8.7 Mathematical Induction

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Objective

- Prove a statement by mathematical induction

Many mathematical facts are established by first observing a pattern, then making a conjecture about the general nature of the pattern, and finally by *proving* the conjecture. In order to *prove* a conjecture, we use existing facts, combine them in such a way that they are relevant to the conjecture, and proceed in a logical manner until the truth of the conjecture is established.

For example, let us make a conjecture regarding the sum of the first n even integers. First, we look for a pattern:

$$\begin{aligned} 2 &= 2 \\ 2+4 &= 6 \\ 2+4+6 &= 12 \\ 2+4+6+8 &= 20 \\ 2+4+6+8+10 &= 30 \\ &\vdots \end{aligned}$$

From the equations above, we can build the following table:

n	Sum of first n even integers
1	2
2	6
3	12
4	20
5	30

The numbers in the "sum" column of the table can be factored as follows: $2 = 1 \cdot 2$, $6 = 2 \cdot 3$, $12 = 3 \cdot 4$, $20 = 4 \cdot 5$, and $30 = 5 \cdot 6$. Noting the values of n to which the factorizations correspond, we make our conjecture:

The sum of the first n even integers is $n(n+1)$.

According to our calculations, this holds for n up to and including 5. But does it hold for all n? To establish the pattern for all values of n, we must *prove* the conjecture. Simply substituting various values of n is not feasible, since you would have to verify the statement for infinitely many n. A more practical proof technique is needed. We next introduce a method called **mathematical induction**, which is typically used to prove statements such as this.

Discover and Learn

What pattern do you observe for the sum of the first n odd integers? $\square$

Mathematical induction

Before giving a formal definition of mathematical induction, we take our discussion of the sum of the first n even integers and introduce some new notation which we will need in order to work with this type of proof.

First, the conjecture is given a name: P_n. The subscript n means that the conjecture depends on n. Stating our conjecture, we have

$$P_n : \text{The sum of the first } n \text{ even integers is } n(n+1)$$

For some specific values of n, the conjecture reads as follows:

$$P_8 : \text{The sum of the first 8 even integers is } 8 \cdot 9 = 72$$
$$P_{12}: \text{The sum of the first 12 even integers is } 12 \cdot 13 = 156$$
$$P_k : \text{The sum of the first } k \text{ even integers is } k(k+1)$$
$$P_{k+1} : \text{The sum of the first } k+1 \text{ even integers is } (k+1)(k+2)$$

We next state the principle of mathematical induction, which will be needed to complete the proof of our conjecture.

The Principle of Mathematical Induction

Let n be a natural number and let P_n be a statement that depends on n. If

1. P_1 is true, and
2. for all positive integers k, P_{k+1} can be shown to be true if P_k is assumed to be true,

then P_n is true for all natural numbers n.

The underlying scheme behind proof by induction consists of two key pieces:

1. Proof of the base case: proving that P_1 is true
2. Use of the induction hypothesis (the assumption that P_k is true) for a general value of k to show that P_{k+1} is true

Taken together, these two pieces (proof of the base case and use of the induction hypothesis) prove that P_n holds for every natural number n.

In proving statements by induction, we often have to take an expression in the variable k and replace k with $k+1$. The next example illustrates that process.

EXAMPLE 1 Replacing k with $k+1$ in an algebraic expression

Replace k with $k+1$ in the following.

a) $3^k - 1$

b) $\dfrac{k(k+1)(2k+1)}{6}$

Solution

a) Replacing k by $k+1$,we obtain

$$3^{k+1} - 1$$

b) Replacing k by $k+1$ and simplifying,

$$\begin{aligned} &\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ =&\frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

□

Check It Out 1

Replace k by $k+1$ in $2k(k+2)$. □

We now return to the conjecture we made at the beginning of this section, and prove it by induction.

EXAMPLE 2 Proving a formula by induction

Prove the following formula by induction:

$$2 + 4 + \cdots + 2n = n(n+1).$$

Solution This is the just the statement that we conjectured earlier, but in the form of an equation. Recall that we denoted that statement by P_n, so we denote the proposed equation by P_n as well.

First, we prove that P_n is true for $n = 1$ (the base case). We do this by replacing every n in P_n with a 1, and then demonstrating that the result is true.

$$P_1 : \qquad 2(1) = 1(1+1)$$

Since $2(1) = 1(1+1)$, we see that P_1 is true.

Next, we state P_k (replace every n in P_n with a k) and assume that P_k is true.

$$P_k : \qquad 2 + 4 + \cdots + 2k = k(k+1)$$

Finally, we state P_{k+1}, and use the induction hypothesis (the assumption that P_k is true) to prove that P_{k+1} holdsas well.

$$P_{k+1} : \qquad 2 + 4 + \cdots + 2k + 2(k+1) = (k+1)(k+2)$$

To prove that P_{k+1} holds, we will start with the expression on the left-hand side of P_{k+1}, and show that it is equal to the expression on the right-hand side.

$$\begin{aligned}
&\boxed{2+4+\cdots+2k}+2(k+1) && \text{Left-hand side of } P_{k+1}\\
&=\boxed{k(k+1)}+2(k+1) && \text{Induction hypothesis}\\
&=k^2+k+2k+2 && \text{Expand}\\
&=k^2+3k+2 && \text{Combine like terms}\\
&=(k+1)(k+2) && \text{Factor}
\end{aligned}$$

We see that the result, $(k+1)(k+2)$, is the expression on the right-hand side of P_{k+1}. Thus, by mathematical induction, P_n is true for all natural numbers n. □

Check It Out 2

Prove the following formula by mathematical induction: $1+3+5+\cdots+(2n-1)=n^2$. □

Example 3 Proving a summation formula by induction

Prove the following formula by induction:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

Solution First, denote the proposed equation by P_n, and prove that it holds for $n = 1$ (the base case). Replacing every n with a 1, we get

$$P_1: \qquad 1 = \frac{1(1+1)}{2}$$

Clearly, this is true, so P_1 holds.

Next, state P_k and assume that P_k is true.

$$P_k: \qquad 1+2+3+\cdots+k = \frac{k(k+1)}{2}.$$

Finally, state P_{k+1}, and use the induction hypothesis (the assumption that P_k is true) to prove that P_{k+1} holdsas well.

$$P_{k+1}: \qquad 1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$$

$$\begin{aligned}
&\boxed{1+2+3+\cdots+k}+k+1 && \text{Left-hand side of } P_{k+1}\\
&=\boxed{\frac{k(k+1)}{2}}+k+1 && \text{Induction hypothesis}\\
&=\frac{k(k+1)+2(k+1)}{2} && \text{Use common denominator}\\
&=\frac{k^2+k+2k+2}{2} && \text{Expand}\\
&=\frac{k^2+3k+2}{2} && \text{Combine like terms}\\
&=\frac{(k+1)(k+2)}{2} && \text{Factor}
\end{aligned}$$

We see that the result, $\frac{(k+1)(k+2)}{2}$, is the expression on the right-hand side of P_{k+1}. Thus, by mathematical induction, P_n is true for all natural numbers n. $\square$

Check It Out 3

Prove by induction: $2+5+8+\cdots+(3n-1)=\frac{1}{2}n(3n+1)$. □

EXAMPLE 4 Proving a formula for partial sums by induction

Prove by induction:

$$1+2+2^2+2^3+\cdots+2^{n-1}=2^n-1.$$

Solution First, denote the proposed equation by P_n, and prove that it holds for $n=1$ (the base case), by replacing every n with a 1.

$$P_1: \qquad 1=2^1-1,$$

It is easy to see that P_1 is true.

Next, state P_k and assume that P_k is true.

$$P_k: \qquad 1+2+2^2+2^3+\cdots+2^{k-1}=2^k-1$$

Finally, state P_{k+1}, and use the induction hypothesis (the assumption that P_k is true) to show that P_{k+1} holdsas well.

$$P_{k+1}: \qquad 1+2+2^2+2^3+\cdots+2^{k-1}+2^k=2^{k+1}-1$$

$\boxed{1+2+2^2+2^3+\cdots+2^{k-1}}+2^k$	Left-hand side of P_{k+1}
$=\boxed{2^k-1}+2^k$	Induction hypothesis
$=2(2^k)-1$	Combine like terms
$=2^{k+1}-1$	Simplify

We see that the result, $2^{k+1} - 1$, is the expression on the right-hand side of P_{k+1}. Thus, by mathematical induction, P_n is true for all natural numbers n. □

Check It Out 4

Prove by induction: $1 + 4 + 4^2 + \cdots + 4^{n-1} = \dfrac{4^n - 1}{3}$ □

You may wonder how one gets the formulas to prove by induction in the first place. Many of these are arrived at by first examining patterns and then coming up with a general formula using various mathematical facts. A complete discussion of how to *obtain* these formulas is beyond the scope of this book.

8.7 Key Points

- The **Principle of Mathematical Induction** is stated as follows: Let n be a natural number and let P_n be a statement that depends on n. If

  1. P_1 is true, and
  2. for all positive integers k, P_{k+1} can be shown to be true if P_k is assumed to be true,

  then P_n is true for all natural numbers n.

8.7 Exercises

SKILLS This set of exercises will reinforce the skills illustrated in this section.

Replace k by $k+1$ in each of the following expressions.

1. $k(k+1)(k+2)$
2. $3^k - 1$
3. $\dfrac{k}{k+1}$
4. $\dfrac{3}{1+k^2}$

In Exercises 5–18, prove the statements by induction.

5. $3+5+\cdots+(2n+1)=n(n+2)$
6. $2+6+10+\cdots+(4n-2)=2n^2$
7. $1+4+7+\cdots+(3n-2)=\dfrac{n(3n-1)}{2}$
8. $5+4+3+\cdots+(6-n)=\dfrac{1}{2}n(11-n)$
9. $7+5+3+\cdots+(9-2n)=-n^2+8n$
10. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$
11. $1^3+2^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$
12. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$
13. $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$.

14. $1 + 3 + 3^2 + \cdots + 3^{n-1} = \dfrac{3^n - 1}{2}$

15. $1 + 5 + 5^2 + \cdots + 5^{n-1} = \dfrac{5^n - 1}{4}$

16. $1 + r + r^2 + \cdots + r^{n-1} = \dfrac{r^n - 1}{r - 1}$, r a positive integer, $r \neq 1$.

17. $3^n - 1$ is divisible by 2.

18. $n^2 + n$ is even.

CONCEPTS This set of exercises will draw on the ideas presented in this section and your general math background.

Induction is not the only method of proving a statement is true. The following problems suggest alternate methods for proving statements.

19. By factoring $n^2 + n$, n a natural number, show that $n^2 + n$ is divisible by 2.

20. By factoring $a^3 - b^3$, a, b positive integers, show that $a^3 - b^3$ is divisible by $a - b$.

21. Prove that $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$ by using the formula for the sum of terms of an arithmetic sequence.

22. Prove that $1 + 4 + 4^2 + \cdots + 4^{n-1} = \dfrac{4^n - 1}{3}$ by using the formula for the sum of terms of a geometric sequence.