M4P58 Modular Forms

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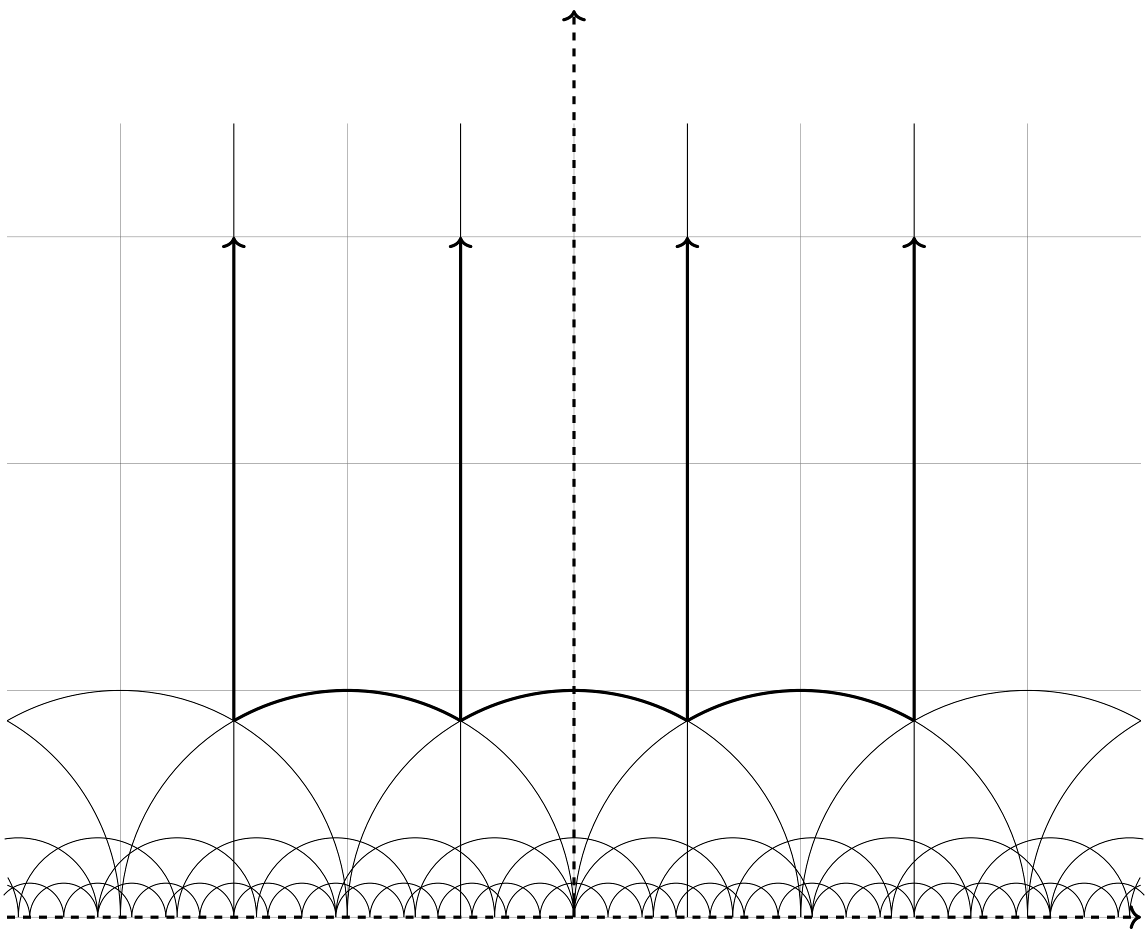


$$\mathcal{D} = \left\{ z \in \mathbb{H} \;\middle|\; \tfrac{1}{2} \le \operatorname{Re} z \le \tfrac{1}{2},\, |z| \ge 1 \right\} \subseteq \mathbb{H}$$

Syllabus

Modular forms of level one. Eisenstein series. Spaces of modular forms of level one. Theta series. Hecke operators of level one. L-functions of level one. Modular forms of higher level. Spaces of modular forms of higher level. Hecke operators of higher level. L-functions of higher level. Oldforms and newforms.

Contents

0 Introduction

Lecture 1
Friday
04/10/19

The following are textbooks.

- J P Serre, A course in arithmetic, 1973
- J Shurman and F Diamond, A first course in modular forms, 2005

Let
$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 \left(1 - q^{11n}\right)^2 = \sum_{n=1} b_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots,$$
and let a_n be the number of solutions modulo n to the elliptic curve
$$E = \left\{(x, y) \in \mathbb{Z} \mid y^2 + y = x^3 - x^2 - 10x - 20\right\}.$$

- Modulo 2, there are $a_2 = 4$ solutions $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$.
- Modulo 3, there are $a_3 = 4$ solutions $(1, 0)$, $(1, -1)$, $(-1, 0)$, $(-1, -1)$.
- Modulo 5, there are $a_5 = 4$ solutions $(0, 0)$, $(0, -1)$, $(1, 0)$, $(1, -1)$.
- Modulo 7, there are $a_7 = 9$ solutions $(1, 3)$, $(2, 2)$, $(2, -3)$, $(-1, 1)$, $(-1, -2)$, $(-2, 1)$, $(-2, -2)$, $(-3, 1)$, $(-3, -2)$.

If $p \ne 11$, then
$$a_p - p = -b_p.$$

The following are some questions.

- What is the relationship between E and f?
- Can we find similar relationships for other E?
- How does one prove something like this?

Let
$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\} \subseteq \mathbb{C}.$$

Then $\mathbb{H}$ has an action of
$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

Modular forms are complex functions on $\mathbb{H}$ with a high degree of symmetry. These functions are symmetric under the action of large discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, in particular
$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} \subseteq \mathrm{SL}_2(\mathbb{R}).$$

Why are these interesting to number theorists? Power series expansions often involve expressions of interest to number theorists. For example,

- Bernoulli numbers,
- divisor functions $\sigma_k(n) = \sum_{d \mid n} d^k$,
- number of points on elliptic curves, and
- traces of Galois representations.

1 Modular forms of level one

1.1 Modular forms

1.1.1 Modular actions

$\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \begin{cases} \dfrac{az+b}{cz+d} & z \ne -\dfrac{d}{c} \\ \infty & z = -\dfrac{d}{c} \\ \dfrac{a}{c} & z = \infty \end{cases}.$$

One checks that this gives a bijection from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$, where the inverse is given by the inverse matrix $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot z\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot z.$$

One obtains a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C} \cup \{\infty\}$. An observation is

$$\operatorname{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \operatorname{Im} \frac{az+b}{cz+d} = \operatorname{Im} \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{\operatorname{Im}(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2}.$$

In particular, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, then

$$\operatorname{Im} \gamma z = \frac{\operatorname{Im} z}{|cz+d|^2}.$$

So $\mathrm{SL}_2(\mathbb{R})$ preserves $\mathbb{H} \cup \{\infty\}$. More generally, if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R},\ ad - bc \ne 0 \right\},$$

then

$$\operatorname{Im} \gamma z = \frac{\det \gamma \operatorname{Im} z}{|cz+d|^2}.$$

So

$$\mathrm{GL}_2(\mathbb{R})_+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R},\ ad - bc > 0 \right\}$$

preserves $\mathbb{H} \cup \{\infty\}$. Lecture 2 Friday 04/10/19

Definition 1.1.1. Let $f : \mathbb{H} \to \mathbb{C}$, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})_+$, and let $k \in \mathbb{Z}$. Define

$$\begin{array}{cccc} f|_{k,\gamma} & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & \det \gamma^{k-1} f(\gamma z)(cz+d)^{-k} \end{array},$$

where $\det \gamma^{k-1}$ is the **fudge factor**, which is one for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, and $(cz+d)^{-k}$ is the **twisted action** on functions.

Check that

$$f|_{k,\mathrm{id}} = f, \qquad \left(f|_{k,\gamma}\right)\Big|_{k,\gamma'} = f|_{k,\gamma\gamma'}.$$

This gives, for each k, a left action of $\mathrm{GL}_2(\mathbb{R})_+$ on functions $\mathbb{H} \to \mathbb{C}$, a **modular action of weight** k. A modular form of weight k will be a sufficiently nice function $f : \mathbb{H} \to \mathbb{C}$ such that $f|_{k,\gamma} = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. That is, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$,

$$f(\gamma z)(cz+d)^{-k} = f(z) \qquad \Longrightarrow \qquad f(\gamma z) = f(z)(cz+d)^k,$$

the **modular transformation law of weight** k.

The following are some observations.

- Let $k = 0$. Then constant functions satisfy $f(\gamma z) = f(z)$. It will turn out that all functions of weight zero are constant.
- Let k be odd, and $\gamma = -\mathrm{id}$. Then $\gamma z = z$ for all z and $cz + d = -1$, so $f(\gamma z) = f(z)(cz+d)^k$ gives $f(z) = f(z)(-1)^k = -f(z)$, so $f(z) = 0$ for all z. So no non-zero functions $f : \mathbb{H} \to \mathbb{C}$ satisfy the modular transformation law of weight k, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, when k is odd.

1.1.2 Review of complex analysis

Let $f : U \to \mathbb{C}$ for $U \subseteq \mathbb{C}$ open, and let $p \in U$.

Definition 1.1.2. f is **holomorphic** at p if

$$f'(p') = \lim_{\epsilon \to 0,\ \epsilon \in \mathbb{C}} \frac{f(p' + \epsilon) - f(p')}{\epsilon}$$

exists for all p' in a neighbourhood of p.

Proposition 1.1.3. *f is holomorphic at p implies that f is continuous and infinitely differentiable at p, that is $f^{(n)}(p)$ exists for all $n \geq 0$. Moreover, we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n = f(p) + f'(p)(z-p) + \frac{f''(p)}{2}(z-p)^2 + \dots,$$

for all z in a neighbourhood of p.

Corollary 1.1.4. *If f is holomorphic and not identically zero on an open set U, then the zeroes of f are isolated on U.*

More generally is the following.

Definition 1.1.5. f is **meromorphic** at p if there exists a neighbourhood U of p and $g, h : U \to \mathbb{C}$ holomorphic on U such that $f = g/h$ on $U \setminus \{p\}$. Such an f has a **Laurent series expansion** at p,

$$f(z) = \sum_{i=-N}^{\infty} c_i (z-p)^i.$$

The smallest i such that $c_i \neq 0$ is denoted by $\mathrm{ord}_p f$, the **order of vanishing** of f at p.

- If $\mathrm{ord}_p f = -n$ for $n > 0$, we say f has a **pole of order** n.
- If $\mathrm{ord}_p f = n$ for $n > 0$, we say f has a **zero of order** n.

Proposition 1.1.6.

- $\mathrm{ord}_p fg = \mathrm{ord}_p f + \mathrm{ord}_p g$, *and*
- $\mathrm{ord}_p (f+g) \geq \min\{\mathrm{ord}_p f, \mathrm{ord}_p g\}$, *with equality if* $\mathrm{ord}_p f \neq \mathrm{ord}_p g$.

If f is holomorphic on $U \setminus \{p\}$ for U a neighbourhood of p, then f may or may not be meromorphic at p.

Example. $f(z) = e^{-1/z^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but not meromorphic at zero.

Theorem 1.1.7. *Let f be holomorphic on $U \setminus \{p\}$, and there exists $n > 0$ such that*

$$\lim_{x \to p} (x-p)^n f(x)$$

exists. Then f is meromorphic on U, and $\mathrm{ord}_p f \geq -n$.

1.1.3 Modular forms

Definition 1.1.8. $f : \mathbb{H} \to \mathbb{C}$ is a **weakly modular function of weight** k if

- f is meromorphic on $\mathbb{H}$, and
- f satisfies the modular transformation law of weight k.

Consider $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, so $\gamma z = z + 1$ and $cz + d = 1$. The modular transformation law gives $f(z+1) = f(z)$. Let

$$\mathbb{D} = \{q \mid |q| < 1\}.$$

Can define a function

$$\begin{array}{ccccc} g & : & \mathbb{D} \setminus \{0\} & \longrightarrow & \mathbb{H} \\ & & q & \longmapsto & f\left(\dfrac{\log q}{2\pi i}\right) \end{array},$$

that is

$$\begin{array}{ccccc} f & : & \mathbb{H} & \longrightarrow & \mathbb{D} \setminus \{0\} \\ & & z & \longmapsto & g\left(e^{2\pi i z}\right) \end{array},$$

and $q = e^{2\pi i z}$, where g is holomorphic or meromorphic on $\mathbb{D} \setminus \{0\}$ if and only if f is holomorphic or meromorphic on $\mathbb{H}$.

Definition 1.1.9. $f : \mathbb{H} \to \mathbb{C}$ is a **modular form of weight** k if

1. f satisfies the modular transformation law of weight k,
2. f is holomorphic on $\mathbb{H}$, and
3. f is holomorphic at ∞, so the function $g : \mathbb{D} \setminus \{0\} \to \mathbb{C}$, which is holomorphic on $\mathbb{D} \setminus \{0\}$ by 2, extends to a holomorphic function on $\mathbb{D}$.

Lecture 3 Monday 07/10/19

Then $q \to 0$ in $\mathbb{D}$ if and only if $\operatorname{Im} z \to +\infty$. Recall that a holomorphic function g on $\mathbb{D} \setminus \{0\}$ extends to a meromorphic function on $\mathbb{D}$ if and only if there exists n such that $\lim_{q \to 0} q^n g(q)$ exists. Then 3 means $g(q)$ is bounded as $q \to 0$ so $f(z)$ is bounded as $\operatorname{Im} z \to +\infty$. For f satisfying 3, $g : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ has a series expansion

$$g(q) = \sum_n a_n q^n = a_0 + a_1 q + \dots$$

in $q = e^{2\pi i z}$. We call this the **q-expansion** for f.

Definition 1.1.10. $f : \mathbb{H} \to \mathbb{C}$ is a **meromorphic modular form of weight** k if the same conditions 1 to 3 hold, but with holomorphic weakened to meromorphic.

Note. If f is only meromorphic at ∞ then a finite number of negative powers of q can appear.

Example.

- The **discriminant**

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

is a modular form of weight twelve.

- The j**-invariant**

$$\mathrm{j}(z) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \dots$$

is a meromorphic modular form of weight zero.

1.2 Eisenstein series

1.2.1 Lattice functions

How can we construct modular forms?

Definition 1.2.1. A **lattice** in $\mathbb{C}$ is an abelian subgroup of $\mathbb{C}$ of the form $\mathbb{Z}w_1 + \mathbb{Z}w_2$, where $w_1, w_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. More generally if V is an $\mathbb{R}$-vector space, a **lattice** L in V is a discrete abelian subgroup of V that spans V over $\mathbb{R}$. For $L \subseteq \mathbb{C}$ a lattice and $\lambda \in \mathbb{C}^\times$, let

$$\lambda L = \{\lambda x \mid x \in L\} \subseteq \mathbb{C}.$$

We say that L and λL are **homothetic**. For $z \in \mathbb{H}$, let

$$\mathrm{L}_{z,1} = \mathbb{Z} + \mathbb{Z}z = \{az + b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

A question is when is $\mathrm{L}_{z,1}$ homothetic to $\mathrm{L}_{z',1}$, and what is a homothety factor?

- Suppose $\mathrm{L}_{z,1} = \lambda \mathrm{L}_{z',1}$. Then there exist a, b, c, d such that

$$\begin{cases} \lambda z' = az + b \\ \lambda = cz + d \end{cases} \qquad \implies \qquad \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{1}$$

  On the other hand there exist a', b', c', d' such that

$$\begin{cases} z = a'\lambda z' + b'\lambda \\ 1 = c'\lambda z' + d'\lambda \end{cases} \qquad \implies \qquad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{2}$$

  By (1) and (2),

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix},$$

  so $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Moreover (1) implies that $z' = (az + b) / (cz + d)$.

- Conversely, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma z = (az + b) / (cz + d)$, so

$$\mathrm{L}_{\gamma z,1} = (cz + d)^{-1} \mathrm{L}_{az+b,cz+d}.$$

  But certainly $\mathrm{L}_{az+b,cz+d} \subseteq \mathrm{L}_{z,1}$. On the other hand if $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ is inverse to γ,

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma'\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \begin{pmatrix} a'(az + b) + b'(cz + d) \\ c'(az + b) + d'(cz + d) \end{pmatrix},$$

  so $z \in \mathrm{L}_{az+b,cz+d}$ and $1 \in \mathrm{L}_{az+b,cz+d}$. So

$$\mathrm{L}_{az+b,cz+d} = \mathrm{L}_{z,1},$$

  so $\mathrm{L}_{\gamma z,1} = (cz + d)^{-1} \mathrm{L}_{z,1}$.

Definition 1.2.2. A **lattice function of weight** k is a function $F : \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$ such that

$$F(\lambda L) = \lambda^{-k} F(L),$$

for all lattices L. Given such an F, can define

$$\begin{array}{rccc} f & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & F(\mathrm{L}_{z,1}) \end{array}.$$

If F has weight k, then

$$f(\gamma z) = F(\mathrm{L}_{\gamma z,1}) = F\left((cz + d)^{-1} \mathrm{L}_{z,1}\right) = (cz + d)^k F(\mathrm{L}_{z,1}) = (cz + d)^k f(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1.2.2 Eisenstein series

Lecture 4 Friday 11/10/19

Definition 1.2.3. For $L \in \mathbb{C}$, define the **Eisenstein series**

$$\mathrm{G}_k(L) = \sum_{w \in L,\ w \neq 0} \frac{1}{w^k}, \qquad \mathrm{g}_k(z) = \mathrm{G}_k(\mathrm{L}_{z,1}) = \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}.$$

Then

$$\mathrm{G}_k(\lambda L) = \sum_{w' \in \lambda L,\ w' \neq 0} \frac{1}{w'^k} = \sum_{w \in L,\ w \neq 0} \frac{1}{(\lambda w)^k} = \lambda^{-k} \mathrm{G}_k(L).$$

Corollary 1.2.4. *g_k satisfies the modular transformation law of weight k.*

The following are some questions.

- Does G_k, or g_k, converge?
- Is g_k holomorphic, or meromorphic, on $\mathbb{H}$?
- Is g_k holomorphic at ∞?
- What is the q-expansion of g_k?

1.2.3 Convergence and holomorphy on $\mathbb{H}$

Definition 1.2.5. Let $U \subseteq \mathbb{C}$ be open. A sequence of functions $f_n : U \to \mathbb{C}$ **converges uniformly on compact sets** to f if for all $C \subseteq U$ compact and $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that for all $n > N$,

$$|f(z) - f_n(z)| < \epsilon, \qquad z \in C.$$

Theorem 1.2.6. *A uniform limit of holomorphic functions is holomorphic. If f_n converges to f uniformly on compact sets and f_n is holomorphic on U, then f is holomorphic on U.*

Theorem 1.2.7. *Let $k \geq 4$. The series $\mathrm{g}_k(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$.*

Proof. Let

$$P_{z,r} = \{az + b \mid a, b \in \mathbb{R},\ \max(|a|, |b|) = r\} \subseteq \mathbb{C},$$

so $P_{z,r} = rP_{z,1}$, and there are $8r$ points on $P_{z,r} \cap \mathrm{L}_{z,1}$. Then

$$\mathrm{g}_k(z) = \sum_{r=1}^{\infty} \sum_{w \in \mathrm{L}_{z,1} \cap P_{z,r}} \frac{1}{w^k}.$$

The function $z \mapsto |z|$ attains a non-zero minimum $\delta(z)$ on $P_{z,1}$, so on $P_{z,1}$, have $|z| > \delta(z)$, so $1/|z|^k < 1/\delta(z)^k$. On $P_{z,r}$, have $|z| > r\delta(z)$, so $1/|z|^k < 1/r^k\delta(z)^k$. Let $C \subseteq \mathbb{H}$ be compact. Then $z \mapsto \delta(z)$ is a continuous function on C and attains a minimum δ_C. For all $z \in C$ and $w \in P_{z,r}$, get $|w| > r\delta_C$, so

$$\frac{1}{|w|^k} < \frac{1}{r^k \delta_C^k}.$$

Thus for $z \in C$, $\mathrm{g}_k(z)$ is dominated by

$$\sum_{r=1}^{\infty} \frac{8r}{r^k \delta_C^k} = \frac{8}{\delta_C^k} \sum_{r=1}^{\infty} \frac{1}{r^{k-1}},$$

which converges absolutely for $k \geq 4$. ☐

Corollary 1.2.8. *$\mathrm{g}_k(z)$ is holomorphic on $\mathbb{H}$.*

1.2.4 q-expansion and holomorphy at ∞

The idea is to understand series of the form

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}.$$

Theorem 1.2.9. *A bounded holomorphic function on all of $\mathbb{C}$ is constant.*

Lemma 1.2.10.

1.
$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

2.
$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Proof.

1. The right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$. Locally around $z = n$, the series looks like

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \dots + \frac{1}{(z-n+1)^2} + \frac{1}{(z-n)^2} + \frac{1}{(z-n-1)^2} + \dots = \frac{1}{(z-n)^2} + h_1(z),$$

where $h_1(z)$ is holomorphic in a neighbourhood of $z = n$. Similarly, the left hand side is meromorphic on $\mathbb{C}$, and the Laurent series near $z = n$ is

$$\frac{\pi^2}{\sin^2 \pi z} = \pi \left(\frac{1}{\pi^2 (z-n)^2} + \frac{1}{3} + \frac{1}{15} \pi^2 (z-n)^2 + \dots \right) = \frac{1}{(z-n)^2} + h_2(z),$$

where $h_2(z)$ is a holomorphic function. So the difference

$$g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, and the Laurent expression around $z = n$ is

$$g(z) = \frac{1}{(z-n)^2} + h_1(z) - \left(\frac{1}{(z-n)^2} + h_2(z) \right) = h_1(z) - h_2(z),$$

so $g(z)$ is holomorphic at $z = n$ for all n. Consider $t \to \pm\infty$ for $z = a + it$. The right hand side is

$$R = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=a-N}^{a+N} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{a-N-1} \frac{1}{(z-n)^2} + \sum_{n=a+N+1}^{\infty} \frac{1}{(z-n)^2} = R_0 + R_- + R_+,$$

where R_0 has finitely many terms that converge to less than $\epsilon/2$ as $t \to \pm\infty$ and $R_- + R_+ < \epsilon/2$ for $N \gg 0$ independent of t, so $R < \epsilon$ converges to zero. Similarly, the left hand side is

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{2\pi^2}{e^{\pi i z} - e^{-\pi i z}} \right| \to 0,$$

so $\lim_{t \to \infty} g(a+it) = 0$. Moreover, $g(z+1) = g(z)$ for all z. Then

$$S = \{ z \in \mathbb{C} \mid n-1 \le \operatorname{Re} z \le n, \ -N \le \operatorname{Im} z \le N \}, \qquad n \in \mathbb{Z}$$

is compact, so $|g(z)|$ attains a maximum in S, so $g(z)$ is bounded in S, so $g(z)$ is bounded in $\mathbb{C}$, so g is constant. Since $\lim_{t \to \infty} g(a+it) = 0$, $g = 0$.

2. Check that the right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Similarly, the left hand side is also meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Comparing derivatives, Lecture 5 Friday 11/10/19

$$-\frac{\pi^2}{\sin^2 \pi z} = -\frac{1}{z^2} - \sum_{n=1}^{\infty} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right),$$

so the difference is constant. Let $z = \frac{1}{2}$. The left hand side is $\pi \cot \frac{\pi}{2} = 0$ and the right hand side is

$$\frac{2}{1} + \left(-\frac{2}{1} + \frac{2}{3} \right) + \left(-\frac{2}{3} + \frac{2}{5} \right) + \dots \to 0, \qquad n \to \infty,$$

so the difference is zero.

□

Thus

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$

Take $\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}$. For $k \ge 2$ even, get

$$-(k-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n,$$

so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Collecting powers of q,

$$\begin{aligned}
\mathrm{g}_k(z) &= \sum_{\substack{m=-\infty \\ (m,n) \ne (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm} && \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n && \sigma_{k-1}(n) = \sum_{d \mid n,\ d>0} d^{k-1}.
\end{aligned}$$

Corollary 1.2.11. $\mathrm{g}_k(z)$ *is holomorphic at* ∞*. In particular,* g_k *is a modular form of weight* k*.*

1.2.5 Bernoulli numbers

Definition 1.2.12. The **Bernoulli numbers** b_k are defined by

$$\sum_{k=0}^{\infty} \mathrm{b}_k \frac{x^k}{k!} = \frac{x}{e^x - 1},$$

a formal power series with rational coefficients.

Then

$$\mathrm{b}_0 = 1, \qquad \mathrm{b}_1 = -\frac{1}{2}, \qquad \mathrm{b}_2 = \frac{1}{6}, \qquad \mathrm{b}_3 = 0, \qquad \mathrm{b}_4 = -\frac{1}{20}, \qquad \dots,$$

where $\mathrm{b}_{2k} \in \mathbb{Q}$ is interesting and $\mathrm{b}_{2k+1} = 0$ for $k \ge 1$.

Proposition 1.2.13. *For all even k,*

$$\zeta(k) = -\mathrm{b}_k \frac{(2\pi i)^k}{2k!}.$$

Proof. On one hand,

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} \mathrm{b}_k \frac{(2\pi i z)^k}{k!}.$$

On the other hand,

$$\begin{aligned}
\pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \frac{2z}{n^2} \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \\
&= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2}{z} \sum_{k=1}^{\infty} \left(\frac{z^2}{n^2}\right)^k = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \\
&= \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} \zeta(2k) z^{2k},
\end{aligned}$$

so

$$\pi i z + \sum_{k=0}^{\infty} \mathrm{b}_k \frac{(2\pi i z)^k}{k!} = \pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

Comparing,

$$\mathrm{b}_{2k} \frac{(2\pi i)^{2k}}{(2k)!} = -2\zeta(2k),$$

get the desired formula. ☐

So

$$\mathrm{g}_k(z) = -\mathrm{b}_k \frac{(2\pi i)^k}{k!} + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Set the **normalised Eisenstein series**

$$\mathrm{E}_k = \frac{\mathrm{g}_k}{2\zeta(k)} = 1 - \frac{2k}{\mathrm{b}_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Example.

$$\mathrm{E}_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad \mathrm{E}_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$\mathrm{E}_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \qquad \mathrm{E}_{12} = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.$$

p is **regular** if $p \nmid \mathrm{h}(\mathbb{Z}[\zeta_p])$ for $\zeta_p^p = 1$.

Theorem 1.2.14. *p is regular if and only if p does not divide the numerator of b_k for $1 \le k < p - 1$.*

Lecture 6 Monday 14/10/19

An observation is if f is modular of weight k and g is modular of weight k', then fg is modular of weight $k + k'$, and if $k = k'$, then $f + g$ is modular of weight k.

Example.

- The discriminant
$$\Delta(z) = \frac{\mathrm{E}_4^3 - \mathrm{E}_6^2}{1728} = q - 24q^2 + 252q^3 + \dots$$
is a modular form of weight twelve.
- The j-invariant
$$\mathrm{j}(z) = \frac{\mathrm{E}_4^3}{\Delta} = \frac{1}{q} + 744 + 196844q + \dots$$
is a meromorphic modular form of weight zero.

1.3 Spaces of modular forms

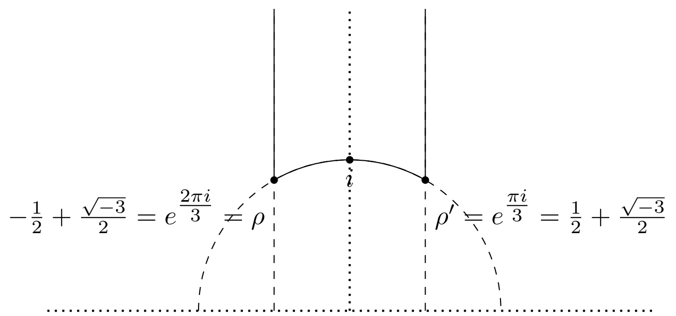
1.3.1 The fundamental domain

The idea is to control the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. If $f : \mathbb{H} \to \mathbb{C}$ satisfies $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and if $D \subseteq \mathbb{H}$ such that D meets every $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathbb{H}$, then f is determined by its values on D.

Definition 1.3.1. Let G be a group acting continuously on a complex analytic space X, such as $X = \mathbb{H}$. A subset $D \subseteq X$ is a **fundamental domain** for the action of G if

- D meets every G-orbit in X,
- the subset $\{x \in D \mid \exists g \in G,\ gx \in D,\ gx \neq x\}$ has measure zero, and
- D is closed in X.

Define
$$\mathcal{D} = \left\{z \in \mathbb{H} \;\middle|\; \tfrac{1}{2} \le \operatorname{Re} z \le \tfrac{1}{2},\, |z| \ge 1\right\} \subseteq \mathbb{H},$$
so



Let
$$\mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}, \qquad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z+1,$$
and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup generated by S and T. We will see later that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.3.2.

1. *For all $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathcal{D}$.*
2. *Suppose $z, z' \in \mathcal{D}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma z = z'$. Then either*
   - $z = z'$,
   - $\operatorname{Re} z = \pm\frac{1}{2}$ *and* $z' = z \mp 1$, *or*
   - $|z| = 1$ *and* $z' = -1/z$.

   In particular, if $z \neq z'$, then z and z' are on the boundary of $\mathcal{D}$.
3. *For $z \in \mathcal{D}$, let Stab_z be the stabiliser of z in $\mathrm{SL}_2(\mathbb{Z})$, that is*
$$\mathrm{Stab}_z = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma z = z\}.$$
   Then $\mathrm{Stab}_z = \{\pm \mathrm{id}\}$ unless
   - $z = i$, *where* $\mathrm{Stab}_z = \{\pm\mathrm{id}, \pm\mathrm{S}\}$,
   - $z = \rho$, *where* $\mathrm{Stab}_z = \left\{\pm\mathrm{id}, \pm(\mathrm{ST}), \pm\left(\mathrm{T}^{-1}\mathrm{S}\right)\right\}$, *or*
   - $z = \rho'$, *where* $\mathrm{Stab}_z = \left\{\pm\mathrm{id}, \pm(\mathrm{TS}), \pm\left(\mathrm{ST}^{-1}\right)\right\}$.

Corollary 1.3.3. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Proof. Fix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathring{\mathcal{D}}$ so $\mathrm{SL}_2(\mathbb{Z})\, z \cap \mathcal{D} = \{z\}$ and $\mathrm{Stab}_z = \{\pm\mathrm{id}\}$. Consider γz. There exists $\gamma' \in \Gamma$ such that $\gamma'\gamma z \in \mathcal{D}$, so $\gamma'\gamma z = z$. So $\gamma'\gamma = \pm\mathrm{id}$, so $\gamma = \pm\gamma'^{-1}$. But $\gamma'^{-1} \in \Gamma$ and $-\mathrm{id} = \mathrm{S}^2 \in \Gamma$, so $\gamma \in \Gamma$. $\square$

Proof of Theorem 1.3.2. Recall that $\operatorname{Im}\gamma z = \operatorname{Im} z / |cz+d|^2$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

1. As c and d vary, $\{cz+d\}$ forms a lattice in $\mathbb{C}$, so there exist only finitely many c and d such that $|cz+d| < 1$. So $\operatorname{Im}\gamma z$ attains a maximum as γ varies over Γ, so there exists $\gamma \in \Gamma$ such that $\operatorname{Im}\gamma z$ is maximal. There exists $n \in \mathbb{Z}$ such that $\mathrm{T}^n\gamma z$ has real part between $-\frac{1}{2}$ and $\frac{1}{2}$. Consider $|\mathrm{T}^n\gamma z|$. If this is less than one, then
$$\operatorname{Im}\mathrm{ST}^n\gamma z = \operatorname{Im}\frac{-1}{\mathrm{T}^n\gamma z} > \operatorname{Im}\mathrm{T}^n\gamma z = \operatorname{Im}\gamma z.$$
Since $\mathrm{ST}^n\gamma \in \Gamma$, this contradicts maximality so $|\mathrm{T}^n\gamma z| \geq 1$, so $\mathrm{T}^n\gamma z \in \mathcal{D}$.

Lecture 7 Friday 18/10/19

2, 3. Let $z, z' \in \mathcal{D}$ such that $\gamma z = z'$. Without loss of generality $\operatorname{Im} z' \geq \operatorname{Im} z$, so $|cz+d| \leq 1$. Note that $|cz+d| \geq \operatorname{Im}(cz+d) \geq \frac{\sqrt{3}}{2}c$, so $c = -1, 0, 1$. Note that can replace γ with $-\gamma$ if convenient.

$c = 0$. $ad = 1$, so can assume $a = d = 1$, so $\gamma z = z + b$. Since $z, z+b \in \mathcal{D}$, $b = \pm 1$ and $\operatorname{Re} z = \mp\frac{1}{2}$.

$c = 1$. Have $|z+d| \leq 1$ and $|z| \geq 1$, so $d = -1, 0, 1$.

$d = 0$. $|z| = 1$, and $\gamma z = (az-1)/z = a - 1/z$. The only possibilities are

* $a = 0$ and $\gamma = \mathrm{S}$,
* $a = 1$ and $\gamma = \mathrm{TS}$, so $z = \rho'$, or
* $a = -1$ and $\gamma = \mathrm{T}^{-1}\mathrm{S}$, so $z = \rho$.

$d = 1$. $z = \rho$, and $\gamma z = ((b+1)z + b)/(z+1) = b + 1 - 1/(z+1)$, so $b = 0$ or $b = -1$.

$d = -1$. $z = \rho'$ is similar.

$c = -1$. Similar.

□

1.3.2 Further review of complex analysis

Recall that on any compact set, a meromorphic function has only finitely many zeroes and poles. If $f(z) = g\left(e^{2\pi i z}\right)$ is meromorphic at ∞, then g is meromorphic on $\mathbb{D} = \{|q| < 1\}$, so zeroes and poles of g are discrete with respect to q, and $\operatorname{Im} z \gg 0$ if and only if $|q| < \epsilon$.

Definition 1.3.4. Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be meromorphic on U. If f has a pole at p, can write
$$f(z) = \sum_{n=\operatorname{ord}_p f < 0}^{\infty} a_n (z-p)^n.$$
The coefficient a_{-1} is called the **residue** $\operatorname{Res}_p f$ of f at p.

Theorem 1.3.5 (Residue theorem)**.** *Let V be a region in $\mathbb{C}$ whose boundary ∂V is a simple closed curve with counterclockwise orientation. Then*
$$\frac{1}{2\pi}\int_{\partial V} f(z)\,\mathrm{d}z = \sum_{p \in V \ \text{pole of } f} \operatorname{Res}_p f.$$

Definition 1.3.6. Let f be meromorphic on $U \subseteq \mathbb{C}$ open. Then the **logarithmic derivative** $\mathrm{d}\log f$ is the function f'/f.

If $f(z) = c_n(z-p)^n + c_{n+1}(z-p)^{n+1} + \dots$ and $c_n \neq 0$, then if $n \neq 0$ then the leading term of f' is $nc_n(z-p)^{n-1}$ and the leading term of f is $c_n(z-p)^n$, so the leading term of f'/f is $n(z-p)^{-1}$. If $n = 0$, then f'/f is holomorphic. So f'/f is meromorphic with simple poles precisely at the points where $\operatorname{ord}_p f \neq 0$, and $\operatorname{Res}_p f'/f$ at such p is $\operatorname{ord}_p f$.

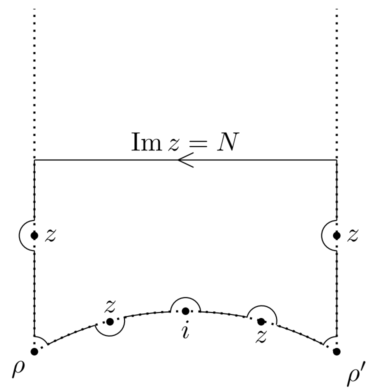
Theorem 1.3.7 (Argument principle)**.**
$$\frac{1}{2\pi i}\int_{\partial V} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{p \in V \ \text{pole of } f} \operatorname{ord}_p f.$$

1.3.3 Controlling modular forms

Theorem 1.3.8 ($k/12$-formula)**.** *Let f be a non-zero meromorphic modular form of weight k. Then*
$$\mathrm{ord}_\infty f + \frac{\mathrm{ord}_\rho f}{3} + \frac{\mathrm{ord}_i f}{2} + \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H},\ p \nsim \{i,\rho\}} \mathrm{ord}_p f = \frac{k}{12}.$$

Lecture 8 Friday 18/10/19

Proof. Consider the closed curve $C_{N,\epsilon}$,



,

where the z's are zeroes or poles of f, and the circles are of radius ϵ. Consider
$$\frac{1}{2\pi i} \int_{C_{N,\epsilon}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H},\ p \nsim \{i,\rho\}} \mathrm{ord}_p f, \qquad \epsilon \to 0.$$
So it suffices to show
$$\lim_{\epsilon \to 0,\ N \to \infty} \frac{1}{2\pi i} \int_{C_{N,\epsilon}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = -\mathrm{ord}_\infty f - \frac{\mathrm{ord}_\rho f}{3} - \frac{\mathrm{ord}_i f}{2} + \frac{k}{12}.$$
The vertical parts of the boundary cancel. Since $f(-1/z) = z^k f(z)$,
$$\mathrm{d}\left(z^k f(z)\right) = \left(k z^{k-1} f(z) + z^k f'(z)\right) \mathrm{d}z,$$
so the integral over the circular part of $\partial \mathcal{D}$ approaches
$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} \, \mathrm{d}z + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)} \, \mathrm{d}z = \frac{1}{2\pi i} \left(\int_\rho^i \frac{f'(z)}{f(z)} \, \mathrm{d}z - \int_\rho^i \frac{f'(-1/z)}{f(-1/z)} \, \mathrm{d}z \right)$$
$$= \frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} - \frac{k z^{k-1} f(z) + z^k f'(z)}{z^k f(z)} \, \mathrm{d}z = -\frac{1}{2\pi i} \int_\rho^i \frac{k}{z} \, \mathrm{d}z = \frac{k}{12}.$$
Since $\mathrm{d}q = 2\pi i q \mathrm{d}z$, the top part is
$$\frac{1}{2\pi i} \int_{\frac{1}{2}+iN}^{\frac{1}{2}-iN} \frac{f'(z)}{f(z)} \, \mathrm{d}z = -\frac{1}{2\pi i} \int_{\text{circle of radius } \epsilon} \frac{g'(q)}{g(q)} \, \mathrm{d}q = -\mathrm{ord}_\infty f.$$
Near i, $f'/f = \mathrm{ord}_i f (z-i)^{-1} + h(z)$, where $h(z)$ is holomorphic such that $h(z) \to 0$ as $\epsilon \to 0$. Then the circle $C_{\epsilon,i}$ of radius ϵ centered at i is
$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon,i}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\text{arc of half circle centered at } i} \frac{\mathrm{ord}_i f}{z-i} \, \mathrm{d}z = -\frac{\mathrm{ord}_i f}{2}.$$
Similarly, at ρ and ρ', get that the circles $C_{\epsilon,\rho}$ and $C_{\epsilon,\rho'}$ of radius ϵ centered at ρ and ρ' are
$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon,\rho}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon,\rho'}} \frac{f'(z)}{f(z)} \, \mathrm{d}z = -\frac{\mathrm{ord}_\rho f}{6},$$
which gives $-\mathrm{ord}_\rho f/3$. □

1.3.4 The space of holomorphic modular forms

Let

$$\mathrm{M}_k = \{\text{holomorphic modular forms of weight } k\},$$
$$\mathrm{S}_k = \{\text{cusp forms of weight } k\} = \{f \in \mathrm{M}_k \mid \mathrm{ord}_\infty f > 0\} \subseteq \mathrm{M}_k.$$

Corollary 1.3.9.

- $\mathrm{M}_k = 0$ *if* $k < 0$, $k = 2$, *or* k *odd.*
- M_0 *are constants.*
- $\mathrm{M}_4 = \mathbb{C}\mathrm{E}_4$, *where* $\mathrm{ord}_\rho \mathrm{E}_4 = 1$ *and no other zeroes.*
- $\mathrm{M}_6 = \mathbb{C}\mathrm{E}_6$, *where* $\mathrm{ord}_i \mathrm{E}_6 = 1$ *and no other zeroes.*
- $\mathrm{M}_8 = \mathbb{C}\mathrm{E}_8$, *where* $\mathrm{ord}_\rho \mathrm{E}_8 = 2$ *and no other zeroes.*
- $\mathrm{M}_{10} = \mathbb{C}\mathrm{E}_{10}$, *where* $\mathrm{ord}_\rho \mathrm{E}_{10} = \mathrm{ord}_i \mathrm{E}_{10} = 1$ *and no other zeroes.*
- $\mathrm{M}_{12} = \mathbb{C}\mathrm{E}_{12} \oplus \mathbb{C}\Delta$, *where* $\mathrm{ord}_\infty \Delta = 1$ *and no other zeroes.*

Corollary 1.3.10. $\Delta : \mathrm{M}_k \to \mathrm{S}_{k+12}$ *is an isomorphism. On the other hand,*

$$\mathrm{M}_k \cong \mathbb{C}\mathrm{E}_k \oplus \mathrm{S}_k, \qquad k \geq 4 \ \text{even},$$

so

$$\mathrm{M}_k \cong \mathbb{C}\mathrm{E}_k \oplus \dots \oplus \mathbb{C}\mathrm{E}_{k-12r}\Delta^r, \qquad k - 12r = 0, 4, 6, 8, 10, 14.$$

Corollary 1.3.11. $\mathrm{E}_4^2 = \mathrm{E}_8$ *and* $\mathrm{E}_4\mathrm{E}_6 = \mathrm{E}_{10}$.

Lecture 9 Monday 21/10/19

So for $k \geq 4$, the set

$$\begin{cases} \mathrm{E}_k, \dots, \mathrm{E}_{k-12\lfloor k/12 \rfloor}\Delta^{\lfloor k/12 \rfloor} & k \not\equiv 2 \mod 12 \\ \mathrm{E}_k, \dots, \mathrm{E}_{14}\Delta^{\lfloor k/12 \rfloor - 1} & k \equiv 2 \mod 12 \end{cases}$$

is a basis for M_k. A variant is to write $k = 4n + 6m$ with $m = 0, 1$ and $n \geq 0$, for $k \geq 4$. Then $\mathrm{M}_k = \mathbb{C}\mathrm{E}_4^n\mathrm{E}_6^m \oplus \mathrm{S}_k$ gives a basis

$$\mathrm{E}_4^n\mathrm{E}_6^m, \dots, \mathrm{E}_4^{n-3\lfloor n/3 \rfloor}\mathrm{E}_6^m\Delta^{\lfloor n/3 \rfloor}$$

for M_k. Since $\Delta = \left(\mathrm{E}_4^3 - \mathrm{E}_6^2\right)/1728$, we see every modular form of weight k is a polynomial in E_4 and E_6, and

$$\Delta \in q + q^2\mathbb{Z}[[q]], \qquad \mathrm{E}_4^n\mathrm{E}_6^m \in 1 + q\mathbb{Z}[[q]], \qquad \mathrm{E}_4^{n-3}\mathrm{E}_6^m\Delta \in q + q^2\mathbb{Z}[[q]], \qquad \dots$$

have integer coefficients.

Corollary 1.3.12. *If the* q*-expansion of* f *has integer coefficients, then* f *is an integer combination of*

$$\mathrm{E}_4^n\mathrm{E}_6^m, \dots, \mathrm{E}_4^{n-3\lfloor n/3 \rfloor}\mathrm{E}_6^m\Delta^{\lfloor n/3 \rfloor}.$$

Notation. $\mathrm{M}_k(\mathbb{Z}) \subseteq \mathrm{M}_k$ consists of modular forms with integer q-expansions.

Theorem 1.3.13. $\mathrm{M}_k(\mathbb{Z})$ *spans* M_k, *and* $f \in \mathrm{M}_k$ *lies in* $\mathrm{M}_k(\mathbb{Z})$ *if and only if* f *is an integral polynomial in* $\mathrm{E}_4, \mathrm{E}_6, \Delta$.

Definition 1.3.14. A **graded ring** is a ring R, together with a direct sum decomposition, as abelian groups,

$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

such that $R_i \cdot R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Example.

- $R = \mathbb{C}[X, Y]$, where R_i are polynomials homogeneous of degree i.
- $R = \bigoplus_{k \in \mathbb{Z}} \mathrm{M}_k$.

Let $\mathbb{C}[X,Y]$ be graded with $\deg X = 4$ and $\deg Y = 6$. Have a homomorphism of graded rings

$$\begin{array}{ccc} \mathbb{C}[X,Y] & \longrightarrow & \bigoplus_{k \in \mathbb{Z}} \mathrm{M}_k \\ (X,Y) & \longmapsto & (\mathrm{E}_4, \mathrm{E}_6) \end{array}.$$

Theorem 1.3.15. *This is an isomorphism of graded rings.*

Proof. This map is surjective, since every $f \in \mathrm{M}_k$ is a polynomial in E_4 and E_6. It remains to show this map is injective. Suppose not. There exists $P(X,Y)$, homogeneous of degree k, such that $P(\mathrm{E}_4, \mathrm{E}_6) = 0$. Write $k = 4n + 6m$ with $m = 0, 1$. If $P = c_0 X^n Y^m + \dots + c_r X^{n-3r} Y^{m+2r}$ where $r = \lfloor n/3 \rfloor$, then

$$c_0 \mathrm{E}_4^n \mathrm{E}_6^m + \dots + c_r \mathrm{E}_4^{n-3r} \mathrm{E}_6^{m+2r} = 0.$$

Dividing by $\mathrm{E}_4^{n-3r} \mathrm{E}_6^{m+2r}$, get $Q\left(\mathrm{E}_4^3/\mathrm{E}_6^2\right) = 0$ where $Q(X) = c_0 X^r + \dots + c_r$. Since the roots of Q are discrete, and $\mathrm{E}_4^3/\mathrm{E}_6^2$ is non-constant, this is impossible. ☐

1.3.5 The space of meromorphic modular forms

Note. The meromorphic modular forms of weight zero form a field.

Example. The j-invariant $\mathrm{j}(z) = \mathrm{E}_4^3/\Delta = 1728\mathrm{E}_4^3/\left(\mathrm{E}_4^3 - \mathrm{E}_6^2\right)$ is a non-constant meromorphic modular form, with a pole of order one at ∞, a zero of order three at ρ, and no other zeroes or poles.

Theorem 1.3.16. j *gives a bijection between* $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ *and* $\mathbb{C}$.

Proof. Given $\lambda \in \mathbb{C}$, want $z \in \mathbb{H}$ such that $\mathrm{j}(z) = \lambda$. Consider $g = j - \lambda$. This is meromorphic of weight zero. This has a pole at ∞, and no other poles, and

$$\mathrm{ord}_\infty g + \frac{\mathrm{ord}_\rho g}{3} + \frac{\mathrm{ord}_i g}{2} + \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H},\ p \nsim \{i, \rho\}} \mathrm{ord}_p g = 0.$$

The only possibilities are

- g has a zero at ρ of order three, and no other zeroes,
- g has a zero at i of order two, and no other zeroes, or
- g has a simple zero somewhere else, and no other zeroes.

In each case, the zero of g is a unique $\mathrm{SL}_2(\mathbb{Z})$-orbit on which $\mathrm{j}(z) = \lambda$. So j is bijective. ☐

Lecture 10 Friday 25/10/19

Theorem 1.3.17. *Every meromorphic modular form of weight zero is a rational function in* j. *That is, the field of meromorphic modular forms is* $\mathbb{C}(\mathrm{j})$.

Proof. Let g be meromorphic of weight zero. Then g has finitely many $\mathrm{SL}_2(\mathbb{Z})$-orbits worth of poles in $\mathbb{H}$. Saw last time that j is holomorphic in $\mathbb{H}$. If p is a pole of g, then $(\mathrm{j}(z) - \mathrm{j}(p))^{n_p}$ is holomorphic on $\mathbb{H}$ and zero at $z = p$. Doing this for all poles, there exists $P \in \mathbb{C}[X]$ such that $P(\mathrm{j})\, g(z)$ is holomorphic on $\mathbb{H}$. Then for some m, $P(\mathrm{j})\, g(z)\, \Delta^m$ is holomorphic of weight $12m$. So it suffices to show if h is holomorphic of weight $12m$, then h/Δ^m is a rational function in j, since if $P(\mathrm{j})\, g(z)\, \Delta^m = h$ then $P(\mathrm{j})\, g(z) \in \mathbb{C}(\mathrm{j})$, so $g(z) \in \mathbb{C}(\mathrm{j})$. Then h is a sum of terms

$$h = \sum_{a,b} c_{a,b} \mathrm{E}_4^a \mathrm{E}_6^b, \qquad c_{a,b} \in \mathbb{C}, \qquad 4a + 6b = 12m.$$

Considering this equation modulo four and modulo three, find $3 \mid a$ and $2 \mid b$, so

$$\frac{h}{\Delta^m} = \sum_{a,b} c_{a,b} \left(\frac{\mathrm{E}_4^3}{\Delta}\right)^{\frac{a}{3}} \left(\frac{\mathrm{E}_6^2}{\Delta}\right)^{\frac{b}{2}}.$$

So it suffices to show E_4^3/Δ and E_6^2/Δ are rational functions in j. Then $\mathrm{j} = \mathrm{E}_4^3/\Delta$, and

$$\frac{\mathrm{E}_6^2}{\Delta} = \frac{1728\mathrm{E}_6^2}{\mathrm{E}_4^3 - \mathrm{E}_6^2} = \frac{1728\left(\mathrm{E}_6^2 - \mathrm{E}_4^3\right) + 1728\mathrm{E}_4^3}{\mathrm{E}_4^3 - \mathrm{E}_6^2} = -1728 + \frac{1728\mathrm{E}_4^3}{\mathrm{E}_4^3 - \mathrm{E}_6^2} = \mathrm{j} - 1728.$$

☐

1.4 Theta series

Let $L \subseteq \mathbb{R}^n$ be a lattice. For $x, y \in L$, $x \cdot y \in \mathbb{R}$. Suppose $x \cdot y \in \mathbb{Z}$ for all $x, y \in L$. A question is for $n \in \mathbb{Z}$, how many $x \in L$ have $x \cdot x = n$? The rough idea is to form the series

$$\sum_{x \in L} q^{x \cdot x} = \sum_{n=0}^{\infty} a_n q^n, \qquad a_n = \# \{x \in L \mid x \cdot x = n\}.$$

We will show that, with some slight modifications, and extra hypotheses on L, this generating function turns out to be a modular form.

1.4.1 Quadratic forms

Fix a lattice $L \subseteq \mathbb{R}^n$, so

$$L = \mathbb{Z} \cdot e_1 \oplus \cdots \oplus \mathbb{Z} \cdot e_n.$$

Given these e_i, form a matrix A such that $A_{ij} = e_i \cdot e_j$.

Note. $A = B^\intercal B$, where B is the matrix whose columns are the e_i, and $|\det B|$ is the **volume** of the parallelogram spanned by e_i, so $\det A = \det B^2 > 0$.

Definition 1.4.1. The **dual lattice** $L^\vee$ is the set of $y \in \mathbb{R}^n$ such that $y \cdot x \in \mathbb{Z}$ for all $x \in L$.

Let $f_1, \ldots, f_n$ be the dual basis to $e_1, \ldots, e_n$, that is the unique set of solutions $f_1, \ldots, f_n$ such that

$$f_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then $L^\vee$ is spanned by the f_i. Clearly $f_i \in L^\vee$ for all i. Conversely, if $y \in L^\vee$, then $y \cdot e_i = a_i \in \mathbb{Z}$, then $y = \sum_{i=1}^n a_i f_i$.

Proposition 1.4.2. *Let $C = A^{-1}$. Then*

$$f_i = \sum_{j=1}^n C_{ij} e_j.$$

Proof.

$$f_i \cdot e_k = \sum_{j=1}^n C_{ij} e_j e_k = \sum_{j=1}^n C_{ij} A_{jk} = (CA)_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}.$$

□

Definition 1.4.3. A lattice L is **self-dual** if $L^\vee = L$ as subsets of $\mathbb{R}^n$.

Proposition 1.4.4. *L is self-dual if and only if the associated matrix A has integer entries and determinant one.*

Proof. Clearly if $L = L^\vee$, then $e_i \cdot e_j \in \mathbb{Z}$, so A has integer entries. Since $L^\vee \subseteq L$, f_i is an integer combination of the e_j, so $C = A^{-1}$ has integer entries. So $\det A = \pm 1$, but already saw $\det A > 0$. Conversely if A has integer entries and determinant one, $C = A^{-1}$ has integer entries. Then A has integer entries implies that $e_i \cdot e_j \in \mathbb{Z}$ for all i and j, so $e_i \in L^\vee$ for all i, so $L \subseteq L^\vee$. Similarly, C has integer entries implies that $L^\vee \subseteq L$. □

If L is self-dual, get an integer-valued **quadratic form**

$$\begin{array}{cccc} \mathrm{Q}_L & : & \mathbb{Z}^n & \longrightarrow & \mathbb{Z} \\ & & (a_1, \ldots, a_n) & \longmapsto & (a_1 e_1 + \cdots + a_n e_n) \cdot (a_1 e_1 + \cdots + a_n e_n) = \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{array}.$$

A question is given m, how often does Q_L represent m?

1.4.2 Fourier analysis

Lecture 11 Friday 25/10/19

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a C^∞ function.

Definition 1.4.5. We will say f is **rapidly decreasing** if for all m,

$$\|x\|^m f(x)| \to 0, \qquad |x| \to \infty,$$

where $|x| = (x \cdot x)^{1/2}$. For $f \in \mathrm{C}^\infty$, rapidly decreasing, define

$$\widehat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i (x \cdot y)} \, \mathrm{d}x : \mathbb{R}^n \to \mathbb{C}.$$

Fact.

- If f is smooth and rapidly decreasing, so is $\widehat{f}$.
- If $f(x) = e^{-\pi(x \cdot x)}$, then $\widehat{f}(x) = f(x)$.
- If f is smooth and rapidly decreasing, and $L \subseteq \mathbb{R}^n$ is a lattice with volume V, then

$$\sum_{x \in L} f(x) = \frac{1}{V} \sum_{x \in L^\vee} \widehat{f}(x).$$

1.4.3 Theta series

A crucial assumption is that L is self-dual. An assumption that can be removed is that L is even, so for all $x \in L$, $\mathrm{Q}_L(x) \in 2\mathbb{Z}$.

Definition 1.4.6. The **theta series** Θ_L is defined by

$$\Theta_L(z) = \sum_{x \in L} q^{\frac{1}{2} x \cdot x} = \sum_{m=0}^{\infty} a_m q^m, \qquad a_m = \#\{x \in \mathbb{Z}^n \mid \mathrm{Q}_L(x) = 2m\}.$$

Theorem 1.4.7. *Θ_L is modular of weight $n/2$.*

Example. Let $\Gamma_8 \subseteq \mathbb{R}^8$ be spanned by

$$e_1 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \qquad e_2 = (1,1,0,0,0,0,0,0),$$

$$e_3 = (-1,1,0,0,0,0,0,0), \qquad e_4 = (0,-1,1,0,0,0,0,0), \qquad e_5 = (0,0,-1,1,0,0,0,0),$$
$$e_6 = (0,0,0,-1,1,0,0,0), \qquad e_7 = (0,0,0,0,-1,1,0,0), \qquad e_8 = (0,0,0,0,0,-1,1,0).$$

Then

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

and

$$\mathrm{Q}_L(z_1, \dots, z_8) = 2\left(z_1^2 + \dots + z_8^2 - z_1 z_3 - z_2 z_4 - z_3 z_4 - z_4 z_5 - z_6 z_7 - z_7 z_8\right).$$

If $L \subseteq \mathbb{R}^n$ is even and self-dual, and Θ_L is modular of weight $n/2$, then the dimension is $\sim n/24$.

Fact. If $L \subseteq \mathbb{R}^n$ is even and self-dual, then $8 \mid n$.

Proof. Serre V.2.1 Corollary 2. $\square$

Proof of Theorem 1.4.7. Know, since L is even, that $\Theta_L(z+1) = \Theta_L(z)$. It suffices to show $\Theta_L(-1/z) = z^{n/2}\Theta_L(z)$. Both sides are holomorphic on $\mathbb{H}$, so it suffices to show

$$\Theta_L\left(-\frac{1}{it}\right) = (it)^{\frac{n}{2}}\,\Theta_L(it).$$

For $t \in \mathbb{R}^\times$, let $L_t = t^{1/2} \cdot L$. Then $L_t^\vee = t^{-1/2} \cdot L = L_{t^{-1}}$, so the volume of L_t is $t^{n/2}$. By the facts,

$$\sum_{x \in L_t} e^{-\pi(x\cdot x)} = t^{-\frac{n}{2}} \sum_{x \in L_{t^{-1}}} e^{-\pi(x\cdot x)},$$

so

$$\sum_{x \in L} e^{-\pi(x\cdot x)t} = t^{-\frac{n}{2}} \sum_{x \in L} e^{-\frac{\pi(x\cdot x)}{t}}.$$

Now return to Θ_L. The left hand side is

$$\Theta_L\left(-\frac{1}{it}\right) = \sum_{x \in L} e^{\frac{1}{2}\cdot 2\pi i\cdot\left(-\frac{1}{it}\right)\cdot(x\cdot x)} = \sum_{x \in L} e^{-\frac{\pi(x\cdot x)}{t}},$$

and the right hand side is

$$\Theta_L(it) = \sum_{x \in L} e^{\frac{1}{2}\cdot 2\pi i\cdot(it)\cdot(x\cdot x)} = \sum_{x \in L} e^{-\pi(x\cdot x)t},$$

so the result follows. □

1.4.4 Asymptotic analysis

Let $L \subseteq \mathbb{R}^n$ be even and self-dual, so $8 \mid n$, and let $\Theta_L = \sum_{m=0}^\infty a_m q^m$, where a_m is the number of ways Q_L represents $2m$, so $a_0 = 1$. Then

$$\Theta_L = \mathrm{E}_{\frac{n}{2}} + g, \qquad \mathrm{E}_{\frac{n}{2}} \sim \sigma_{\frac{n}{2}-1}(m) \sim m^{\frac{n}{2}-1},$$

where g is a cusp form.

Lecture 12 is a problems class. Lecture 12 Monday 28/10/19

Proposition 1.4.8. *Let* Lecture 13 Friday 01/11/19

$$\mathrm{E}_k = \sum_{n=0}^\infty a_n q^n = 1 + C\sum_{n=1}^\infty \sigma_{k-1}(n)\,q^n.$$

Then there exist $A, B \in \mathbb{R}_{>0}$ such that

$$An^{k-1} \le a_n \le Bn^{k-1}.$$

Proof. Set $A = C$. Then

$$\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1} \ge n^{k-1},$$

so $a_n = C\sigma_{k-1}(n) \ge Cn^{k-1}$. Consider

$$\frac{\sigma_{k-1}(n)}{n^{k-1}} = \sum_{d \mid n} \frac{d^{k-1}}{n^{k-1}} = \sum_{d' \mid n} \frac{1}{d'^{k-1}} \le \sum_{n=1}^\infty \frac{1}{n^{k-1}} = \zeta(k-1),$$

so $\sigma_{k-1}(n) \le \zeta(k-1)\,n^{k-1}$. So set $B = C\zeta(k-1)$, so $a_n \le Bn^{k-1}$. □

Theorem 1.4.9 (Hecke)**.** *Let $f = \sum_{n=1}^\infty a_n q^n$ be a cusp form of weight k. Then*

$$|a_n| = \mathrm{O}\left(n^{\frac{k}{2}}\right),$$

that is $|a_n|\,n^{-k/2}$ is bounded as $n \to \infty$.

Proof. f/q is holomorphic on $\mathbb{H}$, so $|f/q|$ is bounded as $q \to 0$, so $|f(z)|/e^{-2\pi \operatorname{Im} z}$ is bounded as $\operatorname{Im} z \to \infty$. That is, there exist $M \in \mathbb{R}$ such that $|f(z)| \le Me^{-2\pi \operatorname{Im} z}$. Consider

$$\phi(z) = |f(z)| \operatorname{Im} z^{\frac{k}{2}},$$

so $\lim_{\operatorname{Im} z \to \infty} \phi(z) = 0$. Note that

$$\phi(\gamma z) = |f(\gamma z)| \operatorname{Im} \gamma z^{\frac{k}{2}} = |f(z)| |cz+d|^k \frac{\operatorname{Im} z^{\frac{k}{2}}}{|cz+d|^{2\frac{k}{2}}} = |f(z)| \operatorname{Im} z^{\frac{k}{2}} = \phi(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then $\phi(z)$ is determined by its values on the standard fundamental domain, so $\phi(z)$ is bounded on $\mathbb{H}$, so $|f(z)| < M' \operatorname{Im} z^{-k/2}$ for some $M' \in \mathbb{R}$. If $z = x + iy$ for y fixed, then by the residue theorem,

$$a_m = \frac{1}{2\pi i} \int_C \frac{f(q)}{q^{m+1}} \, \mathrm{d}q = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x+iy)}{e^{2\pi i (x+iy) m}} \, \mathrm{d}x,$$

where C is a circle around zero, oriented counterclockwise, so

$$|a_m| \le \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|f(x+iy)|}{e^{-2\pi y m}} \, \mathrm{d}x \le \frac{|f(x+iy)|}{e^{-2\pi y m}} \le e^{2\pi y m} M' y^{-\frac{k}{2}}.$$

Set $y = 1/m$. Get $|a_m| \le e^{2\pi} M' m^{k/2}$, so $|a_m|/m^{k/2}$ is bounded. $\square$

Had

$$\Theta_L = \mathrm{E}_{\frac{n}{2}} + g, \qquad \mathrm{E}_{\frac{n}{2}} \sim m^{\frac{n}{2}-1}, \qquad g = \mathrm{O}\left(m^{\frac{n}{4}}\right).$$

Theorem 1.4.10 (Deligne)**.** *Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cusp form of weight k. Then*

$$|a_n| = \mathrm{O}\left(n^{\frac{k-1}{2}} \sigma_0(n)\right).$$

Proof. Very rough sketch of argument.

Ramanujan 1910s. Conjectured by Ramanujan for $f = \Delta$.

Weil 1940s. For an algebraic variety V over $\mathbb{F}_q$, what can we say about $\#V(\mathbb{F}_{q^n})$ for various n? Weil associated to V and $\mathbb{F}_q$ a generating function called the **zeta function** $\zeta_{V,q}(t)$ of V over $\mathbb{F}_q$, conjectured several things about $\zeta_{V,q}$, and proved in the case of curves.

- $\zeta_{V,q}$ is a rational function in t.
- $\zeta_{V,q}$ satisfies a certain symmetry under $t \mapsto 1/t$.
- The **Riemann hypothesis**

$$\zeta_{V,q}(t) = \frac{P_1(t) \dots P_{2d-1}(t)}{P_0(t) \dots P_{2d}(t)}, \qquad \dim V = d,$$

where the roots of $P_i(t)$ have absolute value $q^{i/2}$.

Eichler-Shimura 1950s. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a nice **congruence subgroup**. Then $X_\Gamma = \Gamma \backslash \mathbb{H}$ has the structure of an algebraic curve over $\mathbb{Q}$, with **good reduction** at primes p not dividing $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Eichler, Shimura, and others studied $\zeta_{V,p}$ for $V = X_\Gamma$, and related $\zeta_{V,p}$ to the p-th Fourier coefficients of a basis for forms of weight two and **level** Γ. The **Weil conjectures** bound a_p in terms of $q^{1/2}$.

Deligne 1960s. Deligne showed that in weight k, there exists a **Kuga-Sato variety**, of dimension $k-1$, whose zeta function has a factor coming from modular forms of weight k and level Γ, and showed that if the Weil conjectures, particularly the Riemann hypothesis, holds, then get the coefficient bound.

Deligne 1970s. The Riemann hypothesis in higher dimensions.

$\square$

1.5 Hecke operators

Lecture 14 Friday 01/11/19

Let
$$\Delta = \frac{\mathrm{E}_4^3 - \mathrm{E}_6^2}{1728} = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{m=1}^{\infty} (1 - q^m)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \dots.$$
Then $\tau(n)$ grows roughly like n^6 or $n^{11/2+\epsilon}$. Mordell proved that

- $\tau(mn) = \tau(n)\tau(m)$ if $(m,n) = 1$, and
- $\tau\left(p^{n+1}\right) = \tau(p)\tau(p^n) - p^{11}\tau\left(p^{n-1}\right)$.

Note. If $\mathrm{E}_k = 1 + C\sum_n \sigma_{k-1}(n) q^n$, set
$$\mathrm{E}'_k = \frac{1}{C} + \sum_n \sigma_{k-1}(n) q^n.$$

- If $(m,n) = 1$, then
$$\sigma_{k-1}(nm) = \sum_{d \mid n} \sum_{d' \mid m} (dd')^{k-1} = \left(\sum_{d \mid n} d^{k-1}\right)\left(\sum_{d' \mid m} d'^{k-1}\right) = \sigma_{k-1}(n)\sigma_{k-1}(m).$$
- Since $\sigma_{k-1}(p^n) = 1 + \dots + p^{n(k-1)}$,
$$\begin{aligned} \sigma_{k-1}(p)\sigma_{k-1}(p^n) &= \left(1 + p^{k-1}\right)\left(1 + \dots + p^{n(k-1)}\right) \\ &= 1 + 2p^{k-1} + \dots + 2p^{n(k-1)} + p^{(n+1)(k-1)} \\ &= \sigma_{k-1}\left(p^{n+1}\right) + p^{k-1}\sigma_{k-1}\left(p^{n-1}\right), \end{aligned}$$
so
$$\sigma_{k-1}\left(p^{n+1}\right) = \sigma_{k-1}(p)\sigma_{k-1}(p^n) - p^{k-1}\sigma_{k-1}\left(p^{n-1}\right).$$

1.5.1 Correspondences

Definition 1.5.1. Let X be a set. The **free abelian group on** X, denoted $\mathbb{Z}X$, is the set of finite formal sums
$$\sum_{i=1}^{r} a_i x_i, \qquad a_i \in \mathbb{Z}, \qquad x_i \in X,$$
where x_i are distinct. Add by combining like terms.

Definition 1.5.2. A **correspondence** on X is a homomorphism $\mathbb{Z}X \to \mathbb{Z}X$. Let
$$\mathrm{Corr}\, X = \{\text{correspondences on } X\}.$$
Equivalently, a correspondence associates to each $x \in X$, a finite formal sum
$$\sum_{i=1}^{r} a_i y_i, \qquad a_i \in \mathbb{Z}, \qquad y_i \in X.$$
If X is a finite set $X = \{x_1, \dots, x_r\}$, any correspondence T can be represented, in a unique way, by the matrix M_T such that
$$T x_i = \sum_{j=1}^{r} (M_T)_{ij} x_j,$$
and composition of correspondences is matrix multiplication. Let X be a set, and let
$$\mathrm{Fun}_{\mathbb{C}}\, X = \{\text{functions } X \to \mathbb{C}\}.$$
Then $T \in \mathrm{Corr}\, X$ acts on $\mathrm{Fun}_{\mathbb{C}}\, X$ as follows. If $Tx = \sum_i a_i x_i$ then $(Tf)\, x = \sum_i a_i f(x_i)$. Check $(T \circ T') f = T(T'f)$ etc. Let
$$\mathcal{L} = \{\text{lattices in } \mathbb{C}\}.$$

Example.

- For $\lambda \in \mathbb{C}^\times$, have
$$\begin{array}{rcrcl} \mathrm{R}_\lambda & : & \mathbb{Z}\mathcal{L} & \longrightarrow & \mathbb{Z}\mathcal{L} \\ & & L & \longmapsto & \lambda L \end{array}.$$
- For $n \in \mathbb{Z}_{>0}$, have
$$\begin{array}{rcrcl} \mathrm{T}_n & : & \mathbb{Z}\mathcal{L} & \longrightarrow & \mathbb{Z}\mathcal{L} \\ & & L & \longmapsto & \sum_{L' \subseteq_n L} L' \end{array},$$
the n **Hecke operators**. Note that there are only finitely many $L' \subseteq_n L$, since if $L' \subseteq_n L$, then L' contains $\mathrm{R}_n L$. Then $L/\mathrm{R}_n L \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. The image of L' in $L/\mathrm{R}_n L$ is a subgroup H of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ of order n. The preimage of H in L is L'. Thus there is a bijection
$$\left\{ \text{ subgroups of } L/\mathrm{R}_n L \text{ of order } n \right\} \qquad \leftrightsquigarrow \qquad \left\{ \text{ sublattices of } L \text{ of index } n \right\}.$$

Proposition 1.5.3.

1. $\mathrm{R}_\lambda \mathrm{R}_\mu = \mathrm{R}_{\lambda\mu}$.
2. $\mathrm{R}_\lambda \mathrm{T}_n = \mathrm{T}_n \mathrm{R}_\lambda$.
3. $\mathrm{T}_n \mathrm{T}_m = \mathrm{T}_{nm}$ *if* $(m, n) = 1$.
4. $\mathrm{T}_p \mathrm{T}_{p^n} = \mathrm{T}_{p^{n+1}} + p\mathrm{T}_{p^{n-1}} \mathrm{R}_p$.

Lecture 15 Monday 04/11/19

Corollary 1.5.4. T_p *commute with each other for p prime, also with* R_λ*, and every* T_n *is a polynomial in* T_p *and* R_p *for* $p \mid n$*, so all* T_n *and* R_λ *commute.*

Proposition 1.5.5. *If A is an abelian group of order nm, with $(n, m) = 1$, then A factors uniquely as $B \times C$, where B has order n and C has order m. In particular B is the unique subgroup of A of order n.*

Proof. Write $1 = an + bm$ for $a, b \in \mathbb{Z}$. Have a map
$$\begin{array}{rcl} A & \longleftrightarrow & mA \times nA \\ x & \longmapsto & (mbx, nax) \\ x + y & \longleftarrow & (x, y) \end{array}.$$
Then mA has order n and nA has order m. Clearly inverses on one side, so counting implies isomorphism. ∎

Proof of Proposition 1.5.3.

1. Easy.
2. If $L \in \mathcal{L}$, then
$$\mathrm{R}_\lambda \mathrm{T}_n L = \mathrm{R}_\lambda \sum_{L' \subseteq_n L} L' = \sum_{L' \subseteq_n L} \mathrm{R}_\lambda L' = \sum_{L' \subseteq_n \mathrm{R}_\lambda L} L' = \mathrm{T}_n \mathrm{R}_\lambda L.$$
3. If $L \in \mathcal{L}$, then
$$\mathrm{T}_n \mathrm{T}_m L = \mathrm{T}_n \sum_{L' \subseteq_m L} L' = \sum_{L' \subseteq_m L} \mathrm{T}_n L' = \sum_{L' \subseteq_m L} \sum_{L'' \subseteq_n L'} L''.$$
An observation is $L'' \subseteq_n L' \subseteq_m L$, so $L'' \subseteq_{nm} L$. Then
$$\mathrm{T}_n \mathrm{T}_m L = \sum_{L'' \subseteq_{nm} L} c_{n,m}(L'', L) L'', \qquad c_{n,m}(L'', L) = \#\{L' \in \mathcal{L} \mid L'' \subseteq_n L' \subseteq_m L\}.$$
An observation is that there is a bijection
$$\begin{array}{rcl} \{\text{lattices } L' \mid L'' \subseteq_n L' \subseteq_m L\} & \longleftrightarrow & \{\text{subgroups } H \text{ of } L/L'' \text{ of order } n\} \\ L' & \longmapsto & L'/L'' \subseteq L/L'' \\ \text{preimage of } H \text{ under } L \to L/L'' & \longleftarrow & H \end{array}.$$
Have $(n, m) = 1$, then $c_{n,m}(L'', L) = 1$ so
$$\mathrm{T}_n \mathrm{T}_m L = \sum_{L'' \subseteq_{nm} L} c_{n,m}(L'', L) L'' = \sum_{L'' \subseteq_{nm} L} L'' = \mathrm{T}_{nm} L.$$

4. Similarly, if $L \in \mathcal{L}$, then
$$\mathrm{T}_p \mathrm{T}_{p^r} L = \sum_{L'' \subseteq_{p^{r+1}} L} c_{p,p^r}\left(L'', L\right) L'', \qquad c_{p,p^r}\left(L'', L\right) = \#\left\{L' \in \mathcal{L} \mid L'' \subseteq_p L' \subseteq_{p^r} L\right\}.$$
What is
$$c_{p,p^r}\left(L'', L\right) = \#\left\{\text{subgroups of order } p \text{ in } L/L''\right\}?$$
L/L'' is abelian of order p^{r+1} and is generated by two elements. By the classification of finite abelian groups, every finite abelian group can be written uniquely as
$$\mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_r\mathbb{Z}, \qquad a_1 \mid \cdots \mid a_r,$$
up to isomorphism, and r is the minimal number of generators for such a group. So
$$L/L'' \cong \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}/p^b\mathbb{Z}, \qquad a, b \ge 0, \qquad a + b = r + 1.$$
Case 1. $L/L'' \cong \mathbb{Z}/p^{r+1}\mathbb{Z}$ is cyclic. In this case $c_{p,p^r}\left(L'', L\right) = 1$.
Case 2. $L/L'' \cong \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}/p^b\mathbb{Z}$ with $a, b > 0$. Any subgroup of order p is contained in the subgroup killed by p,
$$p^{a-1}\mathbb{Z}/p^a\mathbb{Z} \times p^{b-1}\mathbb{Z}/p^b\mathbb{Z} \cong \left(\mathbb{Z}/p\mathbb{Z}\right)^2.$$
The $p^2 - 1$ elements of $\left(\mathbb{Z}/p\mathbb{Z}\right)^2$ other than zero each spans a subgroup of order p, and two elements span the same group if and only if they differ by a scalar in $\left(\mathbb{Z}/p\mathbb{Z}\right)^\times$, so there are $\left(p^2 - 1\right) / \left(p - 1\right) = p + 1$ subgroups of order p in $\left(\mathbb{Z}/p\mathbb{Z}\right)^2$. In this case $c_{p,p^r}\left(L'', L\right) = p + 1$.

The latter case occurs if and only if L/L'' maps surjectively to $\left(\mathbb{Z}/p\mathbb{Z}\right)^2 \cong L/\mathrm{R}_p L$, if and only if $\mathrm{R}_p L \supseteq L''$. Thus
$$\begin{aligned} \mathrm{T}_p \mathrm{T}_{p^r} L &= \sum_{L'' \subseteq_{p^{r+1}} L} c_{p,p^r}\left(L'', L\right) L'' = \sum_{L'' \subseteq_{p^{r+1}} L \text{ cyclic}} L'' + (p+1) \sum_{L'' \subseteq_{p^{r+1}} L \text{ not cyclic}} L'' \\ &= \mathrm{T}_{p^{r+1}} L + p \sum_{L'' \subseteq_{p^{r+1}} L \text{ not cyclic}} L'' = \mathrm{T}_{p^{r+1}} L + p \sum_{L'' \subseteq_{p^{r-1}} \mathrm{R}_p L} L'' = \mathrm{T}_{p^{r+1}} L + p \mathrm{T}_{p^{r-1}} \mathrm{R}_p L. \end{aligned}$$

□

1.5.2 Hecke operators

Lecture 16 Friday 08/11/19

If $F : \mathcal{L} \to \mathbb{C}$, then
$$\left(\mathrm{T}_n F\right)(L) = \sum_{L' \subseteq_n L} F\left(L'\right), \qquad \left(\mathrm{R}_\lambda F\right)(L) = F\left(\mathrm{R}_\lambda L\right).$$
Recall that F has weight k if $F\left(\mathrm{R}_\lambda L\right) = \lambda^{-k} F(L)$ for all $\lambda \in \mathbb{C}^\times$, if and only if
$$\mathrm{R}_\lambda F = \lambda^{-k} F, \qquad \lambda \in \mathbb{C}^\times,$$
so
$$\mathrm{R}_\lambda \mathrm{T}_n F = \mathrm{T}_n \mathrm{R}_\lambda F = \mathrm{T}_n \lambda^{-k} F = \lambda^{-k} \mathrm{T}_n F.$$
So the T_n and R_λ preserve lattice functions of weight k. Have a bijection
$$\begin{array}{ccc} \left\{f : \mathbb{H} \to \mathbb{C} \;\middle|\; f(\gamma z) = (cz + d)^k f(z)\right\} & \longrightarrow & \left\{\text{lattice functions } F \text{ of weight } k\right\} \\ f(z) & \longmapsto & F\left(\mathrm{L}_{z,1}\right) \end{array}.$$
On lattice functions of weight k, have
$$\mathrm{T}_p \mathrm{T}_{p^r} = \mathrm{T}_{p^{r+1}} + p^{1-k} \mathrm{T}_{p^{r-1}}.$$
Definition 1.5.6. For $f : \mathbb{H} \to \mathbb{C}$ corresponding to $F : \mathcal{L} \to \mathbb{C}$ of weight k, define $\mathrm{T}_n f$ by
$$\left(\mathrm{T}_n f\right)(z) = n^{k-1}\left(\mathrm{T}_n F\right)\left(\mathrm{L}_{z,1}\right) = n^{k-1} \sum_{L' \subseteq_n \mathrm{L}_{z,1}} F\left(L'\right).$$

On $f : \mathbb{H} \to \mathbb{C}$, T_n satisfy
$$\mathrm{T}_p \mathrm{T}_{p^r} = \mathrm{T}_{p^{r+1}} + p^{k-1} \mathrm{T}_{p^{r-1}}.$$
Need to rewrite $\sum_{L' \subseteq_n \mathrm{L}_{z,1}} F(L')$ in terms of f. Let
$$\mathrm{S}^n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{Mat}_{2 \times 2} \mathbb{Z} \;\middle|\; ad = n, \ a, d > 0, \ 0 \le b < d \right\}.$$

Lemma 1.5.7. *The map*
$$\begin{array}{ccc} \mathrm{S}^n & \longrightarrow & \{\textit{sublattices of } \mathrm{L}_{z,1} \textit{ of index } n\} \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} & \longmapsto & \mathrm{L}_{az+b,d} \end{array}$$
is a bijection.

Proof. For surjectivity, let $L \subseteq_n \mathrm{L}_{z,1}$. Then $\mathrm{L}_{z,1}/L$ is a group of order n. Can consider $1 + L \in \mathrm{L}_{z,1}/L$. Let d be the order of $1 + L$, that is d is the smallest positive integer such that $d \in L$. Then $d \mid n$, so set $a = n/d$. Let $L' = \mathbb{Z} + L$ be the lattice generated by 1 and L. Then $L \subseteq_d L'$ and $L \subseteq_n \mathrm{L}_{z,1}$, so $L' \subseteq_a \mathrm{L}_{z,1}$, so $az \in L'$, so there exists $b \in \mathbb{Z}$ such that $az + b \in L$. Since $d \in L$, without loss of generality can arrange $0 \le b < d$. Now $d \in L$ and $az + b \in L$, so $\mathrm{L}_{az+b,d} \subseteq_n \mathrm{L}_{z,1}$ and $L \subseteq_n \mathrm{L}_{z,1}$, so $L = \mathrm{L}_{az+b,d}$. Thus surjective, and for injectivity, can recover a, b, d from $\mathrm{L}_{az+b,d} \subseteq \mathrm{L}_{z,1}$. ☐

Thus
$$\begin{aligned} \mathrm{T}_n f &= n^{k-1} \sum_{L' \subseteq_n \mathrm{L}_{z,1}} F(L') = n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^n} F(\mathrm{L}_{az+b,d}) \\ &= n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^n} d^{-k} F\left(\mathrm{L}_{\frac{az+b}{d},1}\right) = n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^n} d^{-k} f\left(\frac{az+b}{d}\right). \end{aligned}$$

Theorem 1.5.8. *If $f = \sum_{m=0}^{\infty} c_m q^m$ is modular of weight k, then*
$$\mathrm{T}_n f = \sum_{m=0}^{\infty} \gamma_m q^m, \qquad \gamma_m = \sum_{a \mid (m,n), \ a > 0} a^{k-1} c_{\frac{mn}{a^2}}.$$

Proof.
$$\begin{aligned} \mathrm{T}_n f &= n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^n} d^{-k} f\left(\frac{az+b}{d}\right) = n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^n} \sum_{m=0}^{\infty} d^{-k} c_m e^{2\pi i m \left(\frac{az+b}{d}\right)} \\ &= n^{k-1} \sum_{ad=n, \ a>0} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} d^{-k} c_m q^{\frac{ma}{d}} e^{\frac{2\pi i m b}{d}} = n^{k-1} \sum_{m=0}^{\infty} \sum_{ad=n, \ a>0} d^{-k} c_m q^{\frac{ma}{d}} \sum_{b=0}^{d-1} e^{\frac{2\pi i m b}{d}}. \end{aligned}$$

Then
$$\sum_{b=0}^{d-1} e^{\frac{2\pi i m b}{d}} = \begin{cases} d & d \mid m \\ 0 & d \nmid m \end{cases},$$
so
$$\mathrm{T}_n f = n^{k-1} \sum_{m=0, \ d \mid m}^{\infty} \sum_{ad=n, \ a>0} d^{1-k} c_m q^{\frac{ma}{d}} = \sum_{a \mid n, \ a>0} \sum_{m'=0}^{\infty} a^{k-1} c_{\frac{m'n}{a}} q^{m'a}.$$

Which m' and a give q^m? Need $a \mid (m, n)$ for $a > 0$ and $m'a = m$, so the coefficient is $a^{k-1} c_{mn/a^2}$. The sum of these is
$$\gamma_m = \sum_{a \mid (m,n), \ a>0} a^{k-1} c_{\frac{mn}{a^2}}.$$
☐

Corollary 1.5.9. T_n *preserves* M_k *and* S_k.

In the case $n = p$,

$$\mathrm{T}_p f = \sum_{m=0}^{\infty} \gamma_m q^m, \qquad \gamma_m = \begin{cases} c_{mp} + p^{k-1} c_{\frac{m}{p}} & p \mid m \\ c_{mp} & p \nmid m \end{cases}.$$

1.5.3 Eigenforms

An observation is that the dimensions of $\mathrm{M}_4, \mathrm{M}_6, \mathrm{M}_8, \mathrm{M}_{10}, \mathrm{S}_{12}$ are one, so $\mathrm{E}_4, \mathrm{E}_6, \mathrm{E}_8, \mathrm{E}_{10}, \Delta$ are eigenvectors for T_n for all n.

Definition 1.5.10. A function $f \in \mathrm{M}_k$ is an **eigenform** if there exists $\lambda_n \in \mathbb{C}^\times$ such that $\mathrm{T}_n f = \lambda_n f$ for all $n \in \mathbb{Z}_{>0}$.

Lecture 17
Friday
08/11/19

Proposition 1.5.11. *Let $f \in \mathrm{M}_k$ be an eigenform, with $k > 0$, so $\mathrm{T}_n f = \lambda_n f$ for all n. Then if $f = \sum_m c_m q^m$, we have $c_1 \neq 0$ and $\lambda_n c_1 = c_n$ for all $n \geq 1$. In particular, if $c_1 = 1$, then $c_n = \lambda_n$ for all n.*

Proof.

$$\sum_{m=0}^{\infty} \lambda_n c_m q^m = \lambda_n f = \mathrm{T}_n f = \sum_{m=0}^{\infty} \gamma_m q^m, \qquad \gamma_1 = \sum_{a \mid (1,n)} a^{k-1} c_n = c_n,$$

so $\lambda_n c_1 = c_n$. Suppose $c_1 = 0$. Then $c_n = 0$ for all $n \geq 1$, so f is constant. Since $k \neq 0$, this does not happen. ☐

Corollary 1.5.12. *Recall that $\Delta(z) = \sum_n \tau(n) q^n$. Then*

- $\tau(mn) = \tau(n)\tau(m)$ *if $(m, n) = 1$, and*
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$.

Proof. $\Delta \in \mathrm{S}_{12}$ is one-dimensional, so there exists λ_n such that $\mathrm{T}_n\Delta = \lambda_n\Delta$. Proposition 1.5.11 implies that $\lambda_n = \tau(n)$ for all n. Thus

- $\tau(mn)\Delta = \lambda_{mn}\Delta = \mathrm{T}_{mn}\Delta = \mathrm{T}_m\mathrm{T}_n\Delta = \lambda_m\lambda_n\Delta = \tau(m)\tau(n)\Delta$, and
- $\tau(p^{r+1})\Delta = \mathrm{T}_{p^{r+1}}\Delta = \mathrm{T}_p\mathrm{T}_{p^r}\Delta - p^{11}\mathrm{T}_{p^{r-1}}\Delta = \left(\tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})\right)\Delta$.

☐

In fact, the same argument shows if $f \in \mathrm{M}_k$ for $k > 0$ is an eigenform, with q-coefficient one, a **normalised eigenform**, and $f = \sum_{n=0}^{\infty} c_n q^n$, then

- $c_{nm} = c_n c_m$ if $(n, m) = 1$, and
- $c_{p^{r+1}} = c_p c_{p^r} - p^{k-1} c_{p^{r-1}}$.

Proposition 1.5.13. E_k *is an eigenform for all k.*

Proof. It suffices to show $\mathrm{T}_p\mathrm{E}_k = \lambda_p\mathrm{E}_k$ for all primes p. Recall that E_k is a constant multiple of G_k. Now

$$(\mathrm{T}_p\mathrm{G}_k)(L) = \sum_{L' \subseteq_p L} \sum_{w \in L',\ w \neq 0} \frac{1}{w^k} = \sum_{w \in L,\ w \neq 0} c_w \frac{1}{w_k}, \qquad c_w = \#\{L' \subseteq_p L \mid w \in L'\}.$$

Note that $pL \subseteq L' \subseteq L$. If $w \in pL$, then $w \in L'$ for all $L' \subseteq_p L$, and there are $p + 1$ of these. If $w \notin pL$, then $pL \subsetneq pL + \mathbb{Z}w \subsetneq L$ and $pL \subseteq_{p^2} L$, so $pL \subsetneq_p pL + \mathbb{Z}w$ and $pL + \mathbb{Z}w \subsetneq_p L$. In this case there exists a unique lattice of index p containing w. Thus

$$\begin{aligned}(\mathrm{T}_p\mathrm{G}_k)(L) &= \sum_{w \in L \setminus pL} \frac{1}{w^k} + (p+1) \sum_{w \in pL,\ w \neq 0} \frac{1}{w^k} = \sum_{w \in L,\ w \neq 0} \frac{1}{w^k} + p \sum_{w \in pL,\ w \neq 0} \frac{1}{w^k} \\ &= \mathrm{G}_k(L) + p \sum_{w \in L,\ w \neq 0} \frac{1}{(pw)^k} = \mathrm{G}_k(L) + p^{1-k} \sum_{w \in L,\ w \neq 0} \frac{1}{w^k} = \left(1 + p^{1-k}\right)\mathrm{G}_k(L),\end{aligned}$$

so $\mathrm{T}_p\mathrm{E}_k = \left(1 + p^{k-1}\right)\mathrm{E}_k$. ☐

A question is does M_k have a basis of eigenforms for all k? By linear algebra, there exist nice classes of operators that are guaranteed to admit bases of eigenvectors, such as self-adjoint, or more generally, normal operators.

1.5.4 Hermitian pairings

Let V be a $\mathbb{C}$-vector space and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ a **Hermitian pairing**. That is,

- $\langle \lambda v + w, x \rangle = \lambda \langle v, x \rangle + \langle w, x \rangle$,
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$, and
- $\langle x, x \rangle > 0$ for all $x \neq 0$.

Example. The standard pairing

$$\begin{array}{rcl} \mathbb{C}^n \times \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ \langle z, w \rangle & \longmapsto & \sum_{i=1}^n z_i \overline{w_i} \end{array}.$$

Definition 1.5.14. Let $A : V \to V$ be $\mathbb{C}$-linear, and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ Hermitian. Then the **adjoint** $A^* : V \to V$ is the unique linear map $V \to V$ such that

$$\langle Av, w \rangle = \langle v, A^* w \rangle .$$

We say A is **self-adjoint** if $A^* = A$, and **normal** if A^* commutes with A.

Theorem 1.5.15. *If A is normal, then A has a basis of eigenvectors.*

Lecture 18
Monday
11/11/19

Lemma 1.5.16. $A^{**} = A$.

Proof. For all $v, w \in V$,

$$\langle v, A^{**} w \rangle = \langle A^* v, w \rangle = \overline{\langle w, A^* v \rangle} = \overline{\langle Aw, v \rangle} = \langle v, Aw \rangle ,$$

so $A^{**} w = Aw$ for all $w \in V$. □

Definition 1.5.17. If $W \subseteq V$, let

$$W^\perp = \{ v \in V \mid \forall w \in W, \ \langle v, w \rangle = 0 \} .$$

Proposition 1.5.18. $\operatorname{im} A^* = (\ker A)^\perp$.

Proof. $\langle v, A^* w \rangle = \langle Av, w \rangle = 0$ if $v \in \ker A$. So $\operatorname{im} A^* \subseteq (\ker A)^\perp$, so $\operatorname{rk} A^* \le \operatorname{rk} A$. The same argument with A^* in place of A implies that $\operatorname{rk} A = \operatorname{rk} A^{**} \le \operatorname{rk} A^*$. So $\operatorname{rk} A^* = \operatorname{rk} A$, so $\operatorname{im} A^* = (\ker A)^\perp$. □

In particular, $\operatorname{im} A^* \cap \ker A = \{0\}$ and $\dim \operatorname{im} A^* + \dim \ker A = \operatorname{rk} A^* + n - \operatorname{rk} A = n$. So

$$V = \operatorname{im} A^* \oplus \ker A.$$

Theorem 1.5.19 (Spectral theorem for normal operators)**.** *If A and A^* commute, then A^* is diagonalisable.*

Proof. Induction on $\dim V$. Then $\dim V = 1$ is clear. Let λ be an eigenvalue of A, and let $A' = A - \lambda \operatorname{id}_V$, so $V = \ker A' \oplus \operatorname{im} A'^*$, where $\dim \ker A' > 0$. Then A commutes with A', and $A'^* = A^* - \overline{\lambda} \operatorname{id}_V$, so A commutes with A'^*. So $AA'^* v = A'^* A v$, so A preserves the image of A'^*. The restriction of $\langle \cdot, \cdot \rangle$ to $\operatorname{im} A'^*$ is still Hermitian on $\operatorname{im} A'^*$ and the restriction of A to $\operatorname{im} A'^*$ is still normal, since its adjoint is the restriction of A^* to $\operatorname{im} A'^*$. By induction A is diagonalisable on $\operatorname{im} A'^*$ and scalar on $\ker A'$, so diagonalisable. □

Also the need the following observation.

Proposition 1.5.20. *If $A : V \to V$ and $B : V \to V$ commute, and $V_\lambda = \ker (A - \lambda \operatorname{id}_V)$, then $BV_\lambda = V_\lambda$.*

Proof. If $v \in V_\lambda$, then $ABv = BAv = B\lambda v = \lambda Bv$, so $Bv \in V_\lambda$. □

1.5.5 The Petersson inner product

To apply this to modular forms, we need a bilinear pairing on M_k or S_k. The idea is to show that there exists a pairing $\langle \cdot, \cdot \rangle_k : \mathrm{S}_k \times \mathrm{S}_k \to \mathbb{C}$ such that $\langle \mathrm{T}_n f, g \rangle = \langle f, \mathrm{T}_n g \rangle$ for all n, so T_n are self-adjoint, hence diagonalisable.

Definition 1.5.21. Let $f, g \in \mathrm{S}_k$. The **Petersson inner product of weight** k is

$$\langle f, g \rangle_k = \iint_{\mathcal{D}} f(z) \overline{g(z)} \frac{y^k}{y^2} \, \mathrm{d}x \, \mathrm{d}y = \frac{i}{2} \iint_{\mathcal{D}} f(z) \overline{g(z)} \frac{\operatorname{Im} z^k}{\operatorname{Im} z^2} \, \mathrm{d}z \, \mathrm{d}\overline{z}.$$

Here $z = x + iy$ and $\overline{z} = x - iy$, so

$$\mathrm{d}z \mathrm{d}\overline{z} = (\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) = -2i (\mathrm{d}x \wedge \mathrm{d}y).$$

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$f(\gamma z) \overline{g(\gamma z)} \operatorname{Im} \gamma z^k = f(z) (cz + d)^k \overline{g(z) (cz + d)^k} \frac{\operatorname{Im} z^k}{|cz + d|^{2k}} = f(z) \overline{g(z)} \operatorname{Im} z^k,$$

and

$$\frac{1}{\operatorname{Im} \gamma z^2} \mathrm{d}(\gamma z) \mathrm{d}(\gamma \overline{z}) = \frac{1}{\operatorname{Im} \gamma z^2 |cz + d|^4} \mathrm{d}z \mathrm{d}\overline{z} = \frac{1}{\operatorname{Im} z^2} \mathrm{d}z \mathrm{d}\overline{z},$$

so for all $U \subseteq \mathbb{H}$,

$$\iint_{\gamma(U)} f(z) \overline{g(z)} \frac{\operatorname{Im} z^k}{\operatorname{Im} z^2} \, \mathrm{d}z \, \mathrm{d}\overline{z} = \iint_U f(z) \overline{g(z)} \frac{\operatorname{Im} z^k}{\operatorname{Im} z^2} \, \mathrm{d}z \, \mathrm{d}\overline{z}.$$

Note. This converges for $f, g \in \mathrm{S}_k$, since $f(a + it)$ goes like e^{-t} as $t \to \pm\infty$, and the same for g. If $\langle f, f \rangle = 0$, the integrand vanishes identically, since it lives in $\mathbb{R}_{\geq 0}$. So $f = 0$ on $\mathcal{D}$, hence everywhere. Then

$$\langle \lambda f, g \rangle_k = \lambda \langle f, g \rangle_k, \qquad \langle f, \lambda g \rangle_k = \overline{\lambda} \langle f, g \rangle_k, \qquad \langle f, g \rangle_k = \overline{\langle g, f \rangle}_k.$$

So $\langle \cdot, \cdot \rangle_k$ is Hermitian.

Theorem 1.5.22.

$$\langle \mathrm{T}_n f, g \rangle_k = \langle f, \mathrm{T}_n g \rangle_k, \qquad f, g \in \mathrm{S}_k, \qquad n \in \mathbb{Z}_{\geq 1}.$$

Corollary 1.5.23. *Each T_n is diagonalisable on S_k. Since T_n and T_m commute for all n and m, T_m preserves eigenspaces of T_n for all m. By induction, T_m preserves the simultaneous eigenspaces of T_n for all $n < m$.*

Proposition 1.5.24. *Let $n > \lfloor k/12 \rfloor + 1$. Fix $\lambda_2, \dots, \lambda_n \in \mathbb{C}$. The subspace V of S_k on which*

$$\mathrm{T}_i = \lambda_i, \qquad i = 2, \dots, n$$

is zero or one-dimensional.

Proof. Let $f \in V$, so $f = c_1 q + c_2 q^2 + \dots$. Seen if $\mathrm{T}_i f = \lambda_i f$, then $c_i = \lambda_i c_1$. Also seen that if the first n Fourier coefficients of f vanishes, then $f = 0$, by the $k/12$-formula. So $c_1 \neq 0$ unless $f = 0$. Now if $f, g \in V \setminus \{0\}$, there exists $\lambda \in \mathbb{C}$ such that f and λg have the same q-coefficient, and thus the same first n Fourier coefficients. But then $f - \lambda g = 0$. ☐

Corollary 1.5.25. *S_k admits a basis of eigenforms for all k.*

Proof. Let $n \geq \lfloor k/12 \rfloor + 1$. Can diagonalise S_k with respect to the first n Hecke operators. Any simultaneous eigenspace for these is at most one-dimensional, and preserved by all T_n. So each of these is actually an eigenspace for all T_n. ☐

Lecture 19 Friday 15/11/19

Note. If f and g are eigenforms, and f is not a scalar multiple of g, there exists T_n such that $\mathrm{T}_n f = \lambda_n f$ and $\mathrm{T}_n g = \mu_n g$ with $\lambda_n \neq \mu_n$. Then

$$\langle \mathrm{T}_n f, g\rangle_k = \langle \lambda_n f, g\rangle_k = \lambda_n \langle f, g\rangle_k, \qquad \langle f, \mathrm{T}_n g\rangle_k = \langle f, \mu_n g\rangle_k = \overline{\mu_n} \langle f, g\rangle_k,$$

$$\lambda_n \langle f, f\rangle_k = \langle \mathrm{T}_n f, f\rangle_k = \langle f, \mathrm{T}_n f\rangle_k = \overline{\langle \mathrm{T}_n f, f\rangle_k} = \overline{\lambda_n} \langle f, f\rangle_k.$$

So $\lambda_n = \overline{\lambda_n}$ and $\mu_n = \overline{\mu_n}$. Then $(\lambda_n - \mu_n) \langle f, g\rangle_k = 0$, so $\langle f, g\rangle_k = 0$.

By the formula for T_n on q-expansions, T_n takes a q-expansion with integer coefficients to another such. Saw that the space of modular forms with integral q-expansions is spanned by

$$\mathrm{E}_4^n \mathrm{E}_6^m, \dots, \mathrm{E}_4^{n-3\lfloor n/3\rfloor} \mathrm{E}_6^m \Delta^{\lfloor n/3\rfloor}, \qquad k = 4n + 6m, \qquad n, m > 0,$$

where $m = 0, 1$ is minimal, so the matrix of T_n with respect to this basis has integer entries. Thus the characteristic polynomial of T_n on S_k has integer coefficients, so the eigenvalues of T_n are algebraic integers.

Example. Can ask when modular forms are congruent modulo p. In fact $\mathrm{E}_{12} \equiv \Delta \mod 691$.

Ribet 1970s proved that when an Eisenstein series of suitable weight is congruent modulo p to a cusp form, can use the Galois representation attached to that cusp form to construct elements of ideal class groups of cyclotomic fields.

1.6 L-functions

1.6.1 Dirichlet L-functions

Definition 1.6.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers, usually algebraic integers. The **Dirichlet series** attached to a_n is the formal series

$$\sum_{n=1}^{\infty} a_n n^{-s},$$

thought of as a function of $s \in \mathbb{C}$.

In general, if $|a_n| \leq C n^k$, then the corresponding series converges absolutely for $\operatorname{Re} s > k + 1$.

Example.

- The **Riemann ζ-function** is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

- Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a **primitive character**, that is does not factor through $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times$ for $m \mid N$ such that $m \neq N$. Set

$$a_n = \begin{cases} \chi(n) & (n, N) = 1 \\ 0 & (n, N) \neq 1 \end{cases}.$$

  Then

$$\mathrm{L}(s, \chi) = \sum_{n=1}^{\infty} a_n n^{-s}$$

  is the **Dirichlet L-function** attached to χ.

In both these examples, and many others,

- these series have meromorphic, and often analytic, continuations to all of $\mathbb{C}$,
- there is a **functional equation** relating values at s and $k - s$ for some k, and
- there is an **Euler product**.

Example.

$$\zeta(s) = 2^s \pi^{s-1} \sin \tfrac{\pi s}{2} \Gamma(1-s) \zeta(1-s), \qquad \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \qquad \mathrm{L}(s, \chi) = \prod_{p \nmid N} \frac{1}{1-\chi(p) p^{-s}}.$$

1.6.2 Hecke L-functions

Definition 1.6.2. Let $f = \sum_{n=0}^{\infty} a_n q^n \in \mathrm{M}_k$. Define the **Hecke L-function of weight** k

$$\mathrm{L}(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Example. Let

$$f = \mathrm{E}_k' = \mathrm{b}_k \frac{(-1)^{\frac{k}{2}}}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Then

$$\mathrm{L}(s, f) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-\sigma_{k-1}(p) p^{-s}} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \cdot \frac{1}{1-p^{k-1} p^{-s}} = \zeta(s) \zeta(s-k+1),$$

since $\sigma_{k-1}(mn) = \sigma_{k-1}(m) \sigma_{k-1}(n)$ for $(m, n) = 1$ and $\sigma_{k-1}(p^r) = 1 + \dots + p^{r(k-1)}$.

Lecture 20 Friday 15/11/19

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cusp form. Recall that Hecke implies that $|a_n| \le C n^{k/2}$, so gives absolute convergence of $\mathrm{L}(s, f)$ for $\operatorname{Re} s > k/2 + 1$.

Theorem 1.6.3.

1. *$\mathrm{L}(s, f)$ extends to a holomorphic function on all of $\mathbb{C}$.*
2. *Set*
$$\mathrm{R}(s, f) = \frac{\Gamma(s)}{(2\pi)^s} \mathrm{L}(s, f).$$
Then
$$\mathrm{R}(s, f) = (-1)^{\frac{k}{2}} \mathrm{R}(k-s, f).$$
3. *If f is a normalised eigenform, then*
$$\mathrm{L}(s, f) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1} p^{-2s}}.$$

Definition 1.6.4. The infinite product $\prod_{n=1}^{\infty} (1 + c_n)$ **converges** if

$$\lim_{N \to \infty} \prod_{n=1}^{N} (1 + c_n)$$

converges to a non-zero number, if and only if $\sum_{n=1}^{\infty} \log(1 + c_n)$ converges. Then $\prod_{n=1}^{\infty} (1 + c_n)$ **converges absolutely** if

$$\prod_{n=1}^{\infty} (1 + |c_n|)$$

converges.

Lemma 1.6.5. *$\prod_{n=1}^{\infty} (1 + c_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty} |c_n|$ converges.*

Proof.

$$\sum_{n=1}^{N} |c_n| \le \prod_{n=1}^{N} (1 + |c_n|) \le \prod_{n=1}^{N} e^{|c_n|} \le e^{\sum_{n=1}^{\infty} |c_n|}.$$

□

Proof of Theorem 1.6.3. Recall that
$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, \mathrm{d}t$$
is meromorphic on $\mathbb{H}$, with poles at $\mathbb{Z}_{\le 0}$ and never zero, and satisfies $\Gamma(s+1) = s\Gamma(s)$ so $\Gamma(n) = (n-1)!$. Substituting $t \mapsto 2\pi n t$ in $\Gamma(s)$,
$$\Gamma(s) = \int_0^\infty (2\pi n t)^{s-1} e^{-2\pi n t} (2\pi n) \, \mathrm{d}t = (2\pi n)^s \int_0^\infty t^{s-1} e^{-2\pi n t} \, \mathrm{d}t,$$
so
$$\mathrm{L}(s,f) = \sum_{n=1}^\infty a_n n^{-s} = \sum_{n=1}^\infty a_n \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-2\pi n t} \, \mathrm{d}t.$$
Then
$$\begin{aligned} \mathrm{R}(s,f) = \frac{\Gamma(s)}{(2\pi)^s} \mathrm{L}(s,f) &= \sum_{n=1}^\infty a_n \int_0^\infty t^{s-1} e^{-2\pi n t} \, \mathrm{d}t = \int_0^\infty t^{s-1} \sum_{n=1}^\infty a_n e^{-2\pi n t} \, \mathrm{d}t = \int_0^\infty t^{s-1} f(it) \, \mathrm{d}t \\ &= \int_0^1 t^{s-1} f(it) \, \mathrm{d}t + \int_1^\infty t^{s-1} f(it) \, \mathrm{d}t = \int_1^\infty \left(\frac{1}{t}\right)^{s-1} f\left(\frac{i}{t}\right) \mathrm{d}\left(\frac{1}{t}\right) + \int_1^\infty t^{s-1} f(it) \, \mathrm{d}t \\ &= \int_1^\infty \left(t^{-s-1} (it)^k f(it) + t^{s-1} f(it)\right) \mathrm{d}t = \int_1^\infty f(it) \left((-1)^{\frac{k}{2}} t^{k-s-1} + t^{s-1}\right) \mathrm{d}t. \end{aligned}$$

1. $\mathrm{R}(s,f)$ converges independently of s uniformly for s in a compact subset of $\mathbb{C}$, so it is holomorphic in s, and extends to a holomorphic function on $\mathbb{C}$. Then
$$\mathrm{L}(s,f) = \frac{(2\pi)^s}{\Gamma(s)} \mathrm{R}(s,f),$$
so $\mathrm{L}(s,f)$ is holomorphic since $\Gamma(s)$ is non-vanishing.
2. $\mathrm{R}(s,f)$ is symmetric up to a sign under $s \mapsto k - s$, so
$$\mathrm{R}(s,f) = (-1)^{\frac{k}{2}} \mathrm{R}(k-s,f).$$
3. Now assume f is a normalised eigenform, so $f = \sum_{n=1}^\infty a_n q^n$ with $a_1 = 1$ and $\mathrm{T}_n f = a_n f$. Then $a_{nm} = a_n a_m$ if $(n,m) = 1$, so
$$\mathrm{L}(s,f) = \sum_n a_n n^{-s} = \prod_{p \text{ prime}} \sum_{k=0}^\infty a_{p^k} p^{-ks},$$
a power series in p^{-s}. Fix p, and consider
$$\left(1 - a_p p^{-s} + p^{k-1} p^{-2s}\right) \sum_{k=0}^\infty a_{p^k} p^{-ks}.$$
The p^0 coefficient is $a_1 = 1$, the p^{-s} coefficient is $a_p p^{-s} - a_p p^{-s} = 0$, and the $p^{-(r+1)s}$ coefficient is
$$a_{p^{r+1}} p^{-(r+1)s} - a_p a_{p^r} p^{-(r+1)s} + p^{k-1} a_{p^{r-1}} p^{-(r+1)s} = \left(a_{p^{r+1}} - a_p a_{p^r} + p^{k-1} a_{p^{r-1}}\right) p^{-(r+1)s} = 0,$$
since $a_{p^{r+1}} = a_p a_{p^r} - p^{k-1} a_{p^{r-1}}$. So
$$\mathrm{L}(s,f) = \prod_{p \text{ prime}} \sum_{k=0}^\infty a_{p^k} p^{-ks} = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1} p^{-2s}}.$$

□

Lecture 21 is a problems class. Lecture 21 Monday 18/11/19

2 Modular forms of higher level

2.1 Modular forms

2.1.1 Congruence subgroups

Lecture 22
Friday
22/11/19

$\mathrm{GL}_2(\mathbb{Q})_+$ acts on $\mathbb{H}$ by fractional linear transformations.

Definition 2.1.1. $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is the kernel of $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ for $N \in \mathbb{Z}_{>0}$. Alternatively,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1 \mod N,\ b \equiv c \equiv 0 \mod N \right\}.$$

Note. $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ has finite index.

Definition 2.1.2. $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Q})_+$ is a **congruence subgroup** if Γ contains $\Gamma(N)$ with finite index for some $N \in \mathbb{Z}_{>0}$.

Example. $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(N)$ are congruence subgroups. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\},$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a \equiv d \equiv 1 \mod N,\ c \equiv 0 \mod N \right\},$$

so $\Gamma_1(N)$ is the preimage of $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \subseteq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{Z})$. Then $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups such that

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

Proposition 2.1.3. *Let $\alpha \in \mathrm{GL}_2(\mathbb{Q})_+$, and let Γ be a congruence subgroup. Then $\alpha\Gamma\alpha^{-1}$ is also a congruence subgroup.*

Proof. Need that there exists M with $\Gamma(M) \subseteq \alpha\Gamma\alpha^{-1}$ with finite index. There exists N such that $\Gamma(N) \subseteq \Gamma$. Note that $\Gamma(N) = \mathrm{SL}_2(\mathbb{Q}) \cap (\mathrm{id}_2 + N \operatorname{Mat}_2 \mathbb{Z})$. Consider

$$\alpha\Gamma(N)\alpha^{-1} = \mathrm{SL}_2(\mathbb{Q}) \cap \left(\mathrm{id}_2 + N\alpha \operatorname{Mat}_2 \mathbb{Z}\alpha^{-1}\right).$$

Choose $n \in \mathbb{Z}$ such that $n\alpha$ and $n\alpha^{-1}$ have entries in $\mathbb{Z}$. Then $n^2\alpha^{-1} \operatorname{Mat}_2 \mathbb{Z}\alpha \subseteq \operatorname{Mat}_2 \mathbb{Z}$, so $n^2 \operatorname{Mat}_2 \mathbb{Z} \subseteq \alpha \operatorname{Mat}_2 \mathbb{Z}\alpha^{-1}$, so $Nn^2 \operatorname{Mat}_2 \mathbb{Z} \subseteq N\alpha \operatorname{Mat}_2 \mathbb{Z}\alpha^{-1}$, so

$$\Gamma\left(n^2N\right) = \mathrm{SL}_2(\mathbb{Q}) \cap \left(\mathrm{id}_2 + Nn^2 \operatorname{Mat}_2 \mathbb{Z}\right) \subseteq \mathrm{SL}_2(\mathbb{Q}) \cap \left(\mathrm{id}_2 + N\alpha \operatorname{Mat}_2 \mathbb{Z}\alpha^{-1}\right) = \alpha\Gamma(N)\alpha^{-1}.$$

Similarly, show

$$\alpha\Gamma\left(n^4N\right)\alpha^{-1} \subseteq \Gamma\left(n^2N\right) \subseteq \alpha\Gamma(N)\alpha^{-1}.$$

Since $\Gamma\left(n^4N\right)$ has finite index in $\Gamma(N)$, $\Gamma\left(n^2N\right)$ has finite index in $\alpha\Gamma(N)\alpha^{-1}$. $\square$

Note. Also, if $T = \operatorname{lcm}(M, N)$ then $\Gamma(T) \subseteq \Gamma(M) \cap \Gamma(N)$, so the intersection of two congruence subgroups is a congruence subgroup.

Example. Let $\alpha = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\alpha^{-1} \mathrm{SL}_2(\mathbb{Z})\alpha = \left\{ \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\},$$

and

$$\alpha^{-1} \mathrm{SL}_2(\mathbb{Z})\alpha \cap \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \;\middle|\; ad - bpc = 1 \right\} = \Gamma_0(p).$$

2.1.2 Modular forms

Recall that for $f : \mathbb{H} \to \mathbb{C}$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})_+$, we defined $f|_{k,\alpha}$ by

$$f|_{k,\alpha}(z) = \det \alpha^{k-1} f(\alpha z)(cz+d)^{-k}.$$

Suppose we have a $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$ and $f : \mathbb{H} \to \mathbb{C}$ such that $f|_{k,\gamma} = f$ for all $\gamma \in \Gamma$. Then if $g = f|_{k,\alpha}$, then $g|_{k,\alpha^{-1}\gamma\alpha} = g$, since

$$\left(f|_{k,\alpha}\right)\Big|_{k,\alpha^{-1}\gamma\alpha} = f|_{k,\gamma\alpha} = \left(f|_{k,\gamma}\right)\Big|_{k,\alpha} = f|_{k,\alpha}.$$

Fix $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$ a congruence subgroup.

Definition 2.1.4. A function $f : \mathbb{H} \to \mathbb{C}$ is a **weakly holomorphic or meromorphic modular form of weight k and level Γ** if

- $f|_{k,\gamma} = f$ for all $\gamma \in \Gamma$, and
- f is holomorphic or meromorphic on $\mathbb{H}$.

A question is what condition should we impose at ∞ to get a good theory?

Example. Let $k \geq 4$ and $N \in \mathbb{Z}$, and let

$$\mathrm{E}_k^{0,1}(z) = \sum_{(m,n)\in S^{0,1}} \frac{1}{(mz+n)^k}, \qquad S^{0,1} = \left\{(m,n) \in \mathbb{Z}^2 \setminus \{0\} \;\middle|\; m \equiv 1 \mod N,\ n \equiv 0 \mod N\right\}.$$

Claim that $\mathrm{E}_k^{0,1}(\gamma z) = \mathrm{E}_k^{0,1}(z)$ for $\gamma \in \Gamma(N)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$. Then

$$\begin{aligned}\mathrm{E}_k^{0,1}(\gamma z) &= \sum_{(m,n)\in S^{0,1}} \frac{1}{\left(m\left(\frac{az+b}{cz+d}\right)+n\right)^k} = (cz+d)^k \sum_{(m,n)\in S^{0,1}} \frac{1}{(m(az+b)+n(cz+d))^k} \\ &= (cz+d)^k \sum_{(m,n)\in S^{0,1}} \frac{1}{((ma+nc)z+(mb+nd))^k}.\end{aligned}$$

Since $m \equiv a \equiv d \equiv 1 \mod N$ and $n \equiv b \equiv c \equiv 0 \mod N$, $ma+nc \equiv 1 \mod N$ and $mb+nd \equiv 0 \mod N$, so $(ma+nc, mb+nd) \in S^{0,1}$. Moreover, the map

$$\begin{array}{rcl} S^{0,1} & \longleftrightarrow & S^{0,1} \\ (m,n) & \longmapsto & (ma+nc, mb+nd) \\ (m'a'+n'c', m'b'+n'd') & \longleftarrow & (m',n') \end{array}$$

is a bijection, where $\gamma^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. So $\mathrm{E}_k^{0,1}(\gamma z) = \mathrm{E}_k^{0,1}(z)(cz+d)^k$.

Every congruence subgroup is conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ need not be in Γ. On the other hand, if $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, then Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, so there exists a minimal $\mathrm{n}_\Gamma > 0$ such that $\left(\begin{smallmatrix} 1 & \mathrm{n}_\Gamma \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Then if f is weakly modular of weight k and level Γ, know $f(z+\mathrm{n}_\Gamma) = f(z)$ for all z, so f is a function of q^{1/n_Γ}. Let $g\left(q^{1/\mathrm{n}_\Gamma}\right)$ be a function on $\mathbb{D} \setminus \{0\}$ such that $f(z) = g\left(e^{2\pi i z/\mathrm{n}_\Gamma}\right)$. Then if g is meromorphic on $\mathbb{D}$, can express g as a Laurent series in q^{1/n_Γ}. We say f is **meromorphic at** ∞, and the series for q is its **q-expansion**. Lecture 23 Friday 22/11/19

Example.

- For $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$, $\mathrm{n}_\Gamma = 1$.
- For $\Gamma = \Gamma(N)$, $\mathrm{n}_\Gamma = N$.

2.1.3 A fundamental domain

A question is for $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, can we write down a fundamental domain for Γ? For all $z \in \mathbb{H}$, there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma z \in \mathcal{D}$. For $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, write

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{\gamma_i \in \mathrm{SL}_2(\mathbb{Z})} \pm\gamma_i \cdot \Gamma.$$

Set

$$\mathcal{D}_\Gamma = \bigcup_i \gamma_i^{-1} \cdot \mathcal{D}.$$

Theorem 2.1.5.

1. *For all $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathcal{D}_\Gamma$.*
2. *The subset $\{z \in \mathcal{D}_\Gamma \mid \Gamma \cdot z \cap \mathcal{D}_\Gamma \neq \{z\}\}$ is contained in $\bigcup_{\gamma_i \in \mathrm{SL}_2(\mathbb{Z})} \gamma_i \cdot \partial\mathcal{D}$, the boundary of $\mathcal{D}$, so has measure zero.*

That is, $\mathcal{D}_\Gamma$ is a fundamental domain for Γ.

Proof.

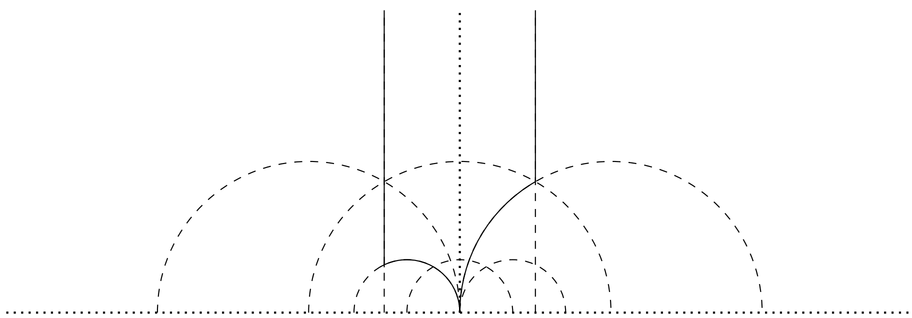
1. Fix $z \in \mathbb{H}$. There exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma z \in \mathcal{D}$. Can write γ as $\pm\gamma_i\gamma'$ for some i and $\gamma' \in \Gamma$. Then $\pm\gamma_i\gamma' z \in \mathcal{D}$, so $\gamma_i\gamma' z \in \mathcal{D}$, so $\gamma' z \in \gamma_i^{-1}\mathcal{D} \subseteq \mathcal{D}_\Gamma$.
2. Let $z \in \bigcup_i \gamma_i^{-1} \cdot \mathring{\mathcal{D}}$. Want $\Gamma \cdot z \cap \mathcal{D}_\Gamma = \{z\}$. Suppose $\gamma z \in \mathcal{D}_\Gamma$ for $\gamma \in \Gamma$. There exist i and j such that $z \in \gamma_i^{-1} \cdot \mathring{\mathcal{D}}$ and $\gamma z \in \gamma_j^{-1} \cdot \mathring{\mathcal{D}}$, so $\gamma_i z, \gamma_j \gamma z \in \mathring{\mathcal{D}}$. So $\gamma_i z = \gamma_j \gamma z$ so $\gamma^{-1}\gamma_j^{-1}\gamma_i z = z$. Then $\mathrm{Stab}_z = \pm \mathrm{id}_2$, so $\gamma_i = \pm\gamma_j\gamma$. Since $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \pm\gamma_i \cdot \Gamma$, this is only possible if $i = j$. Then $\gamma_i = \pm\gamma_i\gamma$, so $\gamma = \pm \mathrm{id}_2$. So $z = \gamma z$.

□

Example. $\Gamma = \Gamma_0(2)$ has index three in $\mathrm{SL}_2(\mathbb{Z})$. The coset representatives are

$$\mathrm{id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : z \mapsto z, \qquad \mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}, \qquad \mathrm{ST} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} : z \mapsto -\frac{1}{z+1},$$

so



A question is for a given Γ and $\mathcal{D}_\Gamma$, what are the ways to escape to ∞ in $\mathcal{D}_\Gamma$? Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Then

$$\mathrm{SL}_2(\mathbb{Z}) \cdot \infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty \right\} = \left\{ \frac{a}{c} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = \mathbb{Q} \cup \{\infty\}.$$

Definition 2.1.6. The set of **cusps** for Γ is the set of Γ-orbits on $\mathbb{Q} \cup \{\infty\}$.

Note. If $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \pm\gamma_i \cdot \Gamma$, then $\{\gamma_i^{-1} \cdot \infty\}$ is a set of representatives for the Γ-orbits on $\mathbb{Q} \cup \{\infty\}$.

Example. Let $\Gamma = \Gamma_0(p)$ for p prime. Then

$$\Gamma \cdot \infty = \left\{ \frac{a}{pc} \;\middle|\; (a, pc) = 1 \right\} \cup \{\infty\}, \qquad \Gamma \cdot 0 = \left\{ \frac{b}{d} \;\middle|\; d \nmid p \right\}.$$

Definition 2.1.7. A weakly modular form f of weight k and level Γ is **holomorphic or meromorphic at all cusps** if for all $\gamma \in \Gamma$, $f|_{k,\gamma}$ is holomorphic or meromorphic at ∞.

Note. Since $f|_{k,\gamma} = f$ for $\gamma \in \Gamma$, it suffices to check on a set of coset representatives for Γ in $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.8. A **modular form of weight k and level** Γ is a weakly modular form of weight k and level Γ that is holomorphic on $\mathbb{H}$ and at all cusps.

2.2 Spaces of modular forms

2.2.1 The space of holomorphic modular forms

Lecture 24 Monday 25/11/19

Let

$$\mathrm{M}_k(\Gamma) = \{\text{holomorphic modular forms of weight } k \text{ and level } \Gamma\},$$
$$\mathrm{S}_k(\Gamma) = \{f \in \mathrm{M}_k(\Gamma) \mid f \text{ vanishes at all cusps}\}.$$

Note. For any $\gamma \in \mathrm{GL}_2(\mathbb{Q})_+$, if $f \in \mathrm{M}_k(\Gamma)$, then $f|_{k,\gamma} \in \mathrm{M}_k\left(\gamma^{-1}\Gamma\gamma\right)$. If we consider the $\mathbb{C}$-vector space $\widetilde{\mathrm{M}_k} = \bigcup_\Gamma \mathrm{M}_k(\Gamma)$, then γ acts on $\widetilde{\mathrm{M}_k}$ by $\gamma \cdot f = f|_{k,\gamma}$. In fact, $\mathrm{GL}_2(\mathbb{Q})_+ \subseteq \mathrm{GL}_2\left(\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}}\right)$ and the action extends to this larger group. If we enlarge $\widetilde{\mathrm{M}_k}$ in a suitable way, the correct group that acts is $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

A question is what can we say about $\dim_{\mathbb{C}} \mathrm{M}_k(\Gamma)$? Assume $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, and fix $f \in \mathrm{M}_k(\Gamma)$. Write

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{j=1}^d \Gamma \cdot \alpha_j, \qquad g = \prod_{j=1}^d f|_{k,\alpha_j}, \qquad d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma].$$

Proposition 2.2.1.

1. *g is independent of the choice of α_i.*
2. *$g \in \mathrm{M}_{kd}$.*

Proof.

1. Suppose I replace α_j' such that $\Gamma \cdot \alpha_j = \Gamma \cdot \alpha_j'$. Then there exists $\gamma \in \Gamma$ such that $\gamma\alpha_j = \alpha_j'$, so $f|_{k,\alpha_j'} = \left(f|_{k,\gamma}\right)\Big|_{k,\alpha_j} = f|_{k,\alpha_j}$. So the product defining g does not change.
2. For $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, $g|_{kd,\alpha} = \prod_{j=1}^d \left(f|_{k,\alpha_j}\right)\Big|_{k,\alpha} = \prod_{j=1}^d f|_{k,\alpha_j\alpha}$. Since $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{j=1}^d \Gamma \cdot \alpha_j$, $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \cdot \alpha = \bigsqcup_{j=1}^d \Gamma \cdot \alpha_j\alpha$. So the elements $\alpha_i\alpha$ are another set of coset representatives for Γ in $\mathrm{SL}_2(\mathbb{Z})$. Since g was independent of the choice of representatives, $g|_{kd,\alpha} = g$.

$\square$

Have

$$\sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})} \frac{1}{e_p} \operatorname{ord}_p g = \frac{kd}{12}, \qquad e_p = \begin{cases} \frac{1}{2} \# \operatorname{Stab}_p \in \{1,2,3\} & p \in \mathbb{H} \\ 1 & p \in \mathbb{Q} \cup \{\infty\} \end{cases},$$

so

$$\frac{kd}{12} = \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})} \frac{1}{e_p} \sum_{j=1}^d \operatorname{ord}_p f|_{k,\alpha_j} = \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})} \frac{1}{e_p} \sum_{j=1}^d \operatorname{ord}_{\alpha_j p} f.$$

As p runs over a set of representatives for $\mathrm{SL}_2(\mathbb{Z})$-orbits, and α_j runs over the coset representatives for Γ in $\mathrm{SL}_2(\mathbb{Z})$, $q = \alpha_j p$ runs over the representatives for Γ-orbits, so

$$\sum_{q \in \Gamma \backslash (\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})} \frac{n_q}{e_q} \operatorname{ord}_q f = \frac{kd}{12}, \qquad n_q = \#\{j \mid \alpha_j q \in \Gamma \cdot q\} \ge 1.$$

Corollary 2.2.2. *If $\operatorname{ord}_\infty f \ge kd/12n_\infty + 1$ for $f \in \mathrm{M}_k(\Gamma)$, then $f = 0$.*

Then

$$\begin{aligned} n_\infty &= \#\{j \mid \alpha_j\infty \in \Gamma \cdot \infty\} = \#\{j \mid \exists \gamma \in \Gamma,\ \alpha_j\infty = \gamma\infty\} = \#\left\{j \mid \exists \gamma \in \Gamma,\ \alpha_j^{-1}\gamma \in \operatorname{Stab}_\infty\right\} \\ &= \#\left\{j \mid \alpha_j^{-1} \cdot \Gamma \subseteq \operatorname{Stab}_\infty \cdot \Gamma\right\} = \#\left(\operatorname{Stab}_\infty \cdot \Gamma / \Gamma\right) = \#\left(\operatorname{Stab}_\infty / \left(\operatorname{Stab}_\infty \cap \Gamma\right)\right) = \mathrm{n}_\Gamma, \end{aligned}$$

since $\operatorname{Stab}_\infty = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{Z}\}$. Since f is a power series in q^{1/n_Γ}, and f is determined by its terms of order at most $kd/12\mathrm{n}_\Gamma$, f is determined by the first $1 + kd/12$ terms of its q-expansion, so

$$\dim_{\mathbb{C}} \mathrm{M}_k(\Gamma) \le 1 + \frac{kd}{12}.$$

2.2.2 The space of meromorphic modular forms

Lecture 25 Friday 29/11/19

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Let $\mathrm{F}(\Gamma)$ be the field of meromorphic modular forms of weight zero and level Γ, and let $\mathrm{F}_N = \mathrm{F}(\Gamma(N))$, so $\mathrm{F}_1 = \mathrm{F}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}(\mathrm{j})$. If $M \mid N$, then $\Gamma(N) \subseteq \Gamma(M)$, so $\mathrm{F}_M \subseteq \mathrm{F}_N$. Then $\mathrm{SL}_2(\mathbb{Z})$ normalises $\Gamma(N)$ so if $f \in \mathrm{F}_N$, then $f|_{0,\alpha}$ is modular for $\alpha^{-1}\Gamma(N)\alpha = \Gamma(N)$ if $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Note. $(fg)|_{0,\alpha} = f|_{0,\alpha} \cdot g|_{0,\alpha}$ and $(f+g)|_{0,\alpha} = f|_{0,\alpha} + g|_{0,\alpha}$.

Then $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ gives an automorphism of F_N fixing F_1. Get an action of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)$ on F_N by field automorphisms and F_1 is the fixed field.

Theorem 2.2.3 (Galois theory)**.** *Let F be a field and G a finite group acting faithfully on F by automorphisms, that is no $g \in G$ acts on F as* id *except $g = \mathrm{id}_G$. Then F is a Galois extension of*

$$F^G = \{x \in F \mid \forall g \in G,\ gx = x\},$$

with Galois group G. In particular $\left[F : F^G\right] = \#G$.

Proposition 2.2.4. $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ *acts faithfully on* F_N*.*

Proof. Use dimension formulae for $\mathrm{M}_k(\Gamma)$ to show that for $k \gg 0$ even, $\dim \mathrm{M}_k(\Gamma(N)) > \dim \mathrm{M}_k(\Gamma)$ for $\Gamma \supsetneq \Gamma(N)$, so there exists $f \in \mathrm{M}_k(\Gamma(N))$ such that the only elements of $\mathrm{SL}_2(\mathbb{Z})$ fixing f lie in $\Gamma(N)$. Then f/E_k lies in F_N but not in $\mathrm{F}(\Gamma)$ for $\Gamma \supsetneq \Gamma(N)$. So f/E_k is not fixed by non-trivial elements of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)$. ☐

Corollary 2.2.5. $\mathrm{F}_N/\mathrm{F}_1$ *is Galois with Galois group* $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$*.*

Then F_N is a finite and algebraic extension of $\mathbb{C}(\mathrm{j})$, of transcendence degree one over $\mathbb{C}$. For Γ arbitrary in $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma \supseteq \Gamma(N)$ for some N, so $\mathrm{F}(\Gamma)$ is the fixed field of $\Gamma/\Gamma(N)$ in F_N. Then $\mathrm{F}(\Gamma)/\mathrm{F}_1$ is not Galois in general, but is algebraic of degree $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$.

Proposition 2.2.6. *There exists a unique smooth and projective algebraic curve $X(\Gamma)$ over $\mathbb{C}$, whose field of rational functions is* $\mathrm{F}(\Gamma)$*.*

Proof. Fix Γ, and let f be a primitive element of $\mathrm{F}(\Gamma)$, that is f generates $\mathrm{F}(\Gamma)$ over F_1. Consider the polynomial

$$P(X) = \prod_{\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_j \Gamma \cdot \alpha_j} \left(X - f|_{0,\alpha_j}\right) = X^d + \frac{G_1(\mathrm{j})}{H_1(\mathrm{j})} X^{d-1} + \dots + \frac{G_d(\mathrm{j})}{H_d(\mathrm{j})} \in \mathrm{F}_1[X], \qquad G_i, H_i \in \mathbb{C}[Y].$$

Let

$$Q(X,Y) = H_1(Y) \dots H_d(Y) \left(X^d + \frac{G_1(Y)}{H_1(Y)} X^{d-1} + \dots + \frac{G_d(Y)}{H_d(Y)}\right) \in \mathbb{C}[X,Y].$$

Then $Q(X,\mathrm{j}) = H_1(\mathrm{j}) \dots H_d(\mathrm{j}) \cdot P(X)$. Since $P(f) = 0$, $Q(f,\mathrm{j}) = 0$. Consider the map

$$\begin{array}{rccc} \phi & : & \mathbb{H} & \longrightarrow & \mathbb{C}^2 \\ & & z & \longmapsto & (f(z), \mathrm{j}(z)) \end{array}.$$

The image is contained in the zero locus of $Q(X,Y)$, and factors through $\Gamma\backslash\mathbb{H}$. The following are some issues.

- This map is not necessarily defined everywhere. To fix, replace $\mathbb{C}^2$ with $\mathbb{CP}^2$. Then ϕ extends to $\Gamma\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}) \to \mathbb{CP}^2$.
- This map is not necessarily injective on $\Gamma\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})$, but will be generically injective since f is primitive.
- This image might be singular. There are standard ways to fix, such as normalisation. When these are fixed, the map becomes injective.

The upshot is to get a complex algebraic curve $X(\Gamma)$ whose function field is $\mathrm{F}(\Gamma)$, whose complex points are in bijection with $\Gamma\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})$. ☐

$\mathrm{M}_k(\Gamma)$ is the space of sections of certain line bundles on $X(\Gamma)$.

2.3 Hecke operators

Lecture 26
Friday
29/11/19

Let $f \in \mathrm{M}_k(\Gamma)$.

1. If $\Gamma' \subseteq \Gamma$, then $f \in \mathrm{M}_k(\Gamma')$.
2. If $\alpha \in \mathrm{GL}_2(\mathbb{Q})_+$, then $f|_{k,\alpha} \in \mathrm{M}_k\left(\alpha^{-1}\Gamma\alpha\right)$.
3. If $\Gamma \subseteq \Gamma'$, can write $\Gamma' = \bigsqcup_{i=1}^d \Gamma \cdot \alpha_i$, then $\sum_{i=1}^d f|_{k,\alpha_i}$ is independent of choices and lives in $\mathrm{M}_k(\Gamma')$.

The rough idea is given $f \in \mathrm{M}_k(\Gamma)$ and $\Gamma' \supseteq \alpha^{-1}\Gamma\alpha$, act on it by α to get a modular form of level $\alpha^{-1}\Gamma\alpha$, using 2, and average to get a modular form of level Γ', using 3. Recall that if $H, K \leq G$ and $g \in G$, then the **double coset** is

$$HgK = \{hgk \mid h \in H, \ k \in K\}.$$

That is, the orbit of G under the action of HxK on G such that $(h,k) \cdot g = hgk^{-1}$.

Definition 2.3.1. Let $f \in \mathrm{M}_k(\Gamma)$, let $\alpha \in \mathrm{GL}_2(\mathbb{Q})_+$, and let Γ' be a congruence subgroup. Then

$$f|_{k,\Gamma\alpha\Gamma'} = \sum_{i=1}^d f|_{k,\alpha_i}, \qquad \Gamma\alpha\Gamma' = \bigsqcup_{i=1}^d \Gamma\alpha_i.$$

The idea is that the α_i are of the form $\alpha\beta_i$ where β_i are a set of coset representatives for $\alpha^{-1}\Gamma\alpha \cap \Gamma'$ in Γ', by the coursework, so

$$\sum_{i=1}^d f|_{k,\alpha_i} = \sum_{i=1}^d \left(f|_{k,\alpha}\right)\Big|_{k,\beta_i},$$

where

- acting by α gets $f|_{k,\alpha}$ modular of level $\alpha^{-1}\Gamma\alpha$,
- so also modular of level $\alpha^{-1}\Gamma\alpha \cap \Gamma$, and
- averaging gets $f|_{k,\Gamma\alpha\Gamma'}$ modular of level Γ'.

So the double coset $\Gamma\alpha\Gamma'$ gives a map between $\mathrm{M}_k(\Gamma)$ and $\mathrm{M}_k(\Gamma')$.

2.3.1 Hecke operators

Recall that

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ \middle| \ a \equiv d \equiv 1 \mod N, \ c \equiv 0 \mod N \right\}.$$

Definition 2.3.2. For a prime $p \nmid N$, define

$$\begin{array}{cccc} \mathrm{T}_p & : & \mathrm{M}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{M}_k(\Gamma_1(N)) \\ & & f & \longmapsto & f|_{k,\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma_1(N)} \end{array}.$$

Recall that for $\mathrm{SL}_2(\mathbb{Z})$ we set

$$\mathrm{T}_p f = p^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^p} d^{-k} f\left(\frac{az+b}{d}\right) = \sum_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^p} f|_{k,\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)}.$$

To show this agrees with our new definition, we need that

$$\mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^p} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

- For the reverse containment, it suffices to show $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{S}^p$ lies in $\mathrm{SL}_2(\mathbb{Z}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \mathrm{SL}_2(\mathbb{Z})$, and

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

- For disjointness, if $\mathrm{SL}_2(\mathbb{Z})\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \mathrm{SL}_2(\mathbb{Z})\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right) \in \mathrm{S}^p$, then $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)^{-1} \in \mathrm{SL}_2(\mathbb{Z})$, so $a = a'$ and $d = d'$. If $a = p$, then $d = 1$ and $b = 0$, and the same holds for b', so equal. If $a = 1$, have
$$\begin{pmatrix} 1 & \frac{b-b'}{p} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}\begin{pmatrix} 1 & b' \\ 0 & p \end{pmatrix}^{-1} \in \mathrm{SL}_2(\mathbb{Z}),$$
so $p \mid b - b'$. Since $0 \le b, b' < p$, $b = b'$.
- It remains to show that $\mathrm{SL}_2(\mathbb{Z})\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\mathrm{SL}_2(\mathbb{Z})$ is the union of $p+1$ left cosets. By the coursework,
$$\begin{aligned} \#\{\text{cosets}\} &= \#\,\mathrm{SL}_2(\mathbb{Z}) / \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}^{-1} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{Z})\right) = \#\,\mathrm{SL}_2(\mathbb{Z}) / \Gamma_0(p) \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)] = [\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) : \text{upper triangular matrices modulo } p]. \end{aligned}$$
For upper triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ of determinant one modulo p, there are $p(p-1)$ possibilities. For $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, there are $p^2 - 1$ possibilities for the first row, the second row cannot be a multiple of the first row, so there are $p^2 - p$ possibilities, and to get determinant one need to rescale the second row, so there are p possibilities left over, so $\#\,\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) = p\left(p^2 - 1\right)$. Thus the index is $p\left(p^2 - 1\right)/p(p-1) = p + 1$.

Extending from T_p to T_n for $(n, N) = 1$, we set
$$\begin{array}{rcccl} \mathrm{T}_n & : & \mathrm{M}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{M}_k(\Gamma_1(N)) \\ & & f & \longmapsto & \sum_{ad=n,\ a \mid d} f|_{k, \Gamma_1(N)\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\Gamma_1(N)} \end{array}.$$

2.3.2 Diamond operators

Lecture 27 Monday 02/12/19

Recall that
$$\Gamma_1(N) \subseteq \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}.$$
Have a surjection
$$\begin{array}{ccc} \Gamma_0(N) & \longrightarrow & (\mathbb{Z}/N\mathbb{Z})^\times \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & d \end{array},$$
where the kernel is $\Gamma_1(N)$. So $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$.

Note. If $f \in \mathrm{M}_k(\Gamma_1(N))$ and $\alpha \in \Gamma_0(N)$, then $f|_{k,\alpha}$ is modular of level $\alpha^{-1}\Gamma_1(N)\alpha = \Gamma_1(N)$. Moreover $f|_{k,\alpha}$ depends only on the class of $\alpha \in \Gamma_0(N)/\Gamma_1(N)$, that is only on the lower right entry of α.

Definition 2.3.3. For $d \in \mathbb{Z}$ such that $(d, N) = 1$, we define the **diamond operator**
$$\begin{array}{rcccl} \langle d \rangle & : & \mathrm{M}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{M}_k(\Gamma_1(N)) \\ & & f & \longmapsto & f|_{k,\alpha} \end{array},$$
where $\alpha \in \Gamma_0(N)$ with lower right entry congruent to d modulo N.

This defines an action of $(\mathbb{Z}/N\mathbb{Z})^\times \cong \Gamma_0(N)/\Gamma_1(N)$ on $\mathrm{M}_k(\Gamma_1(N))$. Since $\langle d \rangle \langle d' \rangle = \langle dd' \rangle = \langle d' \rangle \langle d \rangle$, and operators of finite order on a $\mathbb{C}$-vector space are diagonalisable, $\mathrm{M}_k(\Gamma_1(N))$ splits as a direct sum of simultaneous eigenspaces for the $\langle d \rangle$. Let V be one such eigenspace. Then for each $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, there exists $\chi(d) \in \mathbb{C}^\times$ such that $\langle d \rangle f = \chi(d) f$ for all $f \in V$. Since $\langle d \rangle \langle d' \rangle = \langle dd' \rangle$, $\chi(d)\chi(d') = \chi(dd')$, so χ is a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, that is a **character**.

Definition 2.3.4. For any character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, let $\mathrm{M}_k(\Gamma_1(N), \chi)$ be the subspace of $\mathrm{M}_k(\Gamma_1(N))$ consisting of the forms f such that
$$\langle d \rangle f = \chi(d) f, \qquad d \in (\mathbb{Z}/N\mathbb{Z})^\times.$$
A warning is that this might be zero.

Example. If k is odd and $\chi(-1) = 1$, this space is zero.

We have a direct sum decomposition

$$\mathrm{M}_k(\Gamma_1(N)) \cong \bigoplus_{\chi:(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}} \mathrm{M}_k(\Gamma_1(N), \chi).$$

Proposition 2.3.5. *Let $(n, N) = 1$ and $f \in \mathrm{M}_k(\Gamma_1(N), \chi)$ such that $f = \sum_{m=1}^\infty c_m q^m$. Then*

$$\mathrm{T}_n f = \sum_{m=1}^\infty \gamma_m f, \qquad \gamma_m = \sum_{d \mid (n,m)} \chi(d) d^{k-1} c_{\frac{nm}{d^2}}.$$

In particular, if $\mathrm{T}_n f = \lambda_n f$ for some n with $(n, N) = 1$, then $c_n = \lambda_n c_1$.

2.3.3 The Petersson inner product

Fix $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup.

Definition 2.3.6. For $f, g \in \mathrm{S}_k(\Gamma)$ define the **Petersson inner product of weight k and level Γ**

$$\langle f, g\rangle_{k,\Gamma} = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \iint_{\mathcal{D}_\Gamma} f(z) \overline{g(z)} \frac{y^k}{y^2} \, \mathrm{d}x \, \mathrm{d}y,$$

where $\mathcal{D}_\Gamma$ is a fundamental domain for Γ.

Note. The scaling factor ensures if $\Gamma' \subseteq \Gamma$ and $f, g \in \mathrm{S}_k(\Gamma)$, then $\langle f, g\rangle_{k,\Gamma'} = \langle f, g\rangle_{k,\Gamma}$.

Proposition 2.3.7. *Let $f \in \mathrm{S}_k(\Gamma)$ and $g \in \mathrm{S}_k(\alpha^{-1}\Gamma\alpha)$ for $\alpha \in \mathrm{GL}_2(\mathbb{Q})_+$. Then*

$$\left\langle f|_{k,\alpha}, g\right\rangle_{k,\alpha^{-1}\Gamma\alpha} = \left\langle f, g|_{k,\alpha'}\right\rangle_{k,\Gamma}, \qquad \alpha' = \alpha^{-1}\det\alpha.$$

Proof. Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\alpha^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Set $z' = \alpha z$ and $C = [\mathrm{SL}_2(\mathbb{Z}) : \alpha^{-1}\Gamma\alpha]$. Have $(cz + d)(c'z' + d') = 1$. Then

$$\begin{aligned}
\left\langle f|_{k,\alpha}, g\right\rangle_{k,\alpha^{-1}\Gamma\alpha} &= \frac{1}{C} \iint_{\alpha^{-1}\mathcal{D}_\Gamma} f|_{k,\alpha}(z) \overline{g(z)} \frac{y^k}{y^2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{C} \iint_{\mathcal{D}_\Gamma} f|_{k,\alpha}(\alpha^{-1}z') \overline{g(\alpha^{-1}z')} \frac{\det\alpha^{-k} y'^k |cz+d|^{2k}}{y'^2} \, \mathrm{d}x' \, \mathrm{d}y' \\
&= \frac{1}{C} \iint_{\mathcal{D}_\Gamma} \det\alpha^{k-1} f(z') (cz+d)^{-k} \overline{g(\alpha^{-1}z')} \det\alpha^{-k} |cz+d|^{2k} \frac{y'^k}{y'^2} \, \mathrm{d}x' \, \mathrm{d}y' \\
&= \frac{1}{C} \iint_{\mathcal{D}_\Gamma} \det\alpha^{-1} f(z') \overline{(cz+d)^k} \overline{g(\alpha^{-1}z')} \frac{y'^k}{y'^2} \, \mathrm{d}x' \, \mathrm{d}y' \\
&= \frac{1}{C} \iint_{\mathcal{D}_\Gamma} \det\alpha^{-1} f(z') \overline{(c'z'+d')^{-k}} \left(\det\alpha^{-1}\right)^{1-k} \overline{g|_{k,\alpha^{-1}}(z') (c'z'+d')^k} \frac{y'^k}{y'^2} \, \mathrm{d}x' \, \mathrm{d}y' \\
&= \frac{1}{C} \iint_{\mathcal{D}_\Gamma} \det\alpha^{k-2} f(z') \overline{g|_{k,\alpha^{-1}}(z')} \frac{y'^k}{y'^2} \, \mathrm{d}x' \, \mathrm{d}y' \\
&= \det\alpha^{k-2} \left\langle f, g|_{k,\alpha^{-1}}\right\rangle_{k,\Gamma}.
\end{aligned}$$

Recall $\alpha' = \alpha^{-1}\det\alpha$. Then

$$g|_{k,\lambda\alpha}(z) = \det\lambda\alpha^{k-1} g(\lambda\alpha z)(\lambda cz + \lambda d)^{-k} = \lambda^{2k-2} \det\alpha^{k-1} g(\alpha z)(cz+d)^{-k} \lambda^{-k} = \lambda^{k-2} \, g|_{k,\alpha}(z),$$

so $g|_{k,\alpha'}(z) = \det\alpha^{k-2} \, g|_{k,\alpha^{-1}}(z)$. Thus $\left\langle f|_{k,\alpha}, g\right\rangle_{k,\alpha^{-1}\Gamma\alpha} = \left\langle f, g|_{k,\alpha'}\right\rangle_{k,\Gamma}$. $\square$

Recall that Lecture 28 Friday 06/12/19

$$\begin{array}{cccl} \mathrm{T}_p & : & \mathrm{S}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{S}_k(\Gamma_1(N)) \\ & & f & \longmapsto & f|_{k,\Gamma_1(N)\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma_1(N)} = \sum_i f|_{k,\alpha_i} \end{array}, \qquad \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N) = \bigsqcup_i \Gamma_1(N) \alpha_i.$$

Suppose we can find α_i such that

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \bigsqcup_i \Gamma_1(N)\,\alpha_i, \qquad \Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \bigsqcup_i \alpha_i\Gamma_1(N).$$

Applying the operation $'$ to the latter gives

$$\Gamma_1(N)\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\Gamma_1(N) = \bigsqcup_i \Gamma_1(N)\,\alpha_i'.$$

If $f, g \in \mathrm{S}_k(\Gamma_1(N))$,

$$\begin{aligned}\langle \mathrm{T}_p f, g\rangle_{k,\Gamma_1(N)} &= \sum_i \left\langle f|_{k,\alpha_i}, g\right\rangle_{k,\Gamma}, \qquad \Gamma \subseteq \Gamma_1(N) \cap \bigcap_i \alpha_i^{-1}\Gamma_1(N)\,\alpha_i \cap \bigcap_i \alpha_i'^{-1}\Gamma_1(N)\,\alpha_i' \\ &= \sum_i \left\langle f, g|_{k,\alpha_i'}\right\rangle_{k,\Gamma} = \left\langle f, g|_{k,\Gamma_1(N)\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right)\Gamma_1(N)}\right\rangle_{k,\Gamma} = \left\langle f, g|_{k,\Gamma_1(N)\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right)\Gamma_1(N)}\right\rangle_{k,\Gamma_1(N)}.\end{aligned}$$

For $\mathrm{SL}_2(\mathbb{Z})$,

$$\mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\mathrm{SL}_2(\mathbb{Z}),$$

so

$$\langle \mathrm{T}_p f, g\rangle_{k,\mathrm{SL}_2(\mathbb{Z})} = \langle f, \mathrm{T}_p g\rangle_{k,\mathrm{SL}_2(\mathbb{Z})}, \qquad f, g \in \mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z})),$$

which is Theorem 1.5.22. In general,

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \mathrm{T}_p\langle p\rangle.$$

Will not prove, so see Diamond and Shurman chapter 5. This argument depends on finding α_i such that

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \bigsqcup_i \Gamma_1(N)\,\alpha_i = \bigsqcup_i \alpha_i\Gamma_1(N).$$

Lemma 2.3.8. *Such α_i exist.*

Proof. This is Diamond and Shurman 5.5.1(c). Write

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \bigsqcup_{i=1}^r \Gamma_1(N)\,\gamma_i = \bigsqcup_{j=1}^r \widetilde{\gamma_j}\Gamma_1(N).$$

Claim that for all $1 \le i \le r$, $\Gamma_1(N)\,\gamma_i \cap \widetilde{\gamma_i}\Gamma_1(N) \ne \emptyset$. Suppose otherwise. Then $\Gamma_1(N)\,\gamma_i \subseteq \bigsqcup_{j\ne i}\widetilde{\gamma_j}\Gamma_1(N)$. The right hand side is stable under right multiplication by $\Gamma_1(N)$, so

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \Gamma_1(N)\,\gamma_i\Gamma_1(N) = \bigcup_{\beta\in\Gamma_1(N)} \Gamma_1(N)\,\gamma_i\beta \subseteq \bigsqcup_{j\ne i}\widetilde{\gamma_j}\Gamma_1(N).$$

This is impossible since $\widetilde{\gamma_i}$ is in the left hand side but not the right hand side. For all i, choose α_i such that $\alpha_i \in \Gamma_1(N)\,\gamma_i \cap \widetilde{\gamma_i}\Gamma_1(N)$, so $\Gamma_1(N)\,\alpha_i = \Gamma_1(N)\,\gamma_i$ and $\alpha_i\Gamma_1(N) = \widetilde{\gamma_i}\Gamma_1(N)$. Now,

$$\bigsqcup_{i=1}^r \Gamma_1(N)\,\alpha_i = \bigsqcup_{i=1}^r \Gamma_1(N)\,\gamma_i = \Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_1(N) = \bigsqcup_{i=1}^r \widetilde{\gamma_i}\Gamma_1(N) = \bigsqcup_{i=1}^r \alpha_i\Gamma_1(N).$$

∎

The upshot is

$$\langle \mathrm{T}_p f, g\rangle_{k,\Gamma_1(N)} = \langle f, \langle p\rangle\,\mathrm{T}_p g\rangle_{k,\Gamma_1(N)}, \qquad p \nmid N, \qquad f, g \in \mathrm{S}_k(\Gamma_1(N)).$$

Check, such as by formulas on q-expansions, that T_p and T_q commute for $p, q \nmid N$ prime, and T_p and $\langle d\rangle$ commute. Thus T_p commutes with its adjoint for all p, so T_p is diagonalisable on $\mathrm{S}_k(\Gamma_1(N))$.

2.4 L-functions

2.4.1 Hecke L-functions

Definition 2.4.1. Let $f = \sum_{n=1}^{\infty} a_n q^n \in \mathrm{S}_k(\Gamma_1(N))$. Then the **Hecke L-function of weight k and level $\Gamma_1(N)$** is

$$\mathrm{L}(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

This is absolutely convergent for $\mathrm{Re}\, s \gg 0$, and has a meromorphic continuation and a functional equation. Set

$$\mathrm{R}(s, f) = N^{\frac{s}{2}} \frac{\Gamma(s)}{(2\pi)^s} \mathrm{L}(s, f).$$

Note.

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \Gamma_1(N) \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \Gamma_1(N).$$

Set

$$\begin{array}{cccc} \mathrm{w}_N & : & \mathrm{S}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{S}_k(\Gamma_1(N)) \\ & & f & \longmapsto & i^k N^{1-\frac{k}{2}} \left. f \right|_{k, \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)} \end{array}.$$

The constants are chosen so that $\mathrm{w}_N^2 = \mathrm{id}$, the **Atkin-Lehner involution**. A warning is that this does not commute with T_p and $\langle p \rangle$. In fact $\mathrm{w}_N \mathrm{T}_p \mathrm{w}_N = \langle p \rangle \mathrm{T}_p$ and $\mathrm{w}_N \langle p \rangle \mathrm{w}_N = \langle p \rangle^{-1}$, and

$$\mathrm{R}(s, f) = \mathrm{R}(k - s, \mathrm{w}_N f).$$

If $f \in \mathrm{S}_k(\Gamma_1(N), \chi)$ for $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is an eigenform for all T_p for $p \nmid N$ and $c_1 = 1$, then using

$$\mathrm{T}_p f = \sum_{n=1}^{\infty} c_{np} q^n + \chi(p) c_n q^{np},$$

if $\mathrm{T}_p f = \lambda_p f = \sum_{n=1}^{\infty} \gamma_n q^n$ for $p \nmid N$, then

$$\gamma_n = \begin{cases} c_{np} + \chi(p) p^{k-1} c_{\frac{n}{p}} & p \mid n \\ c_{np} & p \nmid n \end{cases}.$$

The upshot is for $p \nmid m$,

$$c_{p^{k+1} m} = \lambda_p c_{p^k m} + \chi(p) p^{k-1} c_{p^{k-1} m}, \qquad k \ge 1,$$

so

$$\mathrm{L}(s, f) = \prod_{p \nmid N} \frac{1}{1 - \lambda_p p^{-s} + \chi(p) p^{k-1-2s}} \sum_{m \text{ divisible only by primes } l \mid N} c_m m^{-s}.$$

2.4.2 Oldforms and newforms

Lecture 29 Friday 06/12/19

Let $p \nmid N$ and $l \mid N$, and let

$$\begin{array}{ccccc} \mathrm{U}_l & : & \mathrm{S}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{S}_k(\Gamma_1(N)) \\ & & f & \longmapsto & \left. f \right|_{k, \Gamma_1(N) \left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right) \Gamma_1(N)} \end{array}.$$

On q-expansions, if $f = \sum_{n=1}^{\infty} c_n q^n$, then $\mathrm{U}_l f = \sum_{n=1}^{\infty} c_{nl} q^n$. Then U_l commutes with T_p and $\langle d \rangle$, by checking on q-expansions. A problem is that U_l are generally not self-adjoint or even normal. Let $f = \sum_n c_n q^n \in \mathrm{S}_k(\Gamma_1(N))$ be an eigenform for T_p and $\langle d \rangle$. Atkin-Lehner defined

$$\begin{array}{ccccc} \alpha_{N,l} & : & \mathrm{S}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{S}_k(\Gamma_1(Nl)) \\ & & f & \longmapsto & f \end{array}, \qquad \begin{array}{ccccc} \beta_{N,l} & : & \mathrm{S}_k(\Gamma_1(N)) & \longrightarrow & \mathrm{S}_k(\Gamma_1(Nl)) \\ & & f & \longmapsto & z \mapsto f(lz) = \sum_n c_n q^{nl} \end{array}.$$

Then β, a multiple of $f|_{k,\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)}$, is modular of weight k and level $\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\Gamma(N)\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right) \supseteq \Gamma_1(Nl)$. Check that $(\mathrm{T}_p\beta_{N,l})(f) = \beta_{N,l}(\mathrm{T}_p f)$, and similarly for $\langle d\rangle$ for $d \in (\mathbb{Z}/Nl\mathbb{Z})^\times$. Check that $\alpha_{N,l}$ and $\beta_{N,l}$ commute with T_p, $\langle d\rangle$ for $d \in (\mathbb{Z}/Nl\mathbb{Z})^\times$, and U_p for $p \mid N$ and $l \neq p$. Then $\mathrm{U}_l(\beta_{N,l}(f)) = f$ and $\mathrm{U}_l(\alpha_{N,l}(f)) = \mathrm{T}_p f + p^k\chi(p)\beta_{N,l}(f)$, so the image of

$$\begin{array}{rcl} \mathrm{S}_k(\Gamma_1(N))^2 & \longrightarrow & \mathrm{S}_k(\Gamma_1(Nl)) \\ (f,g) & \longmapsto & \alpha_{N,l}f + \beta_{N,l}g \end{array}$$

is stable under T_p, $\langle d\rangle$, U_p, and U_l.

Definition 2.4.2. Define the **oldforms**

$$\mathrm{S}_k(\Gamma_1(N))^{\mathrm{old}} = \sum_{l \nmid N} \left(\alpha_{\frac{N}{l},l}\left(\mathrm{S}_k\left(\Gamma_1\left(\frac{N}{l}\right)\right)\right) + \beta_{\frac{N}{l},l}\left(\mathrm{S}_k\left(\Gamma_1\left(\frac{N}{l}\right)\right)\right)\right),$$

which is stable under T_p, $\langle d\rangle$, and U_l. Define

$$\mathrm{S}_k(\Gamma_1(N))^{\mathrm{new}} = \left(\mathrm{S}_k(\Gamma_1(N))^{\mathrm{old}}\right)^{\perp},$$

the orthogonal complement with respect to $\langle\cdot,\cdot\rangle_{k,\Gamma_1(N)}$, which is stable under T_p and $\langle d\rangle$, and not a priori under U_p for $p \mid N$.

Theorem 2.4.3 (Atkin-Lehner 1979, strong multiplicity one)**.** *Let $0 \neq f \in \mathrm{S}_k(\Gamma_1(N))^{\mathrm{new}}$ and $g \in \mathrm{S}_k(\Gamma_1(N))$. Suppose for all $p \nmid N$, there exist $\lambda_p \in \mathbb{C}$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ such that*

$$\mathrm{T}_p f = \lambda_p f, \qquad \mathrm{T}_p g = \lambda_p g, \qquad \langle d\rangle f = \chi(d) f, \qquad \langle d\rangle g = \chi(d) g.$$

Then g is a scalar multiple of f.

Corollary 2.4.4. *U_p for $p \mid N$ preserves, and is diagonalisable on, $\mathrm{S}_k(\Gamma_1(N))^{\mathrm{new}}$.*

Corollary 2.4.5. *$\mathrm{S}_k(\Gamma_1(N))^{\mathrm{new}}$ breaks up as a direct sum of one-dimensional simultaneous eigenspaces for T_p, U_l, and $\langle d\rangle$ for $(d,N) = 1$.*

Let $f = \sum_n c_n q^n$, so $\mathrm{U}_l f = \sum_n c_{nl} q^n$, and $\mathrm{U}_l f = \lambda_l f$ implies that $c_{nl} = \lambda_l c_n$.

Corollary 2.4.6. *If $f \in \mathrm{S}_k(\Gamma_1(N), \chi)$ is an eigenform for T_p and U_l, then $c_1 \neq 0$.*

Definition 2.4.7. A **newform** is an element of $\mathrm{S}_k(\Gamma_1(N))^{\mathrm{new}}$ with $c_1 = 1$, that is an eigenform for T_p, U_l, and $\langle d\rangle$ for $(d,N) = 1$.

Let $f \in \mathrm{S}_k(\Gamma_1(N), \chi)$ be a newform such that $\mathrm{T}_p f = \lambda_p f$ and $\mathrm{U}_l f = \lambda_l f$. Then

$$\mathrm{L}(s,f) = \prod_{p \nmid N} \frac{1}{1 - \lambda_p p^{-s} + \chi(p) p^{k-1-2s}} \prod_{l \mid N} \frac{1}{1 - \lambda_l l^{-s}}.$$

2.5 Fermat's last theorem

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N, and let

$$a_p = \begin{cases} \#E(\mathbb{F}_p) - p - 1 & p \nmid N \\ 1 & E \text{ has split multiplicative reduction modulo } p \\ -1 & E \text{ has non-split multiplicative reduction modulo } p \\ 0 & E \text{ has additive reduction modulo } p \end{cases}.$$

Let

$$\mathrm{L}(s,E) = \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \prod_{l \mid N} \frac{1}{1 - a_l l^{-s}}.$$

Theorem 2.5.1 (Eichler-Shimura). *Let $f \in \mathrm{S}_2(\Gamma_0(N))$ be a newform with integer coefficients. There exists an elliptic curve $E_f/\mathbb{Q}$ of conductor N such that $\mathrm{L}(s,f) = \mathrm{L}(s,E_f)$.*

A question is that is the converse true? Lecture 30 Monday 09/12/19

Theorem 2.5.2 (Eichler-Shimura, Deligne). *Let $f \in \mathrm{S}_k(\Gamma_0(N),\chi)$ be a newform for $k \ge 2$ such that $\mathrm{T}_l f = a_l f$ for all $l \nmid N$, and let p be a prime. There exists a unique homomorphism $\overline{\rho_{f,p}} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\overline{\mathbb{F}_p}\right)$ such that for all $l \nmid N$, $\overline{\rho_{f,p}}$ is unramified at l, $\mathrm{Tr}\,\overline{\rho_{f,p}}(\mathrm{Frob}_l) \equiv a_l \mod p$, and $\det \overline{\rho_{f,p}}(\mathrm{Frob}_l) \equiv \chi(l)\, l^{k-1} \mod p$.*

Example. If $f \in \mathrm{S}_2(\Gamma_0(N))$ has integer coefficients, then $E_f[p]\left(\overline{\mathbb{Q}}\right) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Then $\rho_{f,p} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{F}_p)$ gives an $\mathbb{F}_p$-linear action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ on $E_f[p]\left(\overline{\mathbb{Q}}\right)$.

A natural question is given $\overline{\rho} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2\left(\overline{\mathbb{F}_p}\right)$, is $\overline{\rho} = \overline{\rho_{f,p}}$ for some newform f? If so, for which (k, N, χ)?

Theorem 2.5.3 (Serre's conjecture 1987 and Khare-Wintenberger theorem 2005). *Let $\overline{\rho} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{F}_p)$ be odd, that is $\det \overline{\rho}(i \mapsto -i) = -1$.*

- *$\overline{\rho} = \overline{\rho_{f,p}}$ for some newform f.*
- *Can take f of weight $k_{\overline{\rho}}$, level $N_{\overline{\rho}}$, and character $\chi_{\overline{\rho}}$, where*
  - *$2 \le k_{\overline{\rho}} \le p$,*
  - *$\det \overline{\rho}(\mathrm{Frob}_l) \equiv \chi_{\overline{\rho}}(l)\, l^{k_{\overline{\rho}}-1} \mod p$, and this condition determines $k_{\overline{\rho}}$ modulo $p-1$ and $\chi_{\overline{\rho}}$,*
  - *$N_{\overline{\rho}}$ is the so-called* ***Artin conductor*** *$\mathrm{N}(\overline{\rho})$ of $\overline{\rho}$ usually, where*
$$\mathrm{v}_l(\mathrm{N}(\overline{\rho})) = \begin{cases} 0 & \overline{\rho} \text{ is unramified at } l \\ 1 & \overline{\rho}^{\mathrm{I}_l} \text{ has dimension one} \\ \ge 2 & \text{otherwise} \end{cases},$$
  - *if $k_{\overline{\rho}} = 2$,*
$$N_{\overline{\rho}} = \begin{cases} \dfrac{\mathrm{N}(\overline{\rho})}{p} & \overline{\rho} \text{ is finite at } p \\ \mathrm{N}(\overline{\rho}) & \overline{\rho} \text{ is not finite at } p \end{cases}.$$

Example. If $\overline{\rho}$ comes from $E/\mathbb{Q}$, then $k_{\overline{\rho}} = 2$, $\chi_{\overline{\rho}}$ is trivial, and $N_{\overline{\rho}} \mid \mathrm{N}(E)$, where $\mathrm{N}(E) = \prod_{l \text{ bad for } E} p^{\mathrm{v}_l}$ is the **conductor** of E, and
$$\mathrm{v}_l(\mathrm{N}(E)) = \begin{cases} 1 & E \text{ has multiplicative reduction} \\ \ge 2 & E \text{ has additive reduction} \end{cases}.$$
Moreover, if $\mathrm{v}_l(\mathrm{N}(E)) = 1$ and $p \mid \mathrm{ord}_l\, \Delta_E$, then $\mathrm{v}_l(N_{\overline{\rho}}) = 0$.

Definition 2.5.4 (Frey 1985). Suppose $p \ge 5$ and $a^p + b^p = c^p$ for a, b, c coprime. Consider
$$y^2 = x(x - a^p)(x + b^p),$$
so $\Delta = 2^s (abc)^p$.

Then $E_{a,b,c}$ has multiplicative reduction modulo l for all l, so $\mathrm{N}(E_{a,b,c}) = \mathrm{rad}\, 2abc$. Let $\overline{\rho} : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}_2(\mathbb{F}_p)$ giving action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ on $E_{a,b,c}[p]\left(\overline{\mathbb{Q}}\right)$. Then $N_{\overline{\rho}} = 2$, $k_{\overline{\rho}} = 2$, and $\chi_{\overline{\rho}}$ is trivial.

Theorem 2.5.5 (Ribet 1986). *If $\overline{\rho}$ comes from any newform, it comes from the level, weight, and character predicted by Serre.*

Corollary 2.5.6. *If $E_{a,b,c}$ is modular, then the corresponding $\overline{\rho}$ comes from a modular form in $\mathrm{S}_2(\Gamma_0(2))$.*

The problem is $\dim \mathrm{S}_k(\Gamma) \le \frac{1}{12} k\, [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, and $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(2)] = 3$, so $\dim \mathrm{S}_2(\Gamma_0(2)) \le \frac{1}{2}$.

Theorem 2.5.7 (Wiles 1995 and Taylor-Wiles 1996). *All elliptic curves over $\mathbb{Q}$ such that $\mathrm{N}(E)$ is square-free are modular.*

Corollary 2.5.8. *Fermat's last theorem holds.*