M4P54 Differential Topology

Lectured by Prof Paolo Cascini
Typed by David Kurniadi Angdinata

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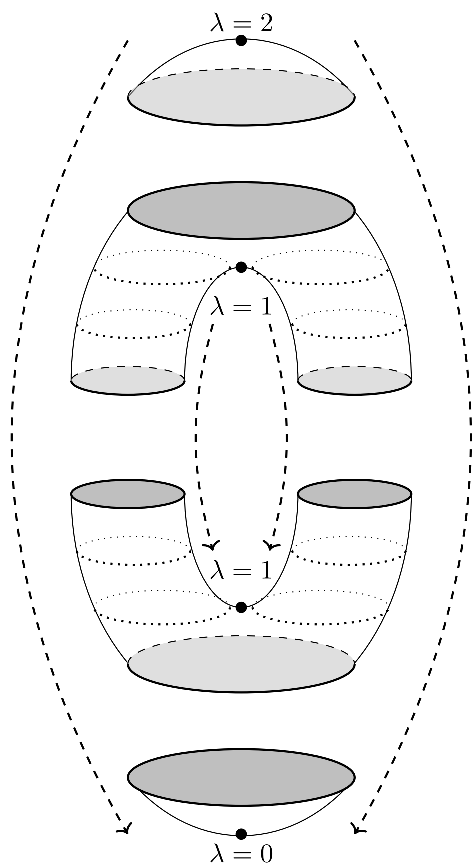

$S^1 \times S^1$

Syllabus

Differential forms on manifolds. Integrations on manifolds. Stokes' theorem. De Rham cohomology. Homotopy invariance. The Mayer-Vietoris sequence. Compactly supported de Rham cohomology. Poincaré duality. Degree of a morphism. CW-complexes. The CW-structure associated to a Morse function. The fundamental theorems of Morse theory. Morse homology. Singular homology. Singular cohomology.

Contents

0 Introduction

Lecture 1
Thursday
09/01/20

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

1 Differential forms on manifolds

1.1 Alternating p-forms on a vector space

Let V be a vector space over $\mathbb{R}$, and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \to \mathbb{R}$ is called an **alternating p-form** if we have

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) = \epsilon(\sigma)\,\omega(v_1, \ldots, v_p), \qquad v_1, \ldots, v_p \in V \qquad \sigma \in \mathcal{S}_p,$$

where $\mathcal{S}_p$ is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ.

Recall that if m is the number of transpositions in a decomposition of σ, then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2.

$$\bigwedge^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$$

is called the **p-th exterior power** of V.

Check that it is a vector space. [1]

Example 1.3.

- $\bigwedge^0 V^* = \mathbb{R}$.
- $\bigwedge^1 V^* = V^* = \mathrm{Hom}(V, \mathbb{R})$, the **dual** of V.

Definition 1.4. Let $\omega_1 \in \bigwedge^p V^*$ and $\omega_2 \in \bigwedge^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \bigwedge^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2(v_1, \ldots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma)\,\omega_1\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right)\omega_2\left(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}\right), \qquad v_1, \ldots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \bigwedge^1 V^*$. Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1)\,\omega_2(v_2) - \omega_1(v_2)\,\omega_2(v_1), \qquad v_1, v_2 \in V.$$

- Assume $\omega_1, \ldots, \omega_p \in \bigwedge^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \ldots, v_p) = \det(\omega_i(v_j))_{i,j=1,\ldots,p}, \qquad v_1, \ldots, v_p \in V.$$

Proposition 1.6. *Let $\omega_i \in \bigwedge^{p_i} V^*$ for $i = 1, 2, 3$.*

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, *assuming* $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 \cdot p_2}\,\omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \to W$ be a linear map between vector spaces over $\mathbb{R}$, and let $\omega \in \bigwedge^p W^*$. Then the **pull-back** $\Phi^*\omega \in \bigwedge^p V^*$ of ω is an alternating p-form on V defined by

$$\Phi^*\omega(v_1, \ldots, v_p) = \omega(\Phi(v_1), \ldots, \Phi(v_p)), \qquad v_1, \ldots, v_p \in V.$$

[1] Exercise

Proposition 1.8. *Given a linear map $\Phi : V \to W$,*

- *the pull-back*
$$\begin{array}{cccc} \Phi^* & : & \bigwedge^p W^* & \longrightarrow & \bigwedge^p V^* \\ & & \omega & \longmapsto & \Phi^* \omega \end{array}$$
is a linear map that preserves exterior products, that is
$$\Phi^* \left(\omega_1 \wedge \omega_2\right) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \qquad \omega_1 \in \bigwedge^p W^*, \qquad \omega_2 \in \bigwedge^q W^*,$$
- *if $\Psi : W \to Z$ is linear then*
$$\left(\Psi \circ \Phi\right)^* \omega = \Phi^* \Psi^* \omega, \qquad \omega \in \bigwedge^p Z^*,$$
- *assuming $V = W$ and $p = \dim V$, then*
$$\Phi^* \omega = \left(\det \Phi\right) \omega, \qquad \omega \in \bigwedge^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n, and let $x \in M$. Then the tangent space $\mathrm{T}_x M$ of M at x is a vector space of dimension n.

Notation 1.9. Let
$$\bigwedge^p \mathrm{T}_x^* M = \bigwedge^p \left(\mathrm{T}_x M\right)^*.$$

Consider the set
$$\bigwedge^p \mathrm{T}^* M = \bigsqcup_{x \in M} \bigwedge^p \mathrm{T}_x^* M,$$
the **p-th exterior bundle** on M. There exists a morphism $\pi : \bigwedge^p \mathrm{T}^* M \to M$ such that
$$\pi^{-1}\left(x\right) = \bigwedge^p \mathrm{T}_x^* M, \qquad x \in M,$$
so $\bigwedge^p \mathrm{T}^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\bigwedge^0 \mathrm{T}^* M = M \times \mathbb{R}$.
- $\bigwedge^1 \mathrm{T}^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

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Definition 1.11. A **differential p-form** ω on M is a smooth section of π, that is it is a smooth morphism $\omega : M \to \bigwedge^p \mathrm{T}^* M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Thus, $\omega\left(x\right) \in \bigwedge^p \mathrm{T}_x^* M$.

Notation 1.12.
$$\Omega^p\left(M\right) = \left\{\text{differential } p\text{-forms } \omega \text{ on } M\right\}, \qquad \Omega^\bullet\left(M\right) = \bigoplus_p \Omega^p\left(M\right).$$

Example 1.13.
$$\Omega^0\left(M\right) \cong \left\{f : M \to \mathbb{R} \ \mathrm{C}^\infty\text{-function}\right\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}\left(M\right) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p\left(M\right)$ and $\omega_2 \in \Omega^q\left(M\right)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}\left(M\right)$ is defined by
$$\omega_1 \wedge \omega_2\left(x\right) = \omega_1\left(x\right) \wedge \omega_2\left(x\right) \in \bigwedge^{p+q} \mathrm{T}_x^* M, \qquad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p\left(M\right)$. Let $F : M \to N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map
$$\mathrm{D}F_x : \mathrm{T}_x M \to \mathrm{T}_{F(x)} N.$$

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$F_x^* \quad : \quad \begin{array}{ccc} \bigwedge^p \mathrm{T}_{F(x)}^* N & \longrightarrow & \bigwedge^p \mathrm{T}_x^* M \\ \omega \left(v_1, \dots, v_p\right) & \longmapsto & \omega \left(\mathrm{D}F_x \left(v_1\right), \dots, \mathrm{D}F_x \left(v_p\right)\right) \end{array}, \qquad \omega \in \bigwedge^p \mathrm{T}_{F(x)}^* N, \qquad v_1, \dots, v_p \in \mathrm{T}_x^* M.$$

Thus, we can define

$$F^* \quad : \quad \begin{array}{ccc} \Omega^p \left(N\right) & \longrightarrow & \Omega^p \left(M\right) \\ \omega \left(x\right) & \longmapsto & F^* \omega \left(F \left(x\right)\right) \end{array}, \qquad \omega \in \Omega^p \left(N\right).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^* \left(\omega_1 \wedge \omega_2\right) = F^* \omega_1 \wedge F^* \omega_2.$$

If $G : N \to P$,

$$\left(G \circ F\right)^* \omega = F^* G^* \omega.$$

1.3 Local description of p-forms

Let M be a manifold of dimension n, let $x_0 \in M$, let (U, ϕ) be a local chart around x_0, and let $(x_1, \dots, x_n)$ be local coordinates around x_0. A basis of $\mathrm{T}_{x_0} M$ is given by

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

A basis of $\mathrm{T}_{x_0}^* M$ is given by

$$\left\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\right\}, \qquad \mathrm{d}x_i \left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

A basis of $\bigwedge^p \mathrm{T}_{x_0}^* M$ is

$$\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p \left(M\right)$ is locally given by

$$\omega \left(x\right) = \sum_{|I| = p} f_I \left(x\right) \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad I = \left(i_1, \dots, i_p\right), \qquad i_1 < \dots < i_p,$$

where f_I is a C^∞-function on U for all I.

Example 1.15. Let $F : M \to N$ be a smooth morphism between manifolds of dimension n, and let $\omega \in \Omega^n \left(N\right)$. Locally,

$$\omega \left(y\right) = f \left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad y \in N,$$

for some $f \in \mathrm{C}^\infty$. By Proposition 1.8,

$$F^* \omega \left(x\right) = \left(f \circ F\right) \left(x\right) \det \mathrm{D}F_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n, \qquad x \in M,$$

where $y_i = \mathrm{p}_i \circ F$ and $\mathrm{p}_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection.

Let $f : M \to \mathbb{R}$ be a smooth function, so $f \in \Omega^0 \left(M\right)$. Locally, the **differential** is

$$\mathrm{d} \quad : \quad \begin{array}{ccc} \Omega^0 \left(M\right) & \longrightarrow & \Omega^1 \left(M\right) \\ f & \longmapsto & \sum_{i=1}^n \frac{\partial}{\partial x_i} f \mathrm{d}x_i \end{array}.$$

Check that $\mathrm{d}f \in \Omega^1 \left(M\right)$, so $\mathrm{d}f$ is a 1-form on M. Alternatively, $\mathrm{d}f = f^* \mathrm{d}x$ for $\mathrm{d}x$ a 1-form on $\mathbb{R}$, or $\mathrm{d}f \left(X\right) = X \left(f\right)$ for any vector field X on M. More in general, let $\omega \in \Omega^p \left(M\right)$. Locally,

$$\omega = \sum_{|I| = p} f_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad f_I \in \mathrm{C}^\infty,$$

so $\mathrm{d}\omega \in \Omega^{p+1} \left(M\right)$. Then the **de Rham differential** is

$$\mathrm{d} \quad : \quad \begin{array}{ccc} \Omega^p \left(M\right) & \longrightarrow & \Omega^{p+1} \left(M\right) \\ \omega & \longmapsto & \sum_{|I| = p} \mathrm{d}f_I \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} \end{array}.$$

Proposition 1.16.

- *The Leibniz rule*
$$\mathrm{d}\left(\omega_1 \wedge \omega_2\right) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad w_1 \in \Omega^p\left(M\right), \qquad \omega_2 \in \Omega^q\left(M\right).$$
- $\mathrm{d}^2 = 0$, *that is*
$$\mathrm{d}\left(\mathrm{d}\omega\right) = 0, \qquad \omega \in \Omega^p\left(M\right).$$
- *Let* $F : M \to N$ *be a smooth morphism between manifolds. Then*
$$F^* \mathrm{d}\omega = \mathrm{d}\left(F^* \omega\right), \qquad \omega \in \Omega^p\left(M\right),$$
so
$$\begin{tikzcd} \Omega^p\left(M\right) \arrow[r, "\mathrm{d}"] & \Omega^{p+1}\left(M\right) \\ \Omega^p\left(N\right) \arrow[u, "F^*"] \arrow[r, "\mathrm{d}"'] & \Omega^{p+1}\left(N\right) \arrow[u, "F^*"'] \end{tikzcd}.$$

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Definition 1.17.

- $\omega \in \Omega^p\left(M\right)$ is **closed** if $\mathrm{d}\omega = 0$.
- $\omega \in \Omega^p\left(M\right)$ is **exact** if there exists $\omega' \in \Omega^{p-1}\left(M\right)$ such that $\mathrm{d}\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = \mathrm{d}\omega'$ then $\mathrm{d}\omega = \mathrm{d}^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n, let $F : M \to M$ be a smooth morphism, and let $\omega \in \Omega^n\left(M\right)$. Then
$$F^*\omega\left(x\right) = \det \mathrm{D}F_x \omega\left(F\left(x\right)\right).$$
Locally, assume $\omega = f \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ for some coordinates $\left(y_1, \dots, y_n\right)$ and $f \in \mathrm{C}^\infty$. Let $\left\{\left(U_\alpha, \phi_\alpha\right)\right\}$ be an atlas of M, where $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. Then
$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n \to \phi_\beta\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n,$$
such that
$$h_{\alpha\beta}^*\omega\left(x\right) = \left(f \circ h_{\alpha\beta}\right)\left(x\right) \det\left(\mathrm{D}h_{\alpha\beta}\right)_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \to \mathbb{R}$ be a C^∞-function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \to h\left(U\right)$ be a diffeomorphism. Then
$$\int_{h^{-1}(D)} f\left(y\right) \mathrm{d}y_1 \dots \mathrm{d}y_n = \int_{h^{-1}(D)} f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n = \int_D \left(f \circ h\right)(x) |\det \mathrm{D}h_x| \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let us assume that $\omega = f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ on U. We define
$$\int_D \omega = \int_D f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad D \subset U.$$
Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as
$$\operatorname{supp} \omega = \overline{\left\{x \in U \mid \omega\left(x\right) \neq 0\right\}}, \qquad \omega\left(x\right) \in \bigwedge^p \mathrm{T}_x^* U.$$
Then ω has **compact support**, if $\operatorname{supp} \omega$ is compact.

Fact. Under this assumption, we can define
$$\int_U \omega = \int_D \omega \in \mathbb{R},$$
which is well-defined. Under the same assumption, if $\phi : V \to U$ is a diffeomorphism, provided that $\det \mathrm{D}\phi_x > 0$, since $\det \mathrm{D}\phi_x \neq 0$ for all x, then
$$\int_U \omega = \int_V \phi^* \omega.$$

1.5 Orientation

If V is a vector space over $\mathbb{R}$ of dimension n, and $B = (b_1, \dots, b_n) \subset V$ and $B' = (b'_1, \dots, b'_n) \subset V$ are ordered bases of V, then B and B' have the **same orientation** if $\det T > 0$ for the linear map

$$\begin{array}{cccc} T & : & V & \longrightarrow & V \\ & & b_i & \longmapsto & b'_i \end{array}.$$

If $\omega \in \bigwedge^n V^*$ for $\omega \neq 0$, then B and B' have the same orientation if and only if $\omega (b_1, \dots, b_n)$ has the same sign as $\omega (b'_1, \dots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. If $\phi : V \to W$ is an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively, we say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W, so Λ_v induces Λ_w. If $V = \mathbb{R}^n$, and $e_i = \begin{pmatrix} 0 \dots 0 & 1 & 0 \dots 0 \end{pmatrix}$, then $e_1, \dots, e_n$ defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : \mathrm{T}_x U \to \mathbb{R}^n$. Let Λ_x^+ be an orientation on $\mathrm{T}_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on $\mathbb{R}^n$. Then $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. On U_α, we define the orientation so that

$$(\mathrm{D}\phi_\alpha)_x : \mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} \phi_\alpha (U_\alpha) \subset \mathbb{R}^n$$

is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α).

We define Λ^+ on M, which is a collection of Λ_x^+ on $\mathrm{T}_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det \mathrm{D} \left(\phi_\beta^{-1} \circ \phi_\alpha\right) > 0$ for all α and β.

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Notation 1.19. For all $p \geq 0$,

$$\Omega_{\mathrm{c}}^p (M) = \{\omega \in \Omega^p (M) \mid \operatorname{supp} M \text{ is compact}\}.$$

If M is compact $\Omega_{\mathrm{c}}^p (M) = \Omega^p (M)$.

Let $\omega \in \Omega_{\mathrm{c}}^n (M)$. Assume $\operatorname{supp} \omega \subset U$ where (U, ϕ) is a chart of M, and $\phi : U \to \phi (U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi (U) \to U$ such that $\left(\phi^{-1}\right)^* \omega \in \Omega_{\mathrm{c}}^n (\phi (U))$, that is $\operatorname{supp} \left(\phi^{-1}\right)^* \omega \subset \phi (U)$. We define

$$\int_M \omega = \int_{\phi(U)} \left(\phi^{-1}\right)^* \omega. \tag{1}$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ). Let $\left(\overline{U}, \overline{\phi}\right)$ be also a positively oriented chart such that $\operatorname{supp} \omega \subset \overline{U}$. We want to show that

$$\int_{\phi(U)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}\left(\overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* \omega.$$

Let $\overline{\phi} \circ \phi^{-1} : \phi \left(U \cap \overline{U}\right) \to \overline{\phi} \left(U \cap \overline{U}\right)$, so

$$\begin{tikzcd} & U \cap \overline{U} \arrow[dl, "\phi"'] \arrow[dr, "\overline{\phi}"] & \\ \mathbb{R}^n \supset \phi\left(U \cap \overline{U}\right) \arrow[rr, "\overline{\phi} \circ \phi^{-1}"'] & & \overline{\phi}\left(U \cap \overline{U}\right) \subset \mathbb{R}^n \end{tikzcd}.$$

Since both charts are positively oriented the determinant of the differential $\mathrm{D} \left(\overline{\phi} \circ \phi^{-1}\right)$ is positive, so

$$\begin{aligned} \int_{\overline{\phi}(U)} \left(\overline{\phi}^{-1}\right)^* \omega &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi} \circ \phi^{-1}\right)^* \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \overline{\phi}^* \left(\overline{\phi}^{-1}\right)^* \omega \\ &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \left(\overline{\phi}^{-1} \circ \overline{\phi}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}(U)} \left(\phi^{-1}\right)^* \omega, \end{aligned}$$

by a property of the pull-back and since $\left(\overline{\phi}^{-1}\right)^* \omega = 0$ outside $\overline{\phi} \left(U \cap \overline{U}\right)$.

1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $\mathcal{U} = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to $\mathcal{U}$ is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that

1. $\operatorname{supp} f_\alpha = \overline{\{x \in M \mid f_\alpha(x) = 0\}} \subset U_\alpha$ for all α,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists an open $U \ni x$ such that $\operatorname{supp} f_\alpha \cap U \neq \emptyset$ for only finitely many α.

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = \mathrm{S}^1 = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| = 1\right\}, \qquad U_1 = \mathrm{S}^1 \setminus \{(1,0)\}, \qquad U_2 = \mathrm{S}^1 \setminus \{(-1,0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos\theta, \sin\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta, \qquad f_2(\cos\theta, \sin\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta.$$

Then f_i is a partition of unity.

Proposition 1.22. *Let M be a manifold, and let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M. Then there exists a partition of unity f_α with respect to $\mathcal{U}$.*

Proof. We omit the proof. □

Proposition 1.23. *Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M, so*

$$\omega(x) \neq 0, \qquad x \in M.$$

Then ω is called a **volume form** on M.

Proof.

$\Longleftarrow$ Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M, that is Λ_x on $\mathrm{T}_x M$ for all $x \in M$. Given an oriented basis $v_1, \dots, v_n$ of $\mathrm{T}_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $\mathrm{T}_x M$ by considering the class of positively oriented ordered basis of $\mathrm{T}_x M$ which is compatible with the choice of an atlas on M. Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. On U_α,

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2, so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

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$\Longrightarrow$ Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α, we consider

$$\omega_\alpha = \phi_\alpha^* \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$, and let $\widetilde{\omega_\alpha} = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega_\alpha} \in \Omega^n(M)$ by extending equal to zero outside U_α. We define $\omega = \sum_\alpha \widetilde{\omega_\alpha} \in \Omega^n(M)$. For all α, since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega_\alpha} \neq 0$, so $\omega \neq 0$.

□

Let M be an orientable manifold of dimension n, and let $\omega \in \Omega^n_{\mathrm{c}}(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\operatorname{supp} \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M, and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\operatorname{supp} f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α, we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi^{-1}\right)^* f_\alpha.$$

Lemma 1.26. *$\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α.*

Proof. Under the assumption that $\operatorname{supp} \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α). Let $\{(U_\alpha, \phi_\alpha)\}$ and $\left\{\left(\overline{U_\alpha}, \overline{\phi_\alpha}\right)\right\}$ be two atlases with positively oriented charts, and let f_α and $\overline{f_\alpha}$ be two partitions of unity with respect to $\{U_\alpha\}$ and $\left\{\overline{U_\alpha}\right\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f_\alpha} = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f_\beta} f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha,\beta} \int_M \overline{f_\beta} f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f_\beta} \omega = \sum_\beta \int_M \overline{f_\beta} \omega.$$

□

Proposition 1.27. *Let M and N be orientable manifolds of dimension n, and let $\omega, \eta \in \Omega_c^n(M)$.*

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal. Let $\overline{M}$ be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ. Then*

$$\int_M \omega = -\int_{\overline{M}} \omega.$$

3. *Positivity. Let ω be the volume form on M. Then*

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance. Let $F : N \to M$ be an orientation preserving diffeomorphism. Then*

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. [2]

2. Exercise. [3]

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α, so

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \qquad g_\alpha > 0.$$

   Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M. Then $\left\{\left(F^{-1}(U_\alpha), \phi_\alpha \circ F\right)\right\}$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\left\{F^{-1}(U_\alpha)\right\}$, so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^*(f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

[2] Exercise
[3] Exercise

1.7 Manifolds with boundary

Denote
$$\mathbb{R}^n_{\geq 0} = (\mathbb{R}_{\geq 0})^n, \qquad \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}^n_+$ be open, and let $F : U \to \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞-function $\widetilde{F} : \widetilde{U} \to \mathbb{R}^m$ where $\widetilde{U} \supset U$ and $\widetilde{U}$ is open. Lecture 6 Tuesday 21/01/20

Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α, there exists a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β,
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \to \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+$$
is a diffeomorphism, so
$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\ \mathbb{R}^n_+ \supset \phi_\alpha (U_\alpha \cap U_\beta) \arrow[rr, "\phi_\alpha \circ \phi_\beta^{-1}"'] & & \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \end{tikzcd}.$$

The **boundary** of M is
$$\partial M = \left\{x \in M \;\middle|\; \exists \alpha, \ \phi_\alpha (x) \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}\right\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M.
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n.

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial D = \mathrm{S}^{n-1}$.
- $M = [0, 1] \times \mathrm{S}^1$ is a manifold with boundary $\partial M = \mathrm{S}^1 \sqcup \mathrm{S}^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1}$. This induces a positive orientation on ∂M.
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open coverings $\mathcal{U} = \{U_\alpha\}$, there exists a partition of unity f_α. This implies that if $\omega \in \Omega^n_{\mathrm{c}} (M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes)**.** *For any manifold with boundary M of dimension n, we have*
$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \in \Omega^n_{\mathrm{c}} (M), \qquad \omega \in \Omega^{n-1}_{\mathrm{c}} (M).$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \to \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M, so

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M \mathrm{d}\left(f_\alpha \omega\right) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right).$$

By Proposition 1.16,

$$\left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right) = \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right).$$

Then $\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)$ is an $(n-1)$-form on $\phi_\alpha\left(U_\alpha\right)$. In coordinates,

$$\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right) = \sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n,$$

where ω_j is a smooth function on $\phi_\alpha\left(U_\alpha\right)$ and

$$\begin{tikzcd} U_\alpha \arrow[r, "\widetilde{\phi_\alpha}"] \arrow[d, "f_\alpha"'] & \phi_\alpha\left(U_\alpha\right) \arrow[dl, "\widetilde{f_\alpha}"] \\ {[0,1]} & \end{tikzcd}.$$

Then

$$\begin{aligned} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) &= \mathrm{d}\left(\sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\widetilde{f_\alpha}\omega_j\right) \mathrm{d}x_k \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha}\omega_j\right) \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha}\omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right),$$

because $\widetilde{f_\alpha} = 0$ outside $\phi_\alpha\left(U_\alpha\right)$. Thus

$$\begin{aligned} \int_M \mathrm{d}\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha}\omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \\ &= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \int_0^\infty \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha}\omega_j\right) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^\infty \dots \widehat{\int_{-\infty}^\infty} \dots \int_{-\infty}^\infty \int_0^\infty (-1)^{j-1} \frac{\partial}{\partial x_j}\left.\left(\widetilde{f_\alpha}\omega_j\right)\right|_{x_n=0} \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \widehat{\mathrm{d}x_j} \dots \mathrm{d}x_1 \\ &= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left.\left(\widetilde{f_\alpha}\omega_j\right)\right|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1, \end{aligned}$$

since $\left.\left(f_\alpha \omega_j\right)\right|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left.\left(\widetilde{f_\alpha}\omega_j\right)\right|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. $\square$

1.9 Applications of Stokes' theorem

Lecture 7 Thursday 23/01/20

Theorem 1.33 (Integration by parts). *Let M be an orientable n-dimensional manifold with boundary, let $\omega \in \Omega_{\mathrm{c}}^p(M)$, let $\eta \in \Omega_{\mathrm{c}}^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}\omega \wedge \eta + (-1)^p \int_M \omega \wedge \mathrm{d}\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}(\omega \wedge \eta) = \int_M (\mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta),$$

by Stokes and the Leibniz rule. □

Theorem 1.34 (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \le 1\},$$

so

$$\partial D = \mathrm{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and let $f : D \to D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \ne x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x. Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \to \partial D$. It is easy to check that g is smooth. Since $\partial D = \mathrm{S}^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \ne 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $\mathrm{d}\omega \in \Omega^n(\partial D)$, which is an n-dimensional manifold, so $\mathrm{d}\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D \mathrm{d}(g^* \omega) = \int_D g^* \mathrm{d}\omega = 0,$$

by Stokes, a contradiction. □

Example 1.35. Recall any exact form is closed, since $\mathrm{d}^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2} \mathrm{d}y - \frac{y}{x^2 + y^2} \mathrm{d}x \in \Omega^1(M).$$

Then ω is closed, since

$$\mathrm{d}\omega = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) \mathrm{d}x \wedge \mathrm{d}y - \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) \mathrm{d}y \wedge \mathrm{d}x = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = \mathrm{d}f, \qquad f \in \Omega^0(M) = \{\mathrm{C}^\infty\text{-function}\}.$$

In particular $\omega = \mathrm{d}f$ on $\mathrm{S}^1 \subset M$. Let

$$\begin{array}{cccc} \gamma & : & [0, 2\pi] & \longrightarrow & \mathrm{S}^1 \\ & & \theta & \longmapsto & (\cos\theta, \sin\theta) \end{array}.$$

Then

$$\int_{\mathrm{S}^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left(\left(\frac{\cos\theta}{\cos^2\theta + \sin^2\theta} \right) \cos\theta \mathrm{d}\theta - \left(\frac{\sin\theta}{\cos^2\theta + \sin^2\theta} \right) (-\sin\theta) \mathrm{d}\theta \right) = \int_0^{2\pi} \mathrm{d}\theta = 2\pi,$$

and

$$\int_{\mathrm{S}^1} \omega = \int_{\mathrm{S}^1} \mathrm{d}f = \int_{\partial \mathrm{S}^1} f = \int_\emptyset f = 0,$$

so ω is not exact.

Proposition 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. ☐

Proposition 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. ☐

Let M be an orientable manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^k(M)$, and let $N \subset M$ be a submanifold of dimension k. We can define

$$\int_M \omega = \int_N i^*\omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^*\omega \in \Omega_{\mathrm{c}}^k(N).$$

Proposition 1.38. *Let M be an oriented manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^k(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- *ω is not exact,*
- *$\omega|_S$ is not exact, and*
- *S is not the boundary of an orientable manifold $N \subset M$ of dimension $k+1$.*

Proof. Exercise. [4] ☐

[4] Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Lecture 8 Monday 27/01/20

Definition 2.1. Let M be a manifold of dimension n, and let $p \geq 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = \mathrm{d}\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$\mathcal{Z}^p(M) = \ker\left(\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathcal{B}^p(M) = \operatorname{im}\left(\mathrm{d} : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $\mathcal{B}^p(M) \subset \mathcal{Z}^p(M)$ for all $p \geq 0$.

Notation. If $p = 0$, then $\mathcal{B}^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$ then $\omega_1 - \omega_2 \in \mathcal{B}^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Definition 2.2. Denote the ***p*-th de Rham cohomology group** as

$$\mathrm{H}^p(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) = \{[\omega] \mid \omega \in \mathcal{Z}^p(M)\}, \qquad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω.

Remark. $\mathrm{H}^p(M)$ is a vector space over $\mathbb{R}$.

Definition 2.3. The ***p*-th Betti number** of M is

$$\mathrm{b}_p(M) = \dim \mathrm{H}^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

Proposition 2.4. *If M is connected then*

$$\mathrm{H}^0(M) = \mathbb{R},$$

that is $\mathrm{b}_0(M) = 1$. *More in general,* $\mathrm{b}_0(M)$ *is the number of connected components of* M*.*

Proof. Assume M is connected. Then $\mathcal{B}^0(M) = 0$, so

$$\begin{aligned}
\mathrm{H}^0(M) = \mathcal{Z}^0(M) &= \left\{f \in \Omega^0(M) \text{ closed}\right\} \\
&= \left\{f \in \Omega^0(M) \;\middle|\; \text{locally } \forall x \in M,\ \frac{\partial}{\partial x_i} f(x) = 0\right\} \\
&= \left\{f \in \Omega^0(M) \text{ locally constant}\right\} = \mathbb{R}.
\end{aligned}$$

$\square$

Example. Let $M = \mathrm{S}^1$. Then $\mathrm{H}^0(M) = \mathbb{R}$.

Proposition 2.5. *Let M be a manifold of dimension n. Then*

$$\mathrm{H}^p(M) = 0, \qquad p \geq n+1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \geq n+1$ because all alternating p-forms for $p \geq n+1$ on an n-dimensional vector space are zero, so $\mathcal{Z}^p(M) = 0$. Thus $\mathrm{H}^p(M) = 0$. $\square$

Proposition 2.6. *Let M be a compact orientable manifold of dimension n without boundary. Then*

$$\mathrm{H}^n(M) \neq 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega^n_{\mathrm{c}}(M)$, by Proposition 1.23. Then ω is closed, because $\mathrm{d}\omega$ is an $(n+1)$-form on M, so $\omega \in \mathcal{Z}^n(M)$. We want to show that $[\omega] \neq 0$ in $\mathrm{H}^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = \mathrm{d}\eta$ where η is an $(n-1)$-form on M, so

$$0 < \int_M \omega = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. $\square$

Proposition 2.7. *Let $G : M \to N$ be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \to \Omega^p(M), \qquad p \geq 0$$

takes closed forms of N to closed forms on M and exact forms of N to exact forms on M.

Proof. By Proposition 1.16, $G^*\mathrm{d} = \mathrm{d}G^*$. If ω is closed then $\mathrm{d}G^*\omega = G^*\mathrm{d}\omega = G^*0 = 0$, so $G^*\omega$ is closed. If $\omega = \mathrm{d}\eta$ is exact then $G^*\omega = \mathrm{d}G^*\eta$ is also exact. $\square$

Thus $G^* : \mathcal{Z}^p(N) \to \mathcal{Z}^p(M)$ and $G^* : \mathcal{B}^p(N) \to \mathcal{B}^p(M)$, so there exists a linear map

$$\begin{array}{cccc} G^* & : & \mathrm{H}^p(N) & \longrightarrow & \mathrm{H}^p(M) \\ & & [\omega] & \longmapsto & [G^*\omega] \end{array}.$$

Corollary 2.8. *Let M and N be diffeomorphic manifolds. Then*

$$\mathrm{H}^p(M) \cong \mathrm{H}^p(N), \qquad p \geq 0,$$

that is $\mathrm{H}^p(M)$ is a diffeomorphic invariant.

Proof. By Proposition 2.7 there exists $F^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ and $\left(F^{-1}\right)^* : \mathrm{H}^p(M) \to \mathrm{H}^p(N)$. By Proposition 1.8,

$$\left(F^{-1}\right)^* F^*\omega = \left(F \circ F^{-1}\right)^* \omega = \mathrm{id}_N^* \omega = \omega, \qquad \omega \in \mathrm{H}^p(N),$$

so $\left(F^{-1}\right)^* \circ F^* = \mathrm{id}_{\mathrm{H}^p(N)}$. Similarly $F^* \circ \left(F^{-1}\right)^* = \mathrm{id}_{\mathrm{H}^p(M)}$, so F^* is an isomorphism. $\square$

2.2 Homotopy invariance

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \to M_1$ be smooth morphisms. Then f_0 and f_1 are **smoothly homotopic equivalent** if there exists a smooth morphism

$$\begin{array}{cccc} H & : & M_0 \times [0,1] & \longrightarrow & M_1 \\ & & (x,0) & \longmapsto & f_0(x) \\ & & (x,1) & \longmapsto & f_1(x) \end{array}, \qquad x \in M_0.$$

A **homotopy** is a smooth morphism $H : M_0 \times [0,1] \to M_1$ where M_0 and M_1 are smooth manifolds.

Lecture 9 Tuesday 28/01/20

Notation 2.10. Let $f_t(x) = H(x,t)$, so $f_t : M_0 \to M_1$ is a smooth morphism. Then f_0 and f_1 are said to be homotopic equivalent, denoted by $f_0 \sim f_1$, and $\sim$ is an equivalence. [5]

Definition 2.11. M_0 and M_1 are **homotopy equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$.

Example 2.12.

- Let $M_0 = \mathbb{R}^n$ and $M_1 = \{0\}$. Then M_0 and M_1 are homotopy equivalent. Let

  $$\begin{array}{cccc} f & : & M_0 & \longrightarrow & M_1 \\ & & x & \longmapsto & 0 \end{array}, \qquad \begin{array}{cccc} g & : & M_1 & \longrightarrow & M_0 \\ & & 0 & \longmapsto & 0 \end{array}.$$

  Then

  $$\begin{array}{cccc} f \circ g & : & M_1 & \longrightarrow & M_1 \\ & & 0 & \longmapsto & 0 \end{array},$$

  so $f \circ g = \mathrm{id}_{M_1}$, and

  $$\begin{array}{cccc} g \circ f & : & M_0 & \longrightarrow & M_0 \\ & & x & \longmapsto & 0 \end{array}.$$

  We want to show that $g \circ f \sim \mathrm{id}_{M_0}$. Define a smooth morphism

  $$\begin{array}{cccc} H & : & M_0 \times [0,1] & \longrightarrow & M_0 \\ & & (x,t) & \longmapsto & tx \end{array}.$$

  Then $H(x,0) = 0 = (g \circ f)(x)$ for all x, and $H(x,1) = x = \mathrm{id}_{M_0}(x)$ for all x, so $g \circ f \sim \mathrm{id}_{M_0}$. More in general $M \subset \mathbb{R}^n$ is called **convex** if for all $x, y \in M$ the segment joining x to y is contained inside M. If M is convex then M is homotopy equivalent to $M \times \{0\}$.

[5] Exercise

- Let $M_0 = \mathbb{R}^2 \setminus \{0\}$ and $M_1 = \mathrm{S}^1$. Then M_0 and M_1 are homotopy equivalent. Let

$$f : \begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ x & \longmapsto & \dfrac{x}{|x|} \end{array}, \qquad g : \begin{array}{ccc} M_1 & \longrightarrow & M_0 \\ x & \longmapsto & x \end{array}.$$

Then

$$f \circ g : \begin{array}{ccc} M_1 & \longrightarrow & M_1 \\ x & \longmapsto & x \end{array},$$

so $f \circ g = \mathrm{id}_{M_1}$, and

$$g \circ f : \begin{array}{ccc} M_0 & \longrightarrow & M_0 \\ x & \longmapsto & \dfrac{x}{|x|} \end{array}.$$

Let

$$H : \begin{array}{ccc} M_0 \times [0,1] & \longrightarrow & M_0 \\ (x,t) & \longmapsto & tx + (1-t)\dfrac{x}{|x|} \end{array}$$

be smooth. Then $H(x,0) = x/|x| = (g \circ f)(x)$ and $H(x,1) = x = \mathrm{id}_{M_0}(x)$, so $g \circ f \sim \mathrm{id}_{M_0}$.

Proposition 2.13. *Let M and N be manifolds, and let $H : M \times [0,1] \to N$ be smooth. Denote*

$$f_t : \begin{array}{ccc} M & \longrightarrow & N \\ x & \longmapsto & H(x,t) \end{array}, \qquad t \in [0,1].$$

Then

$$f_t^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M), \qquad p \geq 0$$

does not depend on t.

Proof. Let $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. The goal is $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$ for all $[\eta] \in \mathrm{H}^p(N)$. Let

$$i_k : \begin{array}{ccc} M & \longrightarrow & M \times [0,1] \\ x & \longmapsto & (x, t_k) \end{array}, \qquad k = 1, 2.$$

Claim that for all p there exists a linear map $h : \Omega^p(M \times [t_1,t_2]) \to \Omega^{p-1}(M)$ such that

$$\mathrm{d}(h(\omega)) + h(\mathrm{d}\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \qquad \omega \in \Omega^p(M \times [0,1]). \tag{2}$$

Step 1. The claim implies Proposition 2.13. Let $\eta \in \Omega^p(N)$ be closed, so $\mathrm{d}\eta = 0$. Then $H^*\eta$ is also closed, so let $\omega = H^*\eta \in \Omega^p(M \times [t_1,t_2])$. Apply h. Then $\mathrm{d}\omega = 0$, so $\mathrm{d}(h(\omega)) = i_2^*\omega - i_1^*\omega$ is exact. Thus

$$f_{t_1}^*[\eta] = \left[f_{t_1}^*\eta\right] = [i_1^*H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^*H^*\eta] = \left[f_{t_2}^*\eta\right] = f_{t_2}^*[\eta],$$

so Proposition 2.13 follows.

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Step 2. The proof of the claim. Let $\omega \in \Omega^p(M \times [t_1,t_2])$. Then for all $(x,t) \in M \times [t_1,t_2]$, $\omega(x,t)$ is an alternating p-form on $\mathrm{T}_{(x,t)}(M \times [t_1,t_2])$. We want an alternating $(p-1)$-form $h(\omega)(x)$ on T_xM. Let $v_1, \dots, v_{p-1} \in \mathrm{T}_xM$. Then

$$h(\omega)(x)(v_1, \dots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x,t)\left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1}\right) \mathrm{d}t$$

is a $(p-1)$-form on M, and $\frac{\partial}{\partial t}$ is a global vector field. Check h is linear. [6] It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates $(x_1, \dots, x_n, t)$ around a point of $M \times [0,1]$. Then

$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \qquad \omega_I = g_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad \omega_J = g_J \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t,$$

where g_I and g_J are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for ω_I and ω_J.

[6] Exercise

ω_I. Let $\omega = g(x,t)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$. Then

$$\mathrm{d}\left(h\left(\omega(x,t)\left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1}\right)\right)\right) = \mathrm{d}(h(0)) = 0,$$

and

$$\begin{aligned} h(\mathrm{d}\omega) &= h\left(\frac{\partial}{\partial t} g(x,t)\,\mathrm{d}t \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}\right) \\ &= \left(\int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x,t)\,\mathrm{d}t\right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} + 0 \\ &= (g(x,t_2) - g(x,t_1))\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} = i_2^*\omega - i_1^*\omega, \end{aligned}$$

so (2) holds.

ω_J. Let $\omega = g(x,t)\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t$. Then

$$\begin{aligned} \mathrm{d}(h(\omega)) &= (-1)^{p-1}\,\mathrm{d}\left(\left(\int_{t_1}^{t_2} g(x,t)\,\mathrm{d}t\right) \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{p-1}}\right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\int_{t_1}^{t_2} g(x,t)\,\mathrm{d}t\right) \mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}t\right) \mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(\mathrm{d}\omega) &= h\left(\sum_{j=1}^n \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t + 0\right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}t\right) \mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{p-1}} = -\mathrm{d}(h(\omega)), \end{aligned}$$

and $i_2^*\omega = i_1^*\omega = 0$, so (2) holds.

□

Corollary 2.14. *Assume M and N are homotopy equivalent. Then there exist isomorphisms*

$$\mathrm{H}^p(N) \to \mathrm{H}^p(M), \qquad p \geq 0.$$

Proof. There exist $f : M \to N$ and $g : N \to M$ such that $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$. By Proposition 2.13 $(g \circ f)^* : \mathrm{H}^p(M) \to \mathrm{H}^p(M)$ coincides with $\mathrm{id}_M^* = \mathrm{id}_{\mathrm{H}^p(M)}$. Then $f^* \circ g^* = (g \circ f)^* = \mathrm{id}_{\mathrm{H}^p(M)}$. Similarly $g^* \circ f^* = \mathrm{id}_{\mathrm{H}^p(N)}$, so g^* and f^* are isomorphisms. □

Definition 2.15. Let M be a manifold. Then M is **smoothly contractible** if M is homotopy equivalent to a point.

Example. $\mathbb{R}^n$ is contractible, by Example 2.12. If $M \subset \mathbb{R}^n$ is convex then M is contractible.

Theorem 2.16 (Poincaré lemma)**.** *If M is a contractible manifold then*

$$\mathrm{H}^p(M) = 0, \qquad p \geq 1.$$

Proof. By Corollary 2.14, there exists an isomorphism $\mathrm{H}^p(M) \to \mathrm{H}^p(\{\text{point}\})$. Then $\{\text{point}\}$ is a zero-dimensional manifold, so by Proposition 2.5, $\mathrm{H}^p(\{\text{point}\}) = 0$ for all $p > 0$. □

Thus $\mathrm{H}^p(\mathbb{R}^n) = 0$ for all $p > 0$, so $\mathbb{R}^n$ is not diffeomorphic to any compact orientable manifold. Lecture 11 Monday 03/02/20

Proposition 2.17. *Let M be a manifold, and let $\omega \in \Omega^p(M)$ be a closed p-form for $p > 0$. Then for all $x \in X$, there exists a neighbourhood $U \ni x$ such that ω is exact on U, that is there exists $\eta \in \Omega^{p-1}(U)$ such that $\omega = \mathrm{d}\eta$ on U.*

Proof. Let (U, ϕ) be a chart around x. I may assume that $V = \phi(U)$ is a ball in $\mathbb{R}^n$. Then U is diffeomorphic to $B = \{z \mid |z - z_0| < r\}$ for some $z_0 \in \mathbb{R}^n$ and $r > 0$, so $\mathrm{H}^p(U) \cong \mathrm{H}^p(B)$ for all $p \ge 0$. Since B is contractible, $\mathrm{H}^p(B) = 0$ for all $p > 0$. The restriction of ω on U gives a class $[\omega] \in \mathrm{H}^p(U) = 0$, so ω is cohomologous to zero on U. Thus ω is exact on U. ☐

Definition 2.18. Let M be a manifold, let $\gamma : [0,1] \to M$ be a continuous or smooth path, and let $x = \gamma(0)$ and $y = \gamma(1)$. A **homotopy of paths** from x to y is a map

$$\begin{array}{cccccc} F & : & [0,1] \times [0,1] & \longrightarrow & M & \\ & & (0,t) & \longmapsto & x & . \\ & & (1,t) & \longmapsto & y & \end{array}$$

Proposition 2.19. *Let γ_0 and γ_1 be homotopic paths on a manifold M, and let $\omega \in \Omega^1(M)$ be closed. Then*

$$\int_0^1 \gamma_0^* \omega = \int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. Lee's introduction to smooth manifolds. The idea is that

$$\int_{\gamma_0} \omega - \int_{\gamma_1} \omega = \int_{\gamma_0 \cup \gamma_1} \omega = \int_{\mathrm{im}\, F} \omega = 0,$$

by Stokes' theorem. ☐

Recall that M is **simply connected**, so $\pi_1(M) = 0$, if any path γ from x to x is homotopic equivalent to a point.

Proposition 2.20. *Let M be a simply connected orientable manifold. Then*

$$\mathrm{H}^1(M) = 0.$$

Proof. Let $\omega \in \Omega^1(M)$ be a closed form. Then claim that ω is exact if and only if $\int_\gamma \omega = 0$ for all loops γ, that is paths from x to x.

- The proof of the claim. Assume that $\omega = \mathrm{d}f$ is exact for $f \in \Omega^0(M)$. By Proposition 2.19,

  $$\int_\gamma \omega = \int_{\text{trivial loop}} \omega = 0.$$

  Assume that $\int_\gamma \omega = 0$ for all loops γ. Fix x. Let

  $$f(y) = \int_x^y \omega.$$

  Since $\int_{\gamma_1 \cup \gamma_2} \omega = 0$, f is well-defined, that is it does not depend on the choice of the path. Then $\mathrm{d}f = \omega$. This can be checked locally, that is in an open set of $\mathbb{R}^n$. Here it follows from the fundamental theorem of calculus.

- The claim implies Proposition 2.20. Being simply connected, any loop inside M is homotopic equivalent to the trivial loop. For all loops γ and for all closed ω, $\int_\gamma \omega = 0$ by Proposition 2.19, so ω is exact. Thus $[\omega] = 0$ in $\mathrm{H}^1(M)$.

☐

2.3 Some homological algebra

Let $C^\bullet$ be a sequence of vector spaces, that is C^k is a vector space for $k \in \mathbb{Z}$.

Definition 2.21. $(C^\bullet, d^\bullet)$ is a **cochain complex** if $C^\bullet$ is a sequence of vector spaces and $d^\bullet$ is a sequence of linear maps $d^k : C^k \to C^{k+1}$ such that the composition $d^{k+1} \circ d^k : C^k \to C^{k+1} \to C^{k+2}$ is zero for all k. Then $d^\bullet$ is the **differential**.

Definition 2.22. The elements of

$$\mathcal{Z}^k(C^\bullet, d^\bullet) = \ker\left(d^k : C^k \to C^{k+1}\right) \subset C^k$$

are called **cocycles**. The elements of

$$\mathcal{B}^k(C^\bullet, d^\bullet) = \mathrm{im}\left(d^k : C^{k-1} \to C^k\right) \subset C^k$$

are called **coboundaries**. Then $d^{k-1} \circ d^k = 0$, so $\mathcal{B}^k \subset \mathcal{Z}^k$. The quotients

$$\mathrm{H}^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$$

are the ***k*-th cohomology groups** of $(C^\bullet, d^\bullet)$.

Definition 2.23. Let $(C^\bullet, d^\bullet)$ and $(D^\bullet, d^\bullet)$ be two cochain complexes. A map $f : (C^\bullet, d^\bullet) \to (D^\bullet, d^\bullet)$ is a sequence of linear maps $f^k : C^k \to D^k$ such that $f^{k+1} \circ d^k = d^k \circ f^k$ for all k, so

$$\begin{tikzcd} \dots \arrow[r] & C^k \arrow[r, "d^k"] \arrow[d, "f^k"] & C^{k+1} \arrow[r, "d^{k+1}"] \arrow[d, "f^{k+1}"] & C^{k+2} \arrow[r] \arrow[d, "f^{k+2}"] & \dots \\ \dots \arrow[r] & D^k \arrow[r, "d^k"] & D^{k+1} \arrow[r, "d^{k+1}"] & D^{k+2} \arrow[r] & \dots \end{tikzcd} .$$

Proposition 2.24. *Let $f : (C^\bullet, d^\bullet) \to (D^\bullet, d^\bullet)$ be a map between cochain complexes. Then there exists a natural induced map*

$$f^k : \mathrm{H}^k(C^\bullet, d^\bullet) \to \mathrm{H}^k(D^\bullet, d^\bullet).$$

Proof. Let $[\omega] \in \mathrm{H}^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet)/\mathcal{B}^k(C^\bullet, d^\bullet)$ for $\omega \in \mathcal{Z}^k(C^\bullet, d^\bullet)$, that is $d^k(\omega) = 0$. I want to check that $f^k(\omega) \in \mathcal{Z}^k(D^\bullet, d^\bullet)$. By definition of maps, $d^k\left(f^k(\omega)\right) = f^{k+1}\left(d^k(\omega)\right) = 0$, so there is a map

$$\mathcal{Z}^k(C^\bullet, d^\bullet) \to \mathcal{Z}^k(D^\bullet, d^\bullet).$$

Now I need to check that if $\omega \in \mathcal{B}^k(C^\bullet, d^\bullet)$ then $f^k(\omega) \in \mathcal{B}^k(D^\bullet, d^\bullet)$. [7] ∎

Lecture 12 Tuesday 04/02/20

Definition 2.25. A sequence of linear maps

$$C^1 \xrightarrow{f^1} C^2 \to \dots \to C^{k-1} \xrightarrow{f^{k-1}} C^k$$

between vector spaces is said to be **exact** if for all i, $\ker f^i = \mathrm{im}\, f^{i-1}$.

Example 2.26.

- A sequence
$$0 \to C^1 \xrightarrow{f^1} C^2$$
is exact if and only if f^1 is injective.
- A sequence
$$C^1 \xrightarrow{f^1} C^2 \to 0$$
is exact if and only if f^1 is surjective.
- An exact sequence
$$0 \to C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \to 0$$
is called a **short exact sequence**. In particular f^1 is injective and f^2 is surjective.

[7] Exercise

- Any long exact sequence
$$C^1 \xrightarrow{f^1} C^2 \to \dots \to C^{q-1} \xrightarrow{f^{q-1}} C^q$$
can be split into short exact sequences

$$\begin{tikzcd}[column sep=small]
\dots \arrow[r] & C^{k-1} \arrow[rr, "f^{k-1}"] \arrow[dr] & & C^k \arrow[rr, "f^k"] \arrow[dr] & & C^{k+1} \arrow[r] & \dots \\
& & \operatorname{im} f^{k-1} = \ker f^k \arrow[ur] \arrow[dr] & & \operatorname{im} f^k = \ker f^{k+1} \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}, \qquad k = 2, \dots, q-1.$$

Lemma 2.27 (Snake lemma)**.** *Consider the commutative diagram*

$$\begin{tikzcd}
& C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 &
\end{tikzcd},$$

such that both the horizontal lines are exact sequences. Then there exists a long exact sequence

$$\ker \alpha_1 \to \ker \alpha_2 \to \ker \alpha_3 \xrightarrow{\delta} \operatorname{coker} \alpha_1 \to \operatorname{coker} \alpha_2 \to \operatorname{coker} \alpha_3.$$

If

$$\begin{tikzcd}
0 \arrow[r] & C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 \arrow[r] & 0
\end{tikzcd},$$

then

$$0 \to \ker \alpha_1 \to \ker \alpha_2 \to \ker \alpha_3 \xrightarrow{\delta} \operatorname{coker} \alpha_1 \to \operatorname{coker} \alpha_2 \to \operatorname{coker} \alpha_3 \to 0.$$

Proof. We are going to construct $\delta : \ker \alpha_3 \to \operatorname{coker} \alpha_1$. Let $x \in \ker \alpha_3$. There exists $y \in C^2$ such that $f^2(y) = x$ because f^2 is surjective. Let $z = \alpha_2(y)$ then

$$g^2(z) = g^2(\alpha_2(y)) = \alpha_3\left(f^2(y)\right) = \alpha_3(x) = 0,$$

since $x \in \ker \alpha_3$. Then $z \in \ker g^2 = \operatorname{im} g^1$, so there exists $w \in D^1$ such that $z = g^1(w)$. The idea is that

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \ker \alpha_1 \arrow[r] \arrow[d] & \ker \alpha_2 \arrow[r] \arrow[d] & \ker \alpha_3 \arrow[d] \arrow[dddll, rounded corners, to path={-- ([xshift=3em]\tikztostart.east) |- ([xshift=-3em]\tikztotarget.west) -- (\tikztotarget)}] & \\
& C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & y \in C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & x \in C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & w \in D^1 \arrow[r, "g^1"'] \arrow[d] & z \in D^2 \arrow[r, "g^2"'] \arrow[d] & D^3 \arrow[d] & \\
& \operatorname{coker} \alpha_1 \arrow[r] \arrow[d] & \operatorname{coker} \alpha_2 \arrow[r] \arrow[d] & \operatorname{coker} \alpha_3 \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}.$$

Define $\delta(x) = [w] \in \operatorname{coker} \alpha^1 = D^1 / \operatorname{im} \alpha^1$. Need to check that δ is well-defined, so $[w]$ does not depend on our choice of w and y. The rest is an exercise. [8] ∎

[8] Exercise

2.4 The Mayer-Vietoris sequence

The idea is that given a manifold M, we may write $M = U \cup V$ with open U and V so that $\mathrm{H}^i(U)$, $\mathrm{H}^i(V)$, and $\mathrm{H}^i(U \cap V)$ are easy to compute, so this will give us $\mathrm{H}^i(M)$. Let M be a manifold, and let U and V be open such that $M = U \cup V$. Assume $U \cap V \neq \emptyset$. Let

$$i_U : U \to M, \qquad i_V : V \to M, \qquad j_U : U \cap V \to U, \qquad j_V : U \cap V \to V$$

be inclusions, and let $i_U^*, i_V^*, j_U^*, j_V^*$ be pull-backs.

Proposition 2.28. *For all p there exist short exact sequences*

$$0 \to \Omega^p(M) \xrightarrow{f} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{g} \Omega^p(U \cap V) \to 0,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. More precisely, if $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$ then $g(\omega_1, \omega_2) = j_V^*\omega_2 - j_U^*\omega_1$.

Proof.

- f is injective. Assume $\omega \in \Omega^p(M)$ such that $f(\omega) = 0$, so $i_U^*\omega = i_V^*\omega = 0$. Since $M = U \cup V$ then $\omega = 0$ on M, so f is injective.
- $\operatorname{im} f = \ker g$. Let $f(\omega) \in \operatorname{im} f$, so $f(\omega) = (i_U^*\omega, i_V^*\omega)$. Then $g(f(\omega)) = j_V^* i_V^*\omega - j_U^* i_U^*\omega = l^*\omega - l^*\omega = 0$, where

$$\begin{tikzcd} & U \arrow[dr, "i_U"] & \\ U \cap V \arrow[ur, "j_U"] \arrow[rr, "l"] \arrow[dr, "j_V"'] & & M \\ & V \arrow[ur, "i_V"'] & \end{tikzcd},$$

so $\operatorname{im} f \subset \ker g$. Now let $(\omega_1, \omega_2) \in \ker g$, so $j_V^*\omega_2 = j_U^*\omega_1$ for $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$. The restriction of ω_2 on $U \cap V$ coincides with the restriction of ω_1 on $U \cap V$. Then define

$$\omega = \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}.$$

Then $f(\omega) = (\omega_1, \omega_2)$, so $\ker g \subset \operatorname{im} f$.
- g is surjective. Let $\eta \in \Omega^p(U \cap V)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Then $\operatorname{supp} f_U \subset U$ and $f_U + f_V = 1$. Let $\eta_1 \in \Omega^p(U)$ be defined by

$$\eta_1 = \begin{cases} f_V \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \operatorname{supp} f_V \end{cases},$$

and let $\eta_2 \in \Omega^p(V)$ be defined by

$$\eta_2 = \begin{cases} f_U \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \operatorname{supp} f_U \end{cases}.$$

Then $g(-\eta_2, \eta_1) = \eta_1|_{U \cap V} + \eta_2|_{U \cap V} = (f_U + f_V) \cdot \eta = \eta$, so $\eta \in \operatorname{im} g$.

$\square$

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Thursday
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Theorem 2.29 (Mayer-Vietoris)**.** *Let M be a manifold, and let U and V be open in M such that $M = U \cup V$ and $U \cap V \neq \emptyset$. Then for all $p \geq 0$ there exists a linear $\delta : \mathrm{H}^p(U \cap V) \to \mathrm{H}^{p+1}(M)$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}^p(M) \arrow[r, "(i_U^*, i_V^*)"] & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \arrow[r, "j_V^* - j_U^*"] & \mathrm{H}^p(U \cap V) \arrow[dll, "\delta"'] & \\ & \mathrm{H}^{p+1}(M) \arrow[r, "(i_U^*, i_V^*)"] & \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V) \arrow[r, "j_V^* - j_U^*"] & \mathrm{H}^{p+1}(U \cap V) \arrow[r] & \dots \end{tikzcd}$$

is exact.

Example 2.30. Let $M = \mathrm{S}^1$, let $N = (0,1)$ and $S = (0,-1)$, and let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $M = U \cup V$ and $U \cap V = M \setminus \{N, S\}$. Then

$$\mathrm{H}^p(U) \cong \mathrm{H}^p(V) \cong \mathrm{H}^p((0,1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \qquad (0,1) \subset \mathbb{R},$$

and

$$\mathrm{H}^p(U \cap V) = \mathrm{H}^p(U \setminus \{S\}) = \mathrm{H}^p\left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) = \begin{cases} \mathbb{R}^2 & p = 0 \\ 0 & p > 0 \end{cases}, \qquad \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \subset \mathbb{R},$$

so

$$\begin{array}{ccccccccccccc} 0 \to & \mathrm{H}^0(M) & \to & \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) & \xrightarrow{\phi} & \mathrm{H}^0(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^1(M) & \to & \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) & \to & \mathrm{H}^1(U \cap V) & \to \dots \\ & \cong & & \cong & & \cong & & & & \cong & & \cong & \\ & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R}^2 & & & & 0 \oplus 0 & & 0 & \end{array}.$$

Thus $\operatorname{im} \phi = \mathbb{R} \subset \mathrm{H}^0(U \cap V) = \mathbb{R}^2$, so $\mathrm{H}^1(M) = \operatorname{coker} \phi = \mathbb{R}^2 / \operatorname{im} \phi \cong \mathbb{R}$.

Remark 2.31. Let

$$0 \to C^1 \to \dots \to C^k \to 0$$

be an exact sequence. Then

$$\sum_k (-1)^k \dim C^k = 0.[9]$$

In our $M = \mathrm{S}^1$ case $1 - 2 + 2 - \dim \mathrm{H}^1(M) = 0$, so $\dim \mathrm{H}^1(M) = 1$. Thus $\mathrm{H}^1(M) \cong \mathbb{R}$.

Example 2.32. Let $M = \mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere. Then

$$\mathrm{H}^p(M) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

By induction on n.

$n = 1$. Ok.

$n > 1$. Let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $U \cap V \neq \emptyset$ and $U \cup V = M$. Then

$$U \cong V \cong \mathbb{R}^n, \qquad U \cap V = V \setminus \{N\} \cong \mathbb{R}^n \setminus \{0\} \cong \mathrm{S}^{n-1},$$

so

$$\begin{array}{ccccccccccc} 0 \longrightarrow & \mathrm{H}^0(M) & \longrightarrow & \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) & \longrightarrow & \mathrm{H}^0(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^1(M) & \longrightarrow & \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) & \longrightarrow \dots \\ & \cong & & \cong & & \cong & & & & \cong & \\ & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} & & & & 0 \oplus 0 & \end{array}.$$

Then $1 - 2 + 1 - \dim \mathrm{H}^1(M) = 0$, so $\dim \mathrm{H}^1(M) = 0$. Thus $\mathrm{H}^1(M) = 0$. Then for $p > 0$

$$\begin{array}{ccccccccc} \dots \longrightarrow & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) & \longrightarrow & \mathrm{H}^p(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^{p+1}(M) & \longrightarrow & \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V) & \longrightarrow \dots \\ & \cong & & & & & & \cong & \\ & 0 \oplus 0 & & & & & & 0 \oplus 0 & \end{array}$$

is exact, so $\mathrm{H}^p(U \cap V) \cong \mathrm{H}^{p+1}(M)$. By induction

$$\mathrm{H}^p(U \cap V) = \mathrm{H}^{p+1}(M) = \begin{cases} \mathbb{R} & p = n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

[9] Exercise

Proof of Theorem 2.29. By Proposition 2.28 for all p

$$\begin{tikzcd} 0 \arrow[r] & \Omega^p(M) \arrow[r] \arrow[d, "\mathrm{d}^p_M"] & \Omega^p(U) \oplus \Omega^p(V) \arrow[r] \arrow[d, "(\mathrm{d}^p_U, \mathrm{d}^p_V)"] & \Omega^p(U \cap V) \arrow[r] \arrow[d, "\mathrm{d}^p_{U\cap V}"] & 0 \\ 0 \arrow[r] & \Omega^{p+1}(M) \arrow[r] & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) \arrow[r] & \Omega^{p+1}(U \cap V) \arrow[r] & 0 \end{tikzcd}$$

are exact. Recall d commutes with the pull-back. By the strong snake lemma,

$$\begin{tikzcd} & \operatorname{coker} \mathrm{d}^{p-1}_M \arrow[r] \arrow[d, "\partial^p_M = \mathrm{d}^p_M"] & \operatorname{coker}\left(\mathrm{d}^{p-1}_U, \mathrm{d}^{p-1}_V\right) \arrow[r] \arrow[d, "(\partial^p_U, \partial^p_V) = (\mathrm{d}^p_U, \mathrm{d}^p_V)"] & \operatorname{coker} \mathrm{d}^{p-1}_{U\cap V} \arrow[r] \arrow[d, "\partial^p_{U\cap V} = \mathrm{d}^p_{U\cap V}"] & 0 \\ 0 \arrow[r] & \ker \mathrm{d}^{p+1}_M \arrow[r] & \ker\left(\mathrm{d}^{p+1}_U, \mathrm{d}^{p+1}_V\right) \arrow[r] & \ker \mathrm{d}^{p+1}_{U\cap V} & \end{tikzcd},$$

which is well-defined, since $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$. By the weak snake lemma again,

$$\ker \partial^p_M \to \ker\left(\partial^p_U, \partial^p_V\right) \to \ker \partial^p_{U\cap V} \xrightarrow{\delta} \operatorname{coker} \partial^p_M \to \operatorname{coker}\left(\partial^p_U, \partial^p_V\right) \to \operatorname{coker} \partial^p_{U\cap V}.$$

Then $\operatorname{coker} \mathrm{d}^{p-1}_M = \Omega^p(M) / \operatorname{im} \mathrm{d}^{p-1}_M$. There exists

$$\mathrm{H}^p(M) = \ker \mathrm{d}^p_M / \operatorname{im} \mathrm{d}^{p-1}_M \xrightarrow{\sim} \ker\left(\Omega^p(M) / \operatorname{im} \mathrm{d}^{p-1}_M \to \ker \mathrm{d}^{p+1}_M\right) = \ker \partial^p_M.$$

Similarly, $\ker\left(\partial^p_U, \partial^p_V\right) \cong \mathrm{H}^p(U) \oplus \mathrm{H}^p(V)$ and $\ker \partial^p_{U\cap V} \cong \mathrm{H}^p(U \cap V)$. There exists

$$\mathrm{H}^{p+1}(M) = \ker \mathrm{d}^{p+1}_M / \operatorname{im} \mathrm{d}^p_M \xrightarrow{\sim} \operatorname{coker}\left(\Omega^p(M) / \operatorname{im} \mathrm{d}^{p-1}_M \to \ker \mathrm{d}^{p+1}_M\right) = \operatorname{coker} \partial^p_M.$$

Similarly, $\operatorname{coker}\left(\partial^p_U, \partial^p_V\right) \cong \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V)$ and $\operatorname{coker} \partial^p_{U\cap V} \cong \mathrm{H}^{p+1}(U \cap V)$. ☐

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Example 2.33. Let $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ be the torus. Then

$$\mathrm{H}^p\left(\mathbb{T}^2\right) = \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & p = 1 \end{cases}.$$ [10]

Definition 2.34. Let M be a manifold, and let $\mathcal{U} = \{U_i\}$ be an open cover of M. Then $\mathcal{U}$ is said to be **good** if for all $I = (i_1, \dots, i_p)$, $U_{i_1} \cap \dots \cap U_{i_p}$ is either $\emptyset$ or contractible.

Lemma 2.35. *Let M be a connected manifold which admits a finite good cover. Then for all $p \ge 0$, $\mathrm{H}^p(M)$ is a finite-dimensional vector space.*

Exercise. Find a counterexample without assuming there exists a finite good cover.

Proof. Let $\mathcal{U}$ be a finite good cover. Define $k = \#\mathcal{U}$. By induction on k.

$k = 1$. $M = U_1$ is contractible, so

$$\mathrm{H}^p(M) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$k > 1$. Assume ok for covers with at most $k - 1$ elements. Let

$$U = \bigcup_{i=1}^{k-1} U_i, \qquad V = U_k.$$

Then $U \cup V = M$ and $U \cap V \ne \emptyset$, so Mayer-Vietoris holds. By induction $\mathrm{H}^p(U)$ and $\mathrm{H}^p(V)$ are finite-dimensional, since $\mathrm{H}^p(U)$ is covered by $k - 1$ of U_i and $\mathrm{H}^p(V)$ is contractible. Then $U \cap V = \bigcup_{i=1}^{k-1}(U_i \cap U_k)$, and $\{U_i \cap U_k\}$ is a good cover of $U \cap V$ with $k - 1$ elements. [11] By induction $\mathrm{H}^p(U \cap V)$ is finite-dimensional. Thus $\mathrm{H}^p(M)$ is also finite-dimensional.

☐

[10] Exercise

[11] Exercise

Fact. Any manifold admits a good cover.

Theorem 2.36. *Let M be a compact connected manifold. Then $\mathrm{H}^p(M)$ is finite-dimensional.*

Proof. Follows from the fact and Lemma 2.35. ☐

2.5 Compactly supported de Rham cohomology

Let M be a manifold, and let $\omega \in \Omega_c^p(M)$. Then $d\omega \in \Omega_c^{p+1}(M)$ and $d^2 = 0$, so

$$\Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \dots.$$

Definition 2.37. The **p-th compactly supported de Rham cohomology group** is

$$\mathrm{H}_c^p(M) = \mathcal{Z}_c^p(M) / \mathcal{B}_c^p(M) = \ker\left(d : \Omega_c^p(M) \to \Omega_c^{p+1}(M)\right) / \operatorname{im}\left(d : \Omega_c^{p-1}(M) \to \Omega_c^p(M)\right).$$

Example. If M is compact, then

$$\mathrm{H}_c^p(M) = \mathrm{H}^p(M), \qquad p \ge 0.$$

Proposition 2.38. *Let M be a non-compact connected manifold. Then*

$$\mathrm{H}_c^0(M) = 0.$$

Recall if M is connected $\mathrm{H}^0(M) = \mathbb{R}$, since $\mathrm{H}^0(M) = \{f \text{ constant on } M\}$.

Proof.

$$\mathrm{H}_c^0(M) = \{f \text{ constant on } M \text{ and with compact support}\}.$$

Since M is non-compact, if $f \in \Omega_c^0(M)$, then $\operatorname{supp} f \subsetneq M$. Thus there exists $x \in M$ such that $f(x) = 0$, so $f \equiv 0$, since f is constant. ☐

Remark 2.39. Let $f : M \to N$ be a smooth morphism between manifolds, and let $\omega \in \Omega_c^p(N) \subset \Omega^p(N)$. Then $f^*\omega \in \Omega^p(M)$, and $\operatorname{supp} f^*\omega \subset f^{-1}(\operatorname{supp}\omega)$, which is not compact in general, so $f^*\omega \notin \Omega_c^p(M)$ in general. If f is **proper**, that is $f^{-1}(K)$ is compact for all compact subsets $K \subset N$, then $f^* : \Omega_c^p(N) \to \Omega_c^p(M)$ is well-defined. If f is a diffeomorphism then f^* induces an isomorphism $\mathrm{H}_c^p(N) \to \mathrm{H}_c^p(M)$. [12]

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Definition 2.40. Let M_0 and M_1 be manifolds without boundary, and let $f_i : M_0 \to M_1$ be smooth morphisms for $i = 0, 1$. Then f_0 and f_1 are **properly smoothly homotopic** if there exists a smooth $H : M_0 \times [0,1] \to M$ such that $H(\cdot, i) = f_i(\cdot)$ for $i = 0, 1$ and H is proper. Then M_0 and M_1 are **properly smoothly homotopically equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$, where the equivalences are properly homotopic.

Notation. $f_t(\cdot) = H(\cdot, t) : M_0 \to M_1$.

Remark 2.41. To say that H is proper is not the same as saying f_t is proper for all t. Find H such that f_t is proper but H is not. A hint is to let $M_0 = M_1 = \mathbb{R}$ and $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $f_t^{-1}(0)$ is bounded for all t but $H^{-1}(0)$ is not. [13]

Proposition 2.42. *If M_0 and M_1 are properly homotopically equivalent then*

$$\mathrm{H}_c^p(M_0) \cong \mathrm{H}_c^p(M_1).$$

Let M be a manifold, and let $i : U \hookrightarrow M$ be an open set. Then there exist linear **push-forwards**

$$i_* : \Omega_c^p(U) \to \Omega_c^p(M), \qquad p \ge 0.$$

Let $\omega \in \Omega_c^p(U)$. Then $\omega = 0$ outside U. We can define

$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}.$$

If $j : V \hookrightarrow U$ and $i : U \hookrightarrow M$, then

$$(i \circ j)_* = i_* \circ j_*.$$

[12] Exercise

[13] Exercise

Lemma 2.43. *Let M be a manifold, and let $i : U \hookrightarrow M$ be an immersion such that U is open. Then for all $p \ge 0$, $i_* : \Omega_c^p(U) \to \Omega_c^p(M)$ commutes with d, that is*

$$\mathrm{d}(i_*\omega) = i_*\mathrm{d}\omega, \qquad \omega \in \Omega_c^p(U).$$

In particular if ω is closed then $i_*\omega$ is closed, and if ω is exact then $i_*\omega$ is exact.

Proof.

$$\mathrm{d}(i_*\omega) = \begin{cases} \mathrm{d}\omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases} = i_*\mathrm{d}\omega.$$

Let ω be closed, so $\mathrm{d}\omega = 0$. Then $\mathrm{d}(i_*\omega) = i_*\mathrm{d}\omega = 0$, so $i_*\omega$ is closed. Similarly for exactness. ☐

Let $U \hookrightarrow M$ be as before. Then there exist

$$i_* : \mathrm{H}_c^p(U) \to \mathrm{H}_c^p(M), \qquad p \ge 0.$$

Proposition 2.44 (Punctured manifolds)**.** *Let M be a manifold of dimension n, let $x \in M$, and let $i : M \setminus \{x\} \hookrightarrow M$. Then*

- *for all $p \ge 2$, $i_* : \mathrm{H}_c^p(M \setminus \{x\}) \to \mathrm{H}_c^p(M)$ is an isomorphism.*
- *for all $p \ge 1$, if M is compact $i_* : \mathrm{H}_c^p(M \setminus \{x\}) \to \mathrm{H}_c^p(M) = \mathrm{H}^p(M)$ is an isomorphism.*

Proof.

- Injectivity.

  $p \ge 2$. Let $\omega \in \Omega_c^p(M \setminus \{x\})$ be closed such that $i_*[\omega] = 0$, so $[i_*\omega] = 0$ in $\mathrm{H}_c^p(M)$. The goal is $[\omega] = 0$. There exists $\eta \in \Omega_c^{p-1}(M)$ such that $i_*\omega = \mathrm{d}\eta$. By the Poincaré lemma there exists $U \subset M$ containing x such that $\mathrm{H}^q(U) = 0$ for all $q \ge 1$. Then $i_*\omega = 0$ in a neighbourhood of x because $\operatorname{supp}\omega \subset M \setminus \{x\}$, so $\mathrm{d}\eta = 0$ in a neighbourhood of x. By taking U smaller we can assume η is closed. Since $p \ge 2$, $[\eta] \in \mathrm{H}^{p-1}(U) = 0$, so η is exact. Then there exists $\sigma \in \Omega^{p-2}(U)$ such that $\eta = \mathrm{d}\sigma$ on U. Let $(U, M \setminus \{x\})$ be an open cover of M, let $\left(f_U, f_{M\setminus\{x\}}\right)$ be a partition of unity, and let $\eta' = \eta - \mathrm{d}(i_*(f_U\sigma))$. On a neighbourhood of x, $\eta' = 0$ because $i_*(f_U\sigma) = \sigma$, so $\operatorname{supp}\eta' \subset M \setminus \{x\}$. Thus $\eta' \in \Omega_c^{p-1}(M \setminus \{x\})$ and $\omega = \mathrm{d}\eta'$, so $[\omega] = 0$.

  $p = 1$. The same proof. Let $\omega \in \Omega_c^1(M \setminus \{x\})$ be closed such that $[i_*\omega] = 0$. There exists $\eta \in \Omega_c^0(M)$ such that $i_*\omega = \mathrm{d}\eta$. By taking an open set $U \subset M$ such that $x \in U$, we may assume $\mathrm{d}\eta = 0$, so $\eta = c$ is constant on U. Let $\eta' = \eta - c$. Then $\eta' = 0$ on U. If M is compact then $\eta' \in \Omega_c^0(M \setminus \{x\})$. Thus $\omega = \mathrm{d}\eta'$, so $[\omega] = 0$.

- Surjectivity.

  $p \ge 1$. Let $[\omega] \in \Omega_c^p(M)$ such that ω is closed. By the Poincaré lemma there exists an open $U \ni x$ such that ω is exact, so there exists $\sigma \in \Omega^{p-1}(U)$ such that $\omega = \mathrm{d}\sigma$. Let $\left(f_U, f_{M\setminus\{x\}}\right)$ be a partition of unity as before, and let $\omega' = \omega - \mathrm{d}(i_*(f_U\sigma))$. Then $\omega' = 0$ in a neighbourhood of x and $[\omega'] = [\omega]$, and $\omega'|_{M\setminus\{x\}} \in \Omega_c^p(M \setminus \{x\})$. Thus $\left[i_*\ \omega'|_{M\setminus\{x\}}\right] = [\omega'] = [\omega]$.

☐

Exercise. Compute $\mathrm{H}_c^1\left(\mathbb{R}^2 \setminus \{0\}\right)$ by hands.

Example 2.45.

$$\mathrm{H}_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}.$$

Recall $\mathbb{R}^n \cong \mathrm{S}^n \setminus \{x\}$ for $x \in \mathrm{S}^n$. By Proposition 2.44, by $M = \mathrm{S}^n$,

$$\mathrm{H}_c^p(\mathbb{R}^n) = \mathrm{H}_c^p(\mathrm{S}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}, \qquad p \ge 1,$$

and $\mathrm{H}_c^0(\mathbb{R}^n) = 0$.

Let M be a manifold such that $M = U \cup V$ for open U and V such that $U \cap V \neq \emptyset$, and let Lecture 16 Thursday 13/02/20

$$\begin{tikzcd} & U \arrow[dr, hook, "i_U"] & \\ U \cap V \arrow[ur, hook, "j_U"] \arrow[dr, hook, "j_V"'] & & M \\ & V \arrow[ur, hook, "i_V"'] & \end{tikzcd}, \qquad \begin{tikzcd} & \Omega^p(U) \arrow[dr, "i_{U*}"] & \\ \Omega^p(U \cap V) \arrow[ur, "j_{U*}"] \arrow[dr, "j_{V*}"'] & & \Omega^p(M) \\ & \Omega^p(V) \arrow[ur, "i_{V*}"'] & \end{tikzcd}, \qquad p \geq 0.$$

Proposition 2.46. *We have a short exact sequence*

$$0 \leftarrow \Omega^p(M) \xleftarrow{i} \Omega^p(U) \oplus \Omega^p(V) \xleftarrow{j} \Omega^p(U \cap V) \leftarrow 0,$$

where $i = i_{U*} + i_{V*}$ and $j = (-j_{U*}, j_{V*})$.

Proof.

- j is injective. Let $\omega \in \Omega^p(U \cap V)$ such that $j(\omega) = 0$, so $j_{U*}\omega = j_{V*}\omega = 0$. Then $\omega = 0$, so j is injective.
- $\ker i = \operatorname{im} j$. Let $\omega \in \Omega^p(U \cap V)$. Then $i(j(\omega)) = i(-j_{U*}\omega, j_{V*}\omega) = -i_{U*}j_{U*}\omega + i_{V*}j_{V*}\omega = 0$, so $\ker i \supset \operatorname{im} j$. Let $(\omega_1, \omega_2) \in \ker i$. Then $i_{U*}\omega_1 + i_{V*}\omega_2 = 0$, so $i_{V*}\omega_1 = -i_{V*}\omega_2$, so $\operatorname{supp}\omega_1 \subset U \cap V$ and $\operatorname{supp}\omega_2 \subset U \cap V$, so there exists $\eta \in \Omega^p(U \cap V)$ such that $j_{U*}\eta = -\omega_1$ and $j_{V*}\eta = \omega_2$, so $(\omega_1, \omega_2) = j(\eta)$, so $\ker i \subset \operatorname{im} j$.
- i is surjective. Let $\omega \in \Omega_c^p(M)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Define $\omega_U = f_U \cdot \omega|_U \in \Omega_c^p(U)$ and $\omega_V = f_V \cdot \omega|_V \in \Omega_c^p(V)$. Then $i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = f_U \cdot \omega + f_V \cdot \omega = (f_U + f_V) \cdot \omega = \omega$.

$\square$

Thus for all p we get

$$\begin{tikzcd} 0 \arrow[r] & \Omega_c^p(U \cap V) \arrow[r] \arrow[d, "\mathrm{d}"] & \Omega_c^p(U) \oplus \Omega_c^p(V) \arrow[r] \arrow[d, "(\mathrm{d},\mathrm{d})"] & \Omega_c^p(M) \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\ 0 \arrow[r] & \Omega_c^{p+1}(U \cap V) \arrow[r] & \Omega_c^{p+1}(U) \oplus \Omega_c^{p+1}(V) \arrow[r] & \Omega_c^{p+1}(M) \arrow[r] & 0 \end{tikzcd}.$$

Theorem 2.47. *There exists $\delta : \mathrm{H}_c^p(M) \to \mathrm{H}_c^{p+1}(U \cap V)$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_c^p(U \cap V) \arrow[r] & \mathrm{H}_c^p(U) \oplus \mathrm{H}_c^p(V) \arrow[r] & \mathrm{H}_c^p(M) \arrow[dll, "\delta"'] & \\ & \mathrm{H}_c^{p+1}(U \cap V) \arrow[r] & \mathrm{H}_c^{p+1}(U) \oplus \mathrm{H}_c^{p+1}(V) \arrow[r] & \mathrm{H}_c^{p+1}(M) \arrow[r] & \dots \end{tikzcd}.$$

Proof. Same proof as Mayer-Vietoris for $\mathrm{H}^p(M)$. $\square$

2.6 Poincaré duality

Let M be an orientable manifold. Then $\mathrm{H}^p(M) \cong \mathrm{H}_c^{n-p}(M)^*$, the dual of $\mathrm{H}_c^{n-p}(M)$.

Proposition 2.48. *Let M be a manifold. Then the bilinear map*

$$\begin{array}{cccc} \cup & : & \mathrm{H}^p(M) \times \mathrm{H}^q(M) & \longrightarrow & \mathrm{H}^{p+q}(M) \\ & & ([\omega], [\eta]) & \longmapsto & [\omega \wedge \eta] \end{array}$$

is well-defined, and

$$[\omega] \cup [\eta] = (-1)^{p \cdot q} [\eta] \cup [\omega].$$

Proof. Follows from the Leibniz rule and Proposition 1.6. $\square$

Lemma 2.49. *Let M be oriented without boundary of dimension n. Then there exists a linear map*

$$\begin{array}{cccccc} \mathrm{I}_M & : & \mathrm{H}^n_\mathrm{c}(M) & \longrightarrow & \mathbb{R} \\ & & [\omega] & \longmapsto & \int_M \omega \end{array},$$

and I_M is surjective.

Then I_M is called **integration**.

Proof. Let $\omega \in \Omega^n_\mathrm{c}(M)$ such that $[\omega] = 0$, so ω is exact. By Stokes $\int_M \omega = 0$, so I_M is well-defined and it is linear. It is enough to show there exists closed $\omega \in \Omega^n_\mathrm{c}(M)$ such that $\int_M \omega \neq 0$. Take a volume form ω_0, which exists because M is oriented. Take $f \in \mathrm{C}^\infty(M)$ for $f \geq 0$ and with compact support. Let $\omega = f \cdot \omega_0 \in \Omega^n_\mathrm{c}(M)$. Then ω is closed because $\Omega^{n+1}_\mathrm{c}(M) = 0$ and $\int_M \omega = \int_M f \cdot \omega_0 > 0$, by definition of volume forms. ☐

Example 2.50. Let $M = \mathrm{S}^n$, and let $\omega \in \Omega^n_\mathrm{c}(M)$ such that $\int_M \omega = 0$. We want to show that ω is exact. Since M is compact, $\mathrm{H}^n_\mathrm{c}(M) = \mathrm{H}^n(M) = \mathbb{R}$. By Lemma 2.49 $\mathrm{I}_M : \mathrm{H}^n_\mathrm{c}(M) \to \mathbb{R}$ is surjective, and $\mathrm{H}^n_\mathrm{c}(M) = \mathbb{R}$, so I_M is injective. Since $\int_M \omega = 0$, $\mathrm{I}_M([\omega]) = 0$, so $[\omega] = 0$. Thus ω is exact.

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Let M be a connected manifold of dimension n. If $\omega_2 \in \mathrm{H}^q_\mathrm{c}(M)$ then $[\omega_1 \wedge \omega_2] \in \mathrm{H}^{p+q}_\mathrm{c}(M)$. Then

$$\cup : \mathrm{H}^p(M) \times \mathrm{H}^q_\mathrm{c}(M) \to \mathrm{H}^{p+q}_\mathrm{c}(M).$$

Let M be an oriented manifold without boundary of dimension n. Then

$$\begin{array}{cccccc} \mathrm{I}_M & : & \mathrm{H}^n_\mathrm{c}(M) & \longrightarrow & \mathbb{R} \\ & & [\omega] & \longmapsto & \int_M \omega \end{array}.$$

Choose $q = n - p$. Then

$$\mathrm{I}_M \circ \cup : \mathrm{H}^p(M) \times \mathrm{H}^{n-p}_\mathrm{c}(M) \to \mathrm{H}^n_\mathrm{c}(M) \to \mathbb{R}.$$

Recall that if $\phi : V \times W \to \mathbb{R}$ is bilinear, then there exists

$$\begin{array}{ccc} V & \longrightarrow & W^* = \mathrm{Hom}(W, \mathbb{R}) \\ v & \longmapsto & \phi_v \end{array}, \qquad \begin{array}{ccccc} \phi_v & : & W & \longrightarrow & \mathbb{R} \\ & & w & \longmapsto & \phi(v, w) \end{array}.$$

Thus, we get

$$\mathrm{H}^p(M) \to \mathrm{H}^{n-p}_\mathrm{c}(M)^*.$$

Poincaré duality says that this is an isomorphism.

Example. Assume M is compact and oriented. Then $\mathrm{H}^p(M) \xrightarrow{\sim} \mathrm{H}^{n-p}(M)$, so

$$\mathrm{b}^p(M) = \mathrm{b}^{n-p}(M).$$

Example 2.51. Let $U \subset \mathbb{R}^n$ be an open subset diffeomorphic to $\mathbb{R}^n$. Then

$$\mathrm{H}^p(U) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \qquad \mathrm{H}^p_\mathrm{c}(U) = \begin{cases} 0 & p < n \\ \mathbb{R} & p = n \end{cases}.$$

We want to show that Poincaré duality holds. We just need to check that Poincaré duality holds for $p = 0$. It is enough to show that $\phi : \mathrm{H}^0(U) \to \mathrm{H}^n_\mathrm{c}(U)^*$ is injective, that is there exists ω such that $\phi(\omega) \neq 0$. Given $\omega \in \mathrm{H}^0(U)$,

$$\begin{array}{ccccc} \phi(\omega) & : & \mathrm{H}^n_\mathrm{c}(U) & \longrightarrow & \mathbb{R} \\ & & \eta & \longmapsto & \int_U \eta \wedge \omega \end{array}.$$

Then $\omega = c$ is a constant function on U, so

$$\begin{array}{ccccc} \phi(\omega) & : & \mathrm{H}^n_\mathrm{c}(U) & \longrightarrow & \mathbb{R} \\ & & \eta & \longmapsto & \int_U c\omega \end{array}.$$

If $c \neq 0$ there exists η such that this map is not zero, so $\phi(\omega) \neq 0$. Thus ϕ is an isomorphism.

We will prove the following.

Theorem 2.52 (Poincaré duality)**.** *Assume that M is an oriented manifold, without boundary, such that there exists a finite open cover $\mathcal{U} = \{U_i\}$ such that $U_{i_1} \cap \dots \cap U_{i_q}$ is $\emptyset$ or diffeomorphic to $\mathbb{R}^n$. Then*

$$\mu_M : \mathrm{H}^p(M) \xrightarrow{\sim} \mathrm{H}^{n-p}_{\mathrm{c}}(M)^*, \qquad p \geq 0, \qquad n = \dim M$$

is an isomorphism.

Any compact manifold M admits such a cover.

Lemma 2.53. *Let*

$$C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3$$

be exact, where C^i are vector spaces of finite dimension. Then there exists

$$\left(C^3\right)^* \xrightarrow{\left(f^2\right)^*} \left(C^2\right)^* \xrightarrow{\left(f^1\right)^*} \left(C^1\right)^*,$$

which is also exact, where $\left(f^1\right)^* \phi = \phi \circ f^1$ and $\left(f^2\right)^* \phi = \phi \circ f^2$.

Proof. By assumption $\ker f^2 = \operatorname{im} f^1$. We want to prove $\ker \left(f^1\right)^* = \operatorname{im} \left(f^2\right)^*$.

- Let $\phi \in \operatorname{im} \left(f^2\right)^*$. Then there exists $\psi \in \left(C^3\right)^*$ such that $\left(f^2\right)^* \psi = \phi$, so $\psi \circ f^2 = \phi$, so $0 = \psi \circ f^2 \circ f^1 = \phi \circ f^1 = \left(f^1\right)^* \phi$, so $\phi \in \ker \left(f^1\right)^*$.
- Let $\phi \in \ker \left(f^1\right)^*$. Then $\phi \circ f^1 = 0$, so $\ker f^2 = \operatorname{im} f^1 \subset \ker \phi$, so there exists $\overline{\phi} : C^2 / \ker f^2 \to \mathbb{R}$, so there exists $\psi : C^3 \to \mathbb{R}$ extending $\overline{\phi}$ such that $\psi \circ f^2 = \phi$, so $\left(f^2\right)^* \psi = \phi$, so $\phi \in \operatorname{im} \left(f^2\right)^*$.

∎

Lemma 2.54 (Five lemma)**.** *Let*

$$\begin{tikzcd} C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r, "f^3"] \arrow[d, "\alpha_3"] & C^4 \arrow[r, "f^4"] \arrow[d, "\alpha_4"] & C^5 \arrow[d, "\alpha_5"] \\ D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 \arrow[r, "g^3"] & D^4 \arrow[r, "g^4"] & D^5 \end{tikzcd},$$

such that the horizontal lines are exact. Suppose

- *α_1 is surjective,*
- *α_5 is injective, and*
- *α_2 and α_4 are isomorphisms.*

Then α_3 is an isomorphism.

Proof. Let $x \in C^3$ such that $\alpha_3(x) = 0$, so if $y = f^3(x)$ then $\alpha_4(y) = 0$. Since α_4 is an isomorphism, $y = 0$. Then $x \in \ker f^3 = \operatorname{im} f^2$, so there exists $z \in C^2$ such that $f^2(z) = x$. Let $w = \alpha_2(z)$ then $g^2(w) = 0$, so $w \in \ker g^2 = \operatorname{im} g^1$. Then there exists $t \in D^1$ such that $g^1(t) = w$. Since α_1 is surjective there exists $s \in C^1$ such that $\alpha_1(s) = t$, so

$$\begin{tikzcd} s \in C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & z \in C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & x \in C^3 \arrow[r, "f^3"] \arrow[d, "\alpha_3"] & y \in C^4 \arrow[r, "f^4"] \arrow[d, "\alpha_4"] & C^5 \arrow[d, "\alpha_5"] \\ t \in D^1 \arrow[r, "g^1"] & w \in D^2 \arrow[r, "g^2"] & 0 \in D^3 \arrow[r, "g^3"] & 0 \in D^4 \arrow[r, "g^4"] & D^5 \end{tikzcd}.$$

We want to show that $f^1(s) = z$, and $\alpha_2\left(f^1(s)\right) = g^1(\alpha_1(s)) = g^1(t) = w = \alpha_2(z)$, so $f^1(s) = z$, since α_2 is injective. Thus $x = f^2(z) = f^2\left(f^1(s)\right) = 0$, so α_3 is injective. Show that α_3 is surjective. [14] ∎

[14] Exercise

Proof of Theorem 2.52. Let $N = \# U$. We proceed by induction on N. Then $N = 1$ is ok, so let $N > 1$. Let Lecture 18 Tuesday 18/02/20

$$U = \bigcup_{i=1}^{N-1} U_i, \qquad V = U_N,$$

so $M = U \cup V$. Both U and V, and $U \cap V$, satisfy Poincaré duality by induction. The idea is to use classical Mayer-Vietoris and compact support Mayer-Vietoris, and the five lemma. By Mayer-Vietoris,

$$\mathrm{H}^{p-1}(U) \oplus \mathrm{H}^{p-1}(V) \xrightarrow{g} \mathrm{H}^{p-1}(U \cap V) \xrightarrow{\delta} \mathrm{H}^p(M) \xrightarrow{f} \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \to \dots,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. By compact support Mayer-Vietoris,

$$\dots \to \mathrm{H}_\mathrm{c}^{n-p}(U) \oplus \mathrm{H}_\mathrm{c}^{n-p}(V) \xrightarrow{i} \mathrm{H}_\mathrm{c}^{n-p}(M) \xrightarrow{\delta_\mathrm{c}} \mathrm{H}_\mathrm{c}^{n-(p-1)}(M) \xrightarrow{j} \mathrm{H}_\mathrm{c}^{n-(p-1)}(U) \oplus \mathrm{H}_\mathrm{c}^{n-(p-1)}(V),$$

where $j = (-j_{U*}, j_{V*})$ and $i = i_{U*} + i_{V*}$. Taking the dual, by Lemma 2.53,

$$\mathrm{H}_\mathrm{c}^{n-(p-1)}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-(p-1)}(V)^* \xrightarrow{j^*} \mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)^* \xrightarrow{\delta_\mathrm{c}^*} \mathrm{H}_\mathrm{c}^{n-p}(M)^* \xrightarrow{i^*} \mathrm{H}_\mathrm{c}^{n-p}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-p}(V)^* \to \dots.$$

We get a diagram

$$\begin{tikzcd}
\mathrm{H}^{p-1}(U) \oplus \mathrm{H}^{p-1}(V) \arrow[r, "g"] \arrow[d, "n_{p-1} \cdot \mu_U \oplus \mu_V"] & \mathrm{H}^{p-1}(U \cap V) \arrow[r, "\delta"] \arrow[d, "n_{p-1} \cdot \mu_{U \cap V}"] & \mathrm{H}^p(M) \arrow[r, "f"] \arrow[d, "n_p \cdot \mu_M"] & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \arrow[r] \arrow[d, "n_p \cdot \mu_U \oplus \mu_V"] & \dots \\
\mathrm{H}_\mathrm{c}^{n-(p-1)}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-(p-1)}(V)^* \arrow[r, "j^*"] & \mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)^* \arrow[r, "\delta_\mathrm{c}^*"] & \mathrm{H}_\mathrm{c}^{n-p}(M)^* \arrow[r, "i^*"] & \mathrm{H}_\mathrm{c}^{n-p}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-p}(V)^* \arrow[r] & \dots
\end{tikzcd},$$

where $n_0 = 1$ and $n_p = (-1)^{p-1} n_{p-1}$. The goal is to show that μ_M is an isomorphism. The idea is by the five lemma, it is enough to show that

1. all the other vertical arrows are isomorphisms, and
2. the diagram is commutative.

We know 1 is ok by induction on N. We need to show 2.

- The first square. We want to show that

  $$\mu_{U \cap V} \circ g = j^* \circ (\mu_U \oplus \mu_V).$$

  Let $\omega_U \in \Omega^{p-1}(U)$ and $\omega_V \in \Omega^{p-1}(V)$ be closed forms. We want to show

  $$\mu_{U \cap V}(g([\omega_U], [\omega_V])) = j^*(\mu_U([\omega_U]), \mu_V([\omega_V])),$$

  in $\mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)^*$, that is we want to show that on any element of $\mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)$ they coincide. Let $\eta \in \Omega_\mathrm{c}^{n-(p-1)}(U \cap V)$. Recall $g = j_V^* - j_U^*$. Then

  $$\int_{U \cap V} g(\omega_U, \omega_V) \wedge \eta = -\int_U \omega_U \wedge j_{U*}\eta + \int_V \omega_V \wedge j_{V^*}\eta,$$

  since $g(\omega_U, \omega_V) = j_V^* \omega_V - j_U^* \omega_U$. Lecture 19 Thursday 20/02/20

- The second square. We want an explicit construction of δ and δ_c. Let $\omega \in \Omega^p(M)$ be a closed form, and let $\{f_U, f_V\}$ be a partition of the unity with respect to $\{U, V\}$. Define

  $$\omega_U = f_U \cdot \omega|_U \in \Omega_\mathrm{c}^p(U), \qquad \omega_V = f_V \cdot \omega|_V \in \Omega_\mathrm{c}^p(V),$$

  so $(\omega_U, \omega_V) \in \Omega_\mathrm{c}^p(U) \oplus \Omega_\mathrm{c}^p(V)$. Recall $i = i_{U*} + i_{V*}$. Then

  $$i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = \omega_U + \omega_V = f_U \cdot \omega + f_V \cdot \omega = \omega.$$

If ω is closed, then $i\left(\mathrm{d}\omega_U, \mathrm{d}\omega_V\right) = \mathrm{d}\left(i_{U*}\omega_U\right) + \mathrm{d}\left(i_{V*}\omega_V\right) = 0$, so $\left(\mathrm{d}\omega_U, \mathrm{d}\omega_V\right) \in \ker i = \operatorname{im} j \subset \Omega_{\mathrm{c}}^{p+1}(U) \oplus \Omega_{\mathrm{c}}^{p+1}(V)$. Since j is injective there exists a unique $\delta_{\mathrm{c}}(\omega) \in \Omega_{\mathrm{c}}^{p+1}(U \cap V)$ such that $j\left(\delta_{\mathrm{c}}(\omega)\right) = \left(\mathrm{d}\omega_U, \mathrm{d}\omega_V\right)$. Since $f_U + f_V = 1$, $\mathrm{d}f_U + \mathrm{d}f_V = 0$, so $\mathrm{d}f_U = -\mathrm{d}f_V$. Then

$$j\left(\delta_{\mathrm{c}}(\omega)\right) = \left(\mathrm{d}\omega_U, \mathrm{d}\omega_V\right) = \left(\mathrm{d}f_U \wedge \omega|_U, \mathrm{d}f_V \wedge \omega|_V\right) = \left(-\mathrm{d}f_V \wedge \omega|_U, \mathrm{d}f_V \wedge \omega|_V\right) = j\left(\mathrm{d}f_V \wedge \omega|_{U \cap V}\right).$$

Since j is injective, $\delta_{\mathrm{c}}(\omega) = \mathrm{d}f_V \wedge \omega|_{U \cap V}$, so $\delta_{\mathrm{c}} : \Omega_{\mathrm{c}}^p(M) \to \Omega_{\mathrm{c}}^{p+1}$. Let η be a form on M. Since $\delta_{\mathrm{c}}(\mathrm{d}\eta) = \mathrm{d}f_V \wedge \mathrm{d}\eta|_{U \cap V} = -\mathrm{d}\delta_{\mathrm{c}}(\eta)$, δ_{c} maps closed forms to closed forms and exact forms to exact forms, so

$$\begin{array}{rcccl} \delta_{\mathrm{c}} & : & \mathrm{H}_{\mathrm{c}}^p(M) & \longrightarrow & \mathrm{H}_{\mathrm{c}}^{p+1}(U \cap V) \\ & & \omega & \longmapsto & \mathrm{d}f_V \wedge \omega|_{U \cap V} \end{array}.$$

By construction, it makes the long exact sequence exact. Similarly

$$\begin{array}{rcccl} \delta & : & \mathrm{H}^p(U \cap V) & \longrightarrow & \mathrm{H}^{p+1}(M) \\ & & \omega & \longmapsto & \begin{cases} \mathrm{d}f_V \wedge \omega & \text{on } U \cap V \\ 0 & \text{otherwise} \end{cases} \end{array}.$$

Now we check that the second square is commutative, that is

$$n_{p-1} \cdot \mu_M\left(\delta\left(\left[\omega_1\right]\right)\right) = n_p \cdot \delta_{\mathrm{c}}^*\left(\mu_{U \cap V}\left(\left[\omega_1\right]\right)\right), \qquad \omega_1 \in \Omega^{p-1}(U \cap V).$$

That is,

$$n_{p-1} \int_M \delta\left(\omega_1\right) \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \delta_{\mathrm{c}}\left(\omega_2\right), \qquad \omega_2 \in \Omega_{\mathrm{c}}^{n-p}(M).$$

For all $\omega_2 \in \Omega_{\mathrm{c}}^{n-p}(M)$,

$$n_{p-1} \int_M \delta\left(\omega_1\right) \wedge \omega_2 = n_{p-1} \int_{U \cap V} \mathrm{d}f_V \wedge \omega_1 \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \mathrm{d}f_V \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \delta_{\mathrm{c}}\left(\omega_2\right).$$

- The third square. To check

$$\left(\mu_U \oplus \mu_V\right) \circ f = i^* \circ \mu_M,$$

so

$$\left(\mu_U \oplus \mu_V\right)\left(f\left([\omega]\right)\right) = i^*\left(\mu_M\left([\omega]\right)\right), \qquad \omega \in \Omega^p(M),$$

in $\mathrm{H}_{\mathrm{c}}^{n-p}(U)^* \oplus \mathrm{H}_{\mathrm{c}}^{n-p}(V)^*$. Let $\eta_U \in \Omega_{\mathrm{c}}^{n-p}(U)$ and $\eta_V \in \Omega_{\mathrm{c}}^{n-p}(V)$. Then

$$\int_U \omega|_U \wedge \eta_U + \int_V \omega|_V \wedge \eta_V = \int_M \omega \wedge i\left(\eta_U, \eta_V\right).$$

□

The following is an easy consequence.

Corollary 2.55. *Let M be an oriented compact connected manifold of dimension n. Then*

$$\mathrm{H}^n(M) = \mathbb{R},$$

and

$$\mathrm{H}_{\mathrm{c}}^p(M \setminus \{x\}) = \mathrm{H}^p(M), \qquad x \in M, \qquad 1 \le p < n.$$

Definition 2.56. The **Euler characteristic** of M is

$$\chi(M) = \sum_{p=0}^n (-1)^p \dim \mathrm{H}^p(M).$$

Corollary 2.57. *If M is a compact oriented manifold of odd dimension then $\chi(M) = 0$.*

Proof. By Poincaré duality, $\dim \mathrm{H}^i(M) = \dim \mathrm{H}^{n-i}(M)$. □

2.7 Degree of a morphism

Let M and N be connected oriented manifolds of dimension n, and let $f : M \to N$ be a proper smooth morphism. Then

$$f^* : \mathrm{H}_{\mathrm{c}}^n(N) \cong \mathbb{R} \to \mathrm{H}_{\mathrm{c}}^n(M) \cong \mathbb{R},$$

by Poincaré duality and connectedness, so

$$f(x) = c \cdot x, \qquad \deg f = c \in \mathbb{R}.$$

Thus

$$\int_M f^*\omega = \deg f \cdot \int_M \omega, \qquad \omega \in \Omega_{\mathrm{c}}^n(M).$$

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Proposition 2.58. *Let M, N, P be connected oriented manifolds of dimension n.*

- *If $f : M \to N$ and $g : N \to P$ are smooth and proper morphisms then*

$$\deg(g \circ f) = \deg f \cdot \deg g$$

- *If f is a diffeomorphism then*

$$\deg f = \begin{cases} 1 & f \text{ is orientation preserving} \\ -1 & \text{otherwise} \end{cases}.$$

- *If $f, g : M \to N$ are smooth proper and properly homotopic equivalent then*

$$\deg f = \deg g.$$

Theorem 2.59 (Mapping degree theorem)**.** *Let $f : M \to N$ be a proper smooth morphism between connected oriented manifolds of dimension n. Then $\deg f \in \mathbb{Z}$.*

Definition 2.60. Let $f : M \to N$ be a smooth morphism. Then $y \in N$ is **regular** if for all $x \in f^{-1}(y)$, $\mathrm{D}f_x$ has maximal rank.

Theorem 2.61 (Preimage theorem)**.** *Let $f : M \to N$ be a smooth morphism, and let $y \in N$ be a regular value. Then $f^{-1}(y)$ is a manifold of dimension $\dim M - \operatorname{rk} \mathrm{D}f_x$ where $x \in f^{-1}(y)$.*

Theorem 2.62 (Implicit function theorem)**.** *Let $f : M \to N$ be a smooth morphism, and let $x \in M$ be such that $\mathrm{D}f_x$ is an isomorphism. Then there exists an open $x \in U \subset M$ such that $f|_U : U \to f(U)$ is an isomorphism.*

Theorem 2.63 (Sard's theorem)**.** *Let $f : M \to N$ be smooth. Then if $Z \subset N$ is the set of regular values of f then $Z \cap f(M)$ is dense in $f(M)$.*

Proof of Theorem 2.59. Recall $\dim M = \dim N$, and if $\omega \in \Omega_{\mathrm{c}}^n(N)$ and if $\mathrm{D}f_x$ is of rank less than n for all x, then $\deg f = 0$. We are done. In particular we may assume there exists x such that $\mathrm{D}f_x$ has rank equal to n. Let $y = f(x)$. By Sard's theorem, we may assume that for all $x \in f^{-1}(y)$, $\mathrm{D}f_x$ has rank n. By the preimage theorem $f^{-1}(y)$ is a manifold of dimension zero, such as $\mathbb{Z} \subset \mathbb{R}$, so

$$f^{-1}(y) = \{x_1, \dots, x_k\}$$

is a finite set, because f is proper. By the implicit function theorem, for all i there exists an open set $U_i \ni x$ such that $f|_U$ is a diffeomorphism and $f(U_i) = U$. Let $\omega \in \Omega_{\mathrm{c}}^n(N)$ be such that $\int_U \omega = 1$ and $\operatorname{supp} \omega \subset U$. Since $f|_{U_i}$ is a diffeomorphism

$$\int_{U_i} f|_{U_i}^* \omega = \operatorname{sgn}(\det \mathrm{D}f_{x_i}) \int_U \omega,$$

and $f|_{U_i}^* \omega$ has support in U_i. Since $\operatorname{supp} f^*\omega \subset \bigcup_i U_i$,

$$\int_M f^*\omega = \sum_{i=1}^k \int_{U_i} f^*\omega = \sum_{i=1}^k \operatorname{sgn}(\det \mathrm{D}f_{x_i}) \int_U \omega = \sum_{i=1}^k \operatorname{sgn}(\det \mathrm{D}f_{x_i}) \int_M \omega,$$

so $\deg f = \sum_{i=1}^k \operatorname{sgn}(\det \mathrm{D}f_{x_i}) \in \mathbb{Z}$, which does not depend on y, if y is a regular point. $\square$

Exercise. Suppose that $f : M \to N$ is a proper morphism between oriented connected manifolds. If $\deg f \neq 0$, then f is surjective.

Example 2.64. Let $M = \mathrm{S}^n = N$, and let

$$\begin{array}{rcccl} f & : & M & \longrightarrow & N \\ & & x & \longmapsto & -x \end{array}$$

be the antipodal map. Claim that $\deg f = (-1)^{n+1}$. Let $i : \mathrm{S}^n \hookrightarrow \mathbb{R}^{n+1}$, let

$$\widetilde{\omega} = x_1 \mathrm{d}x_2 \wedge \dots \wedge \mathrm{d}x_{n+1} \in \Omega^n \left(\mathbb{R}^{n+1}\right),$$

and let $\omega = i_* \widetilde{\omega} \in \Omega^n \left(\mathrm{S}^n\right)$. By Stokes and $\mathrm{S}^n = \partial \mathrm{D}_{n+1}$,

$$\int_{\mathrm{S}^n} \omega = \int_{\mathrm{S}^n} i^* \widetilde{\omega} = \int_{\mathrm{D}_{n+1}} \mathrm{d}\widetilde{\omega} = \int_{\mathrm{D}_{n+1}} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n+1} \neq 0,$$

so f can be extended to

$$\begin{array}{rcccl} \widetilde{f} & : & \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^{n+1} \\ & & x & \longmapsto & -x \end{array}.$$

Then $\widetilde{f} \circ i = i \circ f$ and $\widetilde{f}^* \widetilde{\omega} = (-1)^{n+1} \widetilde{\omega}$, so

$$f^* \omega = f^* i^* \widetilde{\omega} = (i \circ f)^* \widetilde{\omega} = \left(\widetilde{f} \circ i\right)^* \widetilde{\omega} = i^* \widetilde{f}^* \widetilde{\omega} = (-1)^{n+1} i^* \widetilde{\omega} = (-1)^{n+1} \omega.$$

Thus

$$(-1)^{n+1} \int_{\mathrm{S}^n} \omega = \int_{\mathrm{S}^n} f^* \omega = \deg f \int_{\mathrm{S}^n} \omega,$$

so $\deg f = (-1)^{n+1}$.

3 Morse theory

Definition 3.1. Let M be a manifold of dimension n, and let $f : M \to \mathbb{R}$ be smooth. A **critical point** of f is a point $x \in M$ such that $\mathrm{D} f_x = 0$, that is if $x_1, \dots, x_n$ are local coordinates at x, then

$$\frac{\partial}{\partial x_i} f(x) = 0, \qquad i = 1, \dots, n.$$

For such x, we define the **Hessian** of f to be

$$\mathrm{H}_f(x) = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(x) \right).$$

Then x is called **non-degenerate** if $\det \mathrm{H}_f(x) \neq 0$. A function f such that every critical point of f is non-degenerate is called a **Morse function**.

Fact. By Sard's theorem most of the functions satisfy this property.

3.1 Cell decomposition

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Notation. Let $\mathrm{D}_n = \{x \mid |x| \leq 1\} \subset \mathbb{R}^n$ be the unit ball, and let $\mathrm{S}^{n-1} = \partial \mathrm{D}_n$.

Definition 3.2. An n**-cell** is a topological space which is homeomorphic to the open ball $\mathrm{D}_n \setminus \partial \mathrm{D}_n$. A **cell decomposition** of a topological space M is a family F of pairwise disjoint subspaces of M which are n-cells and such that $M = \bigsqcup_{e_i \in F} e_i$. If F is finite, then this is called a **finite cell decomposition**. Let

$$\mathrm{SK}_m M = \bigsqcup_{\dim e_i \leq m} e_i, \qquad m \geq 0.$$

Example 3.3. $\mathrm{S}^1 = \left(\mathrm{S}^1 \setminus \{p\}\right) \sqcup \{p\}$, where $\mathrm{S}^1 \setminus \{p\}$ is a 1-cell and $\{p\}$ is a 0-cell.

Notation 3.4. Let M be a topological space, and let $f_\partial : \mathrm{S}^{n-1} \to M$ be continuous. We construct a new topological space

$$M \cup_{f_\partial} \mathrm{D}_n = M \sqcup \mathrm{D}_n / \sim,$$

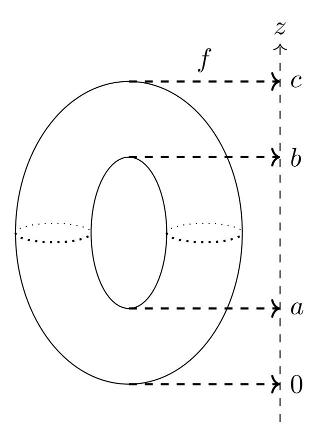
where $M \ni x \sim y \in \mathrm{S}^{n-1} \subset \mathrm{D}_n$ if $f_0(y) = x$. Then $M \cup_{f_\partial} \mathrm{D}_n$ is said to be obtained by **attaching** an n-cell to M via f_∂.

Example. Let $M = \{p\}$, and let $f_\partial : \mathrm{S}^0 = \partial \mathrm{D}_1 = \partial [0,1] \to \{p\}$. Then $\mathrm{S}^1 = M \cup_{f_\partial} \mathrm{D}_1$.

Exercise. If M admits a cell decomposition then also $M \cup_{f_\partial} \mathrm{D}_n$ does.

Example 3.5. Let $M = \mathrm{S}^1 \times \mathrm{S}^1 \subset \mathbb{R}^3$ be the torus

$M =$



,

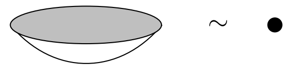
where

$$\begin{array}{cccc} f & : & M & \longrightarrow & \mathbb{R} \\ & & (x,y,z) & \longmapsto & z \end{array}.$$

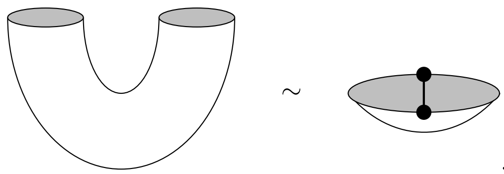
Define

$$\mathrm{S}_h = \{p \in M \mid f(p) \le h\} = f^{-1}((0,h]), \qquad h \ge 0.$$

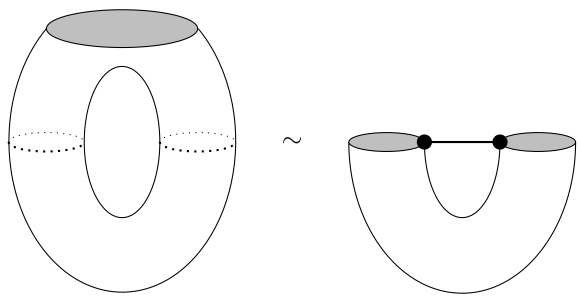
- If $h < 0$ then $\mathrm{S}_h = \emptyset$.
- If $0 < h < a$ then S_h is homotopically equivalent to a 0-cell, so

$\mathrm{S}_h =$  .

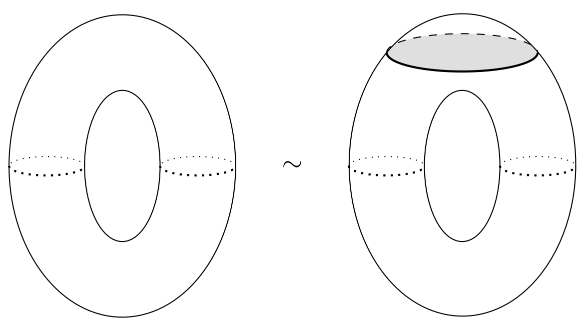
- If $a < h < b$ then S_h is homotopically equivalent to a 1-cell attached to the previous S_h, so

$\mathrm{S}_h =$  .

- If $b < h < c$ then S_h is homotopically equivalent to a 1-cell attached to the previous S_h, so

$\mathrm{S}_h =$  .

- If $h > c$ then S_h is homotopically equivalent to a 2-cell attached to the previous S_h, so

$\mathrm{S}_h =$  .

Thus

$$M = \text{0-cell} \sqcup \text{two 1-cells} \sqcup \text{2-cell}.$$

Given a Morse function $f : M \to \mathbb{R}$, the goal is to study the **level sets** of f,

$$\mathrm{S}_h = f^{-1}((-\infty, h]).$$

Definition 3.6. Let M be a manifold, let $f : M \to \mathbb{R}$ be a Morse function, and let $x \in M$ be critical. Denote

$$\mathrm{Eig}^- \mathrm{H}_f(x) = \{\text{eigenvectors of } \mathrm{H}_f \text{ with negative eigenvalues}\}.$$

Recall that H_f is a symmetric matrix. The **index** of f at x is the dimension of $\mathrm{Eig}^- \mathrm{H}_f(x)$.

Lemma 3.7 (Morse)**.** *Let M be a manifold of dimension n, let $f : M \to \mathbb{R}$ be a Morse function, and let $x_0 \in M$ be a critical point. Then there exist local coordinates $(x_1, \dots, x_n)$ around x_0 such that $x_0 = (0, \dots, 0)$ and*

$$f = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2,$$

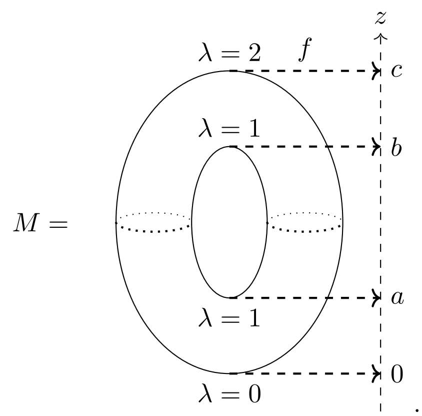
where λ is the index of f at x.

Thus the set of critical points of f is discrete, since locally at critical x_0,

$$f = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$$

has no more critical points.

Example 3.8. Let $f : M \to \mathbb{R}$ be as in Example 3.5. Then



3.2 CW-complexes

Definition 3.9. A topological space M admits a **CW-structure** if there exists a sequence of topological subspaces

$$M^{(0)} \subset \dots \subset M^{(n)},$$

such that

1. $M^{(0)}$ is a discrete subset of M,
2. $M^{(k)}$ is obtained by attaching k-cells to $M^{(k-1)}$, and
3. $V \subset M$ is closed if and only if $V \cap M^{(k)}$ is closed for all k.

Such M is called a **CW-complex**. Then M is a **finite CW-complex** if it is obtained by attaching finitely many cells. In this case 3 is not needed. A **subcomplex** of a CW-complex M is a closed subspace of M which is a union of cells of M. A **closed cell** is the image of D_n in a cell. An **open cell** is the image of $\mathrm{D}_n \setminus \partial \mathrm{D}_n$ in a cell. Open cells are not open in M in general.

Example 3.10.

- $\mathrm{S}^n = \{p\} \cup \mathrm{D}_n = M^{(0)} \cup M^{(n)}$.
- If $M = \mathbb{R}^n$, and $\Lambda = \{\text{integral points in } \mathbb{R}^n\}$, then Λ gives a decomposition of $\mathbb{R}^n$ into n-cubes, which are n-cells, where 0-cells are points of Λ, 1-cells are edges of Λ, etc.
- If $n \neq 4$ and M is a manifold of dimension n, then M is a CW-complex. If $n = 4$, then it is open.

Proposition 3.11. *Let M be a CW-complex. Then*

1. *if $K \subset M$ is a compact subset, then K is contained in a finite union of open cells, and*
2. *the closure of every cell of M is contained in a finite subcomplex of M.*

Proof.

1. We first prove 1. Let $K \subset M$ be a compact subset. We want to show that K only intersects finitely many cells of M. Assume by contradiction that there is an infinite sequence of points $S = \{x_j\} \subset K$ all lying in distinct cells. We claim that $S \cap M^{(n)}$ is closed and discrete for all $n \geq 0$. We proceed by induction on n. For $n = 0$, this follows from the fact that $M^{(0)}$ is closed and discrete. Assume now that $S \cap M^{(n)}$ is closed and discrete. Then, if $\{e_i\}_I$ are the $(n+1)$-cells, then the open cell corresponding to e_i contains at most one $x_j \in S$. Thus $S \cap (\bigcup_i e_i)$ is closed and discrete. It follows that $S \cap M^{(n+1)}$ is closed and discrete, as claimed. Since $S \subset K$, it follows that S is finite, a contradiction.
2. We now prove 2. To this end, we proceed by induction on the dimension n of the cell. For $n = 0$, the result is clear. Assume now that the result is true for any m-cell with $m < n$ and let e_n be an n-cell. In particular, the border K of e_n is the image of S^{n-1} and it is compact. Hence, it is contained in a finite union of open cells of dimension smaller than n by 1. By induction, each of these cells is contained in a finite subcomplex. The union of these subcomplexes is a finite subcomplex containing K. Hence attaching e_n results in a finite subcomplex containing e_n.

□

Corollary 3.12. *Let M be a CW-complex. Then any compact subset of M is contained in a finite subcomplex.*

Proof. Since a finite union of finite subcomplexes is again a finite subcomplex, the result follows immediately from Proposition 3.11. □

3.3 Gradient flows

Lecture 22 Thursday 27/02/20

Definition 3.13. Let M be a manifold, then a **flow** or a **group of diffeomorphisms** of M is the collection of diffeomorphisms $\phi_t : M \to M$ for $t \in \mathbb{R}$ such that there exists $\phi : \mathbb{R} \times M \to M$ with $\phi_t = \phi(t, \cdot)$ and

- $\phi_0 = \mathrm{id}_M$, and
- $\phi_{t+s} = \phi_t \circ \phi_s$.

The **flow line** or the **integral curve** for the flow is

$$\begin{array}{cccc} \gamma_x & : & \mathbb{R} & \longrightarrow & M \\ & & t & \longmapsto & \phi(t, x) \end{array}, \qquad x \in M,$$

where $\gamma_x(0) = x$.

Since $\frac{\mathrm{d}}{\mathrm{d}t} \gamma_x|_{t=0} \in \mathrm{T}_x M$, the flow defines a vector field on M, that is a smooth section of $\mathrm{T}M$. On the other hand, given a vector field, we can find a flow. Let X be a section of $\mathrm{T}M$. There exists ϕ_t, and γ_x, as above such that $\frac{\mathrm{d}}{\mathrm{d}t} \gamma_x|_{t=0} \in \mathrm{T}_x M$, locally around $t = 0$.

Lemma 3.14. *Let M be a manifold, and let X be a smooth compactly supported vector field on M. Then X generates a unique one-parameter group of diffeomorphisms $\phi_t : M \to M$ such that we have*

$$(X \circ \gamma_x)(t) = \frac{\mathrm{d}}{\mathrm{d}t} \gamma_x(t), \qquad x \in M.$$

It can be shown that two distinct flow lines are disjoint. Thus, the manifold M decomposes into a disjoint union of flow lines.

Example. Let $M = \mathbb{R}$, and let $X = x^2 \partial_x$. Find the flow. [15]

[15] Exercise

Definition 3.15. Let M be a manifold. A **Riemannian metric** g on M is a collection of inner products for $\mathrm{T}_x M$ for $x \in M$ given by $g_x : \mathrm{T}_x M \times \mathrm{T}_x M \to \mathbb{R}$, such that for all vector fields X and Y we have that $g_x\left(X\left(x\right), Y\left(x\right)\right) : M \to \mathbb{R}$ is smooth. A **Riemannian manifold** (M, g) is a manifold M and a Riemannian metric g on M.

Then g gives $\mathrm{T}M \xrightarrow{\sim} \Omega^1\left(M\right) = \left(\mathrm{T}M\right)^*$.

Definition 3.16. Let (M, g) be a Riemannian manifold, and let $f : M \to \mathbb{R}$ be a smooth function. The **gradient vector field** ∇f of f is the unique vector field on M such that for all vector fields X,

$$g\left(\nabla f, X\right) = X\left(f\right) = \mathrm{D}f\left(X\right),$$

the derivative of f with respect to X.

Thus,

$$\left\|\nabla f\right\|^2 = g\left(\nabla f, \nabla f\right) = \mathrm{D}f\left(\nabla f\right).$$

In particular $\nabla f\left(x\right) = 0$ if and only if x is a critical point of f, and ∇f is orthogonal to any vector tangent to $f^{-1}\left(c\right)$ for all regular values $c \in \mathbb{R}$. In particular, for all smooth functions $f : M \to \mathbb{R}$, we can take the flow ϕ associated to $-\nabla f$ such that if $\gamma_x\left(t\right) = \phi\left(t, x\right)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_x\left(0\right) = -\nabla f\left(\gamma_x\left(0\right)\right), \qquad \gamma_x\left(0\right) = x.$$

This is called the **gradient flow** of f. The integral curve of this flow are called **gradient flow lines**.

Lemma 3.17. *f decreases along the gradient lines, that is $f\left(\gamma_x\left(t\right)\right)$ is a decreasing function with respect to t.*

Proof.

$$0 \le -\left\|\nabla f\left(\gamma_x\left(t\right)\right)\right\|^2 = \mathrm{D}f_{\gamma_x(t)}\left(-\nabla f\left(\gamma_x\left(t\right)\right)\right) = \mathrm{D}f_{\gamma_x(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma_x\left(t\right)\right) = \frac{\mathrm{d}}{\mathrm{d}t} f\left(\gamma_x\left(t\right)\right).$$

$\square$

Proposition 3.18. *Let M be a compact manifold, and let $f : M \to \mathbb{R}$ be a Morse function. Then every gradient flow line begins and ends with a critical point of f, that is $\lim_{t\to\pm\infty}\gamma_x\left(t\right)$ exist and are critical points of f.*

Proof. First we prove that if the limit exists then it is a critical point of f. For all x, $f\left(\gamma_x\left(\cdot\right)\right) : \mathbb{R} \to \mathbb{R}$ is a bounded function. Then

$$0 = \lim_{t\to\pm\infty}\frac{\mathrm{d}}{\mathrm{d}t} f\left(\gamma_x\left(t\right)\right) = \lim_{t\to\pm\infty} -\left\|\nabla f\left(\gamma_x\left(t\right)\right)\right\|^2,$$

so the limit is critical. The goal is to show that limits exist. Since M is compact and f is a Morse function, the set of critical points is finite. Fix $\epsilon > 0$. Let U be the union of open balls of radius $\epsilon > 0$ around each critical point, so U is open in M. Then $M \setminus U$ is compact, so $\left\|\nabla f\left(\cdot\right)\right\|^2$ admits a minimum inside $M \setminus U$, but it cannot be zero, but $\lim_{t\to\pm\infty}\left\|\nabla f\left(\gamma_x\left(t\right)\right)\right\|^2 = 0$. If $\pm t$ is very large then $\gamma_x\left(t\right) \notin M \setminus U$, so if ϵ is sufficiently small, $\gamma_x\left(t\right)$ is in a ball around a single critical point for $t \to \pm\infty$. This implies that the limit is the critical point. $\square$

3.4 The fundamental theorems of Morse theory

Lecture 23 Monday 02/03/20

Definition 3.19. Let X be a topological space, and let $S \subset X$. Then S is called a **deformation retract** of X if there exists $F : X \times [0,1] \to X$ such that $F\left(x, 0\right) = x$ and $F\left(x, 1\right) \in S$ for all $x \in X$, and $F\left(s, 1\right) = s$ for all $s \in S$.

This implies that S and X are homotopy equivalent.

Theorem 3.20 (First fundamental theorem of Morse theory)**.** *Let M be a manifold, let $f : M \to \mathbb{R}$, let $a < b \in \mathbb{R}$ such that $f^{-1}\left([a,b]\right)$ does not contain any critical point, and let*

$$\mathrm{S}_t = f^{-1}\left(\left(-\infty, t\right]\right), \qquad t \in \mathbb{R}.$$

Then S_a is a deformation retract of S_b.

The idea is that we will use a perturbation of the gradient flow.

Proof. There exists $\epsilon > 0$ such that $f^{-1}([a-\epsilon, b+\epsilon])$ does not contain any critical point. Fix a metric g on M. Define ρ smooth on M such that $\rho(x) \ge 0$ and

$$\rho(x) = \begin{cases} \dfrac{1}{\|\nabla f\|^2} & x \in f^{-1}([a,b]) \\ 0 & x \in M \setminus f^{-1}([a-\epsilon, b+\epsilon]) \end{cases}.$$

Define

$$X(x) = -\rho(x) \nabla f(x).$$

There exists a flow $\phi(t,x)$ induced by X, that is if $\gamma_x(t) = \phi(t,x)$ then $\frac{\mathrm{d}}{\mathrm{d}t} \gamma_x(0) = X(x)$. By definition of ∇f,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_x(t)) = \mathrm{D}f_{\gamma_x(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma_x(t)\right) = g\left(\nabla f, \frac{\mathrm{d}}{\mathrm{d}t}\gamma_x(t)\right) = g(\nabla f, -\rho(\gamma_x(t))\nabla f) = -\rho(\gamma_x(t))\|\nabla f\|^2 \le 0,$$

so $f(\gamma_x(t))$ is decreasing. Moreover for all t such that $f(\gamma_x(t)) \in [a,b]$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_x(t)) = -1,$$

by definition of ρ. If $\gamma_x(s) \in f^{-1}([a,b])$ for all $s \in [0,t]$, by the fundamental theorem of calculus

$$f(\gamma_x(t)) - f(\gamma_x(0)) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma_x(s)) \, \mathrm{d}s = -t,$$

so

$$f(\gamma_x(t)) = f(x) - t.$$

Then

1. if $f(x) \le b$ then $f(\phi_{b-a}(x)) = f(\gamma_x(b-a)) \le a$, and
2. if $f(x) > b$ then $f(\phi_{b-a}(x)) > a$.

1 implies that $\phi_{b-a}(\mathrm{S}_b) \subset \mathrm{S}_a$ and 2 implies that $\phi_{a-b}(\mathrm{S}_a) \subset \mathrm{S}_b$. Recall that $\phi_{a-b} = \phi_{b-a}^{-1}$, so S_a and S_b are diffeomorphic. Now we define

$$\begin{array}{cccc} F & : & \mathrm{S}_b \times [0,1] & \longrightarrow & \mathrm{S}_b \\ & & (x,t) & \longmapsto & \begin{cases} x & f(x) \le a \\ \phi_{t(f(x)-a)}(x) & a \le f(x) \le b \end{cases} \end{array}.$$

Then $F(x,0) = x$, since $\phi_0(x) = x$, and

$$F(x,1) = \begin{cases} x & f(x) \le a \\ \phi_{f(x)-a}(x) = \gamma_x(f(x)-a) & a \le f(x) \le b \end{cases}.$$

In particular if $x \in \mathrm{S}_a$, then $F(x,1) = x$ and for all $x \in \mathrm{S}_b$, $F(x,1) \in \mathrm{S}_a$. □

Theorem 3.21 (Reeb's theorem)**.** *Let M be a compact manifold, and let $f : M \to \mathbb{R}$ be a Morse function. Assume that f admits exactly two critical points. Then M is homeomorphic to a sphere S^n.*

Proof. There exists a unique $x_{\min}$ such that $h_{\min} = f(x_{\min})$ is the minimum and there exists a unique $x_{\max}$ such that $h_{\max} = f(x_{\max})$ is the maximum. Both $x_{\min}$ and $x_{\max}$ are critical points. Then $\lambda(x_{\min}) = n$ and $\lambda(x_{\max}) = 0$. By the Morse lemma, around $x_{\min}$, we can write locally

$$f(x) = h_{\min} + \sum_{i=1}^n x_i^2,$$

for some local coordinates $x_1, \dots, x_n$ such that $x_{\min} = (0, \dots, 0)$. Let $a > h_{\min}$ be sufficiently close to $h_{\min}$. Then

$$\mathrm{S}_a = \left\{ h_{\min} + \sum_{i=1}^n x_i^2 \le a \right\} = \mathrm{D}_n.$$

Similarly there exists $b < h_{\max}$ sufficiently close such that $M \setminus \mathrm{S}_b \cong \mathrm{D}_n$. By Theorem 3.20, since there do not exist critical points in $f^{-1}([a,b])$ we know that $\mathrm{S}_b \cong \mathrm{S}_a \cong \mathrm{D}_n$. We proved that there exist

$$\phi_+ : \mathrm{D}_n^+ \xrightarrow{\sim} H_+, \qquad \phi_- : \mathrm{D}_n^- \xrightarrow{\sim} H_-,$$

where $H_- = \mathrm{S}_b$ and $H_+ = \overline{M \setminus \mathrm{S}_b}$, such that $M = H_+ \cup H_-$ and $\phi_+(\partial \mathrm{D}_n^+) = \phi_-(\partial \mathrm{D}_n^-) = H_+ \cap H_-$, so

$$\begin{tikzcd}
\mathrm{S}^{n-1} = \partial \mathrm{D}_n^+ \arrow[r, phantom, "\subset"] \arrow[drr, "\phi_+"'] & {} & \mathrm{D}_n^+ \arrow[r, "\phi_+"] & H_+ \arrow[dr, hook] & \\
& & H_+ \cap H_- \arrow[ur, hook] \arrow[dr, hook] & & H_+ \cup H_- = M \\
\mathrm{S}^{n-1} = \partial \mathrm{D}_n^- \arrow[r, phantom, "\subset"] \arrow[urr, "\phi_-"] & {} & \mathrm{D}_n^- \arrow[r, "\phi_-"'] & H_- \arrow[ur, hook] &
\end{tikzcd}.$$

The problem is that in general $\phi_+|_{\partial \mathrm{D}_n^+} \ne \phi_-|_{\partial \mathrm{D}_n^-}$. Let

$$f = \left(\phi_+^{-1} \circ \phi_-\right)\big|_{\mathrm{S}^{n-1}} : \partial \mathrm{D}_n^- \to \partial \mathrm{D}_n^+.$$

I want a homeomorphism $F : \mathrm{D}_n^+ \to \mathrm{D}_n^+$ such that $F|_{\partial \mathrm{D}_n^-} = f$. By taking $\phi_+ \circ F$ we obtain that M is obtained by attaching D_n^+ with D_n^- so that they coincide at the boundary with the identity on S^{n-1}, so M is homeomorphic to S^n. The goal is given a homeomorphism $f : \mathrm{S}^{n-1} \to \mathrm{S}^{n-1}$, there exists a homeomorphism $F : \mathrm{D}_n \to \mathrm{D}_n$ such that $F|_{\mathrm{S}^{n-1}} = f$. Indeed, if $v \in \mathbb{R}^n$ such that $|v| = 1$, then let $F(tv) = tf(v)$. We do the same with the inverse. ☐

Lecture 24 Tuesday 03/03/20

Lecture 24 is a problems class.

Lecture 25 Thursday 05/03/20

Theorem 3.22 (Second fundamental theorem of Morse theory)**.** *Let M be a manifold of dimension n, let $f : M \to \mathbb{R}$ be a Morse function, let $x_0 \in M$ be a critical point for f such that if $c = f(x_0)$ then there exists $\epsilon > 0$ such that $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains only one critical point, and let λ be the index of f at x_0. Then if ϵ is sufficiently small, $\mathrm{S}_{c+\epsilon}$ is homotopy equivalent to $\mathrm{S}_{c-\epsilon}$ attached to a λ-dimensional cell.*

Proof. If $\lambda = 0$, then x_0 is a local minimum for d. Around x_0, $\mathrm{S}_{c-\epsilon}$ is empty and $\mathrm{S}_{c+\epsilon}$ is homotopy equivalent to a point. Indeed it is a ball. There exists $U \ni x_0$ such that outside U, $f|_{M \setminus \overline{U}} : M \setminus \overline{U} \to \mathbb{R}$ does not contain any critical points. By the first fundamental theorem $\mathrm{S}_{c-\epsilon} \setminus \overline{U} \cong \mathrm{S}_{c+\epsilon} \setminus \overline{U}$. If $\lambda = n$, then x_0 is a local maximum. Let $U \ni x_0$ be a ball D_n^+, by the Morse lemma. Like in the proof of Reeb's theorem, $\mathrm{S}_{c+\epsilon}$ is homotopy equivalent to $\mathrm{S}_{c-\epsilon}$ attached to D_n^+, so ok. Let $1 \le \lambda \le n-1$. We apply the Morse lemma. There exists $U \ni x_0$ such that on U, there exist coordinates $x_1, \dots, x_n$ such that $x_0 = (0, \dots, 0)$ and

$$f(x) = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2.$$

We just need to study $\mathrm{S}_{c-\epsilon} \cap U$ and $\mathrm{S}_{c+\epsilon} \cup U$. Define

$$B_{\sqrt{2\epsilon}} = \left\{ (x_1, \dots, x_n) \;\middle|\; \sum_{i=1}^{n} x_i^2 \le 2\epsilon \right\},$$

so $B_{\sqrt{2\epsilon}} \subset U$ if ϵ is sufficiently small, and

$$e_\lambda = \left\{ (x_1, \dots, x_n) \;\middle|\; \sum_{i=1}^{\lambda} x_i^2 \le \epsilon, \ x_{\lambda+1} = \dots = x_n = 0 \right\} \cong \mathrm{D}_\lambda.$$

An easy case is $n = 2$ and $\lambda = 1$. Then

$$U \cap f^{-1}(c) = \left\{ x_1^2 - x_2^2 = 0 \right\}, \qquad U \cap e_1 = \left\{ x_1^2 \le \epsilon, \ x_2 = 0 \right\},$$

$$U \cap \mathrm{S}_{c-\epsilon} = \left\{ x_1^2 - x_2^2 \ge \epsilon \right\}, \qquad U \cap \mathrm{S}_{c+\epsilon} = \left\{ x_1^2 - x_2^2 \ge -\epsilon \right\}.$$

We want to perturb f, and obtain $g \leq f$ with the same critical value, and $g = f$ outside U. We define

$$\begin{array}{rccl} \mu & : & \mathbb{R}_{\geq 0} & \longrightarrow & \mathbb{R}_{\geq 0} \\ & & t & \longmapsto & \begin{cases} > \epsilon & t = 0 \\ = 0 & t \geq 2\epsilon \end{cases} \end{array},$$

such that $-1 < \mu'(t) \leq 0$. We define

$$g = \begin{cases} f & \text{outside } U \\ f - \mu(\xi + 2\eta) & \text{in } U \end{cases}, \qquad \xi = x_1^2 + \dots + x_\lambda^2, \qquad \eta = x_{\lambda+1}^2 + \dots + x_n^2.$$

Then $g \leq f$, since $f(x) = f(x_0) - \xi + \eta$, so $g(x) = f(x_0) - \xi + \eta - \mu(\xi + 2\eta)$ inside U, and $g = f$ in the ellipsoid

$$E = \{\xi + 2\eta \leq 2\epsilon\} \subset B_{\sqrt{2\epsilon}} \subset U.$$

In E, $\eta - \xi \leq \frac{1}{2}(\xi + 2\eta) \leq \epsilon$, so $E \subset \mathrm{S}_{c+\epsilon} = \{\eta - \xi \leq \epsilon\}$. Lecture 26 Monday 09/03/20

1. $\mathrm{S}_{c+\epsilon} = g^{-1}((-\infty, c+\epsilon])$. If $x \in f^{-1}((-\infty, c+\epsilon])$, then $x \in g^{-1}((-\infty, c+\epsilon])$. Vice versa, assume that $f(x) > c+\epsilon$, so $x \notin E$, so $g(x) = f(x)$ by definition of g, so $g(x) > c+\epsilon$, so $x \notin g^{-1}((-\infty, c+\epsilon])$.
2. f and g have the same critical points. Since $\mu' \in (-1, 0]$, $\frac{\mathrm{d}g}{\mathrm{d}\xi} = -1 - \mu'(\xi + 2\eta) < 0$ and $\frac{\mathrm{d}g}{\mathrm{d}\eta} = 1 - 2\mu'(\xi + 2\eta) > 1$, so $0 = \mathrm{d}g = \frac{\mathrm{d}g}{\mathrm{d}\xi}\mathrm{d}\xi + \frac{\mathrm{d}g}{\mathrm{d}\eta}\mathrm{d}\eta$, so $\mathrm{d}\xi = \mathrm{d}\eta = 0$, so $\xi = \eta = 0$, so $x = x_0$.
3. $g^{-1}((-\infty, c-\epsilon])$ is a deformation retract of $\mathrm{S}_{c+\epsilon}$. By the first fundamental theorem of Morse theory and 1, we just need to check there do not exist critical values in $[c-\epsilon, c+\epsilon]$. By 2 the only critical point is x_0. Since $\mu(0) > \epsilon$, $g(x_0) = f(x_0) - \mu(0) < c - \epsilon$, so done.
4. $\mathrm{S}_{c-\epsilon} \cup e_\lambda$ is a deformation retract of $g^{-1}((-\infty, c-\epsilon])$. Let H be the closure of $g^{-1}((-\infty, c-\epsilon]) \setminus \mathrm{S}_{c-\epsilon}$. Claim that $e_\lambda \subset H$. Since $\frac{\mathrm{d}g}{\mathrm{d}\xi} < 0$, $g(x) \leq g(x_0)$ for all $x \in e_\lambda$, so $g(x) < c - \epsilon$, but $f(x) = c - \xi + \eta > c - \epsilon$, so $x \notin \mathrm{S}_{c-\epsilon}$ for all $x \in e_\lambda$, so $e_\lambda \subset H$.

   Case 1. Let $\xi < \epsilon$. Then
   $$r_t(x_1, \dots, x_n) = (x_1, \dots, x_\lambda, tx_{\lambda+1}, \dots, tx_n).$$
   If $t = 1$, then $r_1 = \mathrm{id}$. If $t = 0$, then the image is e_λ. Since g is decreasing $g^{-1}((-\infty, c-\epsilon])$ maps to itself.

   Case 2. Let $\epsilon \leq \xi \leq \eta + \epsilon$. Then
   $$r_t(x_1, \dots, x_n) = (x_1, \dots, x_\lambda, l_t x_{\lambda+1}, \dots, l_t x_n), \qquad l_t = t + (1-t)\sqrt{\frac{\xi - \epsilon}{\eta}},$$
   so l_t is continuous in $t \in (0, 1)$. If $t = 1$, then $r_1 = \mathrm{id}$. If $t = 0$, then r_0 maps everything to $\mathrm{S}_{c-\epsilon} = \{f \leq c - \epsilon\} = \{\eta - \xi \geq \epsilon\}$. For all t, $r_t(g^{-1}((-\infty, c-\epsilon])) \subset g^{-1}((-\infty, c-\epsilon])$.

   Case 3. Let $\xi > \eta + \epsilon$, so $x \in \mathrm{S}_{c-\epsilon}$. Then $r_t = \mathrm{id}$ for all t.

   Check that the three retractions coincide at the border of each region.

Thus 3 and 4 imply Theorem 3.22. ☐

Remark 3.23. Let M be a manifold, and let $f : M \to \mathbb{R}$ be a Morse function. Assume that $f^{-1}([c-\epsilon, c+\epsilon])$ is compact. Let $x_1, \dots, x_k$ be the critical points. Assume that $f(x_i) = c$. Then if ϵ is small enough $\mathrm{S}_{c+\epsilon}$ retracts to $\mathrm{S}_{c-\epsilon}$ attached to $e_{\lambda_1}, \dots, e_{\lambda_k}$ where λ_i is the index of x_i.

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The goal is the following.

Theorem 3.24. *Let M be a manifold, and let $f : M \to \mathbb{R}$ be a Morse function such that for all $h \in \mathbb{R}$, $\mathrm{S}_h = f^{-1}((-\infty, h])$ is compact. Then M is a CW-complex obtained by attaching a λ-cell for each critical point of index λ.*

Definition 3.25. Let X and Y be CW-complexes, and let $f : X \to Y$ be continuous. Then f is **cellular** if $f\left(X^{(n)}\right) \subset Y^{(n)}$ for all n.

Theorem 3.26 (Cellular approximation). *Let $f : X \to Y$ be continuous where X and Y are CW-complexes, and let $S \subset X$ be a subcomplex such that $f|_S$ is cellular. Then there exists a cellular $\widetilde{f} : X \to Y$ which is homotopic equivalent to f and such that $\left.\widetilde{f}\right|_S = f|_S$.*

The idea is to work on induction on n.

Theorem 3.27 (Whitehead). *Let X be a topological space, and let $f_1, f_2 : \partial\mathrm{D}_n \to X$ be continuous such that $f_1 \sim f_2$. Then $X \cup_{f_1} \mathrm{D}_n \sim X \cup_{f_2} \mathrm{D}_n$.*

Theorem 3.28 (Hilton). *Let X and Y be topological spaces, let $f : \partial\mathrm{D}_n \to X$ be continuous, and let $h : X \to Y$ be a homotopy equivalence. Then there exists a homotopy equivalence $H : X \cup_f \mathrm{D}_n \to Y \cup_{h \circ f} \mathrm{D}_n$ for $h \circ f : \partial\mathrm{D}_n \to Y$.*

Proof of Theorem 3.24. Let $c_0, c_1, \dots$ be critical values of f such that $c_0 < c_1 < \dots$. For all h there exist only finitely many critical points inside S_h because it is compact. We proceed by induction. Claim that for any $i \geq 0$ there exists $\epsilon_i > 0$ such that $\mathrm{S}_{c_i+\epsilon_i}$ is homotopy equivalent to a CW-complex.

- $i = 0$. $c_0 = \min f$, because by assumption f admits a minimum. There exist $x_1, \dots, x_m \in M$ such that $f(x_i) = c_0$. For all i, $\lambda(x_i) = 0$. If ϵ_0 is small enough, then $\mathrm{S}_{c_0+\epsilon_0}$ is a union of balls around $x_1, \dots, x_m$. Each ball is homotopy equivalent to a point, so $\mathrm{S}_{c_0+\epsilon_0} \sim \{m \text{ points}\}$.
- $i > 0$. Fix c_i. There exists $\epsilon_i > 0$ such that there do not exist critical values in $(c_i - \epsilon_i, c_i + \epsilon_i)$. Let $x_1, \dots, x_m \in M$ be such that $f(x_j) = c_j$ for all j. By the second fundamental theorem of Morse theory, if ϵ_i is sufficiently small then $\mathrm{S}_{c_i+\epsilon_i}$ is homotopy equivalent to $\mathrm{S}_{c_i-\epsilon_i}$ attached to a λ_j-cell for all $j = 1, \dots, m$, where λ_j is the index of f at x_j. By induction, there exists $\epsilon_{i-1} > 0$ such that $\mathrm{S}_{c_{i-1}+\epsilon_{i-1}}$ is homotopy equivalent to a CW-complex. There do not exist critical values between $c_{i-1} + \epsilon_{i-1}$ and $c_{i-1} - \epsilon_{i-1}$. By the first fundamental theorem of Morse theory, $\mathrm{S}_{c_i-\epsilon_i} \sim \mathrm{S}_{c_{i-1}+\epsilon_{i-1}}$. For each critical point x_j,
$$\partial\mathrm{D}_{\lambda_j} \to \mathrm{S}_{c_i-\epsilon_i} \xrightarrow{\sim} \mathrm{S}_{c_{i-1}+\epsilon_{i-1}} \to \mathrm{S}_{c_i+\epsilon_i}.$$
By the Hilton and Whitehead theorems, we may assume that the attachment does not depend on $\partial\mathrm{D}_{\lambda_j}$ and $\mathrm{S}_{c_i-\epsilon_i}$, so $\mathrm{S}_{c_i+\epsilon_i}$ is a CW-complex.

□

3.5 Morse homology

Let M be a manifold, let g be a Riemannian metric, and let $f : M \to \mathbb{R}$ be a Morse function. There is a **Morse homology** depending on g and f, equivalent to **singular homology**, and **de Rham homology**, which does not depend on g and f and is a finite-dimensional version of **Floer homology**. We have a gradient flow. We showed that for all $x \in M$, $\gamma_x(t)$ converges to a critical point for $t \to \infty$ and $t \to -\infty$. Let c be a fixed critical point, and let

$$\mathrm{W}^{\mathrm{s}}(c) = \left\{x \;\middle|\; \lim_{t \to \infty} \gamma_x(t) = c\right\}, \qquad \mathrm{W}^{\mathrm{u}}(c) = \left\{x \;\middle|\; \lim_{t \to -\infty} \gamma_x(t) = c\right\}.$$

Then
$$M = \bigsqcup_c \mathrm{W}^{\mathrm{s}}(c) = \bigsqcup_c \mathrm{W}^{\mathrm{u}}(c),$$
and $\mathrm{W}^{\mathrm{s}}(c)$ and $\mathrm{W}^{\mathrm{u}}(c)$ are homotopy equivalent to balls. Choose two critical points c_1 and c_2 of f such that $\lambda(c_i) = \lambda(c_{i-1}) + 1$. We want that there exist only finitely many lines from c_i to c_{i-1}. This property is called **Morse-Smale**. Let C_k be the free abelian group generated by all the critical points of index k, and let $\partial_k : \mathrm{C}_k \to \mathrm{C}_{k-1}$. Then $\partial_{k-1} \circ \partial_k = 0$. Thus the **Morse homology group** is

$$\mathrm{H}_k(M, f, g) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

4 Singular homology

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Morse homology is isomorphic to singular homology and de Rham cohomology is isomorphic to singular cohomology.

Definition 4.1. An n**-simplex** is

$$\Delta_n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \right\}.$$

The i**-th face** of Δ_n is $\mathrm{F}_i = [e_0, \dots, \widehat{e_i}, \dots, e_n] : \Delta_{n-1} \to \Delta_n$.

Example 4.2. $\partial \Delta_2 = [e_0, e_1] \cup [e_1, e_2] \cup [e_0, e_2]$.

Definition 4.3. Let X be a topological space. An n**-singular simplex** in X is a continuous map $\sigma : \Delta_n \to X$. The i**-th face** of this simplex is $\sigma \circ \mathrm{F}_i : \Delta_{n-1} \to \Delta_n \to X$. The **border map** is

$$\partial \sigma = \partial_n \sigma = \sum_{i=0}^{n} (-1)^i (\sigma \circ \mathrm{F}_i),$$

where σ are $(n-1)$-simplices.

4.1 Singular homology

Definition 4.4. For each $p \geq 0$,

$$\mathrm{C}_p(X) = \mathbb{Z}\{\sigma\ p\text{-singular}\} = \left\{ \sum_\sigma a_\sigma \sigma \text{ finite sum} \;\middle|\; a_\sigma \in \mathbb{Z},\ \sigma : \Delta_p \to X \text{ continuous} \right\}$$

is the free abelian group generated by all the p-simplices on X. Then ∂ induces a linear map

$$\begin{array}{cccc} \partial_p & : & \mathrm{C}_p(X) & \longrightarrow & \mathrm{C}_{p-1}(X) \\ & & \sum_\sigma a_\sigma \sigma & \longmapsto & \sum_\sigma a_\sigma \partial \sigma \end{array}.$$

Lemma 4.5.

$$\begin{array}{cccc} \partial_{p-1} \circ \partial_p & : & \mathrm{C}_p(X) & \longrightarrow & \mathrm{C}_{p-2}(X) \\ & & \sigma & \longmapsto & 0 \end{array}, \qquad p \geq 0.$$

Proof. We need to check that for all p-simplices σ,

$$\begin{aligned} \partial_{p-1} \partial_p \sigma &= \sum_{i<j} (-1)^i (-1)^{j-i} \sigma([e_0, \dots, \widehat{e_i}, \dots, \widehat{e_j}, \dots, e_p]) + \sum_{i<j} (-1)^i (-1)^j \sigma([e_0, \dots, \widehat{e_i}, \dots, \widehat{e_j}, \dots, e_p]) \\ &= 0. \end{aligned}$$

□

Let

$$\mathcal{Z}_p(X) = \ker(\partial_p : \mathrm{C}_p(X) \to \mathrm{C}_{p-1}(X)), \qquad \mathcal{B}_p(X) = \mathrm{im}(\partial_{p+1} : \mathrm{C}_{p+1}(X) \to \mathrm{C}_p(X)).$$

Then the p**-th singular homology group** is

$$\mathrm{H}_p(X) = \mathcal{Z}_p(X) / \mathcal{B}_p(X).$$

Exercise. If X and Y are homeomorphic then

$$\mathrm{H}_p(X) \cong \mathrm{H}_p(Y).$$

Example 4.6. Let $X = \{\text{point}\}$. For all $\sigma : \Delta_p \to X$, σ is constant. Then

$$\mathrm{C}_p(X) = \mathbb{Z} \cdot X = \mathbb{Z}, \qquad p \geq 0$$

is the free abelian group generated by a point, and

$$\begin{array}{cccc} \partial_p & : & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & & \sigma & \longmapsto & \sum_{i=0}^{p} (-1)^i \sigma = \begin{cases} 0 & p \text{ is odd} \\ 1 & p \text{ is even} \end{cases} \end{array},$$

so

$$\dots \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

Thus

$$\mathrm{H}_p(X) = \begin{cases} \mathbb{Z}/0 & p = 0 \\ 0/0 & p > 0 \text{ is even} \\ \mathbb{Z}/\mathbb{Z} & p \text{ is odd} \end{cases} = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0 \end{cases}.$$

Exercise. If $X = X_1 \sqcup \dots \sqcup X_k$ are the connected components then

$$\mathrm{H}_p(X) = \bigoplus_{i=1}^{k} \mathrm{H}_p(X_i).$$

Exercise. Let $f : X \to Y$ be a continuous map, and let $\sigma \in \mathrm{C}_p(X)$. Then

$$f_* \sigma = f \circ \sigma : \Delta_p \to Y.$$

By linearity, there exists $f_* : \mathrm{C}_p(X) \to \mathrm{C}_p(Y)$. Then check that $f_*(\mathcal{Z}_p(X)) \subset \mathcal{Z}_p(Y)$ and $f_*(\mathcal{B}_p(X)) \subset \mathcal{B}_p(Y)$, so f_* induces a linear map in homology,

$$f_* : \mathrm{H}_p(X) \to \mathrm{H}_p(Y), \qquad p \geq 0.$$

4.2 Singular cohomology

We define

$$\mathrm{C}^p(X, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathrm{C}_p(X), \mathbb{R}) = \{\mathbb{R}\text{-linear maps } \mathrm{C}_p(X) \to \mathbb{R}\}, \qquad p \geq 0.$$

We want to define

$$\begin{array}{cccc} \partial^p & : & \mathrm{C}^p(X, \mathbb{R}) & \longrightarrow & \mathrm{C}^{p+1}(X, \mathbb{R}) \\ & & \phi & \longmapsto & \phi \circ \partial_{p+1} : \mathrm{C}_{p+1}(X) \to \mathrm{C}_p(X) \to \mathbb{R} \end{array}.$$

Exercise. $\partial^{p+1} \circ \partial^p = 0$.

For all p, there is a chain complex

$$\mathrm{C}^{p-1}(X, \mathbb{R}) \xrightarrow{\partial^{p-1}} \mathrm{C}^p(X, \mathbb{R}) \xrightarrow{\partial^p} \mathrm{C}^{p+1}(X, \mathbb{R}).$$

Let

$$\mathcal{Z}^p(X, \mathbb{R}) = \ker\left(\partial^p : \mathrm{C}^p(X, \mathbb{R}) \to \mathrm{C}^{p+1}(X, \mathbb{R})\right), \qquad \mathcal{B}^p(X, \mathbb{R}) = \mathrm{im}\left(\partial^{p-1} : \mathrm{C}^{p-1}(X, \mathbb{R}) \to \mathrm{C}^p(X, \mathbb{R})\right).$$

We can define the **p-th singular cohomology group** of X,

$$\mathrm{H}^p(X, \mathbb{R}) = \mathcal{Z}^p(X, \mathbb{R}) / \mathcal{B}^p(X, \mathbb{R}).$$

Exercise. Let $X = \{\text{point}\}$. Then

$$\mathrm{H}^p(X) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}.$$

Let $f : X \to Y$ be a continuous map. The **pull-back map** is defined by

$$\begin{array}{rccc} f^* & : & \mathrm{C}^p(Y,\mathbb{R}) & \longrightarrow & \mathrm{C}^p(X,\mathbb{R}) \\ & & \phi & \longmapsto & f^*\phi \end{array}, \qquad \begin{array}{rccc} f^*\phi & : & \mathrm{C}_p(X) & \longrightarrow & \mathrm{C}_p(Y) \\ & & \sigma & \longmapsto & \phi(f_*\sigma) \end{array}.$$

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If M is a manifold, then

$$\mathrm{C}_p(X) = \mathbb{Z}\{\sigma : \Delta_p \to M \text{ smooth}\}.$$

If $M \cong N$, then

$$\mathrm{H}^p(M,\mathbb{R}) = \mathrm{H}^p(N,\mathbb{R}).$$

Theorem 4.7. *Let $f : M \to N$ be a smooth morphism between manifolds such that f is a homotopy equivalence. Then*

$$f^* : \mathrm{H}^p(N,\mathbb{R}) \xrightarrow{\sim} \mathrm{H}^p(M,\mathbb{R}), \qquad p \geq 0.$$

Example. If M is contractible, then

$$\mathrm{H}^p(M,\mathbb{R}) = \mathrm{H}^p(\{\text{point}\},\mathbb{R}) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}.$$

Theorem 4.8. *Let $M = U \cup V$ be a manifold for U and V open in M such that $U \cap V = \emptyset$. Then there exists $\delta : \mathrm{H}^p(U \cap V,\mathbb{R}) \to \mathrm{H}^{p+1}(M,\mathbb{R})$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}^p(M,\mathbb{R}) \arrow[r] & \mathrm{H}^p(U,\mathbb{R}) \oplus \mathrm{H}^p(V,\mathbb{R}) \arrow[r] & \mathrm{H}^p(U \cap V,\mathbb{R}) \arrow[dll, "\delta"] \\ & \mathrm{H}^{p+1}(M,\mathbb{R}) \arrow[r] & \mathrm{H}^{p+1}(U,\mathbb{R}) \oplus \mathrm{H}^{p+1}(V,\mathbb{R}) \arrow[r] & \mathrm{H}^{p+1}(U \cap V,\mathbb{R}) \arrow[r] & \dots \end{tikzcd}$$

is exact.

4.3 De Rham homomorphism

Let M be a manifold, let $\sigma : \Delta_p \to M$ be a smooth simplex, and let $\omega \in \Omega^p(M)$. Then

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^*\omega.$$

By the linearity we can extend this to $\mathrm{C}_p(X)$. Given ω, we define

$$\begin{array}{rccc} \int \omega & : & \mathrm{C}_p(X) & \longrightarrow & \mathbb{R} \\ & & \sum_\sigma n_\sigma \sigma & \longmapsto & \sum_\sigma n_\sigma \int_\sigma \omega \end{array}.$$

By definition, $\int \omega$ is linear.

Theorem 4.9 (Stokes' theorem)**.** *Let ω be a $(p-1)$-form on M, and let $c \in \mathrm{C}_p(X)$. Then*

$$\int_{\partial c} \omega = \int_c \mathrm{d}\omega,$$

where $\mathrm{d}\omega$ is a p-form and ∂c is considered with orientation.

Let M be a manifold, and let $\omega \in \Omega^p(M)$. Then $\int \omega = \left(c \mapsto \int_c \omega\right) \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{C}_p(M),\mathbb{R}) = \mathrm{C}^p(M,\mathbb{R})$, so

$$\begin{array}{rccc} \mathrm{l}^p & : & \Omega^p(M) & \longrightarrow & \mathrm{C}^p(M,\mathbb{R}) \\ & & \omega & \longmapsto & \int \omega \end{array}$$

is an $\mathbb{R}$-linear map such that $\mathrm{l}^{p+1} \circ \mathrm{d}^p = \partial^p \circ \mathrm{l}^p$, so

$$\begin{tikzcd} \Omega^p(M) \arrow[r, "\mathrm{d}^p"] \arrow[d, "\mathrm{l}^p"'] & \Omega^{p+1}(M) \arrow[d, "\mathrm{l}^{p+1}"] \\ \Omega^p(M,\mathbb{R}) \arrow[r, "\partial^p"'] & \mathrm{C}^{p+1}(M,\mathbb{R}) \end{tikzcd}.$$

Exercise. l^p induces a map $\mathrm{H}^p(M) \to \mathrm{H}^p(M,\mathbb{R})$.

The goal is to show that it is an isomorphism.

Lemma 4.10. *Let $M \subset \mathbb{R}^n$ be an open contractible subset. Then*

$$\mathrm{I}^p : \mathrm{H}^p(M) \to \mathrm{H}^p(M, \mathbb{R}), \qquad p \geq 0$$

is an isomorphism.

Proof. We know that

$$\mathrm{H}^p(M) = \mathrm{H}^p(M, \mathbb{R}) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}.$$

We need to show that $\mathrm{I}^0 : \mathrm{H}^0(M) = \mathbb{R} \to \mathrm{H}^0(M, \mathbb{R}) = \mathbb{R}$ is an isomorphism. Then $\mathrm{H}^0(M)$ is the set of constant functions in $\mathbb{R}$. Take $a \neq 0$. There exists σ such that $\int_\sigma a \neq 0$, so I^0 is not zero. Thus I^0 is surjective, so I^0 is an isomorphism. ∎

Theorem 4.11. *If M is a compact manifold then*

$$\mathrm{I}^p : \mathrm{H}^p(M) \to \mathrm{H}^p(M, \mathbb{R})$$

is an isomorphism.

Proof. Very similar to Poincaré duality. The idea is that M has a finite good cover. There exists $\{U_i\}_{i \in I}$ such that I is finite and for all $i_1 < \dots < i_l$ we have that $U_{i_1} \cap \dots \cap U_{i_l}$ is $\emptyset$ or contractible. We proceed by induction on the number of elements of I. If $\#I = 1$, then Theorem 4.11 follows from Lemma 4.10. If $\#I > 1$, then let

$$U = U_1, \qquad V = \bigcup_{i \neq 1} U_i.$$

Then $U \cup V = M$ and by induction I^p is an isomorphism on U and V, that is $\mathrm{I}^p : \mathrm{H}^p(U) \xrightarrow{\sim} \mathrm{H}^p(U, \mathbb{R})$ and $\mathrm{I}^p : \mathrm{H}^p(V) \xrightarrow{\sim} \mathrm{H}^p(V, \mathbb{R})$. By Mayer-Vietoris both for $\mathrm{H}^p(M)$ and for $\mathrm{H}^p(M, \mathbb{R})$,

$$\begin{tikzcd} \mathrm{H}^{p-1}(U) \oplus \mathrm{H}^{p-1}(V) \arrow[r] \arrow[d, "\mathrm{I}^{p-1}"] & \mathrm{H}^{p-1}(U \cap V) \arrow[r] \arrow[d, "\mathrm{I}^{p-1}"] & \mathrm{H}^p(M) \arrow[r] \arrow[d, "\mathrm{I}^p"] & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \arrow[r] \arrow[d, "\mathrm{I}^p"] & \mathrm{H}^p(U \cap V) \arrow[d, "\mathrm{I}^p"] \\ \mathrm{H}^{p-1}(U, \mathbb{R}) \oplus \mathrm{H}^{p-1}(V, \mathbb{R}) \arrow[r] & \mathrm{H}^{p-1}(U \cap V, \mathbb{R}) \arrow[r] & \mathrm{H}^p(M, \mathbb{R}) \arrow[r] & \mathrm{H}^p(U, \mathbb{R}) \oplus \mathrm{H}^p(V, \mathbb{R}) \arrow[r] & \mathrm{H}^p(U \cap V, \mathbb{R}) \end{tikzcd}.$$

Apply the five lemma. ∎

Lecture 30 is a problems class. Lecture 30 Tuesday 17/03/20