

M4P57 Complex Manifolds

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$$\begin{array}{ccc}
 \bigoplus_{p+q=k} H_{\text{BC}}^{p,q}(X) & := & \bigoplus_{p+q=k} \frac{\ker \left(C^\infty(X, \Omega_X^{p,q}) \xrightarrow{d} C^\infty(X, \Omega_X^{p+1,q+1}) \right)}{\text{im} \left(C^\infty(X, \Omega_X^{p-1,q-1}) \xrightarrow{\partial \bar{\partial}} C^\infty(X, \Omega_X^{p,q}) \right)} \\
 \downarrow 6.39 \sim & & \\
 \bigoplus_{p+q=k} H_D^{p,q}(X) & := & \bigoplus_{p+q=k} \frac{\ker \left(C^\infty(X, \Omega_X^{p,q}) \xrightarrow{\bar{\partial}} C^\infty(X, \Omega_X^{p,q+1}) \right)}{\text{im} \left(C^\infty(X, \Omega_X^{p,q-1}) \xrightarrow{\bar{\partial}} C^\infty(X, \Omega_X^{p,q}) \right)} \\
 \downarrow 6.22 \sim & & \\
 \bigoplus_{p+q=k} \mathcal{H}_{\Delta \bar{\partial}}^{p,q}(X) & := & \bigoplus_{p+q=k} \ker \left(C^\infty(X, \Omega_X^{p,q}) \xrightarrow{\Delta \bar{\partial}} C^\infty(X, \Omega_X^{p,q}) \right) \\
 \downarrow 6.35 \sim & & \\
 \mathcal{H}_\Delta^k(X) & := & \ker \left(C^\infty(X, \Omega_{X,\mathbb{C}}^k) \xrightarrow{\Delta} C^\infty(X, \Omega_{X,\mathbb{C}}^k) \right) \\
 \downarrow 6.18 \sim & & \\
 H_{\text{dR}}^k(X, \mathbb{C}) & := & \frac{\ker \left(C^\infty(X, \Omega_{X,\mathbb{C}}^k) \xrightarrow{d} C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1}) \right)}{\text{im} \left(C^\infty(X, \Omega_{X,\mathbb{C}}^{k-1}) \xrightarrow{d} C^\infty(X, \Omega_{X,\mathbb{C}}^k) \right)} \\
 \downarrow 6.48 \sim & & \\
 \bigoplus_{r \geq 0} L^r H_{\text{dR}}^{k-2r}(X, \mathbb{C})_{\text{prim}} & := & \bigoplus_{r \geq 0} L^r \ker \left(H_{\text{dR}}^{k-2r}(X, \mathbb{C}) \xrightarrow{L^{n-(k-2r)+1}} H_{\text{dR}}^{2n-(k-2r)+2}(X, \mathbb{C}) \right)
 \end{array}$$

Syllabus

Holomorphic functions and differential forms. Complex manifolds and submanifolds. Holomorphic vector bundles and the tautological line bundle. Real and complex tangent bundles. Dolbeault cohomology. Connections and curvature operators. Hermitian metrics and vector bundles. Chern connections and curvature. The first Chern class. De Rham cohomology. Kähler forms and manifolds. Hodge \star and Hodge-de Rham harmonic operators. Lefschetz operators and Kähler identities. Hodge decomposition. Bott-Chern cohomology. Lefschetz decomposition.

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1 Introduction

Lecture 1
Thursday
09/01/20

The following are references.

- O Biquard and A Höring, Kähler geometry and Hodge theory, 2008.
- J P Demailly, Complex analytic and differential geometry, 2012.
- C Voisin, Hodge theory and complex algebraic geometry, 2002.
- R O Wells, Differential analysis on complex manifolds, 1973.
- A Gathmann, Algebraic geometry, 2002
- P Griffiths and J Harris, Principles of algebraic geometry, 1978.

Complex manifolds are manifolds over \mathbb{C}^n .

Example 1.1. \mathbb{C}^1 is a complex manifold. Any open $U \subset \mathbb{C}^n$ is a complex manifold.

Example 1.2. The **sphere** $S^2 \subset \mathbb{R}^3$ is a complex manifold by $S^2 \cong \mathbb{C} \cup \{\infty\} = \mathbb{P}_{\mathbb{C}}^1 = \mathbb{CP}^1$. More in general $\mathbb{P}_{\mathbb{C}}^n$ is a complex manifold for all n .

Example 1.3. The **torus** $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/\mathbb{Z}^2$ is a complex manifold. More in general a $2n$ -dimensional torus \mathbb{C}^n/Λ for a lattice $\Lambda \cong \mathbb{Z}^{2n}$ is a complex manifold.

Example 1.4. Compact Riemannian surfaces of genus $g > 1$, called **hyperbolics**, are all complex manifolds.

Example 1.5. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. The graph of f ,

$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C},$$

is a complex manifold. From Γ_f we can recover f , by

$$f(x) = q(p^{-1}(x) \cap \Gamma_f),$$

where $p, q : \mathbb{C}^2 \rightarrow \mathbb{C}$ are the projections to the first and second factors. This allows us to define f^{-1} . Assume f is bijective. Define

$$\begin{array}{ccc} \tau & : & \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \\ & & (x, y) \longmapsto (y, x) \end{array}$$

Define

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then f^{-1} is the function induced by $\Gamma_{f^{-1}}$. This makes sense even if f is not bijective. Then we get a multivalued function, such as $\log z$ as the inverse of $\exp z$.

Example 1.6. Generalising Example 1.5, we can consider two complex manifolds M and N and we can consider holomorphisms $f : M \rightarrow N$. Given M ,

$$\text{Aut } M = \{f : M \rightarrow M \text{ holomorphic bijective and } f^{-1} \text{ holomorphic}\}.$$

If $M = \mathbb{C}$, there are lots of \mathbb{C}^∞ -functions $\mathbb{C} \rightarrow \mathbb{C}$ but the automorphisms of \mathbb{C} are just affine linear maps. If $M = \mathbb{C}/\mathbb{Z}^2$, then $\text{Aut } M$ is interesting.

Example 1.7. Algebraic geometry is the zeroes of polynomials. That is, fix m , and take polynomials f_1, \dots, f_k in m variables. Define

$$M = \{(x_1, \dots, x_m) \in \mathbb{C}^m \mid f_1(x_1, \dots, x_m) = \dots = f_k(x_1, \dots, x_m) = 0\}.$$

Then M is called an **algebraic variety**. If M is smooth then M is a complex manifold. Fix m , take homogeneous polynomials F_1, \dots, F_k in $m+1$ variables, where F is **homogeneous** if it is the sum of monomials of the same degree. Consider

$$N = \{(x_0, \dots, x_m) \in \mathbb{P}_{\mathbb{C}}^m \mid F_1(x_0, \dots, x_m) = \dots = F_k(x_0, \dots, x_m) = 0\}.$$

Then N is called a **projective variety**. If N is smooth then N is a complex manifold.

The idea is if M is a differentiable manifold, then M contains lots of submanifolds N . This is not true for complex manifolds. There exist complex manifolds without any proper complex submanifolds, which is not true for projective varieties. The following are questions.

- What can we say about the topology of complex manifolds? For example, what is $\pi_1(M)$? What is the cohomology of M ?
- Assume that M and N are complex manifolds which are diffeomorphic. Are they also isomorphic, so there exists a biholomorphism $M \rightarrow N$?

What is next?

- Hodge decomposition theorem. Understand the cohomology of M by using the complex structure.
- Kodaira embedding theorem. Understand when a compact complex manifold is projective.

Note. If $M \subset \mathbb{P}_{\mathbb{C}}^m$ is a compact complex manifold then M is projective.

Example. Let $M = \Gamma_{\exp}$ for $\exp : \mathbb{C} \rightarrow \mathbb{C}$. Then $M \subset \mathbb{C}^2$ but it is not algebraic.

2 Local theory

2.1 Holomorphic functions in several variables

Notation 2.1. Given $z_0 \in \mathbb{C}$ and $r > 0$, the **disc** is

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

and $\partial D(z_0, r)$ is the boundary of $D(z_0, r)$.

Definition 2.2. Let $U \subset \mathbb{C}$, and let $f : U \rightarrow \mathbb{C}$ be a function. Then f is **holomorphic at** $z_0 \in U$ if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Theorem 2.3 (Cauchy). Let $U \subset \mathbb{C}$ be open, let f be holomorphic on U , and let $z_0 \in U$. Assume that if $D = D(z_0, r) \subset U$ then $\overline{D} \subset U$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

Notation 2.4. Fix $z_0 = (z_{01}, \dots, z_{0n}) \in \mathbb{C}^n$ and $R = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$. Then the **polydisc of polyradius** R is

$$D(z_0, R) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid \forall i, |z_i - z_{0i}| < r_i\}.$$

Definition 2.5. Let $U \subset \mathbb{C}^n$ be open, let $f : U \rightarrow \mathbb{C}$ be a continuous function, and let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then f is **holomorphic at** z , if assuming that $D = D(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ then

$$f(z_1, \dots, z_{i-1}, \cdot, z_{i+1}, \dots, z_n) : D(z_i, r_i) \rightarrow \mathbb{C}$$

is holomorphic for all i .

Example 2.6. Any convergent power series in n variables is holomorphic.

The opposite is also true.

Theorem 2.7 (Cauchy). Let $U \subset \mathbb{C}^n$ be an open set, let $f : U \rightarrow \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Assume that if $D = D(z, R)$ for some $R = (r_1, \dots, r_n)$ then $\overline{D} \subset U$. If $z' = (z'_1, \dots, z'_n) \in D$ then

$$f(z') = \frac{1}{(2\pi i)^n} \int_{\partial D(z_1, r_1)} \dots \int_{\partial D(z_n, r_n)} \frac{f(z)}{(z - z'_1) \dots (z - z'_n)} dz_n \dots dz_1.$$

Proof. Use induction on n and Cauchy theorem at each step. □

Corollary 2.8. Let $U \subset \mathbb{C}^n$ be open, let $f : U \rightarrow \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Then there exists $D = D(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ and there exists

$$p(w) = \sum_{m_1, \dots, m_n \geq 0} a_{m_1, \dots, m_n} (w_1 - z_1)^{m_1} \dots (w_n - z_n)^{m_n},$$

such that p is convergent on D and $f(w) = p(w)$ inside D .

Proof. The idea is to use Theorem 2.7 and $1/(1-w) = \sum_{k \geq 0} w^k$. □

Definition 2.9. Let $U \subset \mathbb{C}^n$ be open. Then $f : U \rightarrow \mathbb{C}^m$ is **holomorphic** if $f_i = p_i \circ f$ is holomorphic for any $i = 1, \dots, m$ where $p_i : \mathbb{C}^m \rightarrow \mathbb{C}$ is the i -th projection, so $f = (f_1, \dots, f_m)$.

Fact. If $f : U \rightarrow \mathbb{C}^m$ is holomorphic and $g : V \rightarrow \mathbb{C}^p$ is holomorphic where $V \supset f(U)$ then $g \circ f$ is holomorphic.

Definition 2.10. Let $U \subset \mathbb{C}^n$ be open. A holomorphic function $f : U \rightarrow \mathbb{C}^m$ is **biholomorphic at** $z_0 \in U$ if there exists an open neighbourhood $V \subset U$ of z_0 such that $f : V \rightarrow f(V)$ is bijective and $f^{-1} : f(V) \rightarrow V$ is holomorphic. Then f is **biholomorphic** if f is bijective and f is biholomorphic at any point.

Note. $f(V)$ is automatically open in \mathbb{C}^m if $m = n$.

Example 2.11. Let $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be linear such that $\det \Phi \neq 0$. Then Φ is a biholomorphism.

Example 2.12. Let $U = \mathbb{C} \setminus \{0\}$ and

$$\begin{array}{ccc} f & : & U \longrightarrow U \\ z & \longmapsto & z^2 \end{array}.$$

Check that f is biholomorphic at any point of U but f is not biholomorphic.

Remark. $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Then a holomorphic $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ is also a diffeomorphism $U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$.

Theorem 2.13 (Hartogs). *Let $n \geq 2$, let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $\delta = (\delta_1, \dots, \delta_n)$ such that $\epsilon_i > \delta_i > 0$, let $U = D(0, \epsilon) \setminus \overline{D(0, \delta)}$, and let $f : U \rightarrow \mathbb{C}^m$ be holomorphic. Then there exists a holomorphic $\bar{f} : D(0, \epsilon) \rightarrow \mathbb{C}^m$ such that $\bar{f}|_U = f$.*

Example. Hartogs' theorem is false for $n = 1$. If $f(z) = 1/z$, for all $\epsilon > \delta > 0$, then f cannot be extended.

2.2 Cauchy formula in one variable

Let $\omega = x + iy \in \mathbb{C}$ for $x, y \in \mathbb{R}$, and let $f : U \rightarrow \mathbb{C}$ be C^∞ for some $U \subset \mathbb{C}$. Recall that

$$\frac{\partial}{\partial \omega} f = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f, \quad \frac{\partial}{\partial \bar{\omega}} f = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f.$$

Recall that f is holomorphic if and only if $\frac{\partial}{\partial \bar{\omega}} f = 0$ on U . More in general, let $U \subset \mathbb{C}^n$ be open, let $z_i = x_i + iy_i$, and let $f : U \rightarrow \mathbb{C}$ be a C^∞ -function. Then f is holomorphic if and only if $\frac{\partial}{\partial \bar{z}_i} f = 0$ for all $i = 1, \dots, n$. Let $\omega \in \mathbb{C}$. Since $dx \wedge dy = -dy \wedge dx$, let

$$dA = \frac{i}{2} d\omega \wedge d\bar{\omega} = \frac{i}{2} (dx + idy) \wedge (dx - idy) = dx \wedge dy,$$

which is the Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$.

Proposition 2.14. *Let $f : U \rightarrow \mathbb{C}$ for $U \subset \mathbb{C}$ be a C^∞ -function, and let $D = D(z, r)$ such that $\bar{D} \subset U$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f}{\omega - z} d\omega - \frac{1}{\pi} \int_D \frac{1}{\omega - z} \frac{\partial}{\partial \bar{\omega}} f dA.$$

Proof. Assume $z = 0$. Recall that $f(\omega) = 1/\omega$ is locally integrable around zero, so

$$\int_D \frac{1}{\omega} \frac{\partial}{\partial \bar{\omega}} f dA = \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} \frac{1}{\omega} \frac{\partial}{\partial \bar{\omega}} f dA.$$

Away from zero

$$\begin{aligned} d \left(\frac{f}{\omega} d\omega \right) &= \frac{1}{\omega} df \wedge d\omega + f d \left(\frac{1}{\omega} \right) \wedge d\omega = \frac{1}{\omega} \left(\frac{\partial}{\partial \omega} f d\omega + \frac{\partial}{\partial \bar{\omega}} f d\bar{\omega} \right) \wedge d\omega + f \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \right) d\omega \wedge d\omega \\ &= \frac{1}{\omega} \frac{\partial}{\partial \bar{\omega}} f d\bar{\omega} \wedge d\omega = \frac{2i}{\omega} \frac{\partial}{\partial \bar{\omega}} f dA. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\pi} \int_D \frac{1}{\omega} \frac{\partial}{\partial \bar{\omega}} f dA &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} \frac{1}{\omega} \frac{\partial}{\partial \bar{\omega}} f dA \\ &= \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} d \left(\frac{f}{\omega} d\omega \right) & \frac{1}{\omega} \frac{\partial}{\partial \bar{\omega}} f dA &= \frac{1}{2i} d \left(\frac{f}{\omega} d\omega \right) \\ &= \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \left(\int_{\partial D} \frac{f}{\omega} d\omega - \int_{\partial D(0, \epsilon)} \frac{f}{\omega} d\omega \right) & \text{Stokes' theorem} \\ &= \frac{1}{2\pi i} \left(\int_{\partial D} \frac{f}{\omega} d\omega - 2\pi i f(0) \right) & \lim_{\epsilon \rightarrow 0} \int_{\partial D(0, \epsilon)} \frac{1}{\omega} d\omega &= 2\pi i. \end{aligned}$$

□

If f is holomorphic, then $\frac{\partial}{\partial \bar{\omega}} f = 0$, which implies Theorem 2.3.

2.3 Rank theorem

Let $U \subset \mathbb{C}^n$ be open, and let $f : U \rightarrow \mathbb{C}^m$ be holomorphic. Then the **Jacobian** is

$$J_f = \left(\frac{\partial}{\partial z_i} f_j(z) \right),$$

where $f_j = p_j \circ f$ and $p_j : \mathbb{C}^m \rightarrow \mathbb{C}$ is the j -th projection.

Exercise. Show that the real Jacobian, which is $2n \times 2n$, has non-negative determinants.

Theorem 2.15 (Rank theorem). *Let $z \in U$ such that $r = \text{rk } J_f(z')$ is constant around z . Then there exist open $z \in V \subset U \subset \mathbb{C}^n$ and $f(z) \in W \subset \mathbb{C}^m$ such that $\phi : D(0,1)^n \rightarrow V$ and $\psi : D(0,1)^m \rightarrow W$ are biholomorphisms such that*

$$\eta = \psi^{-1} \circ f \circ \phi : \begin{array}{ccc} D(0,1)^n & \longrightarrow & D(0,1)^m \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_r, 0, \dots, 0) \end{array},$$

so

$$\begin{array}{ccccc} \mathbb{C}^n \supset U & \supset & V \ni z & \xrightarrow{f} & f(z) \in W \subset \mathbb{C}^m \\ & & \uparrow \phi & & \uparrow \psi \\ & & D(0,1)^n & \xrightarrow{\eta} & D(0,1)^m \end{array}.$$

Corollary 2.16 (Inverse function theorem). *Let $f : U \rightarrow \mathbb{C}^n$ be holomorphic for $U \subset \mathbb{C}^n$, and let $z \in U$ such that $\det J_f(z) \neq 0$. Then f is a biholomorphism at z .*

Proof. $\det J_f(z) \neq 0$ if and only if $\text{rk } J_f(z) = n$, so $\text{rk } J_f(z') = n$ around z , since $\det J_f(z)$ is a continuous function. Let ϕ and ψ as in Theorem 2.15. Then $\eta = \psi^{-1} \circ f \circ \phi = \text{id}$, so on V , $f = \psi \circ \phi^{-1}$ is a composition of biholomorphisms, which is a biholomorphism. \square

Remark 2.17. Let $f : U \rightarrow \mathbb{C}^n$ for $U \subset \mathbb{C}^n$. Then $\det J_f(z)$ is a holomorphism, so

$$Z = \{z \in U \mid \det J_f(z) = 0\}$$

is closed.

2.4 Holomorphic differential forms

Let $U \subset \mathbb{C}^n$ be open.

Definition 2.18. A **holomorphic vector field** on U is an expression

$$X = \sum_i a_i \frac{\partial}{\partial z_i},$$

where a_i are holomorphic functions on U .

For all $x \in U$, the **tangent space** is

$$T_x U = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cong \mathbb{C}^n.$$

If $x \in U$, then $X(x) \in T_x U$.

Notation 2.19. Let

$$H^0(U, \mathcal{O}_U) = \{\text{holomorphic functions } f : U \rightarrow \mathbb{C}\}, \quad H^0(U, T_U) = \{\text{holomorphic vector fields on } U\}.$$

Remark. $R = H^0(U, \mathcal{O}_U)$ is a ring and $M = H^0(U, T_U)$ is a module over R . That is, if $X \in H^0(U, T_U)$ and $f \in H^0(U, \mathcal{O}_U)$, then $fX \in H^0(U, T_U)$.

Definition 2.20. Let R be a ring and M be an R -module for $p \geq 1$. The p -th exterior power $\bigwedge^p M$ of M is the R -module $M^{\otimes p}$ with the relations

$$m_1 \otimes \cdots \otimes m_p - \epsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(p)}, \quad m_1, \dots, m_p \in M, \quad \sigma \in \mathcal{S}_p,$$

where $\epsilon(\sigma) = (-1)^m$ is the signature of σ and m is the number of transpositions defining σ . Then $M^* = \text{Hom}_R(M, R)$ is the **dual** of M as an R -module.

Let $R = H^0(U, \mathcal{O}_U)$ and $M = H^0(U, T_U)$.

Definition 2.21. Let $p > 0$. We define a **holomorphic p -form**, as an element of

$$H^0(U, \Omega_U^p) = \bigwedge^p M^*.$$

If $p = 0$, by convention a **holomorphic 0-form** is just an element in R .

Let (z_1, \dots, z_n) be coordinates for U . Recall $\eta \in M$ is given by $\eta = \sum_i a_i \frac{\partial}{\partial z_i}$ for holomorphic functions $a_i \in R$. Then $\omega \in M^*$ is given by the expression

$$\sum_i b_i dz_i, \quad b_i \in R, \quad dz_i \left(\frac{\partial}{\partial z_j} \right) = \delta_{ij}.$$

More in general $\omega \in H^0(U, \Omega_U^p)$ is given by

$$\omega = \sum_{|I|=p} f_I dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad f_I \in R, \quad I = (i_1, \dots, i_p), \quad i_1 < \cdots < i_p,$$

where $dz_{i_1}, \dots, dz_{i_p}$ is an R -basis of $H^0(U, \Omega_U^p)$.

Example.

$$H^0(U, \Omega_U^p) \cong \bigwedge^p H^0(U, \Omega_U^1)$$

is an isomorphism as R -modules. This is not true for complex manifolds in general.

The **exterior product** is

$$\begin{aligned} H^0(U, \Omega_U^p) \otimes H^0(U, \Omega_U^q) &\longrightarrow H^0(U, \Omega_U^{p+q}) \\ \omega_1 \otimes \omega_2 &\longmapsto \omega_1 \wedge \omega_2 \end{aligned},$$

where we just need to define

$$\omega_1 \wedge \omega_2 = f dz_{i_1} \wedge dz_{i_p} \otimes g dz_{j_1} \wedge dz_{j_q} = f g dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q},$$

by linearity. Then $\omega_1 \wedge \omega_2 = 0$ if $\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_q\} \neq \emptyset$, since $dz_i \wedge dz_i = 0$.

Exercise. Check that this definition coincides with the definition in M4P54.

The **exterior derivative** is

$$\begin{aligned} d : H^0(U, \Omega_U^p) &\longrightarrow H^0(U, \Omega_U^{p+1}) \\ f dz_{i_1} \wedge \cdots \wedge dz_{i_p} &\longmapsto \sum_{j=1}^n \frac{\partial}{\partial z_j} f dz_j \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}. \end{aligned}$$

By definition d is \mathbb{C} -linear, but not R -linear. That is,

$$d(a\omega_1 + b\omega_2) = ad\omega_1 + bd\omega_2, \quad \omega_1, \omega_2 \in H^0(U, \Omega_U^p), \quad a, b \in \mathbb{C}.$$

Proposition 2.22. Let $U \subset \mathbb{C}^n$ be open. Then

- the Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in H^0(U, \Omega_U^p), \quad \omega_2 \in H^0(U, \Omega_U^q),$$

- $d^2 = 0$, that is

$$dd\omega = 0, \quad \omega \in H^0(U, \Omega_U^p).$$

Lecture 4
Thursday
16/01/20

Definition 2.23. Let $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ be holomorphic, let $f_i = p_i \circ f : U \rightarrow \mathbb{C}$ where $p_i : \mathbb{C}^m \rightarrow \mathbb{C}$ is the i -th projection, and let $f(U) \subset V \subset \mathbb{C}^m$ be open. Then if

$$\omega = h dz_{i_1} \wedge \cdots \wedge dz_{i_p} \in H^0(V, \Omega_V^p), \quad h \in H^0(U, \mathcal{O}_U),$$

then we can define the **pull-back** of ω ,

$$f^*\omega = h \circ f df_{i_1} \wedge \cdots \wedge df_{i_p} \in H^0(U, \Omega_U^p),$$

since $f_i \in H^0(V, \mathcal{O}_V) = H^0(V, \Omega_V^0)$ implies that $df_i \in H^0(V, \Omega_V^1)$, so

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \subset V \\ & \searrow h \circ f \in H^0(U, \mathcal{O}_U) & \downarrow h \\ & & \mathbb{C} \end{array} .$$

This is linear over \mathbb{C} and over $H^0(U, \mathcal{O}_U)$.

Proposition 2.24. Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, and $W \subset \mathbb{C}^{m'}$ be open, let $f : U \rightarrow \mathbb{C}^m$ and $g : V \rightarrow \mathbb{C}^{m'}$ be holomorphic such that $V \supset f(U)$ and $W \supset g(V)$, and let $\omega \in H^0(V, \Omega_V^p)$ and $\eta \in H^0(W, \Omega_W^q)$. Then

- $f^*(\omega + \eta) = f^*\omega + f^*\eta$ if $p = q$,
- $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$,
- $d(f^*\omega) = f^*d\omega$, and
- $f^*g^*\omega = (g \circ f)^*\omega$.

Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and let $z_i = x_i + iy_i$ for $i = 1, \dots, n$ and $x_i, y_i \in \mathbb{R}$. Then

$$dz_i = dx_i + idy_i,$$

so any holomorphic form is a differentiable form on \mathbb{R}^{2n} . A (p, q) -**form** is a differentiable $(p + q)$ -form of the expression

$$\omega = \sum_{|I|=p, |J|=q} f_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \quad f_{I,J} : U \rightarrow \mathbb{C} \cong \mathbb{R}^2 \in C^\infty,$$

where $d\bar{z}_j = dx_j - idy_j$. We denote

$$C^\infty(U, \Omega_U^{p,q}) = \{\text{differentiable } (p + q)\text{-forms on } U\}.$$

If ω is a (p, q) -form, then the **conjugate** $\bar{\omega}$ of ω is the (q, p) -form defined by

$$\bar{\omega} = \sum_{|I|=p, |J|=q} \overline{f_{I,J}} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q}.$$

3 Complex manifolds

3.1 Complex manifolds

Definition 3.1. A **complex manifold** of dimension n is a connected Hausdorff topological space X , with a countable open cover $\{U_\alpha\}$ of X such that for all α , there exists $\phi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ such that $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$ is a homeomorphism and

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is a biholomorphism for each α and β , so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{C}^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \end{array} .$$

The pair (U_α, ϕ_α) is called a **holomorphic chart**. The set $\{(U_\alpha, \phi_\alpha)\}$ is called a **holomorphic atlas** or a **complex structure**.

Recall X is Hausdorff if for all $x, y \in X$ there exist U and V open in X such that $U \cap V = \emptyset$ and $x \in U$ and $y \in V$.

Example 3.2.

- If $U \subset \mathbb{C}^n$ is an open set then U is a complex manifold. More in general if X is a complex manifold and $U \subset X$ is open then U is a complex manifold. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on X . Then

$$\{(\overline{U_\alpha}, \overline{\phi_\alpha})\} = \{(U_\alpha \cap U, \phi_\alpha|_{\overline{U_\alpha}})\}$$

is a complex structure of X .

- If X and Y are complex manifolds, then $X \times Y$ is a complex manifold.

Example 3.3. The projective space $\mathbb{P}_{\mathbb{C}}^n$ or \mathbb{CP}^n . Let $V^* = \mathbb{C}^{n+1} \setminus \{0\}$, with coordinates (z_0, \dots, z_n) . Define an equivalence on V^* as

$$v_1 \sim v_2 \iff \exists \lambda \in \mathbb{C}, v_1 = \lambda v_2.$$

Check that \sim is an equivalence. Consider the Euclidean topology on V^* . Then there exists an induced topology on

$$X = V^* / \sim = \{[v] \mid v \in V^*\},$$

with quotient map

$$\begin{array}{ccc} q & : & V^* \longrightarrow X \\ & & v \longmapsto [v] \end{array} .$$

Given $v = (z_0, \dots, z_n) \in V^*$ we denote $[v] = [z_0, \dots, z_n]$ such that $z_i \neq 0$ for some i . Two elements $[x_0, \dots, x_n]$ and $[y_0, \dots, y_n]$ of X define the same point if and only if there exists λ such that $x_i = \lambda y_i$ for all i . Let

$$V_i = \{(z_0, \dots, z_n) \in V^* \mid z_i \neq 0\},$$

which is open in V^* , and let $U_i = q(V_i)$, which is open in X , such that $\{U_i\}$ is a cover of X , that is $\bigcup_i U_i = X$. Let

$$H_i = \{(z_0, \dots, z_n) \in V^* \mid z_i = 1\}.$$

Then there exists a homeomorphism

$$\begin{array}{ccc} r_i & : & H_i \longrightarrow \mathbb{C}^n \\ & & (z_0, \dots, z_n) \longmapsto [z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \end{array} ,$$

and let

$$q_i = q|_{H_i} : \begin{array}{ccc} H_i \subset V^* & \longrightarrow & U_i \subset X \\ (z_0, \dots, z_n) & \longmapsto & [z_0, \dots, z_n] \end{array}$$

be also a homeomorphism.

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- q_i is surjective. Take $[x_0, \dots, x_n] \in U_i$. Then $x_i \neq 0$, so choose $\lambda = 1/x_i$. Then

$$[x_0, \dots, x_n] = \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] = q(z_0, \dots, z_n), \quad z_j = \frac{x_j}{x_i},$$

and in particular $z_i = 1$, so there exists $(z_0, \dots, z_n) \in H_i$ such that $q_i(z_0, \dots, z_n) = [x_0, \dots, x_n]$.

- q_i is injective. ¹

For all i , $q_i^{-1} : U_i \rightarrow H_i$ and $r_i : H_i \rightarrow \mathbb{C}^n$ are homeomorphisms, so $\phi_i = r_i \circ q_i^{-1} : U_i \rightarrow \mathbb{C}^n$ is also a homeomorphism. We want to show that (U_i, ϕ_i) define a holomorphic atlas, so

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is a biholomorphism. Consider the case $j = 0$ and $i = 1$. Then $\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \mid x_1 \neq 0\}$, so

$$\begin{aligned} \phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) &\longrightarrow \phi_1(U_0 \cap U_1) \\ (x_1, \dots, x_n) &\longmapsto \left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) \end{aligned}$$

is a biholomorphism. Thus X is a compact complex manifold. If $n = 1$, then $\mathbb{P}_{\mathbb{C}}^1 \cong \mathbb{S}^2$.

Example 3.4. The complex torus. Let

$$\begin{aligned} \Lambda = \mathbb{Z}^{2n} &\longrightarrow \mathbb{C}^n \\ (a_1, \dots, a_n, b_1, \dots, b_n) &\longmapsto (a_1 + ib_1, \dots, a_n + ib_n) \end{aligned}$$

Define an equivalence on \mathbb{C}^n by

$$v_1 \sim v_2 \iff v_1 - v_2 \in \Lambda.$$

Then

$$X = \mathbb{C}^n / \sim,$$

with quotient map $q : \mathbb{C}^n \rightarrow X$ is Hausdorff and compact. Topologically $X \cong [0, 1]^{2n} / \sim$. For each $x \in \mathbb{C}^n$, we can find an open set $U \subset \mathbb{C}^n$ such that $q|_U : U \rightarrow X$ is a homeomorphism. The idea is if $x \in (0, 1)^{2n}$ then we can take $U = (0, 1)^{2n}$. If not, there exists a translation of $\mathbb{C}^n \rightarrow \mathbb{C}^n$ such that the property holds. We define

$$\phi_U = q|_U^{-1} : V \subset \mathbb{C}^n / \Lambda \rightarrow U \subset \mathbb{C}^n, \quad V = q(U).$$

Show that (V, ϕ_U) define a complex structure on X . ² This is also a compact complex manifold. More in general \mathbb{C}^n / Λ for a lattice $\Lambda \cong \mathbb{Z}^{2n}$ is a compact complex manifold.

3.2 Holomorphic functions on complex manifolds

Definition 3.5. Let $f : X \rightarrow Y$ be a continuous morphism between complex manifolds. Then f is **holomorphic** if there exists a complex structure $\{(U_\alpha, \phi_\alpha)\}$ on Y and for all $y \in Y$ there exists a holomorphic chart (V_α, ψ_α) such that $x \in V_\alpha$ and $f(V_\alpha) \subset U_\alpha$ around any point x of $f^{-1}(y)$ and $\phi_\alpha \circ f \circ \psi_\alpha^{-1}$ is holomorphic, so

$$\begin{array}{ccc} X \supset V_\alpha & \xrightarrow{f} & U_\alpha \subset Y \\ \psi_\alpha \downarrow & & \downarrow \phi_\alpha \\ \psi_\alpha(V_\alpha) & \xrightarrow{\bar{f}} & \phi_\alpha(U_\alpha) \end{array}$$

Let $J_f = J_{\bar{f}}$, and let a **holomorphic function** on X be a holomorphic function $f : X \rightarrow \mathbb{C}$.

Exercise 3.6. If X is a compact complex manifold then any holomorphic function $f : X \rightarrow \mathbb{C}$ is constant.

¹Exercise

²Exercise

Definition 3.7. Let $f : X \rightarrow Y$ be a holomorphic function between complex manifolds. Then f is

- a **submersion** if $\dim Y \geq \dim X = r$ and $\text{rk } J_f = r$ at any point,
- an **immersion** if $r = \dim X \leq \dim Y$ and $\text{rk } J_f = r$ at any point, and
- an **embedding** if it is an immersion and $f : X \rightarrow f(X)$ is a homeomorphism.

Example 3.8. Let $f_2, \dots, f_n : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic, and let

$$\begin{aligned} f &: \mathbb{C} \longrightarrow \mathbb{C}^n \\ z &\longmapsto (z, f_2(z), \dots, f_n(z)) \end{aligned}$$

Then f is an embedding.

Example 3.9. Let $X = \mathbb{C}^2 / \Lambda$ for $\Lambda = \mathbb{Z}^4 \subset \mathbb{C}^2$, and let $q : \mathbb{C}^2 \rightarrow X$. Fix $\lambda \in \mathbb{C}$. Let

$$\begin{aligned} f &: \mathbb{C} \longrightarrow \mathbb{C}^2 \\ z &\longmapsto (z, \lambda z) \end{aligned}$$

Then $\tilde{f} = q \circ f : \mathbb{C} \rightarrow X$ is an immersion.

- If $\lambda = 0$ or $\lambda = \frac{1}{2}$, then $\tilde{f}(\mathbb{C})$ is a closed submanifold.
- If λ is general then $\tilde{f}(\mathbb{C})$ is dense inside X , so it is not closed. Thus it is not a complex submanifold of X .

3.3 Complex submanifolds

Definition 3.10. Let $i : X \rightarrow Y$ be an embedding of complex manifolds. If $i(X) \subset Y$ is closed then $i(X)$ is called a **complex submanifold** of Y . The **codimension** of X in Y is $\dim Y - \dim X$.

Theorem 3.11.

1. Let $i : X \rightarrow Y$ be a submanifold of codimension k , and let $n = \dim X$. Then for all $x \in X$, there exists an open neighbourhood $x \in U \subset Y$ and a submersion $f : U \rightarrow D(0, 1)^k \subset \mathbb{C}^k$ such that $X \cap U = f^{-1}(0)$.
2. If $X \subset Y$ is a closed subset such that for all $x \in X$ there exists $U \ni x$ open in Y and a submersion $f : U \rightarrow D(0, 1)^k$ such that $X \cap U = f^{-1}(0)$, then X is a complex submanifold.

Proof.

1. We can assume that if there exists a holomorphic chart (U, ψ) on Y such that $x \in U$ and if $V = i^{-1}(U)$ then there exists $\phi : V \rightarrow \mathbb{C}^n$ such that (V, ϕ) is a holomorphic chart on X . After possibly shrinking U smaller, by the rank theorem, there exist biholomorphic $a : \psi(U) \rightarrow D(0, 1)^{n+k}$ and $b : \phi(U) \rightarrow D(0, 1)^n$ such that the induced morphism is given by

$$\begin{aligned} D(0, 1)^n &\longrightarrow D(0, 1)^{n+k} \\ (z_1, \dots, z_n) &\longmapsto (z_1, \dots, z_n, 0, \dots, 0) \end{aligned}$$

Let

$$\begin{aligned} c &: D(0, 1)^{n+k} \longrightarrow D(0, 1)^k \\ (z_1, \dots, z_{n+k}) &\longmapsto (z_{n+1}, \dots, z_{n+k}) \end{aligned}$$

so

$$\begin{array}{ccccc} Y & \supset & U & \xrightarrow{\phi} & \phi(U) & \xrightarrow{b} & D(0, 1)^n \subset \mathbb{C}^n & \xleftarrow{c} & \mathbb{C}^n \\ \uparrow i & & \uparrow i & & \downarrow & & \downarrow & & \\ X & \supset & V & \xrightarrow{\psi} & \psi(U) & \xrightarrow{a} & D(0, 1)^{n+k} \subset \mathbb{C}^{n+k} & & \end{array}$$

Then f is the composition $c \circ a \circ \psi : U \rightarrow D(0, 1)^n$.

2. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y , and let $V_\alpha = X \cap U_\alpha$ and $\psi_\alpha = \phi_\alpha|_{V_\alpha}$. The goal is to show that $\{(V_\alpha, \psi_\alpha)\}$ defines a complex structure on X . By assumption,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k}$$

is biholomorphic. Let $U' = \phi_\beta(U)$, let $X' = \phi_\beta(X \cap U)$, and let $f' = f \circ \phi_\beta^{-1}$, so

$$\begin{array}{ccccccc} & & & \phi_\alpha(U) & \subset & \phi_\alpha(U_\alpha \cap U_\beta) & \subset \mathbb{C}^{n+k} \\ & & & \nearrow \phi_\alpha & & \uparrow \phi_\alpha \circ \phi_\beta^{-1} & \\ Y & \supset & U_\alpha \cap U_\beta & \supset & U & \xrightarrow{\phi_\beta} & U' \subset \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \\ \uparrow i & & \cup & & \cup & \searrow f & \downarrow f' \\ X & \supset & X \cap U_\alpha \cap U_\beta & \supset & X \cap U & \xrightarrow{f} & X' \subset D(0,1)^k \subset \mathbb{C}^k \end{array}$$

Then $f'^{-1}(0) = \phi_\beta(X \cap U_\alpha \cap U_\beta)$ and f' is also a submersion. By the rank theorem, we may assume that $U' = D(0,1)^{n+k}$ and $f'(z_1, \dots, z_{n+k}) = (z_1, \dots, z_k)$, so $\phi_\beta(X' \cap U_\alpha \cap U_\beta) = f'^{-1}(0)$. Thus

$$(\psi_\alpha \circ \psi_\beta^{-1})(z_1, \dots, z_n) = (\phi_\alpha \circ \phi_\beta^{-1})(z_1, \dots, z_n, 0, \dots, 0)$$

is also a biholomorphism. □

3.4 Examples of complex manifolds

Example 3.12. Let $U \subset \mathbb{C}^n$ be open, let $k \leq n$, let $f_1, \dots, f_k : U \rightarrow \mathbb{C}$, and let

$$V = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}.$$

Assume that $\left(\frac{\partial}{\partial z_j} f_i\right)$ has maximal rank k at any point of U . Then V is a complex submanifold of U . The idea is if $f = (f_1, \dots, f_k) : U \rightarrow \mathbb{C}^k$, then f is a submersion around any point of V , and use Theorem 3.11.

Example 3.13. Let $f : X \rightarrow Y$ be a holomorphism between complex manifolds, and let $W \subset X$ be a submanifold. Then $f|_W : W \rightarrow Y$ is holomorphic.

Exercise 3.14. Let $X = \mathbb{C}^n$. Show that all the compact submanifolds of X are zero-dimensional, that is points.

Exercise 3.15. Let X and Y be compact manifolds. Recall that $X \times Y$ is also a complex manifold. Assume $f : X \rightarrow Y$, so

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

Show that Γ_f is a complex submanifold.

Example 3.16. Let $n, m > 0$, and let

$$\text{Mat}_{n,m} \mathbb{C} = \{(n \times m)\text{-matrices}\} \cong \mathbb{C}^{n \cdot m}.$$

Then $\text{Mat}_{n,m} \mathbb{C}$ is a complex manifold. Let

$$\text{GL}_n \mathbb{C} = \{(n \times n)\text{-matrices } A \mid A \text{ invertible}\}.$$

Then $\text{GL}_n \mathbb{C}$ is a complex manifold, open in $\text{Mat}_{n,n} \mathbb{C}$.

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Example 3.17. Projective manifolds. Let $R = \mathbb{C}[x_0, \dots, x_n]$ be the ring of polynomials, and let $X = \mathbb{P}_{\mathbb{C}}^n$ be the complex projective space. Then $f \in R$ is **homogeneous of degree** d if $f(\lambda x) = \lambda^d f(x)$. Let $q : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_{\mathbb{C}}^n$, let F_1, \dots, F_k be homogeneous polynomials in R , and let

$$V = \{F_1 = \dots = F_k = 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}, \quad W = q(V) \subset \mathbb{P}_{\mathbb{C}}^n,$$

so $q^{-1}(W) = V$, because F_i are homogeneous. Since V is closed in $\mathbb{C}^{n+1} \setminus \{0\}$, W is closed in $\mathbb{P}_{\mathbb{C}}^n$. Claim that if V is a submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$ then W is a compact submanifold of $\mathbb{P}_{\mathbb{C}}^n$. If $\{U_i\}$ is the open covering given by

$$U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\},$$

then it is enough to show that $W \cap U_i$ is a complex submanifold of U_i for all i . Assume $i = n$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then $q(x) = \mathbb{C}^*$ for all $x \in X$ but $\mathbb{P}_{\mathbb{C}}^n \times \mathbb{C}^* \neq \mathbb{C}^{n+1} \setminus \{0\}$. We want to show there exists a biholomorphism

$$\begin{aligned} \phi_n : \quad U_n \times \mathbb{C}^* &\longrightarrow q^{-1}(U_n) = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid x_n \neq 0\} \\ ([x_0, \dots, x_n], t) &\longmapsto \left(\frac{tx_0}{x_n}, \dots, \frac{tx_{n-1}}{x_n}, t \right), \end{aligned}$$

such that

$$\begin{aligned} \phi_n^{-1} : \quad q^{-1}(U_n) &\longrightarrow U_n \times \mathbb{C}^* \\ (y_0, \dots, y_n) &\longmapsto (q(y_0, \dots, y_n), y_n) = ([y_0, \dots, y_n], y_n). \end{aligned}$$

From this, it follows that $V \cap q^{-1}(U_n) \cong (W \cap U_n) \times \mathbb{C}^*$, so the claim follows.

Example 3.18. Plane curves. Let $X = \mathbb{P}_{\mathbb{C}}^2$, let $F \in R[x_0, x_1, x_2]$ be homogeneous of degree d , and let $W = \{F = 0\} \subset \mathbb{P}_{\mathbb{C}}^2$. Then W is a compact complex submanifold if and only if for all $x \in \mathbb{P}_{\mathbb{C}}^2$, $\partial_{x_i} F(x) \neq 0$ for some i .

$d = 1$. W is the projective line, so $F = ax_0 + bx_1 + cx_2$ for a, b, c not all zero. Then W is a complex submanifold. There exists a biholomorphism $\mathbb{P}_{\mathbb{C}}^1 \rightarrow W$.

$d = 2$. W is a conic, so F is a degree two polynomial. Then $F = x_0x_1$ does not define a manifold. If $F = x_0x_1 - x_2^2$, then W is a complex submanifold of X . There exists

$$\begin{aligned} \mathbb{P}_{\mathbb{C}}^1 &\longrightarrow W \subset X \\ [t_0, t_1] &\longmapsto [t_0^2, t_1^2, t_0t_1]. \end{aligned}$$

Check that it is a biholomorphism.³ This is true for any f of degree two such that W is a complex submanifold.

$d \geq 3$. If W is a complex submanifold then we will show that W is not biholomorphic to $\mathbb{P}_{\mathbb{C}}^1$.

3.5 Tangent spaces of complex manifolds

Definition 3.19. Let X be a complex manifold of dimension n , and let $x \in X$. Then there exists a chart (U, ϕ) around x such that $\phi(U) \subset \mathbb{C}^n$. The **holomorphic tangent space** $T_x X$ of X at x , is the vector space over \mathbb{C} generated by

$$\left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle.$$

Let X be a real manifold. The **real tangent space** $T_x^{\mathbb{R}} X$ is the vector space over \mathbb{R} defined by

$$\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\rangle,$$

where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates of \mathbb{R}^{2n} . The **complex tangent space** $T_x^{\mathbb{C}} X$ is the vector space over \mathbb{C} generated by

$$\left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\rangle,$$

a $2n$ -dimensional vector space over \mathbb{C} . Then $T_x^{\mathbb{C}} X = T_x^{\mathbb{R}} X \otimes_{\mathbb{R}} \mathbb{C}$.

³Exercise

3.6 Holomorphic differential forms on complex manifolds

Definition 3.20. Let X be a complex manifold of dimension n , and let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on X . A **holomorphic p -form** on X is the data ω_α , the p -forms on $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that if

$$h_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta),$$

then $h_{\alpha\beta}^* \omega_\beta = \omega_\alpha$ for all α and β .

Notation 3.21. Let

$$\Omega_X^p(X) = H^0(X, \Omega_X^p) = \{\text{holomorphic } p\text{-forms on } X\},$$

$$\mathcal{O}_X(X) = H^0(X, \mathcal{O}_X) = \{\text{holomorphic functions on } X\}.$$

Then $R = \mathcal{O}_X(X)$ is a ring and $M = \Omega_X^p(X)$ is an R -module.

Lemma 3.22. Let $f : X \rightarrow Y$ be holomorphic. Then $f^* : \Omega_Y^p \rightarrow \Omega_X^p(X)$.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y . We can write $f^{-1}(U_\alpha) = \bigcup_{\alpha,\beta} V_{\alpha,\beta}$ where $\{(V_{\alpha,\beta}, \psi_{\alpha,\beta})\}$ is a complex structure on X , so

$$\mathbb{C}^n \xleftarrow{\psi_{\alpha,\beta}} V_{\alpha,\beta} \xrightarrow{f|_{V_{\alpha,\beta}}} U_\alpha \xrightarrow{\phi_\alpha} \mathbb{C}^n.$$

Assume ω is defined by ω_α on $\phi_\alpha(U_\alpha)$. Let

$$\omega_{\alpha,\beta} = \left(\left(\psi_{\alpha,\beta}^{-1} \right)^* \circ f^* \circ \phi_\alpha^* \right) (\omega_\alpha)$$

be a p -form on $\psi_{\alpha,\beta}(V_{\alpha,\beta})$. Check that $\omega_{\alpha,\beta}$ are compatible with respect to the atlas on X .⁴ □

As in the local case, we can define

$$\begin{array}{ccc} \Omega_X^p(X) \otimes \Omega_X^q(X) & \longrightarrow & \Omega_X^{p+q}(X) \\ \omega_1 \otimes \omega_2 & \longmapsto & \omega_1 \wedge \omega_2 \end{array}.$$

Similarly there exists $d : \Omega_X^p(X) \rightarrow \Omega_X^{p+1}(X)$.

⁴Exercise

4 Vector bundles

4.1 Holomorphic vector bundles

Definition 4.1. Let X be a complex manifold. A **holomorphic vector bundle** E of rank r on X is a complex manifold E , a holomorphism $\pi : E \rightarrow X$, and an open covering U_α of X such that there exists a biholomorphism

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^r,$$

such that if $p_\alpha : U_\alpha \times \mathbb{C}^r \rightarrow U_\alpha$ is the projection then $\pi|_{\pi^{-1}(U_\alpha)} = p_\alpha \circ \psi_\alpha$, so

$$\begin{array}{ccc} E & \supset & \pi^{-1}(U_\alpha) \xrightarrow{\psi_\alpha} U_\alpha \times \mathbb{C}^r \\ \pi \downarrow & & \downarrow \pi \swarrow p_\alpha \\ X & \supset & U_\alpha \end{array} .$$

A vector bundle of rank one is called a **line bundle**.

For any $x \in X$, there exists α such that $x \in U_\alpha$, so

$$\begin{array}{ccc} \pi^{-1}(x) & \xrightarrow{\psi_\alpha} & \{x\} \times \mathbb{C}^r \\ \pi \downarrow & & \swarrow p_\alpha \\ x & & \end{array} .$$

Then $E(x) = \pi^{-1}(x)$ is a vector space of rank r over \mathbb{C} . Let $U_\alpha \ni x \in U_\beta$. There exists a biholomorphism

$$\mathbb{C}^r \cong p_\alpha^{-1}(x) \rightarrow p_\beta^{-1}(x) \cong \mathbb{C}^r,$$

because they are both biholomorphic to $\pi^{-1}(x)$, so $g_{\alpha\beta}(x) \in \mathrm{GL}_r \mathbb{C}$ because all the biholomorphisms from $\mathbb{C}^r \rightarrow \mathbb{C}^r$ are linear. The holomorphisms

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}_r \mathbb{C}$$

are called **transition functions**. Then

$$\begin{array}{ccc} p_\alpha^{-1}(x) & \xrightarrow{\mathrm{id}} & p_\alpha^{-1}(x) \\ & \searrow & \swarrow \\ & p_\beta^{-1}(x) & \end{array} ,$$

so

$$(g_{\alpha\beta} \circ g_{\beta\alpha})(x) = x, \quad x \in U_\alpha \cap U_\beta,$$

and

$$\begin{array}{ccc} p_\alpha^{-1}(x) & \xrightarrow{g_{\alpha\gamma}} & p_\gamma^{-1}(x) \\ & \searrow & \swarrow \\ & p_\beta^{-1}(x) & \end{array} ,$$

so

$$(g_{\alpha\beta} \circ g_{\beta\gamma})(x) = g_{\alpha\gamma}(x), \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition 4.2. Let X be a complex manifold, and let E and F be vector bundles on X of rank r and s respectively, with $\pi : E \rightarrow X$ and $\pi' : F \rightarrow X$. A **holomorphic map** $f : E \rightarrow F$ is a holomorphic function $E \rightarrow F$ such that $\pi = \pi' \circ f$ and such that the rank of the induced linear map $E(x) \rightarrow F(x)$ is independent of $x \in X$, so

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array} , \quad \begin{array}{ccc} E(x) = \pi^{-1}(x) & \xrightarrow{f} & \pi'^{-1}(x) = F(x) \\ \pi \searrow & & \swarrow \pi' \\ & x & \end{array} .$$

4.2 Examples of holomorphic vector bundles

Example 4.3. $\pi : E = X \times \mathbb{C}^r \rightarrow X$ is a vector bundle of rank r , called **trivial**.

Example 4.4. Algebra of vector bundles. Let $\pi : E \rightarrow X$ and $\pi' : F \rightarrow X$ be vector bundles on X of rank r and s respectively.

- The **direct sum** $E \oplus F$ is the $(r + s)$ -vector bundle such that

$$(E \oplus F)(x) = E(x) \oplus F(x), \quad x \in X.$$

The idea is to take an open cover which trivialises both E and F . Find the transition function of $E \oplus F$.⁵

- The **tensor product** $E \otimes F$ is the $(r \cdot s)$ -vector bundle such that

$$(E \otimes F)(x) = E(x) \otimes F(x), \quad x \in X.$$

- The **p -th exterior power** of E is the vector bundle $\bigwedge^p E$ such that

$$(\bigwedge^p E)(x) = \bigwedge^p E(x), \quad x \in X.$$

If $p = r = \text{rk } E$ then $\det E = \bigwedge^r E$ is a line bundle on X .

- The **dual** of E is the rank r vector bundle E^* such that

$$E^*(x) = (E(x))^*, \quad x \in X,$$

the dual $\text{Hom}(E(x), \mathbb{C})$ of $E(x)$.

- Let $f : E \rightarrow F$ be a holomorphic map. Then the **kernel** $\ker f$ is a vector bundle such that

$$(\ker f)(x) = \ker f(x) \subset E(x), \quad x \in X.$$

The **cokernel** $\text{coker } f$ is a vector bundle such that

$$(\text{coker } f)(x) = \text{coker } f(x) \subset F(x), \quad x \in X.$$

Example 4.5. Let $X = \mathbb{P}_{\mathbb{C}}^n$, and let

$$\mathcal{O}(-1) = \{(x, v) \mid x = [x_0, \dots, x_n] \in \mathbb{P}_{\mathbb{C}}^n, v = \mu(x_0, \dots, x_n), \mu \in \mathbb{C}\} \subset \mathbb{P}_{\mathbb{C}}^n \times \mathbb{C}^{n+1}.$$

Then $\pi = p_1 : \mathcal{O}(-1) \rightarrow \mathbb{P}_{\mathbb{C}}^n$, so

$$\pi^{-1}([x_0, \dots, x_n]) = \{v = \mu(x_0, \dots, x_n) \mid \mu \in \mathbb{C}\} \cong \mathbb{C}^1.$$

Let $\{U_i\}$ be an open covering of X given by $U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\}$. We define

$$\begin{aligned} \psi_i : \quad & \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C} \\ & ([x_0, \dots, x_n], (v_0, \dots, v_n)) \longmapsto ([x_0, \dots, x_n], v_i) \end{aligned}$$

which is a biholomorphism. Thus $\mathcal{O}(-1)$ is a complex manifold and $\mathcal{O}(-1)$ is a line bundle. The **tautological line bundle** $\mathcal{O}(1)$ is the dual of $\mathcal{O}(-1)$. Let

$$\mathcal{O}(k) = \begin{cases} X \times \mathbb{C} & k = 0 \\ \mathcal{O}(1)^{\otimes k} & k > 0 \\ \mathcal{O}(-1)^{\otimes k} & k < 0 \end{cases}.$$

Then $\mathcal{O}(k) = \mathcal{O}(-k)^*$.⁶ On $\mathbb{P}_{\mathbb{C}}^n$ these are the only line bundles. That is, if \mathcal{L} is a line bundle on $\mathbb{P}_{\mathbb{C}}^1$, there exists $k \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(k)$. Let $X = \mathbb{P}_{\mathbb{C}}^1$, and let E be a line bundle of rank r on X . Then

$$E \cong \bigoplus_{i=1}^r \mathcal{O}(a_i), \quad a_1, \dots, a_r \in \mathbb{Z}.$$

This is false for $X = \mathbb{P}_{\mathbb{C}}^n$, with $n \geq 2$.

⁵Exercise

⁶Exercise

Definition 4.6. Let $f : Y \rightarrow X$ be a holomorphism between complex manifolds, and let E be a vector bundle of rank r on X . Then there exists a vector bundle f^*E of rank r on Y defined by

$$f^*E = \{(y, v) \in Y \times E \mid f(y) = \pi(v)\},$$

the **fibre product** of E and Y over X , such that

$$\begin{array}{ccc} f^*E & \xrightarrow{f'} & E \\ \pi' \downarrow & & \downarrow \pi \\ Y & \xrightarrow{f} & X \end{array}$$

Let $\mathcal{U} = \{U_i\}$ be an open cover of X which trivialises E , so

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times \mathbb{C}^r \\ & \searrow \pi \quad \swarrow p_1 & \\ & U_i & \end{array}$$

Then $\mathcal{U}' = \{f^{-1}(U_i)\}$ is an open covering of Y , so

$$\begin{array}{ccccccc} \pi'^{-1}(f^{-1}(U_i)) & \xrightarrow{f'} & \pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times \mathbb{C}^r & \xrightarrow{p_2} & \mathbb{C}^r \\ \pi' \downarrow & & \downarrow \pi & & & & \\ f^{-1}(U_i) & \xrightarrow{f} & U_i & & & & \end{array},$$

and

$$\begin{aligned} \pi'^{-1}(f^{-1}(U_i)) &= \{(y, v) \in f^{-1}(U_i) \times \pi^{-1}(U_i) \mid f(y) = \pi(v)\} \longrightarrow f^{-1}(U_i) \times \mathbb{C}^r \\ (y, v) &\longmapsto (y, p_2(\psi_i(v))) \end{aligned}$$

is a biholomorphism. Thus f^*E is a vector bundle, where

$$f^*E(y) = \pi'^{-1}(y) = E(f(y)), \quad y \in Y.$$

Notation 4.7. Let $f : Y \rightarrow X$ be a morphism, and let E be a vector bundle on X . Then $f^*E = E|_Y$, mostly used if $f : Y \hookrightarrow X$.

Definition 4.8. Let E be a holomorphic vector bundle on a complex manifold X , and let $\pi : E \rightarrow X$. A **section** of E is a holomorphic function $s : X \rightarrow E$ such that $\pi \circ s = \text{id}_X$.

Example 4.9. Let $E = X \times \mathbb{C}^r$ be the trivial vector bundle of rank r . Fix $v \in \mathbb{C}^r$. Then

$$\begin{aligned} s_v &: X \longrightarrow E \\ x &\longmapsto (x, v) \end{aligned}$$

is a section of E . If v_1, \dots, v_r is a basis of \mathbb{C}^r then s_{v_1}, \dots, s_{v_r} have the property that $s_{v_1}(x), \dots, s_{v_r}(x)$ forms a basis of $E(x)$. Vice versa, assume E is a vector bundle on X of rank r such that there exist sections s_1, \dots, s_r of E such that for all $x \in X$, $s_1(x), \dots, s_r(x)$ is a basis of $E(x)$. Then $E \cong X \times \mathbb{C}^r$, since

$$\begin{aligned} X \times \mathbb{C}^r &\longrightarrow E \\ (x, (v_1, \dots, v_r)) &\longmapsto \sum_i v_i s_i(x) \end{aligned}$$

is a biholomorphism. Then s_1, \dots, s_r is called a **holomorphic frame** for E . Recall that for all $E \rightarrow X$ and for all $x \in X$ there exists an open $U \ni x$ such that $E|_U$ is trivial, so there exists a frame on U for $E|_U$. This is called a **local frame** around x .

Example 4.10. Let X be a complex manifold of dimension n , and let (z_1, \dots, z_n) be coordinates on \mathbb{C}^n . There exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ for $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$. For all $x \in U_\alpha$, $T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} V_\alpha$, and $T_{\phi_\alpha(x)} V_\alpha = \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle$ is a frame of T_{V_α} . Let

$$T_X = \bigcup_{x \in X} T_x X,$$

and let $\pi^{-1} : T_X \rightarrow X$ such that $\pi^{-1}(x) = T_x X$. Then T_X is a holomorphic vector bundle of rank n called the **tangent bundle**, where $\mathcal{U} = \{U_\alpha\}$ and

$$\psi_\alpha : \pi^{-1}(U_\alpha) = T_X|_{U_\alpha} \rightarrow T_{\mathbb{C}^n}|_{V_\alpha} \cong V_\alpha \times \mathbb{C}^r \rightarrow U_\alpha \times \mathbb{C}^r$$

defines the trivialisation. The **cotangent bundle** of X is

$$\Omega_X^1 = T_X^*,$$

and let

$$\Omega_X^p = \bigwedge^p \Omega_X^1, \quad p \geq 1.$$

A holomorphic p -form on X is a section of Ω_X^p .⁷

4.3 Complexification of tangent bundles

Let X be a complex manifold. How to view X as a differentiable manifold? Let V be a vector space of dimension m over \mathbb{R} . An **almost complex structure** on V is a linear map $J : V \rightarrow V$ such that $J^2 = -\text{id}_V$. If V admits an almost complex structure, then V can be seen as a vector space over \mathbb{C} . Let $\lambda = a + ib$ for $a, b \in \mathbb{R}$, and let $v \in V$. Define

$$\lambda v = av + bJ(v).$$

If $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2)v$.⁸ Let $v_1, \dots, v_n \in V$ be a basis over \mathbb{C} . Then

$$v_1, \dots, v_n, J(v_1), \dots, J(v_n)$$

is a basis of V over \mathbb{R} . The idea is to assume that $a_i, b_i \in \mathbb{R}$ such that $\sum_i a_i v_i + \sum_i b_i J(v_i) = 0$, then

$$0 = \sum_i a_i v_i + \sum_i b_i J(v_i) = \sum_i (a_i v_i + b_i J(v_i)) = \sum_i (a_i + ib_i) v_i,$$

so $a_i + ib_i = 0$ for all i . Thus $a_i = b_i = 0$, so $m = 2n$. On a vector space an almost complex structure is a complex structure. Let V be a vector space of dimension $2n$ over \mathbb{R} . Then the **complexification** $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V is a \mathbb{C} -vector space of dimension $2n$ over \mathbb{C} , where

$$\begin{aligned} \lambda & : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ v \otimes \mu & \longmapsto v \otimes \mu \lambda, \quad \lambda \in \mathbb{C}. \end{aligned}$$

Let J be an almost complex structure on V . Then we can extend J to a linear map

$$\begin{aligned} J & : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ v \otimes \mu & \longmapsto J(v) \otimes \mu, \end{aligned}$$

such that $J^2 = -\text{id}_{V_{\mathbb{C}}}$,⁹ so $J^2 + \text{id}_{V_{\mathbb{C}}} = 0$. Thus the eigenvalues of J on $V_{\mathbb{C}}$ are $\pm i$. Let $V^{1,0}$ be the eigenspace for i and $V^{0,1}$ be the eigenspace for $-i$, so

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

The **conjugation**

$$\begin{aligned} \bar{\cdot} & : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ v \otimes \mu & \longmapsto v \otimes \bar{\mu} \end{aligned}$$

on $V_{\mathbb{C}}$ is linear over \mathbb{R} , such that $\overline{V^{1,0}} = V^{0,1}$ and $\overline{V^{0,1}} = V^{1,0}$,¹⁰ so $V^{1,0}$ and $V^{0,1}$ are \mathbb{C} -vector spaces of dimension n .

⁷Exercise

⁸Exercise

⁹Exercise

¹⁰Exercise

Example 4.11. Let $W = \mathbb{C}^n$ with coordinates (z_1, \dots, z_n) , and let $z_j = x_j + iy_j$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ for \mathbb{R}^{2n} . Define

$$J : \begin{array}{ccc} \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \\ (x_1, y_1, \dots, x_n, y_n) & \longmapsto & (-y_1, x_1, \dots, -y_n, x_n) \end{array}.$$

Then $J^2 = \text{id}_{\mathbb{R}^{2n}}$, and J is the **standard almost complex structure** on \mathbb{R}^{2n} . Let $V = \mathbb{R}^{2n}$, so $V_{\mathbb{C}} \cong \mathbb{C}^{2n}$ with complex coordinates $(x_1, y_1, \dots, x_n, y_n)$. Then $V^{0,1}$ is spanned by $x_j - iy_j$ and $V^{1,0}$ is spanned by $x_j + iy_j$, where $\overline{x_j + iy_j} = x_j - iy_j$ for $j = 1, \dots, n$.

Definition 4.12. Let X be a differentiable manifold. A **real, or complex, vector bundle of rank r** is a differentiable manifold E with a smooth morphism $\pi : E \rightarrow X$ such that if $K = \mathbb{R}$, or $K = \mathbb{C}$, then there exists an open covering $\mathcal{U} = \{U_i\}$ of X such that

- for all $x \in X$, the fibre of π , $E(x) = \pi^{-1}(x)$, is a vector space of rank r over K ,
- for all i there exists a diffeomorphism h_i such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{h_i} & U_i \times K^r \xrightarrow{p_2} K^r \\ & \searrow \pi & \swarrow p_1 \\ & U_i & \end{array},$$

and for all x , $p_2 \circ h_i : E(x) \rightarrow K^r$ is an isomorphism of vector spaces.

Pull-backs, sections, exterior powers, tensors, direct sums, frames, etc are the same as holomorphic vector bundles, where holomorphic becomes smooth and biholomorphic becomes diffeomorphic, and for all X there exists a tangent bundle T_X . Assume X is a complex manifold of dimension n . Let T_X be the holomorphic tangent bundle of X . Then X is also a differentiable manifold of dimension $2n$, so let $T_{X,\mathbb{R}}$ be the **real tangent bundle** of X , which is a rank $2n$ vector bundle, and let

$$T_{X,\mathbb{C}} = T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

be the **complex tangent bundle** of X , which is a non-holomorphic complex vector bundle of rank $2n$. Smooth morphisms of real or complex vector bundles are defined similarly as holomorphisms between holomorphic vector bundles such that the rank of the image is constant. Let X be a differentiable manifold of dimension $m = 2n$. Then an **almost complex structure** on X is a smooth morphism between the real tangent bundle $J : T_{X,\mathbb{R}} \rightarrow T_{X,\mathbb{R}}$ such that $J^2 = -\text{id}$. In particular, $J(x) : T_x^{\mathbb{R}} X \rightarrow T_x^{\mathbb{R}} X$ is an almost complex structure for all $x \in X$.

Proposition 4.13. *Let X be a complex manifold. Then the underlying differentiable manifold admits an almost complex structure $J : T_{X,\mathbb{R}} \rightarrow T_{X,\mathbb{R}}$ such that $J^2 = -\text{id}$.*

Proof. Let $x \in X$, and let (U, ϕ) be a complex chart around x such that

$$\phi : \begin{array}{ccc} U & \longrightarrow & V \\ x & \longmapsto & 0 \end{array}.$$

Fix holomorphic coordinates (z_1, \dots, z_n) on U . The tangent bundle of X on U is trivial, with a local frame $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$, so

$$T_X|_U \xrightarrow{\sim} T_V = V \times \mathbb{C}^n.$$

Define $x_i = \text{Re } z_i$ and $y_i = \text{Im } z_i$. Then $(x_1, y_1, \dots, x_n, y_n)$ are smooth coordinates $U \rightarrow \mathbb{R}$ around x , and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ define a local smooth frame of $T_{X,\mathbb{R}}$ on U , so

$$T_{X,\mathbb{R}}|_U \xrightarrow{\sim} T_V = V \times \mathbb{R}^{2n}.$$

In particular, there exists an almost complex structure J_U for $T_V \cong T_{X,\mathbb{R}}|_U$, so

$$J_U : T_{X,\mathbb{R}}|_U \rightarrow T_{X,\mathbb{R}}|_U, \quad J_U^2 = -\text{id}.$$

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Let $f : V \rightarrow V$ be a biholomorphism, so

$$\begin{array}{ccc} & U \cap U' & \\ \phi \swarrow & & \searrow \phi \\ V & \xrightarrow{f} & V \end{array},$$

and let z'_1, \dots, z'_n be local holomorphic coordinates given by

$$z'_i = f_i(z_1, \dots, z_n), \quad f_i = p_i \circ f,$$

where $p_i : \mathbb{C}^n \rightarrow \mathbb{C}$ is the i -th projection. Define

$$x'_i = \operatorname{Re} z'_i = \operatorname{Re} f_i(z_1, \dots, z_n) = u_i(z_1, \dots, z_n), \quad y'_i = \operatorname{Im} z'_i = \operatorname{Im} f_i(z_1, \dots, z_n) = v_i(z_1, \dots, z_n),$$

so $f_j = u_j + iv_j$. The real Jacobian J_f of f is given by the derivatives of u_j and v_j with respect to $x_1, y_1, \dots, x_n, y_n$, a $(2n \times 2n)$ -matrix of $n \times n$ blocks of 2×2 blocks of

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

These define the transition function of $T_{X, \mathbb{R}}$. To show that J extends to X , it is enough to show that J commutes with J_f at each point, so

$$\begin{array}{ccc} T_{X, \mathbb{R}}|_{U \cap U'} & \xrightarrow{J_f} & T_{X, \mathbb{R}}|_{U \cap U'} \\ J \downarrow & & \downarrow J \\ T_{X, \mathbb{R}}|_{U \cap U'} & \xrightarrow{J_f} & T_{X, \mathbb{R}}|_{U \cap U'} \end{array}.$$

Since f_j is holomorphic $\frac{\partial}{\partial \bar{z}_k} f_j = 0$ for all j and k , so the Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k} - \frac{\partial v_j}{\partial y_k} = 0, \quad \frac{\partial v_j}{\partial x_k} + \frac{\partial u_j}{\partial y_k} = 0,$$

or

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix},$$

hold. Since J is the standard almost complex structure on \mathbb{R}^{2n} , where $x_j \mapsto y_j$ and $y_j \mapsto -x_j$,

$$J = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Check that J_f commutes with J .¹¹ □

Corollary 4.14. *Every complex manifold is orientable.*

Proof. We prove that if $T_{X, \mathbb{R}}$ admits an almost complex structure then X is an orientable manifold. For all $x \in X$ choose the orientation on $T_x^{\mathbb{R}} X$, a vector space of dimension $2n$ over \mathbb{R} , given by any ordered basis of the form

$$v_1, \dots, v_n, J(v_1), \dots, J(v_n).$$

Assume that $v_1, \dots, v_n, J(v_1), \dots, J(v_n)$ and $w_1, \dots, w_n, J(w_1), \dots, J(w_n)$ are ordered bases. Show that the determinant of the matrix given by the change of basis is positive.¹² □

¹¹Exercise

¹²Exercise

4.4 Differential forms on complex tangent bundles

Let X be a complex manifold. Then there exists an almost complex structure $J : T_{X,\mathbb{R}} \rightarrow T_{X,\mathbb{R}}$ on X . Then J extends to

$$\begin{aligned} J &: T_{X,\mathbb{C}} \longrightarrow T_{X,\mathbb{C}} \\ v \otimes \mu &\longmapsto J(v) \otimes \mu. \end{aligned}$$

For all x , $J(x)$ has two eigenvalues $\pm i$, so

$$T_{X,\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1},$$

which are complex vector bundles and **eigenbundles**. Locally $T_X^{1,0}$ and $T_X^{0,1}$ are spanned by the frames $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ and $\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}$ respectively. Moreover there exists a conjugation

$$\begin{aligned} T_{X,\mathbb{C}} &\longrightarrow T_{X,\mathbb{C}} \\ v \otimes \mu &\longmapsto v \otimes \bar{\mu} \end{aligned}$$

over \mathbb{R} , such that $\overline{T_X^{1,0}} = T_X^{0,1}$ and $\overline{T_X^{0,1}} = T_X^{1,0}$. Let

$$\Omega_{X,\mathbb{C}}^1 = T_{X,\mathbb{C}}^*$$

be the dual of the complex vector bundle $T_{X,\mathbb{C}}$. Then

$$\Omega_{X,\mathbb{C}}^1 = \Omega_{X,\mathbb{R}}^1 \otimes_{\mathbb{R}} \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1} = (T_X^{1,0})^* \oplus (T_X^{0,1})^*.$$

Exercise. Let V and W be vector spaces. Show that

$$\bigwedge^k (V \oplus W) = \bigoplus_{p+q=k} \bigwedge^p V \otimes \bigwedge^q W$$

is a canonical isomorphism.

Thus,

$$\Omega_{X,\mathbb{C}}^k = \bigwedge^k \Omega_{X,\mathbb{C}}^1 = \bigoplus_{p+q=k} \Omega_X^{p,q}, \quad \Omega_X^{p,q} = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}, \quad k \geq 0,$$

where $\Omega_X^{p,q}$ is a complex vector bundle for any p and q .

Definition 4.15. The sections of $\Omega_X^{p,q}$ are called (p, q) -**forms** on X , or **forms of type** (p, q) .

Locally, let $x \in X$, and let $(U \ni x, \phi)$ be a holomorphic chart for $\phi : U \xrightarrow{\sim} V \subset \mathbb{C}^n$. A (p, q) -form on U can be locally written as

$$\omega = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J = \sum_{|I|=p, |J|=q} \alpha_{I,J} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q},$$

where $\alpha_{I,J}$ are smooth functions on U . Let X be a manifold. If E is a complex vector bundle then

$$C^\infty(X, E) = \{\text{smooth sections of } E\}.$$

The **differential**

$$d : C^\infty(X, \Omega_{X,\mathbb{C}}^k) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1})$$

satisfies the Leibnitz rule and $d^2 = 0$, so $dd\omega = 0$. If $\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p,q})$, then $d\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q+1})$. Assume that locally $\omega = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$. Then

$$d\omega = \sum_{I,J} d\alpha_{I,J} dz_I \wedge d\bar{z}_J, \quad d\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} dz_i + \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} \alpha_{I,J} d\bar{z}_i.$$

Let

$$\partial\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} dz_i \in C^\infty(X, \Omega_X^{1,0}), \quad \bar{\partial}\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} \alpha_{I,J} d\bar{z}_i \in C^\infty(X, \Omega_X^{0,1}).$$

Then $d = \partial + \bar{\partial}$ for smooth functions. Back to $d\omega$. Then

$$d\omega = \sum_{I,J} d\alpha_{I,J} dz_I \wedge d\bar{z}_J = \sum_{I,J} \partial\alpha_{I,J} dz_I \wedge d\bar{z}_J + \sum_{I,J} \bar{\partial}\alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Let

$$\partial\omega = \sum_{I,J} \partial\alpha_{I,J} dz_I \wedge d\bar{z}_J, \quad \bar{\partial}\omega = \sum_{I,J} \bar{\partial}\alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Then $d = \partial + \bar{\partial}$ for ω .

Lemma 4.16. *The linear maps*

$$\partial : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{p+1,q}), \quad \bar{\partial} : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{p,q+1})$$

satisfy the Leibnitz rule. That is, if $\omega \in C^\infty(X, \Omega_X^{p,q})$ and $\eta \in C^\infty(X, \Omega_X^{p',q'})$, then

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta, \quad \bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta.$$

Proof. d satisfies the Leibnitz rule

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{p+q} \omega \wedge d\eta,$$

since $\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q})$, so

$$\begin{aligned} \partial(\omega \wedge \eta) + \bar{\partial}(\omega \wedge \eta) &= (\partial\omega + \bar{\partial}\omega) \wedge \eta + (-1)^{p+q} \omega \wedge (\partial\eta + \bar{\partial}\eta) \\ &= (\partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta) + (\bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta). \end{aligned}$$

Then $\partial(\omega \wedge \eta)$ and $\partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta$ are $(p+1, q)$ -forms, and $\bar{\partial}(\omega \wedge \eta)$ and $\bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta$ are $(p, q+1)$ -forms. Forms of the same type in the decomposition of $d(\omega \wedge \eta)$ must coincide. \square

4.5 Dolbeault cohomology

Lemma 4.17. $\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$.

Proof. Let $\omega \in C^\infty(X, \Omega_X^{p,q})$. Because $d^2 = 0$,

$$0 = d^2\omega = (\partial + \bar{\partial})((\partial + \bar{\partial})\omega) = \partial^2\omega + \partial\bar{\partial}\omega + \bar{\partial}\partial\omega + \bar{\partial}^2\omega.$$

Then $d^2\omega$ is a $(p+q+2)$ -form, $\partial^2\omega$ is a $(p+2, q)$ -form, $\partial\bar{\partial}\omega + \bar{\partial}\partial\omega$ is a $(p+1, q+1)$ -form, and $\bar{\partial}^2\omega$ is a $(p, q+2)$ -form. Forms of the same type in the decomposition of $d^2\omega$ must coincide. \square

Let X be a complex manifold. Fix $p, q \geq 0$. Let

$$\begin{aligned} \mathcal{Z}^{p,q}(X) &= \ker(\bar{\partial} : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{p,q+1})) \\ &= \{\omega \in C^\infty(X, \Omega_X^{p,q}) \mid \bar{\partial}\omega = 0\} \end{aligned}$$

and let

$$\begin{aligned} \mathcal{B}^{p,q}(X) &= \text{im}(\bar{\partial} : C^\infty(X, \Omega_X^{p,q-1}) \rightarrow C^\infty(X, \Omega_X^{p,q})) \\ &= \{\omega \in C^\infty(X, \Omega_X^{p,q}) \mid \exists \eta \in C^\infty(X, \Omega_X^{p,q-1}), \omega = \bar{\partial}\eta\}. \end{aligned}$$

Since $\bar{\partial}^2 = 0$ we have $\mathcal{B}^{p,q}(X) \subset \mathcal{Z}^{p,q}(X)$ for all p and q . The **Dolbeault cohomology group** of X is

$$H^{p,q}(X) = \mathcal{Z}^{p,q}(X) / \mathcal{B}^{p,q}(X).$$

Exercise. Assume X and Y are biholomorphic complex manifolds. Then

$$H^{p,q}(X) = H^{p,q}(Y).$$

If $H^{p,q}(X)$ is finite-dimensional then we define the **Hodge numbers** of X as

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X).$$

Our goal is if X is Kähler and compact

$$\bigoplus_{p+q=k} H^{p,q}(X) = H^{p+q}(X),$$

as the de Rham cohomology. In particular this is true if X is projective. How to compute $H^{p,q}(X)$? We need to use analysis.

Proposition 4.18. *Let X be a complex manifold. Then there exists an isomorphism*

$$H^{p,0}(X) \cong H^0(X, \Omega_X^p) = \{\text{holomorphic sections of } \Omega_X^p\} = \{\text{holomorphic } p\text{-forms on } X\}, \quad p \geq 0.$$

Remark. If X is compact then

$$H^{0,0}(X) = \mathbb{C},$$

because $H^{0,0}(X) = H^0(X, \mathcal{O}_X)$ are constants.

Proof.

$$H^{p,0}(X) = \mathcal{Z}^{p,0}(X) / \mathcal{B}^{p,0}(X) = \mathcal{Z}^{p,0}(X) = \left\{ \omega \in C^\infty(X, \Omega_X^{p,0}) \mid \bar{\partial}\omega = 0 \right\}.$$

Locally $\omega = \sum_{|I|=p} \alpha_I dz_I$. Then

$$\bar{\partial}\omega = \sum_{|I|=p} \bar{\partial}\alpha_I dz_I = \sum_{|I|=p} \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_j} \alpha_I d\bar{z}_j \wedge dz_I,$$

where $d\bar{z}_j \wedge dz_I$ are linearly independent. For all I and for all j , the Cauchy-Riemann equations $\frac{\partial}{\partial \bar{z}_j} \alpha_I = 0$ hold, so for all I , α_I is holomorphic. Then $\omega = \sum_{|I|=p} \alpha_I dz_I$ is a holomorphic p -form, so $\omega \in H^0(X, \Omega_X^p)$. \square

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5 Connection, curvature, and metric

5.1 Connections

Let X be a differentiable manifold, and let E be a complex vector bundle. Then

$$C^\infty(X, E) = \{C^\infty\text{-sections of } E\}.$$

Is there a way to compute the derivatives of these sections?

Definition 5.1. Let X and E be as above. A **connection** of E is a \mathbb{C} -linear map

$$\nabla : C^\infty(X, E) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^1 \otimes E)$$

such that the Leibnitz rule holds, so

$$\nabla(f\omega) = f \cdot \nabla\omega + df \otimes \omega, \quad f \in C^\infty(X), \quad \omega \in C^\infty(X, E).$$

The following is the idea. Let $\omega \in C^\infty(X, E)$. Then

$$\nabla\omega = \sum_i \eta_i \otimes \omega_i,$$

where η_i are 1-forms on X and ω_i are sections of E . Let $x \in X$, and let $v \in T_x X$. Then

$$\nabla_v \omega_x = \sum_i \eta_i(v) \omega_i$$

is a section of E at x . The goal is to study connections locally. Let $x \in X$, and let (U, ϕ) be a chart around x that trivialises E , so $\pi^{-1}(U) = U \times \mathbb{C}^r$ for $\pi : E \rightarrow X$ and $r = \text{rk } E$. Then there exists a frame $s_1, \dots, s_r \in C^\infty(U, E)$ of E on U . Let $\sigma \in C^\infty(X, E)$ be any section. Locally on U we write

$$\sigma \stackrel{U}{=} f = (f_1, \dots, f_r), \quad \sigma = \sum_{i=1}^r f_i s_i, \quad f_1, \dots, f_r \in C^\infty(U).$$

By the Leibnitz rule, on U ,

$$\nabla\sigma = \sum_{i=1}^r \nabla(f_i s_i) = \sum_{i=1}^r (f_i \cdot \nabla s_i + df_i \otimes s_i) \in C^\infty(U, \Omega_{U, \mathbb{C}}^1 \otimes E).$$

Notation. $df = (df_1, \dots, df_r)$.

Then

$$\nabla s_j = \sum_{i=1}^r a_{ij} \otimes s_i, \quad a_{ij} \in C^\infty(U, \Omega_U^1).$$

Notation. $A = (a_{ij})$ is an $(r \times r)$ -matrix with coefficients 1-forms.

With this notation, this becomes

$$\nabla\sigma \stackrel{U}{=} A \cdot f + df.$$

- A depends very much on the choice of the frame.
- Locally on U , ∇ is determined by A .

Consider another chart (U', ϕ') which also gives a trivialisation of E . So we can choose a corresponding frame s'_1, \dots, s'_r . Assume $\sigma \in C^\infty(U \cap U', E)$. Then

$$\sigma \stackrel{U'}{=} f' = (f'_1, \dots, f'_r), \quad \sigma = \sum_{j=1}^r f'_j s'_j, \quad f'_1, \dots, f'_r \in C^\infty(U).$$

Let A' be the matrix with respect to this frame. Then

$$\nabla\sigma \stackrel{U'}{=} A' \cdot f' + df'.$$

Let

$$g : (U \cap U') \times \mathbb{C}^r \rightarrow (U \cap U') \times \mathbb{C}^r$$

be the transition function from the trivialisation of U' to the trivialisation of U . Then $g(x) \in \mathrm{GL}_r \mathbb{C}$ for all $x \in U \cap U'$, and $f = g \cdot f'$. Denote by Dg the differential of g . Then

$$df = d(g \cdot f') = Dg \cdot f' + g \cdot df' = g \cdot (g^{-1} \cdot Dg \cdot f' + df'),$$

by the Leibnitz rule. Thus,

$$\begin{aligned} A' \cdot f' + df' &\stackrel{U'}{=} A \cdot f + df \stackrel{U}{=} A \cdot g \cdot f' + g \cdot (g^{-1} \cdot Dg \cdot f' + df') \stackrel{U}{=} g \cdot ((g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g) f' + df') \\ &\stackrel{U'}{=} (g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g) \cdot f' + df', \end{aligned}$$

so

$$A' = g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g.$$

5.2 Curvature operators

What is ∇^2 ? The idea is that

$$C^\infty(X, E) \xrightarrow{\nabla} C^\infty(X, \Omega_{X, \mathbb{C}}^1 \otimes E) \xrightarrow{\nabla} C^\infty(X, \Omega_{X, \mathbb{C}}^1 \otimes \Omega_{X, \mathbb{C}}^1 \otimes E) \xrightarrow{\wedge} C^\infty(X, \Omega_{X, \mathbb{C}}^2 \otimes E).$$

The **curvature tensor** is

$$\nabla^2 : C^\infty(X, E) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^2 \otimes E).$$

Remark. If X has dimension one, then $\Omega_{X, \mathbb{C}}^2 = 0$, so $\nabla^2 = 0$.

Again for all $x \in X$, take U as above. Let s_1, \dots, s_r be a frame, let $A = (a_{ij})$ be the $(r \times r)$ -matrix of 1-forms, and let DA be the differential of A .

Notation. $A \wedge A = (\sum_{k=1}^r a_{ik} \wedge a_{kj})$ is an $(r \times r)$ -matrix of 2-forms.

Let $\sigma \stackrel{U}{=} (f_1, \dots, f_r) = \sum_i f_i s_i$ on U . Then

$$\begin{aligned} \nabla^2 \sigma &= \nabla(A \cdot f + df) = A \wedge (A \cdot f + df) + d(A \cdot f + df) \\ &= A \wedge A \cdot f + A \wedge df + DA \cdot f - A \wedge df + d^2 f = (A \wedge A + DA) \cdot f \end{aligned}$$

is C^∞ -linear, so $\nabla^2(h\sigma) = h\nabla^2\sigma$. The **curvature operator** is

$$\Theta_\nabla \stackrel{U}{=} A \wedge A + DA,$$

so $\Theta_\nabla(\sigma) = \nabla^2\sigma$.

5.3 Hermitian metrics

Definition 5.2. Let V be a vector space over \mathbb{C} . A **Hermitian inner product** on V is a map

$$\begin{aligned} V \times V &\longrightarrow \mathbb{C} \\ (v, w) &\longmapsto \langle v, w \rangle, \end{aligned}$$

such that

- $\langle v, w \rangle = \overline{\langle w, v \rangle}$,
- it is linear on the first factor, and
- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

Example. $V = \mathbb{C}$ and $\langle z_1, z_2 \rangle = z_1 \cdot \overline{z_2}$.

Definition 5.3. Let X be a manifold, and let E be a complex vector bundle on X . A **Hermitian metric** h , or $\langle \cdot, \cdot \rangle$, on E is a choice of a Hermitian inner product

$$h_x = \langle \cdot, \cdot \rangle_x : E(x) \times E(x) \rightarrow \mathbb{C}, \quad x \in X,$$

such that for any open set $U \subset X$ and for $s, t \in C^\infty(U, E)$, $\langle s(x), t(x) \rangle_x$ is a C^∞ -function with respect to x on U . The pair $(E, \langle \cdot, \cdot \rangle) = (E, h)$ is called a **Hermitian vector bundle**.

Let (E, h) be a Hermitian vector bundle, and let $x \in X$. Locally, let s_1, \dots, s_r be a frame on $U \ni x$. For any $x \in U$, $\langle s_i(x), s_j(x) \rangle_x = h_{ij}(x)$ is a smooth function for all i and j , so

$$H = (h_{ij})_{i,j=1}^r$$

is an $(r \times r)$ -matrix of smooth functions. Let $\sigma, \sigma' \in C^\infty(U, E)$, and let $\sigma \stackrel{U}{=} f = (f_1, \dots, f_r)$ and $\sigma' \stackrel{U}{=} f' = (f'_1, \dots, f'_r)$. Then

$$\langle \sigma(x), \sigma'(x) \rangle_x = f^\top \cdot H \cdot \bar{f}'.$$

Now assume that U' is a different open set with frame (s'_1, \dots, s'_r) . Assume

$$g : (U \cap U') \times \mathbb{C}^r \rightarrow (U \cap U') \times \mathbb{C}^r$$

is the transition function from the trivialisation on U' to the trivialisation on U . Let H' be the matrix of h with respect to s'_1, \dots, s'_r . Then

$$H' = g^\top \cdot H \cdot \bar{g}.$$

Proposition 5.4. Let $\pi : E \rightarrow X$ be a complex vector bundle on X . Then E always admits a Hermitian metric.

Before proving Proposition 5.4, we recall the definition of a partition of the unity.

Definition 5.5. Let M be a manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to \mathcal{U} is a collection of smooth functions $f_\alpha : M \rightarrow [0, 1]$ such that

- $\text{supp } f_\alpha \subset U_\alpha$ for all α , in particular, $f_\alpha = 0$ outside U_α ,
- $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
- for all $x \in M$, there exists an open neighbourhood V of x such that $\text{supp } f_\alpha \cap V \neq \emptyset$ for only finitely many α .

It can be shown that if M is a manifold and $\mathcal{U} = \{U_\alpha\}$ is an open cover of M , then there exists a partition of the unity $\{f_\alpha\}$ with respect to such a cover.

Proof. Let $\mathcal{U} = \{U_i\}$ be an open cover of open sets of X , trivialising E , so $\phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^r$, and let $f_i : X \rightarrow [0, 1]$ be a partition of unity with respect to \mathcal{U} . For each i , consider a Hermitian metric on \mathbb{C}^r . Then there is a Hermitian metric \tilde{h}_i on $U_i \times \mathbb{C}^r$. Let h_i be the Hermitian metric on $E|_{U_i}$ induced by ϕ_i . Take $h = \sum_i f_i h_i$. Check that h defines a Hermitian metric on X .¹³ \square

Let $E \rightarrow X$ be a complex Hermitian vector bundle of rank r . Fix $p, q \geq 0$. There exists a bilinear **cup product**

$$\begin{aligned} \{\cdot, \cdot\} : C^\infty(X, \Omega_{X, \mathbb{C}}^p \otimes E) \times C^\infty(X, \Omega_{X, \mathbb{C}}^q \otimes E) &\longrightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^{p+q}), \\ (\sigma, \tau) &\longmapsto \{\sigma, \tau\} \end{aligned}$$

where $\{\sigma, \tau\}$ is defined as follows. Let $x \in X$, let s_1, \dots, s_r be a frame of E around x , let H be the matrix associated to the Hermitian metric with respect to the frame, and let

$$\sigma = \sum_i \sigma_i \otimes s_i, \quad \tau = \sum_i \tau_i \otimes s_i, \quad \sigma_i \in C^\infty(X, \Omega_{X, \mathbb{C}}^p), \quad \tau_i \in C^\infty(X, \Omega_{X, \mathbb{C}}^q).$$

¹³Exercise

Then we define, around x ,

$$\{\sigma, \tau\} = \sigma^\top \cdot H \cdot \bar{\tau} = \sum_{i,j=1}^r h_{ij} \sigma_i \wedge \bar{\tau}_j.$$

This is uniquely defined, and does not depend on the frame, so it extends to X . In particular $\{\sigma, \tau\}$ is a smooth $(p+q)$ -form.

Definition 5.6. Let E be a complex Hermitian vector bundle on X , and let ∇ be a connection on E . We say that ∇ is **Hermitian**, or **compatible with the metric**, if the Leibnitz rule holds, so we have

$$d\{\sigma, \tau\} = \{\nabla\sigma, \tau\} + (-1)^p \{\sigma, \nabla\tau\}, \quad \sigma \in C^\infty(X, \Omega_{X,\mathbb{C}}^p \otimes E), \quad \tau \in C^\infty(X, \Omega_{X,\mathbb{C}}^q \otimes E).$$

Let $x \in X$, and let s_1, \dots, s_r be a local frame of E . Assume s_1, \dots, s_r is an orthonormal frame around $x \in X$. Let ∇ be a connection compatible with the metric, and let A be the associated matrix with respect to s_1, \dots, s_r . Gram-Schmidt is an algorithm that gives an orthonormal basis of $E(x)$ for all x , which is C^∞ , say s'_1, \dots, s'_r . Then with respect to this frame $H = \text{id}_r$ because $\langle s'_i, s'_j \rangle_x = \delta_{ij}$.

Proposition 5.7. A is anti-autodual, that is

$$\bar{A}^\top = -A.$$

Proof. Let σ and τ be as before, and let $\sigma_1, \dots, \sigma_r$ and τ_1, \dots, τ_r be the components of σ and τ with respect to the frame s_1, \dots, s_r . Then $\{\sigma, \tau\} = \sigma^\top \wedge \bar{\tau}$. Since ∇ is Hermitian, the Leibnitz rule holds, so

$$d\{\sigma, \tau\} = d(\sigma^\top \wedge \bar{\tau}) = d\sigma^\top \wedge \bar{\tau} + (-1)^p \sigma^\top \wedge d\bar{\tau},$$

by the usual Leibnitz rule for d . Then

$$\{\nabla\sigma, \tau\} = \{A \wedge \sigma + d\sigma, \tau\} = \{A \wedge \sigma, \tau\} + \{d\sigma, \tau\} = (A \wedge \sigma)^\top \wedge \bar{\tau} + d\sigma^\top \wedge \bar{\tau} = (-1)^p \sigma^\top \wedge A^\top \wedge \bar{\tau} + d\sigma^\top \wedge \bar{\tau},$$

and

$$\{\sigma, \nabla\tau\} = \sigma^\top \wedge \overline{\nabla\tau} = \sigma^\top \wedge \overline{(A \wedge \tau + d\tau)} = \sigma^\top \wedge \bar{A} \wedge \bar{\tau} + \sigma^\top \wedge d\bar{\tau}.$$

By the Leibnitz rule,

$$\sigma^\top \wedge (A^\top + \bar{A}) \wedge \bar{\tau} = 0.$$

This is true for all σ and τ , so $A^\top + \bar{A} = 0$. □

Exercise. Let s_1, \dots, s_r be any frame, let H be the matrix given by the metric with respect to s_1, \dots, s_r , and let A be the matrix given by the connection with respect to s_1, \dots, s_r where the connection is Hermitian. Then

$$DH = A^\top \cdot H + H \cdot \bar{A},$$

where if $H = (h_{ij})$ then $DH = (dh_{ij})$. A hint is to do the same calculation.

Theorem 5.8. If $E \rightarrow X$ is a complex Hermitian vector bundle, then there exists a connection ∇ compatible with h .

5.4 Holomorphic vector bundles

Proposition 5.9. Let X be a complex manifold, and let $\pi : E \rightarrow X$ be a holomorphic vector bundle of rank r . Then for all $q \geq 0$ there exists a \mathbb{C} -linear map

$$\bar{\partial}_E : C^\infty(X, \Omega_X^{0,q} \otimes E) \rightarrow C^\infty(X, \Omega_X^{0,q+1} \otimes E),$$

which satisfies the Leibnitz rule and $\bar{\partial}_E^2 = 0$. Moreover if σ is a holomorphic section of $\Omega_X^{0,q} \otimes E$ then $\bar{\partial}_E \sigma = 0$.

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The idea is to do it locally in a canonical way, so does not depend on the choice of the trivialisation.

Proof. Let $x \in X$. There exists a holomorphic frame s_1, \dots, s_r of E locally around x in U . Let $\sigma \in C^\infty(X, \Omega_X^{0,q} \otimes E)$. Then locally, $\sigma \stackrel{U}{=} \sum_{i=1}^r f_i \otimes s_i$ where $f_i \in C^\infty(U)$ are $(0, q)$ -forms locally around x . We define

$$\bar{\partial}_E \sigma \stackrel{U}{=} \sum_{i=1}^r \bar{\partial} f_i \otimes s_i \in C^\infty(U, \Omega_U^{0,q+1} \otimes E).$$

We want to show that it can be extended to X . Let $U' \subset X$ be open, let s'_1, \dots, s'_r be a holomorphic frame on U' of E , and let

$$g : (U \cap U') \times \mathbb{C}^r \rightarrow (U \cap U') \times \mathbb{C}^r$$

be the transition map from the trivialisation of U' to the trivialisation of U . Then $\sigma \stackrel{U}{=} \sum_{i=1}^r f'_i \otimes s'_i$, and

$$\bar{\partial}_E \sigma \stackrel{U'}{=} \sum_{i=1}^r \bar{\partial} f'_i \otimes s'_i.$$

Since g is holomorphic, that is $\bar{\partial} g = 0$, this implies that $\bar{\partial}_E$ on U coincides with $\bar{\partial}_E$ on U' . Recall for ∇ the change of frame was

$$A' = g^{-1} \cdot Dg + g^{-1} \cdot A \cdot g,$$

so $\bar{\partial}_E$ extends to X . Let σ be a holomorphic section of $\Omega_X^{0,q} \otimes E$. Then, on U if s_i and f_i are as before, then f_i are holomorphic $(0, q)$ -forms. Thus $\bar{\partial} f_i = 0$, so $\bar{\partial}_E \sigma = 0$. \square

Vice versa if $\nabla : C^\infty(X, E) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^1 \otimes E)$ is a connection and X is a complex manifold, then

$$\Omega_{X,\mathbb{C}}^1 \xrightarrow{\sim} \Omega_X^{1,0} \oplus \Omega_X^{0,1}, \quad \Omega_{X,\mathbb{C}}^1 \otimes E = (\Omega_X^{1,0} \otimes E) \oplus (\Omega_X^{0,1} \otimes E).$$

Then for all σ ,

$$\nabla \sigma = \nabla^{1,0} \sigma + \nabla^{0,1} \sigma,$$

where

$$\nabla^{1,0} : C^\infty(X, E) \rightarrow C^\infty(X, \Omega_X^{1,0} \otimes E), \quad \nabla^{0,1} : C^\infty(X, E) \rightarrow C^\infty(X, \Omega_X^{0,1} \otimes E).$$

Theorem 5.10. *Assume X is a complex manifold and E is a holomorphic Hermitian vector bundle of rank r . Then there exists a unique connection*

$$\nabla_E : C^\infty(X, E) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^1 \otimes E),$$

such that $\nabla_E^{0,1} = \bar{\partial}_E$, which defined in Proposition 5.9, and ∇_E is compatible with h .

Then ∇_E is called the **Chern connection** and ∇_E^2 is called the **Chern curvature**.

Proof. Fix $x \in X$, on $U \ni x$. There exists a local holomorphic frame s_1, \dots, s_r . Let H be the matrix defining the metric h on U , so $H = (h_{ij})$ is an $(r \times r)$ -matrix for $h_{ij} \in C^\infty(U)$. Define the $(r \times r)$ -matrix $\partial H = (\partial h_{ij})$ for $\partial h_{ij} \in C^\infty(U, \Omega_U^{1,0})$. We define

$$A = \bar{H}^{-1} \cdot \partial \bar{H},$$

an $(r \times r)$ -matrix of 1-forms on U . This A will be the matrix defining ∇_E .

- Let $\sigma \stackrel{U}{=} \sum_i f_i s_i \in C^\infty(U, E)$ where $f_i \in C^\infty(U)$. Then

$$\nabla_E \sigma \stackrel{U}{=} A \cdot f + df.$$

Let $A = (a_{ij})$ where by definition of A , a_{ij} are $(1, 0)$ -forms. Thus

$$\nabla_E^{0,1} \sigma = A^{0,1} \cdot f + \bar{\partial} f \stackrel{U}{=} \bar{\partial}_E \sigma.$$

- Recall that ∇ associated to A is compatible with h if and only if $DH = A^\top \cdot H + H \cdot \bar{A}$. Since H is Hermitian, it follows that $H^\top = \bar{H}$, so

$$A^\top \cdot H = \left(\bar{H}^{-1} \cdot \partial \bar{H} \right)^\top \cdot H = (\partial \bar{H})^\top \cdot \left(\bar{H}^{-1} \right)^\top \cdot H = \partial H \cdot H^{-1} \cdot H = \partial H,$$

and

$$H \cdot \bar{A} = H \cdot \overline{\bar{H}^{-1} \cdot \partial \bar{H}} = H \cdot H^{-1} \cdot \bar{\partial} H = \bar{\partial} H.$$

Thus

$$DH = (dh_{ij}) = (\partial h_{ij} + \bar{\partial} h_{ij}) = \partial H + \bar{\partial} \bar{H} = A^\top \cdot H + H \cdot \bar{A},$$

so on U , ∇_E is compatible with h .

- Let ∇ be another connection satisfying $\nabla^{0,1} = \bar{\partial}_E$ and ∇ is compatible with h . As before s_1, \dots, s_r is the local holomorphic frame on U . Let $B = (b_{ij})$ be the $(r \times r)$ -matrix of 1-forms associated to ∇ , and let $B = B^{1,0} + B^{0,1}$, so $b_{ij} = b_{ij}^{1,0} + b_{ij}^{0,1}$. For all $f = (f_1, \dots, f_r)$ if $\sigma = \sum_i f_i s_i$ then

$$\nabla \sigma \stackrel{U}{=} B \cdot f + df,$$

so

$$B^{0,1} \cdot f + \bar{\partial} f \stackrel{U}{=} \nabla^{0,1} \sigma = \bar{\partial}_E \sigma \stackrel{U}{=} \bar{\partial} f.$$

Then for all f , $B^{0,1} \cdot f = 0$, so $B^{0,1} = 0$ and $B = B^{1,0}$. Since ∇ is compatible with h , $DH = B^\top \cdot H + H \cdot \bar{B}$, so $\bar{\partial} \bar{H} = \bar{B}^\top \cdot \bar{H} + \bar{H} \cdot B$. Then

$$B = B^{1,0} = \left(\bar{H}^{-1} \cdot (\bar{\partial} \bar{H} - \bar{B}^\top \cdot \bar{H}) \right)^{1,0} = \bar{H}^{-1} \cdot \partial \bar{H} + \bar{H}^{-1} \cdot 0 \cdot \bar{H} = \bar{H}^{-1} \cdot \partial \bar{H} = A,$$

since $\bar{\partial} \bar{H}^{1,0} = \overline{(dh_{ij})}^{1,0} = (\overline{dh_{ij}})^{1,0} = \partial \bar{h}_{ij}$ and $(\bar{B})^{1,0} = \bar{B}^{1,0} = 0$, so $\nabla = \nabla_E$. We define in the same way ∇_E^U on any open U of X . On $U \cap U'$, by unicity $\nabla_E^U = \nabla_E^{U'}$. Thus ∇_E can be extended to X .

□

Corollary 5.11. *Let X be a complex manifold, let (E, h) be a Hermitian vector bundle on X , let ∇_E be the Chern connection, and let $\Theta_E = \nabla_E^2$ be the Chern curvature. Locally at $x \in U$, let s_1, \dots, s_r be a holomorphic frame, and let A be the matrix associated to ∇_E . Then*

- A is of type $(1, 0)$ and $\partial A = -A \wedge A$,
- $\Theta_E = \bar{\partial} A$ is of type $(1, 1)$, and
- $\bar{\partial} \Theta_E = 0$.

Proof.

- Let H be as above. Recall $A = \bar{H}^{-1} \cdot \partial \bar{H}$ is a $(1, 0)$ -form matrix. Then

$$0 = \partial I = \partial \left(\bar{H} \cdot \bar{H}^{-1} \right) = \bar{H} \cdot \partial \bar{H}^{-1} + \partial \bar{H} \cdot \bar{H}^{-1},$$

so

$$\begin{aligned} \partial A &= \partial \left(\bar{H}^{-1} \cdot \partial \bar{H} \right) = \partial \bar{H}^{-1} \wedge \partial \bar{H} + \bar{H}^{-1} \cdot \partial^2 \bar{H} \\ &= - \left(\bar{H}^{-1} \cdot \partial \bar{H} \cdot \bar{H}^{-1} \right) \wedge \partial \bar{H} = - \left(\bar{H}^{-1} \cdot \partial \bar{H} \right) \wedge \left(\bar{H}^{-1} \cdot \partial \bar{H} \right) = -A \wedge A. \end{aligned}$$

- By 1,

$$\Theta_E = A \wedge A + DA = A \wedge A + \partial A + \bar{\partial} A = \bar{\partial} A.$$

- By 2,

$$\bar{\partial} \Theta_E = \bar{\partial} \bar{\partial} A = 0.$$

□

Lemma 5.12. *Let X be a complex manifold of dimension n , let (E, h) be a Hermitian vector bundle on X of rank r , let ∇_E be the Chern connection compatible with h such that $\nabla_E^{0,1} = \bar{\partial}_E$, and let $\Theta_E = \nabla_E^2$ be the Chern curvature. Locally around $x \in X$, there exists an open neighbourhood $U \ni x$ with local coordinates z_1, \dots, z_n such that $x = (0, \dots, 0)$ and there exists a holomorphic frame s_1, \dots, s_r for E on U such that if H is the matrix associated to the metric with respect to s_1, \dots, s_r then*

1. $H(z) = \text{id} + \mathcal{O}(|z|^2)$, and

2. $\Theta_E(0) \stackrel{U}{=} -\partial\bar{\partial}H(0)$.

1 means $h_{ij} = \delta_{ij} + \mathcal{O}(|z|^2)$, where $(h_{ij} - \delta_{ij})/|z|^2 < C$ for some C .

Proof.

1. Let $U \ni x$ be an open set, let t_1, \dots, t_r be a holomorphic frame for E on U , and let H_1 be the matrix associated to h with respect to t_1, \dots, t_r , so $H_1(0)$ is a Hermitian matrix which gives a metric on $E(x)$. There exists an orthonormal basis of $E(x)$, that is there exists an $(r \times r)$ -matrix $B \in \text{GL}_r \mathbb{C}$ such that

$$B^\top \cdot H_1(0) \cdot \bar{B} = \text{id}.$$

Let $t'_i = B \cdot t_i$, so t'_1, \dots, t'_r is a holomorphic frame. If H_2 is the matrix of h associated to the frame t'_1, \dots, t'_r then

$$H_2(0) = \text{id}, \quad H_2(z) = \text{id} + \mathcal{O}(|z|).$$

The goal is to find a new local frame. We want to apply a change of basis given by the matrix $C(z) = \text{id} + C_0(z)$ where $C_0(z)$ has coefficients linear in z . Recall that with respect to the new frame s_1, \dots, s_r ,

$$H(z) = (\text{id} + C_0^\top) \cdot H_2(z) \cdot (\text{id} + \bar{C}_0).$$

In order to prove 1, we want $DH(0) = 0$. Recall $H_2(0) = \text{id}$. Then

$$DH = DH_2 + D(\text{id} + C_0^\top) \cdot H_2 + H_2 \cdot D(\text{id} + \bar{C}_0) + \mathcal{O}(|z|),$$

so

$$DH(0) = DH_2(0) + DC_0^\top(0) + D\bar{C}_0(0) = (\partial H_2(0) + DC_0^\top(0)) + (\bar{\partial} H_2(0) + D\bar{C}_0(0)).$$

Write

$$C_0 = (c_{ij}), \quad c_{ij} = -\sum_{l=1}^n \frac{\partial}{\partial z_l} (H_2)_{ji}(0) z_l.$$

Then

$$dc_{ij} = -\sum_{l=1}^n \sum_{k=1}^n \frac{\partial}{\partial z_l} (H_2)_{ji}(0) \frac{\partial}{\partial z_k} z_l dz_k = -\sum_{l=1}^n \frac{\partial}{\partial z_l} (H_2)_{ji}(0) dz_l,$$

so

$$DC_0^\top(0) = \partial C_0^\top(0) = -\partial H_2(0).$$

Similarly

$$D\bar{C}_0(0) = \bar{\partial} \bar{C}_0(0) = -\bar{\partial} H_2(0).$$

With this choice, we get $DH(0) = 0$, so $H(z) = \text{id} + \mathcal{O}(|z|^2)$.

2. When we constructed ∇_E , we set $A = \bar{H}^{-1} \cdot \partial \bar{H}$ and we proved $\Theta_E(z) = \bar{\partial} A(z)$ in Corollary 5.11. Since $H(0) = \text{id}$, $DH(0) = 0$, so $\partial H(0) = 0$ and $\bar{\partial} H(0) = 0$. Then

$$\Theta_E(0) = \bar{\partial} A(0) = \bar{\partial} (\bar{H}^{-1} \cdot \partial \bar{H})(0) = \bar{\partial} \bar{H}^{-1}(0) \cdot \partial \bar{H}(0) + \bar{H}^{-1}(0) \cdot \bar{\partial} \partial \bar{H}(0) = \bar{\partial} \partial \bar{H}(0) = -\partial \bar{\partial} \bar{H}(0),$$

since $\partial \bar{\partial} + \bar{\partial} \partial = 0$.

□

Lecture 18
Tuesday
18/02/20

5.5 De Rham cohomology

Given a complex manifold X , we define

$$\begin{aligned}\mathcal{Z}^k(X) &= \ker \left(d : C^\infty(X, \Omega_{X,\mathbb{C}}^k) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^{k+1}) \right), \quad k \geq 0 \\ &= \{ \omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^k) \mid d\omega = 0 \},\end{aligned}$$

and we define

$$\begin{aligned}\mathcal{B}^k(X) &= \operatorname{im} \left(d : C^\infty(X, \Omega_{X,\mathbb{C}}^{k-1}) \rightarrow C^\infty(X, \Omega_{X,\mathbb{C}}^k) \right), \quad k \geq 1 \\ &= \left\{ \omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^k) \mid \exists \eta \in C^\infty(X, \Omega_{X,\mathbb{C}}^{k-1}), \omega = d\eta \right\}.\end{aligned}$$

For convenience, we define $\mathcal{B}^0 = 0$. Since $d \circ d = 0$, it follows that $\mathcal{B}^k(X) \subset \mathcal{Z}^k(X)$ for each $k \geq 0$. Thus, we may define

$$H^k(X, \mathbb{C}) = \mathcal{Z}^k(X) / \mathcal{B}^k(X).$$

The group $H^k(X, \mathbb{C})$ is called the **de Rham cohomology group** of X . If it is finite-dimensional, then their dimension

$$b_k(X) = \dim H^k(X, \mathbb{C})$$

is called the **Betti number** of X .

Remark 5.13. If X and X' are diffeomorphic complex manifolds then

$$H^k(X, \mathbb{C}) \cong H^k(X', \mathbb{C}), \quad k \geq 0.$$

The same result is not true for the Dolbeault cohomology groups.

5.6 Holomorphic line bundles

Let X be a complex manifold, let L be a complex line bundle, and let ∇ be a connection on L . Then Θ_∇ is a C^∞ -linear operator. The idea is that $L^* \otimes L = \operatorname{Hom}(L, L)$, so $\Theta_\nabla \in C^\infty(X, \Omega_{X,\mathbb{C}}^2)$.

Proposition 5.14.

1. The curvature of ∇ defines a global 2-form $\Theta_\nabla \in C^\infty(X, \Omega_{X,\mathbb{C}}^2)$ such that $d\Theta_\nabla = 0$.
2. If ∇' is also a connection then there exists a 1-form η such that $\Theta_{\nabla'} - \Theta_\nabla = d\eta$.

Proof.

1. Let $x \in X$, and let U be open with a non-zero local section $s \in C^\infty(U, L)$. There exists $A = (a)$ with the 1-form a on U representing ∇ . That is, if $\sigma = fs \in C^\infty(U, L)$ where $f \in C^\infty(U)$, then $\nabla\sigma \stackrel{U}{=} f \cdot a + df$ and $\nabla^2\sigma = f \cdot \Theta_\nabla$. Recall that $\Theta_\nabla = a \wedge a + da = da$ is a 2-form on U , since a is a 1-form. Note that $d\Theta_\nabla = d^2a = 0$, so 1 holds. Let $U' \subset X$ be another open set trivialising L , and let

$$g : (U \cap U') \times \mathbb{C} \rightarrow (U \cap U') \times \mathbb{C}$$

be the transition. Recall that if $(a') = A'$ is the matrix representing ∇ with respect to the trivialisation on U , then $a' = g^{-1} \cdot dg + a$, so

$$da' = d(g^{-1} \cdot dg) + da = g^{-2} \cdot dg \wedge dg + g^{-1} \cdot d^2g + da = da,$$

since $dg^{-1} = g^{-2} \cdot dg$.¹⁴ Thus $\Theta_{\nabla'} = da' = da$ does not depend on U , so $\Theta_{\nabla'}$ is a global 2-form on X .

2. Let ∇' be also a connection on L . On U , let b be the 1-form representing ∇' so that $\Theta_{\nabla'} \stackrel{U}{=} db$, so $\Theta_{\nabla'} - \Theta_\nabla \stackrel{U}{=} d(b - a)$. Let U' , g , and a' be as above, and let b' be the 1-form representing ∇' on U' . Then

$$b' - a' = (g^{-1} \cdot dg + b) - (g^{-1} \cdot dg + a) = b - a.$$

Thus $\eta = b - a$ is a global 1-form. □

¹⁴Exercise

Remark 5.15. Thus, if L is a line bundle on a complex manifold X , there exists a 2-form Θ_∇ on X such that $[\Theta_\nabla]$ does not depend on ∇ , and depends only on L , as an element in $H^2(X, \mathbb{C})$, the de Rham cohomology over \mathbb{C} . We can define

$$c_1(L) = \left[\frac{i}{2\pi} \Theta_\nabla \right] \in H^2(X, \mathbb{C}),$$

the **first Chern class** of L . For vector bundles E of rank r on X , then we can define

$$c_1(E) = c_1(\bigwedge^r E),$$

where $\bigwedge^r E$ is a line bundle.

Let X be a complex manifold, and let (L, h) be a Hermitian holomorphic line bundle. Then there exists a unique Chern connection ∇_L compatible with h and such that $\nabla_L^{0,1} = \bar{\partial}_L$. Fix a non-vanishing section $s \in C^\infty(U, L)$. Then $h(x) = \langle s, s \rangle_x : U \rightarrow \mathbb{R}$, because $\langle v, w \rangle = \overline{\langle w, v \rangle}$ and h is positive definite, so

$$\phi = -\log h(x),$$

the **weight** of (L, h) on U with respect to s , is well-defined, and

$$h = e^{-\phi}.$$

Let a be the 1-form defining ∇_L . Recall that

$$a = h^{-1} \cdot \partial h = e^\phi \cdot \partial e^{-\phi} = e^\phi \cdot (-e^\phi) \cdot \partial \phi = -\partial \phi,$$

so

$$\Theta_L = \Theta_{\nabla_L} = da = (\partial + \bar{\partial})(-\partial \phi) = -\bar{\partial} \partial \phi = \partial \bar{\partial} \phi.$$

In particular Θ_L is a $(1, 1)$ -form on X .

Remark. Linear algebra. Let V be a vector space over \mathbb{C} of dimension n . Then $V_{\mathbb{R}}$ is a vector space over \mathbb{R} of dimension $2n$, so $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a vector space over \mathbb{C} of dimension $2n$. There exists a conjugation

$$\begin{array}{ccc} V_{\mathbb{C}} & \longrightarrow & V_{\mathbb{C}} \\ v & \longmapsto & \bar{v} \end{array}.$$

Let $W_{\mathbb{C}} = V_{\mathbb{C}}^* = W^{1,0} \oplus W^{0,1}$. Then

$$\bigwedge^k W_{\mathbb{C}} = \bigoplus_{p+q=k} W^{p,q}, \quad W^{p,q} = \bigwedge^p W^{1,0} \otimes \bigwedge^q W^{0,1}.$$

There exists a conjugation on $\bigwedge^k W_{\mathbb{C}}$. Then the eigenspace with respect to the eigenvalue one via the conjugation is the real forms on V .

Example. Let $V = \mathbb{C}^n$. Then $dz_j + d\bar{z}_j$ is real and $i(dz_j - d\bar{z}_j)$ is real.

Back to L . Then

$$\overline{i\Theta_L} = -i\bar{\partial}\partial\phi = i\partial\bar{\partial}\phi = i\Theta_L,$$

so $\frac{i}{2\pi}\Theta_L$ is a real $(1, 1)$ -form. Thus the first Chern class of a holomorphic line bundle is defined by a real $(1, 1)$ -form.

Remark 5.16. Assume (L, h) is a holomorphic line bundle with $h = e^{-\phi}$ locally at $x \in X$. Then if (L^{-1}, h') is with respect to the induced frame, we can write $h' = e^\phi$.

Definition 5.17. Let (L, h) be a Hermitian holomorphic line bundle on X . Then L is **positive** if for all $x \in X$, $h = e^{-\phi}$ locally at x , such that

$$\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \phi \right)$$

is positive definite, where z_1, \dots, z_n are local coordinates around x .

Example 5.18. Let $L = X \times \mathbb{C}$, let s be the constant section, and let $\phi = 1$ on X . Then $\Theta_L = 0$, so $c_1(L) = 0$.

Example 5.19. The Fubini-Study metric. Let $X = \mathbb{P}_{\mathbb{C}}^n$, let

$$\mathcal{O}(-1) = \{([x], v) \mid [x] \in \mathbb{P}_{\mathbb{C}}^n, v = \lambda x, \lambda \in \mathbb{C}\},$$

and let $U_i = \{[x] \in \mathbb{P}_{\mathbb{C}}^n \mid x_i \neq 0\} \subset \mathbb{P}_{\mathbb{C}}^n$ be a trivialising open set. Then $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}$. Define

$$\phi_i([x_0, \dots, x_n]) = -\log \frac{\sum_j |x_j|^2}{|x_i|^2} \in (0, \infty).$$

Then ϕ_i is well-defined. Claim that it defines h on $\mathcal{O}(-1)$. Let

$$g_{ij} : (U_j \cap U_i) \times \mathbb{C} \rightarrow (U_j \cap U_i) \times \mathbb{C}$$

be the transition $g_{ij} = x_i/x_j$ from U_j to U_i , and let $h_i = e^{-\phi_i}$ on U_i . Then

$$h_j = g_{ij} \cdot h_i \cdot \overline{g_{ij}},^{15}$$

which extends globally to X , is a metric on $\mathcal{O}(-1)$. Let $\mathcal{O}(1)$ be the dual of $\mathcal{O}(-1)$. Define

$$\psi_i = -\phi_i = \log \frac{\sum_j |x_j|^2}{|x_i|^2}.$$

Then ψ_i defines a metric on $\mathcal{O}(1)$. Claim that $\mathcal{O}(1)$ is positive, so

$$\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \phi \right)$$

is positive definite. Let us take $i = 0$, so $z_j = x_j/x_0$ are coordinates on $U_0 \cong \mathbb{C}^n$, and

$$\psi_0 = \log \left(1 + \sum_{j=1}^n |z_j|^2 \right).$$

Then

$$\frac{\partial}{\partial z_k} \left(\frac{\partial}{\partial \bar{z}_l} \psi_0 \right) = \frac{\partial}{\partial z_k} \left(\frac{z_l}{1 + \sum_j |z_j|^2} \right) = \frac{\delta_{kl} (1 + \sum_j |z_j|^2) - z_l \bar{z}_k}{(1 + \sum_j |z_j|^2)^2}.$$

Fix $z \in \mathbb{C}^n$, and let

$$T = \left(\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \psi_0(z) \right).$$

We want to show that T is positive definite. If $n = 1$, then $T = (1 + |z|^2)^{-2} > 0$, so ok. If $n > 1$, let $\langle \cdot, \cdot \rangle$ be the standard inner product on \mathbb{C}^n and let $\|\cdot\|$ be the norm induced by it. For each $w \in \mathbb{C}^n$, we have

$$\langle Tw, w \rangle = \frac{(1 + \|z\|^2) \|w\|^2 - |\langle z, w \rangle|^2}{(1 + \|z\|^2)^2}$$

The Cauchy-Schwarz inequality implies $|\langle z, w \rangle|^2 \leq \|z\|^2 \|w\|^2$. Thus,

$$\langle Tw, w \rangle \geq \frac{\|w\|^2}{(1 + \|z\|^2)} \geq 0.$$

and the equality holds if and only if $w = 0$. Thus T is positive definite and $\mathcal{O}(1)$ is a positive line bundle.

¹⁵Exercise

6 Kähler manifolds

The idea is if (X, ω) is compact Kähler and $k \geq 0$, then

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

6.1 Kähler manifolds

Definition 6.1. Let $V \subset \mathbb{C}^n$ be open. A **positive** real $(1, 1)$ -form on V is a real $(1, 1)$ -form

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{j\bar{k}} dz_j \wedge d\bar{z}_k, \quad (1)$$

such that $(h_{j\bar{k}})$ is positive definite.

If $z_1 = x_1 + iy_1$, then $\frac{i}{2} dz_1 \wedge d\bar{z}_1 = dx_1 \wedge dy_1$. Then ω defines a Hermitian metric on $T_{V, \mathbb{C}} = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\rangle$. Let ω be a positive real $(1, 1)$ -form as in (1). Then ω^n is a real (n, n) -form such that if $z_j = x_j + iy_j$ then

$$\omega^n = \det h_{ij} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

is a volume form.

Definition 6.2. Globally, let X be a complex manifold, and let $\omega \in C^\infty(X, \Omega_X^{1,1})$ be a real $(1, 1)$ -form. Then ω is said to be **positive** if for all $x \in X$, there exists an open $U \ni x$ and there exists a biholomorphism $\phi : U \rightarrow V \subset \mathbb{C}^n$ such that $(\phi^{-1})^* \omega$ is a positive $(1, 1)$ -form on V .

In particular ω^n is a volume form on X , so X is oriented.

Definition 6.3. A complex manifold X is called **Kähler** if there exists a positive real $(1, 1)$ -form ω on X such that $d\omega = 0$. Such ω is called a **Kähler form** on X .

Notation. (X, ω) , where X is a Kähler manifold and ω is a Kähler form.

Example 6.4. Let $X = \mathbb{C}^n$, and let $\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j$. Then ω is Kähler, so \mathbb{C}^n is Kähler.

Example 6.5. Let $X = \mathbb{C}^n / \Lambda$ be the complex torus for a lattice $\Lambda \subset \mathbb{C}^n$. Claim that X is Kähler. Let ω be as in Example 6.4. Consider

$$\begin{aligned} \psi : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ z &\longmapsto z + \lambda \end{aligned}$$

for some fixed $\lambda \in \Lambda$. Then $\psi^* \omega = \omega$, so ω descends to a positive closed real $(1, 1)$ -form on X , that is there exists ω' on X such that $q^* \omega' = \omega$ for $q : \mathbb{C}^n \rightarrow X$. Thus X is Kähler.

Example 6.6. Let $X = \mathbb{P}_{\mathbb{C}}^n$. Recall that if h is the Fubini-Study metric on $\mathcal{O}(1)$, then $i\Theta_h$ is a real positive $(1, 1)$ -form and $d\Theta_h = 0$, so X is Kähler.

Lemma 6.7. Let X be a complex manifold, let ω be a Kähler form on X , and let $i : Y \hookrightarrow X$ be an immersion for a complex submanifold Y . Then $i^* \omega$ is a Kähler form on Y . In particular Y is Kähler.

Proof. Exercise. ¹⁶ □

Corollary 6.8. Let X be a projective manifold. Then X is Kähler.

Proof. X , by definition, is a complex submanifold of $\mathbb{P}_{\mathbb{C}}^n$. By Example 6.6 $\mathbb{P}_{\mathbb{C}}^n$ is Kähler, so X is Kähler by Lemma 6.7. □

Fact. Every compact complex submanifold of $\mathbb{P}_{\mathbb{C}}^n$ is a projective manifold.

Example. Let X be a complex manifold of dimension one. Then X is Kähler. ¹⁷

Lecture 21 is a problems class.

¹⁶Exercise

¹⁷Exercise

Let (X, ω) be compact Kähler. For all $x \geq 1$, $\omega^k = \omega \wedge \cdots \wedge \omega$ is closed by the Leibnitz rule. Claim that $[\omega^k] \neq 0$ in $H^k(X, \mathbb{C})$. Assume $[\omega^k] = 0$, so there exists a $(2k-1)$ -form η such that $\omega^k = d\eta$. Since ω is closed and ω^n is a volume form,

$$0 < \int_X \omega^n = \int_X \omega^{n-k} \wedge d\eta = \int_X d(\omega^{n-k} \wedge \eta) = \int_{\partial X} \omega^{n-k} \wedge \eta = 0,$$

by the Leibnitz rule. Thus

$$H^k(X, \mathbb{C}) \neq 0, \quad k \in 2\mathbb{Z}.$$

Example 6.9. Pick $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$. Then \mathbb{Z} acts on $\mathbb{C}^n \setminus \{0\}$ by

$$\begin{aligned} \mathbb{Z} \times (\mathbb{C}^n \setminus \{0\}) &\longrightarrow \mathbb{C}^n \setminus \{0\} \\ (n, z) &\longmapsto \lambda^n z \end{aligned}.$$

Then $X = \mathbb{C}^n \setminus \{0\} / \mathbb{Z}$ is a **Hopf manifold**. Similarly to the case of complex tori, X can be shown to have a complex structure. Then

$$S^{2n-1} \subset \mathbb{R}^{2n} \setminus \{0\} = \mathbb{C}^n \setminus \{0\} \cong S^{2n-1} \times \mathbb{R}_{>0},$$

so $X \sim S^{2n-1} \times S^1$. Thus if $n \geq 2$, then $H^k(X, \mathbb{C}) = 0$, so X is not Kähler.

6.2 Hodge \star operator

Let V be a vector space over \mathbb{R} of dimension n with an inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$. There is a canonical inner product on $\bigwedge^p V$ for all $p \geq 1$,

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle).$$

In particular there exists a unique up to orientation $\omega \in \bigwedge^n V$ such that $\|\omega\| = 1$. The **Hodge \star operator** is

$$\star : \bigwedge^p V \rightarrow \bigwedge^{n-p} V, \quad p \geq 0,$$

such that

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \omega \in \bigwedge^n V, \quad \alpha, \beta \in \bigwedge^p V.$$

Let e_1, \dots, e_n be an orthonormal basis of V . Then

- $\star 1 = \omega$,
- $\star \omega = 1$,
- $\star e_1 = e_2 \wedge \cdots \wedge e_n$,
- $\star e_i = (-1)^{i-1} e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$, and
- more in general if $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ is ordered such that $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$, and $\mathcal{C}I = \{1, \dots, n\} \setminus I$ such that $\sigma : \{1, \dots, n\} \rightarrow \{I, \mathcal{C}I\}$ is a permutation, then

$$\star e_I = \epsilon(\sigma) e_{\mathcal{C}I},$$

where ϵ is the signature of σ , so

$$\star \star \eta = (-1)^{k(n-k)} \eta, \quad \eta \in \bigwedge^k V.$$

Assume now that V is a complex vector space of dimension n with a Hermitian metric $\langle \cdot, \cdot \rangle$. Then, for each $k \geq 0$, we can extend the Hodge \star operator to $V_{\mathbb{C}}$ to a \mathbb{C} -linear map

$$\star : \bigwedge^k V_{\mathbb{C}} \rightarrow \bigwedge^{2n-k} V_{\mathbb{C}},$$

so that

$$\alpha \wedge \overline{\star \beta} = \langle \alpha, \beta \rangle \omega, \quad \alpha, \beta \in \bigwedge^k V_{\mathbb{C}}.$$

Note. In particular, $\overline{\star \beta} = \star \bar{\beta}$.

Let X be a complex manifold, and let E be a Hermitian holomorphic vector bundle. Recall that we defined

$$\{\cdot, \cdot\} : C^\infty(X, \Omega_{X, \mathbb{C}}^p \otimes E) \times C^\infty(X, \Omega_{X, \mathbb{C}}^q \otimes E) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^{p+q}).$$

Take $p = q = 0$ and $E = \Omega_{X, \mathbb{C}}^k$. If ω is a positive real $(1, 1)$ -form then ω induces a Hermitian metric on $T_{X, \mathbb{C}}$. Locally, let e_1, \dots, e_n be an orthonormal frame of $T_{X, \mathbb{C}}$. Then e_1^*, \dots, e_n^* define a metric on $\Omega_{X, \mathbb{C}}^1$ locally, where $e_i^*(e_j) = \delta_{ij}$. It is easy to check that such a choice is canonical, so the metric on $\Omega_{X, \mathbb{C}}^1$ extends to X . This induces a metric on $\Omega_{X, \mathbb{C}}^k$ for all $k \geq 0$, so there exists a cup product

$$\{\cdot, \cdot\} : C^\infty(X, \Omega_{X, \mathbb{C}}^k) \times C^\infty(X, \Omega_{X, \mathbb{C}}^k) \rightarrow C^\infty(X).$$

Lemma 6.10. *Let (X, ω) be Kähler of dimension n . Then there exists a \mathbb{C} -linear*

$$\star : C^\infty(X, \Omega_{X, \mathbb{C}}^k) \rightarrow C^\infty(X, \Omega_{X, \mathbb{C}}^{2n-k}), \quad k \geq 0,$$

such that

$$\alpha \wedge \star \beta = \{\alpha, \beta\} \omega^n, \quad \alpha, \beta \in C^\infty(X, \Omega_{X, \mathbb{C}}^k).$$

Check that it is defined globally on X . Let E be a vector bundle. Then let

$$C_c^\infty(X, E) = \{s \in C^\infty(X, E) \mid s \text{ has compact support}\}.$$

Let E be Hermitian, and let ω be a positive real $(1, 1)$ -form. Then let

$$(\alpha, \beta)_E = \int_X \{\alpha, \beta\} \omega^n, \quad \alpha, \beta \in C_c^\infty(X, E).$$

Let (X, ω) be Kähler, let E and F be Hermitian vector bundles on X , and let $P : C_c^\infty(X, E) \rightarrow C_c^\infty(X, F)$ be \mathbb{C} -linear. Then the **adjoint** of P is a \mathbb{C} -linear map $P^* : C_c^\infty(X, F) \rightarrow C_c^\infty(X, E)$ such that

$$(P\alpha, \beta)_F = (\alpha, P^*\beta)_E, \quad \alpha \in C_c^\infty(X, E), \quad \beta \in C_c^\infty(X, F).$$

In particular if $d : C_c^\infty(X, \Omega_{X, \mathbb{C}}^k) \rightarrow C_c^\infty(X, \Omega_{X, \mathbb{C}}^{k+1})$ then

$$d^* : C_c^\infty(X, \Omega_{X, \mathbb{C}}^{k+1}) \rightarrow C_c^\infty(X, \Omega_{X, \mathbb{C}}^k).$$

Lemma 6.11. *Let (X, ω) be Kähler of dimension n . Then*

$$d^*\beta = (-1) \star d \star \beta, \quad \beta \in C_c^\infty(X, \Omega_{X, \mathbb{C}}^{k+1}).$$

Proof. Let $\alpha \in C_c^\infty(X, \Omega_{X, \mathbb{C}}^k)$. Then

$$\begin{aligned} (d\alpha, \beta) &= \int_X d\alpha \wedge \star \beta & \{\eta_1, \eta_2\} \omega^n &= \eta_1 \wedge \star \eta_2 \\ &= \int_X d(\alpha \wedge \star \beta) - (-1)^k \int_X \alpha \wedge d\star \beta & \text{the Leibnitz rule} \\ &= (-1)^{k+1} \int_X \alpha \wedge d\star \beta & \text{Stokes} \\ &= (-1)^{(2n-k)k+k+1} \int_X \alpha \wedge \star \star d\star \beta & \star \star \eta = (-1)^{k(2n-k)} \eta \\ &= - \int_X \alpha \wedge \star \star d\star \beta & k^2 - k \text{ is even} \\ &= - \int_X \{\alpha, \star d\star \beta\} \omega^n & \{\eta_1, \eta_2\} \omega^n &= \eta_1 \wedge \star \eta_2 \\ &= - \int_X \{\alpha, \star d\star \beta\} \omega^n & \star \eta &= \star \eta \\ &= -(\alpha, \star d \star \beta). \end{aligned}$$

□

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6.3 Harmonic forms

Definition 6.12. Let (X, ω) be a Kähler manifold. The **Hodge-de Rham operator** is

$$\Delta = dd^* + d^*d : C_c^\infty(X, \Omega_{X, \mathbb{C}}^k) \rightarrow C_c^\infty(X, \Omega_{X, \mathbb{C}}^k).$$

Then $\alpha \in C_c^\infty(X, \Omega_{X, \mathbb{C}}^k)$ is said to be **harmonic** if $\Delta\alpha = 0$. Let

$$\mathcal{H}^k(X) = \{\text{harmonic } k\text{-forms } \alpha \text{ on } X\} = \ker \Delta.$$

Lemma 6.13. Let (X, ω) be Kähler, and let $\alpha \in C_c^\infty(X, \Omega_{X, \mathbb{C}}^k)$. Then α is harmonic if and only if $d\alpha = d^*\alpha = 0$.

Proof. If $d\alpha = d^*\alpha = 0$, then $\Delta\alpha = 0$. Assume $\Delta\alpha = 0$. Then

$$0 = (\Delta\alpha, \alpha) = (dd^*\alpha + d^*d\alpha, \alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha) = \|d^*\alpha\|^2 + \|d\alpha\|^2,$$

so $d\alpha = d^*\alpha = 0$. □

Example 6.14. Let $X = \mathbb{C}^n$, and let $\omega = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i$. Then (X, ω) is Kähler. Let $k = 0$, let $z_j = x_j + iy_j$, and let $f \in C^\infty(X)$ such that $f = f(x_1, \dots, x_n, y_1, \dots, y_n)$. Then

$$\Delta f = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} f + \frac{\partial^2}{\partial y_i^2} f \right) \in C^\infty(X),^{18}$$

which is the Laplacian.

Lemma 6.15. Let (X, ω) be Kähler. Then Δ and \star commute, that is

$$\Delta \star \alpha = \star \Delta \alpha, \quad \alpha \in C^\infty(X, \Omega_{X, \mathbb{C}}^k).$$

Proof. Use $d^* = (-1) \star d \star$ and $\star \star = (-1)^{k(2n-k)}$. □

Lemma 6.16. Let (X, ω) be Kähler. Then Δ is auto-adjoint, that is

$$\{\Delta\alpha, \beta\} \omega^n = \{\alpha, \Delta\beta\} \omega^n, \quad \alpha, \beta \in C^\infty(X, \Omega_{X, \mathbb{C}}^k).$$

Proof.

$$\{\Delta\alpha, \beta\} \omega^n = \{dd^*\alpha + d^*d\alpha, \beta\} \omega^n = \{\alpha, d^*d\beta + dd^*\beta\} \omega^n = \{\alpha, \Delta\beta\} \omega^n.$$

□

Theorem 6.17. Let (X, ω) be a compact Kähler manifold. Then, for all $k \geq 0$,

1. $\mathcal{H}^k(X)$ is a finite-dimensional vector space, and
2. there exist orthogonal decompositions

$$C^\infty(X, \Omega_{X, \mathbb{C}}^k) = \mathcal{H}^k(X) \oplus \Delta C^\infty(X, \Omega_{X, \mathbb{C}}^k) = \mathcal{H}^k(X) \oplus dC^\infty(X, \Omega_{X, \mathbb{C}}^{k-1}) \oplus d^*C^\infty(X, \Omega_{X, \mathbb{C}}^{k+1}),$$

such that

$$\ker d = \mathcal{H}^k(X) \oplus dC^\infty(X, \Omega_{X, \mathbb{C}}^{k-1}), \quad \ker d^* = \mathcal{H}^k(X) \oplus d^*C^\infty(X, \Omega_{X, \mathbb{C}}^{k+1}).$$

Proof.

1. Hard, omit the proof, see Höring.
2. If $\alpha \in \mathcal{H}^k(X)$ and $\beta \in C^\infty(X, \Omega_{X, \mathbb{C}}^k)$, then $(\alpha, \Delta\beta) = (\Delta\alpha, \beta) = 0$. If $\alpha \in C^\infty(X, \Omega_{X, \mathbb{C}}^{k-1})$ and $\beta \in C^\infty(X, \Omega_{X, \mathbb{C}}^{k+1})$, then $(d\alpha, d^*\beta) = (dd\alpha, \beta) = 0$.

□

¹⁸Exercise

Theorem 6.18. *Let (X, ω) be compact Kähler, and let $k \geq 0$. Then*

$$\mathcal{H}^k(X) = H^k(X, \mathbb{C}).$$

Proof. By Theorem 6.17 $\ker d = \mathcal{H}^k(X) \oplus dC^\infty(X, \Omega_{X, \mathbb{C}}^{k-1})$, so

$$\mathcal{H}^k(X) = \ker d / dC^\infty(X, \Omega_{X, \mathbb{C}}^{k-1}) = \mathcal{Z}^k(X) / \mathcal{B}^k(X) = H^k(X, \mathbb{C}).$$

□

Theorem 6.19 (Poincaré duality). *Let (X, ω) be a compact Kähler manifold. Then there exists an isomorphism*

$$H^k(X, \mathbb{C}) \rightarrow H^{2n-k}(X, \mathbb{C}), \quad k \geq 0.$$

Proof. Want to check

$$\star : \mathcal{H}^k(X) \xrightarrow{\sim} \mathcal{H}^{n-k}(X).$$

Given a harmonic k -form α then $\star\alpha$ is a harmonic k -form, since $\Delta\star\alpha = \star\Delta\alpha = \star 0 = 0$, by Theorem 6.13. □

6.4 Harmonic (p, q) -forms

The goal is to study a similar decomposition for (p, q) -forms. Let ω be a positive real $(1, 1)$ -form. Locally at $x \in X$, choose local coordinates z_1, \dots, z_n around x . Then $\Omega_{X, \mathbb{C}}^1$ is spanned by $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$. After a linear change of basis, we may assume that the frame is orthonormal at x , and not locally around x , so

$$\begin{pmatrix} z'_1 \\ \vdots \\ z'_n \end{pmatrix} = A \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix},$$

where A is fixed. Then $h_{ij} = \text{id} + \mathcal{O}(|z|)$, so $\{dz_i, d\bar{z}_j\} = \delta_{ij}$. This implies that if α is a (p, q) -form around x , so

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I, J} dz_I \wedge d\bar{z}_J,$$

then $\star\alpha$ is an $(n-p, n-q)$ -form at the point x . Since \star does not depend on choice of coordinates, $\star\alpha$ is a $(n-p, n-q)$ -form, so

$$\star : C^\infty(X, \Omega_X^{p, q}) \rightarrow C^\infty(X, \Omega_X^{n-p, n-q}).$$

Thus, if α is a (p, q) -form and β is a (p', q') -form such that $p+q = p'+q'$ then $\{\alpha, \beta\} = 0$ unless $p = p'$ and $q = q'$, so

$$C^\infty(X, \Omega_{X, \mathbb{C}}^k) = \bigoplus_{p+q=k} C^\infty(X, \Omega_X^{p, q})$$

is an orthogonal decomposition. Recall that $\partial : C^\infty(X, \Omega_X^{p, q}) \rightarrow C^\infty(X, \Omega_X^{p+1, q})$. Then there exist

$$\partial^* : C^\infty(X, \Omega_X^{p+1, q}) \rightarrow C^\infty(X, \Omega_X^{p, q}), \quad \bar{\partial}^* : C^\infty(X, \Omega_X^{p, q+1}) \rightarrow C^\infty(X, \Omega_X^{p, q}).$$

Like Lemma 6.11, like in the case of d ,

$$\partial^* = (-1) \star \partial \star, \quad \bar{\partial}^* = (-1) \star \bar{\partial} \star.$$

Moreover we define

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

We say that a form α is Δ_∂ -**harmonic** if $\Delta_\partial\alpha = 0$ and $\Delta_{\bar{\partial}}$ -**harmonic** if $\Delta_{\bar{\partial}}\alpha = 0$. As for Δ we have the following.

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Lemma 6.20.

- $\Delta_{\partial}\alpha = 0$ if and only if $\partial\alpha = \partial^*\alpha = 0$.
- $\Delta_{\bar{\partial}}\alpha = 0$ if and only if $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$.

Let

$$\mathcal{H}^{p,q}(X) = \{\text{harmonic } (p,q)\text{-forms } \alpha \mid \Delta_{\bar{\partial}}\alpha = 0\}.$$

Theorem 6.21. Let (X, ω) be a compact Kähler manifold. Then $\mathcal{H}^{p,q}(X)$ are finite-dimensional vector spaces,

$$C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q}(X) \oplus \Delta C^\infty(X, \Omega_X^{p,q}) = \mathcal{H}^{p,q}(X) \oplus \bar{\partial} C^\infty(X, \Omega_X^{p,q-1}) \oplus \bar{\partial}^* C^\infty(X, \Omega_X^{p,q+1}),$$

and

$$\ker \bar{\partial} = \mathcal{H}^{p,q}(X) \oplus \bar{\partial} C^\infty(X, \Omega_X^{p,q-1}), \quad \ker \bar{\partial}^* = \mathcal{H}^{p,q}(X) \oplus \bar{\partial}^* C^\infty(X, \Omega_X^{p,q+1}).$$

Theorem 6.22. The setup is as before. Then

$$\mathcal{H}^{p,q}(X) = H^{p,q}(X) = \ker \bar{\partial} / \text{im } \bar{\partial}.$$

Recall that $\mathcal{H}^k(X) = H^k(X, \mathbb{C})$. The goal is if (X, ω) is Kähler and compact then

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X).$$

6.5 Lefschetz operator

Definition 6.23. Let (X, ω) be Kähler. The **Lefschetz operator** is

$$\begin{aligned} L &: C_c^\infty(X, \Omega_{X,\mathbb{C}}^k) \longrightarrow C_c^\infty(X, \Omega_{X,\mathbb{C}}^{k+2}) \\ \alpha &\longmapsto \alpha \wedge \omega \end{aligned}$$

There exists an adjoint operator

$$\Lambda = L^* : C_c^\infty(X, \Omega_{X,\mathbb{C}}^{k+2}) \rightarrow C_c^\infty(X, \Omega_{X,\mathbb{C}}^k).$$

Lemma 6.24. For all k , if $\beta \in C_c^\infty(X, \Omega_{X,\mathbb{C}}^{k+2})$ then

$$\Lambda\beta = (-1)^k \star L \star \beta.$$

Proof. Let $\alpha \in C_c^\infty(X, \Omega_{X,\mathbb{C}}^k)$. Since $\star\star = (-1)^k$,

$$\begin{aligned} (L\alpha, \beta) &= \int_X \{L\alpha, \beta\} \omega^n = \int_X L\alpha \wedge \star\beta = \int_X \alpha \wedge \omega \wedge \star\beta = \int_X \omega \wedge \alpha \wedge \star\beta \\ &= \int_X \omega \wedge \alpha \wedge \star(-1)^k \star\star\beta = \int_X \left\{ \alpha, (-1)^k \star L \star \beta \right\} \omega^n = \left(\alpha, (-1)^k \star L \star \beta \right), \end{aligned}$$

and $(-1)^k \star L \star$ is the adjoint of L . □

Definition 6.25. Let $p : E \rightarrow X$ and $q : F \rightarrow X$ be holomorphic vector bundles on X . Then a \mathbb{C} -linear

$$P : C^\infty(X, E) \rightarrow C^\infty(X, F)$$

is called a **differential operator of order d** if for all $x \in X$, there exists an open set $U \ni x$, local coordinates z_1, \dots, z_n , a frame s_1, \dots, s_r for E , and a frame t_1, \dots, t_l for F , such that

$$P \left(\sum_{i=1}^r f_i s_i \right) = \sum_{i=1, \dots, l, j=1, \dots, r, I=(i_1, \dots, i_n)} P_{I,i,j} \frac{\partial f_j}{\partial x_I} t_i, \quad f_j \in C^\infty(U),$$

where $P_{I,i,j} = 0$ if $|I| > d$ and $P_{I,i,j} \neq 0$ for some $|I| = d$.

Fact. The definition and the order of P do not depend on the coordinates and on the frames.

Notation 6.26. Let A be an operator of order a , and let B be an operator of order b . Then the **Lie bracket** $[A, B]$ is an operator of order $a + b$ given by

$$[A, B] = AB - (-1)^{a \cdot b} BA.$$

Definition 6.27. Let X be a complex manifold, let v be a vector field, and let ω be a k -form on X . Then $v \lrcorner \omega$, the **contraction** of ω with respect to v , is a $(k - 1)$ -form defined by

$$(v \lrcorner \omega)(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1}),$$

on $C^\infty(X)$.

Example 6.28. Let $U \subset \mathbb{C}^n$ be open. Then T_U is spanned by the frame $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$. Let $I = (i_1, \dots, i_k)$. It is easy to check

$$\frac{\partial}{\partial z_m} \lrcorner dz_I = \begin{cases} 0 & m \notin \{i_1, \dots, i_k\} \\ (-1)^{l-1} dz_{i_1} \wedge \dots \wedge \widehat{dz_{i_l}} \wedge \dots \wedge dz_{i_k} & m = i_l \end{cases}.$$

Exercise. Let $v \in C^\infty(U, T_U)$, and let $\alpha \in C^\infty(U, \Omega_{U, \mathbb{C}}^p)$ and $\beta \in C^\infty(U, \Omega_{U, \mathbb{C}}^q)$. Then

$$v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner \beta).$$

6.6 Kähler identities

The goal is the Hodge decomposition. Want to show if (X, ω) is a Kähler manifold then

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

where $\mathcal{H}^k(X)$ are Δ -harmonic forms and $\mathcal{H}^{p,q}(X)$ are $\Delta_{\bar{\partial}}$ -harmonic forms. We want to compose Δ with $\Delta_{\bar{\partial}}$, by

- Kähler identities on \mathbb{C}^n with the standard metric, and
- showing any Kähler manifold is locally like \mathbb{C}^n with the standard metric.

Example 6.29. Let $U \subset \mathbb{C}^n$, and let $\omega = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i$ be Kähler. With respect to such a metric the frame $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ is orthonormal. Let

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J} dz_I \wedge d\bar{z}_J \in C_c^\infty(U, \Omega_U^{p,q}), \quad \alpha_{I,J} \in C_c^\infty(U).$$

Recall $\partial^* = (-1) \star \partial \star$. From the definition of \star ,

$$\partial^* \alpha = - \sum_{k=1}^n \sum_{|I|=p, |J|=q} \frac{\partial}{\partial z_k} \alpha_{I,J} \frac{\partial}{\partial \bar{z}_k} \lrcorner (dz_I \wedge d\bar{z}_J).^{19}$$

The notation is

$$\frac{\partial}{\partial z_k} \alpha = \sum_{|I|=p, |J|=q} \frac{\partial}{\partial z_k} \alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Thus

$$\partial^* \alpha = - \sum_{k=1}^n \frac{\partial}{\partial \bar{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha.$$

¹⁹Exercise

Lemma 6.30 (Kähler identity on \mathbb{C}^n). *Let $U \subset \mathbb{C}^n$ be open, and let $\omega = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i$. Then*

$$[\bar{\partial}^*, L] = i\partial.$$

Proof. By \mathbb{C} -linearity, we may assume $\alpha = \alpha_{I,J} dz_I \wedge d\bar{z}_J$ for some $|I| = p$ and $|J| = q$. Then $\partial^* \alpha = -\sum_{k=1}^n \frac{\partial}{\partial \bar{z}_k} \lrcorner \frac{\partial}{\partial z_k} \alpha$, so

$$[\bar{\partial}^*, L] \alpha = \bar{\partial}^* L \alpha - L \bar{\partial}^* \alpha = -\sum_{k=1}^n \frac{\partial}{\partial z_k} \lrcorner \frac{\partial}{\partial \bar{z}_k} (\omega \wedge \alpha) + \omega \wedge \left(\sum_{k=1}^n \frac{\partial}{\partial z_k} \lrcorner \frac{\partial}{\partial \bar{z}_k} \alpha \right).$$

Since $\frac{\partial}{\partial \bar{z}_k} (\omega \wedge \alpha) = \omega \wedge \frac{\partial}{\partial \bar{z}_k} \alpha$,

$$[\bar{\partial}^*, L] \alpha = -\sum_{k=1}^n \frac{\partial}{\partial z_k} \lrcorner \left(\omega \wedge \frac{\partial}{\partial \bar{z}_k} \alpha \right) + \omega \wedge \left(\sum_{k=1}^n \frac{\partial}{\partial z_k} \lrcorner \frac{\partial}{\partial \bar{z}_k} \alpha \right).$$

Recall that $v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \lrcorner \beta)$, so

$$\frac{\partial}{\partial z_k} \lrcorner \left(\omega \wedge \frac{\partial}{\partial \bar{z}_k} \alpha \right) = \left(\frac{\partial}{\partial z_k} \lrcorner \omega \right) \wedge \frac{\partial}{\partial \bar{z}_k} \alpha + \omega \wedge \left(\frac{\partial}{\partial z_k} \lrcorner \frac{\partial}{\partial \bar{z}_k} \alpha \right),$$

since $p = 2$. Since $\frac{\partial}{\partial z_k} \lrcorner \omega = i d\bar{z}_k$,

$$\frac{\partial}{\partial z_k} \lrcorner \left(\omega \wedge \frac{\partial}{\partial \bar{z}_k} \alpha \right) = i d\bar{z}_k \wedge \frac{\partial}{\partial \bar{z}_k} \alpha + \omega \wedge \left(\frac{\partial}{\partial z_k} \lrcorner \frac{\partial}{\partial \bar{z}_k} \alpha \right).$$

Thus

$$[\bar{\partial}^*, L] \alpha = \sum_{k=1}^n i d\bar{z}_k \wedge \frac{\partial}{\partial \bar{z}_k} \alpha = i \partial \alpha.$$

□

Theorem 6.31. *Let (X, ω) be a Kähler manifold, and let $x \in X$. There exist local holomorphic coordinates z_1, \dots, z_n around x such that if*

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{jk} dz_k \wedge d\bar{z}_j,$$

then

$$h_{jk} = \delta_{jk} + \mathcal{O}(|z|^2).$$

Remark. Assume that X is a complex manifold and ω is a positive real $(1, 1)$ -form which satisfies (1). Then ω is Kähler, so $d\omega = 0$.²⁰

Proof. Recall there exists a linear change of coordinates such that at x , $h_{jk}(x) = \delta_{jk}$, that is $h_{jk}(z) = \delta_{jk} + \mathcal{O}(|z|)$. Let

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{jk} dz_k \wedge d\bar{z}_j.$$

Then

$$h_{jk} = \delta_{jk} + \sum_{l=1}^n (a_{jkl} z_l + a'_{jkl} \bar{z}_l) + \mathcal{O}(|z|^2), \quad a_{jkl}, a'_{jkl} \in \mathbb{C}.$$

Since h_{jk} is Hermitian $\overline{a_{jkl}} = a'_{kjl}$. Since ω is Kähler, $d\omega = 0$, so $a_{jkl} = a_{ljk}$. Write

$$\xi_k = z_k + \frac{1}{2} \sum_{j,l=1}^n a_{jkl} z_j \bar{z}_l, \quad k = 1, \dots, n.$$

²⁰Exercise

Then ξ_k is holomorphic. Let $\phi(z_1, \dots, z_n) = (\xi_1, \dots, \xi_n)$. Then $d\phi_x = \text{id}$, so around x , $\det d\phi \neq 0$. By the implicit function theorem, ϕ is locally an isomorphism, so ξ_1, \dots, ξ_n are homogeneous holomorphic coordinates at x . Then

$$d\xi_k = dz_k + \frac{1}{2} \sum_{j,l=1}^n a_{jkl} (z_j dz_j + z_l dz_l) = dz_k + \frac{1}{2} \sum_{j,l=1}^n (a_{jkl} + a_{ljk}) z_l dz_j = dz_k + \sum_{j,l=1}^n a_{jkl} z_l dz_j,$$

so

$$\begin{aligned} i \sum_{k=1}^n (d\xi_k \wedge d\bar{\xi}_k) &= i \sum_{k=1}^n dz_k \wedge d\bar{z}_k + i \sum_{j,k,l=1}^n (\overline{a_{jkl} z_l} dz_k \wedge d\bar{z}_j + a_{jkl} z_l dz_j \wedge d\bar{z}_k) + \mathcal{O}(|z|^2) \\ &= i \sum_{j,k=1}^n \delta_{jk} dz_j \wedge d\bar{z}_k + i \sum_{j,k,l=1}^n (a'_{jkl} \bar{z}_l + a_{jkl} z_l) dz_j \wedge d\bar{z}_k + \mathcal{O}(|z|^2) \\ &= i \sum_{j,k=1}^n \left(\delta_{jk} + \sum_{l=1}^n (a'_{jkl} \bar{z}_l + a_{jkl} z_l) \right) dz_j \wedge d\bar{z}_k + \mathcal{O}(|z|^2) \\ &= i \sum_{j,k=1}^n h_{jk} dz_j \wedge d\bar{z}_k = 2\omega. \end{aligned}$$

□

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Then ξ_1, \dots, ξ_n are **normal coordinates** for the Kähler metric.

Theorem 6.32 (Kähler identities). *Let (X, ω) be Kähler. Then*

1. $[\bar{\partial}^*, L] = i\partial$,
2. $[\partial^*, L] = -i\bar{\partial}$,
3. $[\Lambda, \bar{\partial}] = -i\partial^*$, and
4. $[\Lambda, \partial] = i\bar{\partial}^*$.

Proof.

1. $[\bar{\partial}^*, L] = \bar{\partial}^* L - L \bar{\partial}^* = (-1) \star \bar{\partial} \star L + L \star \bar{\partial} \star$. We want to check that 1 holds at $x \in X$. But around x , we may assume that

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{jk} dz_j \wedge d\bar{z}_k, \quad h_{jk} = \delta_{jk} + \mathcal{O}(|z|^2).$$

In the calculation only the first order of ω appears at x , so we can pretend that $X = \mathbb{C}^n$ and ω is the standard metric. We checked that 1 holds on this setting, so 1 holds.

2. Since ω is real $L = \bar{L}$, so 2 holds.
3. Recall Λ is the adjoint of L . Let α and β be (p, q) -forms. Then

$$\begin{aligned} ([\Lambda, \bar{\partial}] \alpha, \beta) &= (\Lambda \bar{\partial} \alpha - \bar{\partial} \Lambda \alpha, \beta) = (\bar{\partial} \alpha, L \beta) - (\Lambda \alpha, \bar{\partial}^* \beta) = (\alpha, \bar{\partial}^* L \beta - L \bar{\partial}^* \beta) \\ &= (\alpha, [\bar{\partial}^*, L] \beta) = (\alpha, i\partial \beta) = (-i\partial^* \alpha, \beta). \end{aligned}$$

4. 3 implies 4, because $\bar{\Lambda} = \Lambda$.

□

Theorem 6.33. *Let (X, ω) be a Kähler manifold. Then*

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

Proof. Since $\partial^2 = 0$,

$$\begin{aligned} \Delta &= dd^* + d^*d \\ &= (\partial + \bar{\partial}) \left(\partial^* + \bar{\partial}^* \right) + \left(\partial^* + \bar{\partial}^* \right) (\partial + \bar{\partial}) & d &= \partial + \bar{\partial} \\ &= \Delta_\partial + \partial\bar{\partial}^* + \bar{\partial}\partial^* + \bar{\partial}\bar{\partial}^* + \partial^*\bar{\partial} + \bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} & \Delta_\partial &= \partial\partial^* + \partial^*\partial \\ &= \Delta_\partial + \partial\bar{\partial}^* + \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} & \bar{\partial}\partial^* &= i\bar{\partial}[\Lambda, \bar{\partial}] = i\bar{\partial}\Lambda\bar{\partial} = -i[\Lambda, \bar{\partial}]\bar{\partial} = \partial^*\bar{\partial} \text{ by 3} \\ &= \Delta_\partial + \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} & \bar{\partial}\bar{\partial}^* &= -i\partial[\Lambda, \bar{\partial}] = -i\partial\Lambda\bar{\partial} = i[\Lambda, \partial]\partial = \bar{\partial}^*\partial \text{ by 4} \\ &= \Delta_\partial - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda - i\Lambda\bar{\partial}\bar{\partial} + i\partial\Lambda\bar{\partial} & \bar{\partial}^* &= -i[\Lambda, \partial] = -i\Lambda\partial + i\partial\Lambda \text{ by 4} \\ &= \Delta_\partial + i[\Lambda, \bar{\partial}]\partial + i\partial[\Lambda, \bar{\partial}] & \bar{\partial}\partial + \partial\bar{\partial} &= 0 \\ &= \Delta_\partial + \partial^*\partial + \partial\bar{\partial}^* & \partial^* &= i[\Lambda, \bar{\partial}] \text{ by 3} \\ &= 2\Delta_\partial & \Delta_\partial &= \partial\partial^* + \partial^*\partial. \end{aligned}$$

Similarly $\Delta = 2\Delta_{\bar{\partial}}$. □

6.7 Hodge decomposition

Lemma 6.34. *If α is a (p, q) -form then $\Delta\alpha$ is a (p, q) -form.*

Proof. Easy. □

Theorem 6.35. *Let (X, ω) be Kähler. Then*

1. *for all $k \geq 0$,*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

2. *for all p and q ,*

$$\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}.$$

Proof.

1. Let $\alpha \in C_c^\infty(X, \Omega_{X,\mathbb{C}}^k)$. Then there is a unique decomposition

$$\alpha = \sum_{p+q=k} \alpha_{p,q}, \quad \alpha_{p,q} \in C_c^\infty(X, \Omega_X^{p,q}),$$

so $\Delta\alpha = \sum_{p+q=k} \Delta\alpha_{p,q}$. Thus

$$\begin{aligned} \alpha \in \mathcal{H}^k(X) &\iff \Delta\alpha = 0 \\ &\iff \forall p, q, p+q=k, \Delta\alpha_{p,q} = 0 \\ &\iff \forall p, q, p+q=k, \Delta_{\bar{\partial}}\alpha_{p,q} = 0 && \text{Theorem 6.33} \\ &\iff \forall p, q, p+q=k, \alpha_{p,q} \in \mathcal{H}^{p,q}(X). \end{aligned}$$

2. Let $\alpha \in C_c^\infty(X, \Omega_X^{p,q})$. Then $\bar{\alpha} \in C_c^\infty(X, \Omega_X^{q,p})$. Thus by Theorem 6.33,

$$\Delta\alpha = 0 \iff \Delta_\partial\alpha = 0 \iff \Delta_{\bar{\partial}}\bar{\alpha} = 0 \iff \Delta\bar{\alpha} = 0.$$

□

Theorem 6.36 (Hodge decomposition). *Let (X, ω) be a compact Kähler manifold. Then*

1. *for all $k \geq 0$,*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

2. *for all p and q ,*

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

Exercise. If $X = \mathbb{C}$ Theorem 6.36 is false.

Proof. $H^k(X, \mathbb{C}) = \mathcal{H}^k(X)$ by Theorem 6.18 and $H^{p,q}(X) = \mathcal{H}^{p,q}(X)$ by Theorem 6.22. \square

Example 6.37. If $X = \mathbb{P}_{\mathbb{C}}^n$, by Mayer-Vietoris,

$$H^k(X, \mathbb{C}) = \begin{cases} \mathbb{C} & k = 0, \dots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

Since $H^{k,k}(X) \neq 0$ for all $k = 0, \dots, n$,

$$H^{p,q}(X) = \begin{cases} \mathbb{C} & p = q \\ 0 & \text{otherwise} \end{cases}.$$

6.8 Bott-Chern cohomology

Let (X, ω) be Kähler. Then for any $k \geq 0$,

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where the de Rham cohomology $H^k(X, \mathbb{C})$ depends on the topology, and the Dolbeault cohomology $H^{p,q}(X)$ depends on the complex structure. The proof depended on

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

where $\mathcal{H}^k(X)$ are Δ -harmonic forms and $\mathcal{H}^{p,q}(X)$ are $\Delta_{\bar{\partial}}$ -harmonic forms. Recall that $\Delta = dd^* + d^*d$ depends on $d^* = (-1)^k \star d \star$, which depends on \star , and \star depends on the metric on ω .

Example. If X is projective there exist many ways for $X \subset \mathbb{P}^N$, so there exist many metrics on X .

The goal is to show that the Hodge decomposition does not depend on ω . Let $\alpha \in C^\infty(X, \Omega_X^{p-1, q-1})$. Then $\partial\bar{\partial}\alpha \in C^\infty(X, \Omega_X^{p,q})$ is closed, since $d\partial\bar{\partial}\alpha = \partial\bar{\partial}\partial\alpha + \bar{\partial}(-\partial\bar{\partial}\alpha) = 0$. Thus we can define the **Bott-Chern cohomology group**

$$H_{BC}^{p,q}(X) = \{\alpha \in C^\infty(X, \Omega_X^{p,q}) \mid d\alpha = 0\} / \partial\bar{\partial}C^\infty(X, \Omega_X^{p-1, q-1}),$$

which is independent of ω . There exists a \mathbb{C} -linear map $\Phi : \{\alpha \in C^\infty(X, \Omega_X^{p,q}) \mid d\alpha = 0\} \rightarrow H^{p+q}(X, \mathbb{C})$. Let $\partial\bar{\partial}\alpha \in \partial\bar{\partial}C^\infty(X, \Omega_X^{p-1, q-1})$. Then $\partial\bar{\partial}\alpha = d\bar{\partial}\alpha = 0$ in $H^{p+q}(X, \mathbb{C})$, so

$$\Phi : H_{BC}^{p,q}(X) \rightarrow H^{p+q}(X, \mathbb{C}).$$

We want to show that there exists a \mathbb{C} -linear map

$$\Phi : H_{BC}^{p,q}(X) \rightarrow H^{p,q}(X).$$

Indeed if $\alpha \in C^\infty(X, \Omega_X^{p,q})$ such that $d\alpha = 0$, then $\partial\alpha = \bar{\partial}\alpha = 0$, so there exists a map $\Phi : \{d\alpha = 0\} \rightarrow H^{p,q}(X)$. Let $\partial\bar{\partial}\alpha \in \partial\bar{\partial}C^\infty(X, \Omega_X^{p-1, q-1})$. Then $\partial\bar{\partial}\alpha = \bar{\partial}(-\partial\alpha) = 0$ in $H^{p,q}(X)$.

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Lemma 6.38 ($\partial\bar{\partial}$ -lemma). *Let (X, ω) be a compact Kähler manifold, and let $\alpha \in C^\infty(X, \Omega_X^{p,q})$ such that*

1. $d\alpha = 0$, and
2. α is ∂ -exact, or $\bar{\partial}$ -exact.

Then there exists $\beta \in C^\infty(X, \Omega_X^{p-1, q-1})$ such that $\alpha = \partial\bar{\partial}\beta$.

Proof. By 1, $\partial\alpha = \bar{\partial}\alpha = 0$. Assume that α is ∂ -exact. Then there exists $\beta \in C^\infty(X, \Omega_X^{p-1, q})$ such that $\alpha = \partial\beta$. Recall, by Theorem 6.21, that

$$C^\infty(X, \Omega_X^{p-1, q}) = \mathcal{H}^{p-1, q}(X) \oplus \bar{\partial}C^\infty(X, \Omega_X^{p-1, q-1}) \oplus \bar{\partial}^*C^\infty(X, \Omega_X^{p-1, q+1}).$$

Then

$$\beta = \beta_1 + \bar{\partial}\beta_2 + \bar{\partial}^*\beta_3, \quad \beta_1 \in \mathcal{H}^{p-1, q}(X), \quad \beta_2 \in C^\infty(X, \Omega_X^{p-1, q-1}), \quad \beta_3 \in C^\infty(X, \Omega_X^{p-1, q+1}),$$

so $\alpha = \partial\beta = \partial\beta_1 + \partial\bar{\partial}\beta_2 + \partial\bar{\partial}^*\beta_3$. The goal is $\partial\beta_1 = \partial\bar{\partial}^*\beta_3 = 0$. Since $\beta_1 \in \mathcal{H}^{p-1, q}(X)$, $\Delta_{\bar{\partial}}\beta_1 = 0$. Since X is Kähler, $\Delta_{\partial} = \Delta_{\bar{\partial}}$, so $\Delta_{\partial}\beta_1 = 0$, if and only if $\partial\beta_1 = \bar{\partial}^*\beta_1 = 0$ by Lemma 6.20. Since X is Kähler, by the Kähler identity $[\Lambda, \partial] = i\bar{\partial}^*$,

$$\bar{\partial}^*\partial = -i[\Lambda, \partial]\partial = -i(\Lambda\partial - \partial\Lambda)\partial = i\partial\Lambda\partial = i\partial[\Lambda, \partial] = i\partial i\bar{\partial}^* = -\partial\bar{\partial}^*.$$

Then $0 = \bar{\partial}\alpha = \bar{\partial}(\partial\bar{\partial}\beta_2 + \partial\bar{\partial}^*\beta_3) = \bar{\partial}\partial\bar{\partial}^*\beta_3 = -\bar{\partial}\bar{\partial}^*\partial\beta_3$, so $\bar{\partial}\bar{\partial}^*\partial\beta_3 = 0$. Recall, by Theorem 6.21, that

$$\ker \bar{\partial}^* = \mathcal{H}^{p, q}(X) \oplus \bar{\partial}^*C^\infty(X, \Omega_X^{p, q+1}).$$

Then $\partial\bar{\partial}^*\beta_3 = -\bar{\partial}^*\partial\beta_3 \in \bar{\partial}^*C^\infty(X, \Omega_X^{p, q+1})$, but $\bar{\partial}^*\partial\bar{\partial}^*\beta_3 = 0$ and $\bar{\partial}\partial\bar{\partial}^*\beta_3 = 0$, so $\partial\bar{\partial}^*\beta_3 \in \mathcal{H}^{p, q}(X)$ is $\Delta_{\bar{\partial}}$ -harmonic. Thus $\partial\bar{\partial}^*\beta_3 = 0$. \square

Theorem 6.39. *Let X be a compact Kähler manifold. There exist isomorphisms*

$$\Phi : H_{\text{BC}}^{p, q}(X) \rightarrow H^{p, q}(X), \quad \Psi : \bigoplus_{p+q=k} H_{\text{BC}}^{p, q}(X) \rightarrow H^k(X, \mathbb{C}).$$

Proof. Φ implies Ψ , so the goal is to prove that Φ is an isomorphism. Let $[\alpha'] \in H^{p, q}(X) \cong \mathcal{H}^{p, q}(X)$. There exists $\alpha \in \mathcal{H}^{p, q}(X)$ such that $[\alpha] = [\alpha']$ and α is $\Delta_{\bar{\partial}}$ -harmonic. Since X is Kähler and $\Delta = 2\Delta_{\bar{\partial}}$, α is Δ -harmonic, so $d\alpha = 0$. Since $H_{\text{BC}}^{p, q}(X) = \{d\alpha = 0\} / \partial\bar{\partial}C^\infty(X, \Omega_X^{p-1, q-1})$, $[\alpha] \in H_{\text{BC}}^{p, q}(X)$, so Φ is surjective. Assume $\Phi([\beta]) = 0$, so $d\beta = 0$, since $\beta \in H_{\text{BC}}^{p, q}(X)$. Then $[\beta] = 0$ inside $H^{p, q}(X) = \ker \bar{\partial} / \text{im } \bar{\partial}$, so $\beta = \bar{\partial}\gamma$, that is β is $\bar{\partial}$ -exact. By the $\partial\bar{\partial}$ -lemma, $\beta = \partial\bar{\partial}\eta$, so $[\beta] = 0$ in $H_{\text{BC}}^{p, q}(X)$. Thus Φ is injective. \square

6.9 Lefschetz decomposition

Let X be a compact Kähler manifold of dimension n . The Hodge numbers are $h^{p, q}(X) = \dim H^{p, q}(X)$, and the **Hodge diamond** is

$$\begin{array}{ccccc} & & h^{0,0}(X) & & \\ & & \vdots & & \\ & h^{1,0}(X) & & h^{0,1}(X) & \\ & \vdots & & \vdots & \\ h^{n,0}(X) & \dots & & \dots & h^{0,n}(X) \\ & \vdots & & \vdots & \\ & h^{n,n-1}(X) & & h^{n-1,n}(X) & \\ & & h^{n,n}(X) & & \end{array}$$

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At each line, the sum corresponds to the Betti numbers $b_0(X), \dots, b_{2n}(X)$, because it is a real manifold of dimension $2n$. What else do we know about these numbers?

- Since we are working with a compact Kähler manifold, $h^{p,p}(X) \neq 0$ for all $p = 0, \dots, n$.
- Since X is connected, $h^{n,n}(X) = 1$ and $h^{0,0}(X) = 1$.
- Since $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and the dimension is the same if we take conjugation, $h^{p,q}(X) = h^{q,p}(X)$.
- Since $H^{p,q}(X) = H^{n-p,n-q}(X)$, and the Hodge \star operator preserves $\bar{\partial}$ -closed, $h^{p,q}(X) = h^{n-p,n-q}(X)$.

The goal is $h^{p,q}(X)$ are increasing. More specifically, we have $h^{p,q}(X) \leq h^{p+1,q+1}(X)$ if $p+1 \leq n$ and $q+1 \leq n$ where $n = \dim X$. If α is a (p,q) -form, then $\alpha \wedge \omega$ is a $(p+1, q+1)$ -form, and this process is injective. Today we study

$$\begin{aligned} L : C^\infty(X, \Omega_X^{p,q}) &\longrightarrow C^\infty(X, \Omega_X^{p+1,q+1}) \\ \alpha &\longmapsto \alpha \wedge \omega \end{aligned}$$

Since ω is real,

$$L : H^k(X, \mathbb{R}) \rightarrow H^{k+2}(X, \mathbb{R}).$$

We need to work with

$$\mathcal{H}^k(X, \mathbb{R}) = \{\alpha \in C^\infty(X, \Omega_{X,\mathbb{R}}^k) \mid \Delta\alpha = 0\}.$$

Corollary 6.40. *In the same context as before, we have an isomorphism*

$$\begin{aligned} \mathcal{H}^k(X, \mathbb{R}) &\longrightarrow H^k(X, \mathbb{R}) \\ \alpha &\longmapsto [\alpha], \quad k \geq 0. \end{aligned}$$

Lemma 6.41. *Let X be a compact Kähler manifold of dimension n .*

1. $[\Delta, L] = 0$.
2. $[L, \Lambda]\alpha = (k-n)\alpha$ for any $\alpha \in C^\infty(X, \Omega_{X,\mathbb{C}}^k)$.

Proof.

1. Recall that $\Delta = 2\Delta_\partial$. Only have to show that L commutes with Δ_∂ . Let α be a k -form on X . Since ω is a closed $(1,1)$ -form on X , $\partial\omega = 0$. Then $\partial L\alpha = \partial(\omega \wedge \alpha) = \omega \wedge \partial\alpha = L\partial\alpha$, so $\partial L = L\partial$. By the Kähler identity, $\partial^*L - L\partial^* = [\partial^*, L] = -i\bar{\partial}$. Then

$$[\Delta_\partial, L] = (\partial\partial^* - \partial^*\partial)L - L(\partial\partial^* - \partial^*\partial) = \partial\partial^*L - \partial^*L\partial - \partial L\partial^* + L\partial^*\partial = -i\partial\bar{\partial} - (-i\bar{\partial}\partial) = 0.$$

2. $[L, \Lambda]$ can be computed pointwise. That is, if we fix $x \in X$ we may assume that there exist coordinates z_1, \dots, z_n such that dz_i is an orthonormal basis with respect to ω , so we may consider a Hermitian vector space V and a real $(1,1)$ -form ω . Proceed by induction on $\dim V = n$. We are working on $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

$n = 1$. $[L, \Lambda](1) = -1$ for a 0-form 1, $[L, \Lambda]\eta = 0$ for a 1-form η , and $[L, \Lambda]\omega = \omega$ for a 2-form ω .²¹

$n > 1$. Let $V = W_1 \oplus W_2$ be an orthogonal decomposition such that $\dim W_1 = n-1$ and $\dim W_2 = 1$. Then $\omega = \omega_1 + \omega_2$ where ω_i is a metric on W_i and $\Lambda = \Lambda_1 + \Lambda_2$ where Λ_i is the adjoint of $L_i = \omega_i \wedge \cdot$. By linear algebra,

$$\bigwedge^k V^* = \left(\bigwedge^k W_1^* \otimes \bigwedge^0 W_2^* \right) \oplus \left(\bigwedge^{k-1} W_1^* \otimes \bigwedge^1 W_2^* \right).$$

Let $\eta = \eta_1 + \eta_2$ for $\eta_1 \in \bigwedge^k W_1^* \otimes \bigwedge^0 W_2^*$ and $\eta_2 \in \bigwedge^{k-1} W_1^* \otimes \bigwedge^1 W_2^*$. We need to check that 2 holds for η_i for $i = 1, 2$. Let $\eta_1 = \eta_1 \cdot 1$ for $\eta_1 \in \bigwedge^k W_1^*$ and $1 \in \bigwedge^0 W_2^*$. By induction,

$$[L, \Lambda]\eta_1 = [L_1, \Lambda_1]\eta_1 + \eta_1 [L_2, \Lambda_2](1) = (k - (n-1))\eta_1 + \eta_1(-1) = (k-n)\eta_1.$$

Similarly, $[L, \Lambda]\eta_2 = (k-n)\eta_2$.²²

□

²¹Exercise

²²Exercise

Proposition 6.42. For all k ,

$$L^{n-k} : \Omega_{X,\mathbb{R}}^k \xrightarrow{\sim} \Omega_{X,\mathbb{R}}^{2n-k}$$

is an isomorphism, where a k -form is mapped to a $(2n - k)$ -form.

Proof. It is enough to show that it is injective, since

$$\mathrm{rk} \Omega_{X,\mathbb{R}}^k = \binom{2n}{k} = \binom{2n}{2n-k} = \mathrm{rk} \Omega_{X,\mathbb{R}}^{2n-k}.$$

Recall that if α is a k -form $[L, \Lambda] \alpha = (k - n) \alpha$. Then

$$[L^r, \Lambda] \alpha = L \left(L^{r-1} \Lambda - \Lambda L^{r-1} \right) \alpha + (\Lambda L - L \Lambda) L^{r-1} \alpha = L [L^{r-1}, \Lambda] \alpha + (2(r-1) + k - n) L^{r-1} \alpha, \quad r > 0.$$

By induction,

$$[L^r, \Lambda] \alpha = (r(k - n) + r(r - 1)) L^{r-1} \alpha. \quad (2)$$

The goal is for all $k \in \{0, \dots, n\}$ and for all $r \in \{0, \dots, n - k\}$, L^r is injective. Double induction.

$r = 0$. $L^0 = \mathrm{id}$, so ok.

$r > 0$. Assume $k \geq 0$. Assume that $L^r \alpha = 0$ for some $\alpha \in C^\infty(X, \Omega_{X,\mathbb{R}}^k)$. Since $r \leq n - k$,

$$p = r(k - n) + r(r - 1) \neq 0.$$

By (2), $L^r \Lambda \alpha = p L^{r-1} \alpha$, so $L^{r-1} (\Lambda \alpha - p \alpha) = 0$. Then L^{r-1} is injective by induction, so $\Lambda \alpha = p \alpha$.

$k \leq 1$. Since $\Lambda \alpha$ is a $(k - 2)$ -form, $\Lambda \alpha = 0$, so $\alpha = 0$.

$k \geq 2$. Let $\beta = \Lambda \alpha$ be a $(k - 2)$ -form. Then $L^{r+1} \beta = p L^r \alpha = 0$, by (2). Since L^{r+1} is injective for all $r + 2 \leq n - (k - 2)$, $\beta = 0$, so $\alpha = 0$, since $L\beta = p\alpha$.

□

Theorem 6.43 (Hard Lefschetz theorem). Let (X, ω) be a compact Kähler manifold of dimension n . Then

$$L^{n-k} : H^k(X, \mathbb{R}) \rightarrow H^{2n-k}(X, \mathbb{R}), \quad L^{n-p-q} : H^{p,q}(X) \rightarrow H^{n-p,n-q}(X)$$

are isomorphisms.

Proof. $H^k(X, \mathbb{R})$ and $H^{2n-k}(X, \mathbb{R})$ have the same dimension because of Poincaré duality. It is enough to show that L^r is injective for all $r \leq n - k$. It is enough to show that

$$L^r : \mathcal{H}^k(X, \mathbb{R}) \rightarrow \mathcal{H}^{2r+k}(X, \mathbb{R})$$

is injective. By Proposition 6.42, it is enough to show that $L^r(\mathcal{H}^k(X, \mathbb{R})) \subset \mathcal{H}^{2r+k}(X, \mathbb{R})$. By 1 of Lemma 6.41, $[\Delta, L] = 0$. □

The goal is the Lefschetz decomposition.

Definition 6.44. Let $k \in \{0, \dots, n\}$. A k -form α on X is called **primitive** if $L^{n-k+1} \alpha = 0$.

Lemma 6.45. α is primitive if and only if $\Lambda \alpha = 0$.

Proof. Recall

$$[L^r, \Lambda] \alpha = (r(k - n) + r(r - 1)) L^{r-1} \alpha.$$

If $r = n - k + 1$ then $L^{n-k+1} \Lambda = \Lambda L^{n-k+1}$. Assume α is primitive, so $L^{n-k+1} \alpha = 0$. Then $L^{n-k+1} \Lambda \alpha = 0$. Since $\Lambda \alpha$ is a $(k - 2)$ -form, L^{n-k+1} is injective for all $n - k + 1 \leq n - (k - 2)$, so $\Lambda \alpha = 0$. Assume $\Lambda \alpha = 0$. Let $r > 0$ be minimal such that $L^r \alpha = 0$. Then $0 = [L^r, \Lambda] \alpha = r(k - n + r - 1) L^{r-1} \alpha$. Since $L^{r-1} \alpha \neq 0$, $k - n + r - 1 = 0$, so $r = n - k + 1$. □

Proposition 6.46. *For all $k \in \{0, \dots, n\}$ and r -forms α , there exists a unique decomposition*

$$\alpha = \sum_{r \geq 0} L^r \alpha_r,$$

where α_r are primitive $(k - 2r)$ -forms.

Proof.

- Existence. By induction on k .

$k \leq 1$. Since $\Lambda \alpha$ is a $(k - 2)$ -form, $\Lambda \alpha = 0$. By Lemma 6.45, α is primitive, so $\alpha = \alpha_0$.

$k \geq 2$. By Proposition 6.42 there exists a $(k - 2)$ -form β such that $L^{n-k+2} \beta = L^{n-k+1} \alpha$, which is a $(k + 2(n - k + 1))$ -form, so $L^{n-k+1}(\alpha - L\beta) = 0$, where $\alpha - L\beta$ is a k -form. By definition, $\alpha_0 = \alpha - L\beta$ is primitive, so $\alpha = \alpha_0 + L\beta$. By induction on k , $\beta = \sum_{r \geq 0} L^r \beta_r$, where β_r are primitive. Then $\alpha = \alpha_0 + \sum_{r \geq 0} L^{r+1} \beta_r$, so existence is ok.

- Unicity. By induction on k .

$k = 0$. $\alpha = \alpha_0$ is ok.

$k > 0$. There exists a primitive β_r such that $\sum_{r \geq 0} L^r \beta_r = 0$. Since β_0 is primitive k -form, $L^{n-k+1} \beta_0 = 0$, so

$$0 = L^{n-k+1} \left(\sum_{r \geq 0} L^r \beta_r \right) = \sum_{r \geq 1} L^{n-k+1+r} \beta_r = L^{r-k+2} \left(\sum_{r \geq 1} L^{r-1} \beta_r \right),$$

where $\sum_{r \geq 1} L^{r-1} \beta_r$ are $(k - 2)$ -forms. By Proposition 6.42, $\sum_{r \geq 1} L^{r-1} \beta_r = 0$. By induction $\beta_r = 0$ for all $r \geq 1$, so $\beta_0 = 0$.

□

Definition 6.47. Let

$$H^{k-2r}(X, \mathbb{R})_{\text{prim}} = \{\text{primitive d-closed } (k - 2r)\text{-forms}\}.$$

Theorem 6.48 (Lefschetz decomposition). *Let (X, ω) be a compact Kähler manifold of dimension n . Then*

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}(X, \mathbb{R})_{\text{prim}}.$$

The same holds over \mathbb{C} .

Remark 6.49. Let (X, ω) be compact Kähler. Then

$$b_{k-2}(X) \leq b_k(X), \quad k = 2, \dots, n,$$

and

$$h^{p-1, q-1}(X) \leq h^{p, q}(X), \quad p, q = 1, \dots, n.$$

Lecture 30 is a problems class.

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