Unit Two

ONE AND TWO DIMENSIONAL MOTION

- **Displacement** of a particle is defined as its change in position in some time interval.
- As it moves from an initial position X_i to a final position X_f, the displacement of the particle is given by

$$\Delta x \equiv x_f - x_i$$

- **Distance** is the length of a path followed by a particle.
- The **average velocity** $\overline{v}_x$ of a particle is defined as the particle's displacement Δx divided by the time interval Δt during which that displacement occurs:

$$\overline{v}_x \equiv \frac{\Delta x}{\Delta t}$$

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- The **average speed** of a particle, a scalar quantity, is defined as the total distance travelled divided by the total time interval required to travel that distance:

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}}$$

- The **instantaneous velocity** v_x equals the limiting value of the ratio $\Delta x/\Delta t$ as Δt approaches zero:

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

- The **instantaneous speed** of a particle is defined as the magnitude of its instantaneous velocity.

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- The **average acceleration** $\overline{a}_x$ *of the particle is defined as the change in velocity* Δv_x *divided by the time interval* Δt *during which that change occurs:*

$$\overline{a}_x \equiv \frac{\Delta v_x}{\Delta t} = \frac{v_{xf} - v_{xi}}{t_f - t_i}$$

- The **instantaneous acceleration** equals the limiting value of the average acceleration as Δt approaches zero.

$$a_x \equiv \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

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One-Dimensional Motion with Constant Acceleration

$$a_x = \frac{v_{xf} - v_{xi}}{t - 0}$$ Where $t_i = 0$ and $t_f = t$

$$v_{xf} = v_{xi} + a_x t \quad \text{.......................}1$$

- *Because velocity at constant acceleration varies linearly in time according to Equation 1 , we can express the average velocity in any time interval as the arithmetic mean of the initial velocity* v_{xi} *and the final velocity* v_{xf} *:*

$$\overline{v}_x = \frac{v_{xi} + v_{xf}}{2} \quad \text{.............. ....}2$$

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- From the definition of average velocity we can also derive the following equation

$$x_f = x_i + \tfrac{1}{2}(v_{xi} + v_{xf})t \quad \text{..................................}3$$

- We can obtain another useful expression for the position of a particle moving with constant acceleration by substituting Equation 1 into Equation 3:

$$x_f = x_i + \tfrac{1}{2}[v_{xi} + (v_{xi} + a_x t)]t$$

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \quad \text{......................}4$$

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- Finally, we can obtain an expression for the final velocity that does not contain time as a variable by substituting the value of *t from Equation 1 into Equation* 3:

$$x_f = x_i + \tfrac{1}{2}(v_{xi} + v_{xf})\left(\frac{v_{xf} - v_{xi}}{a_x}\right) = \frac{v_{xf}^2 - v_{xi}^2}{2a_x}$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i) \quad \text{............. ....}5$$

- Equations 1 through 5 are kinematic equations that may be used to solve any problem involving one-dimensional motion at constant acceleration.

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- Kinematic Equations can be also Derived from Calculus
- The defining equation for acceleration,

$$a_x = \frac{dv_x}{dt}$$

$$dv_x = a_x dt$$ Integrate both sides

$$v_{xf} - v_{xi} = \int_0^t a_x dt$$ For constant acceleration

$$v_{xf} - v_{xi} = a_x \int_0^t dt = a_x(t-0) = a_x t \quad \text{..........1}$$

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- Defining equation for velocity,

$$v_x = \frac{dx}{dt}$$

$$dx = v_x\, dt$$

$$x_f - x_i = \int_0^t v_x\, dt$$

$$v_x = v_{xf} = v_{xi} + a_x t$$

$$x_f - x_i = \int_0^t (v_{xi} + a_x t)\, dt = \int_0^t v_{xi} dt + a_x \int_0^t t dt = v_{xi}(t-0) + a_x\left(\frac{t^2}{2} - 0\right)$$

$$= v_{xi} t + \frac{1}{2} a_x t^2 \quad \text{...........2}$$

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Freely falling objects

- If a body is thrown up or down in the air the effect of the air resistance is neglected, this idealized motion is called free fall.
- The acceleration of freely falling body is acceleration due to gravity. Its denoted by g (g= 9.8 m/s ^{2})
- Let us see the kinematic equation during down and up ward falling

Down ward thrown

$$V_{yf} = V_{yi} + gt$$

$$y_f - y_i = V_{yi} t + \frac{1}{2} g t^2$$

$$V_{yf}^2 = V_{yi}^2 + 2g(y_f - y_i)$$

Up ward thrown

$$V_{yf} = V_{yi} - gt$$

$$y_f - y_i = V_{yi} t - \frac{1}{2} g t^2$$

$$V_{yf}^2 = V_{yi}^2 - 2g(y_f - y_i)$$

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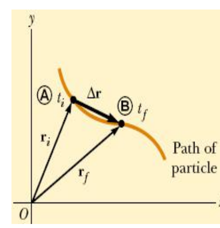
Motion in two dimensions

- **The position vector** of a particle is a vector drawn from the origin of a reference frame to the *xy* position of the particle. For a particle at the Point (*xy*), its position vector **r** is

r = x**i** + y**j**

- At time t_i the particle is at point A, described by position vector r_i. At some later time t_f it is at point B, described by position vector r_f. The particle's change in position is the **displacement vector Δr** :

$$\Delta \mathbf{r} \equiv \mathbf{r}_f - \mathbf{r}_i$$



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- Average velocity of the particle during the time interval Δt as the displacement of the particle divided by the time interval. i.e

$$\bar{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

- The instantaneous velocity **v** is defined as the limit of the average velocity Δr/Δ*t as Δt approaches zero:*

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

- The average acceleration vector is the ratio of the change in the Instantaneous-velocity vector ΔV to the time interval:

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

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- The instantaneous-acceleration vector is the limit of this ratio as Δt approaches zero; in other words, it is the derivative of the velocity vector with respect to time:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

Two-Dimensional Motion with Constant Acceleration

- The position vector for a particle moving in the *xy plane can be written as:*

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \quad \text{...........1}$$

where x, y, and **r** changes with time as the particle moves , while the unit vectors **i** and **j** remain constant.

- the velocity of the particle can be given by:

$$v = \frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = v_x\hat{i} + v_y\hat{j} \quad \text{....................2}$$

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- The final velocity at any time t is:

$$\mathbf{v}_f = (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}}$$
$$= (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}}) + (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t$$
$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$$

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \quad \begin{cases} v_{xf} = v_{xi} + a_x t \\ v_{yf} = v_{yi} + a_y t \end{cases}$$ ....3a **in component form**

- The *x and y coordinates of a particle* moving with constant acceleration are:

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

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- Substituting these expressions into Equation 1 (and labeling the final position vector r_f) *gives*

$$\mathbf{r}_f = (x_i + v_{xi}t + \tfrac{1}{2}a_x t^2)\hat{\mathbf{i}} + (y_i + v_{yi}t + \tfrac{1}{2}a_y t^2)\hat{\mathbf{j}}$$
$$= (x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}}) + (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}})t + \tfrac{1}{2}(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t^2$$
$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2 \quad \begin{cases} x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \\ y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 \end{cases}$$

Example:

A particle starts from the origin at t = 0 with an initial velocity having an x component of 20 m/s and a y component of -15 m/s. The particle moves in the xy plane with an x component of acceleration only, given by $a_x = 4\ \text{m/s}^2$.

(a) Determine the components of the velocity vector at any time and the total velocity vector at any time.

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Solution

- The components of the initial velocity tell us that the particle starts by moving toward the right and downward. The x component of velocity starts at 20 m/s and increases by 4.0 m/s every second. The y component of velocity never changes from its initial $v_{xi} = 20$ m/s, $v_{yi} = -15$ m/s, $a_x = 4\ \text{m/s}^2$, and $a_y = 0$. From Equations 3a

(1) $$v_{xf} = v_{xi} + a_x t = (20 + 4.0t)\ \text{m/s}$$

(2) $$v_{yf} = v_{yi} + a_y t = -15\ \text{m/s} + 0 = -15\ \text{m/s}$$

- Therefore, $\mathbf{v}_f = v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}} = [(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s}$

(B) Calculate the velocity and speed of the particle at t = 5 s.

Solution

With t = 5 s, the result from part (A) gives

$$\mathbf{v}_f = [(20 + 4.0(5.0))\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \text{m/s} = (40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})\ \text{m/s}$$

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- From this result, we have At t = 5 s, $v_{xf} = 40$ m/s and $v_{yf} = -15$ m/s.
- To determine the angle θ

$$\tan\theta = v_{yf}/v_{xf}.$$

(3) $$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \text{m/s}}{40\ \text{m/s}}\right) = -21°$$

where the negative sign indicates an angle of 21 ° below the positive *x axis*.

- The speed is the magnitude of v_f

$$v_f = |\mathbf{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ \text{m/s} = 43\ \text{m/s}$$

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(C) Determine the x and y coordinates of the particle at any time t and the position vector at this time.

Solution

- Because $x_i = y_i = 0$ at t = 0, From Equation 4a

$$x_f = v_{xi}t + \tfrac{1}{2}a_x t^2 = (20t + 2.0t^2)\ \text{m}$$

$$y_f = v_{yi}t = (-15t)\ \text{m}$$

- Therefore, the position vector at any time *t is*

(4) $$\mathbf{r}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} = [(20t + 2.0t^2)\hat{\mathbf{i}} - 15t\hat{\mathbf{j}}]\ \text{m}$$

- At t = 5 s, x = 150 m, y = -75 m, and The magnitude of the displacement of the particle from the origin at t = 5 s is the magnitude of r_f at this time:

$$r_f = |\mathbf{r}_f| = \sqrt{(150)^2 + (-75)^2}\ \text{m} = 170\ \text{m}$$

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Projectile Motion

- **Projectile** is a two dimensional motion which is a body thrown with some initial velocity and then allowed to move under the influence of the gravity alone.
- **Trajectory** is the path followed by the projectile.
- **Angle of departure** is the angle which the direction of the projectile makes with the horizontal.
- **Time of flight** is the time taken by the body to return to the same horizontal plane from it was projected from the instant it is released
- **Maximum height** is the greatest vertical distance by the projectile above the horizontal plane.
- **Horizontal range** is the total horizontal distance travelled by the body during the time of flight.

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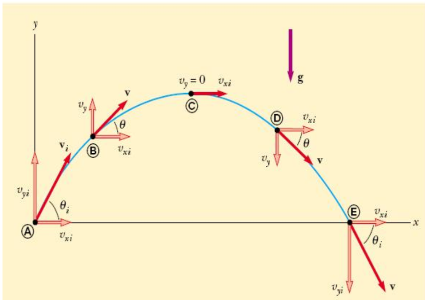
- Assume that a body of mass m is given an initial velocity V_i near the earth's surface.



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- From the definitions of the cosine and sine functions we have

$$\cos\theta_i = v_{xi}/v_i \qquad \sin\theta_i = v_{yi}/v_i \text{ ....1}$$

- Therefore, the initial *x and y components of velocity are*

$$v_{xi} = v_i\cos\theta_i \qquad v_{yi} = v_i\sin\theta_i \text{ ....2}$$

- *We know that*

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$
$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2 \text{ ……………. ..3}$$

- **For the horizontal motion (x- direction),** Substituting the *x component into Eqn (3) with* $x_i = 0$ *and* $a_x = 0$*, we find that*

$$x_f = v_{xi}t = (v_i\cos\theta_i)t \text{ ……4 } \textit{horizontal position component}$$

- For the vertical motion, repeating with the *y component and using* $y_i = 0$ *and* $a_y = -g$*, we obtain*

$$y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = (v_i\sin\theta_i)t - \tfrac{1}{2}gt^2 \text{ .....5 } \textbf{\textit{vertical position}}$$

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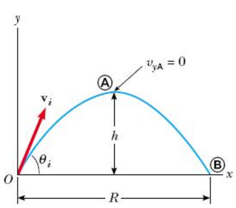
- From eqn (4) substuting t into eqn (5), we get

$$y = (\tan\theta_i)x - \left(\frac{g}{2v_i^2\cos^2\theta_i}\right)x^2 \text{ …..6 equation of trajectory}$$

Horizontal Range and Maximum Height of a Projectile



Here, two points are especially interesting to analyze: the peak point A, which has Cartesian coordinates (R/2, h), and the point B, which has coordinates (R, 0). The distance R is called the horizontal range of the projectile, and the distance h is its maximum height. Now, lets determine h by taking $V_{yA} = 0$

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- At point A, $V_{yA} = 0$ and $t = t_A$

$$v_{yf} = v_{yi} + a_y t$$
$$0 = v_i\sin\theta_i - gt_A$$
$$t_A = \frac{v_i\sin\theta_i}{g} \text{ ....1}$$

Time to reach the maximum height

- Replacing $y = y_A$ with h,

$$h = (v_i\sin\theta_i)\frac{v_i\sin\theta_i}{g} - \frac{1}{2}g\left(\frac{v_i\sin\theta_i}{g}\right)^2$$

$$h = \frac{v_i^2\sin^2\theta_i}{2g} \text{ ....2}$$

maximum height of projectile

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- At point B, the range R is the horizontal position of the projectile at a time that is twice the time at which it reaches its peak, that is, at time $t_B = 2t_A$

$v_{xi} = v_{xB} = v_i\cos\theta_i$ and setting $x_B = R$ at $t = 2t_A$.

$$t = t_B = 2\frac{v_i\sin\theta_i}{g} \text{ ....2}$$

Time of flight

$$R = v_{xi}t_B = (v_i\cos\theta_i)2t_A$$

$$= (v_i\cos\theta_i)\frac{2v_i\sin\theta_i}{g} = \frac{2v_i^2\sin\theta_i\cos\theta_i}{g}$$

But $\sin 2\theta = 2\sin\theta\cos\theta$

$$R = \frac{v_i^2\sin 2\theta_i}{g} \text{ ....3}$$

Range projectile

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Example:- A projectile is projected with intial velocity 98 m/s at an angle of 30.0° with the horizontal. Calculate the time of flight, Range and the maximum height of the projected object

Solution

i). $t = t_B = 2\frac{v_i\sin\theta_i}{g}$

= 9.8 sec

ii). $R = \frac{v_i^2\sin 2\theta_i}{g}$

= 825.9 m

iii). $h = \frac{v_i^2\sin^2\theta_i}{2g}$

= 120.05 m

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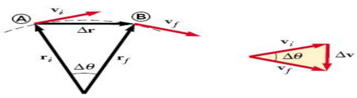
Uniform Circular Motion

- Uniform circular motion is defined as a particle moving in a circular path with constant linear velocity.
- The velocity vector is always tangent to the path of the object and perpendicular to the radius of the circular path.
- For uniform circular motion, the acceleration vector can only have a component perpendicular to the path, which is toward the centre of the circle.



- The particle is at A at time t_i, and its velocity at that time is v_i; it is at B at some later time t_f, and its velocity at that time is v_f. $\mathbf{v}_i$ and $\mathbf{v}_f$ differ only in direction; their magnitudes are the same, that is,

$v_i = v_f = v$
$$\bar{\mathbf{a}} \equiv \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta\mathbf{v}}{\Delta t}$$

- The vector $\Delta\mathbf{v}$ connects the tips of the vectors, representing the vector addition

$$\mathbf{v}_f = \boldsymbol{v}_i + \Delta\boldsymbol{v}.$$

- Because the velocity vector v is always perpendicular to the position vector r. Thus, the two triangles are *similar.* So, we can write a relationship between the lengths of the sides for the two triangles:

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{r} \qquad \text{where } v = v_i = v_f \text{ and } r = r_i = r_f.$$

- The magnitude of the average acceleration over the time interval for the particle to move from A to B:

$$|\bar{\mathbf{a}}| = \frac{|\Delta\mathbf{v}|}{\Delta t} = \frac{v}{r}\frac{|\Delta\mathbf{r}|}{\Delta t}$$

- As A and B approach each other, Δt *approaches zero, and the ratio* $|\Delta r|/\Delta t$ *approaches* the speed v. *In addition, the average acceleration becomes the instantaneous acceleration* at point A.

- Therefore, the magnitude of the acceleration is

$$a_c = \frac{v^2}{r} \qquad \textbf{Centripetal acceleration}$$

- And period of circular motion is given by

$$T \equiv \frac{2\pi r}{v} \qquad \textbf{Period of circular motion}$$

<u>Tangential and Radial Acceleration</u>

- The *total acceleration vector* ***a*** *can be written as the vector sum of* the component vectors:

$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t \qquad \textbf{Total acceleration}$$

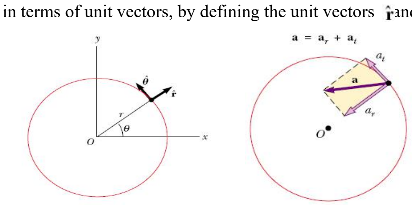
- The tangential acceleration component causes the change in the speed of the particle. This component is parallel to the instantaneous velocity, and is given by

$$a_t = \frac{d|\mathbf{v}|}{dt} \qquad \textbf{Tangential acceleration}$$

- The radial acceleration component arises from the change in direction of the velocity vector and is given by

$$a_r = -a_c = -\frac{v^2}{r} \qquad \textbf{Radial acceleration}$$

- Let's write the acceleration of a particle moving in a circular path in terms of unit vectors, by defining the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$



- Both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ move along with the particle and so vary in time.

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d|\mathbf{v}|}{dt}\hat{\boldsymbol{\theta}} - \frac{v^2}{r}\hat{\mathbf{r}}$$

Exercise:-

An automobile whose speed is increasing at a rate of 0.6 m/s^2 travels along a circular road of radius 20 m. When the instantaneous speed of the automobile is 4 m/s, find

(a) the tangential acceleration component,

(b) the centripetal acceleration component, and

(c) the magnitude and direction of the total acceleration