Analytical Mechanics Report #5

#1| Show that $R(q)$ is orthogonal.

$\Rightarrow$ $R(q)$ - orthogonal means

$$R(q)\,R(q)^T = R(q)^T R(q) = I_n \left(\begin{matrix}\text{identity}\\ \text{matrix}\end{matrix}\right)$$

$$\Rightarrow R(q) = \begin{bmatrix} 2(q_0^2+q_1^2)-1 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3 & 2(q_0^2+q_1^2)-1 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2 & 2(q_2q_3 + q_0q_1) & 2(q_0^2+q_3^2)-1 \end{bmatrix}$$

* Describing the column vector of $R(q)$

$$\Rightarrow a = \begin{bmatrix} 2(q_0^2+q_1^2)-1 \\ 2(q_1q_2 + q_0q_3) \\ 2(q_1q_3 - q_0q_2 \end{bmatrix} = \begin{bmatrix} 2q_0^2 + 2q_1^2 - (q_0^2+q_1^2+q_2^2+q_3^2) \\ 2q_1q_2 + 2q_0q_3 \\ 2q_1q_3 - 2q_0q_2 \end{bmatrix}$$

since :- $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

$$a = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ q_1q_2 + q_1q_2 + q_0q_3 + q_0q_3 \\ q_1q_3 + q_1q_3 - q_0q_2 - q_0q_2 \end{bmatrix}$$

$$a = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\boxed{a = Aq}$$

$$\Rightarrow \quad b = \begin{bmatrix} 2(q_1 q_2 - q_0 q_3) \\ 2(q_0^2 + q_1^2) - 1 \\ 2(q_2 q_3 + q_0 q_1) \end{bmatrix} = \begin{bmatrix} q_1 q_2 + q_1 q_2 - q_0 q_3 - q_0 q_3 \\ 2q_0^2 + 2q_1^2 - (q_0^2 + q_1^2 + q_2^2 + q_3^2) \\ 2q_2 q_3 + 2q_0 q_1 \end{bmatrix}$$

$$b = \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\boxed{b = Bq}$$

$$\Rightarrow C = \begin{bmatrix} 2(q_1q_3 + q_0q_2) \\ 2(q_2q_3 - q_0q_1) \\ 2(q_0^2 + q_3^2) - 1 \end{bmatrix} = \begin{bmatrix} 2q_1q_3 + 2q_0q_2 \\ 2q_2q_3 - 2q_0q_1 \\ 2q_0^2 + 2q_3^2 - (q_0^2 + q_1^2 + q_2^2 + q_3^2) \end{bmatrix}$$

$$C = \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\boxed{c = C q}$$

Now :-

$$R(q) = \begin{bmatrix} Aq & | & Bq & | & Cq \end{bmatrix}$$

$$R(q)^T R(q) = \begin{bmatrix} Aq \\ Bq \\ Cq \end{bmatrix} \begin{bmatrix} Aq & | & Bq & | & Cq \end{bmatrix}$$

$$R(q)^T R(q) = \begin{bmatrix} q^T A^T A q & q^T A^T B q & q^T A^T C q \\ q^T B^T A q & q^T B^T B q & q^T B^T C q \\ q^T C^T A q & q^T C^T B q & q^T C^T C q \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} q_0 & q_3 & -q_2 \\ q_1 & q_2 & q_3 \\ -q_2 & q_1 & -q_0 \\ -q_3 & q_0 & q_1 \end{bmatrix} \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-q_1^2 & q_0 q_1 & q_1 q_3 & -q_1 q_2 \\ q_0 q_1 & 1-q_0^2 & -q_0 q_3 & q_0 q_2 \\ q_1 q_3 & -q_0 q_3 & 1-q_3^2 & q_2 q_3 \\ -q_1 q_2 & q_0 q_2 & q_2 q_3 & 1-q_2^2 \end{bmatrix}$$

$$\Rightarrow B^T B = \begin{bmatrix} 1-q_1^2 & -q_2 q_3 & q_0 q_2 & q_1 q_2 \\ -q_2 q_3 & 1-q_3^2 & q_0 q_3 & q_1 q_3 \\ q_0 q_2 & q_0 q_3 & 1-q_0^2 & -q_0 q_1 \\ q_1 q_2 & q_1 q_3 & -q_0 q_1 & 1-q_1^2 \end{bmatrix}$$

$$\Rightarrow C^T C = \begin{bmatrix} 1-q_3^2 & q_2 q_3 & -q_1 q_3 & q_0 q_3 \\ q_2 q_3 & 1-q_2^2 & q_1 q_2 & -q_0 q_2 \\ -q_1 q_3 & q_1 q_2 & 1-q_1^2 & q_0 q_1 \\ q_0 q_3 & -q_0 q_3 & q_0 q_3 & 1-q_0^2 \end{bmatrix}$$

$$\Rightarrow \quad A^T B = \begin{bmatrix} -q_1 q_2 & -q_1 q_3 & q_0 q_1 & q_1^2 - 1 \\ q_0 q_2 & q_0 q_3 & 1 - q_0^2 & -q_0 q_1 \\ q_2 q_3 & q_3^2 - 1 & -q_0 q_3 & -q_1 q_3 \\ 1 - q_2^2 & -q_2 q_3 & q_0 q_2 & q_1 q_2 \end{bmatrix}$$

$$\boxed{A^T B = B^T A}$$

$$\Rightarrow \quad B^T C = \begin{bmatrix} -q_2 q_3 & q_2^2 - 1 & -q_1 q_2 & q_0 q_2 \\ 1 - q_3^2 & q_2 q_3 & -q_1 q_3 & q_0 q_3 \\ q_0 q_3 & -q_0 q_2 & q_0 q_1 & 1 - q_0^2 \\ q_1 q_3 & -q_1 q_2 & q_1^2 - 1 & -q_0 q_1 \end{bmatrix}$$

$$\boxed{B^T C = C^T B}$$

$$\Rightarrow \quad C^T A = \begin{bmatrix} -q_1 q_3 & q_0 q_3 & q_3^2 - 1 & -q_2 q_3 \\ q_1 q_2 & -q_0 q_2 & -q_2 q_3 & q_2^2 - 1 \\ 1 - q_1^2 & q_0 q_1 & q_1 q_3 & -q_1 q_2 \\ q_0 q_1 & 1 - q_0^2 & -q_0 q_3 & q_0 q_2 \end{bmatrix}$$

$$\boxed{C^T A = A^T C}$$

$$\Rightarrow (A^TA)q = (B^TB)q = (C^TC)q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\Rightarrow (A^TB)q = \begin{bmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{bmatrix} = (B^TA)q$$

$$\Rightarrow (B^TC)q = \begin{bmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{bmatrix} = (C^TB)q$$

$$\Rightarrow (C^TA)q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = (A^TC)q$$

Finally

$$\Rightarrow q^TA^TAq = q^Tq = 1$$

$$\Rightarrow q^TB^TBq = q^Tq = 1$$

$$\Rightarrow q^TC^TCq = q^Tq = 1$$

$$\Rightarrow q^T A^T B q = q^T B^T A q = [q_0 \ q_1 \ q_2 \ q_3] \begin{bmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{bmatrix}$$

$$= 0$$

$$\Rightarrow q^T B^T C q = q^T C^T B q = [q_0 \ q_1 \ q_2 \ q_3] \begin{bmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{bmatrix}$$

$$= 0$$

$$\Rightarrow q^T C^T A q = q^T A^T C q = [q_0 \ q_1 \ q_2 \ q_3] \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$= 0$$

NOW

$$R(q)^T R(q) = \begin{bmatrix} q^T A^T A q & q^T A^T B q & q^T A^T C q \\ q^T B^T A q & q^T B^T B q & q^T B^T C q \\ q^T C^T A q & q^T C^T B q & q^T C^T C q \end{bmatrix}$$

$$R(q)^T R(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $R(q)$ - is orthogonal

#2| Show $\dot{A}q = A\dot{q}$, $\dot{B}q = B\dot{q}$ and $\dot{C}q = C\dot{q}$

$\Rightarrow$ From the first question, we have A, B, and C the follow.

$$A = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix}, \quad \dot{A} = \begin{bmatrix} \dot{q}_0 & \dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 \\ \dot{q}_3 & \dot{q}_2 & \dot{q}_1 & \dot{q}_0 \\ -\dot{q}_2 & \dot{q}_3 & -\dot{q}_0 & \dot{q}_1 \end{bmatrix}$$

$$B = \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix}, \quad \dot{B} = \begin{bmatrix} -\dot{q}_3 & \dot{q}_2 & \dot{q}_1 & -\dot{q}_0 \\ \dot{q}_0 & -\dot{q}_1 & \dot{q}_2 & -\dot{q}_3 \\ \dot{q}_1 & \dot{q}_0 & \dot{q}_3 & \dot{q}_2 \end{bmatrix}$$

$$C = \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix}, \quad \dot{C} = \begin{bmatrix} \dot{q}_2 & \dot{q}_3 & \dot{q}_0 & \dot{q}_1 \\ -\dot{q}_1 & -\dot{q}_0 & \dot{q}_3 & \dot{q}_2 \\ \dot{q}_0 & -\dot{q}_1 & -\dot{q}_2 & \dot{q}_3 \end{bmatrix}$$

$\Rightarrow$ we also have $q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$ $\quad \dot{q} = \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$

$\Rightarrow$ Now :-

$$\Rightarrow \dot{A}q = \begin{bmatrix} \dot{q}_0 & \dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 \\ \dot{q}_3 & \dot{q}_2 & \dot{q}_1 & \dot{q}_0 \\ -\dot{q}_2 & \dot{q}_3 & -\dot{q}_0 & \dot{q}_1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\dot{A}q = \begin{bmatrix} \dot{q}_0 q_0 + \dot{q}_1 q_1 - \dot{q}_2 q_2 - \dot{q}_3 q_3 \\ \dot{q}_3 q_0 + \dot{q}_2 q_1 + \dot{q}_1 q_2 + \dot{q}_0 q_3 \\ -\dot{q}_2 q_0 + \dot{q}_3 q_0 - \dot{q}_0 q_2 + \dot{q}_1 q_3 \end{bmatrix} \quad \text{-----} \; (1)$$

$$\Rightarrow A\dot{q} = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$A\dot{q} = \begin{bmatrix} q_0 \dot{q}_0 + q_1 \dot{q}_1 - q_2 \dot{q}_2 - q_3 \dot{q}_3 \\ q_3 \dot{q}_0 + q_2 \dot{q}_1 + q_1 \dot{q}_2 + q_0 \dot{q}_3 \\ -q_2 \dot{q}_0 + q_3 \dot{q}_1 - q_0 \dot{q}_2 + q_1 \dot{q}_3 \end{bmatrix} \quad \text{-----} \; (2)$$

$\Rightarrow$ From equation (1) and (2), we can show that

$$\boxed{\dot{A}q = A\dot{q}}$$

$\Rightarrow$ The same way we can show that

$$\dot{B}q = B\dot{q} = \begin{bmatrix} -q_3 \dot{q}_0 + q_2 \dot{q}_2 + q_1 \dot{q}_2 - q_0 \dot{q}_3 \\ q_0 \dot{q}_0 - q_1 \dot{q}_2 + q_2 \dot{q}_2 - q_3 \dot{q}_3 \\ q_1 \dot{q}_0 + q_0 \dot{q}_1 + q_3 \dot{q}_2 + q_2 \dot{q}_3 \end{bmatrix}$$

$$\boxed{\dot{B}q = B\dot{q}}$$

$\Rightarrow$ Again for $\dot{C}q = C\dot{q}$

$$\dot{C}q = \begin{bmatrix} q_2\dot{q}_0 + q_3\dot{q}_1 + q_0\dot{q}_2 + q_1\dot{q}_3 \\ -q_1\dot{q}_0 - q_0\dot{q}_1 + q_3\dot{q}_2 + q_2\dot{q}_3 \\ q_0\dot{q}_0 - q_1\dot{q}_1 - q_2\dot{q}_2 + q_3\dot{q}_3 \end{bmatrix}$$

$$\boxed{\dot{C}q = C\dot{q}}$$

#3) Show $Hq = 0$

$$\Rightarrow \quad H = \begin{bmatrix} (B^T C q)^T \\ (C^T A q)^T \\ (A^T B q)^T \end{bmatrix} = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

$$\Rightarrow \quad Hq = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$Hq = \begin{bmatrix} -q_1 q_0 + q_0 q_1 + q_3 q_2 - q_2 q_3 \\ -q_2 q_0 - q_3 q_1 + q_0 q_2 + q_1 q_3 \\ -q_3 q_0 + q_2 q_0 - q_1 q_2 + q_0 q_3 \end{bmatrix}$$

$$Hq = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{Hq = 0}$$

#4) show $\dot{H}\dot{q} = 0$

$$\Longrightarrow \quad \dot{H} = \begin{bmatrix} -\dot{q}_1 & \dot{q}_0 & \dot{q}_3 & -\dot{q}_2 \\ -\dot{q}_2 & -\dot{q}_3 & \dot{q}_0 & \dot{q}_1 \\ -\dot{q}_3 & \dot{q}_2 & -\dot{q}_1 & \dot{q}_0 \end{bmatrix}$$

$$\Longrightarrow \quad \dot{H}\dot{q} = \begin{bmatrix} -\dot{q}_1 & \dot{q}_0 & \dot{q}_3 & -\dot{q}_2 \\ -\dot{q}_2 & -\dot{q}_3 & \dot{q}_0 & \dot{q}_1 \\ -\dot{q}_3 & \dot{q}_2 & -\dot{q}_1 & \dot{q}_0 \end{bmatrix} \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$\dot{H}\dot{q} = \begin{bmatrix} -\dot{q}_1\dot{q}_0 + \dot{q}_0\dot{q}_1 + \dot{q}_3\dot{q}_2 - \dot{q}_2\dot{q}_3 \\ -\dot{q}_2\dot{q}_0 - \dot{q}_3\dot{q}_1 + \dot{q}_0\dot{q}_2 + \dot{q}_1\dot{q}_3 \\ -\dot{q}_3\dot{q}_0 + \dot{q}_2\dot{q}_0 - \dot{q}_1\dot{q}_2 + \dot{q}_0\dot{q}_3 \end{bmatrix}$$

$$\dot{H}\dot{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\dot{H}\dot{q} = 0}$$

#5) Show $H\dot{q} = -\dot{H}q$ and $w = -2\dot{H}q$

$$\Rightarrow H = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

$$\Rightarrow \dot{H} = \begin{bmatrix} -\dot{q}_1 & \dot{q}_0 & \dot{q}_3 & -\dot{q}_2 \\ -\dot{q}_2 & -\dot{q}_3 & \dot{q}_0 & \dot{q}_1 \\ -\dot{q}_3 & \dot{q}_2 & -\dot{q}_1 & \dot{q}_0 \end{bmatrix}$$

$$\Rightarrow \dot{H}q = \begin{bmatrix} -\dot{q}_1 & \dot{q}_0 & \dot{q}_3 & -\dot{q}_2 \\ -\dot{q}_2 & -\dot{q}_3 & \dot{q}_0 & \dot{q}_1 \\ -\dot{q}_3 & \dot{q}_2 & -\dot{q}_1 & \dot{q}_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\dot{H}q = \begin{bmatrix} -\dot{q}_1 q_0 + \dot{q}_0 q_1 + \dot{q}_3 q_2 - \dot{q}_2 q_3 \\ -\dot{q}_2 q_0 - \dot{q}_3 q_1 + \dot{q}_0 q_2 + \dot{q}_1 q_3 \\ -\dot{q}_3 q_0 + \dot{q}_2 q_0 - \dot{q}_1 q_2 + \dot{q}_0 q_3 \end{bmatrix}$$

$$\Rightarrow H\dot{q} = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$H\dot{q} = \begin{bmatrix} -q_1 \dot{q}_0 + q_0 \dot{q}_1 + q_3 \dot{q}_2 - q_2 \dot{q}_3 \\ -q_2 \dot{q}_0 - q_3 \dot{q}_1 + q_0 \dot{q}_2 + q_1 \dot{q}_3 \\ -q_3 \dot{q}_0 + q_2 \dot{q}_0 - q_1 \dot{q}_2 + q_0 \dot{q}_3 \end{bmatrix}$$

$$H\dot{q} = -\begin{bmatrix} q_1\dot{q}_0 - q_0\dot{q}_1 - q_3\dot{q}_2 + q_2\dot{q}_3 \\ q_2\dot{q}_0 + q_3\dot{q}_1 - q_0\dot{q}_2 - q_1\dot{q}_3 \\ q_3\dot{q}_0 - q_2\dot{q}_0 + q_1\dot{q}_2 - q_0\dot{q}_3 \end{bmatrix}$$

$$\therefore \boxed{H\dot{q} = -\dot{H}q}$$

$\Rightarrow$ $W = 2H\dot{q}$, since $H\dot{q} = -\dot{H}q$

$$W = 2(-\dot{H}q)$$

$$\boxed{W = -2\dot{H}q}$$

#6) Show $HH^T = I_{3\times3}$

$$\Longrightarrow HH^T = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}$$

$$HH^T = \begin{bmatrix} q_1^2+q_0^2+q_3^2+q_2^2 & q_1q_2-q_0q_3+q_0q_3-q_1q_2 & q_1q_3+q_0q_2-q_1q_3-q_0q_2 \\ q_1q_2-q_0q_3+q_0q_3-q_1q_2 & q_2^2+q_3^2+q_0^2+q_1^2 & q_2q_3-q_2q_3-q_0q_1+q_0q_1 \\ q_1q_3+q_0q_2-q_1q_3-q_0q_2 & q_2q_3-q_2q_3-q_0q_1+q_0q_1 & q_3^2+q_2^2+q_1^2+q_0^2 \end{bmatrix}$$

Since:- $q_0^2+q_1^2+q_2^2+q_3^2 = 1$

$$HH^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{HH^T = I_{3\times3}}$$

#7) Show $H^T H \dot{q} = \dot{q}$ and $\dot{q} = \frac{1}{2} H^T \omega$

$$\Rightarrow H^T H \dot{q} = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \dot{q}$$

$$H^T H \dot{q} = \begin{bmatrix} 1-q_0^2 & -q_0 q_1 & -q_0 q_2 & -q_0 q_3 \\ -q_0 q_1 & 1-q_1^2 & -q_1 q_2 & -q_1 q_3 \\ -q_0 q_2 & -q_1 q_2 & 1-q_2^2 & -q_2 q_3 \\ -q_0 q_3 & -q_1 q_3 & -q_2 q_3 & 1-q_3^2 \end{bmatrix} \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$H^T H \dot{q} = \begin{bmatrix} (1-q_0^2)\dot{q}_0 - q_0 q_1 \dot{q}_1 - q_0 q_2 \dot{q}_2 - q_0 q_3 \dot{q}_3 \\ -q_0 q_1 \dot{q}_0 + (1-q_1^2)\dot{q}_1 - q_1 q_2 \dot{q}_2 - q_1 q_3 \dot{q}_3 \\ -q_0 q_2 \dot{q}_0 - q_1 q_2 \dot{q}_1 + (1-q_2^2)\dot{q}_2 - q_2 q_3 \dot{q}_3 \\ -q_0 q_3 \dot{q}_0 - q_1 q_3 \dot{q}_2 - q_2 q_3 \dot{q}_2 + (1-q_3^2)\dot{q}_3 \end{bmatrix}$$

$$H^T H \dot{q} = \begin{bmatrix} \dot{q}_0 - q_0^2 \dot{q}_0 - q_0 q_1 \dot{q}_1 - q_0 q_2 \dot{q}_2 - q_0 q_3 \dot{q}_3 \\ \dot{q}_1 - q_0 q_1 \dot{q}_0 - q_1^2 \dot{q}_1 - q_1 q_2 \dot{q}_2 - q_1 q_3 \dot{q}_3 \\ \dot{q}_2 - q_0 q_2 \dot{q}_0 - q_1 q_2 \dot{q}_1 - q_2^2 \dot{q}_2 - q_2 q_3 \dot{q}_3 \\ \dot{q}_3 - q_0 q_3 \dot{q}_0 - q_1 q_3 \dot{q}_2 - q_2 q_3 \dot{q}_2 - q_3^2 \dot{q}_3 \end{bmatrix}$$

$$H^T H \dot{q} = \begin{bmatrix} \dot{q}_0 - q_0(q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3) \\ \dot{q}_1 - q_1(q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3) \\ \dot{q}_2 - q_2(q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3) \\ \dot{q}_3 - q_3(q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3) \end{bmatrix}$$

$\Rightarrow$ From $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, Taking the derivative with respect to time on both side of the eqn

$$\frac{\partial}{\partial t}\left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right) = \frac{\partial}{\partial t}(1)$$

$$2q_0\dot{q}_0 + 2q_1\dot{q}_1 + 2q_2\dot{q}_2 + 2q_3\dot{q}_3 = 0$$

$$\boxed{q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 = 0}$$

Now

$$H^T H \dot{q} = \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$\boxed{H^T H \dot{q} = \dot{q}}$$

$\Rightarrow$ $w = 2H\dot{q}$

$H^T w = 2H^T H\dot{q}$, since $H^T H\dot{q} = \dot{q}$

$$\dot{q} = \left(\frac{1}{2}\right) H^T w$$