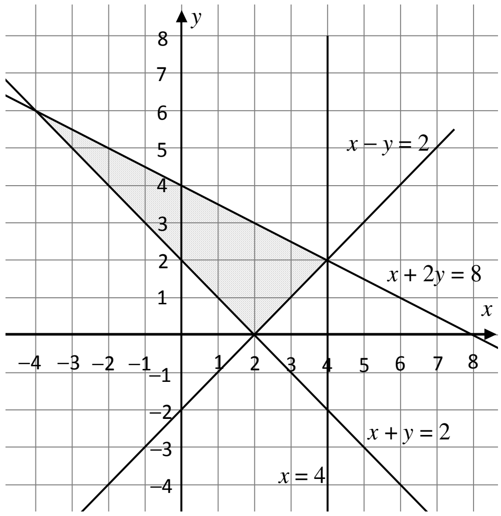


Figure 10.9

c.

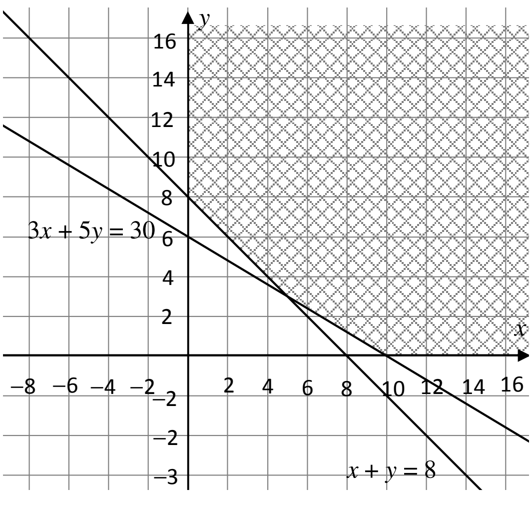


Figure 10.10

d.

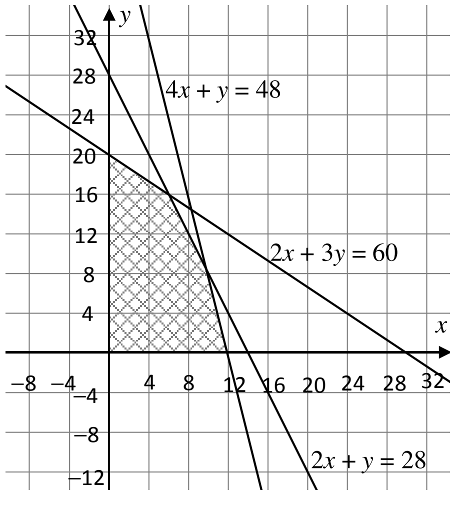


Figure 10.11

10.3 MAXIMUM AND MINIMUM VALUES

Periods Allotted: 5 periods

Competency

At the end of this sub-unit, students will be able to:

- *find maximum and minimum values of a given objective function under given constraints.*

Introduction

Many applications in business and economics involve a process called optimization in which we are asked to find the maximum or the minimum value of a quantity. The function whose maximum or minimum value is to be found under certain conditions is called the objective function. In this text, we consider an optimization strategy called linear programming. In particular, when the objective function depends on two variables (x and y), we can easily solve it using graphical methods in the plane.

Teaching Notes

Before you give the formal definition of maximum and minimum values of a function, make sure that students understand how they can transform the word problem in Group Work 10.1 into a system of linear inequalities.

Answers to Group Work 10.1

You can guide them to first assign variables for the two numbers say x and y. Since the numbers are positive, guide them to realize that $x > 0$ and $y > 0$. From their difference being at most 7, they need to get $x - y \leq 7$. You also need to consider $x + y \geq 15$. Under all these given conditions (constraints), emphasize that the objective is to find x and y whose product xy is the largest (maximum). You can tell them that they should not worry about the exact solution since they can easily solve it after the discussion of the fundamental Theorem of linear programming.

After they formulate the feasible region, it is advisable to discuss what the maximum and the minimum values are and what they mean mathematically. Then you let them discuss Example 1 of the student textbook so as to state the Fundamental Theorem of Linear Programming. Here students need to understand why we use straight lines at fixed values of Z to get the maximum or minimum solution. Right after stating the Fundamental Theorem of Linear Programming, they need to realize that simply checking the objective function values at the corner points is sufficient to determine the maximum or minimum value. Describe this by use of Example 2 of the student textbook. And, by forming groups of students let them do Activity 10.3. This activity will help them to practice solving or finding maximum or minimum values. When they do the activity, let some group members present their work to the whole class and discuss their solutions.

Answers to Activity 10.3

1. You can take the points (1, 1) and $\left(\frac{2}{3}, 1\right)$ inside the solution region with $z = 5$ and $z = 4$, respectively.
2. a. vertices at (0, 0), (5, 0) and (0, 2)

Vertex	$z = 6x + 10y$
(0, 0)	$z = 6(0) + 10(0) = 0$
(5, 0)	$z = 6(5) + 10(0) = 30$
(0, 2)	$z = 6(0) + 10(2) = 20$

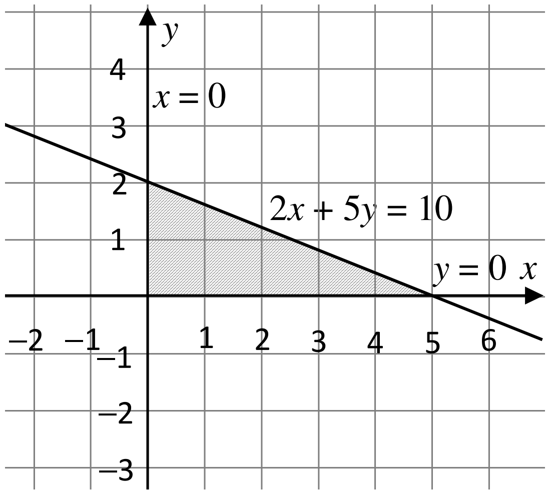


Figure 10.12

Thus, maximum value of z is 30 and minimum is 0.

b. vertices at (0, 0), (36, 0), (24, 8) and (0, 20)

Vertex	$z = 4x + y$
(0, 0)	$z = 4\ (0) + 0 = 0$
(36, 0)	$z = 4(36) + 0 = 144$
(24, 8)	$z = 4(24) + 8 = 104$
(0, 20)	$z = 4\ (0) + 20 = 20$

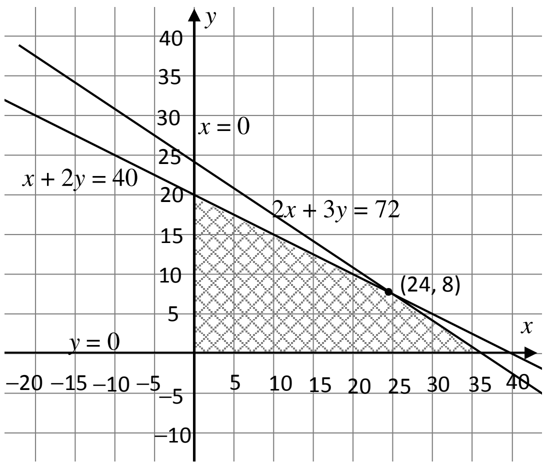


Figure 10.13

Thus, maximum value is 144 and minimum value is 0.

Following the activity which they have done in group, you can give them as many examples and exercises as possible to practice individually. These may, as well, help you to assess students' understanding. While they do the group work, you can let those who finished early do the following problems in class.

a. maximize $P = 2x + y$
subject to $x \geq 0$
$y \geq 0$
$4x + 3y \leq 12$
$5x + 2y \leq 10$

b. maximize $P = x + 3y$
subject to $7x + 4y \leq 28$
$5x + 9y \leq 45$
$5x - 3y \leq 15$
$x, y \geq 0$

Assessment

To make sure that the students are capable of finding maximum or minimum values, you can use exercise 10.3 for the purpose of assessment. It is also possible to give a test or quiz.

Answers to Exercise 10.3

a. vertices at (0, 0), (10, 0), (12, 2) and (0, 8).

Vertex	$z = 2x + 3y$
(0, 0)	$z = 2\,(0) + 3(0) = 0$
(10, 0)	$z = 2(10) + 3(0) = 20$
(12, 2)	$z = 2\,(12) + 3\,(2) = 30$
(0, 8)	$z = 2\,(0) + 3\,(8) = 24$

The maximum value = 30 and minimum value = 0.

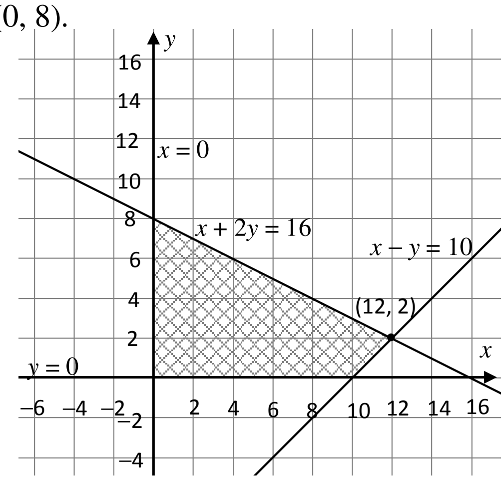


Figure 10.14

b. vertices at (0, 0), (14, 0), (7, 3) and $\left(0, \dfrac{22}{5}\right)$

Vertex	$z = 2x + 3y$
(0, 0)	$z = 2\,(0) + 3(0) = 0$
(14, 0)	$z = 2(14) + 3(0) = 28$
(7, 3)	$z = 2\,(7) + 3\,(3) = 23$
$(0, \dfrac{22}{5})$	$z = 2\,(0) + 3\,(\dfrac{22}{5}) = \dfrac{66}{5}$

The maximum value is 28 and minimum is 0.

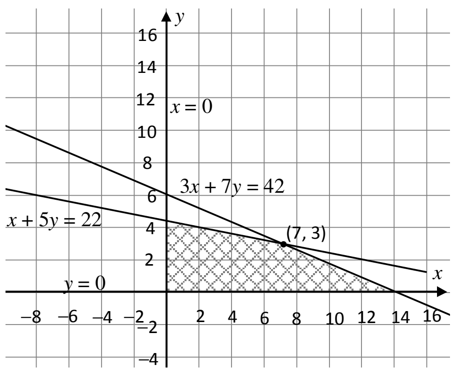


Figure 10.15

c. The point of intersection of $x + 2y = 4$ and $3x + y = 7$ is (2, 1).
The point of intersection of $3x + y = 7$ and $-x + 2y = 7$ is (1, 4).

Vertex	$z = 4x + 2y$
(2, 1)	$z = 4 (2) + 2(1) = 10$
(1, 4)	$z = 4(1) + 2(4) = 12$
(4, 0)	$z = 4 (4) + 2 (0) = 16$

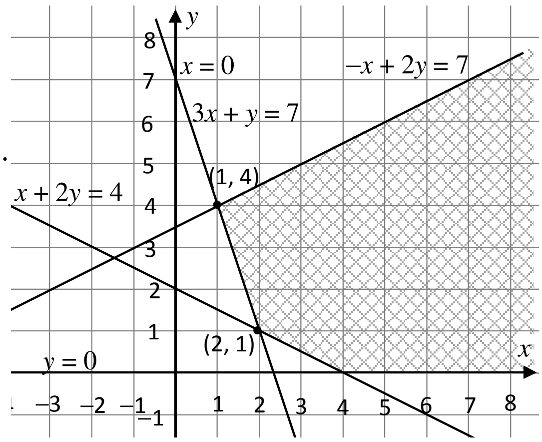


Figure 10.16

The minimum value is 10 at (2, 1).
This problem has no maximum value.

d. vertices at (0, 0), (5, 0), (0, 3) and (4, 1).

	$z = 4x + 5y$
(0, 0)	z = 4 (0) + 5 (0) = 0
(5, 0)	z = 4 (5) + 5 (0) = 20
(0, 3)	z = 4 (0) + 5 (3) = 15
(4, 1)	z = 4 (4) + 5 (1) = 21

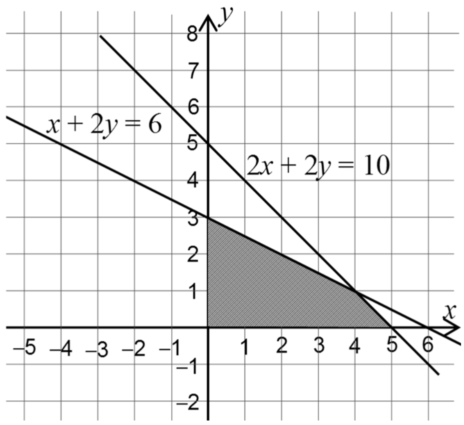


Figure 10.17

The maximum value is 21 and minimum is 0.

e. vertices at (3, 0), (0, 2), (0, 4) and (5, 3).

vertex	z = $4x + 3y$
(3, 0)	z = 4 (3) + 3 (0) = 12
(0, 2)	z = 4 (0) + 3 (2) = 6
(0, 4)	z = 4 (0) + 3 (4) = 12
(5, 3)	z = 4 (5) + 3 (3) = 29

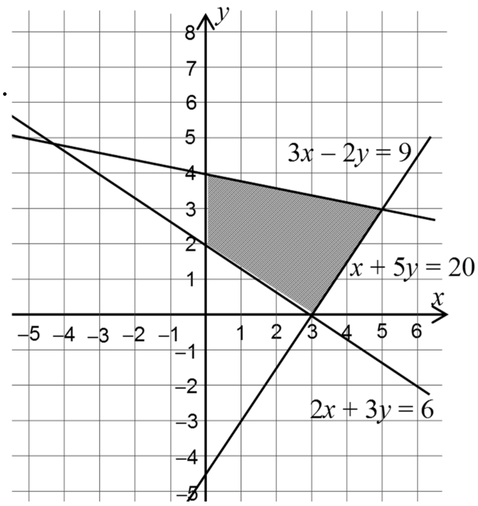


Figure 10.18

The maximum value is 29 and minimum is 6.

f. vertices at (0, 0), (2, 0), (6, 4) and (2, 6).

vertex	$z = 3x + 4y$
(0, 0)	z = 0
(2, 0)	z = 6
(6, 4)	z = 34
(2, 6)	z = 30

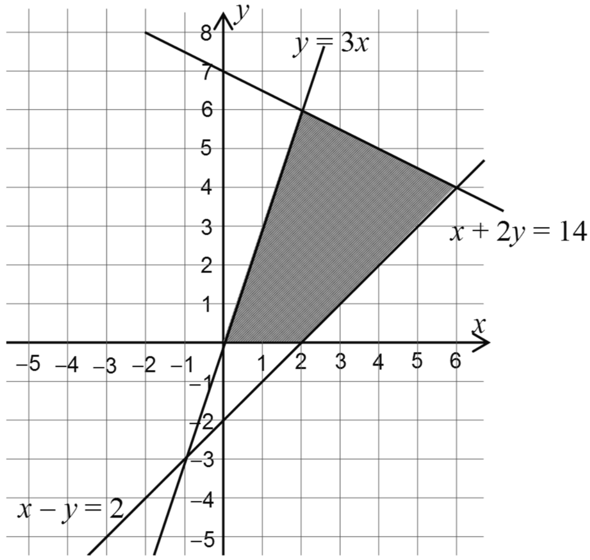


Figure 10.19

The maximum value is 34 and minimum is 0.

10.4 REAL LIFE LINEAR PROGRAMMING PROBLEMS

Periods Allotted: 6 periods

Competencies

At the end of this sub-unit, students will be able to:

- *create linear inequalities from real life problems.*
- *solve real life linear programming problems.*

Introduction

In the examples and exercises given in the student textbook, the objective function and the constraints were given as linear equations and inequalities respectively. But in real life practical problems, both the objective function and the constraints have to be constructed from some given conditions. This sub-unit deals with how real life problems can be formulated as mathematical problems consisting of objective function and constraints.

Teaching Notes

Solving real life linear programming problems involves two basic steps:

1. expressing the problem in terms of objective function and constraints (as linear equalities or inequalities)
2. solve the linear programming problem using the techniques discussed in sub-unit 10.3.

To begin this sub-unit, you can form groups of students and let them do Group Work 10.2. This Group work will help you to see if students can construct an objective function and inequality constraints. It will also help you to guide students to perform the above two steps in the practical examples given in sub-unit 10.4. When the students do the group work, you can select some group members to present their work to the whole class and discuss their work.

Answer to Group work 10.2

1. a. The profit from sales of 6 chairs and 4 tables is $9(6) + 7(4) = 82$ Birr.

   b. If x represents number of chairs and y represents number of tables, then the objective function (in this case profit function) will be $9x + 7y$.

2. The three inequalities and the corresponding region are given by:

   a. $C \geq 2$ b. $W \geq 2$ c. $C + W \leq 10$

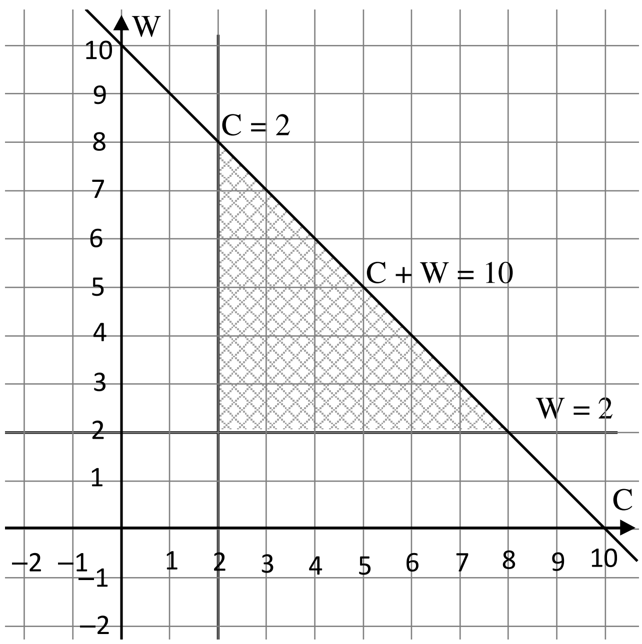


Figure 10.20

You then proceed to discuss the examples given in the student text book with active participation of the students. At the end, you summarize the steps the students can use to solve geometrically real life optimization problems of the examples type.

To enable students work with such optimization problems from their real life you can give them a task as group work where they, for example, can go to a workshop that produces doors and windows, chairs and tables, or beds and cupboards, etc from their surroundings, formulate a linear programming problem and solve it. This may be suitable, however, for clever students. You can also give them the following problems as homework and make them present their results in class.

Chaltu raises two types of animals, namely, hens and goats for commercial purpose. She does not want to raise more than 18 animals. It costs her Birr 6 to raise a hen and Birr 12 to raise a goat. She has Birr 156 available for this purpose. The profit from each hen is Birr 8 and the profit from each goat is Birr 18. How many of each animals should she raise to maximize profit?

Assessment

For the purpose of assessing students' understanding, you can give them several problems as homework. You may also use exercise 10.4 for the purpose of assessment. Since this is the end of the unit, it is also possible to give a test or quiz.

Answers to Exercise 10.4

a. Let x = the number of sheep to be bought.

and y = the number of goats to be bought.

Then $300x + 200y \leq 1700$.

i. The maximum number of goats he can buy is 8 (since 200 (8) = 1600 but 200 (9) = 1800 > 1700) or $200\,y \leq 1700 \Rightarrow y \leq 8.5$

Therefore, the maximum no goats he can buy is 8.

ii. He bought 2 sheep for Birr 600. He is left with Birr 1100 which can buy maximum 5 goats. $200y \leq 1100 \Rightarrow y \leq 5.5$ and hence the maximum number of goats he can buy is 5.

iii. 4 sheep and 3 goats is Birr 1800 which is not possible.

2 sheep and 5 goats is Birr 1600 which is possible.

3 sheep and 4 goats is Birr 1700 which is possible (and is the maximum possible).

b. To define the decision variables, let

x = the number of Table A to be produced.

and y = the number of Table B to be produced.

Summarize in table form as:

	Table A	**Table B**	**Max. hrs**
Cutting time	2	10	112
Assembling time	4	3	54
Profit	60	170	

Then, the objective function is to maximize profit

$$P = 60x + 170y$$

Subject to $2x + 10y \leq 112,$

$4x + 3y \leq 54,$

$x \geq 0,\ y \geq 0$

Vertices at (13.5,0) , (6, 10) and (0, 11.2) with maximum profit of Birr 2060 attained by producing 6 Table A and 10 Table B.

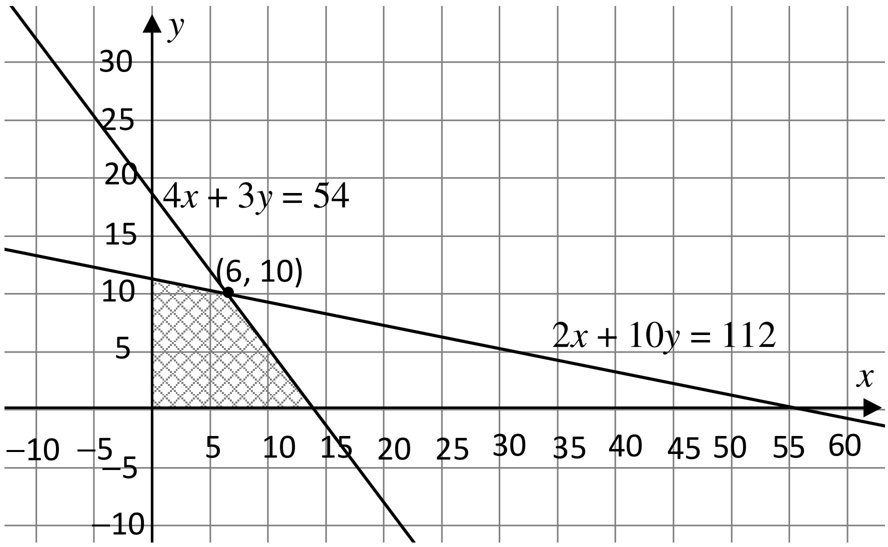


Figure 10.21

c. Let x = number of buses

y = number of vans

Summary:

	Buses	**Vans**
Students	36	6
Supervisors	4	1
Cost	1000	150

Minimize cost $C = 1000\,x + 150y$

Subject to $36x + 6y \ge 420$

$4x + y \le 48,$

$x, y \ge 0$

The vertices are $\left(\frac{35}{3}, 0\right)$, $(12, 0)$ and $(11, 4)$

The minimum cost Birr11, 600 is attained if the officers rent 11 buses and 4 vans.

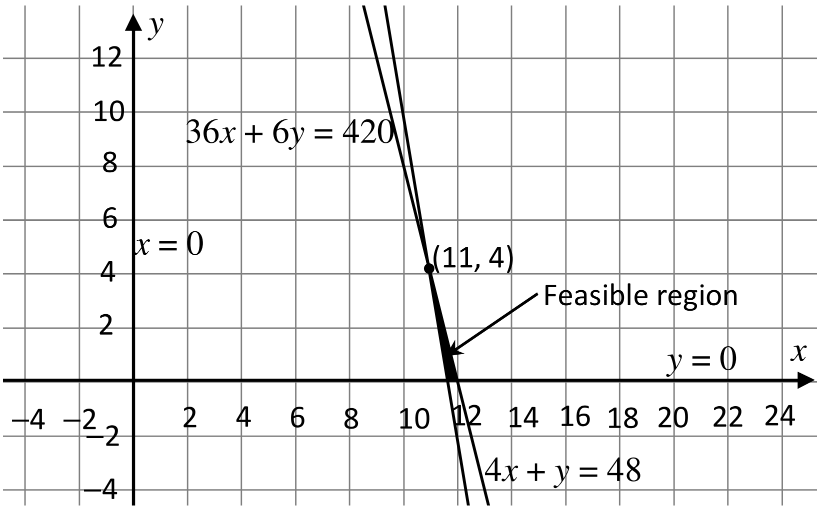


Figure 10.22

Answers to Review Exercises on Unit 10

1. a. $m = \frac{-3}{4}$ b. $m = \tan 45 = 1$ c. $m = \frac{1}{2}$

2. Take two points from each and sketch

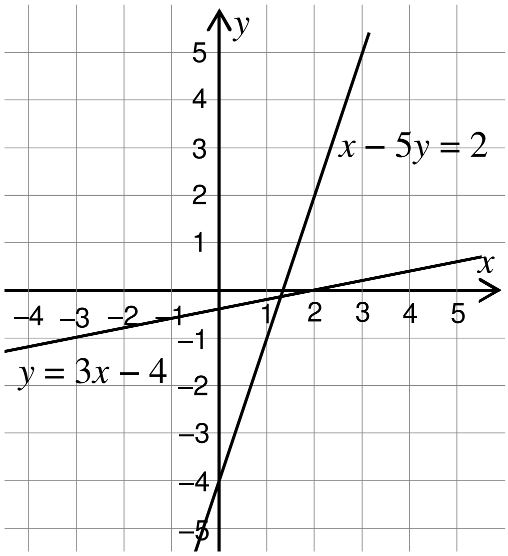


Figure 10.23

3. a.

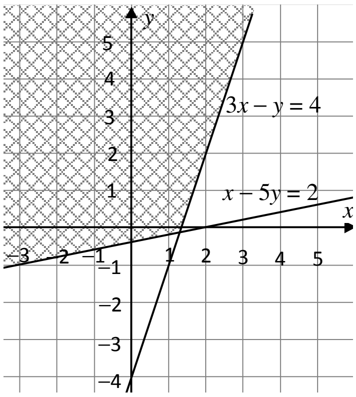


Figure 10.24

b.

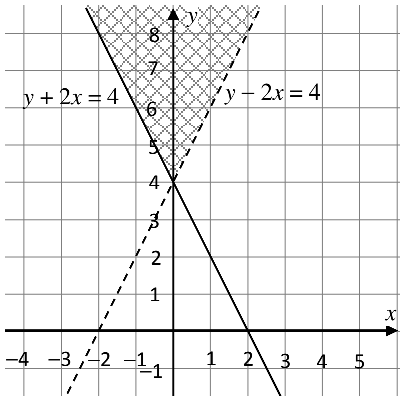


Figure 10.25

c.

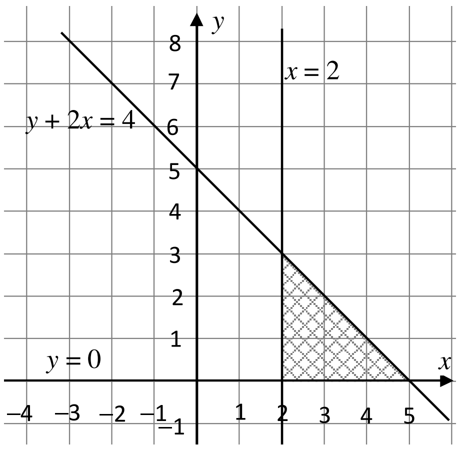


Figure 10.26

d.

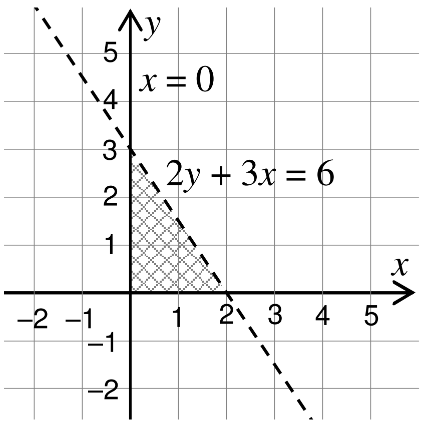


Figure 10.27

4. a. Vertices at (0, 0), (4, 0), (3, 4) and (0, 5).

Maximum value 17 at (3, 4) and minimum value 0 at (0, 0).

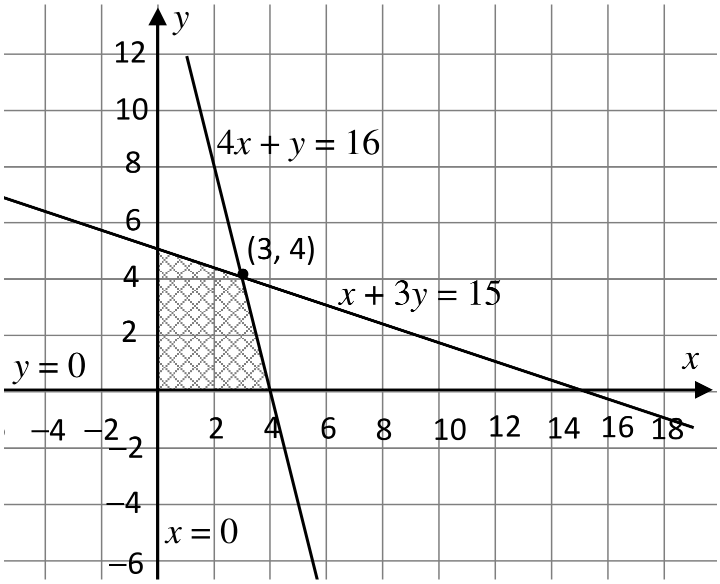


Figure 10.28

b. Vertices at (40, 20), (80, 0) and (0, 100).

Minimum value 140 at (40, 20). No maximum value.

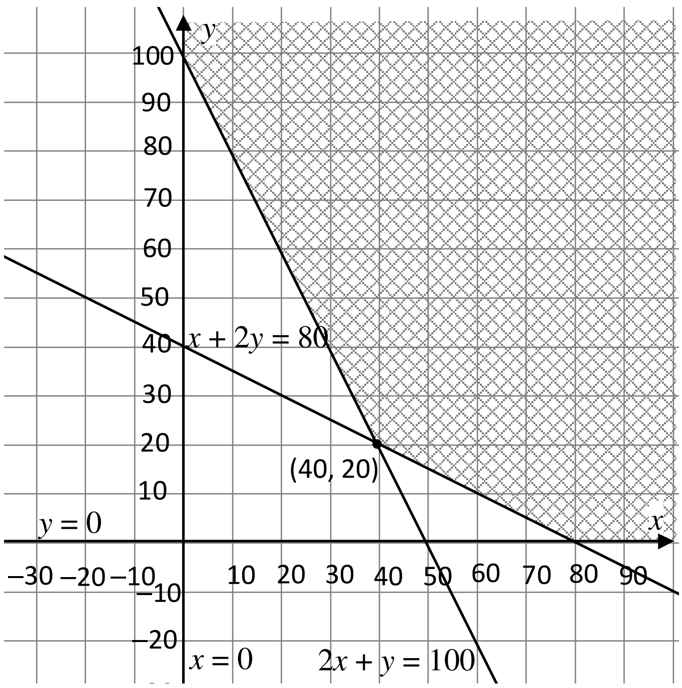


Figure 10.29

c. Maximum value 740 at (60, 20) and minimum value 0 at (0, 0).

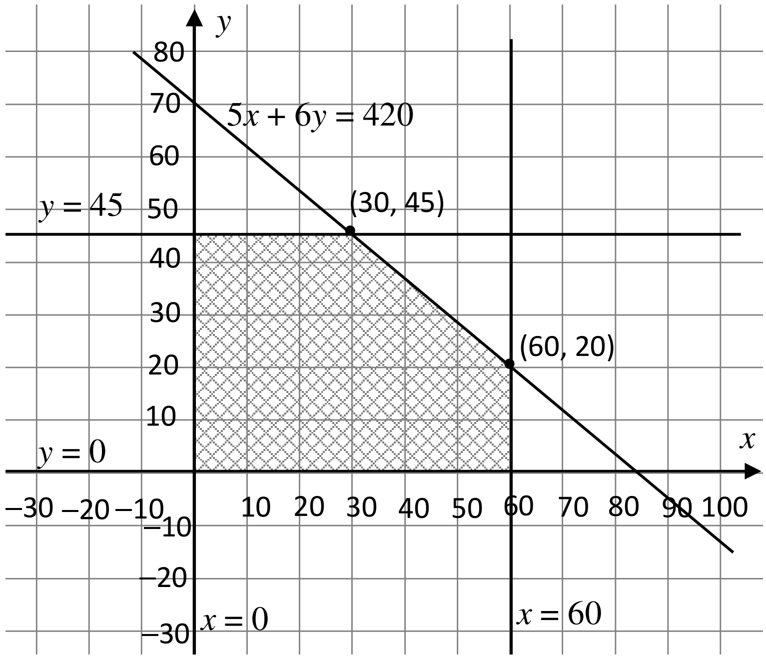


Figure 10.30

d. Vertices at (4, 0), $\left(1, \frac{3}{2}\right)$.

Minimum value is 9 at $\left(1, \frac{3}{2}\right)$.. No maximum value.

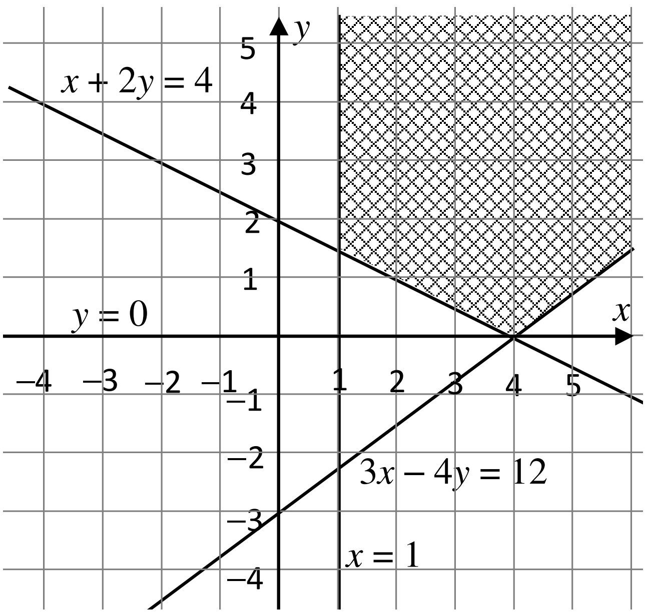


Figure 10.31

5. a. Let x = number of radios of model A

y = number of radios of model B

22 hrs means $22 \times 60 = 1320$ min.

25 hrs means $25 \times 60 = 1500$ min.

Summary:

	Model A	**Model B**	**Max. min.**
Assembly line I	20	10	1320
Assembly line II	10	15	1500
Profit	10	14	

Maximize profit $P = 10x + 14y$

Subject to $20x + 10y \le 1320$

$10x + 15y \le 1500,$

$x, y \ge 0$

Maximum Birr 1416 at (24, 84).

Thus Ahadu's company must produce 24 radios of model A and 84 radios of model B to get maximum profit of Birr 1416.

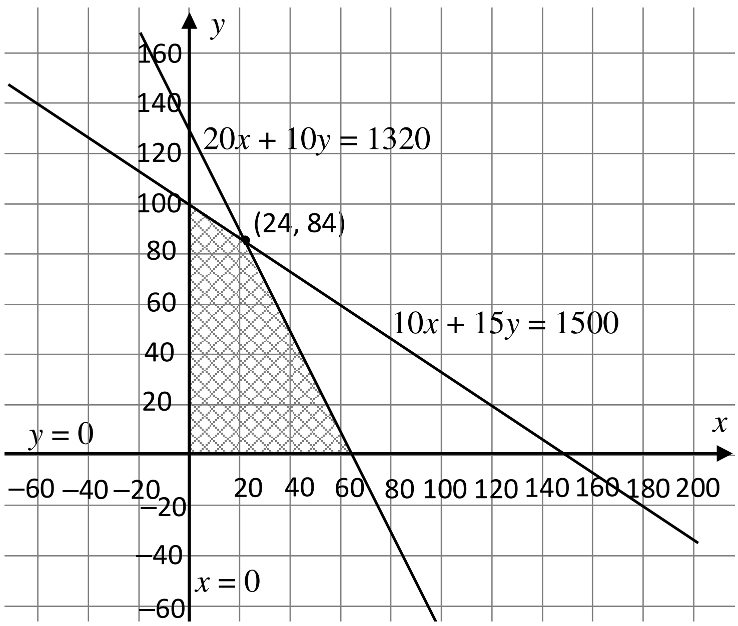


Figure 10.32

b. Let x = the no of bags of Brand X

y = the no of bags of Brand Y

	Brand X	**Brand Y**	**Min. requirement**
units of A	2	1	12
units of B	2	9	36
units of C	2	3	24
Cost	25	20	

Minimize cost $C = 25x + 20y$

Subject to: $2x + y \geq 12$

$2x + 9y \geq 36$

$2x + 3y \geq 24$

$x, y \geq 0$

Vertices at (0, 12), (18, 0) P(3, 6) and Q (9, 2)

Minimum cost is Birr 195 obtained at P(3, 60) ; i.e. 3 bags of Brand X and 6 bags of Brand Y.

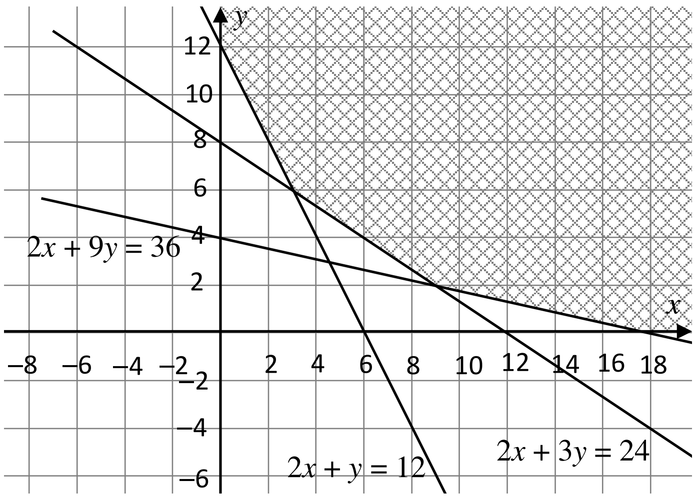


Figure 10.33

UNIT 11 MATHEMATICAL APPLICATIONS IN BUSINESS

INTRODUCTION

The application of mathematics in business involves simple arithmetic operations to compute the profit or loss, or to determine the corresponding percentage. Whenever people are engaged in business directly or indirectly, they are involved with financial institutions such as bank, insurance, etc. Hence, one has to know how to compute the amount of interest obtained or paid for that institution. On the other hand, s/he has to be able to identify and compute the amount of tax that has to be paid for the government.

It is expected that you have to make sure that the students have correctly understood the terms in each section. This can be attained by involving them in solving the examples given in the textbook and then doing activities and exercises following the examples.

To this end, this unit deals with the concepts of basic mathematics in business, compound interest and depreciation, saving, investing and borrowing money, and taxation.

Unit Outcomes

After completing this unit, students will be able to:

- *know common terms related to business.*
- *know basic concepts in business.*
- *apply mathematical principles and theories in practical situations.*

Suggested Teaching Aids in Unit 11

Since this unit is of practical nature where students can utilize samples and discuss real problems, it may require intensive use of teaching aids. From among the many possible teaching aids apart from the students textbook and teacher's guide, the following can also be used as teaching aid for this unit.

- Sample of bank account book of an individual.
- Tax proclamation and regulations of the government of the Federal Democratic Republic of Ethiopia or of your region.
- VAT declaration and collection formats.
- Schedule for the payment of presumptive taxation.
- Sample of payroll sheet of an organization.
- Annual reports of Saving and Credit Associations.
- Annual budget proclaims by the government for a particular year with its details of sources and uses of the funds, etc.

11.1 BASIC MATHEMATICAL CONCEPTS IN BUSINESS

Periods Allotted: 3 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *compare quantities in terms of ratio.*
- *calculate the ratio of increase in price of decrease in price of commodities.*
- *solve problems on proportional variation in business.*
- *solve problems on compound proportion.*
- *find a required percentage of certain given amount.*
- *compute problems on percentage increase or percentage decrease.*

Vocabulary: Ratio, Rate, Rate of change, Rate of increase, Rate of decrease, Proportion, Simple proportion, Compound proportion, Mean proportional, Percentage, Base, Discount, Markup

Introduction

As the topic suggests, this section primarily introduces the basic mathematical concepts associated with business activities. You are expected to raise and discuss concepts like ratio, rate, proportion, compound proportion, percent, commercial discount, mark-up, etc.

Teaching Notes

It is always a good practice to start the lesson by revising important ideas about ratio, rate, proportion and percentage. Raising real life examples of the concepts help the students to have a better understanding of each concept.

Remind students, for example, the concept "ratio" as an expression used to compare two quantities that have the same unit and show them how ratio is written.

For better understanding, you can let the students do Activity 11.1. This activity will help the students mathematically calculate ratio and discuss it.

Answers to Activity 11.1

a. $\frac{20}{40}=\frac{1}{2}=1:2$ b. $\frac{20}{60}=\frac{1}{3}=1:3$

When the students finalize doing and discussing the activity, you can discuss some more examples as presented in the student textbook. Encourage students to identify problems from their surrounding that requires consideration or application of ratio. To this end, give them Exercise 11.1 as class work or homework.

Assessment

To assess whether students have understood the concept of ratio, you can give them practical questions such as: What is the ratio of female students to the total number of students in your section? What is the ratio of male students to female students?, etc. You can also give them exercises that are similar to the examples given in the students textbook. You can also use Exercise 11.1 for assessment purpose.

Answers to Exercise 11.1

1. 4890 Birr, 3260 Birr, 1630 Birr, and 9780 Birr, respectively.
2. The total amount of money shared among the three is 32,704.00 Birr (Hint: The ratio of the division is $\frac{2}{5}:\ \frac{4}{3}:\ 2$, and the common denominator for the ratio is 15. Hence the ratio can be expressed as $\frac{2}{5}:\ \frac{4}{3}:\ 2=\frac{6}{15}:\ \frac{20}{15}:\ \frac{30}{15}$. Therefore, the total part to be allocated is 6 + 20 + 30 = 56)

Owing to the understanding of ratio, the next point for discussion is rate. Rate is another important concept that students should know. Introduce it as a concept which is used to compare two quantities that have different units.

You can use the examples presented in the student textbook for explaining rate. At this moment, it is advisable for the students to recognize that a “ratio” can be a “rate”. At last, use several examples from daily life so that students can differentiate rate from “ratio”. Explain “rate of change” in general and “rate of increase” and “rate of decrease” in particularin relation to selling and buying goods. To explain these concepts, you can use the examples given in the students textbook, and finally let students do exercises on rates of increase and decrease.

For the purpose of assessing students’ understanding, before you pass onto the next point of discussion, you can give them Exercise 11.2 either as class work or homework, and check their results.

Assessment

You can assess your students through several approaches. You can ask oral questions to explain the relation and difference between ratio and rate. You can give them exercises that are similar to the examples given in the students textbook focusing on ratio, rate of change, rate of increase and rate of decrease. You can also use Exercise 11.2 for assessment purpose.

Answers to Exercise 11.2

1. 40 Birr per unit
2. 0.54

Once students have understood the concepts of ratio and rate, another important consideration in business mathematics is proportion. To introduce proportion, raise issues like the following:

> *Three towns A, B and C have a population of 20,000; 332,000 and 40,000 respectively. Suppose town A has 5 medical doctors serving the population in this town. Towns B and C have 8 and 10 medical doctors serving the population of their respective towns. Is the distribution of medical doctors in these three towns fair (proportional) compared to their respective population?*

Assist the students to determine whether a given proportion is true or not and then encourage them to solve more problems on proportion by considering examples from business activities like proportional variation in price and supply of goods to a market.

Following this introduction, you can let the students do Activity 11.2 which will help them apply equality of rates/ratios to determine proportions.

Answers to Activity 11.2

$$\frac{3}{16.5} = \frac{150}{x} \Rightarrow 3x = 16.5(150) \Rightarrow x = \underline{\underline{825 \text{ birr}}}$$

Let some volunteer student do the activity on the board and continue discussing his/her work. Then you give them additional examples from the textbook and from your own sources. Example: If Hamid scored 20 out of 30 in Mathematics and 32 out of 40 in Economics, how do you compare his scores? Are they proportional?, etc.

Following their effort, make sure that the students have understood the concept of proportion. The discussions conducted earlier on ratio, rate and proportion, were assuming change in one variable with respect to the change in another variable. The proportion discussed earlier is also called simple proportion. But, there can be changes attributed to more than one variable. After you make sure of their understanding of simple proportion, guide them through a problem towards compound proportion.

You can also introduce the concept of compound proportion by raising examples which involve more variable quantities like:

> *The amount of interest obtained depends on the amount of money deposited in a bank, length of time it is deposited and the rate of interest payable per year.*

To strengthen their understanding, you can discuss the examples given in the student textbook and let students do more exercises or assignments.

Pursuing the discussion, another important concept which the students need to know is the percentage. This is so because percentage is another concept which needs a treatment not far from the student's life.

Percentage is commonly applied by the students. Some illustrative examples for percentages include:

- *How many students scored correctly 80% in an exam?*
- *What does a 50% discount mean? What does price of a given item has increased by 20% mean?*
- *A refrigerator costs 3800 Birr. If 15% of Value Added Tax (VAT) is added on this sale, then what is the total cost of the refrigerator?*

Raise such issues so that students will be interested in the lesson and think of other real applications of percentages.

To enrich their effort and thinking, you can give them Activity 11.3 and let a student do the answer on the board. Then you discuss his/her answer.

Answers to Activity 11.3

$$x\%\,(60) = 5 \Rightarrow \frac{x}{100}(60) = 5 \Rightarrow x = 8.33\%$$

When you discuss the activity, you can do more examples with the active participation of students. Here, you can use examples of VAT and other taxations that are calculated using percentages.

As a consequence of percentages, you can,as well, let students calculate discount, profit and loss, percentage of increase and decrease. It is quite common to hear discount of sales by some percentage. What it means and how it needs to be calculated will be of interest to the students.

For the purpose of assessing students understanding, you can give them Exercise 11.3 as class work or homework and check their answers.

Assessment

To assess whether students have understood the concepts: ratio, rate, proportion, and percentage, you can group them and ask them to bring practical applications of these concepts and report to the class. You can also give them exercises that are similar to the examples given in the students textbook in a form of quiz. You can also use Exercise 11.3 for assessing students understanding.

Answers to Exercise 11.3

1. 82%
2. 15%

Another important concept that is related to profit is mark-up or mark-down. Here, focus will be given to mark-up only.However, the idea is the same except mark-up treats profit and mark-down may be associated with loss. Mark-up is simply the difference between selling price and cost price. If someone buys an item for Birr 100 and sells it for Birr 112then the mark-up is Birr 12. Though it seems that mark-up is simple, an interesting consequence of it is the mark-up percent which is useful in daily applications and for determining tax rates and the like.

After introducing the mark-up, you can discuss it by giving several examples and exercises. Some illustrative examples are given in the students textbook. For talented students, you can ask questions of the following type.

Example: Suppose Dalelo is paid 10 Dollars for what he did yesterday. Today he is paid Birr 145 for a similar task. If the currency change is \$1 = 13.45Birr, discuss the mark-up of Dalelo's payment. Express this mark-up in percentage.

Assessment

You can use formal or informal assessment techniques like oral questions, group discussions,classwork,etc, to assess the students' level of understanding of this business concept. You can also let them discuss Exercise 11.4.

Answers to Exercise 11.4

1. 45 Birr, 29.03%
2. 53.73%

3. a. 30 b. 11.63%
4. cost = Birr 3320
5. a. 67.2 b. 347.20
6. 736.55
7. 42.86%

11.2 COMPOUND INTEREST AND DEPRECIATION

Periods Allotted: 4 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *calculate payment by installment for a given simple interest arrangement.*
- *calculate the compound interest of a certain amount invested for a given period of time.*
- *apply the formula for compound interest to solve practical problems in business.*
- *compute annuity for a given arrangement in compound interest.*
- *describe what depreciation means and some of its causes.*
- *compute depreciation by using either of the two methods approximately.*

Vocabulary: Principal, Interest, Simple interest, Compound interest, Interest period, Nominal rate, Annuity, Ordinary annuity, Depreciation, Asset, Current asset, Fixed asset, Salvage value.

Introduction

In this sub-unit, the concepts of compound interest and depreciation will be discussed. When money is deposited in a bank for some time, its total amount will increase in the future due to interest. On the other hand, the value of an asset decreases or depreciates due to several reasons as time goes on. These two concepts will be introduced in this sub-unit.

Teaching Notes

Start the lesson by asking if there are students who have ever deposited money in a bank. Or you may simply raise a discussion question as to what banks do with the money deposited by somebody else. Check if there are students who know what interest is and how it is calculated. Then you may revise the concept of simple interest which they have learnt in their previous grades.

Explain terms like “principal”, “rate of interest”, “interest period” and “amount” by using a simple example. Discuss how simple interest is calculated using $I = prt$.

It is also advisable to discuss future value of simple interest, right after they realize how they can calculate the simple interest. This will help them diagnose principal against future value and describe role of simple interest in finding future value. To this end, you can strengthen their understanding by discussing several examples of different type that may require calculating interest, future value, time, and principal. Highlighting examples are given in the student textbook.

11.2.1 Compound Interest

After revising simple interest, ask the students whether they know something about compound interest and what it means. Following their effort and assessing students background in relation to compound interest, you can group the students and let them do Activity 11.4. The purpose of this activity is to enable them calculating compound interest and familiarize them with the terms that are useful in considering compound interest.

Answers to Activity 11.4

$$A = 100\,(1+\frac{0.04}{2})^{2(2)}$$

$$A = \underline{\underline{108.24}}\text{ birr}$$

Pursuant to the activity, continue introducing the concept “compound interest” by considering simple exercise problems. Let students discuss compound interest and its difference from simple interest.

When the students realize the issue of compounding the interest, you can derive the formula for calculating compound interest together with your students. Give them one simple initial problem. Let them calculate the amount for few years,so that they can generalize their findings into $A = P\,(1 + r)^n$.

But you should clearly explain to them that this formula will be modified if interest is compounded several times in a year. If interest is compounded m times per year, then the amount after t years will be $A = P\left(1+\frac{r}{m}\right)^{mt}$. Encourage you students to discuss this change and how it appears.

Here you also need to discuss present value and future value in an amount compounded at stated interest rate and time. Illustrative examples are given on pages 440 – 444 in the student textbook.

Though compounding is one of the common applications in our day-to-day life, it is equally common to consider someone taking a loan may pay fixed amount of money at pre-specified time gap. Example, Helen might have taken a loan of Birr 10,000 and she is required to pay Birr 400 every month. Such regular payments are best discussed with what we call annuity. Annuity in general and ordinary annuity in particular are of interest when discussing compound interest. Introduce ordinary annuity as deposits made at a regular interval and discuss it using several examples. For the purpose of highlighting the concept, an example is given on page 445 of the student textbook. However, you may need to add more examples so that the students can exercise with them.

There could be some students who may need more on such concepts. For those talented students you can also guide them to deal with concepts of sinking fund and floating fund.

Assessment

To assess your students, you can give them Exercise 11.5 as homework so that they can discuss out of class and do the answers. You need to check their answers later on and discuss their answers.

To assess the students understanding on compound interest and related concepts, you can also do any one of the following.

You can give them project work so that they can go to either banks, credit unions, cooperatives, etc, in their surrounding and identify applications of these concepts and report to the class. You can give test/quiz.

Answers to Exercise 11.5

1. 9.3 years
2. a. 18,327.04 Birr b. 1047.04 Birr
3. 1967.15
4. $t = 5.25$
5. 2429.74
6. 25 years

11.2.2 Depreciation

So far, students were discussing interests, specifically compound interest, in which an amount would have been increasing. Example: if we deposit money in a bank at specified interest rate compounded at specified time interval, the amount is increasing as time goes on.

Equally, however, there are quantities that decrease their values as time goes on. To start this, you can guide the students by asking them a question of the following type:

If you want to buy a second hand material, what do you think will its price be? Will it increase or decrease? Why?

For example, if we buy a car for 100,000 Birr now, then the value of the car decreases as time goes on. We say the value of the car depreciates as time goes on. The concept that deals with such a decrease in value is called depreciation.

Thus, depreciation is another important concept that should be discussed in this section. To get into this discussion, you can begin your lesson by defining essential terms such as "Assets" in business and then introduce to them "Fixed Assets". Explain to them that fixed assets are not fixed in value because they wear out at varying rates according to their use over a period of time. Then discuss the concept of "Depreciation" and let them list some of the causes of depreciation.

Note: There could be several reasons that cause depreciation.

Make sure that the students understand depreciation as the fall in value. Then introduce how depreciation should be calculated.

To do so, it will be advisable to let the students identify that there are two types of calculating depreciation: The fixed installment method and Reducing balance method.

Illustrative examples are offered in the students textbook. To help the students practice with these methods on calculating depreciation,you can give them more examples and exercises either as an assignment or project work.You can ask talented students the following questions:

1. Sometimes value of an object, say a car, increases from time to time even if it is used, while the value must have been decreasing. Do you think this is normal? What do think is the problem?
2. List down some examples of physical depreciation and functional depreciation from your surroundings.

Assessment

To assess the students' understanding of these concepts, you can give them homework or assignment. You can also give them chance to discuss if the situation in their surrounding is similar to the theoretical discussion, and try to explain why? You can also give them Exercise 11.6.

Answers to Exercise 11.6

The answers to each of (i), (ii) and (iii) are:

a. The depreciation schedule for the asset using the fixed installment method isshown in the following table.

Number of years	Yearly depreciation	Accumulated depreciation at the end of the years	Book value at the end of the years
0	0	0	100,000
1	18,400	18,400	81,600
2	18,400	36,800	63,200
3	18,400	55,200	44,800
4	18,400	73,600	26,400
5	18,400	92,000	8,000

b. The yearly depreciation and book value using the double reducing balance method is shown in the following table.

Year	Book value at the beginning of the year	Depreciation for the year	Accumulated Depreciation	Book value at the end of the year
1	100,000	40,000	40,000	60,000
2	60,000	24,000	64,000	36,000
3	36,000	14,400	78,400	21,600
4	21,600	8,640	87,040	12,960
5	12,960	5,184	92,224	7,776

11.3 SAVING, INVESTING AND BORROWING MONEY

Periods Allotted: 7 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *list five valid reasons for saving.*
- *explain how saving becomes investment.*
- *list three saving plans.*
- *identify four kinds of financial institutions.*
- *describe three main factors in choosing a particular institution for saving.*
- *compute and solve numerical problems on saving.*

- *identify the four factors that should guide consumers in planning an investment strategy.*
- *explain the differences between stocks and bonds.*
- *describe ways to invest in stock and bond.*
- *compute and solve numerical problems on investment.*
- *describe the advantages and disadvantages of borrowing money.*
- *identify the usual sources of cash loan.*
- *compute the amount and time on the return of loan based on the given agreement.*

Vocabulary: Money, Saving, Investment, Market economy, Stock, Bond, loan.

Introduction

In this sub-unit, several reasons and methods of saving money will be discussed. Most businesses have better information on how to save and invest their money. This requires knowhow about how money can be saved and invested. Different ways of investing money will be discussed in this sub-unit. Finally, reasons and types of borrowing money will be discussed.

Teaching Notes

Before going into the details of saving, investing, and borrowing money, form groups of students and let them do Group Work 11.1. The purpose of this group work is to guide the students discuss issues on who makes most decisions on how much to save and invest. Here, personal decisions, institutional decisions and market based decisions can be among the issues the students need to discuss. The importance of banks and financial markets, and also how saving facilitates growth need to be addressed. Pursuant to their discussion, the students may need to discuss a little about money.

Answers to Group work 11.1

1 Businesses and households.

2 Banks and other financial intermediaries lend money to help people start and expand business. Financial markets such as bond markets and stock markets also play a critical role in channeling personal savings to business that uses these funds to make investments.

3 A key assumption of a market economy is that individuals in households and business have better information about their own circumstances and objectives than government officials. They also have the strongest incentives to make good

investment decision, because they make money from good investments, and lose money with bad investments. However, some kinds of investment decisions relate to public rather than private goods and therefore must be made by government officials.

Before discussing on saving money, it will be nice to discuss what money is and let the students strengthen this idea by doing Activity 11.5

Answers to Activity 11.5

Birr is money because:

- It acts as a medium of exchange
- It is a unit of account
- It is a store of value
- It offers a standard of deferred payment

11.3.1 Saving Money

In this part we will see:

A. Reason for saving B. Planning a saving program

C. Saving as investment D. Saving institutions

A. Reason for Saving

After setting the basis for further discussion, start your lesson by making students discuss what "saving" is. Give them time to reflect their discussion. Along their discussion, you may observe some misconceptions. Some may think saving as "money left over after expenses". Discussing the meaning of saving, explain to them that saving is the most important activity in the personal or family budget. Why do we save? Though some are listed in the student textbook, let students give their own reasons for saving money. Help them to realize that when one plans for saving s/he is planning for her/his dream. Depending on the planned interest, explain to them that there are "short-term" and "long-term" savings. As a consequence, you can leave them with the question: Where and how do we save money?

To make the issue more realistic and help the students reflect their assessment, let them do Group Work 11.2. This will help the students to discuss whether people around them save money and if yes to discuss where they save the money. This also helps to understand some of the saving institutions around them and help them to discuss and explain the reasons on why we save money in detail.

Answers to Group Work 11.2

This group work may not have definite answer. However, the following are possibly among the common ones. Some of them may say "yes, they save money" and they may tell that they save money in a bank.

The seven reasons on why you save money can be elaborated as follows:

1. Save for emergency funds

It is important to have an emergency fund set aside to cover unexpected expenses. This could cover an unexpected car repair or a sudden job loss. Ideally your emergency fund should be about three to six times your monthly expenses.

2. Save for retirement

Another important reason to save money is your retirement. The sooner you start saving for retirement, the less you will have to save in the future. You can put your money to work for you. As you continue to contribute over time you will be earning more interest on the money you have. You should be contributing at least up to your employer's match and eventually you may want to contribute ten to fifteen percent of your gross income.

3. Save for a down payment on a house

A third reason to save money is for a down payment on a house. Your negotiating power goes a lot farther when you have a significant down payment towards your home. You will receive better interest rates, and be able to afford a bigger home. You can determine how much you save towards this each month depending on your circumstances.

4. Save for vacations and other luxury items

A fourth reason to save money is to have fun. You can save for your tour of Lalibela or that of Awash National Park. Additionally you can be saving for large ticket items such as the World Cup Football. Your negotiating power is stronger if you have cash in hand on bigger purchases. Plus you do not want to be paying off your trip to Axum in five years time.

5. Save for a new car

A fifth reason is to purchase a car with cash. You will be amazed at how much money you can free up in your budget if you do not always have a car payment. You can also negotiate the price of the car much lower if you are willing to pay cash at the dealership.

6. Save for sinking funds

A sixth reason is to build up your sinking funds. A sinking fund is money you set aside for future repairs or improvements on your car, home or other possessions. This planning can help you to stop dipping into your emergency fund every time you need to fix your car.

7. Your education

A seventh reason to begin saving money is for your future education. Each year more people return to school to earn their masters or doctorate degrees. You may also consider saving for your child's education when the time comes.

B. Planning a saving Programme

After you discuss Group work 11.2 on why we save money, you can discuss on planning a saving programme. This can be started from Activity 11.6.

Answers to Activity 11.6

In this activity you can give some cases like consider an employee with monthly salary of Birr 1,800 and give chance for the students to manage this salary for retirement, vacations and down payment for a house. Here different figures can be suggested by the students.

C. Saving as investment

The General idea on investment could be discussed in 11.3.2 but here you can stat from Activity 11.7.

Answers to Activity 11.7

Here also you can give a case for the students like the monthly income of an individual is Birr 3,000 and she/he wants to participate in an investment, how could she/he save money to be partner in the investment. Here different ways of saving methods and amount of money to be saved could be indicated.

D. Saving institutions

It will be nice to start this section by letting students do Group work 11.3, here different groups may suggest or include banks, credit associations, Equib, etc. From this group work it will be easier to see the formal saving institutions as stated in the text book.

Assessment

Exercise 11.7 could be given as class work or homework depending on the extra time that we have after the group work.

Answers to Exercise 11.7

a. Commercial banks
b. Commercial banks
c. Credit unions
d. Savings and loan associations
e. Mutual saving banks
f. Mutual saving banks
g. Savings and loan associations
h. Saving and loan associations
i. Commercial banks or saving and loan associations

11.3.2 Investment

To start this section raise the questions like what is investment? After forming groups, let students start this section by doing Group work 11.4 and let some groups present what they discussed to the class, this can tells you how much they know about investment.

Answers to Group work 11.4

The answers that could be given by different groups could vary, but still depending on their surroundings they can give some investment activities in different areas and they could also indicate that there is tight relation between the investors and financial institutions like banks and insurances.

After some groups present their ideas you can explain the concept if investment as the production and purchase of capital goods to produce more goods and services in the future. Then, you may raise conditions that should be considered during investment, like investment strategy types of securities.

To discuss types of securities further, you can use Activity 11.8

Answers to Activity 11.8

Some of the features that distinguish preferred stock from common stocks are

- Preference as to dividends
- Preference as to assets
- The conversion privilege
- The callable feature and
- The voting features

The basic right of stock holders in corporation include:

- To vote at stock holders meetings
- To share in corporate profits in the form of dividends
- To share in corporate assets if the company is liquidated
- To buy additional share from the company if it issues more stock

On type of securities after discussing stocks based on Activity 11.8 you can give emphasis that one should be sure to research a stock before investing. You should understand its products or services, its market, as well as whether it has a sound balance sheet, cash-flow management, and competent directors and managers. You should also consider analysts' projected earnings estimates.

Then you can introduce bonds, to strengthen this idea on bonds you can let your students do Activity 11.9 to see the difference between bonds and stocks.

Answers to Activity 11.9

Major differences between bonds and stocks are as follows:

1. Bonds are commonly issued by corporations and government as a means of raising capital
2. Preferred stock is an ownership security
3. Bond is a debt security
4. Preferred stock has no maturity data
5. Bonds have maturity date
6. In terms of the degree of security, preferred stock ranks behind bonds.
7. Bonds ranks a head of preferred stock.

Because it reduces risk of losses and cost of collecting money.

To be prepared for the next section if you have more time, you can also explain about mutual funds.

Mutual funds

Mutual funds are designed to offer the individual investor diversification and professional money management, even with low investment amounts. A mutual fund pools money from its many investors to purchase securities for the fund's portfolio. As a result, investors typically own a portion of a portfolio that includes many more investments than they could afford to purchase individually — the value of the investor's share of that portfolio increases or decreases based on the value of the investments in the portfolio. Every mutual fund has a specific investment objective. Most mutual funds invest in stocks, bonds, cash equivalents, or a combination of the above. Within those categories, a stock fund may emphasize domestic or foreign equities or stocks from a particular industry sector. A bond fund may concentrate on investments with either long- or short-term maturities, or on government or corporate securities.

A mutual fund distributes its income and capital gains. As the fund buys and sells investments within its portfolio, it distributes any income received from stock dividends or bond interest to the shareholders along with any capital gains from the sale of securities. Be sure to read the prospectus before investing. The prospectus tells you how the fund will invest, how you may purchase shares, how the fund will be administered, and what it will cost you in fees and other expenses.

After discussing bonds and stocks through Activity 11.8 and 11.9 our next discussion will be on how to invest.

To start this discussion you can ask your students on what type of investment they would like and why they choose this type of investment.

As given on the text you can lead your students to the discussion of the categories and the different choices they have and factors that should be used to decide the type of investment. After discussing the short-term and long-term investing vehicles, if you have more time you can also discuss retirement plans

Retirement plans

A number of special plans are designed to create retirement savings, and many of these plans allow you to deposit money directly from your paycheck before taxes are taken out. Employers occasionally will match the amount (or a percentage of that amount) you have withheld from your paycheck up to a certain percentage of your salary. Some of these plans let you withdraw money early without a penalty if you want to buy a home or pay for education. If early withdrawals are not permitted, you may be able to borrow money from the account, or take out low-interest secured loans with your retirement savings as collateral. Rates of return vary on these plans, depending on what you invest in, since you can invest in stocks, bonds, mutual funds, certificate of deposits or any combination.

Investing in stocks

It is worth taking a closer look at stocks, because historically, they have had much better returns than bonds and other investments. Essentially, stock lets you own a part of a business. That ownership is represented by stock -- specialized financial "securities," or financial instruments that are "secured" by a claim on the assets and profits of a company.

Common stock

Common stock is aptly named -- it's the most common form of stock an investor will encounter. This is an ideal investment vehicle for individuals, because anyone can take part; there are absolutely no restrictions on who can purchase common stock -- the young, the old, the savvy, the reckless. Common stock is more than just a piece of paper; it represents a proportional share of ownership in a company -- a stake in a real, living, breathing business. By owning stock -- the most amazing wealth-creation vehicle ever conceived (except for inheriting money from a relative you've never heard of) -- you are a part-owner of a business.

Shareholders "own" a part of the assets of the company and part of the stream of cash those assets generate. As the company acquires more assets and the stream of cash it generates gets larger, the value of the business increases. This increase in the value of the business is what drives up the value of the stock in that business.

Because they own a part of the business, shareholders get a vote to elect the board of directors. The board is a group of individuals who oversee major decisions the company makes.

As with most things in life, the potential reward from owning stock in a growing business has some possible pitfalls. Shareholders also get a full share of the risk inherent in operating the business. If things go bad, their shares of stock may decrease in value. They could even end up being worthless if the company goes bankrupt.

This time, you can use Exercise 11.8 to help the students practice the questions and develop their understanding.

Answers to Exercise 11.8

Allow time for students to respond in writing. Then review.

a.	(I)	b.	(P)	c.	(N: consumption)
d.	(I)	e.	(S)	f.	(I for Ford, P for Sara)
g.	(I)	h.	(P)		

11.3.3 Borrowing Money

Although the students may tell something about borrowing, to help you assess how they understand borrowing you can start from group work 11.5.

Answers to Group Work 11.5

The expected answers in this activity may vary depending on the condition they understand borrowing, sometimes they can understand borrowing as the simple issues of borrowing money from a friend to buy a pen or they can think of their parents borrowing money from their neighbour or relatives.

After the students try Group work 11.5, let some students present their answers to the class. Here you can also help your students to see how one borrows money from an institution for personal use and how one borrows money from an institution for large scale investment purposes.

At last you can help them to relate borrowing money should be a well planned activity, otherwise borrowing will be disadvantageous if it is not planned.

Assessment

Students are to finish their discussion related to money, saving, institutions and borrowing. Before you pass to the next session, it will be good to assess the overall understanding of the students. In order to help you assess the understanding of the students, you can do any of the following as feasible in your school situation. You can give them assignment with list of guiding questions to go to any of the saving institutions around them and report accordingly. You can also give them homework and check their work. To see their general understanding, let the students do Group work 11.6

Answers to Group work 11.6

In this section the students have discussed what overdraft is and they know how to calculate interest on money borrowed from a bank.

To see the two different ways of settling their one million Birr credit, you can give them some clues like when you borrow money from a bank there is a fixed rate of time that you should settle your loan, suppose if you should settle this one million Birr in three years every month you should pay a certain amount of money, and by chance for two or three months if you can’t settle your monthly payment you could face problem on the guarantee you give for the institute.

You can also guide your students in an overdraft the interest you pay could be more but still if your business for two or three months is below your expectation you may not face the problem with the banks.

By doing such discussion you can see how far the students have understand the section.

11.4 TAXATION

Periods Allotted: 4 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *name three types of activities that government performs and examples of each.*
- *explain why governments collect taxes.*
- *describe the basic principles of taxation.*

- *describe the various kinds of taxes.*
- *give their opinion about what "income taxes" mean for them in relation to their future first job.*
- *calculate different types of taxes based on the "rate of tax" in Ethiopia.*

Vocabulary: Tax, Income tax, Turnover tax, Value added tax, Excise tax, Custom duty

Introduction

It is known that governments provide a number of public services. To provide public services, they need money which has to be collected through tax. This sub-unit will discuss different objectives and principles of taxation. Taxes will also be classified as direct and indirect and each one will be discussed.

Teaching Notes

To discuss taxation and related issues, it will be better to give chance for the students to talk about what they know on tax and taxation. To help you guide them, you can form small groups and let students discuss Group Work 11.7 and the following additional issues.

- What kind of public services do governments provide?
- What is taxation?
- Who collects taxes?
- Why do governments collect taxes?
- From where or from what kind of business activities do governments collect taxes?

Answers to Group Work 11.7

1. - To remove inequalities in income and wealth among the people.
   - To achieve economic stability by controlling inflation and deflation.
   - To change people's behaviors by discouraging consumption of harmful products
   - To divert producers attention by imposing heavy tax on non-essential and luxury goods
   - To promote economic growth
2. Income from
   - employment

- rental of buildings
- business
- technical services, games of chance, dividend, casual rental of property, interest, transfer of investment property.

After such questions are properly addressed, you can proceed to the discussion of the objectives and principles of taxation some of which are listed in the student textbook.

Additional points on the earliest functions of government could also be discussed as follows.

One of the earliest functions of government was to set standards to limit the exploitation of the poor (and implicitly limit the supply of labour) by passing laws against child labour. Also to undertake taxation and expenditure measures to redistribute income to achieve a less unequal distribution of wealth.

Secondly, there are some goods called public goods that the private market economy cannot provide. They have the basic quality that if they are made available at all they must be made available equally to all individuals. Since no one can be excluded from their benefits they cannot be produced and sold on a profit making basis. National defence is the most obvious example.

Thirdly, the market economy may not be capable of functioning in a sufficiently stable fashion. Governmental action may be needed to stabilize the economy through appropriate fiscal and monetary policies.

Fourthly, the rate of capital formation may be too high or too low to achieve what is thought to be an acceptable rate of growth. Thus governments may be required to change the parameters that affect the rate of growth.

Clearly the scope of government activity is substantial. To perform all of these functions governments must have extensive source of revenue.

After discussing the activities indicated in the text book, you can give Activity 11.10.

Answers to Activity 11.10

The main objective of this activity is to create awareness on the youth that government is doing so much activities and still have many many more to perform, for these activities to be practical government need to collect taxes and we have to create awareness that the majority of the income for a government is taxation, and if the public failed to pay tax the government cannot implement its plan effectively and the people will not get the necessary public services.

After discussing the principle of taxation as on the text you can also indicate that some taxes are imposed for social and economic purpose, e.g. the highest rates are imposed for the purpose of reducing very large incomes. Taxation has often been increased in order to reduce purchasing power to check demand, or has been reduced to stimulate demand to strength their understanding of the principles.

Form groups and allow the students to work Group work 11.8 after school hours in the library.

Answers to Group Work 11.8

After the students do Group work 11.8, you can give chance to some groups to present their work.

The following are the expected principle.

1. Equity

Similarly situated taxpayers should be taxed similarly. The principle of taxing similar taxpayers similarly is typically described in terms of equity. The concept of horizontal equity provides that two taxpayers with equal abilities to pay should pay the same amount of tax. If a taxpayer has a greater ability to pay than another taxpayer, the concept of vertical equity comes into play, which means that the person with the greater ability to pay should pay more tax.

It is the most important cannon of taxation which embodies the principle of equity or justice. It provides the concept of the equality of sacrifice. The amount of the tax paid is to be in proportion to the respective abilities of the taxpayers. It is not very unreasonable that the rich should contribute to the public expense not only in proportion to their revenue but somewhat more than that proportion. The principle of equity is often viewed as a fairness principle. That is, many people view a tax as fair if taxpayers with the greatest ability to pay have the highest tax burdens.

2. Certainty

The tax rules should specify when the tax is to be paid, how it is to be paid, and how the amount to be paid is to be determined. A person's tax liability should be certain rather than ambiguous. A tax system’s rules must enable taxpayers to determine what is

subject to tax (the tax base) and at what tax rate(s). Taxpayers should be able to determine their tax liabilities with reasonable certainty based on the nature of their transactions. If the transactions subject to tax are easy to identify and value, the principle of certainty is more likely to be attained. On the other hand, if the tax base is dependent on subjective valuations or transactions that are difficult to categorize, the principle of certainty might not be attained. In addition, how the taxes are paid and when the taxes are due should be spelled out in the applicable laws, as well as in the tax forms and instructions.

Certainty is important to a tax system because it helps to improve compliance with the rules and to increase respect for the system. Certainty generally comes from clear statutes as well as timely and understandable administrative guidance that is readily available to taxpayers.

The principle of certainty is closely related to the principle of simplicity. The more complex the tax rules and tax system, the greater the likelihood that the certainty principle will be compromised. The tax paid by each individual should be certain but not arbitrary. The time of payment, the manner of payment and the quantity to pay, should all be clear and plain to the contributor.

3. Convenience

A tax should be due at a time or in a manner that is most likely to be convenient for the taxpayer. For example, a tax on the purchase of goods should be assessed at the time of purchase when the person still has the choice as to whether or not to buy the goods and pay the tax. Convenience of payment is important in helping to ensure compliance with the tax system. The more difficult a tax is to pay the more likely that it will not be paid. Typical payment mechanisms include withholding (such as the withholding of income taxes from employee paychecks) and periodic payments of estimated tax liability. The appropriate payment mechanism should depend on the amount of the liability and ease of collection.

Every tax ought to be levied at the time, or in the manner, in which it is most likely to be convenient for the taxpayers to pay it. A tax upon the rent of land or of houses, payable at the same term at which such rents are usually paid, is levied at the time when it is most likely to be convenient for the contributor to pay; or, when he/she is most likely to have resources to pay.

4. Economy

The costs to collect a tax should be kept to a minimum for both the government and taxpayers. These costs include the administrative cost to the government that is influenced by the number of revenue officers necessary to administer the tax. There are also compliance costs incurred by taxpayers to consider. This principle is also closely related to the principle of simplicity. The more complex a tax, the greater the costs for the government to administer it and the greater the compliance costs for taxpayers to determine their tax liability and report it.

5. Diversity

A tax should be as broad based as possible, resulting in a low percentage rate that offers little incentive for tax evasion and tax avoidance and all the unproductive activities people engage in efforts to lower their taxes or evade or avoid taxes altogether.

6. Productivity

The system should be able to yield enough revenue for the treasury and the government should have no need to resort to deficit financing. The tax system should enable the government to determine how much tax revenue will likely be collected and when. Tax systems should have some level of predictability and reliability to enable the government to determine how much tax revenue is likely be collected and when. This is particularly important to state governments, most of which operate under a balanced budget requirement. Typically, a mix of taxes provides a more stable tax base because different types of taxes are affected differently by changes in the economy.

7. Simplicity

The tax law should be simple so that taxpayers understand the rules and can comply with them correctly and in a cost-efficient manner. Simplicity in the tax system is important both to taxpayers and to those who administer the various taxes. Complex rules lead to errors and disrespect for the system that can reduce compliance. Simplicity is important both to improve the compliance process and to enable taxpayers to better understand the tax consequences of transactions in which they engage in or plan to engage.

8. Neutrality

The effect of the tax law on a taxpayer's decisions as to how to carry out a particular transaction or whether to engage in a transaction should be kept to a minimum. That is, taxpayers should not be unduly encouraged or discouraged from engaging in certain activities or taking certain courses of action primarily due to the effect of the tax law on the activity or action. The primary purpose of a tax is to raise revenue for governmental activities, rather than to influence business and personal decisions.

9. Buoyancy

The tax revenue should have an inherent tendency to increase along with an increase in national income even if the rate and coverage of taxes are not revised.

10. Flexibility

It should be possible for the authorities, without undue delay, to revise the tax structure, both with respect to its coverage and rates, to suit the changing environment of the economy and the treasury.

Before stating the classification of taxes it will be important to give Activity 11.11 as a starter Activity.

Answers to Activity 11.11

In this activity, students can name out some type of taxes they know, like value added tax (VAT), Turn overtax (TOT), etc.

After the activity is done you can discuss the classification of taxes as done in the text book.

If you have some spare time you can also give students a work to calculate the payroll tax that could be paid by the school for administrative workers or teachers, or even you can give a group work to prepare a payroll and calculate the taxes.

Problems like the one in Activity 11.12 can help you to see how far they have understand to calculate taxes.

Answers to Activity 11.12

Ato Dagem's salary was Birr 1,350 and for this he pays Birr 155. When his salary is increased from 1,350 to Birr 1,850 for the first 50 Birr he will pay on the tax classified for Birr 651 – Birr 1,400 which is 15% and for the 50 Birr he will pay $50 \times 15\%$ = Birr 7.50 and for the next Birr 450 he will pay $450 \times 20\%$ = Birr 90 so for the new salary increment he will pay Birr 7.50 + Birr 90 = Birr 97.50. His new net salary is

Birr 1,850 – (Birr 155 + Birr 97.50) = Birr 1597.50

Assessment

Try to always assess the understanding level of your students on the spot. Use oral questions, listen to their group discussions, check their class works and homework, give them project works or assignment, use short quizzes and tests to assess your students learning. Since this is an end of the course, you can also consider this as part of your final exam to assess your students. For the assessment of this section you can give Exercise 11.9 as homework and class work.

Answers to Exercise 11.9

1. a. Taxable income = 850 – 150 = 700 Birr

   Total tax = 50 + 30 = 80 Birr

   b. Taxable income = 2390 – 150 = 2240 Birr

   Total tax = 50 + 112.50 + 190 + 10 = 362.50 Birr

   c. Taxable income = 5400 – 150 = 5250

   Total tax = 50 +112.50 + 190 + 300+435+140 = 1227.50 Birr

2. a. Dividend = 20% of 300,000 = 60,000 Birr

   Tax = 10% of 60,000 = 6,000 Birr

   b. Dividend = 20,000 Birr

   Tax = 2,000 Birr

   c. Dividend = 80,000 Birr

   Tax = 8,000 Birr

3. Tax = 15% of 150,000 = 22,500 Birr.

   Net income = 127,500 Birr

4. Net earnings = 50,000 – 7,500 = 42,500 Birr

5. i. Total prices = 62,500 , 12,000 , 750, respectively

   ii. Total VAT = 9,375 +1800 + 112.50 = 11,287.50 Birr

   iii. Total price = 75,250 + 11,287.50 = 86,537.50 Birr

   iv. a. 2% of 75,250 = 1,505 Birr

   b. 75,250 – 1,505 + 11,287.50 = 85,032.50

6. a. 478,261

   b. 9565.22

   c. 540,434.78

7. a. 8000 + 15% of 8000 = 8000 +1200 = 9,200

   b. Sales before VAT = $\dfrac{12,000}{1.15}$ = 10,435 Birr

   15% of (10,435 – 8000) = 15% (2435) = 365.15 Birr.

8. 5% of 350,000 = 17,500 Birr.

Answers to Review Exercises on Unit 11

1. 9 : 4
2. a. 2 : 3

   b. 1 : 2
3. 7000, 8400, 6000 Birr respectively.
4. 21 workers
5. 6%
6. 220 Birr
7. 180.20 Birr
8. 35.25%
9. 157.50 Birr
10. 1551.66 Birr
11. 9465.12 Birr and 1185.12 Birr
12. 13,277.84 Birr
13. a. 9406.28 Birr and 40,593.72 Birr

    b. 10,156.25 Birr and 30,468.75 Birr

Reference Materials

These days search for a reference is at forefront with authentic supply of electronic references. However, with the assumption that there will be limitations in some parts to over utilize ICT, some hard copy reference materials are listed here that can help develop better learning and teaching of mathematics and these units. These books are selected assuming that they are available in many schools. For those who have access to the internet, e-resources are offered as a supplement to those hard copies, if not essentially preferred. You can also access additional reference materials that are available in you school library. These are simply guides to help you use them as references. However, they are not the only to be prescribed. You can also use the web sites given here for reference and demonstration.

A.W.Goodman & J.S.Ratti (1979). Finite Mathematics With Applications.3rd ed., McMillan Pub.Co.Inc., NewYork

Aufmann, et al (2008). College Algebra and Trigonometry. 6th Ed, John W. Banagan, Houghton, Mifflin Company, USA.

A.W.Bowman et al (1987). Introduction to Statistics: A Computer Illustrated Text. IOP publishing Limited, London, UK.

Bryan H. Bunch, et al (1983). Algebra 1: The Language and Skills of Algebra. McDougal, Little and Company, USA.

Bruce E. Meserve (1983). Fundamental Concepts of Geometry, General Publishing Company, Ltd, Toronto, Canada.

C. Young (2010). Algebra and Trigonometry, 2nd Ed. John Wiley and Sons, Inc. USA.

Daniel T. Finkbeiner III (1978). Introduction to Matrices and Linear Transformations. 3rd ed., W. H. Freeman and Company.

David A Singer (1993). Geometry: Plane and Fancy. Springer-Verlag New York, Inc. USA.

David Cohn (2010). Algebra and Trigonometry. Wadsworth Publishers Company, USA.

Demissu Gemeda and Yismaw Alemu (1996). Basic Mathematics for Business and Social Sciences, Addis Ababa University Press, Addis Ababa.

Ewart Smith (1996). Examples in Mathematics for GCSE Intermediater Tier, 3rd Ed. Stanley Thornes (publishers) LTD, Great Britain.

Gary. L. Musser and William F. Burger (1988). Mathematics for Elementary Teachers: A Contemporary Approach, McMillan Publishing Company, New York, USA.

George Woodbury (2002). An Introduction to Statistics. Cengage Learning, BUXBURY, Thomson Learning, Canada.

Howard Anton (1987). Elementary Linear Algebra. John Wiley & Sons, Inc. USA.

J.E.Kaufmann et al (2009). College Algebra. 7th Ed, Thomson Brooks/Cole publishers, Canada.

J.W.McConnell et al (1996). Algebra. 2nd Ed. Scott Foresman; HarperCollins Publishers, USA.

Kinfegebriel Dessalegn and Zenebe Deneke (1998). The New Guide to Secondary School Mathematics: Algebra and Geometry. Grades 9 and 10. Aster Nega Publishing Enterprise, Addis Ababa.

L Bostock, et al (1996). GCSE Higher Mathematics. Stanley Thornes (publishers) LTD, Great Britain.

R.E. Larson and R.P. Hostetler (). Precalculus, 4th Ed.

Kinfegebriel Dessalegn (2001). Alpha problem solver series mathematics: For Preparatory Program Aster Nega Publishing Enterprise, Addis Ababa.

K.M.Ramachandran (2009). Mathematical Statistics with Applications, Academic Press, USA

M. A. Munem (1982). College Algebra with Applications. Worth Publishers INC. New York, USA.

M.A. Munem, et al (1986). Algebra and Trigonometry with Applications. 2nd Ed, Worth Publishers, INC. New York, USA.

M.A. Munem, et al (1988). Intermediate Algebra, 4th Ed. Worth Publishers, INC. New York, USA.

Marcel Berger (1987). Geometry I. Springer-Verlag Berlin Heidelberg, France.

Michele Audin (2003). Geometry. Springer-Verlag Berlin Heidelberg, New York, Printed in Italy.

Michael J. Crawley (2005). Statistics: An Introduction using R, John Wiley and Sons Ltd, England.

M.R.Spigel (1990). Theory and Problems of College Algebra, Schaum's Outline Series in Mathematics. McGRAW-HILL Publishing Company, USA.

Morris H.D, et al (2001). Probability and Statistics. Addison Wesley Pub. USA

Morris H.D, et al (1973). School Mathematics of East Africa. Cambridge University Press, USA.

N.J. Pullman (1976). Matrix Theory and its Applications. Marcel Dekker Inc. New York.

Raymond A. Barnett, et al (1999). College Algebra with Trigonometry, 6th Ed. WCB/McGRAW-HILL Publishing Company, USA.

R. J. Larsen (2009). An Introduction to Mathematical Statistics and Its Application, 4th Ed. Prinentice Hall, USA

Roger Fenu (2001) Geometry. Springer-Verlag, London Limited, Great Britain.

Roland E. Larson et al (1997). Algebra and Trigonometry, A Graphing Approach, 2nd Ed. Houghton Mifflin Company, New York, USA.

Ruric E. Wheeler (1984). Modern Mathematics: An Elementary Approach 6th Ed. Cole Publishing Company, USA

Streeter Hutchison Helzhe (1998). Intermediate Algebra, 3rd Ed. McGRAW-HILL Publishing Company, USA.

W.A. Wallis et al (2009). Statistics: A New Approach. The Free Press, New York, USA.

Zenebe Deneke and Kinfegebriel Dessalegn (1999). The New Guide Preparatory Program Aster Nega Publishing Enterprise, Addis Ababa.

http://www.coolmath.com

http://www.mhhe.com

http://www.hot.sra.edu/~matsc

http://www.aaamath.com

http://www.homepage.mac.com

http://www.wordmath.com

http://www.geometersketchpad.com

http://www.purplemath.com/modules/linprog.htm

MINIMUM LEARNING COMPETENCIES (MLCs)

No	Content	Minimum Learning Competencies (MLCs)
1	**NUMBER SYSTEM** **The Set of Complex Numbers**	• add complex numbers correctly • subtract complex numbers correctly • describe the closure property of both addition and subtraction • describe the commutative and associative properties of complex numbers • identify the additive identity element in $\mathbb{C}$ • determine the additive inverse of a given complex number • determine the product of two complex numbers • describe five basic properties of multiplication of complex numbers • divide two complex numbers • give reason for each step in the process of division of complex numbers • determine the conjugate of a given complex number • find the Modulus of any given complex number • write the simplified form of expressions involving complex numbers • describe how to set up the Argand Plane • plot the point corresponding to a given complex numbers • identify the complex number that corresponds to a given point in the Argand Plane • represent any complex number in the polar form • determine the modulus and argument of a given complex number
2	**ALGEBRA** **Rational Expression** **Matrices and Determinants**	• define rational expression • identify the universal set of a given rational expression • show the simplified form and the necessary steps in simplify a given rational expression • perform the four fundamental operations on rational expression • decompose rational expressions into sums of partial fractions • solve rational equations • solve rational inequalities by using algebraic method (by considering all possible cases) • solve rational inequality by using the sign chart method — — — — — — —— — — • define matrix • determine the sum and difference of two given matrices of the same order • multiply a matrix by a scalar • describe the properties of multiplication of matrices by scalars. • determine the product of two matrices.

		• describe the properties of the product of two matrices. • determine the transpose of a matrix • determine the determinant of a square matrix of order 2. • determine the minor and cofactor of a given element of a matrix • calculate the determinant of a square matrix of order 3 • describe the properties of determinants • determine inverse of a square matrix • find associated augmented matrix of equations • describe elementary operations on matrices • solve systems of equations in two or three variables using the elementary operations • apply Cramer's rule to solve systems of linear equations — — — — — — — — — —
	Introduction to Linear Programming	◊ ***For social science stream only*** • draw graphs of linear inequalities $y \leq mx + c$, $y \geq mx + c$, $ax + by \leq c$ and $ax + by \geq c$ • find maximum and minimum values of a given objective function under given constraints. • create inequalities from real life examples for linear programming and solve the problem
	Mathematical Applications in Business	◊ ***For social science stream only*** (cont.) • compare quantities in terms of ratio. • calculate the rate of increase and the rate of decrease in price of commodities. • solve problems on proportional variation in business • solve problems on compound proportion • find a required percentage of certain given amount • compute problems on percentage increase or percentage decrease • calculate payment by installment for a given simple interest arrangement. • calculate the compound interest of a certain amount invested for a given period of time. • apply the formula for compound interest to solve practical problems • compute annuity for a give arrangement in compound interest. • describe what depreciation means and some of its causes • compute depreciation by using either of the two methods appropriately. • list five valid reasons for savings. • explain how savings become investment. • list three saving plans. • identify four kinds of financial institutions. • describe three main factors in choosing a particular institution for saving.

		• identify the four factors that should guide consumers in planning an investment strategy • explain the differences between stocks and bond • describe ways to invest in stock and bond — — — — — — — — — — ◊***For social science stream only*** (cont.) • describe the advantages and disadvantages of borrowing money • identify the usual sources of cash loan.. • compute the amount and time on the return of loan based on the or given agreement. • name three types of activities that government performs and give examples of each • explain why government collects taxes • describe the basic principles of taxation • describe the various kinds of taxes • give their opinion what "income taxes" mean for them in relation to their future first job • calculate different types of taxes based on the "rate of tax" in Ethiopia
3	**Rational Functions** **RELATION AND FUNCTION** **Further on Relation and Function**	• define rational function • determine the domain of a given rational function • determine the range of a given rational function • Sketch the graph of a given rational function • determine the intercepts and symmetry of the graph of a given rational function • tell the properties of a given rational function from its graph • use the graphs of rational functions to solve rational inequalities • find out the inverse of a given relation • sketch the graph of a relation and its inverse • define power functions • describe the properties of powers functions in relation to their exponents • determine the domains and ranges of power functions • sketch the graphs of power functions • define Modulus Function (Absolute value Function, • determine the domain and the range of modulus function • sketch the graph of a Modulus function • define the Signum function • determine the domain and range of Signum function • sketch the graph of the Signum function • define the "Greatest Integer Function" • determine the domain and range of the Greatest Integer function • sketch the graph of the Greatest Integer function • define "one-to-one" function

		• identify functions as one-to-one • define "on to' function • identify functions as on to • identify one-to-one correspondence • define the composition of functions • determine the composite function given the component functions • define inverse function • describe the condition for the existence of inverse function • determine inverse function for an invertible function • determine whether two given functions are inverses of each other or not • sketch the graph of the inverse of a function • determine the domain and range of the inverse of a given function • define rational function • determine the domain of a given rational function • determine the range of a given rational function • sketch the graph of a given rational function • determine the intercepts and symmetry of the graph of a given rational function • identify the type asymptote that the graph of a given function may have • tell the properties of a given rational function from its graph • use graphs of rational functions to solve rational inequalities — — — — — — — — — —
	Further on Trigonometric Functions	◊***For Natural Science stream only*** • define and describe the functions sec x, cosec x and cot x • sketch graphs of sec x, cosec x and cot x • define and describe the functions sec x, cosec x and cot x • sketch graphs of sec x, cosec x and cot x • sketch the graphs of $y = a \sin x$ $y = a \sin kx$ $y = a \sin (kx + b)$ $y = a \sin (kx + b) + c$ • list the properties of these graphs. • sketch the graphs of $y = a \cos x$ $y = a \cos kx$ $y = a \cos (kx + b)$ $y = a \cos (kx + b) + c$ • list the properties of these graphs. • apply trigonometric functions to solve problems from fields of science, navigation, engineering, etc

4	**LOGIC** **Mathematical Reasoning**	• explain the difference between "statement" and "open statement" • determine the truth value of a statement • describe the rules for each of the five logical connectives. • use the symbols $\neg$, $\wedge$, $\vee$,$\Rightarrow$ and $\Leftrightarrow$ to make compound statements • determine truth values of compound statements connected by each of the logical connectives. • determine truth values of two or three statements connected by two or three connectives • describe the properties and laws of logical connectives • determine the equivalence of two statements • define "Contradiction" and "Tautology" • determine that a given compound statement is either a contradiction or tautology or neither of them • find the "converse" of a given compound statement • determine the truth value of the converse of a given compound statement • find the "contra -positive" of a given statement • determine the truth value of the contra- positive of a given statement • describe the two types of quantifiers • determine the truth value of statements involving quantifiers • describe what is meant by 'argument' • check the validity of a given argument • use rules of inference to demonstrate the validity of a given argument
5	**STATISTICS AND PROBABILITY** **Statistics and Probability**	• identify qualitative and quantitative data • describe the difference between discrete and continuous variables (data) • identify ungrouped and grouped data • determine class interval (class size) as required to form grouped data from a given ungrouped data • make cumulative frequency table for grouped data (for both discrete and continuous) • described terms related to grouped continuous data, i.e., class limit, class boundary, class interval and class midpoint. • determine class limit, class boundary, class interval and class midpoint for grouped continuous data • find the mean of a given grouped data • find median grouped discrete data • find median for grouped data (continuous variable) • determine the mode of a given grouped data • identify data that are unimodal, bimodal and multimodal • determine the quartiles for a given grouped data • determine the required deciles of a given frequency distribution

		• determine the required percentile of a given frequency distribution • describe the dispersion of data values • find the range of a given data • compute variance for ungrouped data • calculate variance for grouped data • solve problems on variance • calculate standard deviation for grouped data • determine the number of different ways of possible selections from a given sets of objects (by using the multiplication principle) • find the number of ways of selections of mutually exclusive operations (by using the addition principle) • determine the factorial of a given non-negative integer • find the possible ways of arranging objects by using the principle of permutation • compute the possible arrangement of objects around the circle (using the principle of circular permutation) • describe the difference between arrangement of objects and selection of objects. • describe what is meant by "combination of objects" • determine the number of different combinations of ***n*** objects, taken ***r*** at a time. • explain the computational relationship between permutation and combination of objects. • prove simple facts about combination. • solve practical problems on combination of objects • write up to the 6^{th} power of a binomial expression $(x + y)^n$ (i.e. when $n = 0, 1, 2, 3, 4, 5$) in its expanded form by using direct multiplication • describe what they observe in the expansion of $(x + y)^n$ where $n = 0, 1,2,3,4,5$ • describe "Pascal's Triangle" and its use • apply the "Binomial Theorem" in expanding the n^{th} power of binomial terms i.e. $(x + y)^n$, where $n \in \mathbb{Z}^+$ • determine any term in the expanded form of $(x + y)^n$,where $n \in \mathbb{Z}^+$ • solve problems on binomial expansion • describe what is meant by "Random Experiment" • explain what is meant by an outcome of a random experiment • describe what is meant by sample space of a given random experiment • list some of the sample points of a sample space for a given experiment • define "equally likely outcomes" of a given trial in his/her own words. • define "favorable outcomes/ cases" • determine events of a given random experiment • identify sample (elementary) events and compound events

		• determine the number of events of a given sample space • describe the occurrence or nonoccurrence of an event. • explain an event denoted by "**not E**" where "**E**" is a given event • explain events connected by "or" and "and" • describe the simplified forms of events by using the properties of operations on sets • identify exhaustive events • identify mutually exclusive events • describe events that are both exhaustive and mutually exclusive • identify independent events • identify dependent events • describe the axiomatic approach of probability • interpret basic facts in the theory of probability • find probabilities of events based on "Axiomatic" approach • describe the odds in favour of an event or the odds against an event • Find the probability of $\mathbf{E_1} \cup \mathbf{E_2}$ where $\mathbf{E_1}$ and $\mathbf{E_2}$ are events in a random experiment • determine the probability of mutually exclusive events. • find probability of the joint occurrence of independent events (by using rule of multiplication) • describe the outcomes of events using tree diagram • determine the probability of the joint occurrence of dependent events (using multiplication rule) • describe the outcomes of events using tree diagram to determine their probability • identify whether a given events are independent or dependent (by comparing the equation for probability of joint occurrence of independent events)
6	**GEOMETRY** **Coordinate Geometry**	• write different forms of equation of a line • determine the slope, x-intercept and y-intercept of a line from its equation • determine the angle between two intersecting lines on the coordinate plane whose equations are given • determine the distance between a point and a line given on the coordinates plane • name the different types of conic sections • explain how the conic sections are generated (formed) • define circle as a locus and write equation of a circle • find the radius and center of a circle from its equation. • determine whether a given line and circle have a point of intersection • determine the coordinates for the intersection point(s) (if the given line and the given circle intersect)

		• write equation of a tangent line to a given circle. (where the point of tangency is given) • write the standard form of equation of a parabola • draw different types of parabolas • describe some properties of a given parabola • define "ellipse" as a locus (set of points on the plane which satisfy a certain given condition) • write the standard form of equation of an ellipse and sketch ellipse • describe some terms related to ellipses (such as latus rectum, eccentricity, major and minor axes...) • define hyperbola as a locus • write the standard form of equation of a hyperbola • describe related terms to hyperbola (foci, centre, transverse axis, asymptotes, conjugate axis...) • sketch hyperbola based on its given equation • give eccentricity of a given hyperbola • solve problems on hyperbola
	Vectors and Transformation of the Plane	• define a scalar quantity • identify the everyday application of scalars • define a vector quantity • identify the everyday application of vectors • describe the difference between vector and scalar quantities • represent vector by different notations • determine the sum of two or more vectors • determine the difference of two vectors • multiply a vector by a scalar • resolve a given vector in to two components • use unit vectors to determine the column representation of a given vector • determine the magnitude of a vector • find the scalar product (inner product) of two vectors • describe some properties of scalar product of vectors • apply vectors to solve problems on geometry, algebra, mechanics and other related problems • write the parametric equation of a line • write equation of a circle by applying vectors • determine the equation of the tangent line to a circle using vectors • explain what is meant by transformation of the plane • describe the main properties of rigid motion • translate points, lines and circles using vectors • reflect points, lines and circles and some other plane figures • determine the images of a given plane figure when rotated through an angle θ.

Federal Democratic Republic of Ethiopia

Ministry of Education

Mathematics Syllabus

Grade 11

2009

General Introduction

Mathematics learning at the second cycle of secondary education (Grades 11 and 12) should contribute to the students' growth into good, balanced and educated individuals and members of society. At this cycle, they should acquire the necessary mathematical knowledge and develop skills and competencies needed in their further studies, working life, hobbies, and all-round personal development. Moreover the study of mathematics at this level shall significantly contribute to the students' lifelong learning and self development throughout their lives. These aims can be realized by closely linking mathematics learning with daily life, relating theory with practice; paying attention to the practical application of mathematical concepts, theorems, methods and procedures by drawing examples from the fields of agriculture, industry and sciences like physics, chemistry and engineering.

Mathematics study in grades 11 and 12 should be understood as the unity of imparting knowledge, developing abilities and skills and forming convictions, attitudes and habits. Therefore, the didactic-methodical conception has to contribute to all these sides of the educational process and to consider the specifics of students' age, the function of the secondary school level in the present and prospective developmental state of the country, the pre-requisites of the respective secondary school and the guiding principles of the subject mathematics.

In determining the general methodical approach of topics or special teaching methods for single periods, due consideration has always to be given to the **orientation of the main objectives**:

- acquisition of solid knowledge on mathematical concepts, theorems, rules and methods assigned in the syllabus.
- acquisition of reliable capability in working with this knowledge more independently in the field of problem solving.

The main activity for achieving these objectives includes engaging student in mental activity during classroom learning.

- Teaching has to consider students' interest that is related to their range of experience, actual events of the country and local reality so as to help them answer questions originating from daily life
- Problem solving is a suitable means for engaging students in mental activity. This has to be understood as a complex process, including the activities of the teachers and students. A teacher can be engaged in selecting or arranging the problems, planning their use in the classroom and organizing the process of solving problems while the students are engaged in solving the problems and in checking the results gained.
- While planning and shaping classroom learning the teacher has to observe that the application of newly acquired knowledge and capability is a necessary part in the process of complete recognition and solid acquisition of the new subject matter. Thus, application has to be carried out during presentation and stabilization.
- Introduction to a new topic and presentation of new subject matter have to be carried out using knowledge and experiences of students by encouraging students to actively

participate in the teaching learning process by familiarizing students with the new subject matter and help them to understand and appreciate its use.

- Stabilization has to be understood as the fundamental process of mathematics learning. It has to be regarded as a principle of shaping in all stages of teaching and as precondition for mental activity of students and the enhancement of capabilities applying knowledge more independently.
- Within the total process of stabilization, exercising (in relation with revision, deepening and systematization) hold a central position. In mathematics, exercising is to be understood in a wider sense. In the first place, it is aimed at the formation of skills; but it is also oriented towards fixing knowledge (including subject matter dealt with previously) and habituating to certain modes of working and practicing behavior. Furthermore last exercising has to facilitate the development of definite strategies of problems solving, being relatively independent of subject matter. Shaping of exercises has to be concentrated on assigning sufficient time, analyzing the real performances of students overcoming weak points of knowledge and capability, using all fundamental forms of exercises (daily activity, miscellaneous and complex exercises) systematically.
- Under the aspect of Students' preparation for Tertiary Education, it is necessary to prepare them step by step for mastering these demands. So, in shaping the teaching-learning process priority has to be given to methods which promote students' activities of cognition and reduce their mechanical rote learning. Students have to be asked, for instance, to report ways of solving a problem they have used with explanation and reason. Students have to be acquainted with forms of cooperative work between peer groups, with the application of the deductive approach, with preparation of project papers, with seminary instruction and discussion forums about special themes.

The teacher has to observe the following peculiarities of grades 11 and 12 (as compared with mathematics learning of grades 9 and 10).

- deeper penetration into modern and general mathematical theories,
- higher level of abstraction and generalization,
- higher demands with regard to logical strictness in the treatment of subject matter and the exactness of mathematical language (including terminology and symbolism)
- closer relations to neighboring disciplines and ranges of application, especially to Physics, Technology and Agriculture.
- The time allotment for grades 11 and 12 is made for 33 weeks (165 periods). The remaining weeks have to be used for revision, systematization and evaluation.
- Units 1, 2, 3, 4, 5, 6 and 7 of Grades 11 and units 1, 2, 3, 4 and 5 of Grades 12 are common to both natural science and social science stream students, While units 8 and 9 of Grade 11 and units 6 and 7 of Grade 12 are to be offered only to natural science stream students and units 10 and 11of Grades 11 and units 8 and 9 of Grade 12 are only for social science stream students.

Cycle Objectives

Objectives of Mathematics Learning in the Second Cycle of Secondary Education (Grades 11 and 12)

At the end of the second cycle of secondary education, students should be able to:

- apply the mathematical knowledge and capabilities gained to solve problems independently.
- develop mental abilities and high skills and competencies in calculations, especially, in the field of logical thinking, reasoning, proving, defining and use of mathematical language, terminologies and symbols correctly.
- develop an appreciation for the importance of mathematics as a field of study by learning its historical development, scope and its relationship with other disciplines.
- develop scientific outlook and personality characteristics such as working activities with algorithms, exactness, neatness, honesty and carefulness according to self prepared plans for solving problems in line with the needs of the society.

Allotment of Periods

for Units and Sub-units of Mathematics

Grade 11

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
Unit 1: Further on Relations and Functions	1.1 Revision on Relation and Inverse of a Relation 1.1.1 Inverse of a relation 1.1.2 Graphs of inverse relations	2	15
	1.2 Some Additional Types of Functions 1.2.1 Power Functions with their Graphs 1.2.2 Modulus Functions (Absolute Value Functions) 1.2.3 Signum Function 1.2.4 Greatest Integer Function	4	
	1.3 Classification of Functions 1.3.1 One-to-one Functions 1.3.2 Onto Functions	2	
	1.4 Composition of Functions	3	
	1.5 Inverse Functions and their Graphs	4	
Unit 2: Rational Expressions and Rational Functions	2.1 Simplification of Rational Functions 2.1.1 Operations with rational expressions 2.1.2 Decomposition of rational expressions into partial fractions.	4	12
	2.2 Rational Equations	3	
	2.3 Rational Functions and their Graphs 2.3.1 Rational Functions 2.3.2 Graphs of Rational Functions	5	
Unit 3: Coordinate Geometry	3.1 Straight Lines 3.1.1 Angle between two lines on the coordinate plane 3.1.2 Distance between a point and a line on the coordinate plane	3	21
	3.2 Conic Sections 3.2.1 Cone and sections of a cone 3.2.2 Circles 3.2.3 Parabola 3.2.4 Ellipses 3.2.5 Hyperbolas	18	

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
Unit 4: Mathematical Reasoning	4.1 Logic 4.1.1 "Statements and Open Statements" 4.1.2 Fundamental Logical Connectives 4.1.3 Compound Statements 4.1.4 Properties and Laws of Logical Connectives 4.1.5 Contradiction and Tautology 4.1.6 Converse and Contra positive 4.1.7 Quantifiers	13	16
	4.2 Argument and Validity - Rules of Inference	3	
Unit 5: Statistics and Probability	5.1 Statistics 5.1.1 Types of data 5.1.2 Introduction to Grouped data 5.1.3 Measures of Location for Grouped Data 5.1.4 Measures of Dispersion	14	31
	5.2 Probability 5.2.1 Permutations and Combinations 5.2.2 Binomial Theorem 5.2.3 Random Experiment and its outcomes 5.2.4 Events 5.2.5 Probability of an event	17	
Unit 6: Matrices and Determinants	6.1 Matrices	4	22
	6.2 Determinants and their properties	6	
	6.3 Inverse of a square matrix	4	
	6.4 Systems of equations with two or three variable	5	
	6.5 Cramer's Rule	3	
Unit 7: The set of Complex Numbers	7.1 The concept of Complex Numbers	2	13
	7.2 Operations on Complex Numbers	3	
	7.3 Conjugate and Modulus of Complex Numbers	2	
	7.4 Simplification of Complex Numbers	3	
	7.5 Argand Diagram and Polar Representation of Complex Numbers	3	

Unit	*Sub-unit*	*Number of Periods*	
		Sub-unit	*Total*
Unit 8: Vectors and Transformation of the Plane	8.1 Vectors and Scalars 8.2 Representation of Vectors 8.3 Scalar (inner or dot) Product 8.4 Application of Vectors 8.5 Transformations of the plane	3 1 3 5 8	20
Unit 9: Further on Trigonometric Functions	9.1 The Functions sec x, cosec x, and cot x 9.2 Inverse of Trigonometric Functions 9.3 Graphs of Some Trigonometric Functions 9.4 Application of Trigonometric Function	5 4 5 6	20
Unit 10: Introduction to Linear Programming	10.1 Revision on Linear Graphs 10.2 Graphical solution of system of Linear inequalities 10.3 Maximum and Minimum Values 10.4 Real life linear programming problems	2 2 5 6	15
Unit 11: Mathematical Application in Business	11.1 Basic Mathematical Concepts in Business ▪ Ratio ▪ Rate ▪ Proportion ▪ Percentage 11.2 Compound Interest and Depreciation 11.2.1 Compound Interest 11.2.2 Depreciation 11.3 Saving, Investing and Borrowing Money 11.3.1 Saving Money 11.3.2 Investment 11.3.3 Borrowing Money 11.4 Taxation (4 periods)	3 4 7 4	18

Introduction

In relation to the general objectives of the subject matter for this cycle, mathematics study at Grade 11 level should link mathematical theory with practice, paying attention to the applications of mathematical concepts, theorems, methods and procedures in real life situations, by taking application problems and activities in examples from agriculture, industry, business, and other sciences like physics, chemistry, technology etc.

Students' fundamental knowledge and skills and competencies developed unto Grade 10 with regard to relations and functions, working in different number systems, geometry, mathematical reasoning, statistics and probability is stabilized and deepened so that students can apply the knowledge, skills and competencies to solve problems confidently.

New content matters like matrices and determinants, transformation of the plane, linear programming and financial applications of mathematics are introduced and dealt with in relation to prior acquired knowledge and developed competencies. While most of the units are common to natural science and social science streams, two units are special to each of the two streams, Namely Vectors and transformation of the plane and further on trigonometric functions to students of natural science, while linear programming and financial applications of mathematics to social science stream students.

Objectives of Mathematics Learning in Grade 11

After studying Grade 11 Mathematics, students should be able to:

stabilize the fundamental knowledge and competencies acquired and developed up to Grade 10 with regard to:

- calculating in different number systems and working with quantities and variables.
- logic, mapping and functions.
- equations of lines, circles, parabolas, etc.
- mathematical reasoning.
- statistics and probability.
- have deep understanding of functions through learning polynomial, rational, power, modulus, signum, trigonometric functions and how to sketch the graphs of selected representatives of these functions.
- know the concept of vectors, operations on vectors and their rules.
- know the component and co-ordinate representation of vectors and their applications.
- set up vector equations for straight lines and for circles and apply these equations to solve problems from natural science and technology.
- understand the concepts of matrices and determinants and apply these concepts to solve systems of equations.
- use linear programming concept to solve simple maximization problems.
- solve problems involving savings, investment, borrowing, taxation, etc.

Unit 1: Further on Relation and Function (*15 periods*)

Unit outcomes: Students will be able to:

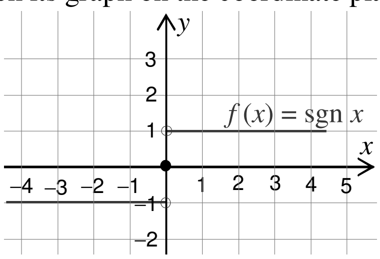
- know specific facts about relations.
- know additional concepts and facts about functions.
- understand methods and principles in composing functions.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • find out the inverse of a given relation	***1. Further on Relation and Function*** **1.1 Revision on Relations** *(2 periods)* **1.1.1 Inverses of a relation**	• You may start the lesson by revising important concepts about a relation such as its domain, range, graphical representation, and given graph of a relation, how to write its formula for the relation. • You can proceed with the lesson by taking a relation with some finite elements, for instance you may take R = {(a, 1), (b, 2), (c, 3) (d, 4)} and ask students to reverse the order of the entries of each ordered pair in R and get another set of ordered pairs. After some similar activities encourage the students to generalize that the inverse of a given by relation R, denoted by R^{-1}, is $R^{-1}\{(y, x): (x, y) \in R\}$ • Let the students practice on how to determine inverses of relations through exercises, in doing so it is better to consider relations that have been expressed by not more than three formulae and they should be selected from relations and function that have been discussed so far. ***e.g.*** $R = \{(x,y): 2x + 3y = 4\}$ $\therefore R^{-1} = \{(y,x) : 2x + 3y = 4\}$ or $R^{-1} = \{(x,y): 2y + 3x = 4\}$ • After taking several relations, discuss with students how to determine their respective domain and ranges as well as the domains and ranges of their inverses. • Encourage students to point out the relationship between the domain of a given relation R and range of R^{-1} and also between range of R and domain of R^{-1} i.e., let them come to the conclusion that: Domain of R = Range of R^{-1} Range of R = Domain of R^{-1}	• Asking oral questions • Ask students to give examples of relation and their inverses • Give students an opportunity to sketch the graphs of the inverses of relations given by themselves. • Give students an opportunities to discuss the inverses of a given relation both individually and in small groups

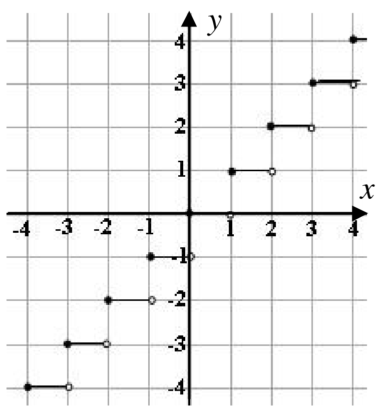
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• Sketch the graph of a relation and its inverse.	**1.1.2 Graphs of Inverse Relations**	• You may consider a relation (expressed by not more than three formulae) and let the students draw the graph of the relation and its inverse on separate coordinate planes and then from these graphs let them determine the domains and ranges of both the relation and its inverse and assert what they concluded, in the previous discussion about their relationship. • By using several examples and exercises let the students practice on drawing graphs of inverse relations. • Now let the students draw the graphs of a given relation and its inverse on the same coordinate plane and ask them to fold the coordinate plane along the line y = x. • By considering such kind of similar activities let the students generalize that, folding the graph of a relation along the line y = x (reflecting the plane on the line y = x) yields the graph of the inverse of the relation.	• Give class activities and home work exercises on drawing graph of inverses of some given relations and check their work.
	1.2 Some Additional types of Functions *(4 periods)*	• You may start the lesson by stating / revising the main points about "Function" that the students had learnt in Grade 9, and then let the students identify functions from a given list of relations. The relations can be given pictorially (Venn-Diagram) or as sets of ordered pairs or using set builder notation (expressed by formula) also allow students to give their own examples of relations which are functions. ***Note:*** *So far the students know functions expressed by one formula and whose domain (except logarithms) is the set of real number and the graphs are continuous but now it is required to introduce functions that are expressed or described by piece wise formula and whose graphs are discontinuous or have jumps or not smooth curve*	• Give exercise problems on identifying functions from a list of relation and let them justify their answer. • Homework' • Test/ Quiz

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• define power functions • describe the properties of powers functions in relation to their exponents • determine the domains and ranges of power functions • sketch the graphs of power functions	**1.2.1 Power Functions with their graph**	• You may begin the lesson with revision of important points about exponential functions and polynomial functions in relation to the exponents. • Introduce the power function by stating its definition as $f(x) = ax^n$ where n is a rational number and $a \in \mathbb{R}$. • By considering different cases for the exponent, i.e. for positive integral exponents, for $n = 1$, for $0 < n < 1$, for $n = 0$ and for negative integral exponents and discuss with your students about the properties of the function. • Based on the above discussion encourage the students to determine the domains and ranges of power functions. • Assist students to make tables of values by considering functions as described in the following way $f(x) = ax^n$ where $n \in \mathbf{Z}^+$, $f(x) = ax^n$ where $n = 1$ $f(x) = ax^n$ where $n = 0$, $f(x) = ax^n$ where $0 < n < 1$ and $f(x) = ax^n$ where $n \in \mathbf{Z}^-$ • Encourage the students to sketch the graph of each power function whose table of values are prepared above.	• Ask students to give you examples of power functions • Ask students to summarize the fundamental properties of a power function • Ask students to sketch graphs of power functions • Give students an opportunity to discuss the behaviour of power functions at some points • Give exercise problems on power function and their graph as class activity or homework and then check their work.
• define Modulus Function (Absolute value Function,	**1.2.2 Modulus Functions (Absolute Value Function)**	• You may start the lesson by revising the concept of absolute value of a number, using examples, that the students had learnt in Grade 9, following this define the modulus function (absolute value function) as $f(x) = \lvert x \rvert = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$	• Ask students to define the absolute value function • Ask students to sketch graphs of absolute value functions either individually of in small groups

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• determine the domain and the range of modulus function • sketch the graph of a Modulus Function		• With the help of the definition allow students to determine the domain and the range of modulus function • Guide students to make table of values of x (say some values between - 4 and 4) and corresponding values of y = f(x) = $\lvert x \rvert$ and assist them to sketch its graph on the coordinate plane and then encourage the students to list main properties of the graph such as: it is continuous in the domain, it passes through and has a sharp corner at the origin, and it is symmetrical with respect to the y-axis.	• Give students some class activities • Home work
• define the signum function	**1.2.3 Signum Function**	• You may begin the lesson with the discussion of a piece wise-defined function that is, $y = f(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x = 0 \\ -1 \text{ for } x < 0 \end{cases}$ and introduce this as a definition of the "**signum function**" and its notation, i.e., "**sgn x**" • you may also state the definition in the alternative way as $y = f(x) = \begin{cases} \dfrac{\lvert x \rvert}{x} & \text{for } x \neq 0 \\ 0 \text{ for } & x = 0 \end{cases}$	• Ask students to define the signum function • Ask students to determine the domain and range of the signum function.
• determine the domain and range of signum function		• Assist students in determining the domain and range of signum function i.e., guide them to describe that the domain is the set of all real numbers and the range is the {-1, 0, 1}	
• Sketch the graph of the signum function		• Guide the student to make table of values of x (say some values between -4 and 4) and the corresponding values of y = f(x) = sgn x and encourage them to sketch its graph on the coordinate plane.	• With the help of the graph you may ask students to describe what peculiar property/properties of signum function that they observe.



<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr>
<td>
• define the "Greatest Integer Function"
• determine the domain and range of the Greatest Integer Function

• Sketch the graph of the Greatest Integer Function
</td>
<td>1.2.4 Greatest integer Function.</td>
<td>
• You may start the lesson by stating the definition of "Greatest Integer Function", with its notation, as:
$f(x) = \lfloor x \rfloor$ or $f(x) = [x]$ where, $[x]$ is defined as
"the greatest integer less than or equal to x."
• Assist the students in finding the value of the function for some numbers x (taken from the domain) by using sufficient examples (like: [3.7] = 3, [-8] = -8, [-2.8] = -3,...) until they familiarize themselves with the concept.
• Discuss with the students how to determine the domain and range of this function, and through the discussion guide them to the conclusion that the domain of the greatest integer is the set of all real numbers R and its range is the set of all integers Z
• Assist students to make table of values for the greatest integer function as follows.
<table>
<tr><td>x</td><td>$-3 \le x < -2$</td><td>$-2 \le x < -1$</td><td>$-1 \le x < 0$</td></tr>
<tr><td>f(x) =[x]
or
y = [x]</td><td>-3</td><td>-2</td><td>-1</td></tr>
</table>
<table>
<tr><td>x</td><td>$0 \le x < 1$</td><td>$1 \le x < 2$</td><td>$2 \le x < 3$</td></tr>
<tr><td>f(x) =[x]
or
y = [x]</td><td>0</td><td>1</td><td>2</td></tr>
</table>
and encourage them to sketch the graph of this function.


</td>
<td>
• Ask students to define the greater integer function

• Ask students to find out the values of the greater integer function for some real numbers
• Ask students to sketch the graph of $f(x) = -[x]$
• Home work
∗ For able (above average) students you may introduce the. Smallest Integer Function as, $f(x) = [x]$ or $f(x) = [x]$ where $[x]$ is defined as the smallest integer greater than or equal to x. e.g. [1.3] = 2 and ask them to draw its graph and describe its domain, range and some of its properties.
</td>
</tr>
</table>

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• define "one-to-one" function • identify functions as one-to-one	**1.3 Classification of Functions** *(2 periods)* **1.3.1 one-to-one function**	• You may begin the lesson with the formal definition of one-to-one function, and by using several examples let the students be well acquainted with the concept and encourage them to identify whether a given function is one-to-one or not. • Also introduce using examples and discuss the horizontal line test as a method of identifying whether a given graph is the graph of a one-to-one function or not. • Allow students to give their own examples of one-to-one functions from their real life (like: marriage relations, usage of tooth brush, etc.)	• Ask students to give examples of one-to-one functions • Ask students to determine whether a give function is one-to-one or not using vertical line test on its graph.
• define "onto' function • identify functions as onto • identify one-to-one correspondence	**1.3.2 onto functions**	• After stating the formal definition of "**onto function**", and by using several examples let the students be familiarized with the notion of onto function. • Assist students to identify a given function as onto function • Finally introduce the definition of **one-to-one correspondence** and discuss with students about this concept with the help of example	• Give exercises on one-to-one and onto functions • Ask students to determine whether a given function is one-to-one correspondence and check their works. • Give home work
• define the composition of function.	**1.4 Composition of functions** *(3 periods)*	• You may begin this lesson with a brief revision of (using several examples) combination of functions (i.e., how to find their sum, difference, product and quotient) that the students had learnt in Grades 9 and 10, in doing so it is better to take sufficient examples for each operation and the functions should be the ones that the students already know	• Give exercise problems on addition, subtraction, multiplication and division of functions.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• You can proceed with the lesson by considering examples like: $f(x) = x^2 + 1$ and asking students to write expressions for $f(k)$, $f(m)$, $f(t)$, $f(u)$, etc. and let them observe each of their steps in doing these.	• Ask about the relationship between the domains and ranges of functions and that of their respective results after combining them.
• determine the composite function given the component functions • determine the domain and the range of a composite function of two given functions.		• Discuss one more example by considering two simple functions like: $h(x) = x^3$ and $g(x) = 2x$ then assist student in determining the composition of these functions say $h(g(x))$, first by evaluating different values for $h(x)$ as follows $h(-2) = (-2)^3$, $h(½) = (½)^3$, $h(m) = m^3$ and then for $h(g(x)) = (g(x))^3 = (2x)^3$ and guide students to state the formal definition of composite function of two functions say g and f (with its notation) as: $(gof)(x)= g(f(x))$	• Give exercise problems on writing expressions for a given function by considering different variables. **e.g.** if $f(x) = x^2+1$ then give expression of $f(a)$, $f(-m)$, $f(w)$...
		• Allow students to practice in determining the composite function given the component functions by using several exercises. You may also ask students to determine one of the component functions, given the composite function and the other component by taking appropriate functions. • Encourage the students to determine the domain and range of a composite function of two given functions. With active participation of students discuss about the relationship between the domains and ranges of the given component functions and their composite function.	• Give exercise problems on determining the composition of two functions, the domains and ranges of the component functions and their composition.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• define inverse function • describe the condition for the existence of inverse function	**1.5 Inverse Functions and their graphs** *(4 periods)*	• You may begin the lesson with revision of important points about inverse of a relation discussed in the first topic of this unit, following this you may consider a linear function, for example: $f = \{(x, y): y = 2x + 3\}$ and ask students to express their opinion on how to form the inverse of f.	• Ask students to find the inverses of functions.
• determine inverse function for an invertible function. • determine whether two given functions are inverses of each other or not.		• After stating the formal definition of "Inverse of a function" and introducing its notation, let the students express what they observe in the connection between inverse of a relation and inverse of a function. • With active participation of students and a) by using examples discuss that not every function has an inverse. Take examples like: $f(x) = x^3 - x + 1$ b) define "Inverse function" by using the concept of composition of function and discuss the condition for the existence of inverse function.	• Ask oral questions during the process of finding the inverse of a given function. • Ask students questions like "Does the inverse of a function always define a function?" and let them justify their answer by giving examples.
• Sketch the graph of the inverse of a function		• Encourage and assist the students to determine the inverse of functions by considering several examples. • After introducing the "Identity Function" namely $f(x) = x$ and explaining why it is called an identify function, discus with students how the knowledge of composition of functions helps in determining whether two given functions are inverses of each other or not, use several examples during your discussion.	• Ask students to formulate functions and find their inverses.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• After revising what the students had learnt in the first topic of this unit about graphs of a relation and its inverse, once more consider a linear function and assist your students to draw the graph of the function and the graph of its inverse using the same coordinate axes.	• Give students opportunities to explain to the class about graphing the inverse of a function.
		• By considering different examples of graphs of several functions and the graphs of their corresponding inverses let the students generalize how the graph of an inverse of a function is obtained from the graph of the function.	• Give exercises problems on sketching the graphs of inverses of functions either individually or in small groups.
• determine the domain and range of the inverse of a given function.		• Assist students in determining the domain and range of the inverses of several functions by using examples and exercises and ask them what kind of connection they observe between the domain and range of a function and that of its inverse. • Ask the opinion of the students on matters like "Is the inverse of a function always a function?"	• As this is the end of unit 1 you can give quiz/Test.

Unit 2: Rational Expressions and Rational Functions *(12 periods)*

Unit outcomes: Students will be able to:

- know methods and procedures in simplifying rational expressions.
- understand and develop efficient methods in solving rational equations and inequalities.
- know basic concept and specific facts about rational functions.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • define rational expression • identify the universal set of a given rational expression • show the simplified form and the necessary steps in simplify a given rational expression.	***2. Rational Expressions and Rational Functions*** **2.1 Simplification of Rational Expressions** *(4 periods)* **2.1.1 Rational Expression**	• You may start the lesson by taking list of different expressions that the students had known so far and with active participation of the students discuss the peculiar properties of each expression and its universal set (i.e. the set under which it is defined) • Proceed the lesson by introducing the definition of a rational expression and elaborate on the stated definition by using several examples of rational expressions. Assist and encourage students to determine the universal set of a given rational expression that is the set under which the given rational expression is defined. • By considering several examples of rational expressions whose numerators and denominators are factor able and have common factor(s), discuss with students on how to write such expressions in their simplified form, in doing so give great emphasis on the fact that the: 1. The universal set of the expression should be determined before any simplification is done. 2. cancellation of common factor can be meaningful and correct if and only if it is done under the assumed universal set and hence the universal set should be given alongside the simplified form (i.e. the end result)	• Ask students to identity rational expressions from a given list of different expressions. • Ask students to determine the universal set of a given rational expression. • Give exercise problems on simplification of rational functions.
• Perform the four fundamental operations on rational expressions.	**2.1.2 Operations with rational expressions**	• By using the rules of addition, subtraction, multiplication and division of rational numbers discuss these operations on rational expressions using several examples. In your discussion emphasize on how to find the least common multiple (LCM) of the denominators of the expressions which should be factorized into prime polynomials (specially into linear expression)	• Give exercise problems on each of the four operations and let the students determine the universal set first and then give the result in its simplified form.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• decompose rational expressions into sums of partial fractions.	**2.1.3 Decomposition of rational expression into partial fractions**.	• In order to make students familiarize themselves with the rules for the operations, let them mention the rule for the corresponding operation alongside each step of their workout, in addition to this let them give the universal set at the beginning and at the end of the workout and let them give the result in its simplified form. • With active participation of students discuss the closure, commutative and associative properties of addition and multiplication of rational expression and the existence of the identify element and inverse of an expression with respect to each of these operations. ***Note:*** *In all the above activities it is better to consider simple expressions to handle for the students, as acquisition of the basic knowledge is essential here.* • Assist students in decomposing rational expression as a sum of partial fractions using several examples.	• Ask students to show the validity of the properties by using examples. • Ask students questions like: "Is every polynomial function a rational function? • Give exercise problems on decomposition of a given rational expression into partial fractions.
• solve rational equations	**2.2 Rational Equations …** *(3 periods)*	• You may begin the lesson with example of simple rational equation and with active participation of the students discuss the steps in finding solutions under the set in which the equation is defined e.g. (a) $\frac{1}{x} = 4(x \neq 0)$ (b) $\frac{x+1}{x-2} = \frac{x-3}{x}$ $(x \neq 0, 2)$: Use sufficient examples similar to the above ones and encourage the students to solve them. • Let the student check their answer is in the universal set and check it by substitution. • Assist students in solving equations involving rational expressions, in this case you may set up exercise problems from real life situations that lead to rational equations.	• Ask students questions like: Solve $\frac{x+1}{x-2} = 1$ and let them give reason for their answers. • Give exercise problems on solving rational equations.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• define rational function. • determine the domain of a given rational function. • determine the range of a given rational function.	**2.3 Rational Functions and their graphs** *(5 periods)* **2.3.1 Rational Functions**	• You may start the lesson by setting an activity in which list of functions, that the students studied so far, are given and ask students to assign name to each function in the list. • Introduce a new function that is defined by rational expression and which is known as "Rational Function" and then state its formal definition and following this with active participation of the students, discuss the definition using several elaborate examples. In this discussion assist and encourage the students to determine the domains and ranges of the rational functions under consideration.	• Give exercise problems on identifying rational functions • Ask students, to determine the value(s) for which a rational function is undefined; to give its domain and range.
• Sketch the graph of a given rational function.	• Graphs of rational functions	• You may start the lesson by giving activities to students to prepare table of values and to plot the corresponding points on the coordinate plane for some simple rational functions like: $f(x) = \frac{1}{x}$, $f(x) = \frac{1}{x+4}$ *and* $f(x) = \frac{x+1}{x-3}$ • Assist students in sketching the graphs of these functions and encourage them to identify the intercepts as well as the symmetry of the graphs they draw. • After a brief description of the meaning of asymptotes and their types let the students determine the vertical or horizontal or oblique asymptote that the graph of a given rational function may have.	• Ask students to prepare table of value for a given rational function first and then let them sketch the graph. • Ask oral questions in your discussion to check whether the students follow or understand the lesson.
• Determine the intercepts and symmetry of the graph of a given rational function.		• With active participation of the students discuss how to determine the type of asymptote that a function may have, nature of the functions near the asymptote and any other property . • That function may have (in this discussion you may use the graph of the function taken as an example above and encourage the students to determine the domain and range of the function from its graph.	

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• identify the type asymptote that the graph of a given function may have. • tell the properties of a given rational function from its graph. • use graphs of rational functions to solve rational inequalities.		• By considering functions of the form $f(x) = \frac{1}{(x-4)^n}$ assist students to generalize the nature of the graphs of these functions when n is odd and when n is even. • In addition to the method discussed in section 2.2 of this unit, discuss with students how the graphs of rational functions are used in solving rational inequalities and encourage students to practice and use this method.	• Give exercise problems on the determination of the asymptotes of the graph of a given rational function. • Ask students to describe the nature of the graph of a given rational function near its asymptotes.

Unit 3: Coordinate Geometry *(21 periods)*

Unit outcomes: Students will be able to:

- understand specific facts and principles about lines and circles.
- know basic concepts about conic sections.
- know methods and procedures in solving problems on conic sections.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • write different forms of equation of a line. • determine the slope, x-intercept and y-intercept of a line from its equation	***3. Coordinate Geometry*** **3.1 Straight line** *(3 periods)* • Revision on equation of a line	• You may begin the topic with a brief revision of equation of a straight line, its slope and intercepts. You can also give activities for the students on identifying parallel, intersecting and perpendicular lines by carefully examining their equations (without drawing).	• Ask students to give examples of linear equations • Give exercises on writing different equations for a line which is shown on the coordinate plane through two given points.
• determine the angle between two intersecting lines on the coordinate plane whose equations are given.	**3.1.1 Angle between two lines on the coordinate plane**	• You may start the lesson by discussing the angle between two non-vertical and two non-perpendicular lines. • Assist students to practise in determining the angle between two lines by using the slopes of the lines.	
• determine the distance between a point and a line given on the coordinates plane.	**3.1.2 Distance between a point and a line on the coordinate plane**	• Encourage students to practice is determining the distance between a point and a line through different examples and exercises. (i.e. the distance(d) of a point (x_1, y_1) from the line $ax + by + c = 0$ is given by: $d = \dfrac{\lvert ax_1 + by_1 + c \rvert}{\sqrt{a^2 + b^2}}$	• Ask students to use the formula and find the distance between a given point and a given line on the coordinates plane.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• name the different types of conic sections • explain how the conic sections are generated (formed)	**3.2 Conic sections** *(18 periods)* **3.2.1 Cone and sections of a cone**	• You may start the lesson by discussing how conic sections are generated, i.e. the formation the four famous curves when two right circular cones (with common vertex and whose altitudes lie on the same line) are sliced or intersected by a plane at different angles. • With active participation of the students, consider the different cases of the intersection of the plane and the pair of cones (arranged as explained above) and discuss on how the conic sections (circle, ellipse, parabola and hyperbola) are generated or formed. Recall that the name "conic section" comes from the "cone" used to generate the curves.	• Ask students oral questions to state the definition of a conic section. • Ask students to give some examples from real life (or their environment) that look like each of the conic sections.
• define circle as a locus • write equation of a circle • find the radius and center of a circle from its equation. • determine whether a given line and circle have a point of intersection or not.	**3.2.2 Circles** • lines and circles	• You may begin with the introduction of the notion of "Locus" as a system of points, lines or curves which satisfies one or more given condition(s). Let the students realize it as a set of points consists of those points (and only those points) whose coordinates satisfy a given equation, then the set of points is the **locus of the equation.**	• Ask students to define the general equation of a circle by the method of completing the square and ask them to interpret this equation of a circle
• determine the coordinates for the intersection	• Equation of a tangent line	• Let students do revision work on writing equations of a circle and determining the center and the radius of circles through examples and exercises. • Assist the students to calculate the	• Give exercise problems on finding the equation of a tangent line

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
point(s) (if the given line and the given circle intersect) • write equation of a tangent line to a given circle. (where the point of tangency is given)		perpendicular distance between the center of a circle and a line, where equations of both the circle and the line are given. • Based on the result they obtained above guide them to determine the number of intersection point(s) of the given circle with the given line. • Let the students determine(find) the point (i.e., its coordinates) of intersection for a circle and line (if they intersect). • Help the students in writing equation of a tangent line to a given circle at the given point.	to a given circle. • Give exercise problems on finding the common point(s) for lines and circles that are intersecting.
• write the standard form of equation of a parabola. • draw different types of a parabolas • describe some properties of a given parabola.	**3.2.3 Parabolas**	• You may start the lesson by defining a parabola as a locus (i.e. a plane curve which is the set of all points equidistant from a fixed point (called focus) and a fixed line (called directrix) in the plane. • With the help of the graph of a given parabola discuss the related terms (directrix, focus, axis, vertex and latus rectum). • Help the students in writing the standard form of equation of a parabola. • Let students practise in drawing the graphs of parabolas by recalling the students' knowledge of some groups of parabolas. • Help students in identifying the orientation of the graph of a parabola (open upward, downward, to the right or to the left) from the equation. • Assist students in the investigation of the properties of parabola through different examples and exercise.	• As a locus or a set of points equidistant from a fixed point (called focus) and a fixed line(caused directrix) on the plane. • Ask students to define a parabola and its different parts. • Ask students to write the standard form of the equation of a parabola. • Give class activities that deal with sketching parabolas. • Give students

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
			opportunities to discuss about some properties of parabola depending up on the coefficients of the highest powers. • Home work
• define "ellipse" as a locus (set of points on the plane which satisfy a certain given condition). • write the standard form of equation of an ellipse. • sketch ellipse. • describe some related terms (latus rectum, eccentricity, major and minor axes,...)	**3.2.4 Ellipses**	• You may start the lesson by defining an ellipse as a locus (i.e. A plane curve which is the set of all points (x,y) the sum of whose distances from two distinct fixed points (called foci) is constant) • With the help of the graph of an ellipse discuss the related terms (foci, vertex, major axis, minor axis, eccentricity and latus rectum) • Help the students in finding the equation of an ellipse based on the given conditions (with the help of examples and exercises). • Let students practise drawing the graphs of ellipses. • Assist student in finding the coordinates of the foci, the vertices, length of major and minor axis, eccentricity and length of latus rectum of an ellipse. • Let students describe some properties of ellipse.	• Ask students to define an ellipse and name its parts. • Ask students to write the equation of an ellipse in the standard form • Ask students to sketch graphs of ellipses given certain conditions • Give students opportunities to discuss about graphs of ellipses • Home work
• define hyperbola as a locus • write the standard form of equation of a	**3.2.5 Hyperbolas**	• You may start the lesson by defining a hyperbola as a locus (i.e., the set of all points (x,y) for which the absolute value of the difference between the distances from two distinct fixed points (called foci) is constant) • Encourage the students to find the	• Ask students to state the definition of a hyperbola • Ask students to sketch

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
hyperbola • describe related terms to hyperbola (foci, centre, transverse axis, asymptotes, conjugate axis...) • sketch hyperbola based on its given equation		equation of a hyperbola based on the given information with the help of examples and exercises. • With the help of the graph of a hyperbola discuss the related terms (focus, centre, transverse axis, conjugate axis, vertex, eccentricity and latus rectum). • Guide the students to practise in drawing the graphs of hyperbola and discuss related terms with their friends in a group.	hyperbolas. • Ask students to name parts of a hyperbola and describe terms related to it. • Ask students to give examples of hyperbolas from their environment
• give eccentricity of a given hyperbola • solve problems on hyperbola.		• Assist students in finding the lengths of the transverse and conjugate axes, the coordinates of the foci and vertices, the eccentricity and length of the latus rectum. • Let students practice describing properties of hyperbola through different examples and exercises.	• Ask students to distinguish type of the conic sections represented by a given equations • Ask students to give examples of conic sections from real life.

Unit 4: Mathematical Reasoning (*16 periods)*

Unit outcomes: Students will be able to:

- know basic concept about mathematical logic.
- know methods and procedures in combining and determining the validity of statements.
- know basic facts about argument and validity.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • explain the difference between "statement" and "open statement" • determine the truth value of a statement.	***4. Mathematical Reasoning*** **4.1 Logic** *(13 periods)* **4.1.1 "Statement" and "Open statement"**	• You may start the lesson by introducing the concepts "statement" and "open statement" using different examples from real life situations and then guide the students come to the definition of "statement" and "open statement". • Assist students to give different examples of "statements" and "open statements" from their daily life. • Guide students to change open statements to statements by substituting numbers or names in place of variables or pronouns and let them determine the truth values of these statements.	• Ask students to give examples of statements and open statements • Ask students to completed the truth values of table with compound statements
• describe the rules for each of the five logical connectives. • use the symbols¬, ∧, ∨, ⇒ and ⇔ to make compound statements	**4.1.2 Fundamental Logical Connectives (operators)** - Negation - Conjunction - Disjunction - Implication and - Bi-implication	• You may begin the lesson with statements that are taken from real life situations and connected by the words "and", "or", "if...., then ....." and "--- if and only if---and let the students determine the validity of the combined statement. • Based on the above discussion introduce the five logical connectives (sometimes they are also called **logical operators**) and tables that define/ describe the rule for the respective connective, in doing so assist students to use the symbols for the connectives that is, ¬ , ∧ , ∨, ⇒ and ⇔accordingly **Note**: *In fact the word "not" denoted by "¬" is applied to a single statement and does not connect two statements, and as a result of this the collective name **"logical operators"** can also be used in place of **"logical connectives"***	• Give for students opportunities to discuss the validity of arguments • Home work • Quiz/ Test

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• determine truth values of compound statements connected by each of the logical connectives.		• Encourage students to determine the truth values of different component statements and of their compound statements connected by each one of these connectives based on the corresponding rule, the statement that you take, as an example, should reflect good ethical and civic values such as patience, obedience, love of work, productivity as well as issues like environmental protection, gender equality HIV/AIDS etc. and statements from geometry and algebra too. • Allow students to give their own similar examples from their day to day activities.	• Let students form compound statements using the logical operators from real life situation and analyse their feedback so as to evaluate their logical thinking.
• determine truth values of two or three statements connected by two or three connectives.	**4.1.3 Compound statements**	• By considering up to three component statements assist students to determine the truth values of their compound statements connected by two or more connectives (use tables of truth values)	• Give exercise problems on combining statements and determining their truth values.
• describe the properties and laws of logical connectives. • determine the equivalence of two statements.	**4.1.4 Properties and laws of logical connectives**	• You may start the lesson by discussing what is meant by "two statements are logically equivalent" using examples. • Guide the students to come to the conclusion about properties of logical connectives (properties like: the commutative and associative properties of both conjunction and disjunction, distributive property, De-Morgan's Law) • Encourage the students to determine whether two given compound statements are equivalent or not by using (applying) the properties of connectives	• Give exercise problems on determining logical equivalence of statements. • Ask students to give examples that justify the validity of the properties of logical operators (connective)

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• define "contradiction and "tautology" • determine that a given compound statement is either a contradiction or tautology or neither of them	**4.1.5 Contradiction and Tautology**	• You may start the lesson by defining "contradiction" and "tautology" and discuss with the students about the application of the definitions in determining whether a given compound statement is a contradiction or tautology or neither of them by using several examples (using tables of truth values)	• Ask students to rewrite (restate) the definition of "contradiction" and "Tautology" in their own words • Give exercise problems on contradiction and tautology.
• find the "converse" of a given compound statement • determine the truth value of the converse of a given compound statement • find the "contra - positive" of a given statement • determine the truth value of the contra- positive of a given statement	**4.1.6 Converse and contra positive**	• You may start the lesson by discussing with students what is meant by "converse of a given compound statement and using several examples explain how to make the converse of a statement. • Assist students how to find the converse of a given statement and encourage them to determine its truth value (i.e. the truth value of the converse) • Let students observe the truth values of a given statement and its converse in such a way that they draw their own conclusion. • By using several examples discuss with the students what is meant by contra positive of a given compound statement and how to make the contrapositive • Encourage your students to determine the truth values of the contrapositive of a given compound statement and let them observe any relation, if it exists, between the truth values of the given compound statement and its contrapositive, so that they can draw conclusion from their observation.	• Give exercise problems on determining the converse of a given statement and its truth value. • Ask students what relation, if there is any, do they observe between the truth values of a given statement and its converse. • Give exercise problems on how the contrapositive of a given statement is determined and give exercises on determining the truth value of the contra- positive of a statement.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
			• Ask students what connection, if there is any, do they observe between the truth values of a statement and its contrapositive.
• describe the two types of quantifiers. • determine the truth value of statements involving quantifiers.	**4.1.7 Quantifiers**	• You may start the lesson by revising important points about "statement" and "open statements" from the lesson of the previous topic. • After introducing **"existential quantifier"** (∃x) and **"universal quantifier"** (∀x) discuss with students how each of these quantifiers can change open statements to statements and hence encourage students to determine the truth value by using sufficient examples. • By taking several examples let the students determine the truth values of statements involving both quantifiers.	• Ask students first, to give statements using only one quantities and then to give its respective truth value. • Give exercise problems on changing open statements to statements by using both quantifiers and determining their truth values.
• describe what is meant by "argument". • check the validity of a given argument.	**4.2 Arguments and validity** *(3 periods)*	• You may start the lesson by considering simple examples from daily life and explain what is meant by "argument" "hypothesis or premises" and "conclusion". You may take examples like: S_1: If he runs fast, he will win the race S_2: He did not win the race S : Therefore he did not run fast.	• Give group/individual activity on setting up a sensible argument from their real life situation.
• use rules of inference to demonstrate the validity of a given argument.	• Rules of Inference	• Thus the above three statements taken together form an argument in which S_1 and S_2 are hypothesis (or premises) and S is the conclusion	• Give exercise problems on identifying the premises and the conclusion of an argument and its validity • Give either class work or home work or quiz (as required)

Unit 5: Statistics and Probability *(31 periods)*

Unit outcomes: Students will be able to:

- know specific facts about types of data.
- know basic concepts about grouped data.
- know principles of counting.
- apply facts and principles in computation of probability.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • identify qualitative and quantitative data • describe the difference between discrete and continuous variables (data)	***5. Statistics and probability*** **5.1 Statistics** *(14 periods)* **5.1.1 Types of data** • Qualitative and quantitative data • Discrete and continuous variables (data)	• You may begin the lesson with a brief revision of the major concepts that the students had studied in Grade 9 statistics • By supporting with sufficient and appropriate examples discuss with students what is meant by **"qualitative data"** and **"quantitative data"** and let the students explain the difference between these types of data • Discuss what is meant by "variable" in statistics i.e. the characteristic which can be measured and expressed in quantitative or numerical terms, since a variable, in statistics, can be either discrete or continuous, with the help of sufficient and elaborate examples introduce the ideas of **"Discrete Variable**" and **" Continuous variable"**, in doing so, with their active participation let the students come to the conclusion that a "discrete variable" can only have observed values at isolated points along a scale of values. These values are generally expressed as an integer (whole numbers) only. Examples of discrete data are (a) the number of persons per households (b) the units of an item in inventory (c) the number of assembled components which are found to be defective. Likewise let the students conclude that a "continuous variable" assume a value at any fractional point along a specified interval of values and hence "continuous data" are generated by the process of measuring. Examples of continuous data are: (a) the weight of each shipment of exported coffee (b) the length of time between successive landings of airplane at Bole Airport.	• Ask students to give their own example of qualitative data and quantitative data. • Let students describe the difference between qualitative and quantitative data with their own words. • Let students describe the difference between discrete data and continuous data and let them give their own example for each kind.

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr>
<td>• identify ungrouped and grouped data
• determine class interval (class size) as required to form grouped data from a given ungrouped data
• make cumulative frequency table for grouped data (for both discrete and continuous)</td>
<td>5.1.2 Introduction to grouped data</td>
<td>• You may begin the lesson with a brief description of "frequency distribution" which is a table in which possible values for a variable are grouped into "classes" and the number of observed values which fall into each class is recorded.
• Following this introduce "grouped data" as those data which are organized in a frequency distribution. You may also explain that we use grouped frequency distribution for the purpose of summarizing a large sample of data.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>•Grouped discrete data</td>
<td>• Consider for instance, the number of patients that a doctor visits per day for 150 working days is given by:
<table>
<tr><td>3</td><td>2</td><td>6</td><td>2</td><td>6</td><td>5</td><td>22</td><td>3</td><td>1</td><td>10</td></tr>
<tr><td>5</td><td>9</td><td>7</td><td>2</td><td>5</td><td>1</td><td>5</td><td>4</td><td>9</td><td>7</td></tr>
<tr><td>25</td><td>19</td><td>8</td><td>2</td><td>5</td><td>8</td><td>10</td><td>16</td><td>15</td><td>5</td></tr>
<tr><td>7</td><td>8</td><td>3</td><td>6</td><td>6</td><td>21</td><td>6</td><td>9</td><td>4</td><td>5</td></tr>
<tr><td>6</td><td>6</td><td>22</td><td>8</td><td>11</td><td>23</td><td>8</td><td>5</td><td>9</td><td>6</td></tr>
<tr><td>8</td><td>7</td><td>5</td><td>10</td><td>16</td><td>11</td><td>13</td><td>1</td><td>7</td><td>3</td></tr>
<tr><td>2</td><td>18</td><td>0</td><td>16</td><td>4</td><td>9</td><td>8</td><td>5</td><td>9</td><td>17</td></tr>
<tr><td>7</td><td>9</td><td>5</td><td>19</td><td>12</td><td>1</td><td>10</td><td>3</td><td>5</td><td>7</td></tr>
<tr><td>13</td><td>18</td><td>8</td><td>7</td><td>8</td><td>7</td><td>7</td><td>13</td><td>0</td><td>5</td></tr>
<tr><td>14</td><td>7</td><td>20</td><td>1</td><td>9</td><td>4</td><td>6</td><td>24</td><td>9</td><td>6</td></tr>
<tr><td>11</td><td>5</td><td>6</td><td>28</td><td>7</td><td>7</td><td>22</td><td>1</td><td>17</td><td>4</td></tr>
<tr><td>11</td><td>8</td><td>1</td><td>4</td><td>12</td><td>13</td><td>9</td><td>23</td><td>14</td><td>5</td></tr>
<tr><td>2</td><td>6</td><td>6</td><td>11</td><td>3</td><td>14</td><td>6</td><td>8</td><td>4</td><td>4</td></tr>
<tr><td>6</td><td>8</td><td>29</td><td>18</td><td>5</td><td>8</td><td>8</td><td>17</td><td>4</td><td>4</td></tr>
<tr><td>5</td><td>18</td><td>7</td><td>3</td><td>11</td><td>23</td><td>20</td><td>10</td><td>6</td><td>6</td></tr>
</table>
As the above list of data is ungrouped, guide students to present it in a grouped frequency distribution or cumulative frequency distribution, and also help them in finding the cumulative frequency as shown below.</td>
<td>• Give students project work to collect and a classify, quantitative data based on issues taken from real life and let them construct and present it in a cumulative frequency distribution.
• This data can be obtained from the class, the school, the Education Bureau, the statistics office, newspaper etc.</td>
</tr>
</table>

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		Note that: It is required to present the frequency distribution consists of five classes, whose approximate interval is given by: $\begin{pmatrix}\text{approximate}\\\text{interval}\end{pmatrix}=\dfrac{\begin{pmatrix}\text{Largest value in}\\\text{ungrouped data}\end{pmatrix}-\begin{pmatrix}\text{Smallest value in}\\\text{ungrouped data}\end{pmatrix}}{\text{Number of class required}}$ $=\dfrac{29-0}{5}=5.8$ ∴ the closest convenient class size is thus 6 ***Table 1*** (see Table 1 below) *Note that the above cumulative frequency distribution is for discrete data.*	• Also ask students to they find from just what inter present presented in their data the frequency distribution table.
• described terms related to grouped continuous data, i.e., class limit, class boundary, class interval and class midpoint.	• Grouped continuous data	• Now consider examples of a frequency distribution like the one given below in which we use continuous data. ***E.g.*** on a certain construction site the weekly wages (in Birr) of 100 labourers taken from a list (i.e. ungrouped data) in which the highest observed wage was 258 birr and the lowest was 142 birr are required to be given in 6 categories (classes) of a frequency distribution as follows (Note that the approximate class interval $=\dfrac{258-142}{6}=19.33$ Birr ∴ The closest class size is 20 birr) ***Table 2:*** (see Table 2 below)	

Table 1

No of patients (class)	No. of visiting days *(f)*	Cumulative frequency *(cf)*
0-5	49	49
6-11	66	66+49 = 115
12-17	16	16+115 = 131
18-23	15	15+131 =146
24-29	4	4+ 146 =150
	Total 150	

Table 2:

Weekly wage (in Birr)	Number of labourers *(f)*
140 - 159	7
160 -179	20
180 - 199	33
200 - 219	25
220 - 239	11
240 - 259	4
	Total 100

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr>
<td>• determine class limit, class boundary, class interval and class midpoint for grouped continuous data.</td>
<td></td>
<td>• With active participation of the students discuss the terms and the corresponding concepts conveyed in these terms related to each class in a frequency distribution of continuous data. So that the students can differentiate and solve any problem on these concepts.
Thus explain terms like:
- The lower and upper class limits which indicate the values included within the class.
- The lower and upper class boundaries or exact limits that are the specific points along the measurement scale (Birr, in our example above) which serve to separate adjoining classes and also describe how they can be determined.
- The class interval which indicates the range of values included within a class, and can be determined by subtracting the lower class boundary from the upper class boundary for the class.
- The class midpoint which can be determined by adding one half of the class interval to the lower boundary of the class. Explain that, for certain summary purposes the values in a class are often represented by this class midpoint.
- By using the frequency distribution of the example taken above (Table 2) you may summarize what has been discussed so far on a table as follows.
Table 3
<table>
<tr><th>Weekly wage (class limits)</th><th>Class Boundaries (Birr)</th><th>Class midpoint</th><th>Number of Labourers</th></tr>
<tr><td>Birr 140-159</td><td>139.59 -159.50</td><td>Birr 149.50</td><td>7</td></tr>
<tr><td>160 - 179</td><td>159.50 -179.50</td><td>169.50</td><td>20</td></tr>
<tr><td>180 - 199</td><td>179.50 -199.50</td><td>189.50</td><td>33</td></tr>
<tr><td>200 - 219</td><td>199.50 -219.50</td><td>209.50</td><td>25</td></tr>
<tr><td>220 - 239</td><td>219.50-239.50</td><td>229.50</td><td>11</td></tr>
<tr><td>240 - 259</td><td>239.50-259.50</td><td>249.50</td><td>4</td></tr>
<tr><td></td><td></td><td>Total</td><td>100</td></tr>
</table>
Note: In general, only one significant decimal digit is expressed in class boundaries as compared with class limits. However, because with monetary units the next more precise unit of measurement after "nearest birr" is usually defined as "nearest cent," in this case two decimal digits are expressed as shown in Table 3 above.</td>
<td>• Give students a frequency distribution (like Table 2) and ask them to find the following
(a) lower and upper class limits and class boundaries
(b) class interval
(c) class midpoint and lastly let them form table like Table 3.</td>
</tr>
</table>

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• find the mean of a given grouped data.	**5.1.3 Measures of Location for Grouped Data** • Mean for grouped data	• You may begin the lesson with a brief revision of the measures of location for ungrouped data (which the students had learnt in Grade 9) • After defining the concept of **"Mean"** or **"Arithmetic Mean"** and clarify it with the help of several examples, discuss with students how to find the "mean" for ungrouped data. Following this, let the students clearly understand that when data have been grouped in a frequency distribution, the midpoint of each class is used as an approximation of all values contained in the class. • Based on the formal definition of "Mean" you already stated and with the help of examples encourage the students to come to the formula for Mean of a grouped data that is given by: $\overline{X} = \frac{\Sigma(fx_c)}{\Sigma f}$ or, more simply $\overline{X} = \frac{\Sigma(fx)}{n}$ • In your discussion emphasize on the fact that, operationally, both formulas indicate that each **"class midpoint"** denoted by X_C or simply X is multiplied by the associated **"class frequency** ***(f)*****",** and all these products are summed up $\left(\sum fX_c\right)$ and then this sum is divided by the total number of observations $\left(\sum f\right)$ or (**n** for sample data) represented in the given frequency distribution. • The following example summarizes what has been said so far. It is taken from the frequency distribution presented in section 5.1.1 (Table 2) **Example**: The mean of the weekly wages (rounded to the nearest Birr) of 100 labourers can be found as follows. It is assumed that the group of the labourers be a sample from a larger population of labourers.	• Give exercise problems on computation of Mean for data given like the following on <table><tr><td>x</td><td>2</td><td>6</td><td>7</td><td>8</td><td>10</td></tr><tr><td>f</td><td>3</td><td>4</td><td>9</td><td>2</td><td>6</td></tr></table> • Give exercise problems on computation of "Mean" for a grouped data (both discreet and continuous variable) • Check whether they correctly apply the formula in their computation • Give Homework exercises.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***

Table 4

Weekly wage (class) Birr	Class midpoint (X)Birr	Number of labourers (f)	fx Birr
140-159	149.50	7	1,046.50
160-179	169.50	20	3,390.50
180-199	189.50	33	6,253.50
200-219	209.50	25	5,237.50
220-239	229.50	11	2,524.50
240-259	249.50	4	998.00
Total		**100**	$\sum fX$=Birr **19,450.00**

$$\therefore Mean = \overline{X} = \frac{\Sigma(fx_C)}{\Sigma f} = \frac{\Sigma(fx)}{n} = \frac{19,450}{100} = 194.50\,birr$$

• find median for grouped discreet data • find Median for grouped data (continuous variable)	• The Median for grouped data	• After stating the formal definition of "Median" and by using examples, give brief revision of median for ungrouped data (like the one discussed in Grade 9) • With active participation of the students discuss the method and procedures that are used to find "median" of a grouped data. • During the discussion guide the students so that in order to find median for grouped data they should follow the procedures very carefully by emphasizing on the following. a) The data should be given in a cumulative frequency distribution. b) Then the class which contains the median value has to be determined first (that means, the class which contains the median is the first class for which the cumulative frequency equals or exceeds one-half of the total number of observations.) c) Once this class (which contains the median value) is identified, the specific value of the median is determined by the formula. $$\textbf{Median} = \mathbf{B_L} + \left(\frac{\frac{n}{2} - cf_B}{f_C}\right)i$$ where: $\mathbf{B_L}$= Lower boundary of the class containing the median **n** = total number of observations in the frequency distribution (N for a population)	• Ask students to find Median of ungrouped data with both odd and even number of observations. • Give data as follows $\begin{array}{|c|c|c|c|c|c|}\hline x & 2 & 5 & 7 & 8 & 10 \\ \hline f & 3 & 4 & 9 & 2 & 6 \\ \hline\end{array}$ • Give exercise problems on determining the Median for grouped data (both discrete and continuous variable) and check whether they apply the formula correctly.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*

cf_B = the cumulative frequency in the class preceding ("coming before") the class containing the median

f_C = the number of observations (frequency) in the class containing the median.

i = the size of the class interval.

The following example summarizes what has been discussed so far. This example is taken from ***Table 4*** above (i.e., the frequency distribution presented for weekly wage of 100 labourers)

Note: Let the students observe how the cumulative frequencies are determined.

Example:

(a) consider the following cumulative frequency distribution taken from Table 4.

Table 5

Weekly wage (class) Birr	No. of labourers (f)	Cumulative frequency (cf)
Birr 140-159	7	7
160-179	20	20 + 7 = 27
180-199	33	33 + 27= 60
200-219	25	25 + 60 = 85
220-239	11	11 + 85 = 96
240-259	4	4 + 96 =100
Total	**100**	

(b) The class containing the median is the class with $\frac{100}{2} = 50^{th}$ value, and hence the first class whose cumulative frequency equals or exceeds 50 is the class with limits Birr 180 - 199.

(c) Thus to determine the specific value of the median the calculation is done within the class 180 - 199.

Hence put

$B_L = 179.50$, $n = 100$, $cf_B = 27$, $f_c = 33$ and $i = 20$ in the formula to get:

$$Med = B_L + \left(\frac{\frac{n}{2} - cf_B}{f_c} \right) i$$

$$= 179.50 + \left(\frac{50 - 27}{33} \right) 20 = 193.44$$

∴ Median = Birr 193.44

Assessment:

- Ask oral question during the discussion
- Give Home work exercises.

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr><td>• determine the mode of a given grouped data.</td><td>• The Mode for a grouped data</td><td>• You may begin the lesson about "mode" with its formal definition followed by examples that explain how to determine the mode of a given ungrouped data. This can be taken mainly as a revision work of what had been discussed in Grade 9.</td><td>• Give class activity to determine the mode of ungrouped data.
• Ask student to determine the mode of the following data
<table><tr><td>x</td><td>2</td><td>5</td><td>7</td><td>8</td><td>10</td></tr><tr><td>f</td><td>3</td><td>4</td><td>9</td><td>2</td><td>6</td></tr></table></td></tr>
<tr><td>• identify data that are unimodal, biomodal and multimodal.</td><td></td><td>• By using sufficient examples introduce the distributions known as "unimodal", "bimodal" and "multimodal"
• With active participation of the students, discuss how to determine the "mode" for a given grouped data by using appropriate and sufficient examples
• In the discussion, emphasize that for a data grouped in frequency distribution, of course, with equal class interval, the class containing the mode is determined first, by identifying the class with the greatest number of observations (or largest frequency) which is also know as "the modal class". Then within this modal class, the mode can be determined with the help of the following formula:
Mode = $B_L + \left(\frac{d_1}{d_1 + d_2}\right) i$
where:
B_L = lower boundary of the modal class (the class containing the mode)
d_1 = the difference between the frequency in the modal class and the frequency in the preceding class
d_2 = the difference between the frequency in the modal class and the frequency in the following (or next) class
i = the size of the class interval</td><td>• Give exercise problem on determining the mode of a given grouped data (discrete and continuous variable)</td></tr>
</table>

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		Example: Refer to the grouped data given in Table 5 above The modal class is the class with limits Birr 180 - 199 Thus put $B_L = 179.50$, $d_1 = 33 - 20 = 13$, $d_2 = 33 - 25 = 8$ and $i = 20$ in the formula to get: $Mode = B_L = \left(\frac{d_1}{d_1 + d_2}\right) i = 179.50 + \left(\frac{13}{13+8}\right)(20)$ $= 191.88$ birr	
• determine the quartiles for a given grouped data	• Quartiles, Deciles and Percentiles for Grouped Data.	• In relation with the median (which divides a given distribution into two halves) introduce the other measures of locations, i.e., **"quartiles"** which divide the data into four quarters**,** "the **deciles"** which divide it into 10 tenths and "the **percentile**" which divide it into 100 parts.	• Give exercise problem on computing quartile, decile and percentile for ungrouped data.
• determine the required deciles of a given frequency distribution		• With active participation of the students and with the help of several examples from ungrouped data let the students realize that, the quartile, deciles and percentiles are very similar to the median in that they also subdivide a distribution of measurements recording to the proportion of frequencies observed.	
• determine the required percentile of a given frequency distribution.		• For the case of grouped data, discuss with students how the formula for the median is modified to the fractional point of interest. In your discussion emphasize that first determining the appropriate class containing the point of interest is important before using the modified formulas, and guide students to come to the formulas. Therefore, formulas in this case are: $Q_1\left(first\ quartile\right) = B_L + \left(\frac{\frac{n}{4} - cf}{f_c}\right) i$ $D_6\left(sixth\ decile\right) = B_L + \left(\frac{\frac{6n}{10} - cf_B}{f_c}\right) i$ $P_{30}\left(\begin{matrix} thirtieth \\ percentile \end{matrix}\right) = B_L + \left(\frac{\frac{70n}{100} - cf_B}{f_c}\right) i$	• Ask student to apply the formula for quartile and compute the first, second, third and fourth quartile for grouped data and ask them what they find about the second quartile in relation to the median. • Give exercise problems on computing certain given decile of grouped data.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		You may consider examples like: **Example:** Referring to Table 5 above determine the values at the (a) third quartile (b) sixth decile and (c) fortieth percentile. **Solution** (a) $Q_3 = B_L + \left(\frac{\frac{3n}{4} - cf_B}{f_c}\right) i$ $= 199.50 + \frac{75-60}{25} 20 = 211.50$ birr Note: the class containing the $\frac{3n}{4}$ or 75th measurement is the class with number of observations (or frequency $f_C = 25$ and limits 200 - 219 Birr, and hence its lower boundary (B_L) is 199.50. (b) $D_6 = B_L + \left(\frac{\frac{6n}{10} - cf_B}{f_c}\right) i$ $= 179.50 + \left(\frac{60-27}{33}\right) 20 = 199.50$ birr (c) $P_{40} = B_L \left(\frac{\frac{40n}{100} - cf_a}{f_c}\right) i$ $= 179.0 + \left(\frac{40-27}{33}\right) 20 = 187.38$ birr	• Give exercise problems on computing some given percentile of a grouped data. • Ask students what they found in their calculation about the relation among the 5th decile, 50th percentile and the median.
• describe the dispersion of data values	**5.1.4 Measures of Dispersion**	• You may begin with introductory discussion on what is meant by "**dispersion**" among values of a given data and precede the discussion by answering question like why we study dispersion. How many types of measures of dispersion are there?	
• find the range of a given data.	•Range	• With active participation of students and considering first example of ungrouped data define "**range**" ***viz,*** the difference between the highest and lowest values for items which have not been grouped in a frequency distribution. and discuss how to compute it. Let students describe what information "range" gives them about the data.	• Give exercise problems n computing the range of grouped data.

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr><td></td><td></td><td>• With active participation of students and considering first example of ungrouped data define "range" Viz, the difference between the highest and lowest values for items which have not been grouped in a frequency distribution. and discuss how to compute it. Let students describe what information "range" gives them about the data.
• Following this with the help of appropriate examples of grouped frequency distribution for both discrete and continuous data discuss with students how to compute "Range" in doing so guide them to come to the formula, i.e., Range = R = B_U(H) - B_L(L) where B_U(H) = upper boundary of the highest class and B_L(L) = the lower boundary of the lowest valued class.</td><td>• Ask them what information does "range" give about the data.

• Ask students to compute the range as well as the variance of data like the ones given below.
<table><tr><td>x</td><td>2</td><td>5</td><td>7</td><td>8</td><td>10</td></tr><tr><td>f</td><td>3</td><td>4</td><td>9</td><td>2</td><td>6</td></tr></table></td></tr>
<tr><td>• Compute variance for ungrouped data</td><td>• Variance

• For ungrouped data</td><td>• With the help of example of ungrouped data and with active participation of students introduce and discuss the methods and procedures in computing "variance". Guide students to come to the formula for variance of ungrouped data i.e. variance = $\frac{\Sigma(x_i - \bar{x})^2}{n}$
where, x_i = the value of the i[th] item as i = 1, 2, 3,... ,n and $\bar{x}$ = the mean of the data,
n = the total number of items in the data..
• Considering exercise problems, encourage and assist student in calculating variance of ungrouped data by using the above formula correctly.</td><td></td></tr>
<tr><td>• calculate variance for grouped data.
• solve problems on variance</td><td>• for grouped data</td><td>• Consider examples for both discrete and continuous grouped data, and discuss with students the methods and procedures (steps) in computation of variance. In the discussion guide students to come to and understand the formula.
Variance = $\frac{\Sigma f_i(x_i - \bar{x})^2}{\Sigma f_i}$
where x_i = mid value of the i[th] class as i = 1, 2, …K (if there are K class intervals)
$\bar{x}$ = the mean of the data
f_i = the frequency of the i[th] class interval</td><td>• Give exercise problems on calculation of variance for grouped data (continuous variable)</td></tr>
</table>

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr><td></td><td></td><td>For example you can take like the following grouped data of discrete variable.

Class	Frequency	Mid Value	f_ix_i	x_i-6	$(x_i-6)^2$	$f_i(x-6)^2$
0-4	4	2	8	-4	16	64
4-8	8	6	48	0	0	0
8-12	2	10	20	4	16	32
12-16	1	14	14	8	64	64
	Σf_i =15		**Σf_ix_i =90**			**$\Sigma[f_i(x_i-6)^2]$ =160**

Mean $= \frac{\Sigma[f_i(x_i)]}{\Sigma f_i} = \frac{90}{15} = 6$

$\therefore$ The required variance $= \frac{\Sigma[f_i(x_i-6)^2]}{\Sigma f_i}$

$= \frac{160}{15} = \frac{32}{3} = 10.67$

(correct to two decimal places)

- Like wise you can also discuss variance of grouped data of continuous variable (series)
- Encourage and assist students in the application of the formula and the steps in calculating variance for grouped data by giving exercise problems to the students.</td><td>

- Ask oral question regarding the steps in computing variance and what kind of information they get about the data from computing variance.</td></tr>
<tr><td>

- Calculate standard deviation for grouped data.</td><td>

- Standard deviation (S.D)</td><td>

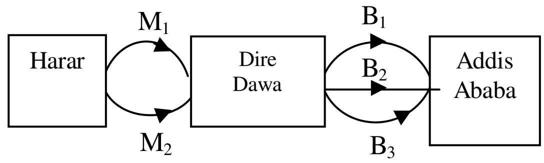
- Start the lesson by defining "**standard Deviation**" as the positive square roof of the variance. For its computation first consider examples of ungrouped data and discuss with students the procedures (steps) in the computation. Encourage students to solve problems on standard deviation by giving them exercise problems.
- Introduce the notation (δ) for "standard deviation" and guide them to come to and apply the formula *Viz*
- Standard Deviation for ungrouped data

$$= \delta\sqrt{\frac{\Sigma(x_i-\overline{x})^2}{n}}$$

- in short = $\delta\sqrt{Variance}$
- Note x_i, $\overline{x}$ and n are as given for variance
- Discuss with students about what standard deviation tells them about the given data by using appropriate examples from practical situations.</td><td>

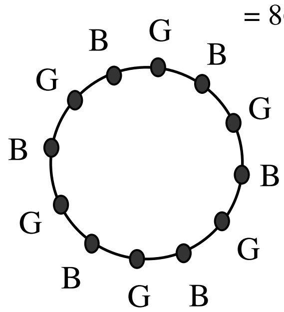
- Ask students to explain their opinion about the statement given below "If the standard deviation is small, there is high degree of uniformity in the observed values (data)".
- Give exercise problems on computation of standard deviation of grouped data and assert the statement given above.</td></tr>
</table>

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
	• Calculation of S.D. for Grouped Data	• Take examples of both types of data, i.e., discrete and continuous variables and discuss with students about the steps in computation of the variance and using the definition of standard deviation assist the students to calculate the standard deviation (δ) and guide them to the formula. $\delta = \sqrt{\frac{\Sigma[f_i(x_i - \overline{x})^2]}{\Sigma f_i}}$ where, x_i, $\overline{x}$ and f_i are as defined in the variance • By using exercise problems assist students to apply the formula correctly.	
	5.2 Probability *(17 periods)* • Revision	• You may start the lesson by revising important ideas about probability discussed in Grade 9. In the revision work you may raise issues like experimental and theoretical approaches of probability and determining probability of simple events. In doing so emphasize on how to find the number of outcomes favourable to the event and total number of possible outcomes.	• Ask oral question on some basic ideas of probability.
	5.2.1 Permutation and combination • Fundamental principle of counting	• With active participation of the students, discuss that finding probability of an event by counting is practical only if the outcomes favourable to the event and the total number of possible out comes are possible to count. • With the help of simple day-to-day activities, introduce the idea of "fundamental principle of counting" which is used to find the number of ways of occurrence (selections) of events in a given order. For the introduction, you may take several examples like the following one: **Example:** Suppose Nuria wants to go from Harar via Dire Dawa to Addis Ababa. There are two Minibuses from Harar to Dire Dawa and 3 Buses from Dire Dawa to Addis Ababa. How many possible ways of selection of cars are there for Nuria to go from Harar to Addis Ababa? Let M stands for Minibus and B stands for Buses.	• Ask students to give number of possible outcomes of an experiment by counting where 3 dies are thrown and let them explain why it is necessary to have an efficient methods of counting.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		 The possible selection are M_1B_1, M_1B_2, M_1B_3, M_2B_1, M_2B_2, M_2B_3	• Give exercise problems on finding the number of possible ways of selection using the fundamental principle of counting (using the principle of multiplication and Addition Principle)
• determine the number of different ways of possible selection from a given sets of objects (by using the multiplication principle)	• Multiplication principle	• Discuss with students that: (a) As the number of objects to be selected from, gets very large, then finding the possible ways of selection one after the other by the method of listing them is tedious and in some cases may not even be possible (b) In most cases we do not want to know the types of selections but what we need is their number only. So state the multiplication principle as "If an event can occur in **m** different ways and for every such choice another event can occur in **n** different ways, then both the event can occur in the given order in $\mathbf{m} \times \mathbf{n}$ different ways". Help students to extend this principle to any number of finite events.	
• find the number of ways of selection of mutually exclusive operations (by using the addition principle)	• Addition principle	• By using examples introduce "The principle of Addition" *viz* if an operation can be performed in **m** different ways and another operation in **n** different ways and the two operation are mutually exclusive (i.e., the performance of one excludes that of the other) then either of the two can be performed in $\mathbf{m} + \mathbf{n}$ ways. For example, A question paper has two parts where one part contains 4 questions and the other 3 questions. Suppose a student has to choose only one question from either part. He can do so in 4 + 3 = 7 ways. Encourage the students to solve problems on the matter discussed.	

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• determine the factorial of a given non-negative integer • find the possible ways of arranging objects by using the principle of permutation	• Permutation	• Define **"factorial n"**, where $n \in \mathbf{N}$ and introduce its notation **(n!)** and use examples to explain how to compute factorial of a given natural number. • By considering examples of situations that involve large and complex outcomes explain that it is necessary to have efficient methods of counting one of which is **"permutation".** With the help of several examples introduce the principle of permutation. • Discuss with students about "permutation" as a means of finding number of arrangements of objects taken some or all objects at a time and introduce its notation i.e., **P(n,r) or** $^{n}P_{r}$ for the number of permutation of **n** distinct objects taken **r** at a time and which is given by **n**! where $0 < r \leq n$ in this case consider some practical problems/ examples on permutation that the students can easily understand and proceed to relatively complex cases accordingly. So you may consider examples like **Example 1.** Five students are contesting an election for 5 places in the executive committee of environmental protection club in their school. In how many ways can their names be listed on the ballot paper. **Solution:** We have to arrange 5 names in 5 places $\therefore$ The number of ways of listing their names on the ballot paper = P(5,5) = 5! = 120 **Example 2.** Find the number of permutation that can be made out of the letters of the word "MATHEMATICS". In how many of these permutations. i) do the words start with C? ii) do all the vowels always occur together? iii) do the vowels never occur together? iv) do the words begin with H and end with S?	• Ask students to compute factorial for some small values of $n \in \mathbf{N}$ like 4!, 5!, 7! and also to evaluate expression like $\frac{5!}{3!2!}$, $\frac{5!\times 6!}{12!3!}$ • Give exercise problems on computing permutation of objects.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		***Note**: You may consider two or three or all of the above four questions given in Example 2 and discuss the solutions thoroughly.*	
• compute the possible arrangement of objects around the circle (using the principle of circular permutation)	•Circular Permutations	• You may begin the lesson with a discussion about the difference in arranging objects in straight line and along the circumference of a circle and allow students to perform simple activity in this regard and let them find the difference. • Assist students in determining the number of arrangements of (n) objects along the circumference of a circle and introduce this as **"circular permutation"** which depends on the relative positions of the objects after we fix the position of **one object** and then arrange the remaining objects in **(n-1)!** possible ways. • Encourage students to come to the formula i.e. **The number of circular permutation of n objects = (n-1)!** and let them apply it in solving problems like the following. **Example:** In how many ways 6 boys and 5 girls dine at a round table, if no two girls are to sit together. **Solution:** First let allot the seats to boys. Now 6 boys can have (6-1)! circular permutation, i.e. the number of permutation in which boys can take their seats = 5! = 120 Next the 5 girls can occupy seats marked (G). There are 6 such seats. This can be done in **${}^6P_5$ = 720 ways** $\therefore$ The required number of ways = 120×720 = 86,400	• Ask students to explain the difference between arrangements objects in a straight line and around a circle. • Give exercise problems on computing number of arrangements of objects on a circle.



Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• describe the difference between arrangement of objects and selection of objects. • describe what is meant by "combination of objects"	•Combination	• You may begin the lesson with the help of simple examples and discussing with students on some revision activities about permutation of objects in which order of arrangements are important and following this consider situations (if possible from examples you have taken above) in which the order of arrangement is not important and let the students explain why they are different, how the numbers of these two kinds of arrangements can be determined. You may consider examples like the following one. **Example**: Five researchers say A, B, C, D and E presented 5 different papers on the strategy of poverty reduction and it is required that only three of them allowed to make speech for national conference on the issue, so in how many different ways can we form a team of 3 speakers. ***Note:*** *To begin with what the students already know, you may list some of the 60 arrangements obtained by permutation, i.e. P(5,3) = 60 for instance (ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC,ADE, AEB, AEC, AED,...)* **Solution:** In the question above arrangements, like: ABC, ACB, BAC, BCA, CAB and CBA all consists of the same team of speakers A, B and C and hence should not be considered as different teams, that means order of the speakers of will not change the teams. Thus the required different teams of speakers asked in the question are ABC, ABD, ABE, ACD, ACE, ADE, BCE, BCE BDE and CDE. Therefore, there are 10 different ways of forming a team which consists of 3 speakers. With the help of examples like the one shown above introduce the concept of **"combination"** as, "the ten groups of speakers listed above are called the combination of 5 speakers taken 3 at a time." In this case emphasize on the fact that	• Ask students to explain about the principle of permutation and that of combination and their difference. • Ask student to find the different number of ways of selecting a certain number of objects out of a given objects.

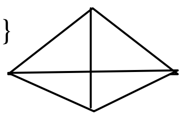
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		(1) When we speak of combination, we don't consider the order. (2) To determine the number of combinations for large number of objects by listing them all (as in the example above) is difficult and boring so it is necessary to have efficient method.	
• determine the number of different combination of n objects taken r at a time.		• After stating the formal definition of "combination" and its denotation as either **C(n, r)** or $^{n}C_{r}$ for combination of n objects taken **r** at a time where $0 < r \leq n$, assist students in solving exercise problems on combination based on the stated definition.	• Give exercise problems on the principle of combination of objects.
• explain the computational relationship between permutation and combination of objects.		• Encourage students to describe the mathematical (computational) relationship-between P(n, r) and C(n, r) by using appropriate and sufficient examples and guide them to come to the relation, $C(n,r) = \frac{P(n,r)}{r!}$, where $0 < r \leq n$	
• prove simple facts about combination.		• Help students to prove some simple facts about combination like: 1) C (n, n) = 1, 2) C (n, 0) = 1, 3) C (n, r) = C (n, n-r), 4) C(n, r) + C (n, r-1) = C(n+1,r)	
• solve practical problems on combination of objects.	• Practical problems on combination	• Assist students in their effort to solve practical or real life problems on combination. You may give exercise problems (beginning with the simpler one to a relatively complex one) on combination like the following ones. ***Note***: *for the following Examples, "Hint" for the solution and the last results are given for checking while the remaining steps are left out for the teacher and students to show.* **Example 1**: In an exam paper there are 12 questions. In how many ways can a student choose eight questions in all if two questions are compulsory. **Solution:** Since 2 questions are compulsory, the student is left with a choice of choosing 6 questions from the remaining 10 questions and this he can do in (C (10, 6) = 210 ways.	• Give several real life problems on the application of the principle that the students have learnt so far.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		Example 2: In how many ways can Bekele invite at least one of his friends out of 5 friends to an art exhibition? **Solution: Hint:** He can invite either one or two or three or four of five $\therefore$ Total number of ways in which he can invite at least one of his friends $= C(5,1)+C(5,2)+C(5,3)+C(5,4)+C(5,5)$ $= 5 + 10 + 10 + 5 + 1 = 31$ **Example 3:** A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected **Solution:** 2 black balls can be selected in $C(5,2) = 10$ ways and 3 red balls can be selected in $C(6,3) = 20$ ways $\therefore$ Total number of selecting 2 black and 3 red balls $= 10 \times 20 = 200$ **Example 4:** A committee of 7 students has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of (i) exactly 3 girls (ii) at least 3 girls (iii) at most 3 girls. **Hint for the solution** (i) When exactly 3 girls are included in the committee, the remaining members will be 4 boys $\therefore$ Total number of ways of forming the committee $= C(4,3) \times C(9,4) = 504$ (ii) At least 3 girls are included means, the committee will consist of either <u>3 girls and 4 boys</u> **or** <u>4 girls and 3 boys</u> $\therefore$ Total number of ways of forming the committee $= [C(4,3) \times C(9,4)] + [C(4,4) \times C(9,3)]$ $= 504 + 84 = 588$ (iii)When at most 3 girls are included, the committee may consist of <u>3 girls and 4 boys</u> **or** <u>2 girls and 5 boys</u> **or** <u>1 girl and 6 boys</u> **or** <u>7 boys (all are boys</u>) $\therefore$The required number of ways of forming the committee $= [C(4,3)\times C(9,4)]+C[(4,2)\times C(9,5)]$ $+[C(4,1)+C(9,6)]+[C(9,7)] =1632.$	

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• write up to the 6^{th} power of a binomial expression $(x + y)^n$ (i.e. when n = 0, 1, 2, 3, 4, 5) in its expanded form by using direct multiplication . • describe what they observe in the expansion of $(x + y)^n$ where n = 0, 1,2,3,4,5	**5.2.2 Binomial Theorem**	• You may start the lesson by revising how the expanded form of the square and cube of a given binomial expression is written, using the distributive property of multiplication over addition. You may consider examples like: $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$ and $(m + n)^3 = (m + n)(m + n)(m + n) = (m + n)(m^2 + 2mn + n^2) = m^3 + 3m^2n + 3mn^2 + n^3$ Following this with active participation of the students discuss the expanded form of the following expressions in such a way that students can observe and describe the pattern in the expansions and the corresponding coefficients. $(x + y)^0 = 1$ $(x + y)^1 = x + y$ $(x + y)^2 = x^2 + 2xy + y^2$ $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ • By using the same method (i.e., actual multiplication) discuss the expanded forms of $(x + y)^4$, $(x + y)^5$ and $(x + y)^6$ and encourage the students to list down all what they observe in the expanded forms such as: 1) The number of terms in the expansions in relation to the index of the binomial 2) The index of the first term in relation to that of the binomial. 3) How the indices in successive terms of the expansion change uniformly 4) The index of the last term of the expanded form in relation to that of the binomial. 5) What the sum of the indices in each term gives 6) How the coefficients of terms equidistant from the beginning and last terms are related 7) If the coefficient of one term is known, how to determine that of the next term	• Give exercise problems on Bionomia expansion (the application Binomial theorem)

Competencies	Contents	Teaching / Learning Activities and Resources	Assessment
• describe "Pascals" Triangle" and its use • apply the "Binomial Theorem" in expanding the n^{th} power of binomial terms i.e. (x + y^n, where n ε $\mathbf{Z}^+$ • determine any term in the expanded form of $(x + y)^n$ where n ε $\mathbf{Z}^+$ • solve problems on binomial expansion		• With the help of the examples discussed above introduce "Pascal's triangle" and discuss with students how it is formed and for what it is used for. • State the "Binomial Theorem" for non-negative integral index (n), i.e. $(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}y + {}^nC_2x^{n-2}y^2$ +--- $+{}^nC_rx^{n-r}y^r$+---+ ${}^nC_ny^n$ and then discuss with students by considering some values for **n** how the list of their observation which they wrote previously are asserted by this theorem and also how the coefficients satisfy conditions described by Pascal's Triangles and how the principle of permutation is applied in determining them. • Encourage students to find any term in the binomial expansion and guide them to state the formula of the "General term" which is used to determine any term of the expansion. • Help students in solving problems on binomial expansion and on the application of "Binomial Theorem" as well as application of the formula for the "General Term".	• Ask students to write the pattern of rascal's triangle up to few rows correctly. • Give exercise problems on writing the expand form of a given binomial with non-negative integral exponent.
• describe what is meant by "Random Experiment". • Explain what is meant by an outcome of a random experiment	**5.2.3 Random Experiments and its outcomes**	• You may start the lesson with brief revision of the concept of probability that the students had learnt in Grade 9. • Proceed the discussion with the introduction of the concept of **"Random Experiment,"** as an experiment, when repeated under identical conditions does not produce the same result or outcomes or as an operation (activity) which produced some well defined results, in doing so from Grade 9 topics use some experiments as an example. • Guide the students to describe what is meant by "an outcome" of a random experiment in their own words, and let them come to the conclusion that, when a random experiment of some kind is performed, then associated with this experiment is the set of possible results which are known as outcomes of the random experiment. • With the help of several examples and active	• Ask students to explain what is meant by Random Experiment with their own words.

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr>
<td>• describe what is meant by sample space of a given random experiment.
• list some of the sample points of a sample space for a given experiment.
• define "equally likely outcomes" of a given trial in his/her own words.
• define</td>
<td></td>
<td>participation of the students discuss on how to list the possible out comes (finite in number) of a given random experiments. As an example you may consider like the following one.
Example: It pair of dice are thrown then find the possible outcomes.
Solution: Here are few out comes of this experiment and the other can be easily determined from the pattern.
<table>
<tr><td>(1,1)</td><td>(1,2)</td><td>(1,3)</td><td>(1,4)</td><td>(1,5)</td><td>(1,6)</td></tr>
<tr><td>(2.,1)</td><td></td><td></td><td></td><td></td><td>(2,6)</td></tr>
<tr><td>(3,1)</td><td></td><td>(3,3)</td><td></td><td></td><td>(3,6)</td></tr>
<tr><td>(4,1)</td><td></td><td></td><td></td><td>(4,5)</td><td>(4,6)</td></tr>
<tr><td>(5,1)</td><td>(5,2)</td><td></td><td></td><td></td><td>(5,6)</td></tr>
<tr><td>(6,1)</td><td>(6,2)</td><td>(6,3)</td><td>(6,4)</td><td>(6,5)</td><td>(6,6)</td></tr>
</table>
• Based on the example (like the above one) you discussed, then define what is meant by "sample space" i.e. when a random experiment is performed then the set consisting of all the possible outcomes of the experiment is called a sample space which is often denoted by (S). Similarly introduce that, each element or member of a sample space is called a "sample point" and give some examples of sample points from your examples.
• In the class discussion by using examples, explain some important concepts which the student may come across in his/her study for instance
(a) outcomes of a trial (performing a random experiment) are said to be equally likely outcomes when there is no reason to expect any one of the outcomes in preference to another.
Example: If a fair die is thrown, then any one of the outcomes 1, 2, 3, 4, 5, 6 can be considered to be equally likely
(b) with the help of appropriate examples</td>
<td>• Ask students to list possible outcomes of an experiment using free diagram (the experiment should have few out comes).
• Call on the students for explanation of terms like "sample space" "sample point", "equally likely outcomes" and "favourable outcomes" with their own words.</td>
</tr>
</table>

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
"favourable outcomes/ cases"		explain what is meant by "**Favourable Cases**" i.e. in a trial, the outcomes which insure the happening of a particular case are said to be cases **favourable** to that particular result we are interested in. You can take examples like. **Example:** In throwing a die, the number of favourable cases for getting an even number is 3 *viz.* 2, 4 and 6 or simply 2, 4 and 6 are favourable outcomes.	
• determine events of a given random experiment.	**5.2.4 Events** • Revision on events	• You may begin the lesson by revising the concept of sample space of a given random experiment and then using simple examples consider situations which ensure the happening of particular condition as a result among the members of the sample space of an experiment. Based on this, define **"an event"** that is, any subset of a sample space and which is commonly denoted by **"E"** and by using this definition encourage your students to list some (if possible all) events of a given random experiment. You may consider examples like the following one.	• Ask students to define "event in probability" by their own words.
• identify sample (elementary) events and compound events.		**Example:** The four faces of a regular tetrahedron are numbered 1, 2, 3 and 4, if it is thrown, and the number on the bottom face (on which it stands) is registered then list the events of this experiment. **Solution:** The sample space = {1, 2, 3, 4} the possible events are {1}, {2}, {3} and {4}  • Encourage students to give their opinion (or let them imagine and say), about events known as "simple or elementary events" and "compound events" and then consolidate their opinion and guide them to come to the conclusion that "elementary event (or simple event)" consists of one sample point whereas "compound event" has more than one sample point. For example the events in the above example are simple events, but if we are interested in the event "getting even numbers", then the event will be compound events, i.e., {2, 4}.	• Give exercise problems to list some events of a given set of outcomes of an experiment.
• determine the number of		• In order to determine the number of events associated with an experiment whose sample	• Ask students to give exhaustive

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
events of a given sample space		space is **S**, you can use the formula from the topic on sets discussed in Grade 9 (i.e., number of subsets of a given set) thus: if n **(S)** = **m** then the number of events = $\mathbf{2^m}$. This number can also be referred as the **exhaustive number of events**, since it is the total number of possible out comes associated with the random experiment. You may use tree diagram to list the sample space of an experiment and encourage your students to practise this method specially to identify compound events of the experiment.	number of events in an experiment whose outcomes are finite and let them explain what this reminds them from set theory.
• describe the occurrence or non occurrence of an event. • explain an event denoted by "not E" where "E" is a given event	•Occurrence or non-occurrence of an event.	• With active participation of the students discuss what is meant by **"an event occurred"** using examples like: **Example:** If a die is thrown, then **S** = {1, 2, 3, 4, 5, 6,}. Let **E** be event of getting an odd number, then **E** = {1,3,5}. Now, in trial, if the outcome is 3, and as 3 ∈ **E** then we say that **E** has occurred. If in another trial, the outcome is 4, then as 4 ∉ **E** we say the event **E** has not occurred (i.e. **not E**) 'You can use the notion of "complement of a set" in order define the event "not E" as : if w is a sample point in **S** (sample space) then **"not E"** = that is **E' = S - E** = {w: w ∈ **S** and w ∉ **E.** You may also ∉ use the Venn - diagram to illustrate the situation pictorially.	• Ask students orally to explain when to say an event occurred or not occurred.
• explain events connected by "or" and "and"	•Algebra of events	• Based on the definition of operations of sets and their properties from the lesson of the previous grades and with active participation of students discuss some condition which can also be used in the study of **probability of events** such as: if E_1, E_2 and E_3 are three events of a sample space S, then: 1. (E_1 or E_2) or ($E_1 \cup E_2$) is the event "either E_1 or E_2 or both" 2. (E_1 and E_2) or ($E_1 \cap E_2$) is the event "both E_1 and E_2" 3. E'_2 or $\overline{E}_2$ or ~ E_2 or E_2^c is the event not E_2"	• Give exercise problems on finding an event which is obtained by combining two or more events.
• describe the simplified forms of		In addition to the above results you should discuss with student events described with a) Commutative and associative properties	• Let the students list some basic properties of

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
events by using the properties of operations on sets		of both the "union" and "intersection" of sets (events) b) De-Morgan's Law (for both union and intersection) c) Distributive property of union over intersection and vise-versa.	combination of events (by using a set theory)
• identify exhaustive events	• Exhaustive Events	• After defining **"Exhaustive Events"** viz a set of events where at least one of them must necessarily occur every time the experiment is performed, discuss with students by considering examples. For instance, if a die is thrown then the events {1}, {2}, {3}, {4}, {5}, {6} are exhaustive events. More generally, the events $\mathbf{E_1}$ $\mathbf{E_2}$, $\mathbf{E_3}\ldots\mathbf{E_n}$ form a set of exhaustive events of a sample space S where $\mathbf{E_1 \cup E_2 \cup E_3 \ldots \cup E_n = S}$ and $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{E_3 \ldots E_n}$ are subsets of S.	• Ask the students to describe exhaustive events in an experiment.
• identify mutually exclusive events	• Mutually Exclusive Events.	• Following the definition of **"Mutually Exclusive Events"** (when events $\mathbf{E_1}$ and $\mathbf{E_2}$ are disjoint, i.e., $\mathbf{E_1 \cap E_2 = \phi}$, which means that $\mathbf{E_1}$ and $\mathbf{E_2}$ have no sample point in common), encourage your students to give some examples of their own and consider more simpler events which elaborate the definition very briefly. For example, if a die is thrown, then the sample space $\mathbf{S}$ = {1, 2, 3, 4, 5, 6}. Let event $\mathbf{E_1}$ (odd numbers) = {1, 3, 5} and let event $\mathbf{E_2}$ (even numbers) = {2, 4, 6}, thus $\mathbf{E_1}$ and $\mathbf{E_2}$ are mutually exclusive because $\mathbf{E_1 \cap E_2 = \phi}$. You may use the Venn diagram as a pictorial representation of the situation	• Let the students give mutually exclusive events of an experiment and let them justify their answers.
• describe events that are both exhaustive and mutually exclusive	Exhaustive and Mutually Exclusive Events.	• By considering sufficient and appropriate examples and active participation of students discuss about events that are both **exhaustive** and **mutually exclusive** and guide student to the generalization that. If $\mathbf{S}$ is the sample space associated with a random experiment and if $\mathbf{E_1, E_2, E_3 \ldots E_n}$ are subsets of $\mathbf{S}$ such that: (i) $\mathbf{E_i} \cap \mathbf{E_j} = \phi$ for $i \neq j$ and (ii) $\mathbf{E_1 \cup E_2 \cup E_3 \cup \text{-----} \cup E_n = S}$ then the collection of the events $\mathbf{E_1}$ $\mathbf{E_2}$, $\mathbf{E_3}, \ldots \mathbf{E_n}$ forms a mutually exclusive and exhaustive system of events.	• Give exercise problems on identifying Exhaustive events, mutually Exclusive events and both Exhaustive and mutually exclusive events from a list of different events.
• identify independent events.	• Independent Events	• State the definition of "**Independent Events**" which means that, the occurrence or non-occurrence of one event does not affect	• Give exercise problems on identifying

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		the probability of the occurrence of the other. For instance, in a simultaneous throw of two coins, event of getting a tail on the first coin and the event of getting a tail on the second coin are independent events. Consider similar examples and discuss with students until they understand the idea.	"independent events" and "dependent" events" from a given list of events of an experiment.
• identify dependent events	• Dependent Events	• Proceed the lesson with introduction of "**Dependent Events**" in relation with independent events by taking examples like: "If a card is drawn from a well shuffled pack of cards and it is replaced before drawing the second card, then the result of the second draw is independent of the first draw. On the other hand, if the first card is not replaced before drawing the second card then the second draw is dependent on the first draw".	
	5.2.5 Probability of an event. • Revision on probability	• You may start the lesson with a brief revision of "Probability" that the students had learnt in Grade 9, i.e. with students discuss "the empirical approach" and "the Classical approach" of probability and by using several examples describe how to find the probability of a given event based on the two approaches.	• Ask oral question about approaches of finding probability that the students had learnt in Grade 9.
• describe the axiomatic approach of probability • interpret basic facts in the theory of probability.	• Axiomatic Approach of Probability	• Introduce the modern theory of probability known as **"Axiomatic approach of probability"** and let the students realize that this approach includes both the Empirical and Classical definitions of probability and overcome the limitation of these two. *You should also make students sit up and take notice that in axiomatic approach, no precise definition of probability is given. Here probability calculations are based on some axioms or postulates.* • Guide students to come to the conclusion and interpret the following basic three facts *With each event* E *we associate a real number* **P(E)** *called the probability of* **E** *with properties*	• Let the students explain about "Axiomatic approach of probability" in their own words.

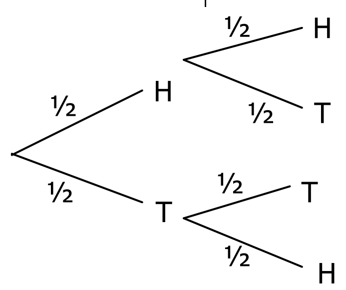
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		(a) $0 \leq P(E) \leq 1$ (b) $P(S) = 1$... where E = S (the sample space) (c) $P(E_i \text{ or } E_j) = P(E_i) + P(E_j)$ if $A_i \cap A_j = \varnothing$ *(i.e.* E_i *and* E_j *are mutually exclusive events)* • By considering P as a function encourage students to conclude that its domain is the set of subsets of S(sample space) and its range is the set of real numbers between 0 and 1 (both inclusive) and **1**) if $E = \varnothing$, then $P(E) = 0$ and if E = S, then $P(E) = P(S) = 1$. Therefore, $0 \leq P(E) \leq 1$ **2**) if **x** is the probability of the occurrence and **y** is the probability of non-occurrence of that event, then $x + y = 1$ • You may consider examples like the following one **Example**: Which of the following cannot be valid assignments of probabilities for outcomes of sample space $S = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ where $w_i \cap w_j = \varnothing$, if $i \neq j$	• Give several problems on concepts of probability based.

	w_1	w_2	w_3	w_4	w_5	w_6	w_7
a	0.1	0.001	0.05	0.03	0.01	0.2	0.6
b	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
c	0.1	0.2	0.3	0.4	0.5	0.6	0.7
d	-0.1	0.2	0.3	0.4	-0.2	0.1	0.3
e	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{13}{14}$

Solution: *The answer and its justification (as a hint) is given for each but further explanation is expected to be given by the teacher.*

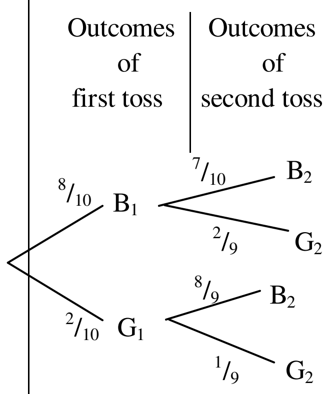
a) Valid because all the 3 properties are satisfied
b) Valid " " " " " " "
c) not valid, because the sum of all the probabilities is 2.8 which is greater than 1 i.e., $0 \leq P(E) \leq 1$ is not satisfied.
d) not valid, because probabilities of w, and w_5 are negative and hence $0 \leq P(E) \leq 1$ is violated
e) not valid, because the sum of all the probabilities, $\frac{17}{7}$, is greater than 1.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• find probabilities of events based on Axiomatic approach		• You should also give exercise problems on computation of probabilities of either simple events or compound events in which any one or more of the principles of counting (Fundament counting or permutation or combination) are applied to find the number of favourable outcomes of the event in question and the number of total outcomes in the respective sample space.	
• describe the odds in favour of an event or the odds against an event	• Odds in Favour of and the Odds Against an Event	• With active participation of the students discuss about the meaning of "**odds in favour of an event**" and "**odds against an event**" by using several examples, let students describe how to find these two expressions and let them also explain the relationship between these expressions of an event and the probability of that event, that means, if **m** and **n** are probability of the occurrence and non occurrence of an event respectively, then the ratio **m: n** is **called the odds in favour** of the event and the ratio **n: m** is **called the odds against the event.** **Example:** The odds against a certain event are 5:7. Find the probability of its occurrence. **Solution:** Let E be the event. Then we are given that n(not E) = 5 and n(E) = 7 $\therefore$ n(S) = n(not E) + n(E) = 5 + 7 = 12 $\therefore P(E) = \frac{n(E)}{n(S)} = \frac{7}{2}$	• Ask student to compute odd in favour or the odd against an event and let them explain the relation between these two ratio.
• Find the probability of $E_1 \cup E_2$ where E_1 and E_2 are events in a random experiment	• The rules of Addition Probabilities	• With the help of set theory, theory of probability and by considering several examples discuss with students how to find the probability of the union of two events, so that the students come to the conclusion that; for two event E_1 and E_2, $P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ ….(-1)	• Give exercise problems on computation of probability by using the rule of addition.
• determine the probability of mutually exclusive events.	• Probability of a mutually exclusive events.	• During the discussion it is better to use the Venn diagram in order to describe the situation very easily • With the help of a few examples discuss with students about the extension of this rule for three events: E_1 , E_2 and E_3	• Give exercise problems on computing probability of mutually exclusive events.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• After reminding students of mutually exclusive events, discuss with them how to find the probability of the union of these events by using the rule of addition (above) and several examples. Let students come to the conclusion that $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ ------- (2) Where E_1 and E_2 are mutually exclusive events.	
	• The Rule of Multiplication of Probabilities.	• Before discussing "The Rules of Multiplication on Probability" briefly revise the situations of "independent events" and "dependent events" by using several examples (as much as possible)	
• find probability of the joint occurrence independent event (by using rule of multiplication) • describe the outcomes of events using tree diagram.	• Probability of independent events	• Following this, introduce "The Rule of Multiplication" which is concerned in determining the probability of the joint occurrence of events E_1 and E_2, since this is the intersection of the events E_1 and E_2; the probability is denoted by $P(E_1 \cap E_2)$. The rule of multiplication for independent events is given by: $P(E_1 \text{ and } E_2)=P(E_1 \cap E_2)=P(E_1)\times P(E_2)\ldots(3)$ • To show the application of this rule use several examples like the following one. (if possible use the tree diagram as a method of portraying the possible events related with sequential trials) **Example:** If a fair coin is tossed twice find the probability that both outcomes will be "heads" **Solution:** Let $E_1 = \{H\}$ and $E_2 = \{H\}$. Since E_1 and E_2 are independent events. The required probability is then, $P(E_1 \text{ and } E_1)=P(E_2 \cap E_2)=P(E_1)=½\times½=¼$ Outcomes of first toss / Outcomes of second toss / Joint event  H and H H and T T and H T and T	• Give exercise problems on probability of independence events.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• Proceed the lesson by introducing the concept of "**conditional probability**" which is employed to designate the probability of occurrence of the related event when two events are dependent and also introduce the expression probability for dependent events, i.e., $\mathbf{P(E_2 \mid E_1)}$ which indicates the probability of the **occurrence event E_2** ***given*** **that event E_1 has already occurred.**	
• determine the probability of the joint occurrence of dependent events (using multiplication rule)	• Probability of dependent events	• Discuss with students that, for dependent event the probability of the joint occurrence of events E_1 and E_2 is the probability of E_1 multiplied by the conditional probability of E_2 given E_1 has occurred, and also explain that an equivalent value is obtained if the two events are reversed in position. Then guide students to come to the conclusion that, the rule of multiplication for dependent events is given by: $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1) \times P(E_2 \mid E_1)$ ----- (4) $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_2) \times P(E_1 \mid E_2)$ -----(5) • Introduce that, either formula (4) or (5) above is often the ***general rule of multiplication*** on probability • To show the application of this rule you may use tree diagram and simple examples like the following one. **Example:** Suppose that a group of 10 students contain eight boys (B) and two girls (G). If two students are chosen randomly with out replacement, then(based on the multiplication rule for dependent events) find the probability that the two students chosen are both boys. *Note: The sequence of possible choice and the probabilities are portrayed by the tree diagram below (the subscripts indicate sequential position of out comes)*	Give exercise problems n computing probability of dependent events.

<table>
<tr><th>Competencies</th><th>Contents</th><th>Teaching / Learning Activities and Resources</th><th>Assessment</th></tr>
<tr>
<td>• describe the outcomes of events using tree diagram to determine their probability</td>
<td></td>
<td>Solution: The probability that both are boys is
$P(B_1 \text{ and } B_2) = P(B_1) \times P(B_2 \mid B_1)$
$= (\frac{8}{10}) \times (\frac{7}{9}) = \frac{56}{90} = \frac{28}{45}$



<table>
<tr><th>Joint event</th><th>Probability of joint event</th></tr>
<tr><td>B_1 and B_2</td><td>$^{56}/_{90}$</td></tr>
<tr><td>B_1 and G_2</td><td>$^{16}/_{90}$</td></tr>
<tr><td>G_1 and B_2</td><td>$^{16}/_{90}$</td></tr>
<tr><td>G_1 and G_2</td><td>$^{2}/_{90}$</td></tr>
<tr><td></td><td>$^{90}/_{90}$</td></tr>
</table>
</td>
<td>• Ask students to show an outcome of a given experiment using tree diagram (to compute probability)</td>
</tr>
<tr>
<td>• identify whether a given events are independent or dependent (by comparing the equation for probability of joint occurrence of independent events.</td>
<td></td>
<td>• Discuss with students that, with out the use of the multiplication rules if the probability of joint occurrence of two events is available directly, then the independence of the two events E_1 and E_2 can be tested by comparing:
$P(E_1 \text{ and } E_2) \stackrel{?}{=} P(E_1) \cdot P(E_2)$
i.e. If they are equal the two events are independent, but if they are not equal the two events are dependent.</td>
<td></td>
</tr>
</table>

Unit 6: Matrices and Determinants *(31 periods)*

Unit outcomes: Students will be able to:

- know basic concepts about matrices.
- know specific ideas, methods and principles concerning matrices.
- perform operation on matrices.
- apply principles of matrices to solve problems.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • define matrix	***6. Matrices and Determinants*** **6.1 Matrices** *(4 periods)* • The concept of matrix	• Assist students to grasp the concept, notation, order, equality, types of matrices, zero matrix, elaborating row matrix, column matrix, square matrix, unit or identity matrix, diagonal matrix, square matrix, upper triangular, lower triangular and comparable matrices by using appropriate examples.	• Different exercise problems are given and the solutions are checked. • Ask students to construct matrices by taking real life examples.
• determine the sum and difference of two given matrices of the same order.	• Addition and subtractio n of matrices	• Define and illustrate the sum and difference of matrices by taking appropriate examples. **Example of Addition of Matrices** Let M = Male F = Female C = Child Ad = Adult Matrix A below shows how many shoes of each type the shop has in stocks. Matrix B below shows the number of shoes of each type it sells in a particular week. A: $\begin{array}{c} C \\ Ad \end{array}\begin{pmatrix} 65 & 42 \\ 111 & 154 \end{pmatrix}$ (columns M, F) B: $\begin{array}{c} C \\ Ad \end{array}\begin{pmatrix} 15 & 21 \\ 19 & 28 \end{pmatrix}$ (columns M, F) Calculate the number of each type of shoe still in stock by the end of the week $Ans\begin{pmatrix} 65 & 42 \\ 111 & 154 \end{pmatrix}-\begin{pmatrix} 15 & 21 \\ 19 & 28 \end{pmatrix}=\begin{pmatrix} 50 & 21 \\ 92 & 126 \end{pmatrix}=\begin{array}{c} C \\ Ad \end{array}\begin{pmatrix} 50 & 21 \\ 92 & 126 \end{pmatrix}$ (columns M, F) 2×2 2×2 $\underline{2\times 2}$	
	• Properties of addition of matrices	• Discuss the main properties of addition of matrices like commutativity, associativity, identity and additive inverse properties through different examples.	
• Multiply a matrix by a scalar • Describe the	• Multiplica tion of a matrix by a scalar.	• Show how to multiply matrices by scalars by taking appropriate real life example like • The marks obtained by Mamo and Nigist (out of 50) in their home examination are	• Give sufficient number of exercise

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
properties of multiplication of matrices by scalars.		given below by a matrix. $\begin{array}{c} \\ Amharic \\ English \\ Science \end{array}\begin{array}{c} \begin{array}{cc} Mamo & Nigist \end{array} \\ \begin{pmatrix} 31 & 37 \\ 40 & 46 \\ 28 & 25 \end{pmatrix} \end{array}$ Now, we want to find their marks in each subject out of 100, then we have: $\begin{array}{c} \\ Amharic \\ English \\ Science \end{array}\begin{array}{c} \begin{array}{cc} Mamo & Nigist \end{array} \\ \begin{pmatrix} 31\times2 & 37\times2 \\ 40\times2 & 46\times2 \\ 28\times2 & 25\times2 \end{pmatrix} \end{array}$ and this can be represented in the matrix form as follows: $\begin{pmatrix} 64 & 74 \\ 80 & 92 \\ 56 & 50 \end{pmatrix}$ We observe that this new matrix is obtained by multiplying each element of the original matrix by 2.	problems on multiplication of matrices by scalars.
• Determine the product of two matrices. • Describe the properties of the product of two matrices.	• Multiplication of two matrices (of order 2×2 and 3×3)	• Discuss the main properties of scalar multiplication of matrices with the help of sufficient number of examples. • Discuss the product of two matrices with the help of sufficient number of examples. **Examples:** Paulos and Meti have a choice of shopping at one of the two supermarkets X and Y. Matrix A shows the type and quantity of certain foods they both wish to buy. Matrix B shows the cost of the items at each of the supermarkets. $A = \begin{array}{c} \\ P \\ M \end{array}\begin{array}{c} \begin{array}{ccc} C & L & Po \end{array} \\ \begin{pmatrix} 2 & 4 & 5 \\ 1 & 7 & 3 \end{pmatrix} \end{array}$ $B = \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} 120 & 110 \\ 55 & 60 \\ 35 & 30 \end{pmatrix} \end{array}\begin{array}{c} \\ C \\ L \\ Po \end{array}$ P M C = Cereal packets L = Loaves of bread Po = Potatoes (kg) (i) Calculate the shopping bill for items at each supermarket (ii) Where should they buy at X or Y?	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		(i) Multiply A by B $\begin{pmatrix} 2 & 4 & 5 \\ 1 & 7 & 3 \end{pmatrix} \times \begin{pmatrix} 120 & 110 \\ 55 & 60 \\ 35 & 30 \end{pmatrix} = \begin{pmatrix} 635 & 610 \\ 610 & 620 \end{pmatrix}$ $2\times3 \quad 3\times2 \quad \underline{2\times2}$ $= \begin{array}{c} \\ P \\ M \end{array}\begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} 635 & 610 \\ 610 & 620 \end{pmatrix} \end{array}$ **NB** $635 = (2\times120) + (4\times55) + (5\times35)$ **Ans**: Paulos should shop at Y Meti should shop at X • Assist students to describe the major properties of the product of two matrices from sufficient number of examples and exercises	
• Determine the transpose of a matrix	• The transpose of a matrix and its properties	• Define the transpose of a matrix using examples. • Discuss the properties of the transpose of a matrix and give examples and exercises on their applications.	• Give exercise problems (including real life) and check solutions.
	6.2 Determinants and their properties *(6 periods)*		
• determine the determinant of a square matrix of order 2.	• Determinants of order 2.	• Define determinant of a square matrix and assist students to determine the determinant of square matrices of order 2 with sufficient examples.	• Give some square matrices of order 2 and ask students to calculate the determinants.
• determine the minor and cofactor of a given element of a matrix	• Minors and cofactors of the elements of matrices.	• Define the minor and cofactor of elements of a matrix and assist students on how to get them using sufficient examples.	• Ask students to determine the minor and cofactor of elements of a matrix.
• calculate the determinate of a square matrix of order 3.	• Determinant of order 3	• Define the determinant of order 3 using co-factors and assist students to apply it through sufficient examples and exercises.	• Ask students to calculate the determinants of some square matrices of order 3.
• describe the properties of determinants.	• Properties of determinants	• Discuss the major properties of determinants with the help of examples and allow students to apply them in exercises.	• Ask students to describe the properties of determinants.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• Determine inverse of a square matrix	**6.3 Inverse of a square matrix** *(4 periods)*	• Define the inverse of a matrix and discuss the uniqueness of the inverse and the invertibility of the transpose of a matrix. Assist students in determining the inverse of a matrix with sufficient examples and exercises.	• Give various exercise problems on determining the inverses of matrices and on checking whether a given matrix is invertible or not.
• Find associated augmented matrix of equations • Perform elementary operations on matrices	**6.4 Systems of equations with two or three variables** *(5 periods)* • Augmented matrix • Elementary operations of matrices.	• Define augmented matrix and assist students to determine the augmented matrix for equations of two or three variables. • Define elementary operations on matrices with row and column operations and discuss some notations of these operations.	• Give exercise problems on determination of the augmented matrices associated with equations of two or three variables.
• Solve systems of equations in two or three variables using the elementary operations.	• Solutions of systems of equations	• Encourage and assist students to solve systems of equations in two or three variables with the help of sufficient number of examples and exercise problems.	• Let students exercise describing elementary operations on matrices and their notations. • Various exercise problems on solving systems of equations in two or three variables using elementary operations are given and solutions are checked.
• Apply Cramer's rule to solve systems of linear equations.	**6.5 Cramer's Rule** *(3 periods)*	• Discuss Cramer's rule for solving systems of linear equations and give examples on how to apply the rule. Let students exercise and applying the rule to solve problems.	• Various exercise problems on the application of the rule are given.

Unit 7: The Set of Complex Numbers *(13 periods)*

Unit outcomes: Students will be able to:

- know basic concepts about complex numbers.
- know general principle of performing operation on complex numbers.
- understand facts and procedures in simplifying complex numbers.
- show the geometric representation of complex numbers on the Argand plane.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • define complex numbers. • identify the real and imaginary parts of a given complex number • determine the equality of two complex numbers	***7. The Set of Complex Numbers*** **7.1 The concept of complex numbers** *(2 periods)* • Definition of complex numbers • Equality of complex numbers	• You may start the lesson by asking oral questions about set of real numbers then ask them to solve simple quadratic equations: $x^2 - 1 = 0$ and $x^2 + 1 = 0$ and let them say some thing about the solutions of the equations. • Also ask them to draw the graphs of $y = x^2 - 1$ and $y = x^2 + 1$ on the same coordinate plane and assist them to explain for the class about the points of intersections of the graphs with the x - axis. • As $x^2 + 1 = 0 \Rightarrow x^2 = -1$ and there is no real number whose square is negative 1, explain the necessity of extension of the set of real number to a bigger set, by introducing a new element (number). Introduce "**imaginary number**" namely $\sqrt{-1} = i$ (read as iota). So with active participation of students discuss show the introduction of $i = \sqrt{-1}$ helps in writing numbers like: $\sqrt{-2} = \sqrt{2\times(-1)} = \sqrt{2}\times\sqrt{-1} = \sqrt{2}\,i = \sqrt{-16}$ $= \sqrt{16\times(-1)} = \sqrt{16}\times\sqrt{-1} = 4i$	• Ask students to give examples of complex numbers. • Give students an opportunities to discuss about the solutions of some quadratic equations whose roots are into real numbers. • Ask students to identify the real and imaginary parts of same complex numbers. • Ask them to write expressions i^5, i^{10}, i^{100} without a powers of i • Give class activities to find the unknowns in $x-3i = 2+12yi$ and $7+2yi = r-10i$

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• describe the set of complex number $\mathbb{C}$ and its relation to the set of real numbers $\mathfrak{R}$.		• Let the students compute some powers of i such as i^2, $i^{.3}$, $i^{.0} = 1$ i^6, i^{12} and i^{23} and ask them what they find and let them describe it with their own words. • Define "**complex number**", i.e. a number which is written in the form $a + b$i where $a, b \in \mathbb{R}$ and i $\sqrt{-1}$ = is called a complex number. Introduce what is meant by the real part and the imaginary part of a given complex number. Following this allow students to determine the equality of two complex number using several examples. Then guide them to come to the conclusion that: $z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2$ i are two complex numbers, $z_1 = z_2$ iff $a_1 = a_2$ and $b_1 = b_2$ • Introduce the set of complex numbers which is denoted by $\mathbb{C}$ and given by $\mathbb{C} = \{z: z = a + bi \text{ where } a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$	• Ask students to describe the relationship between the set of real number $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$
• add complex numbers correctly	**7.2 Operations on Complex Numbers** *(3 periods)* **7.2.1 Addition and subtraction of complex numbers**	• With the help of several examples and active participation of students discuss how to find the sum and difference of complex number. Through the discussion guide students to come to the conclusion that: "if $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ then (a) $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)$ i (b) $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)$ i	• Give exercise problem on addition of complex number like: a) to separate the real and imaginary part of the sum of two complex numbers b) to find the sum $i^{.7} + i^{.10} - i^{13}$
• subtract complex numbers correctly.		• Encourage students to prove some of the basic properties addition and subtraction of complex numbers such as:	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• describe the closure property of both addition and subtraction. • describe the commutative and associative properties of complex numbers. • identify the additive identity element in ℂ. • determine the additive inverse of a given complex number.		i) Closure properties of both addition and subtraction of complex numbers ii) Commutative property of addition iii) Associative property of addition iv) The existence of additive identity (i.e., 0 + 0i) v) The existence of additive inverse (if z = a + bi then -z = - a + (-b)i is the additive inverse of z)	
• determine the product of two complex numbers.	**7.2.2 Multiplication and Division of Complex Numbers** • Multiplication	• With the help of multiplication of binomial expression that the students had learnt and using several examples discuss how to multiply two complex numbers. During the discussion guide students to come to the conclusion that if $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ then $z_1.z_2 = (a_1 + b_1i)(a_2 + b_2i)$ $= a_1a_2 + a_1b_2 i + b_1a_2i + i^2b_1b_2$ $= a_1a_2 + (a_1b_2 + b_1a_2) i - b_1b_2$ ... ($\because i^2 = -1$) $= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1) i$	• Give exercise problems on multiplication of complex number and let them give the product in the form a + bi
• describe five basic properties of multiplication of complex numbers.		• With active participation of the students describe the validity of the following facts about multiplication of complex numbers a) The closure property b) The commutative property c) The associative property d) The existence of multiplicative identity (i.e. 1 + 0i e) The existence of multiplicative inverse	• Ask students to write the squares and cubes of sums and differences of two complex number as well as the difference of the squares of the complex numbers.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		Note: for every non-zero complex number Z = a + bi there is a complex number $\left(\frac{a}{a^2+b^2}\right)+\left(\frac{-b}{a^2+b^2}\right)$ i denoted by $^1/z$ or z^{-1} which is called the **multiplicative inverse** of z such that $z.\frac{1}{z}=1$ (multiplicative identify in ℂ	
• divide two complex numbers	•Division	• Before discussing the division of complex numbers introduce the conjugate of a given complex number that is, if Z = a + bi is a complex number then the complex number denoted by $\overline{z}$ which is given by $\overline{z}$ **= a - bi** is called the **conjugate of Z** • Allow students to find the conjugate of some given complex numbers to practice and understand the concept. Note that z=a + bi ⇒ $\overline{z}$ = a+(-b)i=a-bi Discuss, with active participation of students, on how to perform division on complex number. Let the students observe and practice the application of conjugate in the process of division by using several examples and let the students come to the fact that: if Z_1 = a + bi and Z_2 = c + di ≠ 0 are two complex numbers then $z_1 \div z_2 = z_1 \times \frac{1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di}\times\frac{c-di}{c-di}$ $=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i$	• Give exercise problems on division of complex numbers and check their works.
• give reason for each step in the process of division of complex numbers.		• By giving several exercise problems encourage the students to divide complex numbers and give the results in the form "a + bi" and assist in their work.	• Ask students to justify each step in the process of the division.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• determine the conjugate of a given complex number.	**7.3 Conjugate and Modules of Complex Numbers** *(2 periods)* • Conjugate of complex number	• As it is defined above (in section 7.2 of this unit) revise what is meant by "conjugate of a given complex numbers" and allow students to determine the conjugate of some given complex numbers. • With active participation of the students discuss some basic properties of conjugate of a complex number such as: if $z = a + bi$ is any given complex number then a) $z + \overline{z} = 2a$ c) $(\overline{\overline{z}}) = z$ b) $z.\overline{z} = a^2 + b^2$ and encourage the students to prove the above properties by themselves and allow them to express or justify each of their steps in the proof.	• Ask students to prove the properties of conjugate of complex numbers. • Give exercise problems on the application of the properties of conjugate of complex numbers. • Ask students to prove 1) $z_1+z_2= z_2+ z_1$ 2) $z_1.z_2 = z_2. z_1$
• find the Modulus of any given complex number.	• Modulus of a Complex Number	• Define the "Modulus of a complex Number" i.e. if $z = a + bi$, then the modulus of z denoted by $\lvert z \rvert$ is defined by non-negative real number $\sqrt{a^2+b^2}$ i.e. $\lvert z \rvert = \sqrt{a^2+b^2}$ • By giving exercise problems encourage the students to practice and understand the concept of Modulus of a complex numbers, for instance: given z_1 and z_2, let the students find $\lvert z_1 \rvert$, $\lvert z_2 \rvert$, $\lvert z_1 + z_2 \rvert$, $\lvert z_1 - z_2 \rvert$ and compare what they obtain.	• Given z_1 and z_2 where $z_2 \neq 0$ • Ask students to find $\lvert z_1 \rvert$, $\lvert z_2 \rvert$, $\lvert z_1.z_2 \rvert$, $\lvert z_1 \div z_2 \rvert$, $\lvert z_1+z_2 \rvert$, $\lvert z_1 \div z_2 \rvert$ • Ask students to prove $\lvert z_1.z_2 \rvert = \lvert z_1 \rvert . \lvert z_2 \rvert$
• Write the simplified form of expressions involving complex numbers.	**7.4 Simplification of Complex Numbers** *(3 periods)*	• With the help of the concepts discussed so far, encourage students to simplify expressions involving complex (or imaginary) numbers, for example you may consider expression like: a) $z = \dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$, then find z b) Simplify $[2+\sqrt{-25}]-[3-\sqrt{-216}]+[1-\sqrt{-9}]$ c) Write the following expression in the form a + bi $\dfrac{(3+\sqrt{5}\,i)(3-i\sqrt{5})}{(3+i\sqrt{2}-(\sqrt{3}-i\sqrt{2})}$	• Give exercise problems on simplification of expressions involving complex (or imaginary) numbers.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
	7.5 Argand Diagram and Polar Representation of Complex Numbers *(3 periods)*	• Start the lesson by introducing "Argand Diagram" which is the representation of complex numbers as points in the plane.	• Ask students to plot the point corresponding to a given complex number.
• describe how to set up the Argand Plane.	• Argand Plane	• Set up the Argand plane (the plane representing the complex numbers as points) and with active participation of students discuss that there is a one-to-one correspondence between the set of complex numbers $\mathbb{C}$ and the set of points on the Argand plane and then describe terms related to the representation of complex numbers on the complex plane such as Real axis, Imaginary axis,	• Given a point on the Argand plane, ask students to determine the complex Number that corresponds to the given point.
• Plot the point corresponding to a given complex numbers. • identify the complex number that corresponds to a given point in the Argand Plane.		• Encourage students to plot points corresponding to a given complex numbers after showing them through several examples. Similarly let the students determine the complex number which corresponds to a given point in the Argand plane. Allow students to interpret the physical meaning of Modulus of a complex number by using its representation in the Argand plane	• Ask students questions like "show that the points representing the complex numbers, $1 + i$, $-1 -i$ and $-\sqrt{3} + i\sqrt{3}$ in the Argand plane are the vertices of an equilateral triangles.
• represent any complex number in the polar form	• Polar Representation of a complex Number.	• Discuss the methods and Procedures in representing a given complex numbers on the polar coordinate system. During the discussion define terms related to this second type of representation i.e., terms like "Polar coordinates", and the "principal argument of z(i.e., the value of θ in the interval $-\pi < \theta \leq \pi$) or simply "argument of z"	• Give exercise problems like "To which quadrant each of the following complex numbers belong.

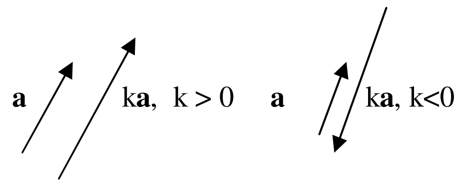
Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• determine the modulus and argument of a given complex number.		• Guide the student to come to the conclusion that: - The argument of all positive real number is Zero. - The argument of all negative real number is π - The argument of all positive imaginary numbers is $\frac{\pi}{2}$ - The argument of all negative imaginary numbers is $-\frac{\pi}{2}$ • Encourage students to solve problems on polar representation of complex number and assist them in their activities. You may consider exercises like **Example:** Convert the complex number -1 -i in the polar form and plot it on the polar coordinate plane.	• Give exercise problems like "To which quadrant each of the following complex numbers belong a) 3 + 5i b) -2 + 3i c) -3i + 4 d) -4i - 6 • Ask students to find the modulus and argument of the complex number $\frac{1+i}{1-i}$

Unit 8: Vectors and Transformation of the Plane *(20 periods)*

Unit outcomes: Students will be able to:

- know basic concepts and procedures about vectors and operation on vectors.
- know specific facts about vectors.
- apply principles and theorem about vectors in solving problems involving vectors.
- know basic concepts about transforming of the plane.
- apply methods and procedures in transforming plane figures.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • define a scalar quantity. • identify the everyday application of scalars.	***8. Vectors and Trans-formation of the Plane*** **8.1 Revision on vectors and scalars** *(3 periods)* • Scalars	• You may start the lesson by revising important points that the students had learnt about scalars in Grade 9. • You may proceed with an activity which deals with the 'concepts' of "scalar quantity" so that students can define scalar as a quantity with size or magnitude only. • Assist students to realize every day examples of scalars like: **Example**: mass 10 kg, time 5 sec, distance 5 km, money 100 Birr, etc.	• Ask students to list out many examples of scalar quantities.
• define a vector quantity. • identify the everyday application of vector.. • describe the difference between a vector and a scalar quantities.	• Vectors	• You may start the topic by reminding the students about vectors that they had learnt in Grade 9. • You may proceed with an activity which deals with the 'concept of vector quantity' so that students can define vector as a quantity with size or magnitude and direction included. • Assist students give to everyday examples of vectors. **Example** Weight, (direction is towards the centers of the earth and whose magnitude is given in Newton(N)).	• Ask students list out many examples of vector quantities. • Ask students to describe the difference between a vector and a scalar quantity through examples.
• represent vector by different notations.	• Representation of a vector	• Discuss the different ways of representing vectors. • Assist students to exercise the different way of representing vectors (Coordinate, column)	• Ask students to determine the different ways of representation of vectors.
• determine the sum of two or more vectors. • determine the difference of two vectors.	• Addition and subtraction of vectors.	• You may start by discussing the addition of vectors using the "**triangular law of addition**" of "**vectors and proceed**" with the **parallelogram law of addition** of vectors.	• Ask students to determine the sum and difference of some pair of vectors.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		• Discuss the commutative and associative properties of addition of vectors with active participation of students. • Using the concept of addition of vectors discuss the difference of two vectors.	
• multiply a vector by a scalar	• Multiplication of vectors by scalars	• Help students to practice through different examples and exercises. • You may start the lesson by introducing multiplication of a vector **a** by a scallar k as k**a** where k**a** is a parallel vector with the same direction for k > 0 and with opposite direction for k < 0. 	• Give exercise problems on scalar multiplication of vectors.
• resolve a given vector in to two components.	**8.2 Representation of vectors** *(1 period)* • Components of vectors	• You may start the lesson with an activity of resolving some force vectors given as a position vector using its X and Y components on the coordinate plane. • Help students to practice component representation of vectors.	• Ask students to resolve some vectors in to their components and check their work.
• use unit vectors to determine the column representation of a given vectors.	• Unit vectors	• Introduce the unit vectors **i** and **j** on the coordinate plane and explain how a given vector is expressed as a sum scalar multiples of them. • Assist students to show how a vector **P** = **x**i + **y**j can be resolved into its horizontal and vertical components **H** = **x**i and **V** = **y**j i.e. **H**= $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and V= $\begin{pmatrix} 0 \\ y \end{pmatrix}$ where the unit vectors are **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	
• determine the magnitude of a vector	• Norm of a vectors	• You may start the lesson by discussing on how to determine the magnitude (or the length) of a given vector P = x**i** + y**j** which is given by $\lvert \mathbf{P} \rvert = \sqrt{x^2 + y^2}$ and allow students to practice through exercises.	• Ask students to determine the length of some vectors.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• find the scalar product (inner product, of two vectors • describe some properties of scalar product of vectors.	**8.3 Scalar (inner or dot) product of vectors** *(3 periods)* • scalar product of vectors. • application of scalar product of vectors	• You may start by stating the definition of scalar product as: i) **a.b** = |**a**| |**b**|cos θ for vectors a and b and angle θ between them and ii) **a.b** = $a_1, b_1, + a_2 b_2$ where **a** = $a_1 + a_2$**j** and b = $b_1 + b_2$ **j** • Discuss some of the properties of scalar product of vectors.	• Give exercise problems on scalar product of vectors and the application.
• apply vectors to solve problems on geometry, algebra, mechanics and other related problems. • write the parametric equation of a line. • write equation of a circle by applying vectors. • determine the equation of the tangent line to a circle using vectors.	**8.4 Application of vector** *(5 periods)* • Vectors and lines. • Vectors and circles • Equations of tangents to circles.	• with active participation of the students discuss on the proof of some theorems from geometry using vector algebra. • Assist students to observe the application of the concept of vector algebra in calculating work done, angle between two vectors and its application to real situations. • With students' active participations derive the parametric vector equation of a line and then assist students in writing the parametric vector equation of a line through different examples and exercises. • With the help of sufficient examples discuss on the use of vectors in writing equation of circles. • Assist students in writing equations of different circles. • Help students to writ the equation of a tangent line to a given circle through examples and exercises.	• Give exercise problems on the application of vector algebra. • Ask students to write the parametric equation of a line. • Give problems on writing equations of tangent to a give circle and check their work.
• explain what is meant by transformation of the plane. • describe the main properties of rigid motion.	**8.5 Transformations of the plane** *(8 periods)*	• You may start the lesson by defining transformation of the plane and rigid motion as a special type of transformation. • With the help of several examples discuss the main properties of rigid motions.	
• Translate points, lines and circles using vectors.	• Translation	• Discuss the effect of translation on the coordinate system. • Assist students to translate points, lines and circles with sufficient examples.	• Give exercise problems on translating some points, lines, circles with given translation.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• Reflect points, lines, circle and some other plane figures.	• Reflection	• You may start the lesson by asking students to express their ideas about reflection while they use plane mirrors. • Discuss the effect of reflection on the coordinate plane. • Assist students in reflecting points, lines, circles and some other plane figures along a given line through examples and exercises.	• Ask students to reflect, points, lines, and some plane figures along given lines.
• Determine the images of a given plane figure when rotated through an angle θ	• Rotation	• You may star the lesson by defining the concept of rotation and with the help of examples discuss rotation of points through 90^0, 180^0 and through any angle θ about the origin. • Discuss the effect of rotation of some plane figures through 90^0, 180^0 clockwise and anti-clockwise directions about the origin and then proceed with rotation through a given angle about the origin. With active participation of students set up the relation between the coordinates of a point and that of its image. • Assist students to determine the images of plane figures after rotating through a given angle θ about a given point (a, b).	• Give exercise problem on rotating points lines and some plane figures through different angles in either direction about a given point.

Unit 9: Further on Trigonometric Functions *(20 periods)*

Unit outcomes: Students will be able to:

- know basic concepts about reciprocal functions.
- sketch graphs of some trigonometrical function.
- apply trigonometric functions to solve related problems.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • define and describe the functions sec x, cosec x and cot x.	***9. Further on Trigonometric Functions*** **9.1 The functions** ***y* = sec *x*, *y* = cosec *x* and *y* = cot *x*** *(5 periods)*	• You may start the lesson by revising the trigonometric function sin *x*, cos *x* and tan *x* and define sec *x*, cosec *x* and cot *x* using a right angled triangle.	• Ask students to re-state the definition of sec x, cosec *x*, and cot *x*. • Give exercise problem on sketching the graph of sec *x*, cosec *x*, and cot *x*.
• Sketch graphs of y = sec x, y = cosec x and y = cot x	• Graphs of *y* = sec *x*, *y* = cosec *x* and *y* = cot *x*	• Let students revise graphs of sin x and cos x first. • Assist students to practice sketching graphs of sec x, cosec x, cot x for different intervals. • Assist students to determine domain and ranges of these functions	
• define the inverse trigonometric functions. • Sketch the graph of the inverse trigonometric function.	**9.2 Inverse of trigonometric functions** *(4 periods)*	• Let students revise about the inverse of function through examples and then introduce and define the inverse trigonometric function. • Allow students distinguish between $\sec x = \dfrac{1}{\cos x}$ and the inverse of cos *x* denoted by $\cos^{-1}x$ $\csc x = \dfrac{1}{\sin x}$ and the inverse of sin *x* which is $\sin^{-1}x$ $\cot x = \dfrac{1}{\tan x}$ and the inverse of tan *x* that is $\tan^{-1}x$ • After revising how reflection along the line $y = x$ helps us to obtain the graph of an inverse from the graph of the function. • Let students practice sketching the graph of the inverse trigonometric functions through reflection in the line $y = x$. • Help students to determine domain and ranges of for the inverse trigonometric function.	• Ask students to re-state the definition of inverse trigonometric function. • Give exercise problems on sketching graph of inverse trigonometric function.

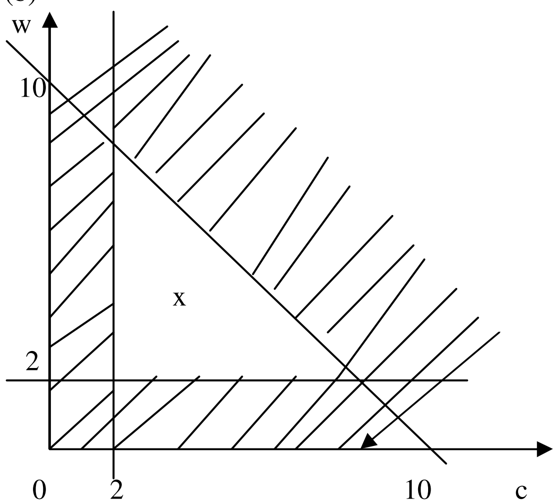
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• Sketch the graphs of $y = a \sin x$, $y = a \sin kx$, $y = a \sin (kx + b)$ and $y = a \sin(kx + b) + c$	**9.3 Graphs of some trigonometric functions** *(5 periods)* • graphs of $y = a \sin x$ $y = a \sin kx$ $y = a \sin(kx+b)$ $y = \sin(kx+b)+c$	• You may start the lesson with an activity in which students are expected to draw graph of $y = \sin x$, $y = 2 \sin x$ and $y = ½ \sin x$ and observe that the graphs of $y = 2 \sin x$ and y = ½ sin x are some transformations of the graph of $y = \sin x$. • Assist students on sketching graphs of (i) $y = a \sin x$, where a = 1, 2, 3, and 4 and some simple fractions such as ½ and ¼ . (ii) $y = a \sin kx$ where a and k = 1, 2, 3 and 4 and simple fractions such as ½ and ¼. (iii) $y = a \sin (kx + b)$ where a, b and k = 1, 2, 3 and 4 and simple fractions such as ½ and ¼. as well as $y = a \sin (kx + b) + c$ where c = 1, 2, 3, -3, -2, and -1	• Give exercise problems of sketching the graphs of $y = a \sin x$ for different values of a. • Give exercise problems on sketching the graphs of $y = a \sin kx$ $y = a \sin (kx + b)$ for different values of a, b and k.
• List the properties of these graphs.		• With active participation of students generalize the properties of $y = a \sin x$, $y = a \sin kx$ and $y = a \sin (kx + b)$	
• Sketch the graphs of $y = a \cos x$, $y = a \cos kx$, $y = a \cos (kx + b)$ $y = a \cos(kx + b) + c$ • List the properties of these graphs.	• graphs of $y = a \cos x$ $y = a \cos kx$ $y = a\cos(kx+b)$ $y = a\cos(kx+b)+c$	• You can follow the same method used above (as used for sine function)	• Ask students sketch the graphs of $y = a \cos x$, $y = a \cos kx$ $y = a \cos (kx + b)$ for different values of a, b and k.
• Apply trigonometric functions to solve problems from fields of science, navigation, engineering etc.	**9.4 Application of trigonometric functions** *(6 periods)*	• Discuss the practical application of trigonometric functions in sciences such as optics, Navigation, Wave motion etc. with active participation of students through sufficient examples.	• Give exercise problems on the application of trigonometric functions and check their works.

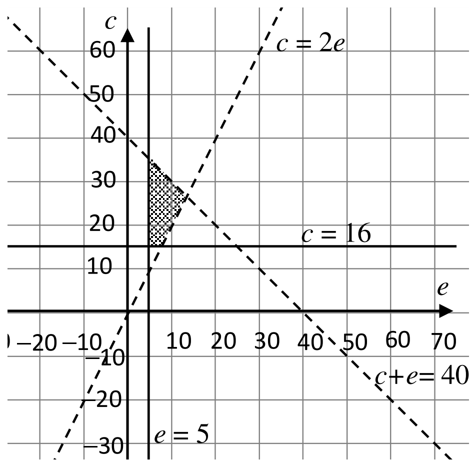
Unit 10: Introduction to Linear Programming *(15 periods)*

Unit outcomes: Students will be able to:

- identify regions of inequality graphs.
- create real life examples of linear programming problems using inequalities and solve them.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to:	***10. Introduction to Linear Programming*** *(4 periods)*	• Describe what linear programme is: A field of mathematics that deals with the problem of finding the maximum and minimum value of a given linear expression, where the variables are subject to certain conditions expressed as linear inequality.	• Give different exercise problems and drawing graphs of linear inequalities and check their works.
	10.1 Revision on Linear Graphs *(2 periods)*	• Draw linear graphs $y = mx + c$ and $ax + by = c$ and vary the values of **m, a, b,** and **c.** • Draw 2 linear graphs of the type $y = mx + c$ and /or $ax + by = c$ using the same axes and vary the values of m, a, b and c.	
• draw graphs of linear inequalities $y \le mx + c$ and $y \ge mx + c$ and $ax + by \ge c$ and $ax + by \le c$	**10.2 Graphical Solutions of System of Linear Inequalities** *(2 periods)*	• Draw and shade boundaries and identify regions of inequalities starting with $x < a$, $x > a$, (broken lines) $x \le a$, $x \ge a$ and similarly for y. • Revise and draw, shade and mark boundaries (broken or unbroken lines) of linear graphs $y \le mx + c$ and $y \ge mx + c$ and /or $ax + by \ge c$ and $ax + by \le c$ and vary the values of $m, a, b,$ and c.	
• find maximum and minimum values of a given objective function under given constraints.	**10.3 Maximum and Minimum value** *(5 periods)*	• Define objective function, and constraints using simple and appropriate example. • Let students exercise on finding maximum or minimum values given an objective function and constraints. E.g. Find the maximum and minimum values of $w = 2x + 3y$ under the constraint $x \ge 0$, $y \ge 0$, $2y + x \le 16$ and $x - y \le 10$.	• Give different exercise problems on finding maximum and minimum values.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• create inequalities from real life examples for linear programming and solve the problem.	**10.4 Real life linear programming problems** *(6 periods)*	• Give real life linear programming problems and show how to solve them with active participation of the students. **Worked example 1** The number of fields a farmer plants with wheat is w and the number of fields with corn is c. The restrictions are that (i) there must be at least 2 fields of corn (ii) there must be at least 2 fields of wheat (iii) not more than 10 fields are to be sown with wheat or corn. a) Construct 3 inequalities from the information given. b) On one pair of axes graph the inequalities and leave unshaded the region which satisfies all the 3 inequalities simultaneously. c) Give two possible arrangements how the farmer should plant. **Answers** a) $c \geq 2$, $w \geq 2$ and $c + w \leq 10$ (b)  c) (4,4) or (5,5) **Worked example 2** Mohammed is employed by a company to do 2 jobs. He repairs cars and also electrical goods. His terms of employment are listed: i) He must be employed up to 40 hours but not 40 hours. ii) He must spend at least 16 hours repairing	• Give various linear programming problems as exercises and follow up students activities. •

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		cars. iii) He must spend at least 5 hours repairing electrical goods. iv) He must spend more than twice as much time mending cars as repairing electrical goods. Let **c** represent hours working with cars and **e** represent hours working with electrical goods. (a) Express the above information using inequalities. (b) Graph the inequalities shading the region which satisfies all the inequalities. (c) Give two possible combinations within the shaded region. **Answers** (a) $c + e < 40$, $c \geq 16$, $e \geq 5$, $c > 2e$ (b)  • Any part of the shaded region except the dotted line would satisfy the inequalities, for example, 8 hours as an electrician and 20 hours mending cars i.e. (8, 20). • Assist and encourage students in solving similar linear programming problems.	

Unit 11: Mathematical Applications in Business *(18 periods)*

Unit outcomes: Students will be able to:

- know common terms related to business.
- know basic concepts in business.
- apply mathematical principles and theories to practical situations.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to:	***11. Mathematical Applications in Business*** **11.1 Basic Mathematical Concepts in Business** *(3 periods)*	• You can start the lesson by revising important ideas about ratio, proportion and percentage • After reminding students about "ratio" i.e., as an expression used to compare two quantities that have the same unit, and explaining how it is written in its simplest form, then discuss with students about its application by using several examples from the field of business.	• Give various exercise problems on calculations of ratio, rate proportion.
• compare quantities in terms of ratio. • calculate the rate of increase and the rate of decrease in price of commodities.	• Ratio • Rate	• Introduce the concept of "rate" which is used to compare two quantities that have different unit and expressed as a fraction and introduce the concept of "unit rate" as well. With the help of several examples taken from daily activities of selling a buying goods, discuss about "the rate of increase" and "the rate of decrease" in the price of goods (commodities). • Encourage and assist students to solve problems range from Local to National current situations involving rate of change in the business sector (or marketing)	
• solve problems on proportional variation in business • solve problems on compound proportion.	• Proportion	• With the help of examples revise the concept of "simple proportion" that the students had learnt in the previous grades. Since it is an expression of the equality of two or more ratios or rates where the degree or comparison is equal, assist the students to determine whether a given proportion is true or not and then encourage them to solve problems on proportion by considering examples from business activities like proportional variation in price and supply of goods to a market.	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• find a required percentage of certain given amount • compute problems on percentage increase or percentage decrease	• Percent	• Following this introduce the notion of "compound proportion" in which one quantity is proportionate to each of several other quantities. You may consider examples which involve units such as time, money, measurements etc. to be introduced into the calculation proportionate to other quantities some directly and others inversely. **Example**: If a man earns 240 birr in 3 weeks working 8 hours per day, how long will it take him to earn 1800 birr working 10 hours per day? **Ans.** 19 weeks *(the work out is left for the teacher to discuss with students)* • By using several example from business revise the notion "percent" and its calculation such as, to find "Amount" when the "Base" and "percent" are given, like wise to find the "Base" when "Amount" and "percent" are given as well as to find "percent" when "Base" and "Amount" are given. • With active participation students discuss different examples of business phenomenon in which the idea of "percent" plays significant role, such as in calculation and expression of "Discount" (i.e. Trade discount, cash discount, Note price") of "Profit and Loss" (Gross profit, Net profit). In doing so describe the meanings of related terms such as "Mark up, Margin" and introduce their formula". Assist students in describing and computing "percentage increase or decrease" in business sector, population, production, industrial development, health etc.	• Give exercise problem on expressing a certain percent a given quantity.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• calculate payment by installment for a given simple interest arrangement. • calculate the compound interest of a certain amount invested for a given period of time. • apply the formula for compound interest to solve practical problems in business.	**11.2 Compound Interest and Depreciation** *(4 periods)* **11.2.1 Compound Interest**	• You may begin the lesson with a brief revision of "simple interest" that the students had learnt in the previous grade ; in doing so give emphasis on the notions conveyed by terms like "principal" "rate of interest" and "interest period" or simply "Time". Use several examples to clarify and remind students about them. Introduce the notion of "Payment by Installment" (or deferred terms) and by using examples discuss how this arrangement of payment is carried out and assist students to solve related problems. • Proceed the lesson by introducing the concept of "compound interest" and by considering simple exercise problems encourage students to calculate and explain the advantages and disadvantages of lending money by comparing "simple interest" and "compound interest" arrangements. • Assist students in computing "Interest" and "Amount" by using exercise problems and guide them to apply the formula that is used for solving problems on compound interest, i.e. $A = P\left(1+\frac{r}{100}\right)^n$ where A = the amount of principal plus interest invested after n years P = the principal sum invested r = the Rate per cent per annum n = the number of years for which the principal is invested • After defining "present value" discuss with students how to compute this value by using examples. During the calculation of compound interest emphasise on how the concept of logarithm is used.	

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• compute annuity for a give arrangement in compound interest.		• Define "Annuity" *viz*, a series of payments at a regular interval and then encourage students to compute annuities for a given compound interest arrangement. You may show to the students a sample of "Account Book" issued for customers/ client by governmental or private bank and let them see and appreciate the application and importance of the concept of "compound interest" in real life situation. You can also take some examples and exercise problems for the students to solve from the sample account book that you brought.	• Give exercise problems on computing - Amount - Principal - Rate - Period of a compound interest arrangements.
• describe what is depreciation mean and some of its causes	**11.2.2 Depreciation**	• You may begin the lesson with the definition of "Assets" in business and then introduce "fixed Assets". As fixed assets, however, are not fixed in value since they wear out at varying rates according to their use over a period of time, discuss with students about the concept of "Depreciation" and let them list some of the causes for it.	• Give real life problems involving compound interest.
• compute depreciation by using either of the two methods appropriately.		• As depreciation is known as the fall in value, discuss with students how to calculate it. Though there are different ways by which depreciation may be calculated, the most commonly used methods are "reducing balance method" and "fixed installment or on-cost method". So by considering several examples to see the application of each method, encourage and assist students in computation of depreciation based on the two methods accordingly and appreciate the application of the notion of geometric progression of depreciation.	
• list five valid reasons for savings.		• You may begin the lesson with a discussion on the concept of "saving" and its importance and misconception about saving i.e. unfortunately, many people think of saving as "money left over after expenses." Savings are, however, probably the most important item (activity) in the personal or family budget. One can be successful at saving money if she/he resolves to set aside part of her/his income for saving first and then live on what is left.	

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• explain how savings become investment. • list three saving plans. • identify four kinds of financial institutions. • describe three main factors in choosing a particular institution for saving • compute and solve numerical problems on saving	**11.3 Saving, Investing and Borrowing Money** *(7 periods)* **11.3.1 Saving Money**	• Assist student in realization of the fact that, when one plan for saving, she/he is planning for her/his dream; and help students to find out that there are two kinds of saving *viz* "short - term" and "Long - term" savings. • Allow student to explain on the following matters regard to savings with their own words **(a) Reasons for saving** - special goals (planning to go to college, after that setting up a business, then marriage...) - emergencies - expensive purchases - recurring expenses **(b) Planning a saving programme** **(c) Saving as investment** **(d) Saving institutions** (save in financial institution to get the most out of saving) and how they work in saving that makes each preferable - Commercial Bank - saving and Loan Associations - Credit Union • Consolidating the students' opinion and give a summary of the above mentioned items in saving. • Give the students some numerical examples and exercises so that they can compute and solve problems on saving.	• Give various exercise problems on computation of depreciation by any of the methods discussed. Let the students discuss in class about saving. • Give simple exercise problem on computation saving.
• identify the four factors that should guide consumers in planning an investment strategy.	**11.3.2 Investment**	• You may start the lesson by discussing with students about events of investment that take place in their environment (if there is any) and in the country. Following this discuss with students about important situation to be considered during "investment" such as: **(a) Investment Strategy** - Investment goal - Knowledge of investment option - Risk - Professional advice	• Ask students to discuss about the type investment that they know

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• explain the differences between stocks and bond. • describe ways to invest in stock and bond. • compute and solve numerical problems on investment		**(b) Types of securities** - Stocks (stock holder), capital gain - Bond (maturity date) **(c) How to invest** - Direct sales of stock - Mutual Fund - Stock broker - Stock exchange • In the above discussion by considering appropriate examples, exercise problems encourage students to do some computation accordingly.	
• describe the advantages and disadvantages of borrowing money	**11.3.3 Borrowing Money**	• You may start the lesson by explaining how "borrowing money" has a long history with human economical development and let students explain their opinion about the advantages and disadvantages of borrowing and then let them come to some important situations to be considered during "borrowing" such as: **a) Why to borrow cash** . identify the purpose	
• identify the usual sources of cash loan. • compute the amount and time on the return of loan based on the given agreement.		**b) When to borrow cash** . identify the time to borrow and how and when it will be returned. **c) From where to get loan** - saving institutions as a sources of Loan - commercial bank - saving and loan associations - credit union ***other sources*** - consumer finance companies - insurance companies - private loan (family members) ▪ Give the students some numerical examples and exercises so that they can problems on "borrowing money" so that they can understand the concept • With active student participation discuss in detail each of the items mentioned above, in doing so, use tangible examples which also involve some calculation on return of cash loan.	• Let the students discuss the advantages and disadvantages of borrowing money.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• give name three types of activities that government performs and examples of each • explain why governments collect taxes. • describe the basic principles of taxation • describe the various kinds of taxes. • give their opinion about "income taxes" mean for them in relation to their future first job. • calculate different types of taxes based on the "rate of tax" in Ethiopia	**11.4 Taxation** *(4 periods)*	• Before describing the concept of "Tax" discuss with students about "What Government Does" in public services, and business activities and guide them to the major three activities, viz. **(1) Government provide public service,** such as, National defense, police and fire protection, Health services, street and park maintenance, sanitation services, High way and bridge construction, public education, Mental hospital, water, gas, and electric system, environmental protection, public transportation etc. **(2) Government regulate business activity** - Protecting consumers, - Making Monetary Policy **(3) Government redistribute income** • Let the students answer question, "to do all the above things from Where Government Gets its Money?" after analysing the students response to the question, introduce the concept of "Taxation" and emphasize on the fact that any responsible person who earns money should pay "tax" to the government based on the law of taxation and this is one of the duties and responsibilities of a citizen and discuss the three types of "Principles of Taxation" *viz*, (a) Taxpayers Identification Principles (b) Tax Rate Principles (c) Payment Principles. • Let the students give some types of taxes they know and then guide them to come to the conclusion that, the most commonly known types are: income tax, sales tax, property taxes, excise taxes, business and license taxes custom duties and tariffs, value added tax (VAT) and let students explain what income taxes mean to them and their future first job. • With a help of examples from each type mentioned above encourage students to calculate "Tax" with appropriate "Rate of Tax" in Ethiopia.	Ask students to list some types of taxes they know • discuss on "Why we pay tax?" • Give exercise problems on computing taxes based real tax rate that applied in Ethiopia.

MATHEMATICS

TEACHER'S GUIDE

GRADE 11

ISBN 978-99944-2-047-0

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA

MINISTRY OF EDUCATION

Price ETB 42.63