Algebraicity of Artin–Hasse–Weil L-series over global function fields

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Abstract

We present a short proof of the analogue of Deligne's period conjecture for the special value of the L-function of an abelian variety over a global function field twisted by an Artin representation.

Deligne's period conjecture is an abstract statement on the special value of the L-function associated to a pure motive with a critical Hodge structure [Del79, Definition 1.3]. Specifically, he conjectures that the L-value is equal to the determinant of a certain period map between its Betti and de Rham realisations, up to non-zero multiples in a number field [Del79, Conjecture 2.8]. This is known for Artin L-functions over $\mathbb{Q}$ [Del79, Proposition 6.7], and has ramifications for the Birch–Swinnerton-Dyer conjecture for abelian varieties over $\mathbb{Q}$ [Del79, Section 4], with numerical evidence for L-functions associated to Jacobians of smooth projective curves over $\mathbb{Q}$ [ECW24, Conjecture 1.1].

In the context of the L-function $L(A, \tau, s)$ of an abelian variety A over a number field K twisted by an Artin representation τ over K, which appear in equivariant refinements of the Birch–Swinnerton-Dyer conjecture [BC24, Conjecture 3.3], Deligne's period conjecture translates to a statement on the algebraicity and Galois equivariance of $L(A, \tau, 1)$ normalised by periods [Eva21, Proposition 4.3.8]. This remains largely open in general, but the case of an elliptic curve over $\mathbb{Q}$ twisted by Artin representations that factor through a false Tate curve extension, such as the trivial representation and primitive Dirichlet characters, is a consequence of the modularity theorem [BD07, Theorem 4.2].

When A is an abelian variety over a global function field K, it is known that $L(A, \tau, s)$ is already a rational function as a formal consequence of the Grothendieck–Lefschetz trace formula, so the aforementioned normalisations by periods are unnecessary. This paper presents a short proof of the analogue of Deligne's period conjecture in this context, which is stated in Theorem 2. To this end, some notational conventions for the formalism of ℓ-adic representations over local fields and global function fields will first be established. Throughout, ℓ will be a fixed prime of $\mathbb{Q}$, and V_ℓ will be a finite-dimensional vector space over a finite extension of $\mathbb{Q}_\ell$, whose choice will not be essential.

Notation 1. Let F be a non-archimedean local field with residue characteristic p. Let I_F denote the inertia subgroup of its Weil group W_F, and let Fr_F denote the inverse of any choice of Frobenius element in W_F. For $\ell \neq p$, an *ℓ-adic representation over F* is a continuous homomorphism $\rho : W_F \to \mathrm{GL}(V_\ell)$, and its *Euler factor* is the inverse characteristic polynomial

$$\mathcal{L}_F(\rho, T) \coloneqq \det(1 - T \cdot \mathrm{Fr}_F \mid \rho^{I_F}),$$

where ρ^{I_F} is the subrepresentation of ρ invariant under I_F.

Now let K be the global function field of a smooth projective absolutely irreducible curve C over $\mathbb{F}_q$ with absolute Galois group G_K. For each place v of K, let K_v denote its completion, and let $\deg v$ denote its residue class degree. For $\ell \neq p$, an *ℓ-adic representation over K* is a continuous homomorphism $\rho : G_K \to \mathrm{GL}(V_\ell)$, and its *formal L-series* is the infinite product

$$\mathcal{L}(\rho, T) \coloneqq \prod_v \frac{1}{\mathcal{L}_{K_v}(\rho, T^{\deg v})},$$

which is a priori only a formal product. Its *L-series* $L(\rho, s)$ is simply $\mathcal{L}(\rho, q^{-s})$, and let $L^{(n)}(\rho, s)$ denote the n-th derivative of $L(\rho, s)$ for all $n \in \mathbb{N}$.

The key example will be the first ℓ-adic cohomology group $\rho_A \coloneqq H^1_{\text{ét}}(A, \mathbb{Q}_\ell)$ of an abelian variety A over K, which is independent of ℓ [GR72, Theorem 4.3], so ℓ is suppressed from notation. Another example is an *Artin representation*, namely a continuous homomorphism $\tau : G_K \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over a number field equipped with the discrete topology, viewed as an ℓ-adic representation over K by some embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. The relevant notions over F are defined analogously. Finally, let $L^{(n)}(A, \tau, s)$ denote $L^{(n)}(\rho_A \otimes \tau, s)$ for all $n \in \mathbb{N}$, including $n = 0$.

For an Artin representation τ, let $\mathbb{Q}(\tau)$ denote the number field generated by the values of $\mathrm{tr}(\tau)$, and let τ^σ denote the representation with character $\sigma \circ \mathrm{tr}(\tau)$ for any $\sigma \in G_{\mathbb{Q}}$. If $(v_i)_i$ is a basis of τ over $\mathbb{Q}$, and (a_{ij}) is the matrix of $g \in G_K$ with respect to this basis, then $(v_i^\sigma)_i$ is a basis of τ^σ over $\mathbb{Q}$, and the matrix of g with respect to this basis is (a_{ij}^σ) [Eva21, Section 2.1.4].

The main result of this paper is as follows.

Theorem 2. Let A be an abelian variety over a global function field K. If τ is an Artin representation over K, then $L^{(n)}(A, \tau, 1) \in \mathbb{Q}(\tau)$ and $L^{(n)}(A, \tau, 1)^\sigma = L^{(n)}(A, \tau^\sigma, 1)$ for any $\sigma \in G_{\mathbb{Q}}$, for all $n \in \mathbb{N}$.

It turns out that the same argument applies to Artin L-series.

Theorem 3. Let τ be an Artin representation over K. Then $L^{(n)}(\tau, 1) \in \mathbb{Q}(\tau)$ and $L^{(n)}(\tau, 1)^\sigma = L^{(n)}(\tau^\sigma, 1)$ for any $\sigma \in G_{\mathbb{Q}}$, for all $n \in \mathbb{N}$.

In what follows, a stronger result on the algebraicity and Galois equivariance of the formal L-series $\mathcal{L}(\rho, T)$ will be proven, which imply the same for the L-series $L^{(n)}(\rho, s)$ for all $n \in \mathbb{N}$ by replacing T with q^{-s}.

To this end, for any field $\mathfrak{F}$ with automorphism group G, define an action of G on the ring of formal power series $\mathfrak{F}[[T]]$ by

$$\left(\sum_{n=0}^{\infty} a_n T^n\right)^g := \sum_{n=0}^{\infty} a_n^g T^n, \qquad g \in G.$$

Evaluating such a power series at an element $f \in \mathfrak{F}$ does not give $\sum_{n=0}^{\infty} a_n^g f^n$ in general, due to potential convergence issues, but the following shows that it does whenever the power series happens to be a rational function.

Lemma 4. Let $\mathfrak{F}$ be a field, and let $P(T) \in \mathfrak{F}[[T]]$ be a power series such that $P(T) = R(T)\,/Q(T)$ for some power series $Q(T) \in \mathfrak{F}[[T]]$ and some polynomial $R(T) \in \mathfrak{F}[T]$. Then $P(T)^\sigma = R(T)^\sigma\,/Q(T)^\sigma$.

Proof. Since $R(T)$ is a polynomial, it suffices to show that $(P(T)\,Q(T))^\sigma = P(T)^\sigma\,Q(T)^\sigma$. Let P_n and Q_n denote the coefficients of the power series $P(T)$ and $Q(T)$ respectively for all $n \in \mathbb{N}$, so that the equality becomes

$$\sum_{n=0}^{\infty}\left(\sum_{i+j=n} P_i Q_j\right)^\sigma T^n = \sum_{n=0}^{\infty} P_n^\sigma T^n \cdot \sum_{n=0}^{\infty} Q_n^\sigma T^n.$$

This is clear since $\sum_{i+j=n} P_i Q_j$ is a finite sum. □

This property applies to formal L-series of ℓ-adic representations over global function fields, as a consequence of the Grothendieck–Lefschetz trace formula.

Proposition 5. Let ρ be an ℓ-adic representation over a global function field $K = \mathbb{F}_q(C)$ that is unramified almost everywhere. Then $\mathcal{L}(\rho, T) \in \overline{\mathbb{Q}_\ell}(T)$.

Proof. This is the standard proof of the rationality of zeta functions [Mil80, Chapter VI, Theorem 13.4], but the argument is sketched here for reference. There is an equivalence of categories between continuous ℓ-adic representations over K that are unramified on an open set U of C and ℓ-adic sheaves that is lisse on U [Mil80, Chapter V, Section 1]. Let $\iota : U \hookrightarrow C$ be any open set at which ρ is unramified, and let $\mathcal{F}_\rho$ be its associated ℓ-adic sheaf that is lisse on U, whose direct image along ι induces étale cohomology groups $H^i := H^i_{\text{ét}}(\overline{C}, \iota_*\mathcal{F}_\rho)$ of the base change $\overline{C}$ of C to $\overline{\mathbb{F}_q}$. Now the Grothendieck–Lefschetz trace formula for ℓ-adic sheaves [Mil80, Chapter VI, Theorem 13.4] says that for all $n \in \mathbb{N}$,

$$\sum_{v \in C(\mathbb{F}_{q^n})} \operatorname{tr}(\mathrm{Fr}_{K_v}^n \mid \rho^{I_{K_v}}) = \sum_{i=0}^{2} (-1)^i \cdot \operatorname{tr}(\mathrm{Fr}_q^n \mid H^i)\,,$$

where Fr_q is the Frobenius in $\mathbb{F}_q$. Dividing both sides by n and exponentiating their generating functions, this equality rearranges to

$$\prod_v \exp \sum_{m=1}^{\infty} \operatorname{tr}(\mathrm{Fr}_{K_v}^m \mid \rho^{I_{K_v}})\,\frac{T^{m \deg v}}{m} = \prod_{i=0}^{2} \exp(\sum_{n=1}^{\infty} \operatorname{tr}(\mathrm{Fr}_q^n \mid H^i)\,\frac{T^n}{n})^{(-1)^i}.$$

An identity in linear algebra [Mil80, Chapter V, Lemma 2.7] shows that

$$\exp \sum_{m=1}^{\infty} \operatorname{tr}(\mathrm{Fr}_{K_v}^m \mid \rho^{I_{K_v}}) \frac{T^{m \deg v}}{m} = \frac{1}{\det(1 - T^{\deg v} \cdot \mathrm{Fr}_{K_v} \mid \rho^{I_{K_v}})},$$

for each place v of K, and that

$$\exp \sum_{n=1}^{\infty} \operatorname{tr}(\mathrm{Fr}_q^n \mid H^i) \frac{T^n}{n} = \frac{1}{\det(1 - T \cdot \mathrm{Fr}_q \mid H^i)}.$$

for $i = 0, 1, 2$. The left hand side becomes $\mathcal{L}(\rho, T)$, while the right hand side expresses it as an alternating product of polynomials $\det(1 - T \cdot \mathrm{Fr}_q \mid H^i)$. $\square$

In particular, $\mathcal{L}(\rho, T)$ is well-defined and respects the action of automorphisms of $\overline{\mathbb{Q}_\ell}$ at $T = q^{-1}$ whenever its denominator does not vanish.

Remark 6. When ρ is the ℓ-adic representation associated to an elliptic curve, Shioda gave an alternative description of $\mathcal{L}(\rho, T)$ in terms of its associated elliptic surface $\mathcal{E}$ [Shi92, Theorem 4]. When $\mathcal{E}$ has at least one singular fibre, he showed that $\mathcal{L}(\rho, T)$ is a polynomial given by

$$\mathcal{L}(\rho, T) = \det(1 - T \cdot \mathrm{Fr}_q \mid W),$$

where W is a subspace of the second ℓ-adic cohomology group $H^2_{\text{ét}}(\mathcal{E}, \mathbb{Q}_\ell(1))$ of $\mathcal{E}$, given as the orthogonal complement of the trivial sublattice of the Neron–Severi group $\mathrm{NS}(\mathcal{E})$ of $\mathcal{E}$ under the cycle class map $\mathrm{NS}(\mathcal{E}) \to H^2_{\text{ét}}(\mathcal{E}, \mathbb{Q}_\ell(1))$. His description has the added benefit that the degree and functional equation of the polynomial $\mathcal{L}(\rho, T)$ can be understood directly from the geometry of $\mathcal{E}$.

The analogue of algebraicity and Galois equivariance can first be proven at the level of Euler factors of local ℓ-adic representations.

Proposition 7. Let ρ be an ℓ-adic representation over a non-archimedean local field F with residue characteristic p, such that $\mathcal{L}_F(\rho, T)$ has coefficients in $\mathbb{Q}$. If τ is an Artin representation over F, then $\mathcal{L}_F(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)[T]$ and $\mathcal{L}_F(\rho \otimes \tau, T)^\sigma = \mathcal{L}_F(\rho \otimes \tau^\sigma, T)$ for any $\sigma \in G_{\mathbb{Q}}$.

Proof. This is similar to the argument by Bouganis–Dokchitser, but they only proved it for Artin twists of elliptic curves over number fields [BD07, Lemma 4.4], so it is repeated here for reference. The first statement follows from the second statement since $\tau^\sigma = \tau$ for any $\sigma \in G_{\mathbb{Q}(\tau)}$, so it suffices to prove the latter. Since $\mathcal{L}_F(\rho \otimes \tau, T)$ has coefficients in $\mathbb{Q}(\tau)$, it suffices to prove it for $\sigma \in \mathrm{Gal}(\mathfrak{L}/\mathfrak{K})$, where $\mathfrak{L}$ is the extension of $\mathbb{Q}(\tau)$ that realises τ and $\mathfrak{K}$ is its subfield fixed by σ. There is an equivalence of categories between ℓ-adic representations over F and complex Weil–Deligne representations of W_F [Del73, Section 8], so ℓ can be replaced with some prime ℓ' that splits in $\mathfrak{K}$ and remains inert in $\mathfrak{L}$, which exists by Chebotarev's density theorem. This gives an isomorphism

$$\phi : \mathrm{Gal}(\mathfrak{L}/\mathfrak{K}) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_{\ell'}(\alpha) / \mathbb{Q}_{\ell'}),$$

where α is the image in $\overline{\mathbb{Q}_{\ell'}}$ of the primitive element of $\mathfrak{K}$ that generates $\mathfrak{L}$. Now let $(v_i)_i$ be a basis of $(\rho \otimes \tau)^{I_F}$ over $\mathbb{Q}_\ell$, and let (a_{ij}) be the matrix of Fr_F with respect to this basis. Then $(v_i^{\phi(\sigma)})_i$ is a basis of $(\rho \otimes \tau^\sigma)^{I_F}$ over $\mathbb{Q}_\ell$, and the matrix of Fr_F with respect to this basis is $(a_{ij}^{\phi(\sigma)})$, so its inverse characteristic polynomial is precisely that of (a_{ij}), but with σ applied. $\square$

The corresponding statement for formal L-series follows from the local statements, by rewriting the infinite product of local Euler factors into a power series with coefficients indexed by effective Weil divisors, and applying rationality.

Theorem 8. Let ρ be an ℓ-adic representation over a global function field $K = \mathbb{F}_q(C)$ that is unramified almost everywhere, such that $\mathcal{L}_{K_v}(\rho, T)$ has coefficients in $\mathbb{Q}$ for each place v of K. If τ is an Artin representation over K, then $\mathcal{L}(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)(T)$ and $\mathcal{L}(\rho \otimes \tau, T)^\sigma = \mathcal{L}(\rho \otimes \tau^\sigma, T)$ for any $\sigma \in G_\mathbb{Q}$.

Proof. For each place v of K, let $a_{v,n}(\tau)$ denote the coefficients of the power series $\mathcal{L}_{K_v}(\rho \otimes \tau, T)^{-1}$ for all $n \in \mathbb{N}$. By Lemma 4 for $P(T) = \mathcal{L}_{K_v}(\rho \otimes \tau, T)^{-1}$, Proposition 7 translates into $a_{v,n}(\tau) \in \mathbb{Q}(\tau)$ and $a_{v,n}(\tau)^\sigma = a_{v,n}(\tau^\sigma)$ for any $\sigma \in G_\mathbb{Q}$. Now for an effective Weil divisor $D = \sum_v n_v[v]$ on C, let $a_D(\tau)$ denote the finitely-supported product $\prod_v a_{v,n_v}(\tau)$, so that $a_D(\tau) \in \mathbb{Q}(\tau)$ and $a_D(\tau)^\sigma = a_D(\tau^\sigma)$ for any $\sigma \in G_\mathbb{Q}$. A rearrangement gives

$$\mathcal{L}(\rho \otimes \tau, T) = \prod_v \sum_{n=0}^\infty a_{v,n}(\tau)\, T^{n \deg v} = \sum_{m=0}^\infty \sum_D a_D(\tau)\, T^m,$$

where the sum ranges over effective Weil divisors D on C of degree precisely m. This is a finite sum, so that $\sum_D a_D(\tau) \in \mathbb{Q}(\tau)$ and $(\sum_D a_D(\tau))^\sigma = \sum_D a_D(\tau^\sigma)$, which proves the second statement and that $\mathcal{L}(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)[[T]]$. The first statement follows from Proposition 5 that $\mathcal{L}(\rho \otimes \tau, T) \in \overline{\mathbb{Q}_\ell}(T)$, using the theory of Hankel determinants [Bou03, Chapter IV.4, Exercise 1]. $\square$

In particular, these apply to $\rho = \rho_A$, which proves Theorem 2, but also to the trivial representation $\rho = \mathbb{1}$, which proves Theorem 3.

Remark 9. Using Proposition 5, Burns–Kakde–Kim proves the algebraicity and Galois equivariance of $L^{(n)}(A, \tau, s)$ up to finitely many local Euler factors away from an open set U of C [BKK18, Proposition 2.2], by directly arguing that the action of Fr_q is preserved under an isomorphism

$$H^i_{\text{ét},c}(\overline{U}, \mathcal{F}_{\rho_A} \otimes \mathcal{F}_\tau^\sigma) \cong H^i_{\text{ét},c}(\overline{U}, \mathcal{F}_{\rho_A} \otimes \mathcal{F}_\tau)^\sigma,$$

for any $\sigma \in G_\mathbb{Q}$, where both sides are compactly-supported étale cohomology groups of the base change $\overline{U}$ of U to $\overline{\mathbb{F}_q}$. The remaining finitely many local Euler factors can be handled separately by Proposition 7, which gives an alternative proof for Theorem 2 independent from Theorem 8.

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References

[BC24] David Burns and Daniel Macias Castillo. On refined conjectures of Birch and Swinnerton-Dyer type for Hasse–Weil–Artin L-series. *Memoirs of the American Mathematical Society*, 297(1482):v+156, 2024.

[BD07] Thanasis Bouganis and Vladimir Dokchitser. Algebraicity of L-values for elliptic curves in a false Tate curve tower. *Mathematical Proceedings of the Cambridge Philosophical Society*, 142(2):193–204, 2007.

[BKK18] David Burns, Mahesh Kakde, and Wansu Kim. On a refinement of the Birch and Swinnerton-Dyer conjecture in positive characteristic, 2018.

[Bou03] Nicolas Bourbaki. *Algebra II. Chapters 4–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, english edition, 2003.

[Del73] Pierre Deligne. Les constantes des équations fonctionnelles des fonctions L. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, volume 89 of *Lecture Notes in Math.*, pages 501–597. Springer, Berlin-New York, 1973.

[Del79] Pierre Deligne. Valeurs de fonctions L et périodes d'intégrales. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, volume XXXIII of *Proc. Sympos. Pure Math.*, pages 313–346. Amer. Math. Soc., Providence, RI, 1979.

[ECW24] Robert Evans, Daniel Macias Castillo, and Hanneke Wiersema. Numerical evidence for a refinement of Deligne's period conjecture for Jacobians of curves. *Experimental Mathematics*, 0(0):1–18, 2024.

[Eva21] Robert Evans. *Artin-twists of abelian varieties: periods, L-values and arithmetic*. PhD thesis, King's College London, 2021.

[GR72] Alexander Grothendieck and Michele Raynaud. Modeles de neron et monodromie. In *Groupes de Monodromie en Géométrie Algébrique*, pages 313–523. Springer Berlin Heidelberg, 1972.

[Mil80] James Stuart Milne. *Étale cohomology*, volume No. 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1980.

[Shi92] Tetsuji Shioda. Some remarks on elliptic curves over function fields. *Astérisque*, 209:12, 99–114, 1992.