

# A blueprint for the Birch and Swinnerton-Dyer conjecture in Lean

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# Introduction

Let  $E_K$  be an elliptic curve over a number field  $K$ .

## Conjecture (Birch–Swinnerton-Dyer)

Assume that  $L(E_K, s)$  has meromorphic continuation at  $s = 1$ .

1. The order of vanishing of  $L(E_K, s)$  at  $s = 1$  is equal to  $\text{rk}(E_K)$ .
2. The group  $\text{III}(E_K)$  is finite.
3. The leading term of  $L(E_K, s)$  at  $s = 1$  satisfies

$$\lim_{s \rightarrow 1} \frac{L(E_K, s)}{(s - 1)^{\text{rk}(E_K)}} = \frac{\Omega(E_K) \cdot \text{Reg}(E_K) \cdot \#\text{III}(E_K) \cdot \text{Tam}(E_K)}{\delta_K \cdot \#\text{tor}(E_K)^2},$$

where  $\delta_K$  is the absolute discriminant of  $K$ .

In this talk, I will describe each of these invariants in detail.

Note that this generalises to abelian varieties over global fields.

## Weierstrass equations

An *elliptic curve*  $E_F$  over a field  $F$  is a smooth projective curve of genus one over  $F$  with a distinguished point  $\mathcal{O}$  over  $F$ .

By the Riemann–Roch theorem,  $E_F$  is isomorphic to a curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

for some  $a_i \in F$  such that  $\Delta \neq 0$ , and  $\mathcal{O}$  is its unique point at infinity.

In `mathlib`, a *Weierstrass curve* over  $F$  is a tuple  $(a_1, a_2, a_3, a_4, a_6) \in F^5$ , and an elliptic curve is a Weierstrass curve such that  $\Delta \neq 0$ .

A *point* over  $F$  is either  $\mathcal{O}$  or an affine point  $(x, y) \in F^2$  satisfying

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

and a nonsingularity condition.

## The group law

With this definition, addition on points is given by explicit rational functions, where associativity is known to be *computationally difficult*: generic associativity is an equality of polynomials with 26,082 terms!

### Formalisation (A.–Xu, 2022)

*The type of nonsingular  $F$ -points  $E_F(F)$  is an abelian group.*

It suffices to show that the homomorphism  $E_F(F) \rightarrow \text{Cl}(F[E_F])$  mapping  $(x, y)$  to  $[(X - x, Y - y)]$  is injective. If it were not, then there are polynomials  $f, g \in F[X]$  such that  $(X - x, Y - y) = (f + gY)$ . Then

$$\deg(\text{Nm}(f + gY)) = \begin{cases} \max(2 \deg(f), 2 \deg(g) + 3), \\ \dim_F(F[E_F]/(f + gY)), \end{cases}$$

which give a contradiction.

# The Tate module

I attempted to formalise the isomorphism  $E_F(F^s)[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$  in 2023.

Silverman defines polynomials  $\psi_n, \phi_n, \omega_n \in F^s[X, Y]$  and *claims* that there is a computational proof for the multiplication-by- $n$  formula

$$[n](x, y) = \left( \frac{\phi_n(x)}{\psi_n^2(x)}, \frac{\omega_n(x, y)}{\psi_n^3(x, y)} \right).$$

Computing  $\deg(\phi_n) = n^2$  and  $\deg(\psi_n^2) = n^2 - 1$ , and proving that  $(\phi_n, \psi_n^2) = 1$ , imply that  $\# \ker[n] = n^2$ , and the result follows formally.

## Formalisation (A.–Wu–Xu, 2026?)

*For any  $\ell \neq \text{char}(F)$ , the  $\ell$ -adic Tate module  $T_\ell E_{F^s}$  defines a two-dimensional Galois representation  $\rho_{E_F, \ell} : G_F \rightarrow \text{GL}(T_\ell E_{F^s})$ .*

The proof is much trickier than he claims!

# The L-function

Let  $E_K$  be an elliptic curve over a number field  $K$ .

The **Euler factor** of  $E_K$  at a finite place  $v$  of  $K$  is

$$L_v(E_K, s) := \det(1 - \rho_{E_K, \ell}^{\vee I_v}(\phi_v) \cdot q_v^{-s}),$$

where  $\ell \nmid q_v$  is any prime number.

The **L-function** of  $E_K$  is

$$L(E_K, s) := \prod_p \frac{1}{L_v(E_K, s)},$$

where the product runs over all finite places  $v$  of  $K$ .

Assuming an appropriate modularity conjecture for  $E_K$  over  $K$ , the L-function has analytic continuation to all of  $\mathbb{C}$ .

## Non-archimedean local fields

Let  $E_F$  be an elliptic curve over a non-archimedean local field  $F$  with normalised valuation  $v$ , valuation ring  $R$ , and residue field  $k$ .

By the valuative criterion for properness, there is a *reduction map*

$$\widetilde{(\cdot)} : E_F \xleftarrow{\sim} E_R \rightarrow E_k,$$

which induces a map on points  $E_F(F) \rightarrow \widetilde{E_F}(k)$ .

Note that this generalises to the fraction field  $F$  of a Bézout domain  $R$  with  $k := R/m$  for any maximal ideal  $m$  of  $R$ .

Say that  $E_F$  is *minimal* if  $v(\Delta) \in \mathbb{N}$  is minimal subject to  $a_i \in R$ . Any elliptic curve over  $F$  is isomorphic to one that is minimal.

If  $E_K$  is an elliptic curve over a number field  $K$  with  $\text{Cl}(K) = 1$ , then  $E_K$  is isomorphic to an elliptic curve that is minimal everywhere.

## Reduction types

Say that  $E_F$  is

- *good* if  $\widetilde{E}_F$  is elliptic,
- *split multiplicative* if  $\widetilde{E}_F$  is nodal with tangent over  $k$ ,
- *non-split multiplicative* if  $\widetilde{E}_F$  is nodal with tangent not over  $k$ , and
- *additive* if  $\widetilde{E}_F$  is cuspidal.

Let  $E_K$  be an elliptic curve over a number field  $K$ . Then

$$L_v(E_K, s) = \begin{cases} 1 - a_v q_v^{-s} + q_v^{1-2s} & \text{if } E_{K_v} \text{ is good,} \\ 1 - q_v^{-s} & \text{if } E_{K_v} \text{ is split multiplicative,} \\ 1 + q_v^{-s} & \text{if } E_{K_v} \text{ is non-split multiplicative,} \\ 1 & \text{if } E_{K_v} \text{ is additive,} \end{cases}$$

where  $a_v := 1 + q_v - \#\widetilde{E}_{K_v}(k_v)$  is the trace of Frobenius of  $E_K$  at  $v$ .



# Tamagawa numbers

The **Tamagawa number** of  $E_F$  is

$$\text{Tam}(E_F) := [E_F(F) : E_F^0(F)],$$

where  $E_F^0(F)$  is the subgroup of  $E_F(F)$  with nonsingular reduction.

Let  $E_K$  be an elliptic curve over a number field  $K$ , and let

$$\omega := \frac{dx}{2y + a_1x + a_3}.$$

For each place  $v$  of  $K$ , let  $\omega_v$  be a non-zero invariant differential of a minimal elliptic curve isomorphic to  $E_{K_v}$ . Then its **Tamagawa number** is

$$\text{Tam}(E_K) := \prod_v \text{Tam}(E_{K_v}) \cdot \left| \frac{\omega_v}{\omega} \right|_v,$$

where the product runs over all finite places  $v$  of  $K$ .

## Complex fields

Let  $E_{\mathbb{C}}$  be an elliptic curve over  $\mathbb{C}$  given by  $y^2 = x^3 + Ax + B$ .

There is a  $\mathbb{C}$ -lattice  $\Lambda_{A,B}$  that is unique up to homothety such that

$$\begin{aligned}\mathbb{C}/\Lambda_{A,B} &\longrightarrow E_{\mathbb{C}}(\mathbb{C}) \\ z &\longmapsto (\wp(z), \tfrac{1}{2}\wp'(z))\end{aligned}$$

is an isomorphism of complex Lie groups.

The **period** of  $E_{\mathbb{C}}$  is

$$\Omega(E_{\mathbb{C}}) := \int_{\mathbb{C}/\Lambda_{A,B}} 2dx dy = \int_{E_{\mathbb{C}}(\mathbb{C})} \omega \wedge \overline{\omega},$$

which is just the area of  $\Lambda_{A,B}$ .

See Silverman's *Advanced Topics in the Arithmetic of Elliptic Curves*.

## Real fields

Let  $E_{\mathbb{R}}$  be an elliptic curve over  $\mathbb{R}$ . Then there is an isomorphism

$$E_{\mathbb{R}}(\mathbb{R}) \cong \begin{cases} S^1 & \text{if } \Delta < 0 \\ S^1 \oplus C_2 & \text{if } \Delta > 0 \end{cases}$$

of real Lie groups.

The **period** of  $E_{\mathbb{R}}$  is

$$\Omega(E_{\mathbb{R}}) := \int_{E_{\mathbb{R}}(\mathbb{R})} \omega.$$

If  $E_K$  is an elliptic curve over a number field  $K$ , its **period** is

$$\Omega(E_K) := \prod_v \Omega(E_{K_v}),$$

where the product runs over all infinite places  $v$  of  $K$ .

# The Mordell–Weil theorem

Let  $E_K$  be an elliptic curve over a number field  $K$ .

## Theorem (Mordell–Weil)

$E_K(K)$  is finitely generated.

By the structure theorem of finitely generated abelian groups,

$$E_K(K) \cong \operatorname{tor}(E_K) \oplus \mathbb{Z}^{\operatorname{rk}(E_K)}.$$

where  $\operatorname{tor}(E_K)$  is the **torsion subgroup** and  $\operatorname{rk}(E_K)$  is the **rank**.

The torsion subgroup can be computed via the reduction map.

The rank is conjecturally the order of vanishing of  $L(E_K, s)$  at  $s = 1$ .

## Naïve heights

The proof that  $E_K(K)$  is finitely generated reduces to a proof of the *weak Mordell–Weil theorem* that  $E_K(K)/n$  is finite and the existence of a *naïve height*  $h : E_K(K) \rightarrow \mathbb{R}$  satisfying the following.

- For all  $Q \in E_K(K)$ , there exists  $C_1 \in \mathbb{R}$  such that for all  $P \in E_K(K)$ ,

$$h(P + Q) \leq 2h(P) + C_1.$$

- There exists  $C_2 \in \mathbb{R}$  such that for all  $P \in E_K(K)$ ,

$$n^2 h(P) \leq h(nP) + C_2.$$

- For all  $C_3 \in \mathbb{R}$ , the set  $\{P \in E_K(K) : h(P) \leq C_3\}$  is finite.

For instance, when  $K = \mathbb{Q}$ ,

$$\begin{array}{rcl} h : & E_{\mathbb{Q}}(\mathbb{Q}) & \longrightarrow \mathbb{R} \\ & (n/d, y) & \longmapsto \log \max(|n|, |d|) . \\ & \mathcal{O} & \longmapsto 0 \end{array}$$

## Canonical heights

Any naïve height defines the *canonical height*  $\widehat{h} : E_K(K) \rightarrow \mathbb{R}$  given by

$$\widehat{h}(P) := \lim_{n \rightarrow \infty} \frac{h([2^n]P)}{4^n},$$

which is independent of the choice of naïve height.

This is a quadratic form on  $E_K(K)$ , with associated bilinear pairing

$$\langle P, Q \rangle := \frac{1}{2}(\widehat{h}(P + Q) - \widehat{h}(P) - \widehat{h}(Q)).$$

The **regulator** of  $E_K$  is

$$\text{Reg}(E_K) := \left| \det(\langle P_i, P_j \rangle)_{i,j=0}^{\text{rk}(E_K)} \right|,$$

where  $\{P_n\}_{n=0}^{\text{rk}(E_K)}$  is any  $\mathbb{Z}$ -basis of  $E_K(K)/\text{tor}(E_K)$ .

# Galois cohomology

For any field  $F$ , multiplication by  $n \in \mathbb{Z}$  gives

$$0 \rightarrow E_F[n] \rightarrow E_F \rightarrow E_F \rightarrow 0,$$

which induces a long exact sequence that truncates to

$$0 \rightarrow E_F(F)/n \rightarrow H^1(F, E_F[n]) \rightarrow H^1(F, E_F)[n] \rightarrow 0.$$

Applying this to  $F = K$  and to  $F = K_v$  for each place  $v$  of  $K$  gives

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_K(K)/n & \longrightarrow & H^1(K, E_K[n]) & \longrightarrow & H^1(K, E_K)[n] \longrightarrow 0 \\
 & & \downarrow & & \downarrow & \searrow \sigma & \downarrow \tau[n] \\
 0 & \rightarrow & \prod_v E_K(K_v)/n & \rightarrow & \prod_v H^1(K_v, E_K[n]) & \rightarrow & \prod_v H^1(K_v, E_K)[n] \rightarrow 0.
 \end{array}$$

Note that  $H^1(K, E_K[n])$  is not finite in general.

## The weak Mordell–Weil theorem

The  $n$ -Selmer group  $\text{Sel}_n(E_K) := \ker \sigma$  and the **Tate–Shafarevich group**

$$\text{III}(E_K) := \ker \left( \tau : H^1(K, E_K) \rightarrow \prod_v H^1(K_v, E_K) \right)$$

fit in a short exact sequence

$$0 \rightarrow E_K(K)/n \rightarrow \text{Sel}_n(E_K) \rightarrow \text{III}(E_K)[n] \rightarrow 0.$$

The weak Mordell–Weil theorem then reduces to showing that

$$\text{Sel}_n(E_K) \subseteq \text{Sel}_n(K, S) \times \text{Sel}_n(K, S),$$

where  $\text{Sel}_n(K, S)$  is the  $n$ -Selmer group of  $K$  unramified outside an explicit finite set  $S$  of bad places of  $K$ , which is finite since

$$0 \rightarrow \mathcal{O}_{K,S}^\times / (\mathcal{O}_{K,S}^\times)^n \rightarrow \text{Sel}_n(K, S) \rightarrow \text{Cl}_S(K)[n] \rightarrow 0.$$