M3P21 Geometry II: Algebraic Topology

Lectured by Dr Christian Urech
Typed by David Kurniadi Angdinata

Spring 2019

$$
\begin{tikzcd}
 & & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & & \\
 & \cdots \arrow[rr, "\partial"] & & A'_{n+1} \arrow[rr, "\partial"] \arrow[dd, "i'"] & & A'_n \arrow[rr, "\partial"] \arrow[dd, "i'"] & & A'_{n-1} \arrow[rr, "\partial"] \arrow[dd, "i'"] & & \cdots \\
\cdots \arrow[rr, "\partial"] & & A_{n+1} \arrow[rr, "\partial"] \arrow[ur, "\alpha"] \arrow[dd, "i"] & & A_n \arrow[rr, "\partial"] \arrow[ur, "\alpha"] \arrow[dd, "i"] & & A_{n-1} \arrow[rr, "\partial"] \arrow[ur, "\alpha"] \arrow[dd, "i"] & & \cdots & \\
 & \cdots \arrow[rr, "\partial"] & & B'_{n+1} \arrow[rr, "\partial"] \arrow[dd, "i'"] & & B'_n \arrow[rr, "\partial"] \arrow[dd, "j'"] & & B'_{n-1} \arrow[rr, "\partial"] \arrow[dd, "i'"] & & \cdots \\
\cdots \arrow[rr, "\partial"] & & B_{n+1} \arrow[rr, "\partial"] \arrow[ur, "\beta"] \arrow[dd, "j"] & & B_n \arrow[rr, "\partial"] \arrow[ur, "\beta"] \arrow[dd, "j"] & & B_{n-1} \arrow[rr, "\partial"] \arrow[ur, "\beta"] \arrow[dd, "j"] & & \cdots & \\
 & \cdots \arrow[rr, "\partial"] & & C'_{n+1} \arrow[rr, "\partial"] \arrow[dd] & & C'_n \arrow[rr, "\partial"] \arrow[dd] & & C'_{n-1} \arrow[rr, "\partial"] \arrow[dd] & & \cdots \\
\cdots \arrow[rr, "\partial"] & & C_{n+1} \arrow[rr, "\partial"] \arrow[ur, "\gamma"] \arrow[dd] & & C_n \arrow[rr, "\partial"] \arrow[ur, "\gamma"] \arrow[dd] & & C_{n-1} \arrow[rr, "\partial"] \arrow[ur, "\gamma"] \arrow[dd] & & \cdots & \\
 & & & 0 & & 0 & & 0 & & \\
 & & 0 & & 0 & & 0 & & &
\end{tikzcd}
$$

$$\Downarrow$$

$$
\begin{tikzcd}
\cdots \arrow[r, "\partial"] & \mathrm{H}_{n+1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_{n+1}(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_{n+1}(C) \arrow[d, "\gamma_*"] \arrow[ddl, "\partial"] & & & \\
\cdots \arrow[r, "\partial"] & \mathrm{H}_{n+1}(A') \arrow[r, "i_*"] & \mathrm{H}_{n+1}(B') \arrow[r, "j_*"] & \mathrm{H}_{n+1}(C') \arrow[ddl, "\partial"] & & & \\
 & & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_n(C) \arrow[d, "\gamma_*"] \arrow[ddl, "\partial"] & & \\
 & & \mathrm{H}_n(A') \arrow[r, "i_*"] & \mathrm{H}_n(B') \arrow[r, "j_*"] & \mathrm{H}_n(C') \arrow[ddl, "\partial"] & & \\
 & & & \mathrm{H}_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_{n-1}(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_{n-1}(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & \cdots \\
 & & & \mathrm{H}_{n-1}(A') \arrow[r, "i_*"] & \mathrm{H}_{n-1}(B') \arrow[r, "j_*"] & \mathrm{H}_{n-1}(C') \arrow[r, "\partial"] & \cdots
\end{tikzcd}
$$

Syllabus

Homotopy and homotopy type. Cell complexes. Basic constructions of the fundamental group. Seifert-van Kampen theorem. Covering spaces. Δ-complexes. Simplicial homology. Singular homology. Homotopy invariance. Exact sequences and excision. Mayer-Vietoris sequences. Degree.

Contents

0 Introduction

0.1 Introduction

Lecture 1 Friday 11/01/19

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$?

We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Some underlying geometric notions

0.2.1 Homotopy and homotopy type

Let X and Y be topological spaces and $\mathrm{I} = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times \mathrm{I} \to Y$. For every $t \in \mathrm{I}$ we obtain a continuous map

$$\begin{array}{cccc} f_t & : & X & \longrightarrow & Y \\ & & x & \longmapsto & f_t(x) = F(x,t) \end{array}.$$

Definition. Two continuous maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy $F : X \times \mathrm{I} \to Y$ such that

$$f_0(x) = F(x,0), \qquad f_1(x) = F(x,1), \qquad x \in X.$$

We write $f_0 \cong f_1$. This is an equivalence relation. [1]

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(X) = A$ and $r|_A = \mathrm{id}_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \to \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{array}{cccc} F & : & X \times \mathrm{I} & \longrightarrow & A \\ & & (x,t) & \longmapsto & f_t(x) \end{array},$$

such that $f_0 = \mathrm{id}_X$ and $f_1 : X \to A$ is the deformation retraction.

Example. The closed n-dimensional **n-disc**

$$\mathrm{D}^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}$$

deformation retracts to $(0, \ldots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. Then $t = 1$ implies that $f_1 = \mathrm{id}_{\mathrm{D}^n}$ and $t = 0$ implies that $f_0 : \mathrm{D}^n \to (0, \ldots, 0)$.

[1] Exercise

Example. Let S^n be the n-**sphere**,

$$\partial \mathrm{D}^{n+1} = \mathrm{S}^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $\mathrm{S}^n \times \mathrm{I}$ deformation retracts to $\mathrm{S}^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is that if X is a topological space, and $f : X \to \{p\}$ for $p \in X$ is a deformation retraction of X to p, then X is path-connected. Indeed, if $F : X \times \mathrm{I} \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{array}{ccc} \mathrm{I} & \longrightarrow & X \\ t & \longmapsto & F(x, t) \end{array}$$

that connects x to p. This implies that not all retractions are deformation retractions.

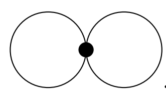
Example. A retraction that is not a deformation retraction. Take a space that is not path-connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. Then $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path-connected.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that $fg \cong \mathrm{id}_Y$ and $gf \cong \mathrm{id}_X$. If there exists a homotopy equivalence between X and Y, then X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. *A deformation retraction $f : X \to A$ is a homotopy equivalence.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = \mathrm{id}_A$ and $if = f \cong \mathrm{id}_X$ by definition. $\square$

Example. The disc with two holes is equivalent to



.

Example. $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.2.2 Cell complexes

Example. The **torus** $\mathrm{S}^1 \times \mathrm{S}^1$ is the union of a point, two open intervals, and the open disc $\mathring{\mathrm{D}}^2$.

These are called **cells**. Can think of discs D^n glued together. Lecture 2 Tuesday 15/01/19

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n-discs $\{\mathrm{D}^n_\alpha\}$ together with continuous maps $\phi_\alpha : \partial \mathrm{D}^n_\alpha \to X^{n-1}$, such that
$$X^n = X^{n-1} \sqcup \bigsqcup_\alpha \mathrm{D}^n_\alpha / \sim,$$
where $x \sim \phi_\alpha(x)$ for all $x \in \partial \mathrm{D}^n_\alpha$ for all α.
- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Remark.

- As a set,
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} e_{\alpha}^n,$$
where each e_{α}^n is homeomorphic to an open n-disc. These e_{α}^n are called the ***n*-cells** of X.

- If $X = X^m$ for some m, then X is called **finite-dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X.

Example. The following are CW-complexes.

$$[0,1], \qquad \mathbb{R}, \qquad \mathrm{S}^1, \qquad \text{a graph}, \qquad \mathrm{S}^n = \mathrm{D}^n / \partial \mathrm{D}^n.$$

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $\mathrm{S}^1 \times \mathrm{S}^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(\mathrm{S}^n) = 0$ if n is odd and $\chi(\mathrm{S}^n) = 2$ if n is even.
- $\chi\left(\mathrm{S}^1 \times \mathrm{S}^1\right) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where V is the number of vertices of P, E is the number of edges of P, and F is the number of faces of P. Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0,1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1 The fundamental group

1.1 Basic constructions

1.1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : \mathrm{I} \to X$, where $\mathrm{I} = [0, 1]$.

Definition. Two paths f_0 and f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{array}{rccc} F & : & \mathrm{I} \times \mathrm{I} & \longrightarrow & X \\ & & (s, t) & \longmapsto & f_t(s) \end{array},$$

such that

$$\begin{array}{lll} f_t(0) = f_0(0), & f_t(1) = f_0(1), & t \in \mathrm{I}, \\ F(s, 0) = f_0(s), & F(s, 1) = f_1(s), & s \in \mathrm{I}. \end{array}$$

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints. Let $f_0, f_1 : \mathrm{I} \to X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1 - t) f_0(s) + t f_1(s).$$

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1.*

Proof.

- f is homotopic to f.
- If f_0 is homotopic to f_1 by a homotopy f_t, then f_1 is homotopic to f_0 by the homotopy f_{1-t}.
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t, then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}.$$

Then

$$\begin{array}{rccc} H & : & \mathrm{I} \times \mathrm{I} & \longrightarrow & X \\ & & (s, t) & \longmapsto & h_t(s) \end{array}$$

is continuous because its restriction to the closed subsets $\mathrm{I} \times \left[0, \frac{1}{2}\right]$ and $\mathrm{I} \times \left[\frac{1}{2}, 1\right]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

$\square$

Let X be a topological space and $\mathrm{I} = [0, 1]$. If $f : \mathrm{I} \to X$ is a path, $[f]$ is the class of all paths on X homotopic to f. Lecture 3 Wednesday 16/01/19

Definition. Let $f, g : \mathrm{I} \to X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$\begin{array}{ccc} \mathrm{I} \times \mathrm{I} & \longrightarrow & X \\ (s, t) & \longmapsto & (f_t \cdot g_t)(s) \end{array}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. $\square$

Remark. Let $\phi : [0,1] \to [0,1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : \mathrm{I} \to X$ is a path, then $f\phi \cong f$. This is a **reparametrisation**. Define

$$\phi_t(s) = (1-t)\phi(s) + ts,$$

then $f\phi_t$ is a homotopy between $f\phi$ and f.

For $x \in X$, let the **constant path** at x be

$$\begin{array}{cccc} \mathrm{c}_x & : & \mathrm{I} & \longrightarrow & X \\ & & s & \longmapsto & x \end{array}.$$

For a path $f : \mathrm{I} \to X$, define

$$\begin{array}{cccc} f^{-1} & : & \mathrm{I} & \longrightarrow & X \\ & & s & \longmapsto & f(1-s) \end{array}.$$

Lemma 1.3. *Let $f, g, h : \mathrm{I} \to X$ be paths. Then*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,
2. $f \cdot \mathrm{c}_{f(1)} \cong f$ *and* $\mathrm{c}_{f(0)} \cdot f \cong f$, *and*
3. $f \cdot f^{-1} \cong \mathrm{c}_{f(0)}$ *and* $f^{-1} \cdot f \cong \mathrm{c}_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h)\phi = f \cdot (g \cdot h)$, where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\ s - \frac{1}{4} & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2s - 1 & s \in \left[\frac{3}{4}, 1\right] \end{cases},$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in \left[0, \frac{1}{2}\right] \\ 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}, \qquad \chi(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

3. Define

$$H(s,t) = \begin{cases} f(\max\{1 - 2s, t\}) & s \in \left[0, \frac{1}{2}\right] \\ f(\max\{2s - 1, t\}) & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

H is continuous, and

$$H(s,0) = f^{-1} \cdot f, \qquad H(s,1) = \mathrm{c}_{f(1)}.$$

The inverse is similar.

∎

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : \mathrm{I} \to X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of **homotopy classes** $[f]$ of loops $f : \mathrm{I} \to X$ with basepoint x_0.

Proposition 1.4. *$\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $\mathrm{c}_{x_0} : \mathrm{I} \to X$, the constant path at x_0.*

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. ∎

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$, since X is convex, so all loops are homotopic to each other.

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply-connected, because all maps $f : \mathrm{I} \to X$ are continuous, so path-connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : \mathrm{I} \to X$ is continuous implies that f is constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path-component of X. Let $h : \mathrm{I} \to X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{array}{cccccc} \beta_h & : & \pi_1(X, x_1) & \longrightarrow & \pi_1(X, x_0) \\ & & [f] & \longmapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

This is well-defined by Lemma 1.2.

Proposition 1.5. $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ *is an isomorphism.*

Proof. It is a homomorphism, since

$$\beta_h[f \cdot g] = \left[h \cdot f \cdot g \cdot h^{-1}\right] = \left[h \cdot f \cdot h^{-1}\right]\left[h \cdot g \cdot h^{-1}\right] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[\mathrm{c}_{x_1}] = [\mathrm{c}_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. $\square$

If X is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply-connected** if it is path-connected and $\pi_1(X) = 0$.

Proposition 1.6. X *is simply-connected if and only if there exists a unique homotopy class of paths between any two points of* X*.*

Proof.

$\Longrightarrow$ There exists a path between any two points. Let f and g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. Then $f \cdot g^{-1} \cong g \cdot g^{-1}$, so
$$f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g.$$

$\Longleftarrow$ X is path-connected. Then $x_1 = x_0$, so all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$.

$\square$

1.1.2 The fundamental group of the circle

The goal is to show that $\pi_1\left(\mathrm{S}^1\right) \cong \mathbb{Z}$.

Lecture 4 Friday 18/01/19

Definition. A **covering space** of a space X is a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there is an open $x \in U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \widetilde{U_j}$, where $\widetilde{U_j} \subseteq \widetilde{X}$ is open,
- $\widetilde{U_i} \cap \widetilde{U_j} = \emptyset$ if $i \neq j$, and
- $p|_{\widetilde{U_j}} : \widetilde{U_j} \to U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The $\widetilde{U_j}$ are called **sheets**.

Example.

$$\begin{array}{cccccc} p & : & \mathbb{R} & \longrightarrow & \mathrm{S}^1 \\ & & s & \longmapsto & (\cos 2\pi s, \sin 2\pi s) \end{array}.$$

Definition. Let $p : \widetilde{X} \to X$ be a covering space. A **lift** of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, so

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}.$$

Proposition 1.7 (Unique lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ of f such that $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ for some $y \in Y$ and if Y is connected, then $\widetilde{f_1} = \widetilde{f_2}$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \widetilde{U_j}.$$

Let $\widetilde{U_1}$ be the sheet such that $\widetilde{f_1}(y) \in \widetilde{U_1}$, and let $\widetilde{U_2}$ be the sheet such that $\widetilde{f_2}(y) \in \widetilde{U_2}$. Let $N \subseteq Y$ be open and $y \in N$ such that $\widetilde{f_1}(N) \subseteq \widetilde{U_1}$ and $\widetilde{f_2}(N) \subseteq \widetilde{U_2}$. We have $p\widetilde{f_1} = p\widetilde{f_2}$. Then $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ if and only if $\widetilde{U_1} = \widetilde{U_2}$, if and only if $\widetilde{f_1}\Big|_N = \widetilde{f_2}\Big|_N$. Let

$$A = \left\{y \in Y \;\middle|\; \widetilde{f_1}(y) = \widetilde{f_2}(y)\right\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ implies that $A = Y$. ☐

Proposition 1.8 (Homotopy lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $F : Y \times \mathrm{I} \to X$ be a continuous map such that there exists a lift $\widetilde{f_0} : Y \times \{0\} \to \widetilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\widetilde{F} : Y \times \mathrm{I} \to \widetilde{X}$ of F such that $\widetilde{F}\Big|_{Y \times \{0\}} = \widetilde{f_0}$.*

Proof. Let $y_0 \in Y$ and $t \in \mathrm{I}$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq \mathrm{I}$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I implies that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\widetilde{F}\Big|_{N \times \mathrm{I}}$ of $F|_{N \times \mathrm{I}}$.

- $\widetilde{F}\Big|_{N \times [0,0]} = \widetilde{f_0}\Big|_{N \times [0,0]}$ exists.
- Assume the lift has been constructed on $N \times [0, t_i]$. Let $\widetilde{U_i} \subseteq \widetilde{X}$ be such that $p|_{\widetilde{U_i}} : \widetilde{U_i} \to U_i$ such that $\widetilde{F}(y_0, t_i) \subseteq \widetilde{U_i}$. After shrinking N, may assume $\widetilde{F}(N \times \{t_i\}) \subseteq \widetilde{U_i}$. Define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1} : U_i \to \widetilde{U_i}$.

After finitely many steps we obtain a lift $\widetilde{F} : N \times \mathrm{I} \to \widetilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F|_{N_y \times \mathrm{I}} : N_y \times \mathrm{I} \to X$ lifts. For all $y \in Y$, $\{y\} \times \mathrm{I}$ is connected and can be lifted, so Proposition 1.7 implies that the lift of $N \times \mathrm{I}$ is unique. Thus there is a unique lift $\widetilde{F} : Y \times \mathrm{I} \to \widetilde{X}$. ☐

Example. Let X be a topological space and A be discrete. Then $p : X \times A \to X$ is a covering space. This is the **trivial covering**. Show the unique lifting property and the homotopy lifting property for the trivial covering. [2]

Corollary 1.9. *Let $f : \mathrm{I} \to X$ be a path, $f(0) = x_0$, and $p : \widetilde{X} \to X$ be a covering space. For each $\widetilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : \mathrm{I} \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x_0}$.*

Proof. Proposition 1.8 for Y a point. ☐

[2] Exercise

Theorem 1.10. *Let $x_0 = (1, 0) \in \mathrm{S}^1$. Then $\pi_1\left(\mathrm{S}^1, x_0\right)$ is the infinite cyclic group generated by the homotopy class of the loop*

$$\begin{array}{cccc} \omega & : & \mathrm{I} & \longrightarrow & \mathrm{S}^1 \\ & & s & \longmapsto & (\cos 2\pi s, \sin 2\pi s) \end{array}.$$

Remark.

- $[\omega]^n = [\omega_n]$, where
$$\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s).$$
-
$$\begin{array}{cccc} p & : & \mathbb{R} & \longrightarrow & \mathrm{S}^1 \\ & & s & \longmapsto & (\cos 2\pi s, \sin 2\pi s) \end{array}$$
is a covering space.
- ω_n lifts to
$$\begin{array}{cccc} \widetilde{\omega_n} & : & \mathrm{I} & \longrightarrow & \mathbb{R} \\ & & s & \longmapsto & ns \end{array},$$
such that $\widetilde{\omega_n}(0) = 0$ and $\widetilde{\omega_n}(1) = n$.

Proof of Theorem 1.10.

- If $f : \mathrm{I} \to \mathrm{S}^1$ is a loop at x_0, then the homotopy lifting property implies that there exists a lift $\widetilde{f} : \mathrm{I} \to \mathbb{R}$ such that $\widetilde{f}(0) = 0$. Since $p\left(\widetilde{f}(1)\right) = f(1) = x_0$, then $\widetilde{f}(1) = n$ for some $n \in \mathbb{Z}$. Then $\widetilde{\omega_n} : \mathrm{I} \to \mathbb{R}$ is another path such that $\widetilde{\omega_n}(0) = 0$ and $\widetilde{\omega_n}(1) = n$, so $\widetilde{f} \simeq \widetilde{\omega_n}$. Let $F : \mathrm{I} \times \mathrm{I} \to \mathbb{R}$ be a homotopy equivalence between $\widetilde{f}$ and $\widetilde{\omega_n}$. Then $pF : \mathrm{I} \times \mathrm{I} \to \mathrm{S}^1$ gives a homotopy between $p\widetilde{f} = f$ and $p\widetilde{\omega_n} = \omega_n$.
- Let $m, n \in \mathbb{Z}$ and assume $\omega_m \simeq \omega_n$. Let $F : \mathrm{I} \times \mathrm{I} \to \mathrm{S}^1$ be a homotopy. Then
$$F(0, t) = \omega_m(t), \qquad F(1, t) = \omega_n(t), \qquad F(s, 0) = F(s, 1) = x_0, \qquad s, t \in \mathrm{I}.$$
The unique lifting property implies that $\widetilde{\omega_n}, \widetilde{\omega_m} : \mathrm{I} \to \mathbb{R}$ are unique lifts such that $\widetilde{\omega_n}(0) = 0 = \widetilde{\omega_m}(0)$. The homotopy lifting property implies that F lifts uniquely to a homotopy $\widetilde{F} : \mathrm{I} \times \mathrm{I} \to \mathbb{R}$ between $\widetilde{\omega_n}$ and $\widetilde{\omega_m}$, and $\widetilde{F}(s, 1) \in \mathbb{Z}$ for all $s \in \mathrm{I}$. Thus $\widetilde{F}(s, 1) = n = m$, so $\omega_m \simeq \omega_n$ if and only if $n = m$.

$\square$

Lecture 5 Tuesday 22/01/19

Lecture 5 is a problems class.

Lecture 6 Wednesday 23/01/19

Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proof. May assume $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$. Assume p has no roots in $\mathbb{C}$. For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{array}{cccc} f_r & : & \mathrm{I} & \longrightarrow & \mathbb{C} \\ & & s & \longmapsto & \dfrac{p\left(re^{2\pi i s}\right)/p(r)}{\left|p\left(re^{2\pi i s}\right)/p(r)\right|} \end{array},$$

so $|f_r(s)| = 1$. Then $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. Then f_0 is the constant loop at 1, and $f_r(s)$ depends continuously on r, so $f_r \simeq f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1\left(\mathrm{S}^1\right)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \dots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \dots + |a_n|)\left|z^{n-1}\right| \geq \left|a_1 z^{n-1}\right| + \dots + |a_n| \geq \left|a_1 z^{n-1} + \dots + a_n\right|.$$

Hence, for $0 \leq t \leq 1$ the polynomial

$$p_t(z) = z^n + t\left(a_1 z^{n-1} + \dots + a_n\right)$$

has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t\left(re^{2\pi i s}\right)/p_t(r)}{\left|p_t\left(re^{2\pi i s}\right)/p_t(r)\right|}.$$

Then $F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1\left(\mathrm{S}^1\right)$. Theorem 1.10 implies that $n = 0$, so p is constant. $\square$

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X and Y be path-connected topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1 (X \times Y, (x_0, y_0)) \cong \pi_1 (X, x_0) \times \pi_1 (Y, y_0).$$

Proof. A map

$$\begin{array}{cccccc} f & : & Z & \longrightarrow & X \times Y \\ & & z & \longmapsto & (g (z), h (z)) \end{array}$$

is continuous if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous. For $Z = \mathrm{I}$,

$$\{ \text{ loops in } X \times Y \text{ based } (x_0, y_0) \} \qquad \leftrightsquigarrow \qquad \{ \text{ loops in } X \text{ based } x_0 \} \times \{ \text{ loops in } Y \text{ based } y_0 \}.$$

Two loops

$$\begin{array}{cccccc} f_1 & : & \mathrm{I} & \longrightarrow & X \times Y \\ & & s & \longmapsto & (g_1 (s), h_1 (s)) \end{array}, \qquad \begin{array}{cccccc} f_2 & : & \mathrm{I} & \longrightarrow & X \times Y \\ & & s & \longmapsto & (g_2 (s), h_2 (s)) \end{array}$$

are homotopic if and only if $g_1 \cong g_2$ and $h_1 \cong h_2$, so there is a bijection

$$\pi_1 (X \times Y, (x_0, y_0)) \cong \pi_1 (X, x_0) \times \pi_1 (Y, y_0).$$

Then $f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. $\square$

Example. The torus $\mathrm{S}^1 \times \mathrm{S}^1$ has

$$\pi_1 \left(\mathrm{S}^1 \times \mathrm{S}^1\right) \cong \pi_1 \left(\mathrm{S}^1\right) \times \pi_1 \left(\mathrm{S}^1\right) \cong \mathbb{Z}^2.$$

1.1.3 Induced homomorphisms

Let X and Y be topological spaces, $x_0 \in X$, and $\phi : X \to Y$. An observation is that ϕ induces a homomorphism

$$\begin{array}{cccccc} \phi_* & : & \pi_1 (X, x_0) & \longrightarrow & \pi_1 (Y, \phi (x_0)) \\ & & [f] & \longmapsto & [\phi f] \end{array}.$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0, then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1. Moreover, $\phi (f \cdot g) = (\phi f) \cdot (\phi g)$ and ϕ maps the constant path at x_0 to the constant path at $\phi (x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. *Let $\psi : X \to Y$ and $\phi : Y \to Z$ be continuous maps between topological spaces, $x_0 \in X$, and*

$$\psi_* : \pi_1 (X, x_0) \to \pi_1 (Y, \psi (x_0)), \qquad \phi_* : \pi_1 (Y, \psi (x_0)) \to \pi_1 (Z, \phi\psi (x_0)),$$
$$(\phi\psi)_* : \pi_1 (X, x_0) \to \pi_1 (Z, \phi\psi (x_0)).$$

   Then $(\phi\psi)_* = \phi_*\psi_*$.

2. *Let $\mathrm{id}_X : X \to X$ be the identity then*

$$(\mathrm{id}_X)_* : \pi_1 (X, x_0) \to \pi_1 (X, x_0)$$

   is the identity.

Proof.

1. Let $f : \mathrm{I} \to X$ be a loop at x_0, then

$$(\phi\psi)_* ([f]) = [(\phi\psi) f] = [\phi (\psi f)] = \phi_* ([\psi f]) = \phi_*\psi_* ([f]).$$

2. $(\mathrm{id}_X)_* ([f]) = [\mathrm{id}_X f] = [f]$.

$\square$

These two observations yield in particular that if $\phi : X \to Y$ is a homeomorphism with inverse $\psi : Y \to X$, then

$$\phi_* : \pi_1 (X, x_0) \to \pi_1 (Y, \phi (x_0))$$

is an isomorphism with inverse ψ_*.

Lecture 7
Friday
25/01/19

Proposition 1.14. *Let $\phi : X \to Y$ be a homotopy equivalence. Then*

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : \mathrm{I} \to X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{array}{cccc} \beta_h & : & \pi_1(X, x_1) & \longrightarrow & \pi_1(X, x_0) \\ & & [f] & \longmapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

Lemma 1.15. *Let $\phi_t : X \to Y$ be a homotopy and $x_0 \in X$. Define the path*

$$\begin{array}{ccccc} h & : & \mathrm{I} & \longrightarrow & Y \\ & & s & \longmapsto & \phi_s(x_0) \end{array}, \qquad h(0) = \phi_0(x_0), \qquad h(1) = \phi_1(x_0).$$

Then $\phi_{0*} = \beta_h \phi_{1*}$, that is the following diagram commutes.

$$\begin{tikzcd} & \pi_1(Y, \phi_1(x_0)) \arrow[d, "\beta_h", "\sim"'] \\ \pi_1(X, x_0) \arrow[ur, "\phi_{1*}"] \arrow[r, "\phi_{0*}"'] & \pi_1(Y, \phi_0(x_0)) \end{tikzcd}.$$

Proof. For $t \in \mathrm{I}$, define the path

$$\begin{array}{ccccc} h_t & : & \mathrm{I} & \longrightarrow & X \\ & & s & \longmapsto & h(ts) \end{array}, \qquad h_t(0) = \phi_0(x_0), \qquad h_t(1) = h(t) = \phi_t(x_0).$$

Let f be a loop at x_0. Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t. So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \qquad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$\phi_{0*}([f]) = [\phi_0 f] = \left[h \cdot (\phi_1 f) \cdot h^{-1}\right] = \beta_h([\phi_1 f]) = \beta_h \phi_{1*}([f]).$$

□

Lemma 1.15 implies in particular the following.

Corollary 1.16. *If $\psi : X \to X$ is continuous and $\psi \cong \mathrm{id}_X$, then*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[d, "\beta_h", "\sim"'] \\ \pi_1(X, x_0) \arrow[ur, "(\mathrm{id}_X)_*"] \arrow[r, "\psi_*"'] & \pi_1(X, \psi(x_0)) \end{tikzcd},$$

so $\psi_* = \beta_h$ hence an isomorphism. □

Proof of Proposition 1.14. Let $\phi : X \to Y$ be a homotopy equivalence. Let $\psi : Y \to X$ be a homotopy inverse of ϕ, that is $\phi\psi \cong \mathrm{id}_Y$ and $\psi\phi \cong \mathrm{id}_X$. Then

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \phi\psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above implies that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. □

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path-connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0, then we can write

$$[f] = [h_1] \dots [h_m],$$

such that the h_i are loops at x_0, and each contained in a single A_{α_i}.

Proof. f is continuous, so for all $s \in \mathrm{I}$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α. We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. Then I is compact, so a finite number of such intervals cover I, so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i. Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. Then $f \cong f_1 \cdot \dots \cdot f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0 and g_m be the constant loops at x_0. Then $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong \left(g_0 \cdot f_1 \cdot g_1^{-1}\right) \cdot \dots \cdot \left(g_{m-1} \cdot f_m \cdot g_m^{-1}\right),$$

so $[f] = [h_1] \dots [h_m]$. □

Lecture 8 Tuesday 29/01/19

Example. Möbius strip M deformation retracts to S^1. Thus $\phi : M \to \mathrm{S}^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1(\mathrm{S}^1) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in \mathrm{S}^1$ because $\pi_1(\mathrm{S}^1) \ncong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $\mathrm{S}^1 \subseteq \mathrm{D}^2$. Assume there is a retraction $r : \mathrm{D}^2 \to \mathrm{S}^1$, consider the embedding $i : \mathrm{S}^1 \hookrightarrow \mathrm{D}^2$. Then $ri = \mathrm{id}_{\mathrm{S}^1}$. Thus

$$\begin{array}{ccccc} \pi_1(\mathrm{S}^1) & \xrightarrow{i_*} & \pi_1(\mathrm{D}^2) & \xrightarrow{r_*} & \pi_1(\mathrm{S}^1) \\ \wr\Vert & & \wr\Vert & & \wr\Vert \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array},$$

so $r_* i_* (\pi_1(\mathrm{S}^1)) = 0$ but $r_* i_* = (ri)_* = \mathrm{id}_{\pi_1(\mathrm{S}^1)}$, a contradiction.

Theorem 1.18 (Brouwer fixed point theorem)**.** *Let $h : \mathrm{D}^2 \to \mathrm{D}^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in \mathrm{D}^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in \mathrm{D}^2$. Define $r : \mathrm{D}^2 \to \mathrm{S}^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1. Then r is continuous, and if $x \in \mathrm{S}^1$, then $r(x) = x$, so r is a retraction, a contradiction. □

Lemma 1.17 implies that if $U_1, U_2 \subseteq X$ are open and path-connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path-connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as

$$[f] = [g_1][h_1] \dots [g_n][h_n],$$

such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2. In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$i_{1*} : \pi_1(U_1, x_0) \to \pi_1(X, x_0), \qquad i_{2*} : \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Lemma 1.17 implies that $i_{1*}(\pi_1(U_1, x_0)) \cup i_{2*}(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. *$\pi_1(\mathrm{S}^n) = 0$ if $n \geq 2$.*

Proof. Let

$$U_1 = \mathrm{S}^n \setminus \{(1, 0, \dots, 0)\}, \qquad U_2 = \mathrm{S}^n \setminus \{(-1, 0, \dots, 0)\}.$$

Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. Then $U_1 \cup U_2 = \mathrm{S}^n$ and $U_1 \cap U_2$ is path-connected. Let $x_0 \in U_1 \cap U_2$. Then $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 implies that $\pi_1(\mathrm{S}^n, x_0)$. ☐

1.2 Seifert-van Kampen theorem

1.2.1 Free products with amalgamation

Definition. If S is a set, then F_S is the **free group** on S. We can write any group G as a quotient of some free group F_S, $G = \mathrm{F}_S / \langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq \mathrm{F}_S$, the smallest normal subgroup of F_S containing R. We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G.

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0 if it satisfies the following universal property. For every group G and all homomorphisms $h_1' : G_1 \to G$ and $h_2' : G_2 \to G$ such that $h_1' f_1 = h_2' f_2$, there exists a unique homomorphism $\alpha : H \to G$ such that $h_1' = \alpha h_1$ and $h_2' = \alpha h_2$, so

$$\begin{tikzcd}
G_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "h_1"] \arrow[ddr, bend left, "h_1'"] & \\
G_2 \arrow[r, "h_2"'] \arrow[drr, bend right, "h_2'"'] & H \arrow[dr, dashed, "\exists!\alpha"] & \\
& & G
\end{tikzcd}.$$

Theorem 1.20. *Given $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 \underset{G_0}{*} G_2$.*

Proof. Worksheet 2. ☐

Lecture 9 Wednesday 30/01/19

$G_0 = \{\mathrm{id}\}$ is the **free product**. We write $G_1 * G_2$ instead of $G_1 \underset{\{\mathrm{id}\}}{*} G_2$. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$, with injections $G_i \hookrightarrow G_1 * G_2$ for $i = 1, 2$. More generally,

$$G_1 \underset{G_0}{*} G_2 \cong G_1 * G_2 / N.$$

where N is the normal closure of the set

$$\left\{ f_1(g) \, f_2(g)^{-1} \;\middle|\; g \in G_0 \right\} \subseteq G_1 * G_2.$$

1.2.2 The Seifert-van Kampen theorem

Theorem 1.21 (Seifert-van Kampen)**.** *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected and let $x_0 \in U_1 \cap U_2$. Then*

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0) \cong \pi_1(U_1, x_0) * \pi_1(U_2, x_0)/N,$$

where N is the normal closure of the set

$$\left\{ j_{1*}(\omega) j_{2*}(\omega)^{-1} \;\middle|\; \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and $j_i : U_1 \cap U_2 \hookrightarrow U_i$, so

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & U_1 \arrow[d, hook, "j_1"] \\ U_2 \arrow[r, hook, "j_2"'] & X \end{tikzcd} \qquad \implies \qquad \begin{tikzcd} \pi_1(U_1 \cap U_2, x_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(U_1, x_0) \arrow[d, "j_{1*}"] \\ \pi_1(U_2, x_0) \arrow[r, "j_{2*}"'] & \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0) \end{tikzcd}.$$

Proof of Theorem 1.21. Appendix A.1. □

Lecture 10 Friday 01/02/19

Theorem 1.22 (Seifert-van Kampen, strong version)**.** *Let X be a path-connected topological space such that*

- $X = \bigcup_\alpha A_\alpha$,
- $A_\alpha, A_\alpha \cap A_\beta, A_\alpha \cap A_\beta \cap A_\gamma$ *are open and path-connected for all* α, β, γ, *and*
- $x_0 \in \bigcap_\alpha A_\alpha$.

Then

$$\pi_1(X, x_0) \cong \underset{\alpha}{*} \pi_1(A_\alpha, x_0)/N,$$

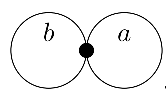
where $N \subseteq \underset{\alpha}{*} \pi_1(A_\alpha, x_0)$ is the normal closure of the set

$$\left\{ (i_{\alpha\beta})_*(\omega) (i_{\beta\alpha})_*(\omega)^{-1} \;\middle|\; \omega \in \pi_1(A_\alpha \cap A_\beta) \right\},$$

and $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ is the inclusion.

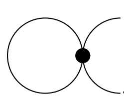
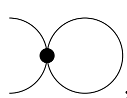
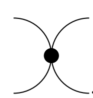
Example. Let $S^1 \vee S^1$ be the wedge product. Fix $x \in S^1$ and $y \in S^1$. Then

$$S^1 \vee S^1 = S^1 \sqcup S^1/x \sim y =$$



.

Let

$A =$  , $B =$  , $A \cap B =$  .

Then $\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$, $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$, and $\pi_1(A \cap B) = \{\mathrm{id}\}$, and $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(S^1 \vee S^1) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a,b\}}.$$

More generally, let $X = S^1_{a_1} \vee \dots \vee S^1_{a_n}$. Induction implies that

$$\pi_1(X) = \mathbb{Z} * \dots * \mathbb{Z} \cong F_{\{a_1, \dots, a_n\}}.$$

Similarly, let $X = \bigvee_{\alpha \in \Lambda} S^1_\alpha$. Strong version of van Kampen implies that

$$\pi_1(X) = \underset{\alpha \in \Lambda}{*} \mathbb{Z} = F_\Lambda.$$

Example. Let T be a torus and $x_0 \in T$. Let

$$A = T \setminus \{\text{small closed disc } D\}, \qquad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- A is homotopy equivalent to $\mathrm{S}^1 \vee \mathrm{S}^1$, so $\pi_1(A) \cong \mathrm{F}_{\{a,b\}}$.
- B is homeomorphic to D^2, so $\pi_1(B) = \{\mathrm{id}\}$.
- $A \cap B$ is homotopy equivalent to S^1, so $\pi_1(A \cap B) \cong \mathbb{Z}$.

Then $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(T) \cong \pi_1(A) / \langle\langle i_*(\pi_1(A \cap B))\rangle\rangle,$$

where $i : A \cap B \hookrightarrow A$. Then

$$\begin{array}{cccccc} i_* & : & \pi_1(A \cap B) = \langle \omega \rangle & \longrightarrow & \pi_1(A) \\ & & \omega & \longmapsto & aba^{-1}b^{-1} \end{array},$$

so

$$\pi_1(T) \cong \mathrm{F}_{\{a,b\}} / \left\langle\left\langle aba^{-1}b^{-1}\right\rangle\right\rangle = \left\langle a, b \mid aba^{-1}b^{-1}\right\rangle \cong \mathbb{Z}^2.$$

1.2.3 Applications to CW-complexes

Let X be a path-connected topological space. Let Y be the space obtained by attaching 2-cells $\left\{e_\alpha^2\right\}$ to X along maps $\phi_\alpha : \partial\mathrm{D}^2 = \mathrm{S}^1 \to X$. Consider the loops

$$\begin{array}{cccccc} \phi'_\alpha & : & \mathrm{I} & \longrightarrow & X \\ & & s & \longmapsto & \phi_\alpha(\cos 2\pi s, \sin 2\pi s) \end{array},$$

based at $\phi'_\alpha(0)$. Let γ_α be a path from x_0 to $\phi'_\alpha(0)$ for each α. Then $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is a loop at x_0. After attaching e_α^2, the loop $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is homotopic to the constant loop at x_0. Let $N \subseteq \pi_1(X, x_0)$ be the normal closure of all the elements of the form $\left[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}\right]$. The inclusion $i : X \hookrightarrow Y$ yields

$$i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0),$$

and $N \subseteq \ker i_*$.

Proposition 1.23. *This inclusion $i : X \hookrightarrow Y$ induces a surjection*

$$i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0),$$

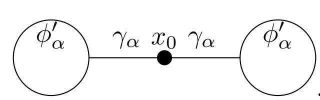
and $\ker i_* = N$, so

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0) / N.$$

Proof. Construct a space Z from Y by attaching a strip $\mathrm{I} \times \mathrm{I}$ to Y by identifying the lower edge $\mathrm{I} \times \{0\}$ with the path γ_α and the right edge $\{1\} \times \mathrm{I}$ with an arch on e_α^2. Attach all the left edges of the strips with each other. Then Z deformation retracts to Y. Choose a point $y_\alpha \in e_\alpha^2$ for each α, such that y_α is not contained in X or in the attached strip. Let

$$A = Z \setminus \bigcup_\alpha \{y_\alpha\}, \qquad B = Z \setminus X.$$

- A deformation retracts to X.
- B is homotopy equivalent to a point.
- $A \cap B$ is homotopy equivalent to

$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} =$ 

.

Then $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(Y) \cong \pi_1(Z) = \pi_1(A) / \langle\langle j_*(\pi_1(A \cap B))\rangle\rangle,$$

where $j : A \cap B \hookrightarrow A$ is the inclusion. So $\langle\langle j_*(\pi_1(A \cap B))\rangle\rangle$ is exactly N. Thus $\pi_1(A) = \pi_1(X)$. ☐

Lecture 11
Tuesday
05/02/19

Corollary 1.24. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

Proof. Let $G = \langle \{g_\alpha\} \mid \{r_\beta\} \rangle$ be a presentation of G, that is $G = \mathrm{F}_{\{g_\alpha\}} / \langle\langle \{r_\beta\} \rangle\rangle$. Seen last time that $\pi_1\left(\bigvee_{g_\alpha} \mathrm{S}^1_{g_\alpha}\right) = \mathrm{F}_{\{g_\alpha\}}$. Each word r_β defines a loop in $\bigvee_{g_\alpha} \mathrm{S}^1_{g_\alpha}$. Attach 2-cells to $\bigvee_{g_\alpha} \mathrm{S}^1_{g_\alpha}$ along the loops defined by the relations $\{r_\beta\}$. Call this new CW-complex Y. Proposition 1.23 implies that

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0) / \langle\langle \{r_\beta\} \rangle\rangle \cong \mathrm{F}_{\{g_\alpha\}} / \langle\langle \{r_\beta\} \rangle\rangle \cong G.$$

$\square$

Remark. Let $X = \bigcup_n X^n$ be a CW-complex, path-connected. Proposition 1.23 can be used to show the following two facts.

- The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism $\pi_1\left(X^1\right) \to \pi_1(X)$.
- The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1\left(X^2\right) \to \pi_1(X)$.

1.3 Covering spaces

1.3.1 Lifting properties

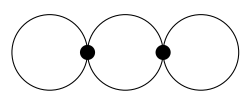
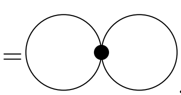
Let X be a topological space. Recall that a covering space is $p : \widetilde{X} \to X$ such that each $x \in X$ has an open neighbourhood U such that

$$p^{-1}(U) = \bigcup_\alpha \widetilde{U_\alpha},$$

where U_α are pairwise disjoint and $p|_{\widetilde{U_\alpha}} : \widetilde{U_\alpha} \to U$ is a homeomorphism for all α.

Example.

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathrm{S}^1 \\ s & \longmapsto & (\cos 2\pi s, \sin 2\pi s) \end{array}, \qquad \begin{array}{ccc} \mathrm{S}^1 & \longrightarrow & \mathrm{S}^1 \\ z & \longmapsto & z^n \end{array},$$

 $\to \mathrm{S}^1 \vee \mathrm{S}^1 =$  .

Let $f : Y \to X$ be a continuous map. A lift of f is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, where $p : \widetilde{X} \to X$ is a covering space. Let Y be connected.

- **Unique lifting property** states that if two lifts $\widetilde{f_1}$ and $\widetilde{f_2}$ of f coincide at one point, then they coincide on all of Y.
- **Homotopy lifting property** states that if $f_t : Y \to X$ is a homotopy and $\widetilde{f_0} : Y \to \widetilde{X}$ is a lift of f_0 then there exists a unique homotopy $\widetilde{f_t} : Y \to \widetilde{X}$ of $\widetilde{f_0}$ that lifts f_t.

Remark.

- If Y is a point, this is called the **path lifting property**. Let $f : \mathrm{I} \to X$ be a path with $f(0) = x_0$. If $\widetilde{x_0} \in p^{-1}(x_0)$, then there is a unique path $\widetilde{f} : \mathrm{I} \to \widetilde{X}$ lifting f and starting at $\widetilde{x_0}$.
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints $f_t(0)$ and $f_t(1)$ define constant paths as t varies.

Fix $x_0 \in X$ and $\widetilde{x_0} \in \widetilde{X}$ such that $p(\widetilde{x_0}) = x_0$, so

$$p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0).$$

To every element in $\pi_1(X, x_0)$ we can associate a homotopy class of paths in $\widetilde{X}$ starting at $\widetilde{x_0}$.

Proposition 1.25.

1. *$p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$ is injective.*
2. *$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x_0}$ are loops.*

Proof.

1. Let $\widetilde{f_0} : \mathrm{I} \to \widetilde{X}$ be a loop at $\widetilde{x_0}$ such that $\left[\widetilde{f_0}\right] \in \ker p_*$, so $p\widetilde{f_0} = f_0$ is homotopic to the constant loop at x_0. Let $f_t : \mathrm{I} \to X$ be a homotopy between f_0 and the constant loop. Homotopy lifting property and remark implies that f_t lifts to a homotopy $\widetilde{f_t}$ of paths between $\widetilde{f_0}$ and the constant loop, so $\left[\widetilde{f_0}\right] = \mathrm{id} \in \pi_1\left(\widetilde{X}, \widetilde{x_0}\right)$ and p_* is injective.
2. Let $f : \mathrm{I} \to X$ be a loop at x_0 that lifts to a loop $\widetilde{f}$ at $\widetilde{x_0}$. Then $p\widetilde{f} = f$, so $p_*\left(\left[\widetilde{f}\right]\right) = [f]$. On the other hand, if $f : \mathrm{I} \to X$ is a loop at x_0 such that there exists a loop $\widetilde{f} : \mathrm{I} \to \widetilde{X}$ at $\widetilde{x_0}$ with $p_*\left(\left[\widetilde{f}\right]\right) = [f]$, then f is homotopic to $p\widetilde{f}$. Homotopy lifting property implies that there exists a loop $\widetilde{f}' : \mathrm{I} \to \widetilde{X}$ at x_0 such that $p\widetilde{f}' = f$.

$\square$

Lecture 12 Wednesday 06/02/19

Let $p : \widetilde{X} \to X$ be a covering space. Let $U \subseteq X$ be an evenly covered neighbourhood of $x \in X$. Let

$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} \widetilde{U_\alpha}.$$

Then the cardinality $\left|p^{-1}(x)\right|$ of $p^{-1}(x)$ is exactly the cardinality of $|\Lambda|$. The set of sheets is in bijection with $p^{-1}(x)$. So the cardinality of $p^{-1}(x)$ is locally constant. If X is connected, the cardinality of $p^{-1}(x)$ is constant.

Notation. Let X and Y be topological spaces, $x \in X$, and $y \in Y$. A continuous map

$$f : (X, x) \to (Y, y)$$

is a continuous map $f : X \to Y$ such that $f(x) = y$.

Proposition 1.26. *Let X and $\widetilde{X}$ be path-connected and*

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

be a covering space. Then the number of sheets of p equals the index of $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ in $\pi_1(X, x_0)$.

Proof. Let g be a loop in X at x_0 and $\widetilde{g}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x_0}$. Let $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ and let $[h] \in H$. Then $h \cdot g$ lifts to a path $\widetilde{h} \cdot \widetilde{g}$ in $\widetilde{X}$ starting at $\widetilde{x_0}$ with the same endpoint as $\widetilde{g}$, because $\widetilde{h}$ is a loop, by Proposition 1.25. Define

$$\begin{array}{cccc} \Phi & : & \{\text{cosets of } H \text{ in } \pi_1(X, x_0)\} & \longrightarrow & p^{-1}(x_0) \\ & & H[g] & \longmapsto & \widetilde{g}(1) \end{array},$$

so Φ is well-defined. Want to show that Φ is bijective.

- Φ is surjective because $\widetilde{X}$ is path-connected. Let $\widetilde{g}$ be a path in $\widetilde{X}$ from $\widetilde{x_0}$ to any point $\widetilde{x_0'} \in p^{-1}(x_0)$, then $g = p \cdot \widetilde{g}$ and $\Phi(H[g]) = \widetilde{x_0'}$.
- Φ is injective, since if $\Phi(H[g_1]) = \Phi(H[g_2])$ then the lift $\widetilde{g_1} \cdot \widetilde{g_2}^{-1}$ of $g_1 \cdot g_2^{-1}$ defines a loop in $\widetilde{X}$ at $\widetilde{x_0}$. Proposition 1.25 implies that $[g_1][g_2]^{-1} \in H$, so $H[g_1] = H[g_2]$.

$\square$

We say that a topological space X has a certain property (P) **locally** if for each point $x \in X$ and each neighbourhood U of x there is an open neighbourhood $V \subseteq U$ having this property (P).

Example. X is locally path-connected or X is locally simply-connected.

Proposition 1.27. *Let*

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

be a covering space and

$$f : (Y, y_0) \to (X, x_0)$$

a continuous map, where Y is path-connected and locally path-connected. Then there is a lift

$$\widetilde{f} : (Y, y_0) \to \left(\widetilde{X}, \widetilde{x_0}\right)$$

if and only if

$$f_* \left(\pi_1 \left(Y, y_0\right)\right) \subseteq p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_0}\right)\right),$$

so

$$\begin{tikzcd} & \left(\widetilde{X}, \widetilde{x_0}\right) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}.$$

Proof.

$\implies$ Clear, because $f = p\widetilde{f}$ implies $f_* = p_*\widetilde{f}_*$.

$\impliedby$ Assume

$$f_* \left(\pi_1 \left(Y, y_0\right)\right) \subseteq p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_0}\right)\right).$$

For each $y \in Y$ choose a path γ from y_0 to y, so $f\gamma$ is a path in X from x_0 to $f(y)$. By path lifting, we can lift $f\gamma$ to a path $\widetilde{f\gamma}$ in $\widetilde{X}$ starting at $\widetilde{x_0}$. Define the map

$$\begin{array}{cccc} \widetilde{f} & : & (Y, y_0) & \longrightarrow & \left(\widetilde{X}, \widetilde{x_0}\right) \\ & & y & \longmapsto & \widetilde{f\gamma}(1) \end{array}.$$

- This map is well-defined, that is does not depend on the choice of γ. Let γ' be another path from y_0 to y. Then $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$ is a loop at x_0 and

  $$[h_0] \in f_* \left(\pi_1 \left(Y, y_0\right)\right) \subseteq p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_0}\right)\right).$$

  Proposition 1.25 implies that can lift h_0 to a loop $\widetilde{h_0}$ at $\widetilde{x_0}$. The first half of $\widetilde{h_0}$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}^{-1}$, so $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. Thus $\widetilde{f}$ is well-defined.
- We have $p\widetilde{f} = f$, so $\widetilde{f}$ lifts f.
- It remains to show that $\widetilde{f}$ is continuous. Let $y \in Y$ and let U be an evenly covered neighbourhood of $f(y)$. Let $\widetilde{U}$ be the sheet above U such that $\widetilde{f}(y) \in \widetilde{U}$, so $p|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Let $V \subseteq Y$ be a path-connected neighbourhood of y such that $f(V) \subseteq U$. Fix a path γ from y_0 to y. Let $y' \in V$ be arbitrary and η be a path from y to y', so $\gamma \cdot \eta$ is a path from y_0 to y'. Then $(f\gamma) \cdot (f\eta)$ is a path in U from x_0 to $f(y')$, and $\widetilde{f\eta} = \left(p|_{\widetilde{U}}\right)^{-1} f\eta$, so $\widetilde{f}\Big|_V = \left(p|_{\widetilde{U}}\right)^{-1} f$. Thus $\widetilde{f}\Big|_V : V \to \widetilde{U}$ is continuous, so $\widetilde{f}$ is continuous.

$\square$

1.3.2 The classification of covering spaces

Lecture 13 Friday 08/02/19

Definition. A covering space $p : \widetilde{X} \to X$ is a **universal cover** if $\widetilde{X}$ is simply-connected.

Definition. A topological space X is **semilocally simply-connected** if each $x \in X$ has a neighbourhood U such that

$$i_* : \pi_1(U, x) \to \pi_1(X, x)$$

is trivial, where $i : U \hookrightarrow X$ is the inclusion.

Example. Let $X = \bigcup_n C_n \subseteq \mathbb{R}^2$ be the **Hawaiian earrings**, where $C_n \subseteq \mathbb{R}^2$ is the circle of radius $1/n$ and centre $(1/n, 0)$. Then X is not semilocally simply-connected.

Proposition 1.28. *If $p : \widetilde{X} \to X$ is a universal cover, then X is semilocally simply-connected.*

Proof. Let $U \subseteq X$ be an evenly covered neighbourhood of $x_0 \in X$, $\widetilde{U} \subseteq \widetilde{X}$ be a sheet over U, and $\gamma \subseteq U$ be a loop at x_0, so γ lifts to a loop $\widetilde{\gamma} \subseteq \widetilde{U}$ at $\widetilde{x_0}$. Then $\widetilde{\gamma}$ is homotopic to the constant loop at $\widetilde{x_0}$. Composing this homotopy with p implies that γ is homotopic to the constant loop at x_0 in X, so

$$\pi_1(U, x_0) \to \pi_1(X, x_0)$$

is trivial. ☐

Theorem 1.29. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there exists a universal cover $p : \widetilde{X} \to X$.*

Remark. If

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

is a universal cover, each point $\widetilde{x} \in \widetilde{X}$ can be joined to $\widetilde{x_0}$ by a unique homotopy class of paths, by Proposition 1.6.

$$\left\{\text{points in } \widetilde{X}\right\} \leftrightsquigarrow \left\{[\gamma] \;\middle|\; \gamma \text{ is a path in } \widetilde{X} \text{ starting at } \widetilde{x_0}\right\} \leftrightsquigarrow \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

by the homotopy lifting property.

Proof. Let $x_0 \in X$, and

$$\widetilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}, \qquad \begin{array}{cccc} p & : & \widetilde{X} & \longrightarrow & X \\ & & [\gamma] & \longmapsto & \gamma(1) \end{array}.$$

Have to

1. give $\widetilde{X}$ a topology,
2. show that $p : \widetilde{X} \to X$ is a covering, and
3. show that $\widetilde{X}$ is simply-connected.

Recall that a **basis** for a topology on a set Y is a collection $\mathcal{B}$ of subsets such that

- $Y = \bigcup_{U \in \mathcal{B}} U$, and
- if $U_1, U_2 \in \mathcal{B}$ and $y \in U_1 \cap U_2$ then there exists $V \in \mathcal{B}$ such that $y \in V$ and $V \subseteq U_1 \cap U_2$.

A basis defines a topology on Y, by $A \subseteq Y$ is open if and only if A is the union of elements of $\mathcal{B}$. A map $f : Z \to Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathcal{B}$.

1. Let $\mathcal{U}$ be the collection of all path-connected open sets $U \subseteq X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Then $X = \bigcup_{U \in \mathcal{U}} U$ because X is semilocally simply-connected. Let $U_1, U_2 \in \mathcal{U}$ and $y \in U_1 \cap U_2$, and let $y \in V \subseteq U_1 \cap U_2$ be path-connected and open. Then

$$\begin{tikzcd} V \arrow[r, hook] & U_1 \arrow[r, hook] & X \end{tikzcd},$$

$$\begin{tikzcd} \pi_1(V) \arrow[r] \arrow[rr, bend right, "\text{trivial}"'] & \pi_1(U_1) \arrow[r, "\text{trivial}"] & \pi_1(X) \end{tikzcd}$$

so $V \in \mathcal{U}$, so $\mathcal{U}$ is a basis for the topology on X. For $U \in \mathcal{U}$ and γ a path in X from x_0 to a point in U, we define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\} \subseteq \widetilde{X}.$$

$U_{[\gamma]}$ only depends on the class $[\gamma]$, so $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is bijective. Surjective because U is path-connected and injective because all paths η in U with the same endpoint are homotopic. Claim that $\{U_{[\gamma]}\}$ forms a basis on $\widetilde{X}$.

- $\bigcup_{U \in \mathcal{U},\ \gamma} U_{[\gamma]} = \widetilde{X}$, because $\bigcup_{U \in \mathcal{U}} U = X$.
- Observe that if $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$. If $\gamma' = \gamma \cdot \eta$ for η a path in U, then elements in $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$, so $U_{[\gamma']} \subseteq U_{[\gamma]}$. The elements in $U_{[\gamma]}$ have the form

$$[\gamma \cdot \mu] = \left[\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu\right] = \left[\gamma' \cdot \eta^{-1} \cdot \mu\right],$$

so $U_{[\gamma]} \subseteq U_{[\gamma']}$. Consider $U_{[\gamma]}$ and $V_{[\gamma']}$ and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, so $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Let $W \in \mathcal{U}$ such that $W \subseteq U \cap V$ and such that $\gamma''(1) \in W$, so $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$. This proves the claim.

2. $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is a homeomorphism. It is bijective, let $V_{[\gamma']} \subseteq U_{[\gamma]}$ be an element of the basis, so $p\left(V_{[\gamma']}\right) = V \in \mathcal{U}$. Then $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. Thus $p : \widetilde{X} \to X$ is continuous. If $U \in \mathcal{U}$, then

$$p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]},$$

so $p : \widetilde{X} \to X$ is a covering space.

3. Let $\widetilde{x_0} \in \widetilde{X}$ be the class of the constant path at x_0. Let $[\gamma] \in \widetilde{X}$ be arbitrary. Then $\gamma : [0,1] \to X$ and $\gamma(0) = x_0$. Let γ_t be the path in X defined by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0,t] \\ \gamma(t) & s \in [t,1] \end{cases}.$$

Then

$$\begin{array}{rcccc} \widetilde{\gamma} & : & \mathrm{I} & \longrightarrow & \widetilde{X} \\ & & t & \longmapsto & [\gamma_t] \end{array}$$

is a path in $\widetilde{X}$ from $\widetilde{x_0}$ to $[\gamma]$, so $\widetilde{X}$ is path-connected. Recall that $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ consists of the classes of loops at x_0 in X that lifts to loops in $\widetilde{X}$ at $\widetilde{x_0}$. Let $[\gamma] \in p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$. Then γ lifts to a loop at $\widetilde{x_0}$ by $t \mapsto [\gamma_t]$. Because it is a loop we have $\widetilde{x_0} = [\gamma_1] = [\gamma]$, so γ is homotopic to the constant loop. Thus $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \{\mathrm{id}\}$, so $\widetilde{X}$ is simply-connected.

$\square$

Lecture 14 is a problems class. Lecture 14 Tuesday 12/02/19

Lecture 15 Wednesday 13/02/19

Let $p : \widetilde{X} \to X$ be a covering space, so $p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_0} \right) \right) \subseteq \pi_1 (X, x_0)$.

Proposition 1.30. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subseteq \pi_1 (X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_* (\pi_1 (X_H, \widetilde{x_0})) = H$ for some basepoint x_0.*

Proof. Let $\widetilde{X}$ be as constructed above. Define $X_H = \widetilde{X}/\sim$, where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $\left[\gamma \cdot (\gamma')^{-1} \right] \in H$. This is an equivalence relation.

- $[\gamma] \sim [\gamma]$ because $\mathrm{id} \in H$.
- $[\gamma] \sim [\gamma']$ implies that $[\gamma'] \sim [\gamma]$ because H contains all its inverses.
- $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ implies that $[\gamma] \sim [\gamma'']$ because H is closed under product.

Then

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{X}/\sim = X_H \arrow[dl, "p"] \\ X & \end{tikzcd}.$$

Let $U_{[\gamma]}$ and $U_{[\gamma']}$ be basis neighbourhoods. If $[\gamma] \sim [\gamma']$ then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$, so p is a covering space, and $p^{-1}(U) = \bigcup_\gamma U_{[\gamma]}$. Let $\widetilde{x_0} \in X_H$ be the equivalence class of the constant path c_{x_0} at x_0. Let γ be a loop in X at x_0 such that $[\gamma] \in p_* (\pi_1 (X_H, \widetilde{x_0}))$. Again $t \mapsto [\gamma_t]$ is a lift of γ at $\widetilde{x_0}$. Then

$$t \mapsto [\gamma_t] \text{ is a loop in } X_H \quad \iff \quad [\gamma_1] = [\gamma] = [\mathrm{c}_{x_0}] \text{ in } X_H \quad \iff \quad [\gamma] \sim [\mathrm{c}_{x_0}] \quad \iff \quad \gamma \in H.$$

□

Definition. We say that two covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ are **isomorphic** if there exists a homeomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ such that

$$\begin{tikzcd} \widetilde{X_1} \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X_2} \arrow[dl, "p_2"] \\ & X & \end{tikzcd}.$$

Proposition 1.31. *Let X be path-connected and locally path-connected and $x_0 \in X$. Two path-connected covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ are isomorphic via an isomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ mapping a basepoint $\widetilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x_2} \in p_2^{-1}(x_0)$ if and only if*

$$p_{1*} \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1} \right) \right) = p_{2*} \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2} \right) \right).$$

Proof.

$\implies$ If

$$f : \left(\widetilde{X_1}, \widetilde{x_1} \right) \to \left(\widetilde{X_2}, \widetilde{x_2} \right)$$

is an isomorphism, then $p_1 = p_2 f$, so

$$p_{1*} \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1} \right) \right) \subseteq p_{2*} \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2} \right) \right),$$

and $p_2 = p_1 f^{-1}$, so

$$p_{2*} \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2} \right) \right) \subseteq p_{1*} \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1} \right) \right).$$

$\Longleftarrow$ Assume
$$p_{1*}\left(\pi_1\left(\widetilde{X_1},\widetilde{x_1}\right)\right) = p_{2*}\left(\pi_1\left(\widetilde{X_2},\widetilde{x_2}\right)\right).$$
By lifting criterion in Proposition 1.27, we can lift p_1 to a continuous map
$$\widetilde{p_1} : \left(\widetilde{X_1},\widetilde{x_1}\right) \to \left(\widetilde{X_2},\widetilde{x_2}\right),$$
and p_2 to a continuous map
$$\widetilde{p_2} : \left(\widetilde{X_2},\widetilde{x_2}\right) \to \left(\widetilde{X_1},\widetilde{x_1}\right),$$
so $p_1\widetilde{p_2} = p_2$ and $p_2\widetilde{p_1} = p_1$.
$$\begin{tikzcd}
\left(\widetilde{X_1},\widetilde{x_1}\right) \arrow[rr, bend left, "\widetilde{p_1}"] \arrow[dr, "p_1"'] & & \left(\widetilde{X_2},\widetilde{x_2}\right) \arrow[ll, bend left, "\widetilde{p_2}"] \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}.$$
Then $\widetilde{p_1}\widetilde{p_2}$ fixes the point $\widetilde{x_2} \in \widetilde{X_2}$. By the unique lifting property in Proposition 1.7, $\widetilde{p_1}\widetilde{p_2} = \mathrm{id}_{\widetilde{x_2}}$. Similarly, $\widetilde{p_2}\widetilde{p_1} = \mathrm{id}_{\widetilde{x_1}}$, so $\widetilde{p_1}$ is an isomorphism.

$\square$

Lecture 16 Friday 15/02/19

Fix $x_0 \in X$, $\widetilde{x_1} \in p_1^{-1}(x_0)$, and $\widetilde{x_2} \in p_2^{-1}(x_0)$. A **basepoint preserving isomorphism**
$$f : \left(\widetilde{X_1},\widetilde{x_1}\right) \to \left(\widetilde{X_2},\widetilde{x_2}\right)$$
is an isomorphism such that $f(\widetilde{x_1}) = \widetilde{x_2}$.

Theorem 1.32 (Galois correspondence)**.** *Let X be path-connected, locally path-connected, and semilocally simply-connected, and $x_0 \in X$. Then*

1. *there is a bijection*
$$\left\{ \begin{array}{c} \textit{path-connected covering spaces } p : \left(\widetilde{X},\widetilde{x_0}\right) \to (X, x_0) \\ \textit{up to basepoint preserving isomorphisms} \end{array} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{array}{c} \textit{subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\},$$
2. *if we ignore the basepoints, this correspondence gives a bijection*
$$\left\{ \begin{array}{c} \textit{path-connected covering spaces } p : \widetilde{X} \to X \\ \textit{up to isomorphisms} \end{array} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{array}{c} \textit{conjugacy classes of subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}.$$

Proof.

1. To a covering space
$$p : \left(\widetilde{X},\widetilde{x_0}\right) \to (X, x_0),$$
we associate the subgroup
$$p_*\left(\pi_1\left(\widetilde{X},\widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0).$$
Proposition 1.30 and Proposition 1.31 show that this is well-defined on the isomorphism classes and it is bijective.

2. Let $p : \widetilde{X} \to X$ be a covering space and $\widetilde{x_1}, \widetilde{x_2} \in p^{-1}(x_0)$. Let
$$H_i = p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_i} \right) \right) \subseteq \pi_1 (X, x_0), \qquad i = 1, 2.$$
Let $\widetilde{\gamma}$ be a path from $\widetilde{x_1}$ to $\widetilde{x_2}$. Let $\gamma = p\widetilde{\gamma}$ be a loop at x_0. Let $[f] \in \pi_1 (X, x_0)$. Then $[f] \in H_1$ if and only if the lift $\widetilde{f}$ is a loop at $\widetilde{x_1}$. Then $\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma}$ is a loop at $\widetilde{x_2}$, so
$$p_* \left(\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma} \right) = \gamma^{-1} \cdot f \cdot \gamma,$$
so $[\gamma]^{-1} [f] [\gamma] \in H_2$. Thus $[\gamma]^{-1} H_1 [\gamma] \subseteq H_2$. Similarly, $[\gamma] H_2 [\gamma]^{-1} \subseteq H_1$. Conversely, let $H_1 \subseteq \pi_1 (X, x_0)$ as above and $[\delta] \in \pi_1 (X, x_0)$ be an arbitrary element. Let $\widetilde{\delta}$ be a lift of δ such that $\widetilde{\delta}(0) = \widetilde{x_0}$ and define $\widetilde{x_3} = \widetilde{\delta}(1)$. Then the same construction yields
$$p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_3} \right) \right) = [\delta]^{-1} H_1 [\delta].$$

$\square$

1.3.3 Deck transformations and group actions

Definition. Let $p : \widetilde{X} \to X$ be a covering space. A **deck-transformation** is an isomorphism from $\widetilde{X}$ to itself.
$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}.$$
The group of deck-transformations is denoted by $\mathrm{G}\left(\widetilde{X}\right)$.

Example.

- Let
$$\begin{array}{cccccc} p & : & \mathbb{R} & \longrightarrow & \mathrm{S}^1 \subseteq \mathbb{C} \\ & & t & \longmapsto & e^{2\pi i t} \end{array}.$$
Then $f : \mathbb{R} \to \mathbb{R}$ such that $p(f(t)) = p(t)$ if and only if $e^{2\pi i f(t)} = e^{2\pi i t}$, if and only if $f(t) = t + n$, so $\mathrm{G}(\mathbb{R}) \cong \mathbb{Z}$.

- Let
$$\begin{array}{cccccc} p & : & \mathrm{S}^1 & \longrightarrow & \mathrm{S}^1 \\ & & z & \longmapsto & z^n \end{array}.$$
Then $\mathrm{G}\left(\mathrm{S}^1\right) \cong \mathbb{Z}/n\mathbb{Z}$.

An observation is that if $\widetilde{X}$ is path-connected then $f \in \mathrm{G}\left(\widetilde{X}\right)$ is uniquely determined by where it sends a single point.
$$\begin{tikzcd} \widetilde{X} \arrow[rr, bend left, "f'"] \arrow[rr, "f"'] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}.$$
If $f(x) = f'(x)$ for a single x, by unique lifting $f = f'$. So the identity is the only deck-transformation with a fixed point.

Definition. A covering space $p : \widetilde{X} \to X$ is **normal**, or **regular**, or **Galois**, if for each $x \in X$ and every pair $\widetilde{x}, \widetilde{x'} \in p^{-1}(x)$ there is an $f \in \mathrm{G}\left(\widetilde{X}\right)$ such that $f(\widetilde{x}) = \widetilde{x'}$.

Example.

- $p : \mathbb{R} \to \mathrm{S}^1$ is normal.
- $p : \mathrm{S}^1 \to \mathrm{S}^1$ is normal.

Proposition 1.33. *Let*

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

be a path-connected covering space, and X be path-connected and locally path-connected. Then $p : \widetilde{X} \to X$ is normal if and only if

$$H = p_* \left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$$

is a normal subgroup.

Proof. Let $\widetilde{x_1} \in p^{-1}(x_0)$, let $\widetilde{\gamma}$ be a path from $\widetilde{x_0}$ to $\widetilde{x_1}$ and $\gamma = p(\widetilde{\gamma})$. Then $[\gamma]$ conjugates H to $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_1}\right)\right)$ so $[\gamma] H [\gamma]^{-1} = H$, if and only if $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_1}\right)\right)$, by Proposition 1.31 if and only if $f(\widetilde{x_0}) = \widetilde{x_1}$. So $\mathrm{G}\left(\widetilde{X}\right)$ acts transitively on $p^{-1}(x_0)$ if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup. Let $x_0' \in X$ be another point and h a path from x_0 to $\widetilde{x_0}$. Let $\widetilde{h}$ be a lift of h such that $\widetilde{h}(0) = \widetilde{x_0}$. Set $\widetilde{x_0} = \widetilde{h}(1)$ and $p\left(\widetilde{x_0'}\right) = x_0'$. Then

$$\begin{tikzcd} \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \arrow[r, "\beta_{\widetilde{h}}"] \arrow[d, "p_*"'] & \pi_1\left(\widetilde{X}, \widetilde{x_0'}\right) \arrow[d, "p_*"] \\ \pi_1(X, x_0) \arrow[r, "\beta_h"'] & \pi_1(X, x_0') \end{tikzcd}.$$

Thus $H \subseteq \pi_1(X, x_0)$ is normal if and only if

$$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0'}\right)\right) \subseteq \pi_1(X, x_0')$$

is normal, as before if and only if $\mathrm{G}\left(\widetilde{X}\right)$ acts transitively on $p^{-1}(x_0')$. □

Lecture 17 Tuesday 19/02/19

Proposition 1.34. *Let*

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

be a covering space, X be path-connected and locally path-connected, and $\widetilde{X}$ be path-connected. Let $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normaliser of H. Then $\mathrm{G}\left(\widetilde{X}\right)$ is isomorphic to $N(H)/H$. In particular,

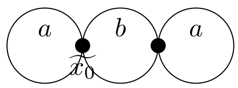
- *if $\widetilde{X}$ is normal, then*
$$\mathrm{G}\left(\widetilde{X}\right) \cong \pi_1(X, x_0)/H,$$
- *if $\widetilde{X}$ is the universal cover, then*
$$\mathrm{G}\left(\widetilde{X}\right) \cong \pi_1(X, x_0).$$

Proof. Read the proof of this in Hatcher. [3] □

[3] Exercise

Example. Let $X = \mathrm{S}^1 \vee \mathrm{S}^1$, so $\pi_1(X) = \mathrm{F}_{\{a,b\}}$. Then the following are covering spaces.

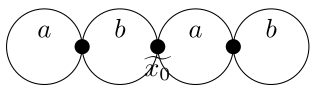
- A normal covering space

$\widetilde{X} =$ 

$$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \left\langle a, b^2, bab^{-1} \right\rangle \overset{2}{\subseteq} \mathrm{F}_{\{a,b\}}.$$

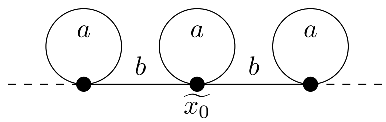
  In general, a two-oriented graph is a covering space of X.

- Not a normal covering space

$\widetilde{X} =$ 

$$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \left\langle b^2, bab^{-1}, a^2, aba^{-1} \right\rangle.$$

- A normal covering space

$\widetilde{X} =$ 

$$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \left\langle b^n a b^{-n} \mid n \in \mathbb{Z} \right\rangle.$$

  The universal cover is a tree.

Example. Let $T = \mathrm{S}^1 \times \mathrm{S}^1$, so $\pi_1(T) = \mathbb{Z}^2$. This is abelian, so all covering spaces are normal. The universal cover is

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathrm{S}^1 \times \mathrm{S}^1 \\ (s,t) & \longmapsto & \left(e^{2\pi i s}, e^{2\pi i t}\right) \end{array},$$

since $\mathbb{R}^2$ is simply-connected. Check that it is a covering space. [4] More generally, if $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ are covering spaces then

$$\begin{array}{ccc} \widetilde{X} \times \widetilde{Y} & \longrightarrow & X \times Y \\ (x,y) & \longmapsto & (p(x), q(y)) \end{array}$$

is again a covering space. For example,

$$\begin{array}{ccc} \mathrm{S}^1 \times \mathrm{S}^1 & \longrightarrow & \mathrm{S}^1 \times \mathrm{S}^1 \\ (z_1, z_2) & \longmapsto & (z_1^n, z_2^m) \end{array}.$$

Example. Worksheet 3 exercise 7. Let

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim = \mathrm{S}^n / \sim$$

be the **projective n-space**, the space of all lines through the origin in $\mathbb{R}^{n+1}$, where $x \sim -x$. Let $p : \mathrm{S}^n \to \mathbb{RP}^n$ be the quotient map. Claim that this is a covering space. Let $[x] \in \mathbb{RP}^n$. Then $p^{-1}([x]) = \{\pm x\}$. Let U be an open neighbourhood of x such that $U \cap (-U) = \emptyset$, so $p(U) = \{[x] \mid x \in U\}$. Then $p^{-1}(p(U)) = U \cup (-U)$ is open and disjoint. Thus $p|_U : U \to p(U)$ is a homeomorphism, so it is a covering space.

- $n \geq 2$ implies that S^n is simply-connected, so $\mathrm{S}^n \to \mathbb{RP}^n$ is a universal cover. Then

$$\{\mathrm{id}\} = p_*(\pi_1(\mathrm{S}^n)) \overset{2}{\subseteq} \pi_1(\mathbb{RP}^n),$$

  so $|\pi_1(\mathbb{RP}^n)| = 2$. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

- $n = 1$ implies that $\mathbb{RP}^1 = \mathrm{S}^1$, so

$$\begin{array}{ccccc} p & : & \mathrm{S}^1 & \longrightarrow & \mathrm{S}^1 \\ & & z & \longmapsto & z^2 \end{array}$$

  is a covering space.

[4] Exercise

2 Homology

Lecture 18 Wednesday 20/02/19

Higher homotopy groups $\pi_n(X, x_0)$ are groups of basepoint preserving homotopies of continuous $\phi : \mathrm{I}^n \to X$ such that $\phi(\partial \mathrm{I}^n) = x_0$.

Example.

$$\pi_1(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \pi_2(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_3(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{otherwise} \end{cases}, \qquad \pi_i\left(\mathrm{S}^2\right) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 4, 5 \\ \mathbb{Z}/12\mathbb{Z} & i = 6 \end{cases}.$$

Homology is more suitable. The following is the plan.

- Simplicial homology.
- Singular homology.
- Technical machinery to show that they coincide.
- Applications.

2.1 Δ-complexes

Definition. Let $m, n \ge 0$.

- An **n-simplex** in $\mathbb{R}^m$ is the convex hull of a set V of $n+1$ points in $\mathbb{R}^m$ that are not all contained in an affine $(n-1)$-dimensional subspace of $\mathbb{R}^m$.
- The **standard n-simplex** is the convex hull of the standard basis $\{e_1, \dots, e_{n+1}\}$ in $\mathbb{R}^{n+1}$,
$$\left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \ge 0,\ x_0 + \dots + x_n = 1\right\}.$$
- An **ordered n-simplex** is an n-simplex with an ordering on the vertices. We denote it by $[v_0, \dots, v_n]$, where $v_0, \dots, v_n$ are the vertices in ascending order.
- The **standard ordered n-simplex** is the ordered n-simplex
$$[e_1, \dots, e_{n+1}]$$
in $\mathbb{R}^{n+1}$. It is denoted by Δ^n.
- Let $[v_0, \dots, v_{n+1}]$ be an n-simplex in $\mathbb{R}^m$ and let $L \subseteq \mathbb{R}^m$ be the affine subspace spanned by $v_0, \dots, v_n$. Then there exists a unique affine morphism
$$\begin{array}{ccc} L & \longrightarrow & \mathbb{R}^{n+1} \\ v_i & \longmapsto & e_{i+1} \end{array}, \qquad i = 0, \dots, n.$$
This gives a homeomorphism from $[v_0, \dots, v_n]$ to Δ^n that preserves this ordering.
- For $n \ge 1$, the **faces** of an ordered n-simplex $[v_0, \dots, v_n]$ are the ordered $(n-1)$-simplices
$$[v_0, \dots, \widehat{v_i}, \dots, v_n].$$
$\widehat{v_i}$ means we omit the vertex v_i.
- The union of all the faces of a simplex Δ is the **boundary** $\partial\Delta$.
- The **interior** of Δ is $\mathring{\Delta} = \Delta \setminus \partial\Delta$.

Example. Let $\Delta^2 = [e_1, e_2, e_3]$. Then $\partial\Delta^2 = [e_1, e_2] \cup [e_1, e_3] \cup [e_2, e_3]$.

Definition. Let X be a topological space. A **Δ-complex structure** on X is a collection of continuous maps
$$\sigma_\alpha : \Delta^{\mathrm{n}(\alpha)} \to X, \qquad \alpha \in A, \qquad \mathrm{n}(\alpha) \in \mathbb{N},$$
such that

1. the restriction $\sigma_\alpha|_{\mathring{\Delta}^{\mathrm{n}(\alpha)}}$ is injective for all $\alpha \in A$ and for each $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha\left(\mathring{\Delta}^{\mathrm{n}(\alpha)}\right)$,
2. the restriction of σ_α to a face of $\Delta^{\mathrm{n}(\alpha)}$ is equal to σ_β for some $\beta \in A$ and $\mathrm{n}(\beta) = \mathrm{n}(\alpha) - 1$, and
3. $U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{\mathrm{n}(\alpha)}$ for all $\alpha \in A$.

An observation is that $\sigma : \bigsqcup_{\alpha \in A} \Delta^{\mathrm{n}(\alpha)} \to X$ induced by the σ_α is a quotient map, since it is surjective by 1 and $U \subseteq X$ is open if and only if $\sigma^{-1}(U)$ is open by 3.

Remark. One can show that an X with a Δ-complex structure is a CW-complex.

Example.

- Torus or Klein bottle is two Δ^2, three Δ^1, and one Δ^0.
- S^2 is a tetrahedron.
- **Dunce hat**, by identifying all the three faces of the standard 2-simplex with each other, is one Δ^2, one Δ^1, and one Δ^0.

2.2 Simplicial homology

2.2.1 Simplicial homology

Let X be a Δ-complex. The group of **n-chains** $\Delta_n(X)$ is the free abelian group on the n-simplices $\sigma_\alpha : \Delta^{\mathrm{n}(\alpha)} \to X$, where $\mathrm{n}(\alpha) = n$. So an element in $\Delta_n(X)$ is of the form
$$\sum_{\alpha \in A,\ \mathrm{n}(\alpha) = n} c_\alpha \cdot \sigma_\alpha, \qquad c_\alpha \in \mathbb{Z},$$
where all but finitely many of the c_α are zero.

Example. Let K be a Klein bottle.

- $\Delta_0(K) = \{n \cdot v \mid n \in \mathbb{Z}\} = \mathbb{Z} \cdot v \cong \mathbb{Z}$.
- $\Delta_1(K) = \{n_1 \cdot a + n_2 \cdot b + n_3 \cdot c \mid n_1, n_2, n_3 \in \mathbb{Z}\} = \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \cong \mathbb{Z}^3$.
- $\Delta_2(K) = \{n_1 \cdot U + n_2 \cdot V \mid n_1, n_2 \in \mathbb{Z}\} = \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V \cong \mathbb{Z}^2$.
- $\Delta_n(K) = 0$ for $n \geq 3$.

Similarly for a torus T.

Lecture 19 Friday 22/02/19

Define the **boundary homomorphism** by
$$\begin{array}{cccc} \partial_n & : & \Delta_n(X) & \longrightarrow & \Delta_{n-1}(X) \\ & & \sigma_\alpha & \longmapsto & \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \end{array}.$$
Moreover, we define $\partial_0 = 0$.

Example. Let $\sigma : [v_0, v_1, v_2, v_3] \to X$. Then
$$\partial_3(\sigma) = \sigma|_{[v_1, v_2, v_3]} - \sigma|_{[v_0, v_2, v_3]} + \sigma|_{[v_0, v_1, v_3]} - \sigma|_{[v_0, v_1, v_2]}.$$

Lemma 2.1. *The composition*
$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$
is the zero map.

Proof. Let $\sigma : [v_0, \dots, v_n] \to X$ be an n-simplex. Then

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]},$$

so

$$(\partial_{n-1} \circ \partial_n)(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]} = 0.$$

If $n = 1$, clear. $\square$

2.2.2 Algebraic situation

A **chain complex** of abelian groups is a diagram $(C_\bullet, \partial)$ of the form

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

where the C_i are abelian groups and the ∂_n are group homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n. Then ∂_n are **boundary homomorphisms**. The elements in C_n are n**-chains**. Let

$$\mathrm{Z}_n = \ker \partial_n \subseteq C_n, \qquad \mathrm{B}_n = \operatorname{im} \partial_{n+1} \subseteq C_n.$$

The elements in Z_n are **cycles** and the elements in B_n are **boundaries**. Since $\partial_{n+1} \circ \partial_n = 0$, we have that $\mathrm{B}_n \subseteq \mathrm{Z}_n$. The n**-th homology group** of this chain complex is defined by

$$\mathrm{H}_n(C_\bullet, \partial) = \mathrm{Z}_n / \mathrm{B}_n.$$

So, by Lemma 2.1

$$\dots \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. The n**-th simplicial homology group** is

$$\mathrm{H}_n^\Delta(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Example. Let $X = \mathrm{S}^1$. Then

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \cong & & \cong & & \cong & & \\ & & 0 & & \mathbb{Z} & & \mathbb{Z} & & \end{array}.$$

- $\ker \partial_0 = \mathbb{Z}$ and $\operatorname{im} \partial_1 = 0$, so $\mathrm{H}_0^\Delta(X) \cong \mathbb{Z}$.
- $\ker \partial_1 = \Delta_1(X)$ and $\operatorname{im} \partial_2 = 0$, so $\mathrm{H}_1^\Delta(X) \cong \mathbb{Z}$.
- $\mathrm{H}_n^\Delta(X) = 0$ if $n \geq 2$.

Example. Let T be a torus. Then

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial_4} & \Delta_3(T) & \xrightarrow{\partial_3} & \Delta_2(T) & \xrightarrow{\partial_2} & \Delta_1(T) & \xrightarrow{\partial_1} & \Delta_0(T) & \xrightarrow{\partial_0} & 0 \\ & & \cong & & \cong & & \cong & & \cong & & \\ & & 0 & & \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V & & \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c & & \mathbb{Z} \cdot v & & \end{array}.$$

- $\ker \partial_0 = \mathbb{Z}$ and $\operatorname{im} \partial_1 = 0$, so $\mathrm{H}_0^\Delta(T) \cong \mathbb{Z}$.
- $\partial_2(U) = a + b - c$ and $\partial_2(V) = a + b - c$, and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$.

  $$\ker \partial_1 = \Delta_1(T), \qquad \operatorname{im} \partial_2 = \mathbb{Z} \cdot (a + b - c),$$

  so $\mathrm{H}_1^\Delta(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $\mathrm{H}_2^\Delta(T) \cong \mathbb{Z}$. [5]

Lecture 20 is a problems class. Lecture 20 Tuesday 26/02/19

[5] Exercise

2.3 Singular homology

2.3.1 Singular homology

Lecture 21 Wednesday 27/02/19

A **singular n-simplex** in a topological space X is a continuous map $\sigma : \Delta^n \to X$. Let $\mathrm{C}_n(X)$ be the free abelian group on the set of all singular simplices in X, that is the elements in $\mathrm{C}_n(X)$ are finite formal sums

$$\sum_i n_i \sigma_i, \qquad n_i \in \mathbb{Z},$$

where $\sigma_i : \Delta^n \to X$ are singular n-simplices. The elements in $\mathrm{C}_n(X)$ are called **singular n-chains**. Define a **boundary map**

$$\begin{array}{cccc} \partial_n & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_{n-1}(X) \\ & & \sigma & \longmapsto & \sum_{i=0}^n (-1)^i \sigma|_{[v_1, \dots, \widetilde{v_i}, \dots, v_n]} \end{array},$$

for a singular n-simplex σ. Extend it linearly to $\mathrm{C}_n(X)$.

Lemma 2.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. The same proof as for Lemma 2.1. □

We obtain a chain complex

$$\dots \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \mathrm{C}_0(X) \xrightarrow{\partial_0} 0.$$

Remark. Often we write ∂ instead of ∂_n.

We define the **n-th singular homology group** by

$$\mathrm{H}_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

An observation is that if X and Y are homeomorphic then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.

Proposition 2.3. *Let X be a topological space and $X = \bigcup_\alpha X_\alpha$ be the decomposition into its path-components. Then*

$$\mathrm{H}_n(X) \cong \bigoplus_\alpha \mathrm{H}_n(X_\alpha).$$

Proof. A singular n-simplex $\sigma : \Delta^n \to X$ has a path-connected image. So

$$\mathrm{C}_n(X) = \bigoplus_\alpha \mathrm{C}_n(X_\alpha).$$

The boundary maps ∂_n preserve this decomposition, so $\partial_n(\mathrm{C}_n(X_\alpha)) \subseteq \mathrm{C}_{n-1}(X_\alpha)$ implies that $\ker \partial_n$ and $\operatorname{im} \partial_{n+1}$ split as well as direct sums, so

$$\mathrm{H}_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1} \cong \bigoplus_\alpha \mathrm{H}_n(X_\alpha).$$

□

Proposition 2.4. *If X is a path-connected, and as always $X \neq \emptyset$, topological space, then*

$$\mathrm{H}_0(X) \cong \mathbb{Z}.$$

Hence for X arbitrary $\mathrm{H}_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component.

Proof. $\partial_0 = 0$, so $\mathrm{H}_0(X) = \mathrm{C}_0(X) / \operatorname{im} \partial_1$. Define

$$\begin{array}{cccc} \epsilon & : & \mathrm{C}_0(X) & \longrightarrow & \mathbb{Z} \\ & & \sum_i n_i \sigma_i & \longmapsto & \sum_i n_i \end{array}.$$

Then ϵ is surjective. It is enough to show that $\ker \epsilon = \operatorname{im} \partial_1$. This implies by the isomorphism theorem $\mathrm{H}_0(X) \cong \mathbb{Z}$. Let $\sigma : \Delta^1 \to X$ be a 1-simplex. Then

$$\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]},$$

so $\epsilon(\partial_1(\sigma)) = 0$, so $\operatorname{im} \partial_1 \subseteq \ker \epsilon$. On the other hand, $\epsilon(\sum_i n_i \sigma_i) = 0$ implies that $\sum_i n_i = 0$. The σ_i correspond to points $\sigma_i([v])$ in X. Choose a basepoint $x_0 \in X$ and let

$$\begin{array}{cccc} \sigma_0 & : & \Delta^0 & \longrightarrow & X \\ & & \Delta^0 & \longmapsto & x_0 \end{array}$$

be the singular 0-simplex. Let τ_i be a path from x_0 to $\sigma_i([v])$. Consider τ_i as a singular 1-simplex $\tau_i : [v_0, v_1] \to X$. We have $\partial_1 \circ \tau_i = \sigma_i - \sigma_0$, so

$$\partial_1\left(\sum_i n_i \tau_i\right) = \sum_i n_i(\sigma_i - \sigma_0) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i.$$

Thus $\ker \epsilon \subseteq \operatorname{im} \partial_1$. □

Proposition 2.5. *If X is a point, then*

$$\mathrm{H}_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}.$$

Proof. For each n there exists a unique singular n-simplex $\partial_n : \Delta^n \to X$, so $\mathrm{C}_n(X) \cong \mathbb{Z}$ for all n. Then

$$\partial_n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases},$$

so $\partial_n = 0$ if n is odd and ∂_n is an isomorphism if n is even, and

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \begin{sabstack}\end{sabstack} \| \wr & & \| \wr & & \\ \dots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & 0 \end{array},$$

so $\mathrm{H}_n = \ker \partial_n / \operatorname{im} \partial_{n+1} = 0$ if $n \geq 1$ and $\mathrm{H}_0(X) \cong \mathbb{Z}$. □

2.3.2 Reduced homology groups

The **reduced homology groups** $\widetilde{\mathrm{H}_n}(X)$ are the homology groups of the **augmented chain complex**

$$\dots \xrightarrow{\partial_2} \mathrm{C}_1(X) \xrightarrow{\partial_1} \mathrm{C}_0(X) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0,$$

where ϵ is as in proof of Proposition 2.4. Then

$$\mathrm{H}_n(X) \cong \widetilde{\mathrm{H}_n}(X), \qquad n \geq 1.$$

Seen in the proof of Proposition 2.4 that ϵ is surjective and $\epsilon \circ \partial_1 = 0$, so $\operatorname{im} \partial_1 \subseteq \ker \epsilon$, so ϵ induces a surjective homomorphism

$$\phi_\epsilon : \mathrm{H}_0(X) = \mathrm{C}_0(X) / \operatorname{im} \partial_1 \to \mathbb{Z}.$$

Then $\ker \phi_\epsilon = \ker \epsilon / \operatorname{im} \partial_1 = \widetilde{\mathrm{H}_0}(X)$, so $\mathrm{H}_0(X) / \widetilde{\mathrm{H}_0}(X) \cong \mathbb{Z}$, so

$$\mathrm{H}_0(X) \cong \widetilde{\mathrm{H}_0}(X) \oplus \mathbb{Z}.$$

2.4 Homotopy invariance

Lecture 22 Friday 01/03/19

Let $(A_\bullet, \partial)$ and $(B_\bullet, \partial)$ be two chain complexes. A **chain map** $f : (A_\bullet, \partial) \to (B_\bullet, \partial)$ is a collection of homomorphisms $f_n : A_n \to B_n$ such that $\partial \circ f_n = f_{n+1} \circ \partial$, that is the following diagram commutes.

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \dots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \dots \end{array}.$$

If X and Y are topological spaces and $f : X \to Y$ is a continuous map define the homomorphisms

$$\begin{array}{cccc} f_\# & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(Y) \\ & & \sigma : \Delta^n \to X & \longmapsto & f \circ \sigma : \Delta^n \to Y \end{array},$$

and extend it linearly to $\mathrm{C}_n(X)$. Then

$$(f_\# \circ \partial)(\sigma) = f_\# \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \right) = \sum_{i=0}^n (f \circ \sigma)|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} = (\partial \circ f_\#)(\sigma),$$

so $f_\# \circ \partial = \partial \circ f_\#$, so $f_\#$ defines a chain map

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial} & \mathrm{C}_{n+1}(X) & \xrightarrow{\partial} & \mathrm{C}_n(X) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(X) & \xrightarrow{\partial} & \dots \\ & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\ \dots & \xrightarrow{\partial} & \mathrm{C}_{n+1}(Y) & \xrightarrow{\partial} & \mathrm{C}_n(Y) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(Y) & \xrightarrow{\partial} & \dots \end{array}.$$

$f_\#$ maps cycles to cycles, since $\alpha \in \mathrm{C}_n(X)$ such that $\partial \circ \alpha = 0$, so

$$(\partial \circ f_\#)(\alpha) = (f_\# \circ \partial)(\alpha) = 0.$$

$f_\#$ maps boundaries to boundaries, since

$$f_\# \circ (\partial \circ \beta) = \partial \circ (f_\# \circ \beta).$$

$f_\#(\ker \partial_n) \subseteq \ker \partial_n$ and $f_\#(\operatorname{im} \partial_{n+1}) \subseteq \operatorname{im} \partial_{n+1}$, so $f_\#$ induces a homomorphism

$$f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

The following are observations.

- $X \xrightarrow{g} Y \xrightarrow{f} Z$, so $(f \circ g)_\# = f_\# \circ g_\#$, since

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z,$$

  so $f \circ (g \circ \sigma) = (f \circ g) \circ \sigma$, so $(f \circ g)_* = f_* \circ g_*$.
- $(\mathrm{id}_X)_* = \mathrm{id}_{\mathrm{H}_n(X)}$.

Theorem 2.6. *If two continuous maps $f, g : X \to Y$ are homotopic, then*

$$f_* = g_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

Corollary 2.7. *If $f : X \to Y$ is a homotopy equivalence, then*

$$f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a continuous map such that $f \circ g \cong \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Then $f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}$. Similarly $g_* \circ f_* = \mathrm{id}$, so f_* is an isomorphism. $\square$

Example.

$$\mathrm{H}_n\left(\mathbb{R}^k\right) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \widetilde{\mathrm{H}_n}\left(\mathbb{R}^k\right) = 0.$$

Proof of Theorem 2.6. Let $F : X \times \mathrm{I} \to Y$ be a homotopy from f to g and $\sigma : \Delta_n \to X$ be a singular n-simplex. Consider the map

$$\Delta^n \times \mathrm{I} \xrightarrow{\sigma \times \mathrm{I}} X \times \mathrm{I} \xrightarrow{F} Y.$$

Then $\Delta^n \times \mathrm{I}$ is not a simplex. But we can subdivide $\Delta^n \times \mathrm{I}$ into $(n+1)$ simplices. In general, we can decompose $\Delta^n \times \mathrm{I}$ into $n+1$ distinct $(n+1)$-simplices

$$[v_0, \dots, v_i, w_i, \dots, w_n], \qquad i = 0, \dots, n.$$

Define **prism-operators**

$$\begin{array}{rccc} P & : & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_{n+1}(Y) \\ & & \sigma & \longmapsto & \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_n]} \end{array},$$

for $\sigma : \Delta^n \to X$ a singular n-simplex, so

$$\begin{tikzcd} \dots \arrow[r, "\partial"] & \mathrm{C}_{n+1}(X) \arrow[r, "\partial"] & \mathrm{C}_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & \mathrm{C}_{n-1}(X) \arrow[r, "\partial"] \arrow[dl, "P"] & \dots \\ \dots \arrow[r, "\partial"] & \mathrm{C}_{n+1}(Y) \arrow[r, "\partial"] & \mathrm{C}_n(Y) \arrow[r, "\partial"] & \mathrm{C}_{n-1}(Y) \arrow[r, "\partial"] & \dots \end{tikzcd}.$$

Claim that

$$\partial \circ P = g_\# - f_\# - P \circ \partial,$$

if and only if $g_\# - f_\# = \partial \circ P + P \circ \partial$. The claim implies the theorem, since if $\alpha \in \mathrm{C}_n(X)$ is a cycle, then

$$g_\#(\alpha) - f_\#(\alpha) = (\partial \circ P)(\alpha) + (P \circ \partial)(\alpha) = (\partial \circ P)(\alpha),$$

so $g_\#(\alpha) - f_\#(\alpha)$ is a boundary. Thus $g_\#(\alpha)$ and $f_\#(\alpha)$ are in the same homology class, so $g_*([\alpha]) = f_*([\alpha])$, where $[\alpha]$ is the homology class of α. Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Then

$$\begin{aligned} (\partial \circ P)(\sigma) &= \partial \left(\sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_n]} \right) \\ &= \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} \\ &\quad + \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]}. \end{aligned}$$

If $i = j$ the two sums cancel except for

$$F \circ (\sigma \times \mathrm{id})|_{[\widehat{v_0},w_0,\dots,w_n]} = g \circ \sigma = g_\#(\sigma), \qquad -F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_n,\widehat{w_n}]} = -f \circ \sigma = -f_\#(\sigma).$$

The terms with $i \ne j$ sum up to $(P \circ \partial)(\sigma)$, since we have

$$\begin{aligned} (P \circ \partial)(\sigma) &= \sum_{j < i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]}. \end{aligned}$$

$\square$

Lecture 23 Tuesday 05/03/19

Remark. One can show that there are also induced homomorphisms

$$f_* : \widetilde{\mathrm{H}_n}(X) \to \widetilde{\mathrm{H}_n}(Y)$$

invariant under homotopy. [6]

[6] Exercise

2.5 Exact sequences and excision

2.5.1 Exact sequences

Let $A \subseteq X$ be a subspace. What is the relationship between $\mathrm{H}_n(A), \mathrm{H}_n(X), \mathrm{H}_n(X/A)$?

Definition. A sequence of group homomorphisms of abelian groups

$$\dots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} \dots$$

is **exact** at A_n if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$. The sequence is **exact** if it is exact at A_n for all n.

An observation is if the sequence is exact, then

- $\alpha_n \alpha_{n+1} = 0$, so exact sequences are chain complexes, and
- the homology groups of this chain complex are all trivial.

Example.

- $0 \to A \xrightarrow{\alpha} B$ is exact if and only if $\ker \alpha = 0$, if and only if α is injective.
- $A \xrightarrow{\alpha} B \to 0$ is exact if and only if $\operatorname{im} \alpha = B$, if and only if α is surjective.
- $0 \to A \xrightarrow{\alpha} B \to 0$ is exact if and only if α is an isomorphism.
- $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact if and only if α is injective, β is surjective, and $\ker \beta = \operatorname{im} \alpha$, hence β induces an isomorphism
$$C \cong B / \operatorname{im} \alpha = B/A.$$
This is called a **short exact sequence**.

Definition. Let X be a topological space and $A \subseteq X$. Then A is a **strong deformation retract** of X if there exists a retraction $r : X \to A$ such that r is homotopic to the identity, and $F : \mathrm{I} \times X \to X$ continuous such that

$$F(0, x) = x, \qquad F(1, x) = r(x), \qquad F(t, a) = a, \qquad x \in X, \qquad a \in A, \qquad t \in \mathrm{I}.$$

Let X be a topological space and $A \subseteq X$ a non-empty closed subspace. Then (X, A) is called a **good pair** if A has a neighbourhood in X that strongly deformation retracts to A.

Example.

- $\left(\mathrm{D}^n, \mathrm{S}^{n-1}\right)$ is a good pair, since S^{n-1} is a deformation retract of $\mathrm{D}^n \setminus \{0\}$.
- Let $A = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq [0, 1]$ then $([0, 1], A)$ is not a good pair.

Theorem 2.8. *Let (X, A) be a good pair, then there is an exact sequence*

$$\dots \to \widetilde{\mathrm{H}_1}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}_1}(X) \xrightarrow{j_*} \widetilde{\mathrm{H}_1}(X/A) \xrightarrow{\partial} \widetilde{\mathrm{H}_0}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}_0}(X) \xrightarrow{j_*} \widetilde{\mathrm{H}_0}(X/A) \to 0,$$

where $i : A \hookrightarrow X$ is the inclusion and $j : X \to X/A$ is the quotient.

Corollary 2.9.

$$\widetilde{\mathrm{H}_i}(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

Proof. $\left(\mathrm{D}^n, \mathrm{S}^{n-1}\right)$ is a good pair. Let $n > 0$. Recall that $\mathrm{D}^n / \mathrm{S}^{n-1} \cong \mathrm{S}^n$, so

$$\dots \longrightarrow \widetilde{\mathrm{H}_i}\left(\mathrm{S}^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_i}\left(\mathrm{D}^n\right)} \xrightarrow{j_*} \widetilde{\mathrm{H}_i}\left(\mathrm{S}^n\right) \xrightarrow{\partial} \widetilde{\mathrm{H}_{i-1}}\left(\mathrm{S}^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_{i-1}}\left(\mathrm{D}^n\right)} \xrightarrow{j_*} \widetilde{\mathrm{H}_{i-1}}\left(\mathrm{S}^n\right) \longrightarrow \dots .$$

Then $\widetilde{\mathrm{H}_i}\left(\mathrm{S}^n\right) \cong \widetilde{\mathrm{H}_{i-1}}\left(\mathrm{S}^{n-1}\right)$ for $i > 0$, so

$$\dots \longrightarrow \widetilde{\mathrm{H}_1}\left(\mathrm{S}^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_1}\left(\mathrm{D}^n\right)} \xrightarrow{j_*} \widetilde{\mathrm{H}_1}\left(\mathrm{S}^n\right) \xrightarrow{\partial} \widetilde{\mathrm{H}_0}\left(\mathrm{S}^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_0}\left(\mathrm{D}^n\right)} \xrightarrow{j_*} \widetilde{\mathrm{H}_0}\left(\mathrm{S}^n\right) \longrightarrow 0 .$$

$n > 0$ and $i > 0$, so $\widetilde{\mathrm{H}_i}\left(\mathrm{S}^n\right) \cong \widetilde{\mathrm{H}_{i-1}}\left(\mathrm{S}^{n-1}\right)$, and $\widetilde{\mathrm{H}_0}\left(\mathrm{S}^n\right) = 0$. We know that $\widetilde{\mathrm{H}_0}\left(\mathrm{S}^0\right) \cong \mathbb{Z}$ and $\widetilde{\mathrm{H}_n}\left(\mathrm{S}^0\right) = 0$, by Proposition 2.3 and Proposition 2.5. Doing induction on n,

$$\widetilde{\mathrm{H}_i}\left(\mathrm{S}^n\right) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

$\square$

Corollary 2.10. *There exists no retraction $r : \mathrm{D}^n \to \partial \mathrm{D}^n$.*

Proof. Assume there exists such an $r : \mathrm{D}^n \to \partial \mathrm{D}^n$. Let $i : \partial \mathrm{D}^n \to \mathrm{D}^n$. Then $ri = \mathrm{id}_{\partial \mathrm{D}^n}$, so $r_* i_* = (ri)_* = \mathrm{id}$, so

$$\underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{Z}}}{\widetilde{\mathrm{H}_{n-1}}\left(\partial \mathrm{D}^n\right)} \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_{n-1}}\left(\mathrm{D}^n\right)} \xrightarrow{r_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{Z}}}{\widetilde{\mathrm{H}_{n-1}}\left(\partial \mathrm{D}^n\right)} .$$

Thus $i_* = 0$ and $r_* = 0$, a contradiction. $\square$

Theorem 2.11 (Brouwer fixed point theorem)**.** *Every continuous map $f : \mathrm{D}^n \to \mathrm{D}^n$ has a fixed point.*

Proof. Assume there exists a fixed point then construct as in dimension two a retraction $\mathrm{D}^n \to \partial \mathrm{D}^n$, a contradiction to Corollary 2.10. $\square$

2.5.2 Relative homology groups

Let X be a topological space and $A \subseteq X$ be a subspace. Define

$$\mathrm{C}_n(X, A) = \mathrm{C}_n(X) / \mathrm{C}_n(A).$$

Let $\partial : \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ be the boundary map then $\partial(\sigma : \Delta^n \to A) \in \partial(\mathrm{C}_n(A)) \subseteq \mathrm{C}_{n-1}(A)$. So ∂ induces a homomorphism

$$\partial : \mathrm{C}_n(X, A) \to \mathrm{C}_{n-1}(X, A),$$

such that $\partial \circ \partial = 0$. This gives a chain complex

$$\dots \to \mathrm{C}_{n+1}(X, A) \xrightarrow{\partial} \mathrm{C}_n(X, A) \xrightarrow{\partial} \mathrm{C}_{n-1}(X, A) \to \dots .$$

- The homology groups $\mathrm{H}_n(X, A)$ of this complex are the **relative homology groups**.
- The **relative n-chains** are $\mathrm{C}_n(X, A)$.
- The **relative n-cycles** are $\ker \partial \subseteq \mathrm{C}_n(X, A)$, of the form $[\alpha]$ for $\alpha \in \mathrm{C}_n(X)$ such that $\partial(\alpha) \in \mathrm{C}_{n-1}(A)$.
- The **relative n-boundaries** are $\operatorname{im} \partial \subseteq \mathrm{C}_n(X, A)$, of the form $[\alpha]$ for $\alpha \in \mathrm{C}_n(X)$ such that $\alpha = \partial\beta + \gamma$ for $\beta \in \mathrm{C}_{n+1}(X)$ and $\gamma \in \mathrm{C}_n(A)$.

Lecture 24
Wednesday
06/03/19

A **short exact sequence of chain complexes** is

$$0 \to (A_\bullet, \partial) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \to 0,$$

for i and j chain maps, where

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$$

is a short exact sequence for all n, so

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "\partial"] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r, "\partial"] \arrow[d, "i"] & \dots \\
\dots \arrow[r, "\partial"] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r, "\partial"] \arrow[d, "j"] & \dots \\
\dots \arrow[r, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

A short exact sequence of chain complexes always yields a **long exact sequence** of homology groups

$$\dots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{j_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{j_*} \mathrm{H}_{n-1}(C) \to \dots.$$

This is the **zig-zag lemma**. First we construct the **connecting map** $\partial : \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$. Let $c \in C_n$ be a cycle.

- j is surjective, so $c = j(b)$ for some $b \in B_n$.
- $j(\partial(b)) = \partial(j(b)) = \partial c = 0$, so $\partial b \in \ker j \subseteq B_{n-1}$, so $\partial(b) = i(a)$ for some $a \in A_{n-1}$, by exactness.
- $\partial(a) = 0$, since $i(\partial(a)) = \partial(i(a)) = \partial(\partial(b)) = 0$ and i is injective, so $\partial(a) = 0$.

$$\begin{tikzcd}
 & a \in A_{n-1} \arrow[d, "i"] \\
b \in B_n \arrow[r, "\partial"] \arrow[d, "j"] & \in \partial(b) \in B_{n-1} \\
c \in C_n &
\end{tikzcd}.$$

Define

$$\begin{array}{cccc} \partial & : & \mathrm{H}_n(C) & \longrightarrow & \mathrm{H}_{n-1}(A) \\ & & [c] & \longmapsto & [a] \end{array}.$$

This is well-defined.

- a is uniquely determined by $\partial(b)$ because i is injective.
- If we choose b' instead of b, then $j(b') = j(b)$, so $j(b' - b) = j(b') - j(b) = 0$, so $b' - b \in \ker j = \operatorname{im} i$, hence $b' - b = i(a')$ for some $a' \in A_n$, so $b' = b + i(a')$. If we replace b by $b' = b + i(a')$ this corresponds to replacing a by $a + \partial(a')$, because

  $$i(a + \partial(a')) = i(a) + i(\partial(a')) = \partial(b) + \partial(i(a')) = \partial(b + i(a')),$$

  and $[a] = [a + \partial(a')]$.
- A different choice of c in its homology class has the form $c + \partial(c')$ for some $c' \in C_{n+1}$. Let $b' \in B_{n+1}$ such that $j(b') = c'$. Then

  $$c + \partial(c') = c + \partial(j(b')) = j(b) + j(\partial(b')) = j(b + \partial(b')),$$

  so b is replaced by $b + \partial(b')$ but $\partial(b) = \partial(b + \partial b')$, so $\partial(b)$ is unchanged and hence a is unchanged.

The map $\partial : \mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$ is a homomorphism, since if $\partial([c_1]) = [a_1]$ and $\partial([c_2]) = [a_2]$ via elements b_1 and b_2 in B_n, then

$$j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2, \qquad i(a_1 + a_2) = i(a_1) + i(a_2) = \partial(b_1) + \partial(b_2) = \partial(b_1 + b_2),$$

so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$.

Theorem 2.12. *The sequence*

$$\dots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B) \xrightarrow{j_*} \mathrm{H}_n(C) \xrightarrow{\partial} \mathrm{H}_{n-1}(A) \xrightarrow{i_*} \mathrm{H}_{n-1}(B) \xrightarrow{j_*} \mathrm{H}_{n-1}(C) \to \dots$$

is exact.

Proof. Diagram chase, see Hatcher. ☐

Let i be the inclusion and j be the quotient.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{\partial} & \mathrm{C}_n(A) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(A) & \xrightarrow{\partial} & \dots \\
 & & \downarrow i & & \downarrow i & & \\
\dots & \xrightarrow{\partial} & \mathrm{C}_n(X) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(X) & \xrightarrow{\partial} & \dots \\
 & & \downarrow j & & \downarrow j & & \\
\dots & \xrightarrow{\partial} & \mathrm{C}_n(X,A) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(X,A) & \xrightarrow{\partial} & \dots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}.$$

This diagram commutes, so this is a short exact sequence of chain complexes. Zig-zag gives a long exact sequence of homology groups

$$\dots \to \mathrm{H}_1(A) \xrightarrow{i_*} \mathrm{H}_1(X) \xrightarrow{j_*} \mathrm{H}_1(X,A) \xrightarrow{\partial} \mathrm{H}_0(A) \xrightarrow{i_*} \mathrm{H}_0(X) \xrightarrow{j_*} \mathrm{H}_0(X,A) \to 0.$$

What is $\partial : \mathrm{H}_n(X,A) \to \mathrm{H}_{n-1}(A)$? If $[a] \in \mathrm{H}_n(X,A)$ is represented by a cycle $\alpha \in \mathrm{C}_n(X)$, then $\partial([\alpha])$ is the class of the cycle $\partial(\alpha)$, so $\partial([\alpha]) = [\partial(\alpha)]$. We also obtain a short exact sequence of the augmented chain complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{C}_1(A) & \longrightarrow & \mathrm{C}_0(A) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{C}_1(X) & \longrightarrow & \mathrm{C}_0(X) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{C}_1(X,A) & \longrightarrow & \mathrm{C}_0(X,A) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array},$$

so if $A \neq \emptyset$, then $\widetilde{\mathrm{H}_n}(X,A) = \mathrm{H}_n(X,A)$ for all n. We also have a long exact sequence

$$\dots \to \widetilde{\mathrm{H}_n}(A) \to \widetilde{\mathrm{H}_n}(X) \to \widetilde{\mathrm{H}_n}(X,A) \to \widetilde{\mathrm{H}_{n-1}}(A) \to \widetilde{\mathrm{H}_{n-1}}(X) \to \widetilde{\mathrm{H}_{n-1}}(X,A) \to \dots.$$

An observation is if $x_0 \in X$ then $\mathrm{H}_n(X,x_0) \cong \widetilde{\mathrm{H}_n}(X)$ for all n. Another observation is that a continuous map $f : X \to Y$ such that $f(A) \subseteq B$ induces a chain map

$$f_\# : \mathrm{C}_n(X,A) \to \mathrm{C}_n(Y,B),$$

since $f_\# : \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ maps $\mathrm{C}_n(A)$ to $\mathrm{C}_n(B)$ so it is well-defined on the quotient, and hence homomorphisms

$$f_* : \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B).$$

This is functorial, so $(f \circ g)_* = f_* \circ g_*$.

Lecture 25
Friday
08/03/19

Definition. A **homotopy** between two maps

$$f, g : (X, A) \to (Y, B)$$

is a continuous map $F : \mathrm{I} \times X \to Y$ such that

$$F(0, x) = f(x), \qquad F(1, x) = g(x), \qquad F(s, a) \in B, \qquad x \in X, \qquad s \in \mathrm{I}, \qquad a \in A.$$

Proposition 2.13. *If*

$$f, g : (X, A) \to (Y, B)$$

are homotopic, then

$$f_* = g_* : \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B).$$

Proof. Analogous to proof of Theorem 2.6. Prism operator $P : \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)$ maps $\mathrm{C}_n(A)$ to $\mathrm{C}_n(B)$ so it induces a map

$$P' : \mathrm{C}_n(X) / \mathrm{C}_n(A) \to \mathrm{C}_{n+1}(Y) / \mathrm{C}_{n+1}(B),$$

and $\partial P' + P' \partial = g_\# - f_\#$, so $f_* = g_*$. ☐

Let (X, A, B) be a triple for X a topological space and $B \subset A \subset X$, so

$$(A, B) \to (X, B) \to (X, A).$$

There is a short exact sequence of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_n(A, B) & \longrightarrow & \mathrm{C}_n(X, B) & \longrightarrow & \mathrm{C}_n(X, A) & \longrightarrow & 0 \\ & & \cong & & \cong & & \cong & & \\ & & \mathrm{C}_n(A) / \mathrm{C}_n(B) & & \mathrm{C}_n(X) / \mathrm{C}_n(B) & & \mathrm{C}_n(X) / \mathrm{C}_n(A) & & \end{array},$$

so there is a long exact sequence

$$\dots \to \mathrm{H}_n(A, B) \to \mathrm{H}_n(X, B) \to \mathrm{H}_n(X, A) \to \mathrm{H}_{n-1}(A, B) \to \mathrm{H}_{n-1}(X, B) \to \mathrm{H}_{n-1}(X, A) \to \dots.$$

2.5.3 Excision

Theorem 2.14 (Excision)**.** *Let X be a topological space and $Z \subset A \subset X$ be subspaces such that the closure $\overline{Z}$ of Z is contained in the interior $\mathring{A}$ of A. Then the inclusion*

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces isomorphisms

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} \mathrm{H}_n(X, A),$$

for all n. Equivalently, let $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$. Then the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces isomorphisms

$$\mathrm{H}_n(B, A \cap B) \xrightarrow{\sim} \mathrm{H}_n(X, A),$$

for all n.

Why equivalent? Set $B = X \setminus Z$ and $Z = X \setminus B$. Then $A \cap B = A \setminus Z$ and $\overline{Z} = X \setminus \mathring{B}$. Then $\overline{Z} \subseteq \mathring{A}$ if and only if $X = \mathring{A} \cup \mathring{B}$.

Proof. Hatcher page 119 to 124. ☐

Proposition 2.15. *Let (X, A) be a good pair. Then the quotient map*

$$q : (X, A) \to (X/A, A/A)$$

induces isomorphisms

$$q_* : \mathrm{H}_n(X, A) \xrightarrow{\sim} \mathrm{H}_n(X/A, A/A) \cong \widetilde{\mathrm{H}_n}(X/A),$$

for all n.

Proof. Let $V \subseteq X$ be a neighbourhood of A that strongly deformation retracts to A. Then (V, A) is homotopy equivalent to (A, A), so

$$\mathrm{H}_n(V, A) \cong \mathrm{H}_n(A, A) = 0.$$

The triple (X, V, A) where $A \subset V \subset X$ induces a long exact sequence

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}_n(V, A) & \longrightarrow & \mathrm{H}_n(X, A) & \longrightarrow & \mathrm{H}_n(X, V) & \longrightarrow & \mathrm{H}_{n-1}(V, A) & \longrightarrow & \dots \\ & & \wr\| & & & & & & \wr\| & & \\ & & 0 & & & & & & 0 & & \end{array},$$

so

$$\mathrm{H}_n(X, A) \cong \mathrm{H}_n(X, V).$$

The same with the triple $(X/A, V/A, A/A)$, so again

$$\mathrm{H}_n(V/A, A/A) \cong \mathrm{H}_n(A/A, A/A) = 0.$$

This gives a long exact sequence

$$\mathrm{H}_n(X/A, A/A) \cong \mathrm{H}_n(X/A, V/A).$$

Consider the diagram

$$\begin{array}{ccccc} \mathrm{H}_n(X, A) & \xrightarrow{\sim} & \mathrm{H}_n(X, V) & \xleftarrow[\alpha]{\sim} & \mathrm{H}_n(X \setminus A, V \setminus A) \\ \downarrow q_* & & \downarrow q_* & & \sim \downarrow j \\ \mathrm{H}_n(X/A, A/A) & \xrightarrow{\sim} & \mathrm{H}_n(X/A, V/A) & \xleftarrow[\beta]{\sim} & \mathrm{H}_n(X/A \setminus A/A, V/A \setminus A/A) \end{array}.$$

- This diagram commutes.
- $q : X \to X/A$ induces a homeomorphism $X \setminus A \to X/A \setminus A/A$, so j is an isomorphism.
- α and β are isomorphisms by the excision theorem.

Thus

$$q_* : \mathrm{H}_n(X, A) \to \mathrm{H}_n(X/A, A/A)$$

is an isomorphism. ☐

Proof of Theorem 2.8. Long exact sequence of pair (X, A) with reduced homology

$$\dots \to \widetilde{\mathrm{H}_n}(A) \to \widetilde{\mathrm{H}_n}(X) \to \widetilde{\mathrm{H}_n}(X, A) \to \widetilde{\mathrm{H}_{n-1}}(A) \to \widetilde{\mathrm{H}_{n-1}}(X) \to \widetilde{\mathrm{H}_{n-1}}(X, A) \to \dots,$$

so

$$\widetilde{\mathrm{H}_n}(X, A) = \mathrm{H}_n(X, A) \cong \widetilde{\mathrm{H}_n}(X/A),$$

by last time. ☐

Lecture 26
Tuesday
12/03/19

Corollary 2.16. *Let $\{X_\alpha\}$ for $\alpha \in A$ be a collection of topological spaces and $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair, for all $\alpha \in A$. Let $\bigvee_\alpha X_\alpha$ be the wedge sum with respect to the points x_α. Then there is an isomorphism*

$$\widetilde{\mathrm{H}_n}\left(\bigsqcup_\alpha X_\alpha\right) \cong \bigoplus_\alpha \widetilde{\mathrm{H}_n}(X_\alpha) \xrightarrow{\sim} \widetilde{\mathrm{H}_n}\left(\bigvee_\alpha X_\alpha\right).$$

Proof. $(X,A) = \left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\}\right)$ is a good pair, so Proposition 2.15 implies that

$$\mathrm{H}_n(X,A) \cong \mathrm{H}_n\left(\bigvee_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\} / \bigsqcup_\alpha \{x_\alpha\}\right) \cong \widetilde{\mathrm{H}_n}\left(\bigvee_\alpha X_\alpha\right),$$

and

$$\mathrm{H}_n(X,A) \cong \bigoplus_\alpha \mathrm{H}_n(X_\alpha, x_\alpha) \cong \bigoplus_\alpha \widetilde{\mathrm{H}_n}(X_\alpha).$$

$\square$

Example.

$$\widetilde{\mathrm{H}_n}\left(\mathrm{S}^1 \vee \mathrm{S}^1\right) \cong \widetilde{\mathrm{H}_n}\left(\mathrm{S}^1\right) \oplus \widetilde{\mathrm{H}_n}\left(\mathrm{S}^1\right) \cong \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}.$$

$$\widetilde{\mathrm{H}_n}\left(\mathrm{S}^1 \vee \mathrm{S}^1 \vee \mathrm{S}^2\right) \cong \widetilde{\mathrm{H}_n}\left(\mathrm{S}^1\right) \oplus \widetilde{\mathrm{H}_n}\left(\mathrm{S}^1\right) \oplus \widetilde{\mathrm{H}_n}\left(\mathrm{S}^2\right) \cong \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

Recall that

$$\mathrm{H}_n^\Delta\left(\mathrm{S}^1 \times \mathrm{S}^1\right) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

We will see that singular and simplicial homology coincide in Appendix A.2, so $\mathrm{S}^1 \vee \mathrm{S}^1 \vee \mathrm{S}^2$ and $\mathrm{S}^1 \times \mathrm{S}^1$ have isomorphic homology groups, but they are not homotopy equivalent.

Theorem 2.17 (Invariance of dimension)**.** *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, non-empty. If U and V are homeomorphic, then $m = n$.*

Proof. For $x \in U$ set $A = \mathbb{R}^m \setminus \{x\}$ and $B = U$. Excision implies that

$$\mathrm{H}_k(U, U \setminus \{x\}) \cong \mathrm{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}).$$

Long exact sequence of a pair implies that

$$\begin{array}{ccccccccc} \dots \longrightarrow & \widetilde{\mathrm{H}_k}(\mathbb{R}^m) & \longrightarrow & \widetilde{\mathrm{H}_k}(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) & \longrightarrow & \widetilde{\mathrm{H}_{k-1}}(\mathbb{R}^m \setminus \{x\}) & \longrightarrow & \widetilde{\mathrm{H}_{k-1}}(\mathbb{R}^m) & \longrightarrow \dots, \\ & \cong & & & & & & \cong & \\ & 0 & & & & & & 0 & \end{array}$$

so $\mathrm{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{\mathrm{H}_{k-1}}(\mathbb{R}^m \setminus \{x\})$. Then $\mathbb{R}^m \setminus \{x\}$ deformation retracts to S^{m-1}, so

$$\mathrm{H}_k(U, U \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = m \\ 0 & \text{otherwise} \end{cases}.$$

Similarly

$$\mathrm{H}_k(V, V \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}.$$

Let $h : U \to V$ be a homeomorphism then this induces isomorphisms

$$h_* : \mathrm{H}_k(U, U \setminus \{x\}) \to \mathrm{H}_k(V, V \setminus \{h(x)\}),$$

for all k, so $m = n$. $\square$

2.5.4 Naturality

Proposition 2.18 (Naturality of connecting homomorphisms)**.** *Let*

$$(A_\bullet, \partial), (B_\bullet, \partial), (C_\bullet, \partial), (A'_\bullet, \partial), (B'_\bullet, \partial), (C'_\bullet, \partial)$$

be chain complexes. Consider a commutative diagram of chain maps

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "i"] \arrow[d, "\alpha"] & B_\bullet \arrow[r, "j"] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A'_\bullet \arrow[r, "i'"] & B'_\bullet \arrow[r, "j'"] & C'_\bullet \arrow[r] & 0 \end{tikzcd},$$

where the rows are short exact sequences. Then the induced diagram

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_n(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & \mathrm{H}_{n-1}(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & \mathrm{H}_{n-1}(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & \mathrm{H}_{n-1}(C) \arrow[r] \arrow[d, "\gamma_*"] & \dots \\ \dots \arrow[r] & \mathrm{H}_n(A') \arrow[r, "i'_*"] & \mathrm{H}_n(B') \arrow[r, "j'_*"] & \mathrm{H}_n(C') \arrow[r, "\partial"] & \mathrm{H}_{n-1}(A') \arrow[r, "i'_*"] & \mathrm{H}_{n-1}(B') \arrow[r, "j'_*"] & \mathrm{H}_{n-1}(C') \arrow[r] & \dots \end{tikzcd}$$

is commutative.

Proof. The first two squares commute by functoriality.

$$\begin{array}{cccc} \partial & : & \mathrm{H}_n(C) & \longrightarrow & \mathrm{H}_{n-1}(A) \\ & & [c] & \longmapsto & [a] \end{array},$$

so

$$\begin{tikzcd} & a \in A_{n-1} \arrow[d, "i"] \\ b \in B_n \arrow[r, "\partial"] \arrow[d, "j"] & \in \partial(b) \in B_{n-1} \\ c \in C_n & \end{tikzcd}.$$

Then $\gamma(c) = \gamma(j(b)) = j'(\beta(b))$ and $i'(\alpha(a)) = \beta(i(a)) = \beta(\partial(b)) = \partial(\beta(b))$, so

$$\begin{tikzcd} & \alpha(a) \in A'_{n-1} \arrow[d, "i'"] \\ \beta(b) \in B'_n \arrow[r, "\partial"] \arrow[d, "j'"] & \in \partial(\beta(b)) \in B'_{n-1} \\ \gamma(c) \in C'_n & \end{tikzcd},$$

so $\partial[\gamma(c)] = [\alpha(a)]$ and hence $\partial(\gamma_*[c]) = \alpha_*[a] = \alpha_*(\partial[c])$. ☐

2.6 Mayer-Vietoris sequences

2.6.1 The Mayer-Vietoris sequence

The main ingredient of the proof of the excision theorem is **barycentric subdivision**. Let X be a topological space and $\mathcal{U} = \{U_i\}$ be a collection of subspaces whose interiors form an open cover of X. Define $\mathrm{C}_n^{\mathcal{U}} \subseteq \mathrm{C}_n(X)$ as the subgroup of all chains of the form $\sum_i n_i\sigma_i$ such that the image of σ_i is contained in some $U_j \in \mathcal{U}$. Then $\partial : \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ satisfies $\partial\left(\mathrm{C}_n^{\mathcal{U}}(X)\right) \subseteq \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ so the $\mathrm{C}_n^{\mathcal{U}}(X)$ define a chain complex. Let $\mathrm{H}_n^{\mathcal{U}}(X)$ be the homology groups with respect to this chain complex.

Proposition 2.19. *The inclusion* $i : \mathrm{C}_n^{\mathcal{U}}(X) \hookrightarrow \mathrm{C}_n(X)$ *induces isomorphisms* $\mathrm{H}_n^{\mathcal{U}}(X) \cong \mathrm{H}_n(X)$ *for all* n.

Proof. Hatcher page 119. ☐

Notation. If $\mathcal{U} = \{A, B\}$ we write $\mathrm{C}_n(A+B)$ instead of $\mathrm{C}_n^{\mathcal{U}}(X)$.

Theorem 2.20 (Mayer-Vietoris sequence). *Let X be a topological space, $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$, and*
$$i_1 : A \cap B \hookrightarrow A, \qquad i_2 : A \cap B \hookrightarrow B, \qquad j_1 : A \hookrightarrow X, \qquad j_2 : B \hookrightarrow X$$
be inclusions. Then there is an exact sequence
$$\dots \to \mathrm{H}_1(A \cap B) \xrightarrow{\Phi} \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) \xrightarrow{\Psi} \mathrm{H}_1(X) \xrightarrow{\partial} \mathrm{H}_0(A \cap B) \xrightarrow{\Phi} \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) \xrightarrow{\Psi} \mathrm{H}_0(X) \to 0,$$
where $\Phi(x) = (i_{1*}(x), -i_{2*}(x))$, $\Psi(x, y) = j_{1*}(x) + j_{2*}(y)$, and ∂ is the connecting homomorphism.

Proof. Let a sequence of chain complexes be
$$0 \to \mathrm{C}_n(A \cap B) \xrightarrow{\phi} \mathrm{C}_n(A) \oplus \mathrm{C}_n(B) \xrightarrow{\psi} \mathrm{C}_n(A + B) \to 0,$$
where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$.

- ϕ is injective.
- $\operatorname{im} \phi \subseteq \ker \psi$.
- If $(x, y) \in \ker \psi$, then $y = -x$, and $x \in \mathrm{C}_n(A)$ and $y \in \mathrm{C}_n(B)$, so $x \in \mathrm{C}_n(A \cap B)$, so $\ker \psi \subseteq \operatorname{im} \phi$.
- ψ is surjective by the definition of $\mathrm{C}_n(A + B)$.

So this is a short exact sequence of chain complexes. This induces a long exact sequence of homology groups
$$\begin{array}{ccccccccccc} \dots \to \mathrm{H}_1(A \cap B) & \xrightarrow{\Phi} & \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) & \xrightarrow{\Psi} & \mathrm{H}_1^{A+B}(X) & \xrightarrow{\partial} & \mathrm{H}_0(A \cap B) & \xrightarrow{\Phi} & \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) & \xrightarrow{\Psi} & \mathrm{H}_0^{A+B}(X) \to 0 \\ & & & & \cong & & & & & & \cong \\ & & & & \mathrm{H}_1(X) & & & & & & \mathrm{H}_0(X) \end{array},$$
by barycentric division. ∎

Lecture 27 Wednesday 13/03/19

If $A \cap B \neq \emptyset$ we can augment these chain complexes and obtain a short exact sequence between these augmented chain complexes
$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & \mathrm{C}_0(A \cap B) \arrow[r, "\phi"] \arrow[d, "\epsilon"] & \mathrm{C}_0(A) \oplus \mathrm{C}_0(B) \arrow[r, "\psi"] \arrow[d, "\epsilon"] & \mathrm{C}_0(A + B) \arrow[r] \arrow[d, "\epsilon"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}.$$
This induces a long exact sequence of homology groups
$$\dots \to \widetilde{\mathrm{H}_1}(A \cap B) \xrightarrow{\Phi} \widetilde{\mathrm{H}_1}(A) \oplus \widetilde{\mathrm{H}_1}(B) \xrightarrow{\Psi} \widetilde{\mathrm{H}_1}(X) \xrightarrow{\partial} \widetilde{\mathrm{H}_0}(A \cap B) \xrightarrow{\Phi} \widetilde{\mathrm{H}_0}(A) \oplus \widetilde{\mathrm{H}_0}(B) \xrightarrow{\Psi} \widetilde{\mathrm{H}_0}(X) \to 0.$$
This is the Mayer-Vietoris sequence for reduced homology groups.

Note. This is the same as in the non-reduced case, but we need to assume that $A \cap B \neq \emptyset$.

An observation is that if $A \cap B$ is path-connected, then $\widetilde{\mathrm{H}_0}(A \cap B) = 0$, so we have an exact sequence
$$\begin{array}{ccccccccc} \dots \longrightarrow \widetilde{\mathrm{H}_1}(A \cap B) & \xrightarrow{\Phi} & \widetilde{\mathrm{H}_1}(A) \oplus \widetilde{\mathrm{H}_1}(B) & \xrightarrow{\Psi} & \widetilde{\mathrm{H}_1}(X) & \xrightarrow{\partial} & \widetilde{\mathrm{H}_0}(A \cap B) & \longrightarrow & \dots \\ & & & & & & \cong & & \\ & & & & & & 0 & & \end{array}.$$
Thus
$$\mathrm{H}_1(X) \cong \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) / \Phi(\mathrm{H}_1(A \cap B)).$$
This is the abelianised version of the theorem of Seifert-van Kampen.

Example. Let $X = \mathrm{S}^n \subseteq \mathbb{R}^{n+1}$ and let $x \in \mathrm{S}^n$. Define $A = \mathrm{S}^n \setminus \{x\}$ and $B = \mathrm{S}^n \setminus \{-x\}$. Then A and B are contractible, so $\widetilde{\mathrm{H}_n}(A) = \widetilde{\mathrm{H}_n}(B) = 0$ for all n, and $A \cap B$ deformation retracts to S^{n-1}. Mayer-Vietoris implies that

$$\begin{array}{ccccccccc} \dots \longrightarrow & \widetilde{\mathrm{H}_i}(A) \oplus \widetilde{\mathrm{H}_i}(B) & \longrightarrow & \widetilde{\mathrm{H}_i}(X) & \longrightarrow & \widetilde{\mathrm{H}_{i-1}}(A \cap B) & \longrightarrow & \widetilde{\mathrm{H}_{i-1}}(A) \oplus \widetilde{\mathrm{H}_{i-1}}(B) & \longrightarrow \dots \\ & \cong & & & & \cong & & \cong & \\ & 0 & & & & \widetilde{\mathrm{H}_{i-1}}(\mathrm{S}^{n-1}) & & 0 & \end{array},$$

so $\widetilde{\mathrm{H}_i}(\mathrm{S}^n) \cong \widetilde{\mathrm{H}_{i-1}}(\mathrm{S}^{n-1})$ for $n \ge 1$. We know $\widetilde{\mathrm{H}_0}(\mathrm{S}^0) \cong \mathbb{Z}$ and $\widetilde{\mathrm{H}_0}(\mathrm{S}^n) = 0$ for $n \ge 1$, so induction and knowledge on $\mathrm{H}_n(\mathrm{S}^0)$ implies that

$$\widetilde{\mathrm{H}_k}(\mathrm{S}^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}.$$

Example. Let $U, V \subseteq \mathbb{R}^n$ be two path-connected open subsets such that $U \cup V = \mathbb{R}^n$. Then $U \cap V$ is path-connected as well. It is enough to show that $\mathrm{H}_0(U \cap V) \cong \mathbb{Z}$, if and only if $\widetilde{\mathrm{H}_0}(U \cap V) = 0$. Then $U \cap V \ne \emptyset$ because $\mathbb{R}^n$ is connected, and U and V are open, so $\mathring{U} = U$ and $\mathring{V} = V$, so $\mathring{U} \cup \mathring{V} = \mathbb{R}^n$. Mayer-Vietoris long exact sequence for reduced homology groups implies that

$$\begin{array}{cccccccccc} \dots \longrightarrow & \widetilde{\mathrm{H}_1}(\mathbb{R}^n) & \longrightarrow & \widetilde{\mathrm{H}_0}(U \cap V) & \longrightarrow & \widetilde{\mathrm{H}_0}(U) \oplus \widetilde{\mathrm{H}_0}(V) & \longrightarrow & \widetilde{\mathrm{H}_0}(\mathbb{R}^n) & \longrightarrow & 0 \\ & \cong & & & & \cong & & \cong & & \\ & 0 & & & & 0 & & 0 & & \end{array},$$

since $\mathbb{R}^n$ is contractible, so $\widetilde{\mathrm{H}_k}(\mathbb{R}^n) = 0$ for all k, and $\widetilde{\mathrm{H}_0}(U) = \widetilde{\mathrm{H}_0}(V) = 0$, because U and V are path-connected. Thus $\widetilde{\mathrm{H}_0}(U \cap V) = 0$.

2.6.2 Classical applications

Definition. Let X and Y be topological spaces. A continuous map $\phi : X \to Y$ is an **embedding** if it is a homeomorphism to its image.

Example. If X is compact and Y is Hausdorff, and $\phi : X \to Y$ is a continuous and injective map, then ϕ is an embedding, since $\phi : X \to \phi(X)$ is continuous and bijective and $\phi(X)$ is Hausdorff, so worksheet 1 implies that ϕ is a homeomorphism $X \to \phi(X)$.

Proposition 2.21.

1. *Let $h : \mathrm{D}^k \to \mathrm{S}^n$ be an embedding, then $\widetilde{\mathrm{H}_i}(\mathrm{S}^n \setminus h(\mathrm{D}^k)) = 0$ for all i.*
2. *Let $h : \mathrm{S}^k \to \mathrm{S}^n$ be an embedding, with $k < n$, then*

$$\widetilde{\mathrm{H}_i}(\mathrm{S}^n \setminus h(\mathrm{S}^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \textit{otherwise} \end{cases}.$$

Corollary 2.22. *Let $h : \mathrm{S}^1 \to \mathrm{S}^2$ be an embedding. Then $\mathrm{S}^2 \setminus h(\mathrm{S}^1)$ consists of exactly two path-components.*

Proof. $\widetilde{\mathrm{H}_0}(\mathrm{S}^2 \setminus h(\mathrm{S}^1)) \cong \mathbb{Z}$ by Proposition 2.21. ☐

Corollary 2.23 (Jordan curve theorem)**.** *Let $h : \mathrm{S}^1 \to \mathbb{R}^2$ be an embedding. Then $\mathbb{R}^2 \setminus h(\mathrm{S}^1)$ consists of exactly two path-components.*

Proof. $\mathbb{R}^2$ is homeomorphic to $\mathrm{S}^2 \setminus \{x\}$, by stereographic projection. ☐

Similarly, $\mathbb{R}^n \setminus h(\mathrm{S}^{n-1})$ consists of exactly two path-components.

Proof of Proposition 2.21. Lecture 28 Friday 15/03/19

1. Induction on k.

   $k = 0$. $\mathrm{S}^n \setminus h\left(\mathrm{D}^0\right) \cong \mathbb{R}^n$, so $\widetilde{\mathrm{H}_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{D}^n\right)\right) = 0$ for all n.

   $k-1 \mapsto k$. Let $h : \mathrm{D}^k \to \mathrm{S}^n$ be an embedding. Replace D^k by I^k. For a contradiction, assume there is a cycle α in $\mathrm{S}^n \setminus h\left(\mathrm{I}^k\right)$ that is not a boundary in $\mathrm{S}^n \setminus h\left(\mathrm{I}^k\right)$. Claim that there is a nested sequence of intervals
   $$[0,1] = I_0 \supseteq I_1 \supseteq \dots,$$
   such that I_i is of length $\frac{1}{2^i}$ and such that α is a cycle in $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times I_i\right)$ but not a boundary in $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times I_i\right)$. Let $A = \mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left[0, \frac{1}{2}\right]\right)$ and $B = \mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left[\frac{1}{2}, 1\right]\right)$, so $A \cap B = \mathrm{S}^n \setminus h\left(\mathrm{I}^k\right)$ and $A \cup B = \mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left\{\frac{1}{2}\right\}\right)$. Induction hypothesis implies that $\widetilde{\mathrm{H}_j}\left(A \cup B\right) = 0$ for all j. Mayer-Vietoris implies that
   $$\dots \longrightarrow \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_{j+1}}\left(A \cup B\right)} \longrightarrow \widetilde{\mathrm{H}_j}\left(A \cap B\right) \xrightarrow{\sim} \widetilde{\mathrm{H}_j}\left(A\right) \oplus \widetilde{\mathrm{H}_j}\left(B\right) \longrightarrow \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_j}\left(A \cup B\right)} \longrightarrow \dots,$$
   so
   $$\widetilde{\mathrm{H}_j}\left(\mathrm{S}^n \setminus h\left(\mathrm{I}^k\right)\right) \cong \widetilde{\mathrm{H}_j}\left(\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left[0, \tfrac{1}{2}\right]\right)\right) \oplus \widetilde{\mathrm{H}_j}\left(\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left[\tfrac{1}{2}, 1\right]\right)\right).$$
   Hence α is a cycle but not a boundary in $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left[0, \frac{1}{2}\right]\right)$ or $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \left[\frac{1}{2}, 1\right]\right)$. This gives us I_1. Iterating, this proves the claim. By induction, α is a boundary of some cycle β in $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \{x\}\right)$ for any $x \in \mathrm{I}$, so in particular, for $\{x\} = \bigcap_i I_i$. Then $\beta = \sum_i n_i \sigma_i$ is a sum of finitely many singular simplices. The images of the σ_i are compact. But $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times I_i\right)$ form an open cover of $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times \{x\}\right)$. So, by compactness, β is a chain in $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times I_i\right)$ for some i. Thus α is a boundary in $\mathrm{S}^n \setminus h\left(\mathrm{I}^{k-1} \times I_i\right)$, a contradiction.

2. Induction on k.

   $k = 0$. $\mathrm{S}^n \setminus h\left(\mathrm{S}^0\right) \cong \mathrm{S}^{n-1} \times \mathbb{R}$, so
   $$\widetilde{\mathrm{H}_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^0\right)\right) \cong \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{otherwise} \end{cases}.$$

   $k-1 \mapsto k$. Let $h : \mathrm{S}^k \to \mathrm{S}^n$ be an embedding and $\mathrm{S}^k = \mathrm{D}^k_+ \cup \mathrm{D}^k_-$. Let $A = \mathrm{S}^n \setminus h\left(\mathrm{D}^k_+\right)$ and $B = \mathrm{S}^n \setminus h\left(\mathrm{D}^k_-\right)$, so 1 implies that $\widetilde{\mathrm{H}_i}\left(A\right) = 0$ and $\widetilde{\mathrm{H}_i}\left(B\right) = 0$ for all i, and $A \cap B = \mathrm{S}^n \setminus h\left(\mathrm{S}^k\right)$ and $A \cup B = \mathrm{S}^n \setminus h\left(\mathrm{S}^{k-1}\right)$. Mayer-Vietoris implies that
   $$\dots \to \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_{i+1}}\left(A\right) \oplus \widetilde{\mathrm{H}_{i+1}}\left(B\right)} \to \widetilde{\mathrm{H}_i}\left(A \cup B\right) \xrightarrow{\sim} \widetilde{\mathrm{H}_{i+1}}\left(A \cap B\right) \to \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{\mathrm{H}_i}\left(A\right) \oplus \widetilde{\mathrm{H}_i}\left(B\right)} \to \dots,$$
   by 1, so
   $$\widetilde{\mathrm{H}_{i+1}}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^{k-1}\right)\right) \cong \widetilde{\mathrm{H}_i}\left(\mathrm{S}^n \setminus h\left(\mathrm{S}^k\right)\right) \cong \begin{cases} \mathbb{Z} & i+1 = n-(k-1)-1 \\ 0 & \text{otherwise} \end{cases},$$
   by induction.

$\square$

Lecture 29 is a problems class. Lecture 29 Tuesday 19/03/19

2.7 Degree

Lecture 30 Wednesday 20/03/19

Let $n \geq 1$. We have seen that $\mathrm{H}_n(\mathrm{S}^n) \cong \langle a \rangle \cong \mathbb{Z}$. Let $f : \mathrm{S}^n \to \mathrm{S}^n$ be a continuous map, so $f_* : \mathrm{H}_n(\mathrm{S}^n) \to \mathrm{H}_n(\mathrm{S}^n)$ is a homomorphism. Then f_* is given by $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$ depending only on f. This integer is the **degree** of f.

Proposition 2.24. *The following are observations.*

1. $\deg \mathrm{id}_{\mathrm{S}^n} = 1$.
2. *If f is not surjective, then* $\deg f = 0$.
3. *If $f \cong g$, then $f_* = g_*$, so* $\deg f = \deg g$.
4. $\deg fg = \deg f \deg g$. *In particular, if f is a homotopy equivalence, then* $\deg f = \pm 1$.
5. *Let*
$$\begin{array}{cccc} R_i & : & \mathrm{S}^n & \longrightarrow & \mathrm{S}^n \\ & & (x_1, \dots, x_i, \dots, x_{n+1}) & \longmapsto & (x_1, \dots, -x_i, \dots, x_{n+1}) \end{array}$$
be the reflection map. Then $\deg R_i = -1$.
6. *The antipodal map*
$$\begin{array}{cccc} -\mathrm{id}_{\mathrm{S}^n} & : & \mathrm{S}^n & \longrightarrow & \mathrm{S}^n \\ & & x & \longmapsto & -x \end{array}$$
has degree $(-1)^{n+1}$.
7. *If $f : \mathrm{S}^n \to \mathrm{S}^n$ has no fixed points, then* $\deg f = (-1)^{n+1}$.

Hopf implies that if $\deg f = \deg g$ then $f \cong g$.

Proof. 1 and 3 are clear.

2. Let $x_0 \in \mathrm{S}^n \setminus f(\mathrm{S}^n)$. So f factors as $f = i \circ f'$, where
$$\mathrm{S}^n \xrightarrow{f'} \mathrm{S}^n \setminus \{x_0\} \xhookrightarrow{i} \mathrm{S}^n.$$
$\mathrm{H}_n(\mathrm{S}^n \setminus \{x_0\}) = 0$ since $\mathrm{S}^n \setminus \{x_0\}$ is contractible, so $f_* = i_* \circ f'_* = 0$.

4. $(fg)_* = f_* g_*$, and there exists $g : \mathrm{S}^n \to \mathrm{S}^n$ such that $fg \cong \mathrm{id}_{\mathrm{S}^n}$, so
$$\deg f \deg g = \deg fg = \deg \mathrm{id}_{\mathrm{S}^n} = 1.$$

5. It is enough to show it for $i = 1$. Induction on n.

$n = 1$. $R_1(x_1, x_2) = (-x_1, x_2)$. Then $\omega : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ implies that $R_1([\omega]) = -[\omega]$, so $\deg R_1 = -1$.

$n - 1 \mapsto n$. Claim that there is an isomorphism $\phi : \mathrm{H}_n(\mathrm{S}^n) \xrightarrow{\sim} \mathrm{H}_{n-1}(\mathrm{S}^{n-1})$ such that

$$\begin{tikzcd} \mathrm{H}_n(\mathrm{S}^n) \arrow[r, "\phi"] \arrow[d, "R_{1*}"] & \mathrm{H}_{n-1}(\mathrm{S}^{n-1}) \arrow[d, "R_{1*}"] \\ \mathrm{H}_n(\mathrm{S}^n) \arrow[r, "\phi"] & \mathrm{H}_{n-1}(\mathrm{S}^{n-1}) \end{tikzcd}$$

commutes. Let

$$N = (0, \dots, 0, 1), \qquad S = (0, \dots, 0, -1), \qquad U = \mathrm{S}^n \setminus \{N\}, \qquad V = \mathrm{S}^n \setminus \{S\},$$

so $R_1(U) = U$ and $R_1(V) = V$. There is a commutative diagram of chain maps

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{C}_\bullet(U \cap V) \arrow[r] \arrow[d, "R_{1\#}"] & \mathrm{C}_\bullet(U) \oplus \mathrm{C}_\bullet(V) \arrow[r] \arrow[d, "R_{1\#} \oplus R_{1\#}"] & \mathrm{C}_\bullet(U + V) \arrow[r] \arrow[d, "R_{1\#}"] & 0 \\ 0 \arrow[r] & \mathrm{C}_\bullet(U \cap V) \arrow[r] & \mathrm{C}_\bullet(U) \oplus \mathrm{C}_\bullet(V) \arrow[r] & \mathrm{C}_\bullet(U + V) \arrow[r] & 0 \end{tikzcd}.$$

This induces a commutative diagram

$$\begin{tikzcd} \mathrm{H}_n\left(\mathrm{S}^n\right) \arrow[r, "\partial"] \arrow[d, "R_{1*}"] & \mathrm{H}_{n-1}\left(U \cap V\right) \arrow[d, "R_{1*}"] & \mathrm{H}_{n-1}\left(\mathrm{S}^{n-1}\right) \arrow[l, "i_*"'] \arrow[d, "R_{1*}"] \\ \mathrm{H}_n\left(\mathrm{S}^n\right) \arrow[r, "\partial"] & \mathrm{H}_{n-1}\left(U \cap V\right) & \mathrm{H}_{n-1}\left(\mathrm{S}^{n-1}\right) \arrow[l, "i_*"'] \end{tikzcd},$$

where

$$\begin{array}{cccc} i & : & \mathrm{S}^{n-1} & \longrightarrow & U \cap V \\ & & (x_1, \dots, x_n) & \longmapsto & (x_1, \dots, x_n, 0) \end{array}$$

is a homotopy equivalence. Then i_* is an isomorphism because i is a homotopy equivalence and ∂ is an isomorphism as seen last week. The first square commutes by naturality and the second square commutes by functoriality.

6. $-\mathrm{id}_{\mathrm{S}^n} = R_1 \dots R_{n+1}$, so

$$\deg -\mathrm{id}_{\mathrm{S}^n} = \deg R_1 \dots \deg R_{n+1} = (-1)^{n+1}.$$

7. If $f(x) \neq x$ for all $x \in \mathrm{S}^n$, then the line segment from $f(x)$ to $-x$ defined by

$$t \mapsto (1-t) f(x) - tx$$

does not pass through the origin. Define

$$f_t(x) = \frac{(1-t) f(x) - tx}{|(1-t) f(x) - tx|},$$

so f_t is a homotopy from f to $-\mathrm{id}_{\mathrm{S}^n}$. Thus

$$\deg f = \deg -\mathrm{id}_{\mathrm{S}^n} = (-1)^{n+1}.$$

□

Proposition 2.25. *If n is even, then $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely by homeomorphisms on S^n.*

Proof. Let G be a group acting freely by homeomorphisms on S^n, so $G \subseteq \mathrm{Homeo}\,\mathrm{S}^n$. So for $f \in G$, $\deg f = \pm 1$ by 4, and $\deg fg = \deg f \deg g$ for all $f, g \in G$ by 3, so the degree defines a homeomorphism $d : G \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. The action is free, so if $g \in G \setminus \{\mathrm{id}\}$, then g has no fixed points, so 7 and n even implies that $\deg g = (-1)^{n+1} = -1$. Then $\ker d = \{\mathrm{id}\}$, so d is injective, so $G = \{\mathrm{id}\}$ or $G \cong \mathbb{Z}/2\mathbb{Z}$. □

Definition. A **vector field** on S^n is a continuous map $v : \mathrm{S}^n \to \mathbb{R}^{n+1}$ such that for each $x \in \mathrm{S}^n$, $v(x)$ is **tangent** to S^n at x, that is $v(x)$ and x are orthogonal.

Theorem 2.26 (Hairy ball theorem)**.** S^n *admits a continuous vector field $v : \mathrm{S}^n \to \mathbb{R}^{n+1}$ that is nowhere zero if and only if n is odd.*

Proof. If $v(x) \neq 0$ for all $x \in \mathrm{S}^n$, let

$$\begin{array}{cccc} v' & : & \mathrm{S}^n & \longrightarrow & \mathbb{R}^{n-1} \\ & & x & \longmapsto & \dfrac{v(x)}{|v(x)|} \end{array}.$$

Define

$$f_t(x) = \cos(t\pi)\, x + \sin(t\pi)\, v'(x).$$

Then $f_t(x) \in \mathrm{S}^n$ for all $x \in \mathrm{S}^n$ and for all $t \in \mathrm{I}$, so f_t is a homotopy from $\mathrm{id}_{\mathrm{S}^n}$ to $-\mathrm{id}_{\mathrm{S}^n}$, so

$$1 = \deg \mathrm{id}_{\mathrm{S}^n} = \deg -\mathrm{id}_{\mathrm{S}^n} = (-1)^{n+1}.$$

Thus n is odd. Conversely, if $n = 2k-1$,

$$v(x_1, \dots, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$$

is a vector field on S^n. □

A Proofs

A.1 The Seifert-van Kampen theorem

Proof of Theorem 1.21. Consider the natural homomorphism

$$\Phi : \pi_1 (U_1, x_0) * \pi_1 (U_2, x_0) \to \pi_1 (X, x_0).$$

Φ is surjective by Lemma 1.17, and $N \subseteq \ker \Phi$. Want to show that $N = \ker \Phi$. A **factorisation** of an element $[f] \in \pi_1 (X, x_0)$ is a formal product $[f_1] \dots [f_k]$ such that

- each f_i is a loop at x_0 in one of the U_i and $[f_i] \in \pi_1 (U_i, x_0)$ is its homotopy class, and
- the loop $f_1 \cdot \dots \cdot f_k$ is homotopic to f in X.

A factorisation of $[f]$ is a word in $\pi_1 (U_1, x_0) * \pi_1 (U_2, x_0)$ that is mapped to $[f]$ by Φ. Two factorisations of $[f]$ are **equivalent** if they are related by finitely many of the following two moves.

- If $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1 (U_i, x_0)$, exchange $[f_i] [f_{i+1}]$ with $[f_i \cdot f_{i+1}]$. These are the relations in $\pi_1 (U_i, x_0) * \pi_1 (U_i, x_0)$.
- If f_i is a loop in $U_1 \cap U_2$, consider $[f_i]$ as an element in $\pi_1 (U_1, x_0)$ instead of $\pi_1 (U_2, x_0)$, and vice versa. These are the relations in $\pi_1 (U_1, x_0) * \pi_1 (U_2, x_0) / N$.

Given $[f] \in \pi_1 (X, x_0)$, we want to show that any two factorisations of $[f]$ are equivalent. Let $[f_1] \dots [f_k]$ and $[f_1'] \dots [f_l']$ be two factorisations of $[f]$, so the two loops $f_1 \cdot \dots \cdot f_k$ and $f_1' \cdot \dots \cdot f_k'$ are homotopic. Let $F : \mathrm{I} \times \mathrm{I} \to X$ be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \qquad 0 = t_0 < \dots < t_n = 1,$$

such that $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $F (R_{ij}) \subseteq U_1$ or $F (R_{ij}) \subseteq U_2$. May assume $0 = s_0 < \dots < s_m = 1$ subdivides the products $f_1 \cdot \dots \cdot f_k$ and $f_1' \cdot \dots \cdot f_l'$. Relabel the R_{ij} to $R_1, \dots, R_{mn}$.

$mn - m + 1$	$\dots$	mn
$\vdots$	$\ddots$	$\vdots$
1	$\dots$	m

A path γ in $\mathrm{I} \times \mathrm{I}$ from left to right gives a loop $F|_\gamma$ in X at x_0. Let γ_r be the path separating the first r rectangles from the others, so

$$F|_{\gamma_0} \cong f_1 \cdot \dots \cdot f_k, \qquad F|_{\gamma_{mn}} = f_1' \cdot \dots \cdot f_l'.$$

Let v be a grid point. Choose a path g_v in X from x_0 to $F (v)$, such that g_v is contained in $U_1 \cap U_2$ if $F (v) \in U_1 \cap U_2$ and in a single U_i otherwise. This gives us a factorisation of $\left[F|_{\gamma_r} \right]$ into loops only contained in U_1 or U_2. The factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F|_{\gamma_r}$ and $F|_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_i. $\square$

A.2 The equivalence of simplicial and singular homology

Lemma A.1 (Five lemma)**.** *Consider the following diagram of abelian groups*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}.$$

If the rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism.

Proof. It is enough to show

- if β and δ are surjective and ϵ is injective, then γ is surjective, and
- if β and δ are injective and α is surjective, then γ is injective.

$\square$

Let $n \geq 1$. Then

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \widetilde{\mathrm{H}_n}(\Delta^n/\partial\Delta^n) \cong \widetilde{\mathrm{H}_n}(\mathrm{S}^n) \cong \mathbb{Z},$$

and $\mathrm{H}_0\left(\Delta^0, \partial\Delta^0\right) \cong \mathbb{Z}$.

Lemma A.2. $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ *is generated by the class of the cycle $i_n : \Delta^n \to \Delta^n$.*

Proof. i_n is a cycle. Induction on n.

$n = 0$. $\mathrm{H}_0\left(\Delta^0, \emptyset\right)$ is generated by $[i_0]$.

$n - 1 \mapsto n$. Let $\Lambda \subseteq \partial\Delta^n$ be the union of all but one of the $(n-1)$-dimensional faces of Δ^n. Then Δ^n strongly deformation retracts to Λ, so

$$\mathrm{H}_i(\Delta^n, \Lambda) = \mathrm{H}_i(\Lambda, \Lambda) = 0.$$

Long exact sequence of the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ implies that

$$\dots \longrightarrow \underset{\substack{\cong \\ 0}}{\mathrm{H}_n(\Delta^n, \Lambda)} \longrightarrow \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\sim} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \longrightarrow \underset{\substack{\cong \\ 0}}{\mathrm{H}_{n-1}(\Delta^n, \Lambda)} \longrightarrow \dots .$$

Note that $\partial\Delta^n/\Lambda$ is homeomorphic to $\Delta^{n-1}/\partial\Delta^{n-1}$, which are good pairs, so

$$\begin{aligned} \mathrm{H}_n(\Delta^n, \partial\Delta^n) &\cong \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \cong \widetilde{\mathrm{H}_{n-1}}(\partial\Delta^n/\Lambda) \\ &\cong \widetilde{\mathrm{H}_{n-1}}\left(\Delta^{n-1}/\partial\Delta^{n-1}\right) \cong \mathrm{H}_{n-1}\left(\Delta^{n-1}, \partial\Delta^{n-1}\right). \end{aligned}$$

One can check that $[i_n]$ maps to $[\pm i_{n-1}]$ along these isomorphisms, so induction implies that $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ is generated by $[i_n]$.

$\square$

Let X be a topological space with a Δ-complex structure, so there is a simplicial chain complex

$$\dots \to \Delta_{n+1}(X) \to \Delta_n(X) \to \Delta_{n-1}(X) \to \dots.$$

Every simplicial chain complex can be viewed as a singular n-chain, so we obtain an inclusion of chain complexes $\Delta_\bullet(X) \to \mathrm{C}_\bullet(X)$.

Theorem A.3. *This inclusion of chain complexes induces an isomorphism* $\mathrm{H}_n^\Delta(X) \xrightarrow{\sim} \mathrm{H}_n(X)$ *for all* n.

Proof. We only consider the case where the Δ-complex structure on X is finite-dimensional, that is $\Delta_m(X) = 0$ for all $m > k$, and the maximal such k is $\dim X$. Induction on k, the dimension.

$k = 0$. $\mathrm{H}_n^\Delta(X) \cong \mathrm{H}_n(X)$ for X points.

$k-1 \mapsto k$. Let X^l be the l-skeleton of X consisting of all simplices of dimension at most l. Then $\mathrm{H}_n^\Delta\left(X^k, X^{k-1}\right)$ are the homology groups of the chain complex

$$\dots \to \Delta_{k+1}\left(X^k\right)/\Delta_{k+1}\left(X^{k-1}\right) \to \Delta_k\left(X^k\right)/\Delta_k\left(X^{k-1}\right) \to \Delta_{k-1}\left(X^k\right)/\Delta_{k-1}\left(X^{k-1}\right) \to \dots,$$

so

$$\mathrm{H}_n^\Delta\left(X^k, X^{k-1}\right) = \begin{cases} 0 & n \ne k \\ \text{free abelian group with basis the } k\text{-simplices of } X & n = k \end{cases}.$$

The short exact sequence of chain complexes

$$0 \to \Delta_n\left(X^{k-1}\right) \to \Delta_n\left(X^k\right) \to \Delta_n\left(X^k\right)/\Delta_n\left(X^{k-1}\right) \to 0$$

gives a long exact sequence

$$\begin{tikzcd} \to \mathrm{H}_{n+1}^\Delta\left(X^k, X^{k-1}\right) \arrow[r] \arrow[d, "\alpha"] & \mathrm{H}_n^\Delta\left(X^{k-1}\right) \arrow[r] \arrow[d, "\beta"] & \mathrm{H}_n^\Delta\left(X^k\right) \arrow[r] \arrow[d, "\gamma"] & \mathrm{H}_n^\Delta\left(X^k, X^{k-1}\right) \arrow[r] \arrow[d, "\delta"] & \mathrm{H}_{n-1}^\Delta\left(X^{k-1}\right) \to \arrow[d, "\epsilon"] \\ \to \mathrm{H}_{n+1}\left(X^k, X^{k-1}\right) \arrow[r] & \mathrm{H}_n\left(X^{k-1}\right) \arrow[r] & \mathrm{H}_n^\Delta\left(X^k\right) \arrow[r] & \mathrm{H}_n\left(X^k, X^{k-1}\right) \arrow[r] & \mathrm{H}_{n-1}\left(X^{k-1}\right) \to \end{tikzcd},$$

which commutes by naturality, where β and ϵ are isomorphisms by induction. Consider the continuous map

$$\Phi : \bigsqcup_\alpha \left(\Delta_\alpha^k, \partial\Delta_\alpha^k\right) \to \left(X^k, X^{k-1}\right).$$

This induces an isomorphism

$$\mathrm{H}_n\left(X^k, X^{k-1}\right) \cong \mathrm{H}_n\left(\bigsqcup_\alpha \Delta_\alpha^k, \bigsqcup_\alpha \partial\Delta^k\right) = \bigoplus_\alpha \mathrm{H}_n\left(\Delta_\alpha^k, \partial\Delta_\alpha^k\right),$$

which is the free abelian group on $i_{n\alpha} : \Delta_\alpha^n \to \Delta_\alpha^n$ by Lemma A.2, so α and δ are isomorphisms. Thus five lemma implies that γ is an isomorphism.

∎