Unit Four

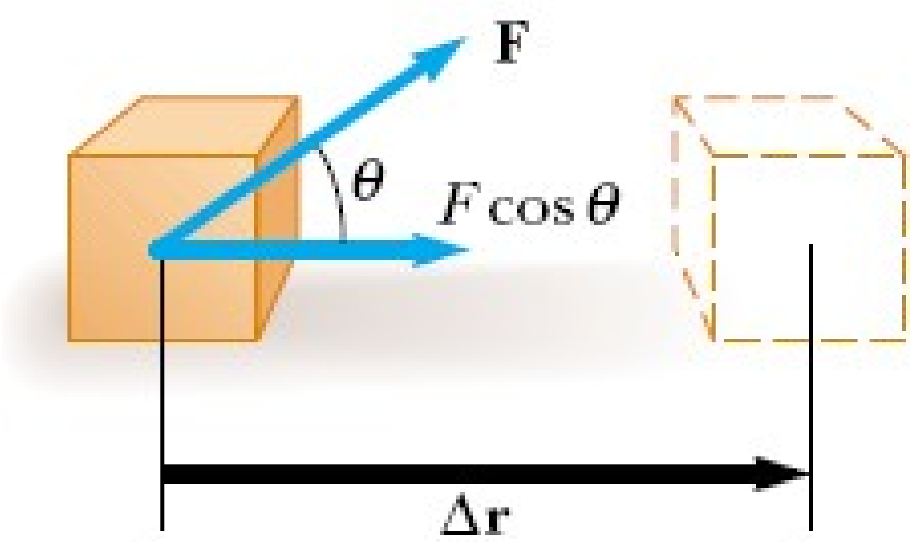
4. Work and energy

4.1. Work done by a constant force

- Work is energy exerted on the object.
- Energy is the capacity to do work.
- Work done by a constant force:Is defined as the product of the component of the force in the direction of the
- displacement and the magnitude of the displacement.
  - Consider an object that undergoes a displacement along a straight line while acted on by a constant force F that makes an angle with the direction of the displacement.

- Work done by a constant force:Is defined as the product of the component of the force in the direction of the displacement and the magnitude of the displacement.

$$W = (F\cos\theta)\, s = \vec{F} \cdot \vec{s} = \begin{cases} +Fs & \text{if } \theta = 0^\circ \\ 0 & \text{if } \theta = 90^\circ \\ -Fs & \text{if } \theta = 180^\circ \end{cases}$$



$$W = F \Delta r \cos\theta$$

W = **F** . °**r** = °**r** . **F**

- The SI unit of work is the Newton . Meter(Nm) is called Joule

A particle moving in the *xy* plane undergoes a displacement $\Delta\mathbf{r} = (2.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}})$ m as a constant force $\mathbf{F} = (5.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})$ N acts on the particle.

a. Calculate the magnitudes of the displacement and the force.
b. Calculate the work done by **F**.

Solution

a $$\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(2.0)^2 + (3.0)^2} = \boxed{3.6\,\mathrm{m}}$$

$$F = \sqrt{{F_x}^2 + {F_y}^2} = \sqrt{(5.0)^2 + (2.0)^2} = \boxed{5.4\,\mathrm{N}}$$

b. $$W = \mathbf{F}\cdot\Delta\mathbf{r} = [(5.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\,\mathrm{N}]\cdot[(2.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}})\,\mathrm{m}]$$

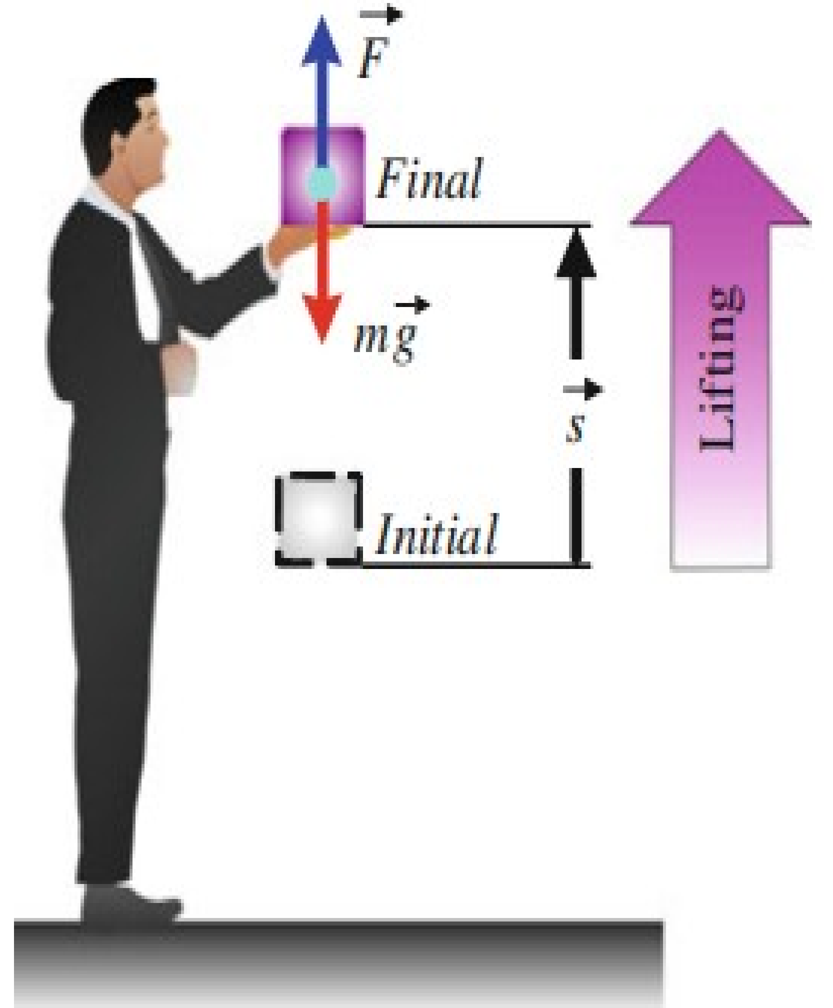
$$= (5.0\hat{\mathbf{i}}\cdot 2.0\hat{\mathbf{i}} + 5.0\hat{\mathbf{i}}\cdot 3.0\hat{\mathbf{j}} + 2.0\hat{\mathbf{j}}\cdot 2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}\cdot 3.0\hat{\mathbf{j}})\mathrm{N}\cdot\mathrm{m}$$

$$= [10 + 0 + 0 + 6]\,\mathrm{N}\cdot\mathrm{m} = \boxed{16\,\mathrm{J}}$$

Work Done by a Weight

- Consider a block of mass **m** to be lifted up with almost zero acceleration (i.e., a ≈ 0) by a constant force **F** applied by a person. While in motion, the force F and the weight **mg** will be oppositely directed but equal in magnitude, i.e. **F = - mg** .
- The work done by F is:

$$W_F = \vec{F} \cdot \vec{s} = Fs\cos 0^\circ = Fs = mgs$$



- Also, we can calculate the work done by the gravitational force (**mg**) as follows:

$$W_g = m\vec{g} \bullet \vec{s} = m\,g\,s\cos 180^\circ = -m\,g\,s$$

- Thus, we conclude that:

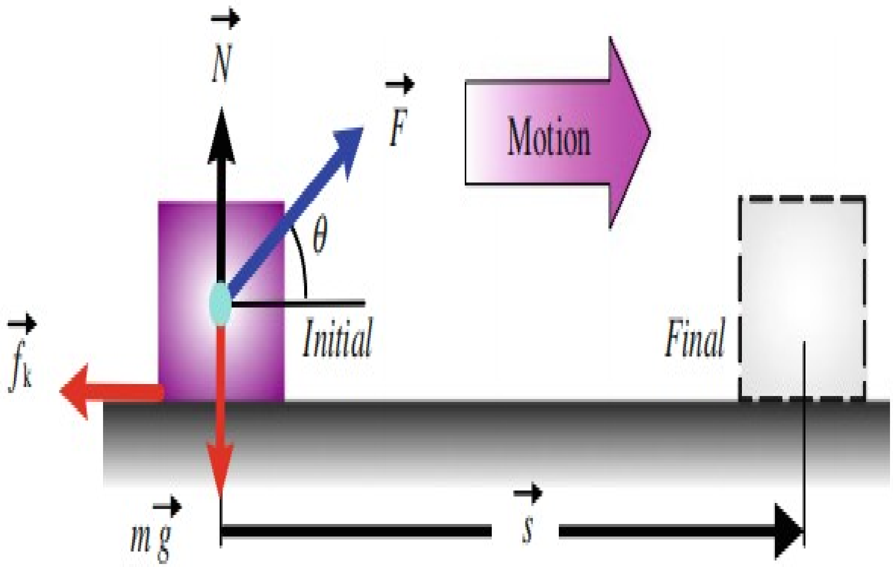
$$\boxed{W_F = m\,g\,s} \quad \text{and} \quad \boxed{W_g = -m\,g\,s} \quad \text{(Lifting case)}$$

- The net work $W_F + W_g$ done on the block is zero, as expected, because the net force on the block is zero. This is not, of course, to say that it takes no work to lift a block through a vertical height **s**.
- In such a context, we do not refer to the net work, but to the work done by the person.

Work Done by Friction

- A common example in which the work is always negative is the work done by friction. When a block slides over a rough surface due to an applied force F, the work done by the frictional force f_k while the block undergoes displacement s is:

$$W_f = \vec{f}_k \cdot \vec{s} = f_k\, s \cos 180^\circ$$
$$= -f_k\, s$$



- one can easily find the work done by gravity, the normal force, and the applied force as follows:

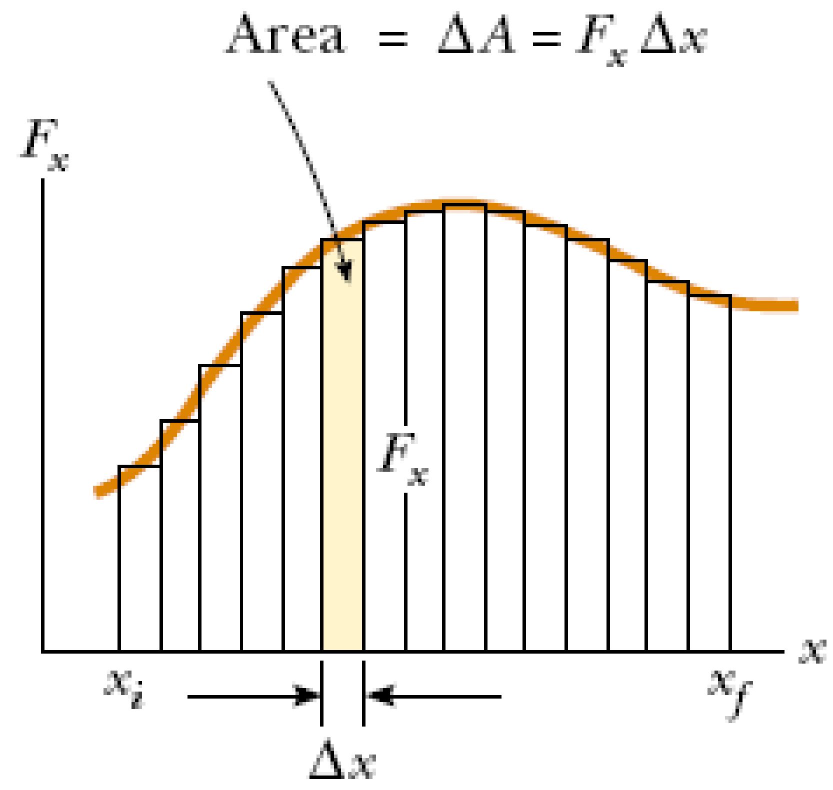
$$W_g = m\vec{g} \bullet \vec{s} = m\,g\,s\,\cos 90^\circ = 0$$

$$W_N = \vec{N} \bullet \vec{s} = N\,s\,\cos 90^\circ = 0$$

$$W_F = \vec{F} \bullet \vec{s} = F\,s\cos\theta$$

4.2. Work done by a variable force

- Consider a particle being displaced along the x axis under the action of a force that varies with position.

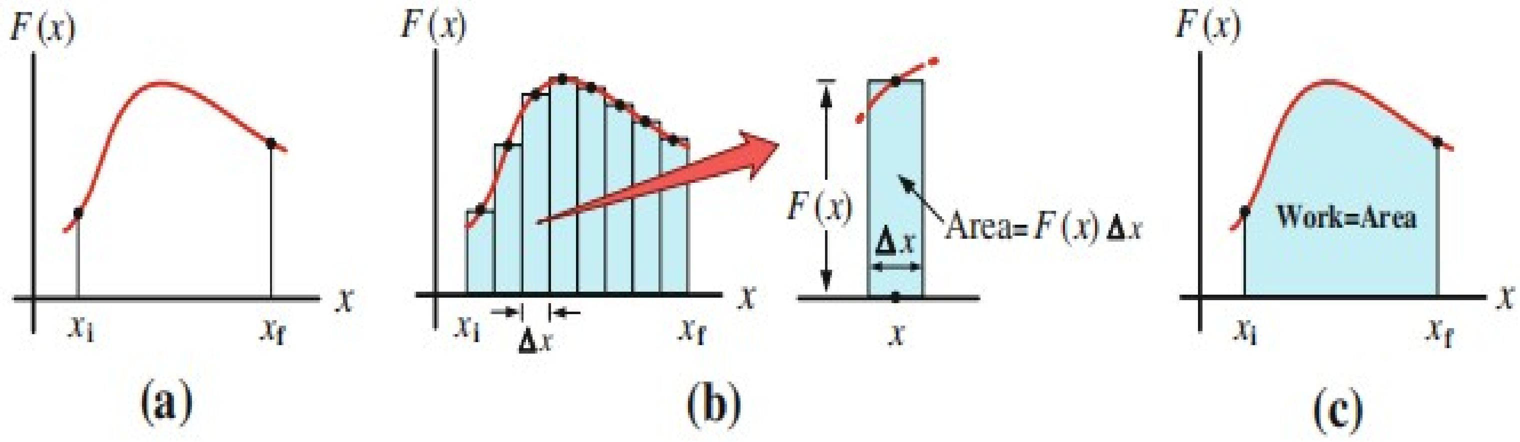


- If we imagine that the particle undergoes a very small displacement °x, the x component F_x of the force is approximately constant o $W \approx F_x \Delta x$ ll interval;
- If we imagine that the F_x versus x curve is divided into a large number of such intervals, the total work done for the displacement from x_i to x_f is approximately equal to the sum of a large numbe :

$$W \approx \sum_{x_i}^{x_f} F_x \Delta x$$

- If the size of the displacements is allowed to approach

$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F(x)\,\Delta x = \int_{x_i}^{x_f} F(x)\,dx$$



Example: Find the work done by an applied force on a spring

$$W = \int_{x_i}^{x_f} kx\,dx = \tfrac{1}{2}k{x_f}^2 - \tfrac{1}{2}k{x_i}^2$$

4.3 Kinetic energy and Work Energy Theorem

- The energy of a body due to its motion is called **kinetic energy**. Kinetic energy is a scalar quantity and has the same unit as work
- In general, the kinetic energy *K of a particle of mass m moving with a speed v is defined* as:

$$K = 1/2\ mv^2$$

- Consider a particle of mass m, moving with acceleration a = a(x) along the x-axis under the effect of a net force F(x) that points along this axis. Thus, according to Newton’s second law of motion we have F(x) = ma. The work done by this net force on the particle as it moves from an initial positi n x_f can be found as follov

$$W = \int_{x_i}^{x_f} F(x)\, dx = \int_{x_i}^{x_f} m\, a\, dx$$

$$m\, a\, dx = m \frac{dv}{dt}\, dx$$

- We can write the quantity ma dx in the last

- Since v is a function of time, then we can use the “chain rule” to have:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\,v$$

- Substituting this result back in to the above respective equation:

$$W = \int_{v_i}^{v_f} m\,v\,dv = m\int_{v_i}^{v_f} v\,dv = m\left[\tfrac{1}{2}v^2\right]_{v_i}^{v_f} = \tfrac{1}{2}m\,v_f^2 - \tfrac{1}{2}m\,v_i^2$$

$$\boxed{W = \Delta K = K_f - K_i}$$

- This equation is known as the **work-energy theorem**.

The work kinetic energy theorem can be stated as follows

" The net work done by the net force acting on the particle equal to the change in kinetic energy of the particles".

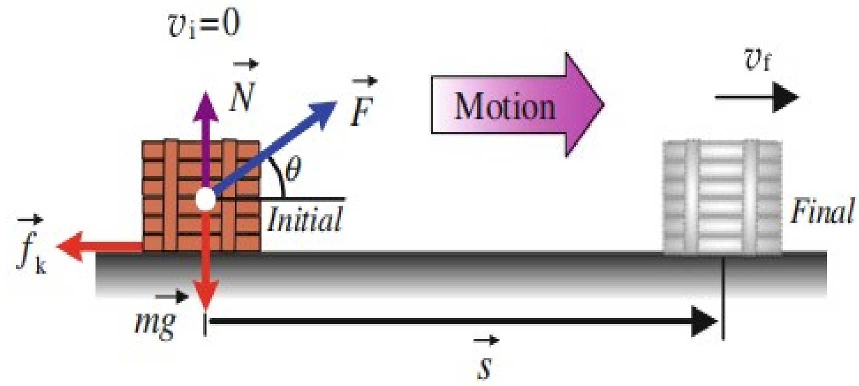
Example: A box of mass m = 10 kg is initially at rest on a rough horizontal surface, where the coefficient of kinetic friction between the box and the surface is $\mu_k = 0.2$. The box is then pulled horizontally by a force F = 50 N that makes an angle $\theta = 60^\circ$ with the horizontal.

(a) Using the work-energy theorem find the speed v_f of the box after it moves a distance of 4 m.

(b) Repeat part (a) using Newtonian mechanics.

Solution: $W_g = W_N = 0$. The work done by the applied force is:

$$W_F = \vec{F} \bullet \vec{s} = F s \cos\theta = (50\,\mathrm{N})(4\,\mathrm{m})(\cos 60^\circ) = 100\,\mathrm{J}$$



The magnitude of the frictional force is $f_k = \mu_k N$, where $N = mg - F\sin\theta$. Therefore, the work done by friction is:

$$W_f = \vec{f}_k \bullet \vec{s} = -f_k\, s = -\mu_k (mg - F\sin\theta)\, s$$

$$= -0.2 \times [(10\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) - (50\,\mathrm{N})(0.866)](4\,\mathrm{m})$$

$$= -43.76\,\mathrm{J}$$

Thus, the net work done on the box is:

$W_{net} = W_F + W_g + W_N + W_f = 100J + 0 + 0 + (-43.76\ J) = 56.24\ J$

Applying the work-energy theorem with $v_i = 0$ gives:

$W_{net} = K_f - K_i = 1/2mv_f^2$

$\Rightarrow$ = = 3.35 m/s

(b) Applying Newton’s second law in the component form, then for the horizontal component, we find that:

∑F

$$a = \frac{F\cos\theta - \mu_k(mg - F\sin\theta)}{m}$$

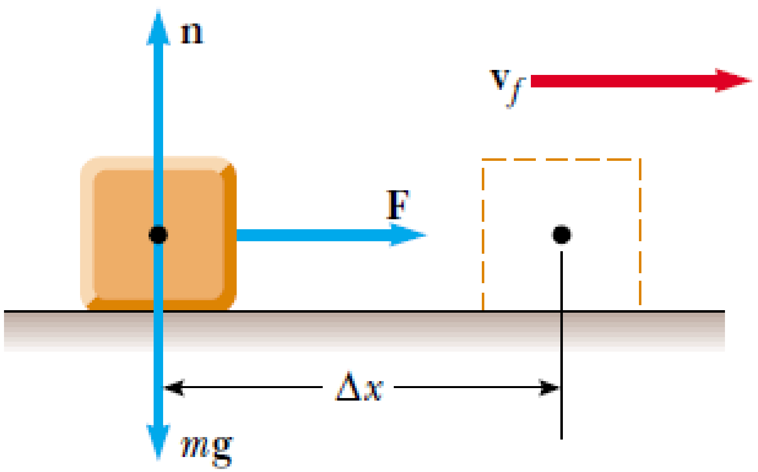
Th

$$= \frac{(50\,\mathrm{N})(\cos 60°) - 0.2 \times [(10\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) - (50\,\mathrm{N})(0.866)]}{10\,\mathrm{kg}} = 1.406\,\mathrm{m/s^2}$$

be given by

$$v_f = \sqrt{2as} = \sqrt{2 \times (1.406\,\mathrm{m/s^2})(4\,\mathrm{m})} = 3.35\,\mathrm{m/s}$$

Excercise: A 6.0-kg block initially at rest is pulled to the right along a horizontal, frictionless surface by a constant horizontal force of 12 N. Find the speed of the block after it has moved 3.0 m.



4.4 Conservative and non-conservative forces

- Conservative forces have two important properties:-

1. The work done by a conservative force on a particle moving between any two points is **independent** of the path taken by the particle.
2. The work done by a conservative force on a particle moving through any closed path is **zero**.

Example: gravitational force, restoring force in a spring, etc

- W_g and W_s depends only on the initial and the final points of the objects and hence independent of the path.
- A force is **non-conservative** if it does not satisfy properties 1 and 2 for conservative forces.

Example: friction force

4.5 Conservative force and potential energy

- Potential Energy (U) is the energy associated with position or shape of a particle.
- Example: *gravitational potential energy* and *elastic potential energy*.
- Because work done by conservative force is a function only of a particles initial and final coordinates, we can define a potential energy function (U) such that the work done by a conserv

$$W_c = \int_{x_i}^{x_f} F_x \, dx = -\Delta U$$

ecrease in the potentia

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x \, dx$$

$$U_f(x) = -\int_{x_i}^{x_f} F_x \, dx + U_i$$

- If the point of application of the force undergoes an infinitesimal displacement d_x :

$$dU = -F_x\,dx$$

$$\boxed{F_x = -\frac{dU}{dx}}$$

Elastic potential energy

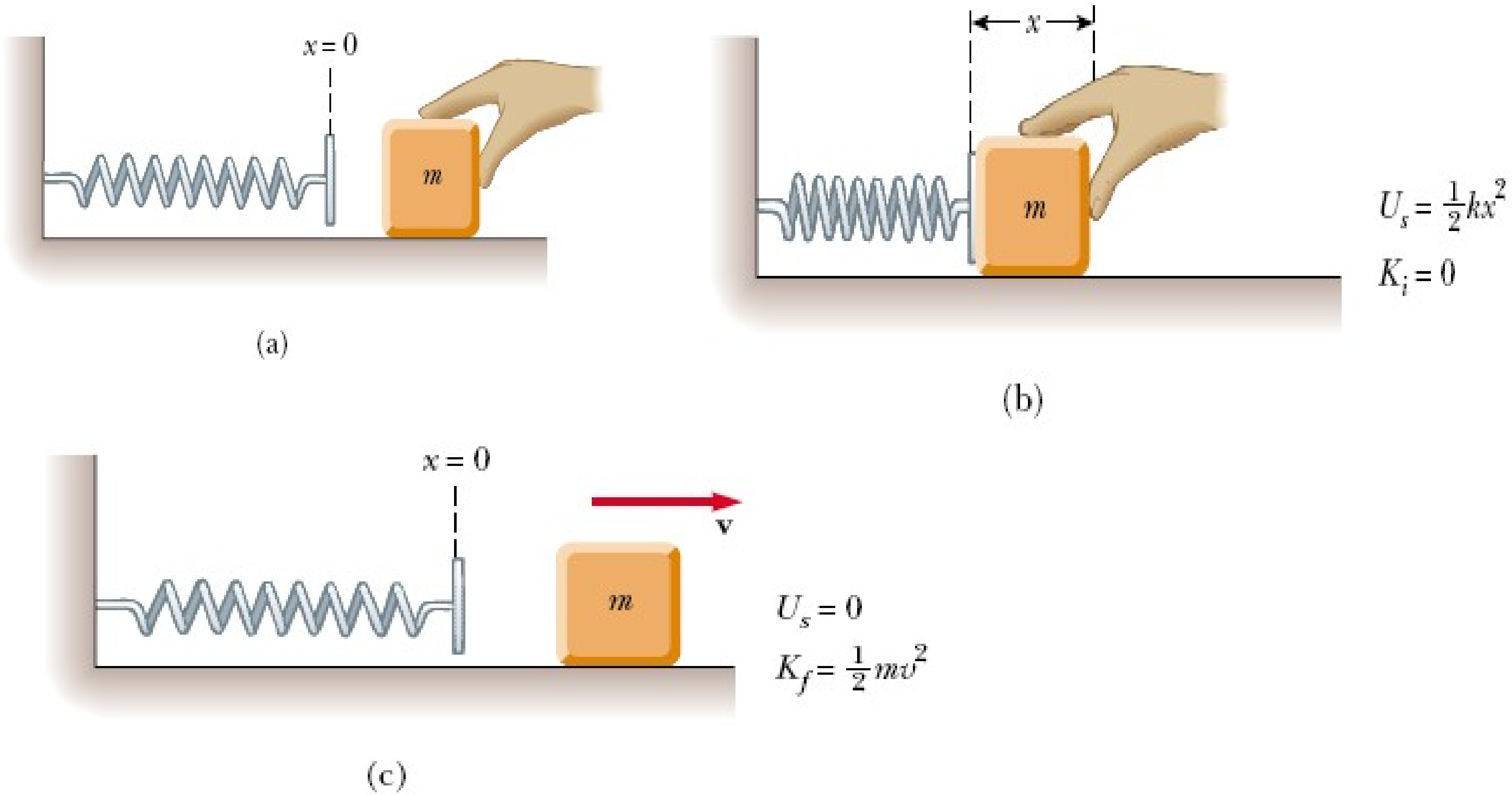
- Consider a system consisting of a block plus a spring.
- The work done by an external applied force F_{app} on a system consisting of a block connected to the spring is

$$W_{F_{\text{app}}} = \int_{x_i}^{x_f} F_{\text{app}}\,dx = \int_{x_i}^{x_f} kx\,dx = \tfrac{1}{2}k{x_f}^2 - \tfrac{1}{2}k{x_i}^2$$

The elastic potential energy of the system can be thought of as the energy stored in the **deformed spring** (one that is either compressed or stretched from its equilibrium position)

- When a block is pushed against the spring (Fig. b) and the spring is compressed a distance x, the elastic potential energy stored in the spring is

$$U_s \equiv \frac{1}{2}kx^2.$$



- When the block is released from rest, the spring exerts a force on the block and returns to its original length. The stored elastic potential energy is transformed into kinetic energy of the block (Fig. c).

- The elastic potential energy stored in a spring is zero whenever the spring is undeformed ($x = 0$).

- Energy is stored in the spring only when the spring is either stretched or compressed.

- The elastic potential energy is a maximum when the spring has reached its maximum compression or extension (that is, when $|x|$ is a maximum).

- Because the elastic potential energy is proportional to x^2, we see that U_s is always positive in a deformed spring.

Example:-

Apotential energy function for a two dimensional force is given by U=3x³y-7x. Find the force that acts at the point (x,y)

Solution

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial\left(3x^3y - 7x\right)}{\partial x} = -\left(9x^2y - 7\right) = 7 - 9x^2y$$

$$F_y = -\frac{\partial U}{\partial y} = -\frac{\partial\left(3x^3y - 7x\right)}{\partial y} = -\left(3x^3 - 0\right) = -3x^3$$

Thus, the force acting at the point $(x,\ y)$ is $\mathrm{F} = F_x\hat{\mathrm{i}} + F_y\hat{\mathrm{j}} = \boxed{\left(7 - 9x^2y\right)\hat{\mathrm{i}} - 3x^3\hat{\mathrm{j}}}$.

4.6 Conservation of mechanical energy

- The total mechanical energy E is defined as the sum of kinetic energy and potential energy. $E = K + U$
- The sum of kinetic and potential energies at a given point remains constant. This is an example of the Principle of the conservative of mechanical energy.
- We can state the principle of conservative of mechanical as

“ Energy is neither created nor destroyed, it can simply transferred from one to another and the total amount of E remains constant”.

Example:-

A stone of mass m = 0.5 kg is thrown vertically up ward from the surface of the earth with a speed of 40 m/s. Here ignore air friction.

a. What is the mechanical energy at the moment thrown.

b. What is the mechanical energy when it reaches its maximum height

c. What is the kinetic, gravitational potential and mechanical energy at 3 sec. later.

Given **Solution**

$v = 40$ m/s a). $E = K+U = \frac{1}{2} mv^2 + mgh = \frac{1}{2} \times 0.5 \times (40 \text{ m/s})^2 + 0 = 400J$

$m = 0.5$ kg b). $h_{max} = v^2 /2g = 80$ m, $E = K + U = 0 + mgh = 400J$

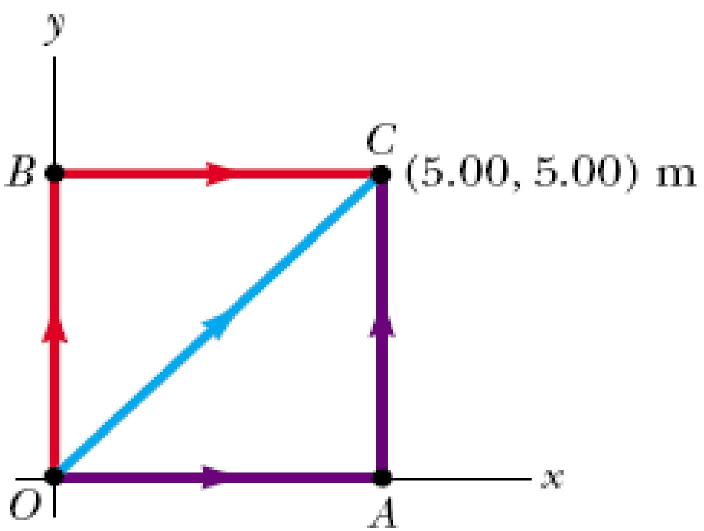
$t = 3$ sec c). $V = V_i - gt$ $y = V_i t - \frac{1}{2} gt^2$

$= 30$ m/s $y = 75$ m

$K = \frac{1}{2} mv^2 = 25J$ $U_g = mgy = 375J$

$E = K + U = 400J$

2. A 4 kg particle moves from the origin to position C, having coordinates x = 5m and y = 5m. One force on the particle is the gravitational force acting in the negative y direction. Calculate the work done by the gravitational force in going from O to C along (a) (



Solution

$$F_g = mg = (4.00\text{ kg})(9.80\text{ m/s}^2) = 39.2\text{ N}$$

(a) Work along OAC = work along OA + work along AC

$$= F_g(\text{OA})\cos 90.0° + F_g(\text{AC})\cos 180°$$

$$= (39.2\text{ N})(5.00\text{ m}) + (39.2\text{ N})(5.00\text{ m})(-1)$$

$$= \boxed{-196\text{ J}}$$

(b) W along OBC = W along OB + W along BC

$$= (39.2\text{ N})(5.00\text{ m})\cos 180° + (39.2\text{ N})(5.00\text{ m})\cos 90.0°$$

$$= \boxed{-196\text{ J}}$$

(c) Work along OC $= F_g(\text{OC})\cos 135°$

$$= (39.2\text{ N})(5.00 \times \sqrt{2}\text{ m})\left(-\frac{1}{\sqrt{2}}\right) = \boxed{-196\text{ J}}$$

4.7 Power

- The rate at which work is done on a particle is called power.
- If an external force is applied to an object and if the work done by this force in the time interval Δt *is W, then the average power during this interval is defined as*

$$\overline{\mathscr{P}} \equiv \frac{W}{\Delta t}$$

- An instantaneous power p as the limiting value of the average power as Δt *approaches zero:*

$$\mathscr{P} \equiv \lim_{\Delta t \to 0} \frac{W}{\Delta t} = \frac{dW}{dt}$$

P =dW/dt but W = **F**. d**s**

dW = **F** . d**s**

$$P = \frac{\mathbf{F} \cdot d\mathbf{s}}{dt}$$

P = **F** . **V**

- The SI of power is Joule per second (J/s) is called Watt (W)

Example:-

1. The electric motor of a model train accelerates the train from rest to 0.620 m/s in 21.0 ms. The total mass of the train is 875 g. Find the average power delivered to the train during the acceleration.

Given

V_i =0

V_f =0.62 m/s

M= 875g = 0.875kg

Solution

$p_{av} = w/\Delta t = mv^2/2\Delta t$

$= 8.01$ w

2. A 700-N Marine in basic training climbs a 10.0-m vertical rope at a constant speed in 8.00 s. What is his power output?

Given

F=mg= 700N

h= 10m

t= 8 sec

Solution

p = w/t = mgh\t

p = 875 w