Report 2 for Advanced Topics for Robot Mechanism

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January 16, 2023

1. Introduction

The aim of the report is to study the kinematics and dynamics of the manipulator. A 6DOF robot with a 6 revolute joint has been used for this purpose. The manipulator contains four revolution joints. In this report, the forward kinematics has been performed, DH (Denavit–Hartenberg) parameters has been used to obtain the detailed mathematical solution. The velocity relation and force relation has been done. The assumed manipulator and its endpoint 3D motion drawn in Fig1 as a hand-drawn punch picture using the technique of technical illustration has been realized.

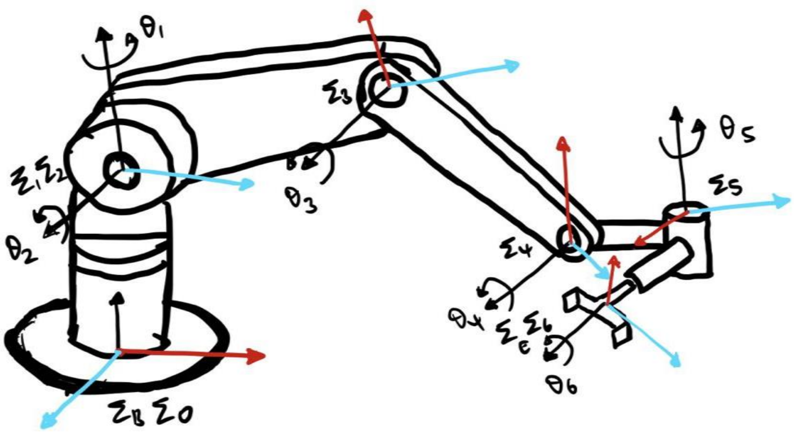

Fig1. 6DOF Elbow manipulator

2. DH Notation and Robot Structure

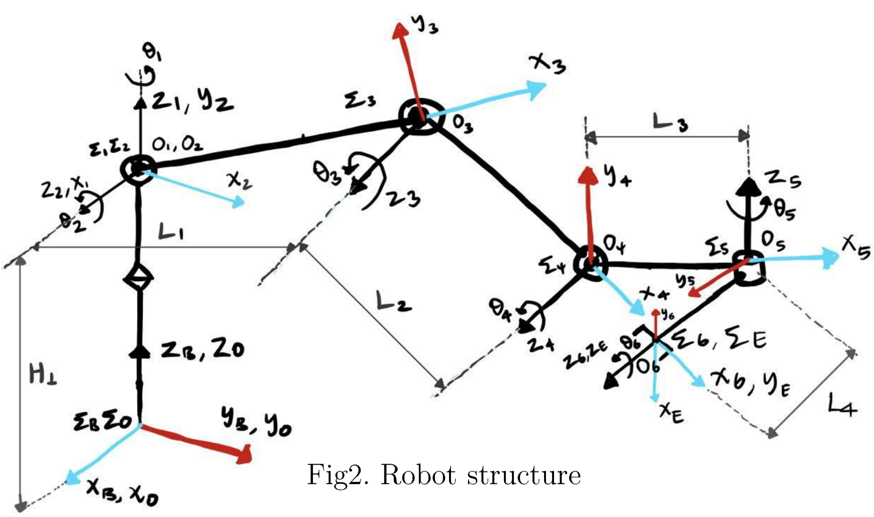

Fig2. Robot structure

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The four link parameters are demonstrated in the above hand-draw punch picture Fig2 which are associated with each link coordinate system. These parameters are also known as DH link parameters. These are illustrated in the Table [1] below:

Table 1. DH link parameters

$\boldsymbol{Link(i)}$	$\boldsymbol{a_{i-1},(m)}$	$\boldsymbol{\alpha_{i-1},(rad)}$	$\boldsymbol{d_i,(m)}$	$\boldsymbol{\theta_i,(rad)}$
1	0	0	$H_1 = 0.5$	θ_1
2	0	0	0	θ_2
3	$L_1 = 0.5$	$\frac{\pi}{2}$	0	θ_3
4	$L_2 = 0.4$	0	0	θ_4
5	$L_3 = 0.3$	0	0	θ_5
6	$L_4 = 0.3$	$\frac{-\pi}{2}$	0	θ_6
7	0	$\frac{\pi}{2}$	0	0

3. Kinematic Equation Derivation of this Manipulator

The parameters mentioned above in Table 1. are used for kinematic modelling of the robot.

$$^{B}T_E = {}^{0}T_1{}^{1}T_2{}^{2}T_3{}^{3}T_4 \dots\dots {}^{n}T_E$$

$$^{i-1}T_i = T_T(x_{i-1}, a_{i-1})T_R(x_{i-1}, \alpha_{i-1})T_T(z_i, d_i)T_T(z_i, \theta_i)$$

The transformation matrix for a link (i) is described as follows:

$$^{i-1}T_i = \begin{bmatrix} C_{\theta_i} & -S_{\theta_i} & 0 & a_{i-1} \\ C_{\alpha_{i-1}}S_{\theta_i} & C_{\alpha_{i-1}}C_{\theta_i} & -S_{\alpha_{i-1}} & -S_{\alpha_{i-1}}d_i \\ S_{\alpha_{i-1}}S_{\theta_i} & S_{\alpha_{i-1}}C_{\theta_i} & C_{\alpha_{i-1}} & C_{\alpha_{i-1}}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$where: \ C_{\theta_i} = \cos\theta_i \,, S_{\theta_i} = \sin\theta_i$$

The transformation matrix between the base and joint 1 is $\boldsymbol{^{0}T_1}$:

$$^{0}T_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & H_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix between joint 1 and joint 2 is $\boldsymbol{^{1}T_2}$:

$$
{}^{1}T_2 = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The transformation matrix between joint 2 and joint 3 is $\boldsymbol{{}^{2}T_3}$:

$$
{}^{2}T_3 = \begin{bmatrix} C_3 & -S_3 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The transformation matrix between joint 3 and joint 4 is $\boldsymbol{{}^{3}T_4}$:

$$
{}^{3}T_4 = \begin{bmatrix} C_4 & -S_4 & 0 & L_2 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The transformation matrix between joint 4 and joint 5 is $\boldsymbol{{}^{4}T}_5$:

$$
{}^{4}T_5 = \begin{bmatrix} C_5 & -S_5 & 0 & L_3 \\ S_5 & C_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The transformation matrix between joint 5 and joint 6 is $\boldsymbol{{}^{5}T_6}$:

$$
{}^{5}T_6 = \begin{bmatrix} C_6 & -S_6 & 0 & L_4 \\ 0 & 0 & -1 & 0 \\ S_6 & -C_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Finally, the transformation matrix between the end effector and link 6 is $\boldsymbol{{}^{6}T_E}$:

$$
{}^{6}T_E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Now, the resultant transformation matrix between the base axis and the end effector axis can be found by multiplying each matrix:

$$
{}^{B}T_E = {}^{B}T_0 {}^{0}T_1 {}^{1}T_2 {}^{2}T_3 {}^{3}T_4 \dots\dots {}^{n}T_E
$$

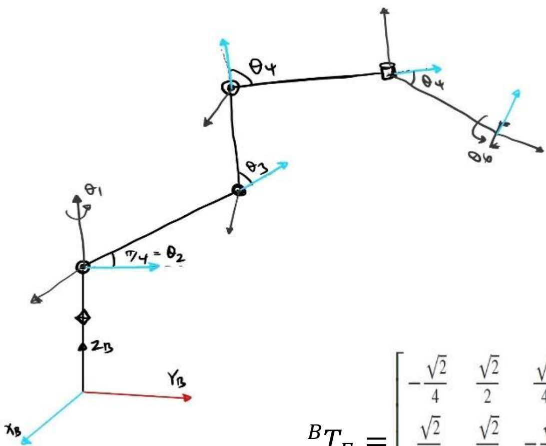
$$
\left(we\ can\ avoid\ \boldsymbol{{}^{B}T_0} becauseitisanidentitymatrix.\right)
$$

For simplification let’s first find the resultant transformation between the base and joint 2 as follow:

$$
{}^{B}T_E = {}^{B}T_0 {}^{0}T_1 {}^{1}T_2 {}^{2}T_3 {}^{3}T_4 {}^{4}T_5 {}^{5}T_6 {}^{6}T_E
$$

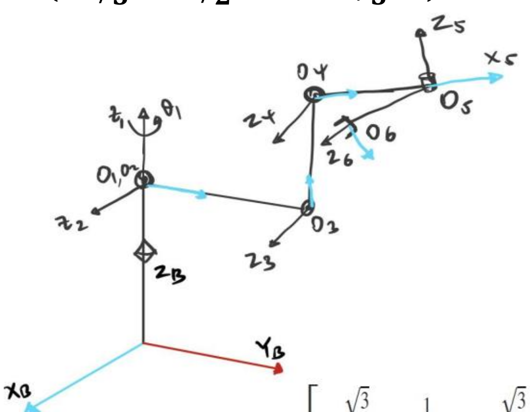
$$
\boxed{{}^{B}T_E = \begin{bmatrix} -S_{12}S_6 + C_{12}C_{345}C_6 & -S_{345}C_{12} & S_{12}C_6 + S_6C_{12}C_{345} & C_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) \\ S_{12}C_{345}C_6 + S_6C_{12} & -S_{12}S_{345} & S_{12}S_6C_{345} - C_{12}C_6 & S_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) \\ S_{345}C_6 & C_{345} & S_{345}S_6 & H_1 + S_3L_2 + S_{34}L_3 + S_{345}L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}
$$

Case 1: $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{\theta_3}, \boldsymbol{\theta_4}, \boldsymbol{\theta_5}, \boldsymbol{\theta_6})[\boldsymbol{rad}] = \left(0, {}^{\boldsymbol{\pi}}/_{\boldsymbol{4}},\ {}^{\boldsymbol{\pi}}/_{\boldsymbol{3}},\ {}^{-\boldsymbol{\pi}}/_{\boldsymbol{2}},\ {}^{-\boldsymbol{\pi}}/_{\boldsymbol{3}}, {}^{\boldsymbol{\pi}}/_{\boldsymbol{6}}\right)$



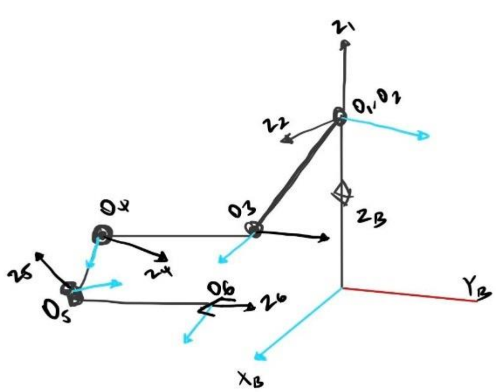
$$ {}^{B}T_{E} = \begin{bmatrix} -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0.15+0.2\sqrt{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Case 2: $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{\theta_3}, \boldsymbol{\theta_4}, \boldsymbol{\theta_5}, \boldsymbol{\theta_6})[\boldsymbol{rad}] = \left({}^{-\boldsymbol{\pi}}/_{\boldsymbol{3}}, 0,\ {}^{\boldsymbol{\pi}}/_{\boldsymbol{2}},\ 0,\ {}^{-4\boldsymbol{\pi}}/_{\boldsymbol{3}}, 0\right)$



$$ {}^{B}T_{E} = \begin{bmatrix} -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} & 0.25-0.05\sqrt{3} \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} & -\frac{\sqrt{3}\cdot(0.5-0.1\sqrt{3})}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Case 3: $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{\theta_3}, \boldsymbol{\theta_4}, \boldsymbol{\theta_5}, \boldsymbol{\theta_6})[\boldsymbol{rad}] = \left({}^{-\boldsymbol{\pi}}/_{\boldsymbol{2}}, {}^{-\boldsymbol{\pi}}/_{\boldsymbol{6}},\ {}^{\boldsymbol{\pi}}/_{\boldsymbol{4}},\ {}^{-\boldsymbol{\pi}}/_{\boldsymbol{4}},\ {}^{-\boldsymbol{\pi}}/_{\boldsymbol{2}}, -{}^{\boldsymbol{\pi}}/_{\boldsymbol{3}}\right)$



$$ {}^{B}T_{E} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{4} & -0.4-0.1\sqrt{2} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} & \frac{1}{4} & -\frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0.2\sqrt{2}+0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

4. Velocity relation of the Manipulator

Here, lets derive the velocity relation for the manipulator using the general manipulator case:

$$\vartheta = \begin{bmatrix} \dot{p} \\ \omega \end{bmatrix} \epsilon\, R^6, \begin{cases} \dot{p}(\epsilon\, R^3): translationalvelocity \\ \omega(\epsilon\, R^3): angularvelocity \end{cases}$$

Then, the velocity relation and the manipulator Jacobian represent as follows:

$$\vartheta = J_\vartheta \dot{q}$$

$$J_\vartheta = \begin{bmatrix} J_{\vartheta_1} & J_{\vartheta_2} & J_{\vartheta_3} & J_{\vartheta_4} & J_{\vartheta_5} & J_{\vartheta_6} \end{bmatrix} \epsilon\, R^{6\times 6}\ in\ this\ case$$

The Jacobian of the *i-th* column can be found by the following equation:

$$J_{\vartheta_i} = \begin{bmatrix} J_{\vartheta_{Li}} \\ J_{\vartheta_{Ri}} \end{bmatrix} = \begin{bmatrix} e_{z_i} \times p_{ip} \\ e_{z_i} \end{bmatrix}$$

$$where: e_{z_i} = R_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ and\ p_{ip} = p_p - p_i$$

$$p_p = \begin{bmatrix} C_{12}(L_1 + C_3 L_2 + C_{34} L_3 + C_{345} L_4) \\ S_{12}(L_1 + C_3 L_2 + C_{34} L_3 + C_{345} L_4) \\ H_1 + S_3 L_2 + S_{34} L_3 + S_{345} L_4 \end{bmatrix}$$

$$*\ in\ our\ case\ all\ joints\ are\ revolute\ so\ we\ use\ the\ above\ formula$$

Let's find the Jacobian matrix for **joint 1**:

$$^0T_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & H_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_{z_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, p_1 = \begin{bmatrix} 0 \\ 0 \\ H_1 \end{bmatrix}$$

$$J_{\vartheta_1} = \begin{bmatrix} e_{z_1} \times p_{1p} \\ e_{z_1} \end{bmatrix} = \begin{bmatrix} -S_{12}(L_1 + C_3 L_2 + C_{34} L_3 + C_{345} L_4) \\ C_{12}(L_1 + C_3 L_2 + C_{34} L_3 + C_{345} L_4) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's find the Jacobian matrix for **joint 2**:

$$^0T_2 = {}^0T_1\, {}^1T_2$$

$$^0T_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & 0 \\ S_{12} & C_{12} & 0 & 0 \\ 0 & 0 & 1 & H_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_{z_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 0 \\ H_1 \end{bmatrix}$$

$$J_{\vartheta_2} = \begin{bmatrix} e_{z_2} \times p_{2p} \\ e_{z_2} \end{bmatrix} = \begin{bmatrix} -S_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) \\ C_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's find the Jacobian matrix for **joint 3**:

$$^0T_3 = {}^0T_1{}^1T_2{}^2T_3$$

$$^0T_3 = \begin{bmatrix} C_{12}C_3 & -S_3C_{12} & S_{12} & C_{12}L_1 \\ S_{12}C_3 & -S_{12}S_3 & -C_{12} & S_{12}L_1 \\ S_3 & C_3 & 0 & H_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_{z_3} = \begin{bmatrix} S_{12} \\ -C_{12} \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} C_{12}L_1 \\ S_{12}L_1 \\ H_1 \end{bmatrix}$$

$$J_{\vartheta_3} = \begin{bmatrix} e_{z_3} \times p_{3p} \\ e_{z_3} \end{bmatrix} = \begin{bmatrix} C_{12}(-S_3L_2 - S_{34}L_3 - S_{345}L_4) \\ S_{12}(-S_3L_2 - S_{34}L_3 - S_{345}L_4) \\ C_3L_2 + C_{34}L_3 + C_{345}L_4 \\ S_{12} \\ -C_{12} \\ 0 \end{bmatrix}$$

Following just like the above methods the result of Jacobian matrix for **Joint 4:**

$$J_{\vartheta_4} = \begin{bmatrix} e_{z_4} \times p_{4p} \\ e_{z_4} \end{bmatrix} = \begin{bmatrix} C_{12}(-S_{34}L_3 - S_{345}L_4) \\ S_{12}(-S_{34}L_3 - S_{345}L_4) \\ C_{34}L_3 + C_{345}L_4 \\ S_{12} \\ -C_{12} \\ 0 \end{bmatrix}$$

The result of Jacobian matrix for **Joint 5:**

$$J_{\vartheta_5} = \begin{bmatrix} e_{z_5} \times p_{5p} \\ e_{z_5} \end{bmatrix} = \begin{bmatrix} -C_{12}S_{345}L_4 \\ -S_{12}S_{345}L_4 \\ C_{345}L_4 \\ S_{12} \\ -C_{12} \\ 0 \end{bmatrix}$$

The result of Jacobian matrix for **Joint 6:**

$$J_{\vartheta_6} = \begin{bmatrix} e_{z_6} \times p_{6p} \\ e_{z_6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -S_{345}C_{12} \\ -S_{345}S_{12} \\ C_{345} \end{bmatrix}$$

The manipulator **Jacobian** becomes:

$$J_\vartheta = \begin{bmatrix} J_{\vartheta_1} & J_{\vartheta_2} & J_{\vartheta_3} & J_{\vartheta_4} & J_{\vartheta_5} & J_{\vartheta_6} \end{bmatrix}$$

$$\boxed{J_\vartheta = \begin{bmatrix} -S_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) & -S_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) & C_{12}(-S_3L_2 - S_{34}L_3 - S_{345}L_4) & C_{12}(-S_{34}L_3 - S_{345}L_4) & -C_{12}S_{345}L_4 & 0 \\ C_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) & C_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) & S_{12}(-S_3L_2 - S_{34}L_3 - S_{345}L_4) & S_{12}(-S_{34}L_3 - S_{345}L_4) & -S_{12}S_{345}L_4 & 0 \\ 0 & 0 & C_3L_2 + C_{34}L_3 + C_{345}L_4 & C_{34}L_3 + C_{345}L_4 & C_{345}L_4 & 0 \\ 0 & 0 & S_{12} & S_{12} & S_{12} & -S_{345}C_{12} \\ 0 & 0 & -C_{12} & -C_{12} & -C_{12} & -S_{345}S_{12} \\ 1 & 1 & 0 & 0 & 0 & C_{345} \end{bmatrix}}$$

Now the **velocity relation** of the manipulator can be expressed as follow:

$$\vartheta = J_\vartheta \dot{q}$$

$$\vartheta = J_\vartheta \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix}$$

Prove of velocity relation with different method:

We have the position matrix from the transformation matrix, using that information we can derive the velocity relation by taking the partial differentiation of the position vector with respect to the joint variable. Then we can prove that the previous answer is correct as follow:

$$p_E = \begin{bmatrix} C_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) \\ S_{12}(L_1 + C_3L_2 + C_{34}L_3 + C_{345}L_4) \\ H_1 + S_3L_2 + S_{34}L_3 + S_{345}L_4 \end{bmatrix}$$

$$\vartheta = \begin{bmatrix} \frac{\partial p_{E,X}}{\partial \theta_1} & \frac{\partial p_{E,X}}{\partial \theta_2} & \frac{\partial p_{E,X}}{\partial \theta_3} & \frac{\partial p_{E,X}}{\partial \theta_4} & \frac{\partial p_{E,X}}{\partial \theta_5} & \frac{\partial p_{E,X}}{\partial \theta_6} \\ \frac{\partial p_{E,Y}}{\partial \theta_1} & \frac{\partial p_{E,Y}}{\partial \theta_2} & \frac{\partial p_{E,Y}}{\partial \theta_3} & \frac{\partial p_{E,Y}}{\partial \theta_4} & \frac{\partial p_{E,Y}}{\partial \theta_5} & \frac{\partial p_{E,Y}}{\partial \theta_6} \\ \frac{\partial p_{E,Z}}{\partial \theta_1} & \frac{\partial p_{E,Z}}{\partial \theta_2} & \frac{\partial p_{E,Z}}{\partial \theta_3} & \frac{\partial p_{E,Z}}{\partial \theta_4} & \frac{\partial p_{E,Z}}{\partial \theta_5} & \frac{\partial p_{E,Z}}{\partial \theta_6} \end{bmatrix}$$

By taking the partial differential coefficient we can obtained the manipulator Jacobian as follow:

$$J_\vartheta = \begin{bmatrix} -S_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & -S_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & C_{12}(-S_3L_2-S_{34}L_3-S_{345}L_4) & C_{12}(-S_{34}L_3-S_{345}L_4) & -C_{12}S_{345}L_4 & 0 \\ C_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & C_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & -S_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & S_{12}(-S_{34}L_3-S_{345}L_4) & -S_{12}S_{345}L_4 & 0 \\ 0 & 0 & C_3L_2+C_{34}L_3+C_{345}L_4 & C_{34}L_3+C_{345}L_4 & C_{345}L_4 & 0 \end{bmatrix}$$

Here we can drive and get similar velocity relation and manipulator Jacobian as the previous method except for the angular velocity term. This way we can prove our velocity relation equation.

Case 1: $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)[rad] = \left({}^{-\pi}/_2, {}^{-\pi}/_6, {}^{\pi}/_4, {}^{-\pi}/_4, {}^{-\pi}/_2, -{}^{\pi}/_3\right)$

$$J_{\vartheta_1} = \begin{bmatrix} e_{z_1} \times p_{1p} \\ e_{z_1} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} \\ -0.4-0.1\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{\vartheta_2} = \begin{bmatrix} e_{z_2} \times p_{2p} \\ e_{z_2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} \\ -0.4-0.1\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{\vartheta_3} = \begin{bmatrix} e_{z_3} \times p_{3p} \\ e_{z_3} \end{bmatrix} = \begin{bmatrix} 0.1+0.1\sqrt{2} \\ -\frac{\sqrt{3}(-0.2\sqrt{2}-0.2)}{2} \\ 0.2\sqrt{2}+0.3 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$J_{\vartheta_4} = \begin{bmatrix} e_{z_4} \times p_{4p} \\ e_{z_4} \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1\sqrt{3} \\ 0.3 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$J_{\vartheta_5} = \begin{bmatrix} e_{z_5} \times p_{5p} \\ e_{z_5} \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1\sqrt{3} \\ 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$J_{\vartheta_6} = \begin{bmatrix} e_{z_6} \times p_{6p} \\ e_{z_6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

Finally, the Jacobian matrix becomes:

$$J_\vartheta = \begin{bmatrix} \frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} & \frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} & 0.1+0.1\sqrt{2} & 0.1 & 0.1 & 0 \\ -0.4-0.1\sqrt{2} & -0.4-0.1\sqrt{2} & -\frac{\sqrt{3}(-0.2\sqrt{2}-0.2)}{2} & 0.1\sqrt{3} & 0.1\sqrt{3} & 0 \\ 0 & 0 & 0.2\sqrt{2}+0.3 & 0.3 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vartheta = J_\vartheta \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} = J_\vartheta \begin{bmatrix} 2 \\ 3.2 \\ 1.1 \\ 0.75 \\ 2.7 \\ 3 \end{bmatrix}$$

$$\vartheta = \begin{bmatrix} 0.11\sqrt{2}+0.455+2.6\sqrt{3}\cdot\left(0.2\sqrt{2}+0.8\right) \\ -2.08-0.52\sqrt{2}-0.55\sqrt{3}\left(-0.2\sqrt{2}-0.2\right)+0.345\sqrt{3} \\ 0.22\sqrt{2}+0.555 \\ \frac{3}{2}-2.275\sqrt{3} \\ 2.275+\frac{3\sqrt{3}}{2} \\ 5.2 \end{bmatrix}$$

Case 2: $(\boldsymbol{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6})[\boldsymbol{rad}] = \left(0, {}^{\boldsymbol{\pi}}/_{\boldsymbol{4}},\ {}^{\boldsymbol{\pi}}/_{\boldsymbol{3}},\ {}^{\boldsymbol{-\pi}}/_{\boldsymbol{2}},\ {}^{\boldsymbol{-\pi}}/_{\boldsymbol{3}}, {}^{\boldsymbol{\pi}}/_{\boldsymbol{6}}\right)$

Similarly with the previous case let's find the final Jacobian and the velocity of each joint.

$$J_\vartheta = \begin{bmatrix} -\frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & -\frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & \frac{\sqrt{2}\cdot(0.35-0.2\sqrt{3})}{2} & 0.175\sqrt{2} & 0.1\sqrt{2} & 0 \\ \frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & \frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & \frac{\sqrt{2}\cdot(0.35-0.2\sqrt{3})}{2} & 0.175\sqrt{2} & 0.1\sqrt{2} & 0 \\ 0 & 0 & 0.2+0.15\sqrt{3} & 0.15\sqrt{3} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vartheta = \begin{bmatrix} -2.6\sqrt{2}\cdot\left(0.15\sqrt{3}+0.7\right)+0.55\sqrt{2}\cdot\left(0.35-0.2\sqrt{3}\right)+0.40125\sqrt{2} \\ 0.55\sqrt{2}\cdot\left(0.35-0.2\sqrt{3}\right)+0.40125\sqrt{2}+2.6\sqrt{2}\cdot\left(0.15\sqrt{3}+0.7\right) \\ 0.22+0.2775\sqrt{3} \\ 3.775\sqrt{2} \\ -0.775\sqrt{2} \\ 5.2 \end{bmatrix}$$

5. Static force relation of the Manipulator

Here, let's use the duality between kinematics and statics to derive the static force relation using the velocity relation. The statics force relation utilizing the duality can be obtained as follow:

$$\boxed{\tau = {J_\vartheta}^T F}$$

$$where\colon\ \tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix}, F = \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix}, \qquad n - is\ moment(in\ Nm)$$

$$\begin{aligned} &{J_\vartheta}^T \\ &= \begin{bmatrix} -S_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & C_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & 0 & 0 & 0 & 1 \\ -S_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & C_{12}(L_1+C_3L_2+C_{34}L_3+C_{345}L_4) & 0 & 0 & 0 & 1 \\ C_{12}(-S_3L_2-S_{34}L_3-S_{345}L_4) & S_{12}(-S_3L_2-S_{34}L_3-S_{345}L_4) & C_3L_2+C_{34}L_3+C_{345}L_4 & S_{12} & -C_{12} & 0 \\ C_{12}(-S_{34}L_3-S_{345}L_4) & S_{12}(-S_{34}L_3-S_{345}L_4) & C_{34}L_3+C_{345}L_4 & S_{12} & -C_{12} & 0 \\ -C_{12}S_{345}L_4 0 & -S_{12}S_{345}L_4 & C_{345}L_4 & S_{12} & -C_{12} & 0 \\ 0 & 0 & 0 & -S_{345}C_{12} & -S_{345}S_{12} & C_{345} \end{bmatrix} \end{aligned}$$

Prove of static force relation of the manipulator with different method:

The main result of the static force relation can also be derived using the principle of virtual work. The virtual displacement of the end effector,$\boldsymbol{\delta d}$, expressed by $\boldsymbol{\Sigma_0}$ corresponding to ϑ and the virtual displacement of the joints δq, satisfies:

$$\delta d = J_\vartheta \delta q$$

Also, on the other hand, from the principle of virtual work we have:

$$(\delta q)^T \tau = (\delta d)^T \begin{bmatrix} f_E \\ n_E \end{bmatrix}$$

Hence, from the above two equations we obtained:

$$\tau = {J_\vartheta}^T \begin{bmatrix} f_E \\ n_E \end{bmatrix}$$

Here we can drive and get similar static force relation of the manipulator as the previous method. This way we can prove our static force relation equation.

case 1: $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{\theta_3}, \boldsymbol{\theta_4}, \boldsymbol{\theta_5}, \boldsymbol{\theta_6})[\boldsymbol{rad}] = \left({}^{-\pi}/_{2}, {}^{-\pi}/_{6}, {}^{\pi}/_{4}, {}^{-\pi}/_{4}, {}^{-\pi}/_{2}, -{}^{\pi}/_{3}\right)$

$$
{J_\vartheta}^T = \begin{bmatrix}
\frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} & -0.4-0.1\sqrt{2} & 0 & 0 & 0 & 1 \\
\frac{\sqrt{3}\cdot(0.2\sqrt{2}+0.8)}{2} & -0.4-0.1\sqrt{2} & 0 & 0 & 0 & 1 \\
0.1+0.1\sqrt{2} & -\frac{\sqrt{3}(-0.2\sqrt{2}-0.2)}{2} & 0.2\sqrt{2}+0.3 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
0.1 & 0.1\sqrt{3} & 0.3 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
0.1 & 0.1\sqrt{3} & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0
\end{bmatrix}
$$

$$\tau = {J_\vartheta}^T \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix} = {J_\vartheta}^T \begin{bmatrix} 30 \\ 20 \\ 50 \\ 0.2 \\ 0.5 \\ 0.15 \end{bmatrix}$$

$$
\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix}
-7.85-2.0\sqrt{2}+15\sqrt{3}\cdot\left(0.2\sqrt{2}+0.8\right) \\
-7.85-2.0\sqrt{2}+15\sqrt{3}\cdot\left(0.2\sqrt{2}+0.8\right) \\
-0.1\sqrt{3}-10\sqrt{3}\left(-0.2\sqrt{2}-0.2\right)+18.25+13.0\sqrt{2} \\
1.9\sqrt{3}+18.25 \\
3.25+1.9\sqrt{3} \\
0.1+0.25\sqrt{3}
\end{bmatrix}
$$

Case 2: $(\boldsymbol{\theta_1}, \boldsymbol{\theta_2}, \boldsymbol{\theta_3}, \boldsymbol{\theta_4}, \boldsymbol{\theta_5}, \boldsymbol{\theta_6})[\boldsymbol{rad}] = \left(0, {}^{\boldsymbol{\pi}}\!/_{\boldsymbol{4}},\ {}^{\boldsymbol{\pi}}\!/_{\boldsymbol{3}},\ {}^{-\boldsymbol{\pi}}\!/_{\boldsymbol{2}},\ {}^{-\boldsymbol{\pi}}\!/_{\boldsymbol{3}}, {}^{\boldsymbol{\pi}}\!/_{\boldsymbol{6}}\right)$

$$J_{\vartheta}{}^{T} = \begin{bmatrix} -\frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & \frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & 0 & 0 & 0 & 1 \\ -\frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & \frac{\sqrt{2}\cdot(0.15\sqrt{3}+0.7)}{2} & 0 & 0 & 0 & 1 \\ \frac{\sqrt{2}\cdot(0.35-0.2\sqrt{3})}{2} & \frac{\sqrt{2}\cdot(0.35-0.2\sqrt{3})}{2} & 0.2+0.15\sqrt{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0.175\sqrt{2} & 0.175\sqrt{2} & 0.15\sqrt{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0.1\sqrt{2} & 0.1\sqrt{2} & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{bmatrix} = \begin{bmatrix} -5\sqrt{2}\cdot\left(0.15\sqrt{3}+0.7\right)+0.15 \\ -5\sqrt{2}\cdot\left(0.15\sqrt{3}+0.7\right)+0.15 \\ -0.15\sqrt{2}+25\sqrt{2}\cdot\left(0.35-0.2\sqrt{3}\right)+10.0+7.5\sqrt{3} \\ 8.6\sqrt{2}+7.5\sqrt{3} \\ 4.85\sqrt{2} \\ 0.35\sqrt{2} \end{bmatrix}$$

6. Equation of Motion for 3DOF Manipulator

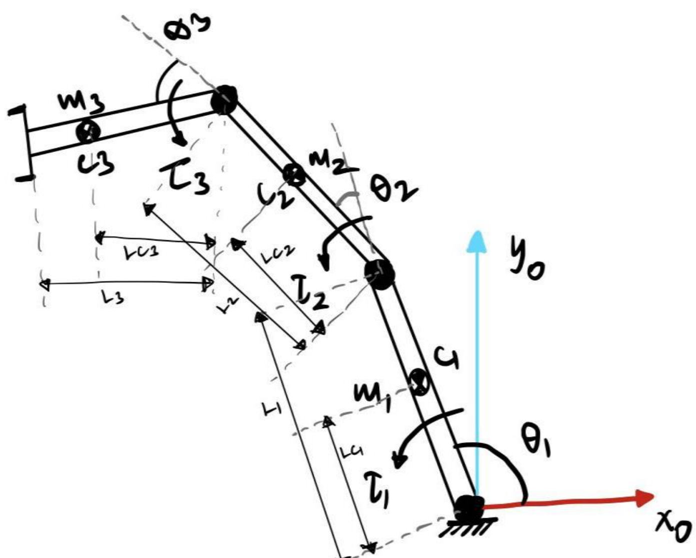


Fig. 3DOF of Manipulator

Step 1: The joint variable $\theta_i (i = 1, 2,3)$ is defined as the generalized coordinate, and the joint driving force $\tau_i (i = 1, 2,3)$ is defined as the generalized force.

Step 2:

Step 2a: The mass center position of the i-th link p_{ci} and its velocity $\dot{p}_{ci} (i = 1,2,3)$ can be obtained as follows.

$$p_{c1} = \begin{bmatrix} Lc_1 \cos(\theta_1) \\ Lc_1 \sin(\theta_1) \\ 0 \end{bmatrix}$$

$$p_{c2} = \begin{bmatrix} L_1 \cos(\theta_1) + Lc_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + Lc_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$p_{c3} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + Lc_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) + Lc_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{bmatrix}$$

$$\dot{p}_{c1} = \begin{bmatrix} -Lc_1 \sin(\theta_1)\dot{\theta}_1 \\ Lc_1 \cos(\theta_1)\dot{\theta}_1 \end{bmatrix}$$

$$\dot{p}_{c2} = \begin{bmatrix} -L_1 \sin(\theta_1)\dot{\theta}_1 - Lc_2 \left(\dot{\theta 2} + \dot{\theta}_1\right) \sin(\theta 2 + \theta_1) \\ L_1 \cos(\theta_1)\dot{\theta}_1 + Lc_2 \left(\dot{\theta 2} + \dot{\theta}_1\right) \cos(\theta 2 + \theta_1) \\ 0 \end{bmatrix}$$

$$\dot{p}_{c3} = \begin{bmatrix} -L_1 \sin(\theta_1)\dot{\theta}_1 - L_2 \left(\dot{\theta 2} + \dot{\theta}_1\right) \sin(\theta 2 + \theta_1) - Lc_3 \left(\dot{\theta 2} + \dot{\theta 3} + \dot{\theta}_1\right) \sin(\theta 2 + \theta 3 + \theta_1) \\ L_1 \cos(\theta_1)\dot{\theta}_1 + L_2 \left(\dot{\theta 2} + \dot{\theta}_1\right) \cos(\theta 2 + \theta_1) + Lc_3 \left(\dot{\theta 2} + \dot{\theta 3} + \dot{\theta}_1\right) \cos(\theta 2 + \theta 3 + \theta_1) \\ 0 \end{bmatrix}$$

Step 2b: Using step 2a. the kinetic energy $\boldsymbol{K}$ and the potential energy $\boldsymbol{P}$ of the manipulator can be calculated as follows:

$$\boldsymbol{K = K_1 + K_2 + K_3}$$

$$K = \frac{I_1 \dot{\theta}_1^2}{2} + \frac{I_2\left(\dot{\theta 2} + \dot{\theta}_1\right)^2}{2} + \frac{I_3\left(\dot{\theta 2} + \dot{\theta 3} + \dot{\theta}_1\right)^2}{2} + \frac{Lc_1^2 m_1 \dot{\theta}_1^2}{2}$$
$$+ \frac{m_2\left(L_1^2\dot{\theta}_1^2 + 2L_1 Lc_2 \cos(\theta 2)\dot{\theta 2}\dot{\theta}_1 + 2L_1 Lc_2 \cos(\theta 2)\dot{\theta}_1^2 + Lc_2^2\dot{\theta 2}^2 + 2Lc_2^2\dot{\theta 2}\dot{\theta}_1 + Lc_2^2\dot{\theta}_1^2\right)}{2}$$
$$+ \frac{m_3\left(\left(L_1 \sin(\theta_1)\dot{\theta}_1 + L_2\left(\dot{\theta 2} + \dot{\theta}_1\right)\sin(\theta 2 + \theta_1) + Lc_3\left(\dot{\theta 2} + \dot{\theta 3} + \dot{\theta}_1\right)\sin(\theta 2 + \theta 3 + \theta_1)\right)^2 + \left(L_1 \cos(\theta_1)\dot{\theta}_1 + L_2\left(\dot{\theta 2} + \dot{\theta}_1\right)\cos(\theta 2 + \theta_1) + Lc_3\left(\dot{\theta 2} + \dot{\theta 3} + \dot{\theta}_1\right)\cos(\theta 2 + \theta 3 + \theta_1)\right)^2\right)}{2}$$

$$\boldsymbol{P = P_1 + P_2 + P_3}$$

$$P = Lc_1 g m_1 \sin(\theta_1) + g m_2 (L_1 \sin(\theta_1) + Lc_2 \sin(\theta 2 + \theta_1)) + g m_3 (L_1 \sin(\theta_1) + L_2 \sin(\theta 2 + \theta_1) + Lc_3 \sin(\theta 2 + \theta 3 + \theta_1))$$

Step 2c: Using step 2b. the Lagrangian $\boldsymbol{L}$ can be calculated as follows:

$$\boldsymbol{L = K - P}$$

$$\begin{aligned}
L = {} & \frac{I_1 \dot{\theta}_1^2}{2} + \frac{I_2 (\dot{\theta} 2 + \dot{\theta}_1)^2}{2} + \frac{I_3 (\dot{\theta} 2 + \dot{\theta} 3 + \dot{\theta}_1)^2}{2} + \frac{Lc_1^2 m_1 \dot{\theta}_1^2}{2} - Lc_1 g m_1 \sin(\theta_1) - g m_2 (L_1 \sin(\theta_1) + Lc_2 \sin(\theta 2 + \theta_1)) \\
& - g m_3 (L_1 \sin(\theta_1) + L_2 \sin(\theta 2 + \theta_1) + Lc_3 \sin(\theta 2 + \theta 3 + \theta_1)) \\
& + \frac{m_2 \left(L_1^2 \dot{\theta}_1^2 + 2 L_1 Lc_2 \cos(\theta 2) \dot{\theta} 2 \dot{\theta}_1 + 2 L_1 Lc_2 \cos(\theta 2) \dot{\theta}_1^2 + Lc_2^2 \dot{\theta} 2^2 + 2 Lc_2^2 \dot{\theta} 2 \dot{\theta}_1 + Lc_2^2 \dot{\theta}_1^2 \right)}{2} \\
& + \frac{\begin{gathered} m_3 \left(\left(L_1 \sin(\theta_1) \dot{\theta}_1 + L_2 (\dot{\theta} 2 + \dot{\theta}_1) \sin(\theta 2 + \theta_1) + Lc_3 (\dot{\theta} 2 + \dot{\theta} 3 + \dot{\theta}_1) \sin(\theta 2 + \theta 3 + \theta_1) \right)^2 \right. \\ \left. + \left(L_1 \cos(\theta_1) \dot{\theta}_1 + L_2 (\dot{\theta} 2 + \dot{\theta}_1) \cos(\theta 2 + \theta_1) + Lc_3 (\dot{\theta} 2 + \dot{\theta} 3 + \dot{\theta}_1) \cos(\theta 2 + \theta 3 + \theta_1) \right)^2 \right) \end{gathered}}{2}
\end{aligned}$$

Step 3: Substituting the Lagrangian L calculated in step 2c into the Lagrangian equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i, \quad i = 1, 2, \cdots, n$$

And the following set of multiple equations of motion can be obtained:

$$\begin{aligned}
& I_1 \ddot{\theta}_1 + I_2 (\ddot{\theta} 2 + \ddot{\theta}_1) + I_3 (\ddot{\theta} 2 + \ddot{\theta} 3 + \ddot{\theta}_1) + Lc_1^2 m_1 \ddot{\theta}_1 + Lc_1 g m_1 \cos(\theta_1) + g m_2 (L_1 \cos(\theta_1) + Lc_2 \cos(\theta 2 + \theta_1)) \\
& + g m_3 (L_1 \cos(\theta_1) + L_2 \cos(\theta 2 + \theta_1) + Lc_3 \cos(\theta 2 + \theta 3 + \theta_1)) \\
\boldsymbol{\tau_1} = {} & + m_2 \left(L_1^2 \ddot{\theta}_1 - L_1 Lc_2 \sin(\theta 2) \dot{\theta} 2^2 - 2 L_1 Lc_2 \sin(\theta 2) \dot{\theta} 2 \dot{\theta}_1 + L_1 Lc_2 \cos(\theta 2) \ddot{\theta} 2 + 2 L_1 Lc_2 \cos(\theta 2) \ddot{\theta}_1 + Lc_2^2 \ddot{\theta} 2 + Lc_2^2 \ddot{\theta}_1 \right) \\
& + m_3 \Big((L_1 \sin(\theta_1) + L_2 \sin(\theta 2 + \theta_1) + Lc_3 \sin(\theta 2 + \theta 3 + \theta_1)) \Big(L_1 \sin(\theta_1) \ddot{\theta}_1 + L_1 \cos(\theta_1) \dot{\theta}_1^2 + L_2 (\dot{\theta} 2 + \dot{\theta}_1)^2 \cos(\theta 2 + \theta_1) \\
& + L_2 (\ddot{\theta} 2 + \ddot{\theta}_1) \sin(\theta 2 + \theta_1) + Lc_3 (\dot{\theta} 2 + \dot{\theta} 3 + \dot{\theta}_1)^2 \cos(\theta 2 + \theta 3 + \theta_1) + Lc_3 (\ddot{\theta} 2 + \ddot{\theta} 3 + \ddot{\theta}_1) \sin(\theta 2 + \theta 3 + \theta_1) \Big) \\
& - (L_1 \cos(\theta_1) + L_2 \cos(\theta 2 + \theta_1) + Lc_3 \cos(\theta 2 + \theta 3 + \theta_1)) \Big(L_1 \sin(\theta_1) \dot{\theta}_1^2 - L_1 \cos(\theta_1) \ddot{\theta}_1 + L_2 (\dot{\theta} 2 + \dot{\theta}_1)^2 \sin(\theta 2 + \theta_1) - L_2 (\ddot{\theta} 2 + \ddot{\theta}_1) \cos \\
& (\theta 2 + \theta_1) + Lc_3 (\dot{\theta} 2 + \dot{\theta} 3 + \dot{\theta}_1)^2 \sin(\theta 2 + \theta 3 + \theta_1) - Lc_3 (\ddot{\theta} 2 + \ddot{\theta} 3 + \ddot{\theta}_1) \cos(\theta 2 + \theta 3 + \theta_1) \Big) \Big)
\end{aligned}$$

$$\begin{aligned}
& I_2 \ddot{\theta} 2 + I_2 \ddot{\theta}_1 + I_3 \ddot{\theta} 2 + I_3 \ddot{\theta} 3 + I_3 \ddot{\theta}_1 + L_1 L_2 m_3 \sin(\theta 2) \dot{\theta}_1^2 + L_1 L_2 m_3 \cos(\theta 2) \ddot{\theta}_1 + L_1 Lc_2 m_2 \sin(\theta 2) \dot{\theta}_1^2 + L_1 Lc_2 m_2 \cos(\theta 2) \ddot{\theta}_1 + L_1 Lc_3 m_3 \sin \\
\boldsymbol{\tau_2} = {} & (\theta 2 + \theta 3) \dot{\theta}_1^2 + L_1 Lc_3 m_3 \cos(\theta 2 + \theta 3) \ddot{\theta}_1 + L_2^2 m_3 \ddot{\theta} 2 + L_2^2 m_3 \ddot{\theta}_1 - 2 L_2 Lc_3 m_3 \sin(\theta 3) \dot{\theta} 2 \dot{\theta} 3 - L_2 Lc_3 m_3 \sin(\theta 3) \dot{\theta} 3^2 - 2 L_2 Lc_3 m_3 \sin(\theta 3) \dot{\theta} 3 \dot{\theta}_1 \\
& + 2 L_2 Lc_3 m_3 \cos(\theta 3) \ddot{\theta} 2 + L_2 Lc_3 m_3 \cos(\theta 3) \ddot{\theta} 3 + 2 L_2 Lc_3 m_3 \cos(\theta 3) \ddot{\theta}_1 + L_2 g m_3 \cos(\theta 2 + \theta_1) + Lc_2^2 m_2 \ddot{\theta} 2 + Lc_2^2 m_2 \ddot{\theta}_1 + Lc_2 g m_2 \cos \\
& (\theta 2 + \theta_1) + Lc_3^2 m_3 \ddot{\theta} 2 + Lc_3^2 m_3 \ddot{\theta} 3 + Lc_3^2 m_3 \ddot{\theta}_1 + Lc_3 g m_3 \cos(\theta 2 + \theta 3 + \theta_1)
\end{aligned}$$

$$\begin{aligned}
& I_3 \ddot{\theta} 2 + I_3 \ddot{\theta} 3 + I_3 \ddot{\theta}_1 + L_1 Lc_3 m_3 \sin(\theta 2 + \theta 3) \dot{\theta}_1^2 + L_1 Lc_3 m_3 \cos(\theta 2 + \theta 3) \ddot{\theta}_1 + L_2 Lc_3 m_3 \sin(\theta 3) \dot{\theta} 2^2 + 2 L_2 Lc_3 m_3 \sin(\theta 3) \dot{\theta} 2 \dot{\theta}_1 + L_2 Lc_3 m_3 \sin \\
\boldsymbol{\tau_3} = {} & (\theta 3) \dot{\theta}_1^2 + L_2 Lc_3 m_3 \cos(\theta 3) \ddot{\theta} 2 + L_2 Lc_3 m_3 \cos(\theta 3) \ddot{\theta}_1 + Lc_3^2 m_3 \ddot{\theta} 2 + Lc_3^2 m_3 \ddot{\theta} 3 + Lc_3^2 m_3 \ddot{\theta}_1 + Lc_3 g m_3 \cos(\theta 2 + \theta 3 + \theta_1)
\end{aligned}$$

Step 4: Combining the set of equations obtained in step 3, the single equation of motion can be obtained as follow:

$$M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) = \tau$$

$$\begin{cases} M(\theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \\ c(\theta, \dot{\theta}) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ g(\theta) = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \end{cases}$$

7. Conclusion

In this small report, 6DOF manipulator that moves in a 3D space studied. The kinematics equation of this manipulator derived by using the DH parameter method. Next, velocity relation and static force relation has been studied for the 6DOF manipulator. In addition, the dynamics of 3DOF manipulator has been studied. And further each calculation has been shown and verified using hand drawn picture and position and rotation vector verification.