UNIT-ONE

VECTORS AND VECTOR SPACES

UNIT OBJECTIVES:

At the end of this unit each student will able to:

- Know representation of vectors geometrically
- Understand operation of vectors algebraically and geometrically.
- Realize properties of operations on vectors.
- Learn about projection of a vector along another vector.
- Understand the dot product and cross product of two vectors.
- Know the norm /magnitude of a vector.
- Develop parametric and non-parametric equation of a given line.
- Develop the parametric form and normal form of equation of a plane.
- Understand about vector spaces and subspaces
- Identify base and dimension of vector space.

1.1 Scalars and Vectors in $\Re^2$ and $\Re^3$

Many physical quantities, such as area, length, mass, and temperature, are completely described once the magnitude of the quantity is given. Such quantities are called ***scalars***. Other physical quantities are not completely determined until both a magnitude and a direction are specified. These quantities are called ***vectors***. For example, wind movement is usually described by giving the speed and direction, say 20 mph northeast. The wind speed and wind direction form a vector called the wind ***velocity***. Other examples of vectors are ***force*** and ***displacement***. In this unit our goal is to review some of the basic theory of vectors in two and three dimensions

Definition of Points In n-Space

Definition 1.1: If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(a_1, a_2, a_3, \ldots, a_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by $\mathbb{R}^n$.

When $n = 2$ or 3, it is customary to use the terms ***ordered pair*** and ***ordered triple***, respectively, rather than *ordered 2-tuple* and *ordered 3-tuple*. When $n = 1$, each ordered n-tuple consists of one real number, so $\mathbb{R}^n$ may be viewed as the set of real numbers. It is usual to write $\mathbb{R}$ *rather* than $\mathbb{R}^1$ for this set. It might have occurred to you in the study of 3-space that the symbol (a_1, a_2, a_3) has two different geometric interpretations: it can be interpreted as a point, in which case, $a_1, a_2, and\, a_3$ are the coordinates (Figure 1.1.1*a*), or it can be interpreted as a vector, in which case $a_1, a_2, and\, a_3$ are the components (Figure 1.1.1*b*). It follows, therefore, that an ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ can be viewed either as a "generalized point" or as a "generalized vector"—the distinction is mathematically unimportant. Thus we can describe the 5-tuple (−2, 4, 0, 1, 6) either as a point in $\mathbb{R}^5$or as a vector in $\mathbb{R}^5$.

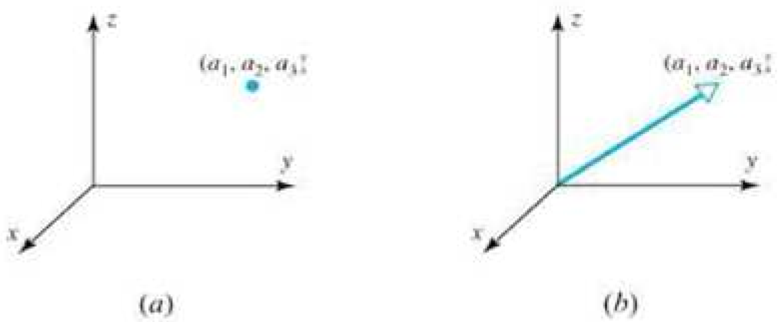


Figure 1.1: The ordered triple (a_1, a_2, a_3) can be interpreted geometrically as a point or as a vector

Definition 1.2: a) Two vectors $u = (u_1, u_2, u_3, \ldots, u_n)$ and $v = (v_1, v_2, v_3, \ldots, v_n)$ in $\mathbb{R}^n$ are called ***equal*** if $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$.

b) The ***sum*** $u+v$ is defined by $u+v=(u_1+v_1, u_2+v_2, \ldots, u_n+v_n)$

c) If k is any scalar, the ***scalar multiple*** ku is defined by $ku=(ku_1, ku_2, ku_3, \ldots, ku_n)$ The operations of addition and scalar multiplication in this definition are called the ***standard operations*** on $\mathbb{R}^n$.

d) The ***zero vector*** in $\mathbb{R}^n$ is denoted by $\mathbf{0}$ and is defined to be the vector $\mathbf{0} = (0,0, \ldots 0)$.

e) If $u=(u_1, u_2, u_3, \ldots, u_n)$ is any vector in $\mathbb{R}^n$, then the ***negative*** (or ***additive inverse***) of u is denoted by $-u$ and is defined by $-u=(-u_1, -u_2, -u_3, \ldots, -u_n)$.

f) The ***difference*** of vectors in $\Re^n$ is defined by $v-u=v+(-u)$ or, in terms of components $v-u=(v_1-u_1, v_2-u_2, v_3-u_3, \ldots, v_n-u_n)$.

Vectors in n-space; Geometric interpretation in 2-and 3 spaces

In this section, vectors in 2-space and 3-space will be introduced geometrically, arithmetic operations on vectors will be defined, and some basic properties of these arithmetic operations will be established.

Geometric Vectors

Vectors can be represented geometrically as directed line segments or arrows in 2-space or 3-space. The direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude. The tail of the arrow is called the **initial point** of the vector, and the tip of the arrow the **terminal point**. Symbolically, we shall denote vectors in lowercase boldface type (for instance, **a**, **k**, **v**, **w**, and **x**). When discussing vectors, we shall refer to numbers as **scalars**. For now, all our scalars will be real numbers and will be denoted in lowercase italic type (for instance, a, k, v, w, and x).If, as in Figure 1.2.1a, the initial point of a vector **v** is A and the terminal point is B, we write $\boldsymbol{v} = \overrightarrow{AB}$

Vectors with the same length and same direction, such as those in Figure 1.2 b, are called **equivalent**. Since we want a vector to be determined solely by its length and direction, equivalent vectors are regarded as **equal** even though they may be located in different positions. If **v** and **w** are equivalent, we write $\boldsymbol{v} = \boldsymbol{w}$

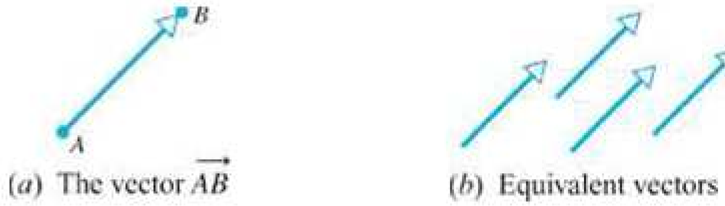


Figure 1.2

Definition 1.3: If $\boldsymbol{v}$ and $\boldsymbol{w}$ are any two vectors, then the ***sum*** $\boldsymbol{v} + \boldsymbol{w}$ is the vector determined as follows: Position the vector $\boldsymbol{w}$ so that its initial point coincides with the terminal point of $\boldsymbol{v}$. The vector $\boldsymbol{v} + \boldsymbol{w}$ is represented by the arrow from the initial point of $\boldsymbol{v}$ to the terminal point of $\boldsymbol{w}$ (Figure 1.3 *a*).

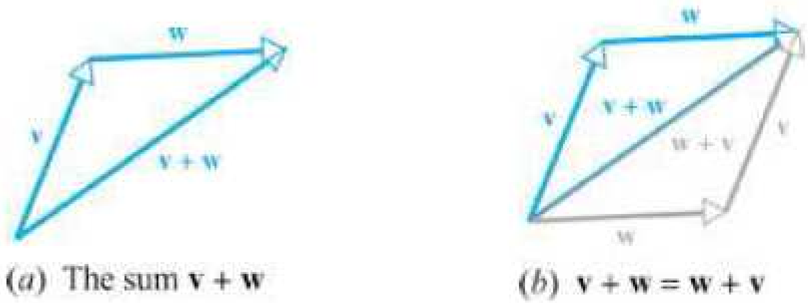


figure 1.3

In Figure 1.3*b* we have constructed two sums, (color arrows) and $\boldsymbol{w} + \boldsymbol{v}$ (gray arrows). It is evident that $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$ and that the sum coincides with the diagonal of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$ when these vectors are positioned so that they have the same initial point.

The vector of length zero is called the ***zero vector*** and is denoted by $\mathbf{0}$. We can easily see that

$$\mathbf{0} + \boldsymbol{v} = \boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$$

for every vector $\boldsymbol{v}$. Since there is no natural direction for the zero vector, we shall agree that it can be assigned any direction that is convenient for the problem being considered. If $\boldsymbol{v}$ is any nonzero vector, then $-v$, the ***negative*** of $\boldsymbol{v}$, is defined to be the vector that has the same magnitude as $\boldsymbol{v}$ but is oppositely directed (Figure 1.2.3). This vector has the property $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$ (Why?) In addition, we define $-\mathbf{0} = \mathbf{0}$.

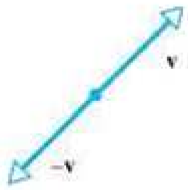


Figure 1.4

The negative of $\boldsymbol{v}$ has the same length as $\boldsymbol{v}$ but is oppositely directed(Figure1.4). Subtraction of vectors is defined as follows:

Definition 1.4: If $\boldsymbol{v}$ and $\boldsymbol{w}$ are any two vectors, then the ***difference*** of $\boldsymbol{w}$ from $\boldsymbol{v}$ is defined by

$$\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-\boldsymbol{w}) \quad \text{(Figure1.5}a\text{).}$$

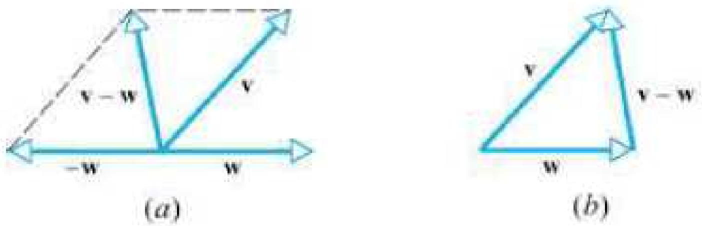


Figure 1.5

To obtain the difference $\boldsymbol{v} - \boldsymbol{w}$ without constructing $-\boldsymbol{w}$, position $\boldsymbol{v}$ and $\boldsymbol{w}$ so that their initial points coincide; the vector from the terminal point of $\boldsymbol{w}$ to the terminal point of $\boldsymbol{v}$ is then the vector $\boldsymbol{v} - \boldsymbol{w}$ (Figure 1.5*b*).

Definition 1.5: If $\boldsymbol{v}$ is a nonzero vector and k is a nonzero real number (scalar), then the ***product*** $k\boldsymbol{v}$ is defined to be the vector whose length is $|k|$ times the length of $\boldsymbol{v}$ and whose direction is the same as that of $\boldsymbol{v}$ if $k > 0$ and opposite to that of $\boldsymbol{v}$ if $k < 0$. We define $k\boldsymbol{v} = \boldsymbol{0}$ if $k = 0$ or $\boldsymbol{v} = \boldsymbol{0}$.

Figure 1.6 illustrates the relation between a vector $\boldsymbol{v}$ and the vector $\frac{1}{2}\boldsymbol{v}, -\boldsymbol{v}, 2\boldsymbol{v}$ and $-3\boldsymbol{v}$**.** Note that the vector $(-1)\boldsymbol{v}$ has the same length as $\boldsymbol{v}$ but is oppositely directed. Thus $(-1)\boldsymbol{v}$ is just the negative of $\boldsymbol{v}$; that is, $(-1)\boldsymbol{v} = \boldsymbol{v}$

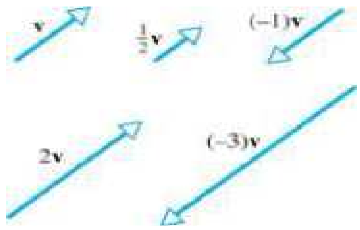


Figure 1.6

A vector of the form $k\boldsymbol{v}$ is called a ***scalar multiple*** of ***v***. As evidenced by Figure 1.6, vectors that are scalar multiples of each other are parallel. Conversely, it can be shown that nonzero parallel vectors are scalar multiples of each other. We omit the proof.

Vectors in Coordinate Systems

Problems involving vectors can often be simplified by introducing a rectangular coordinate system. For the moment we shall restrict the discussion to vectors in 2-space (the plane). Let ***v*** be any vector in the plane, and assume, as in Figure 1.7, that ***v*** has been positioned so that its initial point is at the origin of a rectangular coordinate system. The coordinates (v_1, v_2) of the terminal point of ***v*** are called the ***components of v***, and we write $v = (v_1, v_2)$

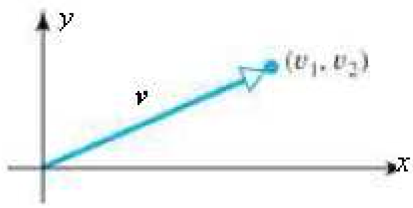


Figure 1.7: v_1 and v_2 are the components of ***v.***

If equivalent vectors, ***v*** and ***w***, are located so that their initial points fall at the origin, then it is obvious that their terminal points must coincide (since the vectors have the same length and direction); thus the vectors have the same components. Conversely,vectors with the same components are equivalent since they have the same length and the same direction. In summary, two vectors $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ are equivalent if and only if $v_1 = w_1$ and $v_2 = w_2$.

The operations of vector addition and multiplication by scalars are easy to carry out in terms of components. As illustrated in Figure 1.8, if $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ then

$$\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2) \qquad (1)$$

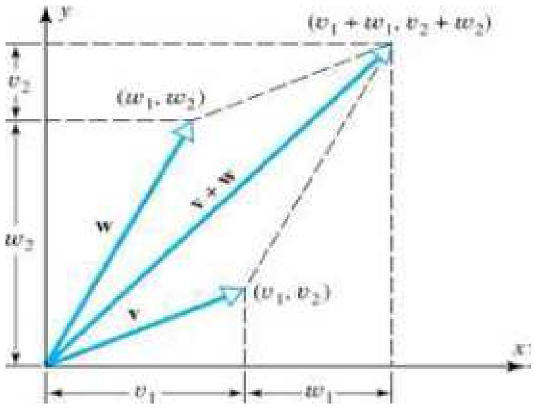


Figure 1.8

If $v = (v_1, v_2)$ and k is any scalar, then by using a geometric argument involving similar triangles, it can be shown that

$$k\boldsymbol{v} = (kv_1, kv_2) \tag{2}$$

(Figure1.9). Thus, for example, if $\boldsymbol{v} = (\mathbf{1}, \mathbf{-2})$ and $\boldsymbol{w} = (\mathbf{7}, \mathbf{6})$, then

$$\mathbf{v} + \mathbf{w} = (1, -2) + (7, 6) = (1 + 7, -2 + 6) = (8, 4)$$

and

$$4\boldsymbol{v} = 4(1, -2) = \big(4(1), 4(-2)\big) = (4, -8)$$

Since, $\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-1)\boldsymbol{w}$, it follows from Formulas 1 and 2 that

$$\boldsymbol{v} - \boldsymbol{w} = (\boldsymbol{v_1} - \boldsymbol{w_1}, \boldsymbol{v_2} - \boldsymbol{w_2}). \text{ (Verify.)}$$

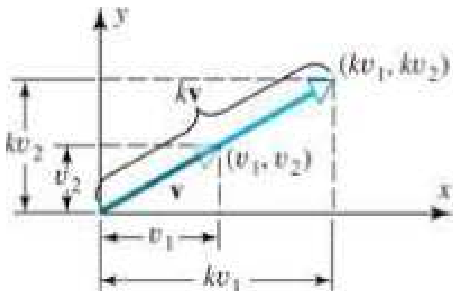


Figure 1.9

Vectors in 3-Space

Just as vectors in the plane can be described by pairs of real numbers, vectors in 3-space can be described by triples of real numbers by introducing a ***rectangular coordinate*** system. To construct such a coordinate system, select a point O, called the ***origin***, and choose three mutually perpendicular lines, called ***coordinate axes***, passing through the origin. Label these axes x, y, and z, and select a positive direction for each coordinate axis as well as a unit of length for measuring distances (Figure 1.10a).Each pair of coordinate axes determines a plane called a ***coordinate plane***. These are referred to as the xy ***-plane***, the xz ***-plane***,and the yz ***-plane***. To each point P in 3-space we assign a triple of numbers (x, y, z), called the ***coordinates of*** P, as follows: Pass three planes through P parallel to the coordinate planes, and denote the points of intersection of these planes with the three coordinate axes by X, Y, and Z (Figure 1.10b). The coordinates of P are defined to be the signed lengths $x = OX, y = OY$ and $z = OZ$.

In Figure 1.11a we have constructed the point whose coordinates are (4, 5, 6) and in Figure 1.11b the point whose coordinates are (-3, 2, -4)

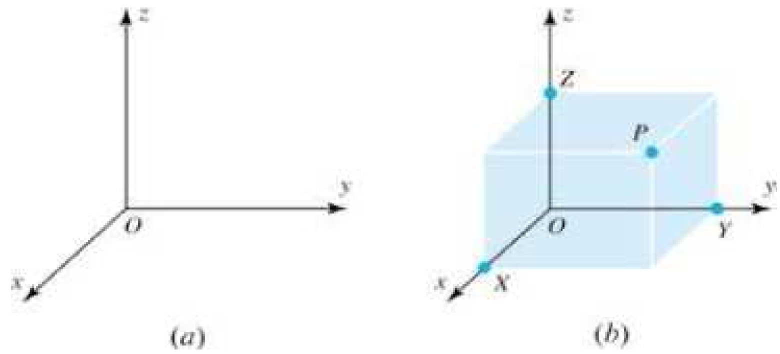


Figure 1.10

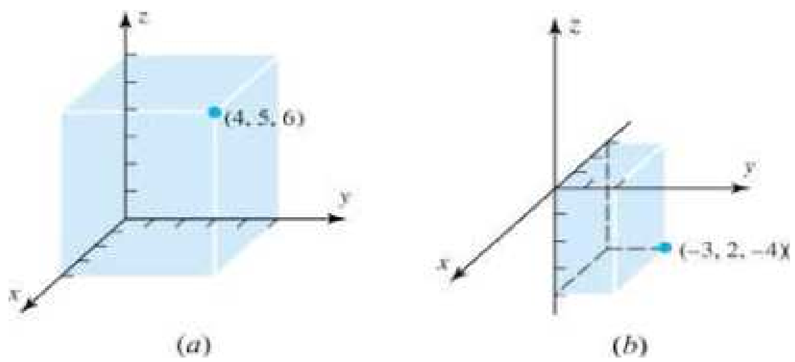


Figure 1.11

Rectangular coordinate systems in 3-space fall into two categories: left-***handed*** and ***right-handed***. A right-handed system has the property that an ordinary screw pointed in the positive direction on the z-axis would be advanced if the positive x-axis were rotated 90° toward the positive y-axis (Figure 1.12a); the system is left-handed if the screw would be retracted (Figure 1.12b).

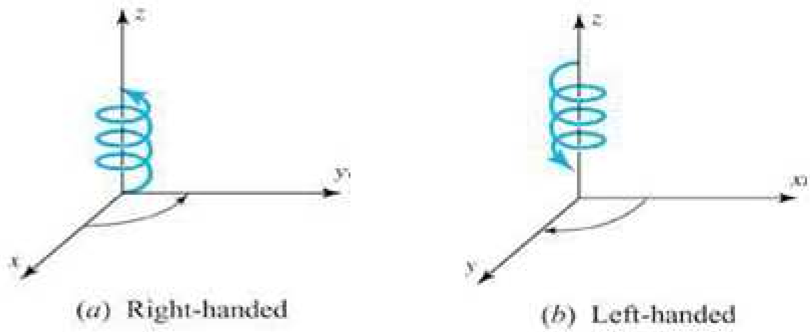


Figure 1.12

✍ **Remark:** In this module we shall use only right-handed coordinate systems.

If, as in Figure 1.13, a vector $\boldsymbol{v}$ in 3-space is positioned so its initial point is at the origin of a rectangular coordinate system, then the coordinates of the terminal point are called the ***components*** of $\boldsymbol{v}$, and we write $\mathbf{v} = (v_1, v_2, v_3)$

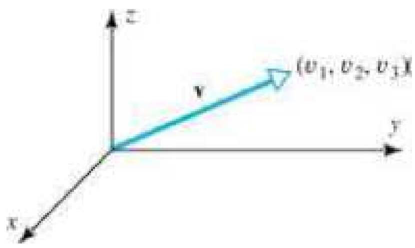


Figure 1.13

If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are two vectors in 3-space, then arguments similar to those used for vectors in a plane can be used to establish the following results

- *$\boldsymbol{v}$ and $\boldsymbol{w}$ are equivalent if and only if*

$$\boldsymbol{v_1} = \boldsymbol{w_1}, \boldsymbol{v_2} = \boldsymbol{w_2} \text{ and} \boldsymbol{v_3} = \boldsymbol{w_3}$$

- $\boldsymbol{v} + \boldsymbol{w} = (\boldsymbol{v_1} + \boldsymbol{w_2}, \boldsymbol{v_2} + \boldsymbol{w_2}, \boldsymbol{v_3} + \boldsymbol{w_3})$

- $k\boldsymbol{v} = (k\boldsymbol{v_1}, k\boldsymbol{v_2}, k\boldsymbol{v_3})$ where k is any scalar.

Example 1 : Vector Computations with Components

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v} + \mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4), \qquad -\mathbf{w} = (-4, -2, -1),$$
$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (-3, -5, 1)$$

Sometimes a vector is positioned so that its initial point is not at the origin. If the vector $\overrightarrow{P_1P_2}$ has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

That is, the components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point. This may be seen using Figure 1.14.

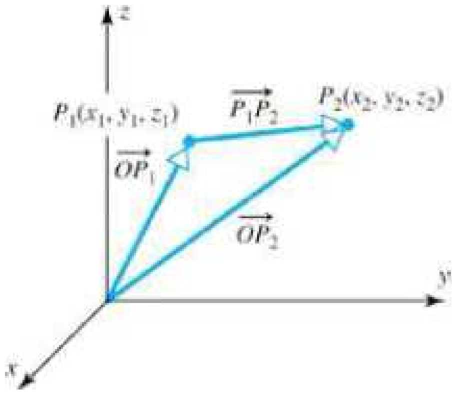


Figure 1.14

The vector is the difference of vectors and , so

$$\overrightarrow{P_1P_2} = \overrightarrow{OP}_2 - \overrightarrow{OP}_1 = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Example 2: Finding the Components of a Vector

The components of the vector with initial point and terminal point are

$$\mathbf{v} = (7 - 2, 5 - (-1), (-8) - 4) = (5, 6, -12)$$

In 2-space the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

1.2 Addition and scalar multiplication

Properties of Vector Operations

The following theorem lists the most important properties of vectors in 2-space and 3-space.

Theorem 1.6: Properties of Vector Arithmetic

*If **u**, **v**, and **w** are vectors in 2- or 3-space and k and l are scalars, then the following relationships hold.*

(a) $u+v=v+u$

(b) $(u+v)+w=u+(v+w)$

(c) $u+0=0+u=u$

(d) $u+(-u)=0$

(e) $k(lu)=(kl)u$

(f) $k(u+v)=ku+kv$

(g) $(k+l)u=ku+lu$

(h) $1u=u$

Before discussing the proof, we note that we have developed two approaches to vectors: ***geometric***, in which vectors are represented by arrows or directed line segments, and ***analytic***, in which vectors are represented by pairs or triples of numbers called components. As a consequence, the equations in Theorem 1.6 can be proved either geometrically or analytically. To illustrate, we shall prove part (*b*) both ways. The remaining proofs are left as exercises.

Proof of part (b) (analytic) we shall give the proof for vectors in 3-space; the proof for 2-space is similar. If, $\mathbf{u}=(u_1,u_2,u_3)$,and $\mathbf{w}=(w_1,w_2,w_3)$ then

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= [(u_1,u_2,u_3)+(v_1,v_2,v_3)]+(w_1,w_2,w_3)\\ &= (u_1+v_1,u_2+v_2,u_3,\ +v_3)+(w_1,w_2,w_3)\\ &= ([u_1+v_1]+w_1,[u_2+v_2]+w_2,[u_3+v_3]+w_3)\\ &= (u_1+[v_1+w_1],u_2+[v_2+w_2],u_3+[v_3+w_3])\\ &= (u_1,u_2,u_3)+(v_1+w_1,v_2+w_2,v_3+w_3)\\ &= \mathbf{u}+(\mathbf{v}+\mathbf{w})\end{aligned}$$

Proof of part (b) (geometric) Let ***u***, ***v***, and ***w*** be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$ as shown in Figure 1.15. Then $\mathbf{v}+\mathbf{w}=\overrightarrow{QS}$ and $\mathbf{u}+(\mathbf{v}+\mathbf{w})=\overrightarrow{PS}$

Also, $\mathbf{u}+\mathbf{v}=\overrightarrow{PR}$ and $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\overrightarrow{PS}$'

Therefore, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

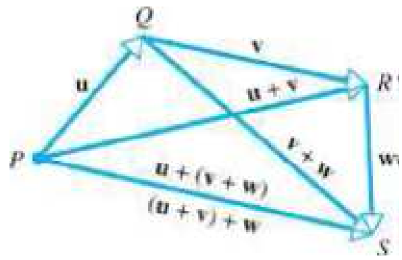


Figure 1.15 The vectors $\mathbf{u}+(\mathbf{v}+\mathbf{w})$ and $(\mathbf{u}+\mathbf{v})+\mathbf{w}$ are equal

Remark : In light of part (b) of this theorem, the symbol $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ is unambiguous since the same sum is obtained no matter where parentheses are inserted. Moreover, if the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are placed "tip to tail," then the sum $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ is the vector from the initial point of $\boldsymbol{u}$ to the terminal point of $\boldsymbol{w}$ (Figure 1.15).

Properties of Vector Operations in n-Space

The most important arithmetic properties of addition and scalar multiplication of vectors in $\Re^n$ are listed in the following theorem. The proofs are all easy and are left as exercises

Theorem 1.7 : Properties of Vectors in $\mathbb{R}^n$

If $u=(u_1,u_2,u_3,\ldots,u_n)$, $v=(v_1,v_2,v_3,\ldots,v_n)$, *and* $w=(w_1,w_2,w_3,\ldots,w_n)$ *are vectors in* $\mathbb{R}^n$ *and k and m are scalars, then:*

(a) $u+v=v+u$

(b) $u+(v+w)=(u+v)+w$

(c) $u+0=0+u=u$

(d) $u+(-u)=0$ *; that is,* $u-u=0$

(e) $k(mu)=(km)u$

(f) $k(u+v)=ku+kv$

(g) $(k+m)u=ku+mu$

(h) $1u=u$

Theorem 1.7 enables us to manipulate vectors in $\mathbb{R}^n$ without expressing the vectors in terms of components. For example, to solve the vector equation $x+u=v$ for $\boldsymbol{x}$, we can add $-u$ to both sides and proceed as follows:

$$
\begin{aligned}
&(x+u)+(-u)=v+(-u)\\
&\Rightarrow x+(u-u)=v-u\\
&\Rightarrow x+0=v-u\\
&\Rightarrow x=v-u
\end{aligned}
$$

The reader will find it instructive to name the parts of Theorem 1.2 that justify the last three steps in this computation

1.3 Scalar product

1.3.1 Magnitude of a Vector

The magnitude of a vector which is a length of a vector and also called norm of a vector. The ***length*** of a vector ***u*** is often called the ***norm*** of ***u*** and is denoted by $\|\mathbf{u}\|$. It follows from the Theorem of Pythagoras that the norm of a vector $\mathbf{u} = (u_1, u_2)$ in 2-space is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$$ (Figure 1.16a).

Let $\mathbf{u} = (u_1, u_2, u_3)$ be a vector in 3-space. Using Figure 1.16*b* and two applications of the Theorem of Pythagoras, we obtain

$$\|\mathbf{u}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (OS)^2 + (RP)^2 = u_1^2 + u_2^2 + u_3^2$$

Thus

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

A vector of norm 1 is called a ***unit vector***.

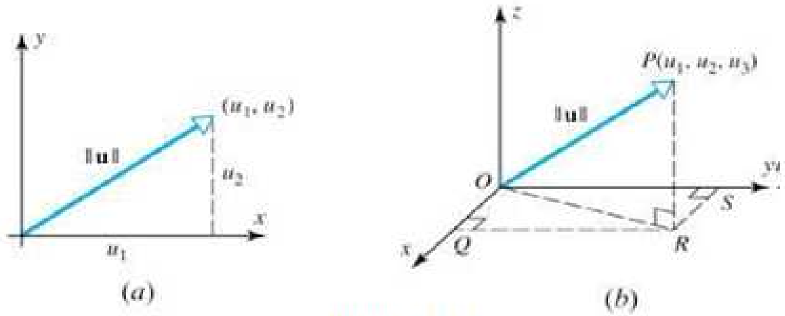


Figure 1.16

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points in 3-space, then the ***distance*** *d* between them is the norm of the vector

$\overrightarrow{P_1P_2}$ (Figure1.3.2). Since $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ it follows from 2 that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Similarly, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in 2-space, then the distance between them is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

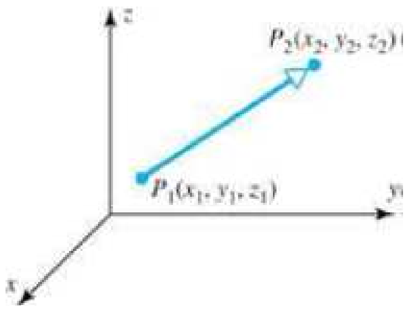


Figure 1.17

The distance between P_1 and P_2 is the norm of the vector $|\overrightarrow{P_1P_2}|$.

Example 1: Finding Norm and Distance

The norm of the vector $\mathbf{u} = (-3, 2, 1)$ is $\|\mathbf{u}\| = \sqrt{(-3)^2 + (2)^2 + (1)^2} = \sqrt{14}$

The distance *d* between the points $P_1(2, -1, -5)$ and $P_2(4, -3, 1)$ is

$d = \sqrt{(4-2)^2 + (-3+1)^2 + (1+5)^2} = \sqrt{44} = 2\sqrt{11}$

From the definition of the product ku, the length of the vector ku is $|k|$ times the length of ***u***.

Expressed as an equation, this statement says that

$$\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$$

This useful formula is applicable in both 2-space and 3-space.

Exercise

1. *Find the norm of* **v**.

(a) $\mathbf{v} = (4, -3)$

(b) $\mathbf{v} = (-7, 2, -1)$

(c) $\mathbf{v} = (0, 6, 0)$

2. *Find the distance between* P_1 *and* P_2.

(a) $P_1(3, 4), P_2(5, 7)$

(b) $P_1(7, -5, 1), P_2(-7, -2, -1)$

3. *Let* $\mathbf{v} = (-1, 2, 5)$. *Find all scalars k such that* $\|k\mathbf{v}\| = 4$.

4. *(a) Show that if* **v** *is any nonzero vector, then* $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ *is a unit vector.*

(b) Use the result in part (a) to find a unit vector that has the same direction as the vector $\mathbf{v} = (3, 4)$

(c) Use the result in part (a) to find a unit vector that is oppositely directed to the vector $\mathbf{v} = (-2, 3, -6)$.

1.3.2 Angle between two vectors

Scalar Product of Vectors: It is also called dot product.

Let ***u*** and ***v*** be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so that their initial points coincide. By the ***angle between u and v***, we shall mean the angle θ determined by ***u*** and ***v*** that satisfies $0 \le \theta \le \pi$ (Figure1.3.3).

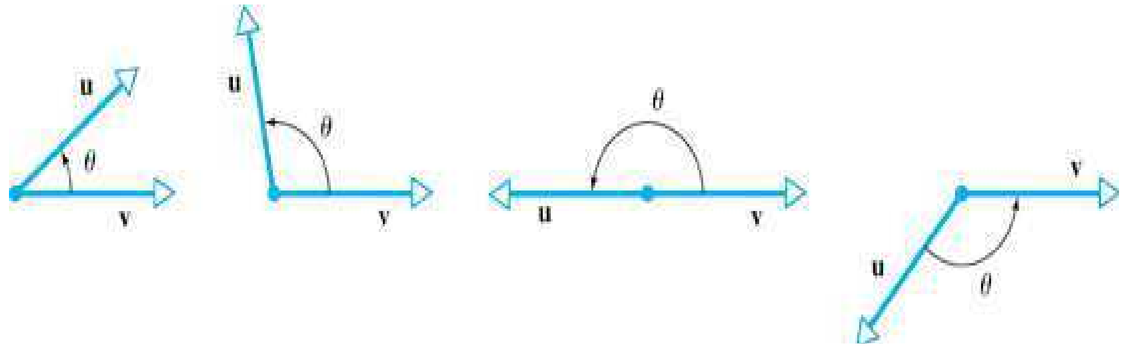


Figure 1.17: The angle θ between ***u*** and ***v*** satisfies $0 \le \theta \le \pi$.

Definition 1.8 : If ***u*** and ***v*** are vectors in 2-space or 3-space and θ is the angle between ***u*** and ***v***, then the ***dot product*** or ***Euclidean inner product*** $\boldsymbol{u.v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases} \qquad (1)$$

Example 1: Dot Product

As shown in Figure 1.19, the angle between the vectors $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$ is 45°. Thus

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = (\sqrt{0^2 + 0^2 + 1^2})(\sqrt{0^2 + 2^2 + 2^2})\left(\frac{1}{\sqrt{2}}\right) = 2$$

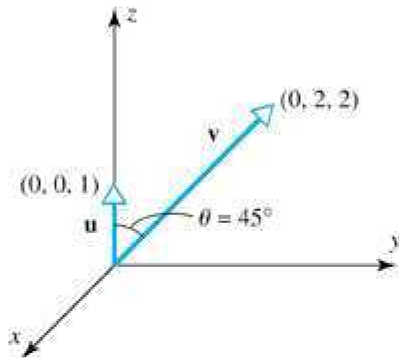


Figure 1.19

Component Form of the Dot Product

For purposes of computation, it is desirable to have a formula that expresses the dot product of two vectors in terms of the components of the vectors. We will derive such a formula for vectors in 3-space; the derivation for vectors in 2-space is similar.

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be two nonzero vectors. If, as shown in Figure1.20 , θ is the angle between ***u*** and ***v***, then the law of cosines yields

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \qquad (2)$$

Since $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$, we can rewrite 2 as

$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

or

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2)$$

Substituting

$$\|\mathbf{u}\|^2 = u_1^2 + u_3^2 + u_3^2, \qquad \|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

And

$$\|\mathbf{v} - \mathbf{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2$$

we obtain, after simplifying,

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \qquad (3)$$

Although we derived this formula under the assumption that ***u*** and ***v*** are nonzero, the formula is also valid if $\boldsymbol{u} = 0$ or $\boldsymbol{v} = 0$ (verify).

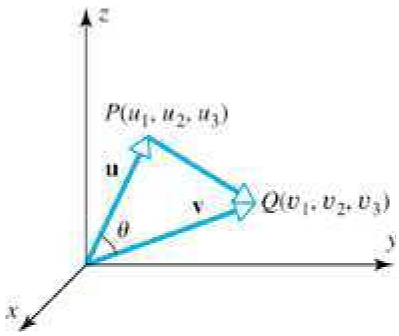


Figure 1.20

If $\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2})$ and $\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2})$ are two vectors in 2-space, then the formula corresponding to 3 is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 \tag{4}$$

Finding the Angle Between Vectors

If ***u*** and ***v*** are nonzero vectors, then Formula 1 can be written as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{5}$$

Example 2: Dot Product Using (3)

Consider the vectors $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (1, 1, 2)$. Find $U.V$ and determine the angle θ between ***u*** and ***v***.

Solution: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = (2)(1) + (-1)(1) + (1)(2) = 3$

For the given vectors we have $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{6}$, so from (5),

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

Thus, $\theta = 60^0$. The following theorem shows how the dot product can be used to obtain information about the angle between two vectors; it also establishes an important relationship between the norm and the dot product.

Theorem 1.9 : *Let **u** and **v** be vectors in 2- or 3-space.*

(a) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$; *that is,* $\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$

(b) *If the vectors **u** and **v** are nonzero and θ is the angle between them, then*

θ is acute	if and only if	$\mathbf{u}\cdot\mathbf{v}>0$
θ is obtuse	if and only if	$\mathbf{u}\cdot\mathbf{v}<0$
$\theta=\pi/2$	*if and only if*	$\mathbf{u}\cdot\mathbf{v}=0$

Proof (a): Since the angle θ between *v* and *v* is 0, we have

$$\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|\|\mathbf{v}\|\cos\theta=\|\mathbf{v}\|^2\cos 0=\|\mathbf{v}\|^2$$

Proof (b): since θ satisfies $0\le\theta\le\pi$, it follows that θ is acute if and only if $\cos\theta>0$, that θ is obtuse if and only if $\cos\theta<0$, and that $\theta=\dfrac{\pi}{2}$ if and only if $\cos\theta=0$. But $\cos\theta$ has the same sign as $u.v$ since $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta, \|u\|>0$, and $\|v\|>0$.

Thus, the result follows.

Example 3: Finding Dot Products from Components

If $\mathbf{u}=(1,-2,3)$, $\mathbf{v}=(-3,4,2)$, and $\mathbf{w}=(3,6,3)$, then

$$\mathbf{u}\cdot\mathbf{v}=(1)(-3)+(-2)(4)+(3)(2)=-5$$
$$\mathbf{v}\cdot\mathbf{w}=(-3)(3)+(4)(6)+(2)(3)=21$$
$$\mathbf{u}\cdot\mathbf{w}=(1)(3)+(-2)(6)+(3)(3)=0$$

Therefore, ***u*** and ***v*** make an obtuse angle, ***v*** and ***w*** make an acute angle, and ***u*** and ***w*** are perpendicular.

Orthogonal Vectors

Perpendicular vectors are also called ***orthogonal*** vectors. In light of the Theorem above *b*, two *nonzero* vectors are orthogonal if and only if their dot product is zero. If we agree to consider ***u*** and ***v*** to be perpendicular when either or both of these vectors is $\mathbf{0}$, then we can state without exception that *two vectors **u** and **v** are orthogonal* (*perpendicular*) *if and only if* $u.v=0$. To indicate that ***u*** and ***v*** are orthogonal vectors, we write $\mathbf{u}\perp\mathbf{v}$.

Example 4: A Vector Perpendicular to a Line

Show that in 2-space the nonzero vector $\mathbf{n}=(a,b)$ is perpendicular to the line $ax+by+c=0$.

Solution: Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be distinct points on the line, so that

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \end{aligned} \qquad (6)$$

Since the vector $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$ runs along the line (Figure 1.21), we need only show that ***n*** and $\overrightarrow{P_1P_2}$ are perpendicular. But on subtracting the equations in (6), we obtain $a(x_2 - x_1) + b(y_2 - y_1) = 0$ which can be expressed in the form $(a, b) \cdot (x_2 - x_1, y_2 - y_1) = 0$ or $\mathbf{n} \cdot \overrightarrow{P_1P_2} = 0$

Thus ***n*** and $\overrightarrow{P_1P_2}$ are perpendicular.

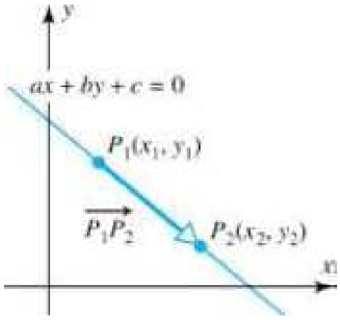


Figure 1.21

The following theorem lists the most important properties of the dot product. They are useful in calculations involving vectors.

Theorem 1.10 : Properties of the Dot Product

*If **u**, **v**, and **w** are vectors in 2-or 3-space and k is a scalar, then*

(a) $u.v = v.u$

(b) $u.(v + w) = u.v + u.w$

(c) $k(u.v) = (ku)v = u.(kv)$

(d) $v.v > 0$ *if* $v \neq 0$, *and* $v.v = 0$ *if* $v = 0$

Proof: We shall prove (*c*) for vectors in 3-space and leave the remaining proofs as exercises. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$; then

$$\begin{aligned} k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1v_1 + u_2v_2 + u_3v_3) \\ &= (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3 \\ &= (k\mathbf{u}) \cdot \mathbf{v} \end{aligned}$$

Similarly, $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

1.3.3 An Orthogonal Projection

In many applications it is of interest to "decompose" a vector ***u*** into a sum of two terms, one parallel to a specified nonzero vector ***a*** and the other perpendicular to ***a***. If ***u*** and ***a*** are positioned so that their initial points coincide at a point *Q*, we can decompose the vector ***u*** as follows (Figure 1.3.7): Drop a perpendicular from the tip of ***u*** to the line through ***a***, and construct the vector $\mathbf{w}_1$ from *Q* to the foot of this perpendicular. Next form the difference

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

As indicated in Figure 1.3.7, the vector $\mathbf{w}_1$ is parallel to ***a***, the vector $\mathbf{w}_2$ is perpendicular to ***a***, and $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$

The vector $\mathbf{w}_1$ is called the ***orthogonal projection of u on a*** or sometimes the ***vector component of u along a***. It is denoted by $\mathrm{proj}_{\mathbf{a}}\mathbf{u}$ (7)

The vector $\mathbf{w}_2$ is called the ***vector component of u orthogonal to a***. Since we have $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$, this vector can be written in notation 7 as $\mathbf{w}_2 = \mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u}$

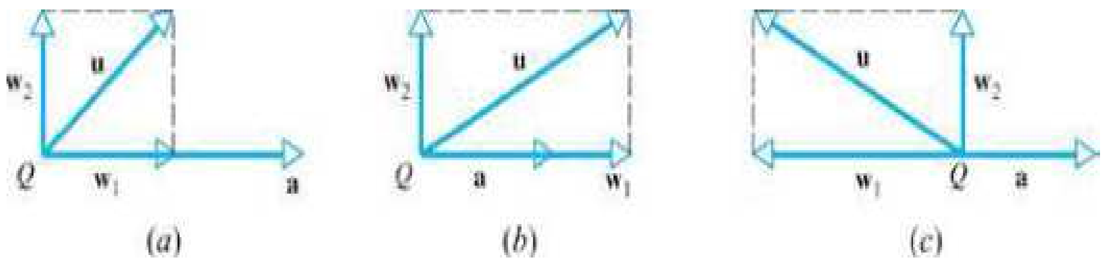


Figure 1.22

The vector ***u*** is the sum of $\mathbf{w}_1$ and $\mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to ***a*** and $\mathbf{w}_2$ is perpendicular to ***a***.

The following theorem gives formulas for calculating $\mathrm{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u}$

Theorem 1.11 :

*If **u** and **a** are vectors in 2-space or 3-space and if* $\mathbf{a} \neq \mathbf{0}$ *, then*

$$\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \qquad \text{(vector component of u along a)}$$

$$\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \qquad \text{(vector component u orthogonal to a)}$$

Proof: Let $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$. Since $\mathbf{w}_1$ is parallel to ***a***, it must be a scalar multiple of ***a***, so it can be written in the form $\mathbf{w}_1 = k\mathbf{a}$. Thus

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = k\mathbf{a} + \mathbf{w}_2 \qquad (8)$$

Taking the dot product of both sides of 8 with ***a*** and using the above Theorems yields

$$\mathbf{u} \cdot \mathbf{a} = (k\mathbf{a} + \mathbf{w}_2) \cdot \mathbf{a} = k\|\mathbf{a}\|^2 + \mathbf{w}_2 \cdot \mathbf{a} \qquad (9)$$

But $\mathbf{w}_2 \cdot \mathbf{a} = 0$ since $\mathbf{w}_2$ is perpendicular to ***a***; so 9 yields $k = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}$

Since $\text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{w}_1 = k\mathbf{a}$, we obtain $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$

Example5: Vector Component of *u* Along *a*

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.

Solution:

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$
$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus the vector component of ***u*** along ***a*** is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{15}{21}(4, -1, 2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

and the vector component of ***u*** orthogonal to ***a*** is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$$

As a check, the reader may wish to verify that the vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and ***a*** are perpendicular by showing that their dot product is zero.

A formula for the length of the vector component of ***u*** along ***a*** can be obtained by writing

$$\begin{aligned}\|\text{proj}_{\mathbf{a}}\mathbf{u}\| &= \left\|\frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}\right\| \\ &= \left|\frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\right|\|\mathbf{a}\| && \leftarrow \text{Formula (5) of Section 3.2} \\ &= \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| && \leftarrow \text{Since } \|\mathbf{a}\|^2 > 0\end{aligned}$$

Which yields

$$\|\mathrm{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} \tag{10}$$

If.θ denotes the angle between ***u*** and ***a***, then $\mathbf{u}\cdot\mathbf{a} = \|\mathbf{u}\|\|\mathbf{a}\|\cos\theta$, so 10 can also be written as

$$\|\mathrm{proj}_{\mathbf{a}}\mathbf{u}\| = \|\mathbf{u}\||\cos\theta| \tag{11}$$

(Verify.) A geometric interpretation of this result is given in Figure 1.23.

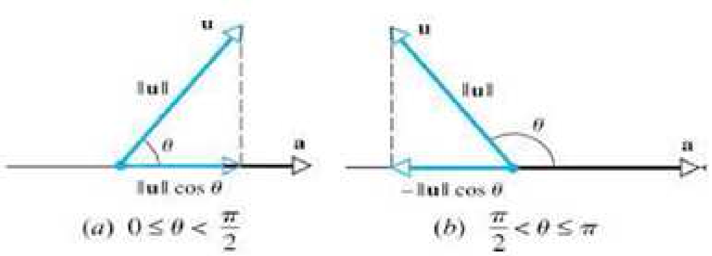


Figure 1.23

As an example, we will use vector methods to derive a formula for the distance from a point in the plane to a line.

1.3.4 Direction Angles and Direction Cosines

The **direction angles** of a nonzero vector ***a*** are the angles $\alpha, \beta,$ and γ (in the interval $[0, \pi]$ that ***a*** makes with the positive x ,y, and z -axes. (See Figure 1.24)

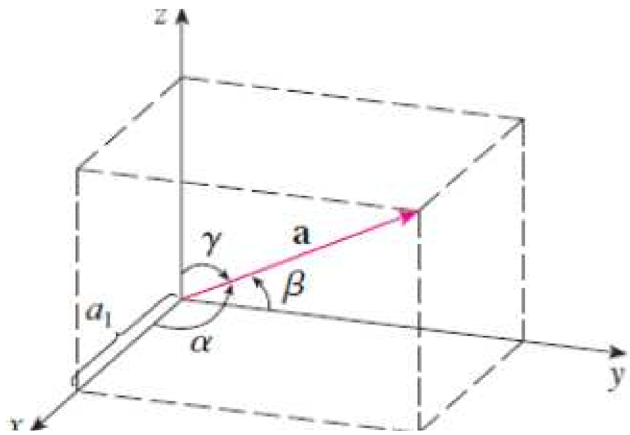


Figure 1.24

The cosines of these direction angles $\cos\alpha, \cos\beta,$ and $\cos\gamma$, are called the **direction**

cosines of the vector **a**. Using the formula of angles between two vectors , we obtain

$$\cos\alpha = \frac{a\cdot i}{\|a\|\|i\|} = \frac{a_1}{\|a\|}, \cos\beta = \frac{a_2}{\|a\|}, \text{and } \cos\gamma = \frac{a_3}{\|a\|}$$

Example : Find the direction angles of the vector $a = (1,2,3)$

Solution: Since $\|a\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$ then

$$\cos\alpha = \frac{1}{\sqrt{14}}, \cos\beta = \frac{2}{\sqrt{14}}, \text{and } \cos\gamma = \frac{3}{\sqrt{14}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^0, \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^0 \; and \; \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^0$$

Exercise

1. *Find* $\mathbf{u}\cdot\mathbf{v}$.
   - *(a)* $\mathbf{u} = (2, 3), \mathbf{v} = (5, -7)$
   - *(b)* $\mathbf{u} = (-2, 2, 3), \mathbf{v} = (1, 7, -4)$
2. *In each part of Exercise 1, find the cosine of the angle* θ *between* ***u*** *and* ***v***.
3. *Determine whether* ***u*** *and* ***v*** *make an acute angle, make an obtuse angle, or are orthogonal.*
   - *(a)* $\mathbf{u} = (6, 1, 4), \mathbf{v} = (2, 0, -3)$
   - *(b)* $\mathbf{u} = (0, 0, -1), \mathbf{v} = (1, 1, 1)$
4. *Find the orthogonal projection of* ***u*** *on* ***a***.
   - *(a)* $\mathbf{u} = (-1, -2), \mathbf{a} = (-2, 3)$
   - *(b)* $\mathbf{u} = (1, 0, 0), \mathbf{a} = (4, 3, 8)$
5. *In each part of Exercise 4, find the vector component of* ***u*** *orthogonal to* ***a***.

1.4 Cross product

In many applications of vectors to problems in geometry, physics, and engineering, it is of interest to construct a vector in 3-space that is perpendicular to two given vectors. In this section we shall show how to do this.

Definition 1.12 : The **cross product** $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$, unlike the dot product, is a vector. For this reason it is also called the **vector product**. Note that $\mathbf{a} \times \mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are *three-dimensional* vectors.

Definition 1.13: If $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and **b**= $\langle b, b_2, b_3 \rangle$, then the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector

$$\boldsymbol{a} \times \boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

This may seem like a strange way of defining a product. The reason for the particular form of Definition 1.13 is that the cross product defined in this way has many useful properties, as we will soon see. In particular, we will show that the vector is perpendicular to both and . In order to make Definition 1.13 easier to remember, we use the notation of determinants.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example, $$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows :

$\boxed{2}$ $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 2 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0-4) - 2(6+5) + (-1)(12-0) = -38$$

If we now rewrite Definition 1 using second-order determinants and the standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

3

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

In view of the similarity between Equations 2 and 3, we often write

4

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 4 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 2, we obtain Equation 3. The symbolic formula in Equation 4 is probably the easiest way of remembering and computing cross products.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k} \\ &= (-15-28)\mathbf{i} - (-5-8)\mathbf{j} + (7-6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k} \end{aligned}$$

EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} \\ &= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0} \end{aligned}$$

The second method is slightly easier; however, many textbooks don't cover this method as it will only work on 3×3 determinant. This method says to take the determinant as listed above in equation 4 and then copy the first columns onto the end as shown below.

$$\boldsymbol{a} \times \boldsymbol{b} = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}\right.$$

We now have three diagomals that move from left to right and three diagonals that move from right to left. We multiply along each diagonal and add those that move from left to right and subtract those that move from right to left.

Example If $a = \langle 2, 1, -1 \rangle$ and $b = \langle -3, 4, 1 \rangle$ compute each of the following.

(a) $a \times b$

(b) $b \times a$

Solution

(a) Here is the computation for this one.

$$a \times b = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & 1 & -1 & 2 & 1 \\ -3 & 4 & 1 & -3 & 4 \end{array}\right.$$

$$= i\,(1)(1) + j\,(-1)(-3) + k\,(2)(4) - j\,(2)(1) - i\,(-1)(4) - k\,(1)(-3)$$
$$= 5i + j + 11k$$

(b) And here is the computation for this one.

$$b \times a = \left|\begin{array}{ccc|cc} i & j & k & i & j \\ -3 & 4 & 1 & -3 & 4 \\ 2 & 1 & -1 & 2 & 1 \end{array}\right.$$

$$= i\,(4)(-1) + j\,(1)(2) + k\,(-3)(1) - j\,(-3)(-1) - i\,(1)(1) - k\,(4)(2)$$
$$= -5i - j - 11k$$

One of the most important properties of the cross product is given by the following theorem.

Theorem 1.14: The vector $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Proof: In order to show that $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$, we compute their dot product as follows:

$$
\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\
&= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\
&= 0
\end{aligned}
$$

Theorem 1.15: If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ $(so\ 0 \leq \theta \leq \pi)$, then

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|sin\theta$$

Proof: From the definitions of the cross product and length of a vector, we have

$$
\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\
&\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta
\end{aligned}
$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Since a vector is completely determined by its magnitude and direction, we can now say that $\boldsymbol{a} \times \boldsymbol{b}$ is the vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, whose orientation is determined by the right hand rule,whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$.

In fact, that is exactly how physicists *define* $\boldsymbol{a} \times \boldsymbol{b}$.

7 **COROLLARY** Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

Proof: Two nonzero vectors **a** and **b** are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$, so $|\mathbf{a} \times \mathbf{b}| = 0$ and therefore $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

The geometric interpretation of Theorem 1.15 can be seen by looking at Figure 2. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\boldsymbol{a}|$, altitude $|\boldsymbol{b}|sin\theta$

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a}\times\mathbf{b}|$$

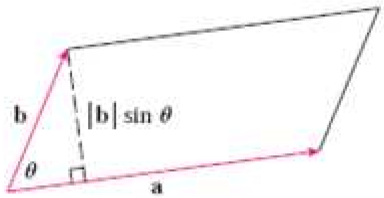


Figure 1.25

Thus we have the following way of interpreting the magnitude of a cross product.

> The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Example 3: Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore perpendicular to the plane through P, Q, and R. We know from (12.2.1) that

$$\overrightarrow{PQ} = (-2-1)\,\mathbf{i} + (5-4)\,\mathbf{j} + (-1-6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1-1)\,\mathbf{i} + (-1-4)\,\mathbf{j} + (1-6)\,\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5-35)\,\mathbf{i} - (15-0)\,\mathbf{j} + (15-0)\,\mathbf{k} = -40\,\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}$$

So the vector $\langle -40, -15, 15\rangle$ is perpendicular to the given plane. Any nonzero scalar multiple of this vector, such as $\langle -8, -3, 3\rangle$, is also perpendicular to the plane.

Example 4: Find the area of the triangle with vertices

$P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution: In Example 3 we computed that $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle -40, -15, 15\rangle$

The area of the parallelogram with adjacent sides PQ and PR is the length of this cross product:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

If we apply Theorems 5 and 6 to the standard basis vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ $using$ $\theta = \frac{\pi}{2}$ we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

- Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

- So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra do hold for cross products. The following theorem summarizes the properties of vector products.

Theorem 1.16: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

Proof Of Property 5

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$\boxed{9}$ $$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)\\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1\\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3\\ &= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\end{aligned}$$

There is an important difference between the dot product and cross product of two vectors—the dot product is a scalar and the cross product is a vector. The following theorem gives some important relationships between the dot product and cross product and also shows that $\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

Theorem 1.17:- Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(a) $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ ($\mathbf{u}\times\mathbf{v}$ **is orthogonal to u**)

(b) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$ ($\mathbf{u}\times\mathbf{v}$ **is orthogonal to v**)

(c) $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$ (**Lagrange's identity**)

(d $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (**relationship between cross and dot products**)

(e) $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ (**relationship between cross and dot products**)

Proof (a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) &= (u_1, u_2, u_3)\cdot(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\\ &= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0\end{aligned}$$

Proof (b) Similar to (*a*).

Proof (c) Since

$$\|\mathbf{u}\times\mathbf{v}\|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \qquad (1)$$

and

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \qquad (2)$$

the proof can be completed by "multiplying out" the right sides of 1 and 2 and verifying their equality.

Proof (d) and (e) As an Exercise.

Example 5: $\mathbf{u}\times\mathbf{v}$ Is Perpendicular to *u* and to *v*

Consider the vectors $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$

Then we can show that $\mathbf{u}\times\mathbf{v} = (2, -7, -6)$

Since $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = (1)(2) + (2)(-7) + (-2)(-6) = 0$

and $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=(3)(2)+(0)(-7)+(1)(-6)=0$

$\mathbf{u}\times\mathbf{v}$ is orthogonal to both ***u*** and ***v***, as guaranteed by the Theorem above.

Exercise

1. *Let* $\mathbf{u}=(3,2,-1)$, $\mathbf{v}=(0,2,-3)$, *and* $\mathbf{w}=(2,6,7)$. *Compute*

(a) $\mathbf{v}\times\mathbf{w}$

(b) $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$

(c) $\mathbf{u}\times(\mathbf{v}-2\mathbf{w})$

2.*Find a vector that is orthogonal to both* ***u*** *and* ***v****.*

(a) $\mathbf{u}=(-6,4,2)$, $\mathbf{v}=(3,1,5)$

(b) $\mathbf{u}=(-2,1,5)$, $\mathbf{v}=(3,0,-3)$

1.5 Lines and planes

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form. Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0,y_0,z_0)$ on Land the direction of L . In three dimensions the direction of a line is conveniently described by a vector, so we let be a vector parallel to L . Let $P(x,y,z)$ be an arbitrary point on L and let $\mathbf{r_0}$ and **r** be the position vectors of P_0 and P(that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If **a** is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1.6.1,then the Triangle Law for vector addition gives $\mathbf{r}=\mathbf{r}_0+\mathbf{a}$. But, since **a** and **v** are parallel vectors, there is a scalar t such that **a** = t**v**. Thus

$$\boxed{\mathbf{r}=\mathbf{r}_0+t\mathbf{v}} \qquad (1)$$

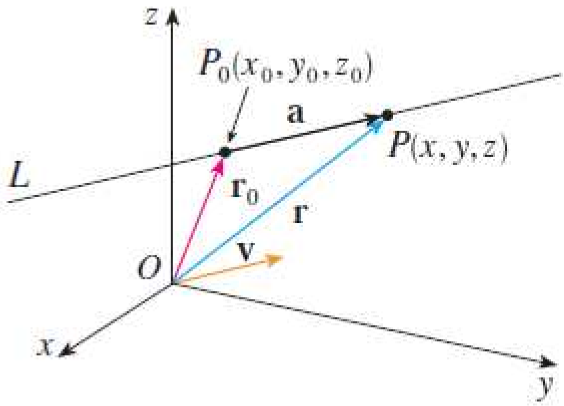


Figure 1.26

which is a **vector equation** of L . Each value of the **parameter** t gives the position vector **r** of a point on L. In other words, as t varies, the line is traced out by the tip of the vector **r**. As Figure 1.6.2 indicates, positive values of t correspond to points on L that lie on one side of P_0 , whereas negative values of t correspond to points that lie on the other side of P_0 . If the vector **v** that gives the direction of the line L is written in component form as $v=(a,b,c)$, then we have $tv=(ta,tb,tc)$. We can also write $r=(x,y,z)$ and $r_0=(x_0,y_0,z_0)$, so the vector equation (1) becomes $(x,y,z)=(x_0+ta, y_0+tb, z_0+tc)$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$\boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct} \quad (2)$$

Where $t \in \Re$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $v=(a,b,c)$. Each value of the parameter t gives a point (x,y,z) on L .

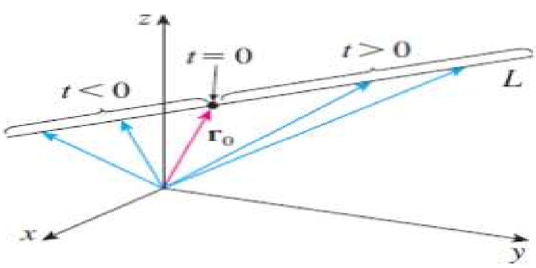


Figure 1.27

Example 1:

(a) Find a vector equation and parametric equations for the line that passes through thepoint $(5,1,3)$ and is parallel to the vector $i+4j-2k$.

(b) Find two other points on the line.

Solution:

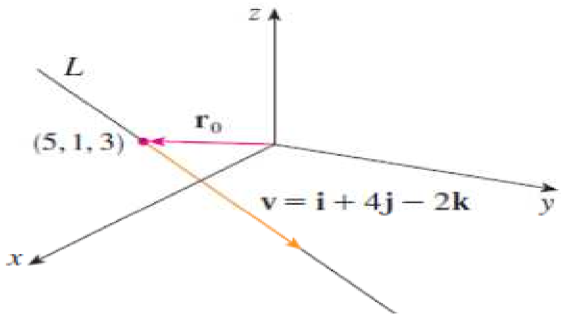


Figure 1.28

(a) Here $r_0 = (5,1,3) = 5i + j + 3k$ and $v = i + 4j - 2k$, so the vector equation(1) becomes $r = (5i + j + 3k) + t(i + 4j - 2k) \Rightarrow r = (5+t)i + (1+4t)j + 3k) + (3-2t)k$

Or Parametric equations are $x = 5 + t, y = 1 + 4t, z = 3 - 2t$

(b) Choosing the parameter value t= 1 gives x=6, y=5, and z=1 so $(6,5,1)$ is a point on the line. Similarly t=-1, gives the point $(4,-3,5)$.

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. Forinstance, if, instead of $(5,1,3)$, we choose the point $(6,5,1)$ in Example 1, then the parametric equations of the line become $x = 6 + t, y = 5 + 4t, z = 1 - 2t$ Or, if we stay with the point $(5,1,3)$ but choose the parallel vector $2i + 8j - 4k$, we arrive at the equations $x = 5 + 2t, y = 1 + 8t, z = 3 - 4t$ In general, if a vector $v = (a,b,c)$ is used to describe the direction of a line L , then the numbers *a*, *b*, and *c* are called **direction numbers** of *L*. Since any vector parallel to **v** could also be used, we see that any three numbers proportional to *a*, *b*, and *c* could also be used as a set of direction numbers for L. Another way of describing a line L is to eliminate the parameter from Equations 2. If none of *a*, *b*, or *c* is 0, we can solve each of these equations for t, equate the results, and obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad (3)$$

These equations are called **symmetric equations** of L. Notice that the numbers a, b, andc that appear in the denominators of Equations 3 are direction numbers of L , that is, components of a vector parallel to L . If one of a, b, or c is 0, we can still eliminate t . For instance, if a = 0, we could write the equations of L as

$x = x_0, \frac{y - y_0}{b} = \frac{z - z_0}{c}$ This means that L lies in the vertical plane x= x_0 .

Example 2:

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$.

(b) At what point does this line intersect the xy-plane?

Solution:

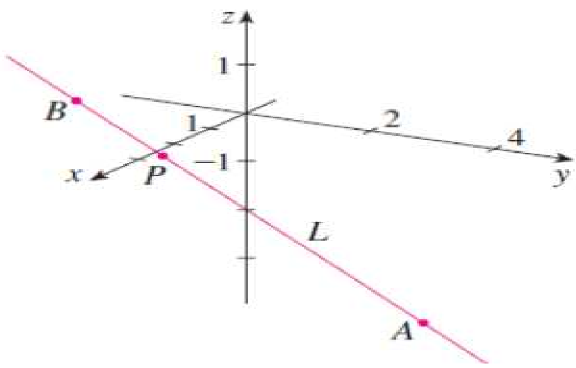


Figure 1.29

(a) We are not explicitly given a vector parallel to the line, but observe that the vector **v** with representation line AB is parallel to the line and $v = (3-2,-1-4,1-(-3)) = (1,-5,4)$

Thus direction numbers are a=1, b=-5, and c=4 . Taking the point $(2,4,-3)$as P_0 , we see that parametric equations (2) are $x = 2 + t, y = 4 - 5t, z = -3 + 4t$ and symmetric equations (3) are $\frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$

(b) The line intersects the xy-plane when z = 0 , so we put z =0 in the symmetric equations and obtain $\frac{x-2}{1}=\frac{y-4}{-5}=\frac{3}{4}$ This gives $x=\frac{11}{4}$ *and* $y=\frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4},\frac{1}{4},0\right)$.

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0, y_1 - y_0,$ and $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}$$

The vector equation of a line through the (tip of the) vector $\mathbf{r}_0$ in the direction of a vector **v** is $\boldsymbol{r} = \boldsymbol{r_0} + t\boldsymbol{v}$. If the line also passes through (the tip of) $\boldsymbol{r}_1$, then we can take $\boldsymbol{v} = \boldsymbol{r_1} - \boldsymbol{r_0}$ and so its vector equation is $r = r_0 + t(r_1 - r_0) = (1-t)r_0 + tr_1$

The line segment from $\boldsymbol{r_1}$ to $\boldsymbol{r_0}$ is given by the parameter interval $0 \le t \le 1$.

4 The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

PLANES

Although a line in space is determined by a point and a direction, a plane in space is more difficult to describe. A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\boldsymbol{n}$ that is orthogonal to the plane. This orthogonal vector $\boldsymbol{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\boldsymbol{r_1}$ and $\boldsymbol{r_0}$ be the position vectors of P_0 and P . Then the vector $\boldsymbol{r_1} - \boldsymbol{r_0}$ is represented by a line P_0P. (See Figure below.) The normal vector $\boldsymbol{n}$ is orthogonal to every vector in the given plane. In particular, $\boldsymbol{n}$ is orthogonal to $\boldsymbol{r_1} - \boldsymbol{r_0}$ and so we have $\boldsymbol{n}.(\boldsymbol{r_1} - \boldsymbol{r_0}) = \mathbf{0}$ which can be rewritten as $\boldsymbol{n}.\boldsymbol{r} = \boldsymbol{n}.\boldsymbol{r_0}$ And this Equation is called a **vector equation of the plane**. To obtain a scalar equation for the plane, we write $\boldsymbol{n} = (a, b, c)$, $\boldsymbol{r} = (x, y, z)$, and $\boldsymbol{r_0} = (x_0, y_0, z_0)$.

Then the above vector equation becomes $(a,b,c)\cdot(x-x_0, y-y_0, z-z_0)=0$ or $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ .And this Equation is the scalar equation of the plane through with normal vector $n=(a,b,c)$.

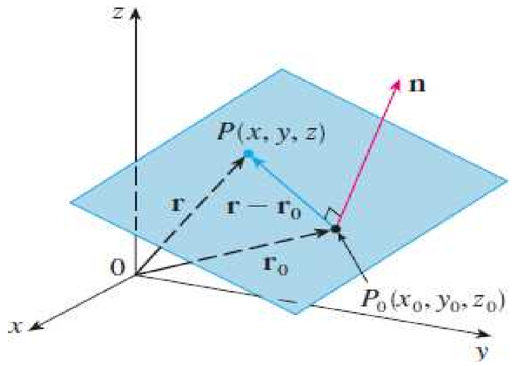


Figure 1.30

Example-1: Find an equation of the plane through the point $(2,4,-1)$ with normal Vector **n** $=(2,3,4)$. Find the intercepts and sketch the plane.

Solution: Putting $a=2, b=3, c=4, x_0=2, y_0=4, z_0=-1$ in the sclalar equation of the plane we see that an equation of the plane is $2(x-2)+3(y-4)+4(z+1)=0$ or $2x+3y+4z=12$

To find the x-intercept we set y =z = 0 in this equation and obtain x = 6 . Similarly, they -intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure below)

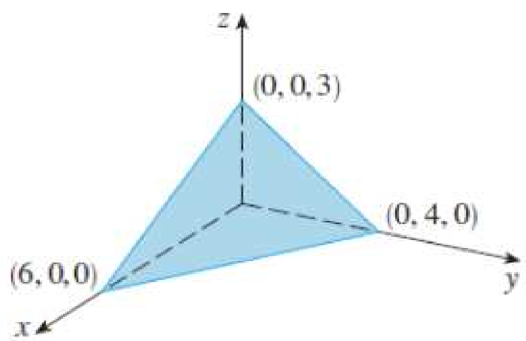


Figure1.31

Example-2: Find an equation of the plane that passes through the points $P(1,3,2), Q(3,-1,6)$ and $R(5,2,0)$

Solution: The vectors **a** and **b** corresponding to line PQ and line PR are **a**= $(2,-4,4)$, **b** $=(4,-1,-2)$.Since both **a** and **b** lie in the plane, their cross product **a**x**b** is orthogonal to the plane and can be taken as the normal vector.

Thus, **n = a**x**b** = $\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$ = 12i+20j+14k With the point $P(1,3,2)$ and the normal vector **n**, an equation of the plane is $12(x-1)+20(y-3)+14(z-2)=0 \Rightarrow 6x+10y+7z=50$

Note : Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x+2y-3z=4$ and $2x+4y-6z=3$ are parallel because their normal vectors are $=(1,2,-3)$ and $=(2,4,-6)$ and $=2$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

1.6 Vector space; Subspaces

Definition 1.18 : ***Definition of a Vector Space***

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every u, v and w in V and for every scalar (an element of a field F) c and d, then V is called vector space over a field F

Addition:

A-1 u + v is in V ------------------------- Closure under addition

A-2 u + v = v + u---------------------------Commutative Property

A-3 u + (v+ w) = (u + v) + w --------------Associative property

A-4 V has a zero vector 0 such that ----Existence of Additive Identity for every u in V, u+0 = u

A-5 For every u in V, there is ----------- Existence of additive inverse a vector in V denoted by-u Such that u + (-u) =0 where 0 is the identity element

Scalar Multiplication:

A-6 cu is in V ------------------ Closure under scalar multiplication

A-7- c (u + v) = cu + cv ---------- Distributive property

A-8- (c+d) u=cu + du -------- Distributive Property

A-9- c (du) = (cd) u = d (cu) -------- Associative Property

A-10- 1 (u) = u, where $1\in F$ --- -- Scalar Identity

Remark

1. It is important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and the two operations. When you refer to a vector space V, be sure that all four entities are clearly stated or understood,

2- A vector space together with the defined operation addition is Abelian group.

Class Activity-

1- Is a vector space V can be empty set? What ever your answer comes with justification

2 - Identify the difference between vector space axioms 7 and 8 (A-7 & A-8)

3 - Is the zero vectors invertible under vector addition? If so, find its additive inverse

Examples of different models of a vector space

Throughout mathematics one encounters many examples of mathematical objects that can be added to each other and multiplied b y numbers. Geometric vectors and vectors in n-space studied in the preceding chapter are examples of such objects. Other examples are real-valued functions, complex numbers, and polynomials over the real. In this chapter we discuss a general mathematical concept which includes these examples and many more.

Throughout this chapter, by a “field” we shall mean either the field of rational numbers, we shall, how ever, use a neutral letter k, since it is necessary to deal with each of these fields. The elements of k will also be called numbers or scalars.

Definition 1.19: A vector space v over a field is a set of objects which can be added and multiplied by elements of such that the following axioms are satisfied
Axiom 1: (closure under addition)

For all $v_1,\ v_2 \in v,$ there exists a unique element of v called the
Sum of v_1 *and* v_2, *denoted* $v_1 + v_2$

Axiom 2: (closure under multiplication by elements of k)
For every $v \in V$ *and all* $a \in k$,
there exists an element of called the product of a and v, *denoted* av.

Axiom 3: (commutative law for addition)
For all $v_1,\ v_2 \in v$, *we have* $v_1 + v_2 = v_2 + v_1$

Examples of Vectorspaces

Example-1 Let **V=** $\mathbb{R}^+$ ***and*** $\boldsymbol{b} \in \mathbb{R}$ Define: $x_1 + x_2 = x_1x_2 = \forall x_1, x_2 \in R^+$, usual product in $\mathbb{R}$ and $\boldsymbol{bx} = \boldsymbol{x^b}$, $\forall \boldsymbol{x} \in \mathbb{R}^+$, then show that V is a vector space over $\mathbb{R}$ with 1 as the zero element.

Solution:

1. Let $x_1, x_2 \in V = \Re^+$ then $x_1 + x_2 = x_1x_2 \in \Re^+$

$\Rightarrow x_1 + x_2 \in V$, Since $\Re^+$ is closed under usual multiplication

$\Rightarrow V$ is closed under vector addition

2. Let $x_1, x_2 \in V = \Re^+$ then $x_1 + x_2 = x_1 x_2 = x_2 x_1$ (Why?)

$= x_2 + x_1$

$\Rightarrow x_1 + x_2 = x_2 + x_1 \ \ \forall x_1, x_2 \in R^+$

$\Rightarrow$Vector addition is commutative

3. Let $x_1, x_2, x_3 \in V$ then

$(x_1 + x_2) + x_3 = (x_1 x_2) + x_3$ by definition of addition in V

$= (x_1x_2)x_3$, by definition of addition in V

$= x_1(x_2 x_3)$, (Why?)

$= x_1(x_2 + x_3)$, by definition of ' +' in V

$= x_1 + (x_2 + x_3)$, by definition of '+' in V

Thus, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3) \forall x_1, x_2, x_3 \in V$

$\Rightarrow$Vector addition is associative.

4. Suppose e is a zero vector in V then

$$e + x = x = x + e, \forall x \in V$$

Now, $e + x = x$

$\Rightarrow e.x = x,$ By definition of '+' in V

$\Rightarrow e = 1$, (How?)

$\Rightarrow$ The zero vector in V $= \Re^+$ is 1

Therefore, $\exists 1 \in V$ such that $1 + x = x = x + 1 \forall x \in V$

$\Rightarrow$ Additive identity exists in V

5. Let $x \in V = R^+$ suppose $y \in V$ is

An additive inverse of x then

$$x + y = 1 = y + x$$

But, $x + y = 1$

$\Rightarrow x.y = 1$

$\Rightarrow y = 1/x$ --- (Why?)

Hence, $\forall x \in V \;\; \exists y = 1/x \in V$ such that $x + y = 1$

$\Rightarrow$ Every element in V has an additive inverse

6. Let $x. \in V$ and $a \in \Re$ then $ax = x^a \in V$ (why?)

$\Rightarrow ax \in V \quad \forall a \in \Re$ And $\forall x \in V$

$\Rightarrow V$ is closed under scalar multiplication.

7. Let $a \in \Re$ and $x, y \in V$ then

$a(x + y) = a(x.y)$ By definition of '+' in V

$= (x.y)^a$ by definition of '.' in V

$= x^a . y^a$ (Why?)

$= x^a + y^a$ by definition of ' +' in V= $\Re^+$

$= ax + ay$ by definition of '.' in V $= \Re^+$

Therefore, $a(x + y) = ax + ay \quad Va \in \Re$ and $\forall x, y \in V$

8. Let $a, b \in \Re$ and $x \in V = \Re^+$ then

(a+ b) $x = x^{a+b}$, by definition of scalar multiplication in V

$= x^a x^b$, (Why)?

$= x^a + x^b$, by definition of '+' in V

$= ax + bx$, by definition of '.' In V

$\Rightarrow (a+b)x = ax + bx \quad \forall a, b \in \Re$ And $\forall x \in V$

9. Let $a, b \in \Re$ and $x \in V = \Re^+$ then

$a(bx) = a(x^b)$, by definition of '.' in V

$= (x^b)^a$, by definition of '.' in V

$= x^{ba}$, Why?

$= x^{ab}$

$= (ab)x$

$\Rightarrow a(bx) = (ab)x \quad \forall a, b \in \Re$ And $\forall x \in V$

10. Let $x \in V = \Re^+$ then

$1.x = x^1 = x$

$\Rightarrow 1.x = x \quad \forall x \in V$

Hence, by 1-10 we have V=$\Re^+$ is a vector space over $\Re$ with the given addition and scalar multiplication defined on it.

Example-2 Let V $= \{(x, 2x) : x \in \Re\}$ then show that V is a vector space over $\Re$ with Standard operations (componentwise addition and multiplication)

Solution:

1. Let $u, v \in V$ then $u = (x, 2x)$ for some $x \in \Re$ and $v = (y, 2y)$ for some $y \in \Re$.

   Now, u+v $= (x, 2x) + (y, 2y)$

$= (x + y, 2x + 2y)$, By definition '+' in V

$= (x + y, 2(x + y))$

$= (z, 2z)$, Where $z = x + y \in \Re$

$\Rightarrow u + v = (z, 2z)$ For some $z = x + y \in \Re$

$\Rightarrow u + v \in V$, by definition of the set V

$\Rightarrow V$ is closed under vector addition.

2. Let $u = (x, 2x), v = (y, 2y) \in V$ then

$u + u = (x, 2x) + (y, 2y)$

$= (x + y, 2x + 2y)$

$= (y + x, 2y + 2x)$; Since addition is commutative in $\Re$

$= (y, 2y) + (x, 2x)$, by definition of ' +' in V

$= v + u$

$\Rightarrow u + v = v + u \quad \forall u, v \in V$

Therefore vector addition is commutative in V

3. Let $u = (x,2x)$, $v = (y,2y), w = (z,2z) \in V$ then

$(u + v) + w = [(x,2x) + (y,2y)] + (z,2z)$

$= (x + y, 2x + 2y) + (z,2z)$, By definition of '+' in V

$= ((x + y) + z, (2x + 2y) + 2z)$, by definition of '+' in V

$= (x + (y + z), 2x + (2y + 2z))$; Since the usual addition is associative in $\Re$

$= (x,2x) + (y + z, 2y + 2z)$; By definition '+' in V

$= (x,2x) + [(y,2y) + (z,2z)]$; By definition of '+' in V

$= u + (v + w)$

$\Rightarrow (u + v) + w = u + (v + w) \quad \forall u, v, w \in V$

Therefore the defined addition is associative in V

4. Since $0 \in \Re$ we have

$0 = (0,2.0) = (0,0) \in V$ And $\forall u = (x,2x) \in V$,

$0+u = (0,0) + (x,2x) = (0 + x, 0 + 2x)$

$= (x,2x)$

$= u$

$\Rightarrow \quad 0 + u = u \quad \forall u \in V$

Therefore additive identity element exists in V and $0 = (0,0)$ is the additive identity element in V

5. Let $u = (x,2x) \in V$ then $-u = (-x,-2x) \in V$ and

$u+-u = (x,2x) + (-x,-2x)$

$= (x + -x, 2x + -2x)$

$= (0,0)$

$\Rightarrow u + -u = 0$

$\Rightarrow \forall u \in V$, $\exists -u \in V$ Such that $u + -u = 0$

That is, every element in V has an additive inverse.

6. Let $a \in \Re$ and $u = (x, 2x) \in V$ then

$a.\ u = a(x, 2x)$

$= (ax, 2ax)$, by definition of scalar multiplication

$= (y, 2y);$ Where $y = ax \in \Re$

$\Rightarrow a.\ u = (y, 2y)$ For some $y = ax \in \Re$

$\Rightarrow au \in V$, why?

Therefore V is closed under scalar multiplication.

7. Let $a \in \Re$ and $u = (x, 2x), v = (y, 2y) \in V$ then $a(u + v) = a((x, 2x) + (y, 2y))$

$= a(x + y, 2x + 2y),$ By definition of ' +' in V

$= (a(x + y), a(2x + 2y))$ by definition of scalar multiplication in V

$= (ax + ay, 2ax + 2ay),$ Why?

$= (ax, 2ax) + (ay, 2ay),$ by definition of scalar multiplication in V

$= a(x,\ 2x) + a(y,\ 2y)$

$= au + av$

$\Rightarrow a(u + v) = au + av \qquad \forall a \in \Re \forall u, v \in V$

8. Let $a, b \in \Re$ and $u = (x, 2x) \in V$ then $(a + b)u = (a + b)(x, 2x)$

$= ((a + b)x, (a + b)2x)$; By definition of scalar multiplication in V

$= (ax + bx, 2ax + 2bx)$; Why?

$= (ax,\ 2ax) + (bx, 2bx)$; By definition of '+' in V

$= a(x, 2x) + b(x, 2x)$; Why?

$= au + bu$

$\Rightarrow (a + b)u = au + bu \qquad \forall a, b \in \Re$ And $\forall u \in V$

9. Let $a, b \in \Re$ and $u = (x, 2x) \in V$ then

$a(bu) = a(b\ (x,\ 2x))$

$= a(bx, 2bx)$; By definition of scalar multiplication in V

$= (a(bx), a(2bx))$; Why?

$$=((ab)x,(ab)2x)\text{; Why?}$$

$$= ab(x,2x)\text{; Why?}$$

$$= (ab)u$$

$\Rightarrow a(bu) = (ab)u \quad \forall a,b \in \Re$ And $\forall u \in V$

10. Let $u = (x,2x) \in V$ then

$$1.u = 1(x,2x) = (1.x,1.2x)$$

$$= (x,2x)$$

$$= u$$

$\Rightarrow 1.u = u \forall u \in V$

Therefore, by 1-10 we've V is a vector space over $\Re$

➢ Examples of sets, which are, not vector space

Example-1 : show that the set of rational numbers Q is not a vector space with standard operations over $\Re$

Solution Since $\sqrt{2} \in \Re$ and $1 \in Q$ such that

$\sqrt{2}.1 = \sqrt{2} \notin Q$ We've Q is not closed under scalar multiplication.

Example-2: Let V=$\Re^3$, the set of ordered triple of real numbers, then. Show that V is not a vector space over $\Re$ with the standard operation of addition and the following non-standard definition of scalar multiplication:

$$a(x_1, x_2, x_3) = (ax_1, ax_2, 0)$$

Solution: Since $1 \in \Re$ we've u = $(1,1,1) \in V$ and $1.(1,1,1) = (1.1,1.1,0) = (1,1,0)$

$\Rightarrow 1.(1,1,1) = (1,1,0) \neq (1,1,1)$ = u

$\Rightarrow 1.(1,1,1) \neq (1,1,1)$ = u

$\Rightarrow 1.u \neq u$

$\Rightarrow V$ Doesn't satisfy one of the vector space axioms

$\Rightarrow V$ is not a vector space over $\Re$

Infact, Show that V satisfies the first nine axioms.

Remark

A single failure of one of the ten vector space axioms suffices to show that a set is not a vector space. That is to show that a set V is not a vector space we need only find one thing wrong regardless of how many of the ten axioms are satisfied.

Subspaces

Definition 1.20: ***(definition of subspace of a vector space)***

Let V be a vector space over a field F and W be a subset of V, then we say that W is a subspaces of V if W is itself a vector space over F under the operations of addition and scalar multiplication in V.

Class Activity

1. Let V= the set of all real valued continuous functions defined on $[0,1]$ and W be the set of real valued differentiable functions on $[0,1]$ then show that

A) V is a vector space over R with standard operations

B) W is a subspace of V

2. Let $V = \Re^4$ *and* $W = \{(0, x, y, z); x, y, z \in \Re\}$ *then show that*

A) V with the standard operations is a vector space over Q.

B) W is a subspace of V.

To establish that a set W is a vector space we must verify all the ten vector space axioms. However if W is a subset of a larger vector space V (and the operations defined on W are the same as those defined on V), then most of the ten properties are inherited from the larger space and need no verification. The following theorem tells us that it is sufficient to test for closure in order to establish a subset of a vector space is a subspace.

Theorem 1.21*: Test for a subspace*

Let V be a vector space over a field F and W be an empty subset of V, then W is a subspace of V over F if and only if the following closure conditions hold:

i) if u and v are in W, then u+v is in W

ii) If u is inW and $\beta \in F$, then β u is W

Proof: Let V be a vector space over a field F and $\phi \neq W \subseteq V$

$(\Rightarrow)$ Suppose W is a subspace of V over F then by definition of subspace we have Wi a vector space over F under the induced operations and hence w is closed under addition and scar multiplication.

$\Rightarrow u+v \in W \quad \forall u,v \in W$ And $\beta u \in W \quad \forall \beta \in F, \forall u \in w$

$\Rightarrow$ i) and ii) hold.

$(\Leftarrow)$ Suppose i) and ii) hold then W is closed under addition and scalar multiplication.

$\Rightarrow$ A-1 and A-6 are satisfied ----- (Δ)

Also, let u, v, w$\in W$ and a, b$\in F$ then

u,v,w$\in V$ and a,b$\in$ F ;Since $W \subset V$

$\Rightarrow$ A-2, A-3, A-7, A-8, A-9, and A-10 are satisfied automatically, by definition of V --------$(\Delta\Delta)$

Furthermore, since w$\neq \phi$ we've $\exists u_1 \in W$ and hence $\beta u_1 \in W \quad \forall \beta \in F$; by II

In particular, for $\beta = 0$ we get $0u_1 \in W$

$\Rightarrow 0 \in W$; Since $0u_1 = 0$ by theorem 2-2

$\Rightarrow W$ Has additive identity element; why?

$\Rightarrow A-4$ is satisfied -------$(\Delta\Delta\Delta)$

In addition, because W is closed under scalar multiplication, if follows that

$\forall u \in W \quad (-1)u \in W$

$\Rightarrow -U \in W$; by theorem 2-2

$\Rightarrow$every element in W is invertible with respect to the induced additive operation

$\Rightarrow A-5$ is satisfied --------- $(\Delta\Delta\Delta\Delta)$

Thus, by$(\Delta),(\Delta\Delta),(\Delta\Delta\Delta)$ and $(\Delta\Delta\Delta\Delta)$ we have W is a vector space over F.

$\Rightarrow W$ is a subspace of V over F.

Remark:

1. If W is a subspace of a vector space V over a field F then both W and V must have the same zero vector O. Justify!
2. Since a subspace of a vector space is itself a vector space, it must contain the zero vector.

Infact, the simplest subspace of a vector space is the one consisting of only the zero

Vector = $\{0\} = W = \{0\}$

Definition 1.22: Let V be a vector space over a field F then

A) V and $\{0\}$ are called improper (trivial) subspaces of V.

B) Subspaces of V other than V and (0) are called proper (non trivial) Subspaces of V

Observe that Improper subspaces of a vector space always exist while the proper Subspace of vector spaces may not exist.

Example (1) Let V be the set of all real valued functions defined on $\Re$ and

W = $\{f : \Re \to \Re / f \ \ is \ \ even \ \ function\}$ then show that

A) With the standard operations, is a vector space over $\Re$

B) W is a subspace of V over $\Re$

Solution A) Left as an exercise to you.

B) Let V= $\{f / f : \Re \to \Re \ is \ afunction\}$ and

W = $\{f : \Re \to \Re / is \ even \ \ function\}$ then

W = $\{f : \Re \to \Re / f(x) = f(-x) \forall x \in \Re\}$

I) Clearly, $W \subseteq V$; why?

Also, consider the function f_0 that has a value of zero for all x.

That is, $f_0(x) = 0, x$ is any real number. Then $f_0 : \Re \to \Re \ and \ \ f_0 \ (x) = 0 = f_0 \ (-x)$

$\Rightarrow f_0 \in W$

$\Rightarrow W \neq \phi$. Thus, $\phi \neq W \subseteq V$

II) Let $F, G \in W$ then $F, G : \Re \to \Re$ Such that both are even functions and hence $F + G : \Re \to \Re$ such that F+G is even function,

Since (F+G) $(x) = F(x) + G(x) = F(-x) + G(-x)$; why?

$$= (F + G)(-x) \forall x \in \Re$$

Hence, W is closed under the induced vector addition.

III) Let $a \in \Re$ and $f \in W$ then

$af : \Re \to \Re$ And $(af)(x) = af(x)$; Why?

$$= af(-x); \text{Why?}$$

$$= (af)(-x) \forall x \in \Re$$

$\Rightarrow af \in W$

$\Rightarrow W$ is closed under the induced scalar multiplication.

Therefore, by I), II), III) and test for subspace theorem we have W is a subspace of V over ℝ.

Example 2: Show that the set of first quadrant vector $W = \{(x, y) : x \geq 0 and y \geq 0\}$, with the standard operations, is not a subspace of $\Re^2$ over $\Re$.

Solution: To see this, we note that (1, 2) is in W, but the scalar multiple (-1)(1,2) = (-1,-2) is not in W. Therefore, W is not closed under scalar multiplication and hence W is not a subspaces of $\Re^2$ over $\Re$. Infact, one can show that W is nonempty and closed under addition

---

Exercise:

1. *Determine whether the set W is a subspace of $\Re^3$ over $\Re$ with standard operations. Justify your answer*

   a) $W = \{(a, b, 2a - 3b) : a, b \in \Re\}$

   b) $W = \{(r, r - t, t) : r, t \in \Re\}$

   c) $W = \{(x, y, xy) : x, y \in \Re\}$

   d) $W = \{(x, y, 3) : x, y \in \Re\}$

2. *Which of the following subset of $C(-\infty, \infty)$ are subspaces of $C(-\infty, \infty)$ over $\Re$?*

a) The set of all real valued nonnegative functions: $f(x) \geq 0 \forall x \in \Re$

b)The set of all real valued even functions: $f(x) = f(-x)\ \forall x \in \Re$

c) The set of all real valued odd functions: $f(-x) = -f(x) \forall x \in \Re$

d) The set of all real valued constant functions: $f(x) = c \forall x \in \Re$

e) The set of all functions such that: $f(0) = 0; 0 \in \Re$

f) The set of all functions such that: $f(0) = 1; 0,1 \in \Re$

3. *Prove that a non empty set W is a subspace of a vector space V over field F if and only if* $ax + by$ *is an element of* $W \forall x, y \in W$ *and* $\forall a, b \in F$.
4. *Give an example showing that the union of two subspace of a vector space V over a filed F is not necessarily a subspace of V over F.*

1.7 Linear Dependence and independence; Basis of a vector space

Spanning Sets

Definition 1.23*: Definition of linear combination of vectors*

Let V be a vector space over a field F and $v, x_1, x_2, ---, x_n$ be vectors in V then we say that v is a liner combination of $x_1, x_2, ---,$ and x_n if v can be expressed as

$$v = \beta_1 x_1 + \beta_2 x_2 + --- + \beta_n x_n; \text{ Where } \beta_i \in F, 1 \leq i \leq n$$

Example 1: Example of linear combinations

Let $v_1, = (1,3,1), v_2 = (0,1,2)$ and $v_3 = (1,0,-5)$ be vectors in $\Re^3$ then since $v_1 = 3v_2 + v_3 = 3(0,1,2) + (1,0,-5) = (1,3,1)$ .We have v_1 is a linear combination of v_2 and v_3 .

Example 2: Finding a liner combination

Let $v = (1,1,1), x_1 = (1,2,3), x_2 = (0,1,2)$ and $x_3 = (-1,0,1)$ be vectors in $\Re^3$ then write a linear combination of x_1, x_2 and x_3 .

Solution: We need to find scalars β_1, β_2 and β_3 such that $v = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$

But $v = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$

$\Rightarrow (1,1,1) = \beta_1(1,2,3) + \beta_2(0,1,2) + \beta_3(-1,0,1)$

$$= (\beta_1 - \beta_3, 2\beta_1 + \beta_2, 3\beta_1 + 2\beta_2 + \beta_3)$$

Then, by equating the corresponding components, we arrive at the following system of linear equations.

$$\begin{cases} \beta_1 - \beta_3 = 1 \\ 2\beta_1 + \beta_2 = 1 \\ 3\beta_1 + 2\beta_2 + \beta_3 = 1 \end{cases}$$

$\Rightarrow \beta_1 = 1 + r, \beta_2 = -1 - 2r, \beta_3 = r$ This is an infinite number of solutions, by Gauss-Jordan elimination method

For instance, to obtain one solution, we could let r =1, then $\beta_1 = 2, \beta_2 = -3$ and $\beta_3 = 1$

Hence; $v = 2x_1 - 3x_2 + x_3$. other choices for r would yield other ways to write v as a linear combination of x_1, x_2 and x_3.

Definition 1.24 : ***Definition of spanning set of a vector space***

Let $S = \{x_1, x_2, ---, x_n\}$ be a subset of a vector space V over a field F. The set S is called a spanning set (or a generator set) of V if every vector in V can be expressed as a linear combination of vectors in S. That is, if $\forall v \in V \quad \exists \beta_1, \beta_2, ---, \beta_n \in F$ such that $v = \sum_{i=1}^{n} \beta_i x_i$. in such cases we say that S spans (or generates) V over F.

Example 3: Examples of spanning sets

a) Let $x_1 = (1,0,0), x_2 = (0,1,0)$ and $x_3 = (0,0,1)$ be vectors in $\Re^3$ then since

$\forall v = (x, y,, z) \in \Re^3, \quad v = (x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$= xx_1 + yx_2 + zx_3$ We have $S = \{x_1, x_2, x_3\}$ spans or generates $\Re^3$

b) The set $S = \{1, x, x^2\}$ generates of P_2, since any polynomial $p(x) = a + bx + cx^2$ in P_2 can be written as $p(x) = a(1) + b(x) + c(x^2) = a + bx + cx^2$

Note: The spanning sets given in example -3 are called the standard spanning sets of $\mathbb{R}^3$ and p_2,respectively

Definition 1.25: Let V be a vector space over a filed F and S be non empty subset of V then the set of all finite linear combination of elements of S is called span of S and span of S is denoted by span (S). That is span

$$(s) = \left\{ \sum_{i=1}^{n} \beta_i \, x_i : \beta_i \in F \;\; and \;\; x_i \in S \;\; \forall_i \;\; such\ that\ 1 \le i \le n \right\}$$

Theorem 1.26: Span(S) is a subspace of V

Let V be a vector space over a field F and S is nonempty subset of V then span(S) is a subspace of V.

Moreover, span(s) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span (s)

Proof: Exercise

Linear Dependence and Independence of vectors

Definition 1.27: Let V be a vector space over a field F and S $= \{x_1, x_2, ---, x_n\}$ be subset of V then S is called linearly independent if the vector equation $\sum_{i=1}^{n} \beta_i x_i = 0$ has only the trivial solution. If there are also non-trivial solutions, then S is called linearly dependent.

That is,

a) If $\sum_{i=1}^{n} \beta_i x_i = 0 \Rightarrow \beta_i = 0 \ \forall i \ (1 \le i \le n)$, then S is a linearly independent set.

b) If $\exists \beta_1, \beta_2, ---, \beta_n \in F$ not all zero such that $\sum_{i=1}^{n} \beta_i \ x_i = 0$ then S is linearly dependent set

Remark: **Testing for Linear Independence and Linear Dependence**

A. Let S = $\{u_1, u_2, ---u_n\}$be a set of vectors in a vector space V over a field F. To determine whether S is linearly independent or linearly dependent, perform the following steps.

- Step-1: From the vector equation $\sum_{i=1}^{n} \beta_i u_i = O$, write a homogenous system of linear equations in the variables $\beta_1, \beta_2, ---, \beta_n$.
- Step-2: Use Gauss ian elimination method to solve the system for $\beta_1, \beta_2, ---, \text{ and } \beta_n$.
- Step-3: If the system has only the trivial solution, $\beta_1 = O = \beta_2 = ---- = \beta_n$, then the set S is linearly independent. If the system also has non trivial solutions, then S is linearly dependent.

B. Every set of vectors in a vector space is either linearly independent or linearly dependent. Thus a single test suffices.

C. We say that an infinite set S is linearly independent if every finite subset of S is a linearly independent set.

Theorem 1.28: A Property of Linearly Dependent Sets

Let $S = \{x_1, x_2, ---, x_n\}, n \geq 2$ be a subset of a vector space V over a field F then S is Linearly dependent if and only if at least one of the vectors x_i can be written as a linear combination of the other vectors in S.

Proof: Exercise

Example-4

Let $S = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}$ then show that S is linearly dependent subset o P_2

Solution:

Let $f_1 = 1 + x - 2x^2$, $f_2 = 2 + 5x - x^2$ *and* $f_3 = x + x^2$ *then one can observe that*

$f_2 = 2f_1 + 3f_3$ And hence f_2 is a linear combination of f_1 and f_3

Thus, by theorem 2.6 we have S is linearly dependent.

Example 5:

Let V be a vector space over a field F and u, v $\in V$ then show that u and v are linearly dependent if and only if one is a scalar multiple of the other.

Solution:

It is a immediate consequence of theorem 2.6. It is a simple test for determining Whether two vectors are linearly dependent

Exercise :

1. *Determine which vectors u, v, and w can be written as linear combinations of the vectors in S.*

   a) $S = \{(1,2,-2),(2,-1,1)\}, u = (1,17,-17), v = (3,-\tfrac{2}{3},\tfrac{2}{3})$ *and* $w = (8,-4,3)$

   b) a) $S = \{(2,0,7),(2,4,5),(2,-12,13)\} u = (4,-20,24), v = (-1,0,0)$ *and* $w = (6,24,9)$

2. *Determine whether the given set S spans* R^3*. If the set does not span* R^3*, describe the subspace that it does span.*

   a) $S = \{(1,0,1),(1,1,0),(0,1,1)\}$

   b) $S = \{(1,0,3),(2,0,-1),(4,0,5),(2,0,6)\}$

   c) $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$

   d) $S = \{(1,2,3),(0,1,2),(-1,0,1)\}$

3. *For which values of r are the following sets linearly independent?*

   a) $S = \{(r,0,0),(0,1,0),(0,0,1)\}$

   b) $S = \{(r,r,r),(r,1,0),(r,0,1)\}$

4. *Determine whether the set* $S = \{y^2 - 2y, y^3 + 8, y^3 - y^2, y^3 - y^2, y^2 - 4\}$ *Spans or generates* p_3

Bases and Dimension of a Vector space

Basis of a vector space

Definition 1.29: ***Definition of Basis***

A set of vectors $S = \{x_1, x_2, ---, x_n\}$ in a vector space V over a field F is called basis for V if

i) S spans V and

ii) S is linearly independent

Remark:

1. The above definition tells us that a basis has two features. A basis S must have enough vectors to span V, but not so many vectors that one of them could be written as a linear combination of the other vectors in S.
2. An infinite set S is said to be a basis for V if
   i- S spans V and ii- S is linearly independent. That is, if every element of V is a finite linear combination of elements of S and every finite subset of S is linearly independent. Thus a basis of a vector space V may consists of infinite number of vectors.

Class Activity:

1. Show that $s\{(1, 0, 0), (0, 1, 0), (0, 0, 1\}$ *is a basis for* R^3*. S is called the standard basis for* R^3*.*

2. Let $e_1 = (1, 0, ---, 0), e_2 = (0, 1, 0, ---, 0), ---, e_n = \{(0, 0, ---, 1)\}$

then show that

S= $\{e_1, e_2, ---, e_n\}$ *is a basis for* R^n *S is called a standard basis for* R^n*.*

3. Show that

a) $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ *is a non standard basis for* R^3

b) $S = \{(1, 1), (1, -1)\}$ *is a non standard basis for* R^2

c) $S = \{1, x, x^2, ---, x^n\}$ *is a standard basis for* P^n

Theorem 1.30: Uniqueness of Basis Representation

Let V be a vector space over field F and $s = \{x_1, x_2, ---x_n\}$ be a basis for V then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Proof : Exercise

Theorem 1.31: ***Basis and Linear Dependence***

If $S=\{v_1,v_2,---,v_n\}$ is a basis for a vector space V over a field F, then every set containing more than n vectors in V is linearly dependent. That is, the maximum number of linearly independent vectors in V is n.

Proof: Exercise

Example-3: Linearly Dependent sets in R^3, P_3 & R^n

a) Since R^3 has a basis consisting of three vectors, the set

$\{(1,2,-1),(1,1,0),\ (2,3,0),(5,9,-1)\}$ must be linearly dependent.

b) Since P_3 has a basis consisting of four vectors, the set

$\{1,\ 1+x,\ 1-x,\quad 1+x+x^2\quad ,1-x+x^2\}$ must be linearly dependent.

c) Since, R^n has the standard basis

Containing of n vectors, it follows from theorem 2-8 that every set of vectors in R^n congaing more than n vectors must be linearly dependent.

Theorem 1.32: Number of Vectors in a Basis

If a vector space V over a field F has one basis with n vectors, then every basis for V has n vectors.

Proof:

Let $S_1=\{x_1,x_2,---,x_n\}$ **and** $S_2=\{y_1,y_2,---,y_m\}$ be basis for V over F then since S_1 is a basis and S_2 is linearly independent we get $m\le n---(*)$ Also, since S_2 is a basis for V and S_1 is linearly independent we get $n\le m---(**)$

Now, Combing (*) and (**) we have $m\le n\ \ and\ n\le m$ which intern implies m=n.

Example-4

a) Since the standard basis for R^3, $\beta=\{(1,0,0),(0,1,0),(0,0,1)\}=\{i,\ j,\ k\}$ has three vectors we have the set $\{(3,2,1),(7,-1,4)\}$ is not a basis for R^3.

b) Since the standard basis for $p_3,\beta=\{1,x,x^2,x^3\}$ has four elements we have the set $\{x+2,x^2,x^3-1,3x+1,x^2-2x+3\}$ is not a basis for P_3.

The Dimension of a Vector Space

Definition 1.33*: Definition of dimension of a vector space*

If a vector space V over a field F has a basis consisting of n vectors, then the number n is called the dimension of V, denoted by dim (V)=n. If V consists of the zero vector alone, then the dimension of V is defined as zero. In general, let V be a vector space, over a filed F. Suppose there is some whole number n such that V contains a set of n vectors that are linearly independent, while every set of (n+1) vectors in V is linearly dependent then V is called finite dimensional and n is called the dimension of V.

Example -5

a) Since $\beta = \{e_{1,} e_2, e_3, ---, e_n\}$ form a basis for R^n over R we have dim (R^n) = n

b) Since $\beta = \{1, x, x^2, ---, x^n\}$ form a basis for P_n over R we have dim (P_n) = n+1

Theorem 1.34: *Basis Tests in an n-Dimensional space*

Let V be a vector space over a field F with dimension n

a) If $\beta = \{x_1 x_2, ---x_n\}$ is a linearly independent set of vectors in V, then S is a basis for V over F .

b) If $\beta = \{v_1, v_2, ---, v_n\}$ spans V, then S is a basis for V over F.

Proof: Left as an exercise

Exercise:

1. *Determine whether* β *is a basis for the indicated vector space*
   a) $\beta = \{(-1,2,0,0), (2,0,-1,0), (3,0,0,4), (0,0,5,0)\}$ *for* R^4 *over R*
   b) $\beta = \{(1, i\}$ *for* $\mathbb{C}$ *overR*
   c) $\beta = \{(1,5,3), (0,1,2), (0,0,6)\}$ *for* R^3 *overR*
2. *Find all subsets of the set*
   a) $S = \{(1,0),0,1), (1,1)\}$ *that form a basis for* R^2 *over R*
   b) $S = \{(1,3,-2), (-4,1,1), (-2,7,-3), (2,1,1)\}$ *that form a basis for* R^3 *over* $\Re$
3. *Find a basis for*
   a) -R^2 *over R that includes the vector (1,1)*

b) -R^3 over R that includes the set

$S = \{(1,0,2),(0,1,1)\}$

4. *Let* $w_1 = \{(2t,t) : t \in R\}$

$w_2 = \{(0,t) : t \in R\}$

$w_3 = \{(2r,r,-r) : r \in R\}$

$w_4 = \{(2s-t,s,t) : s,t \in R\}$

a) Show that w_1 and w_2 are subspaces of R^2 over R

b) Show that w_3 and w_4 are subspaces of R^3 over R

SUMMARY OF CHAPTER-ONE

- vectors are physical quantities that have both magnitude and direction, and thus can not be described or represented by a single real number. Geometirically vectors are represented by a directed line segment or an arrow. We define a vector in n-space to be an n-tuple of numbers: $(x_1,\ x_2, ---, x_n)$

  $Thus,\ \Re^n = \{(x_1,\ x_2, ---, x_n\ :\ x_i \in \Re\ \ for\ all\ i\ \ such\ that\ 1 \le i \le n\}$

- For any vectors u, v and w and any scalars r and s we have:

  a) $u + v = v = u$

  b) $u + (v + w) = (u + v) + w$

  c) $v + 0 = v = 0 + v$

  d) $v + (-v) = 0$

  e) $(r + s)v = rv + sv$

  f) $r(u + v) = ru + rv$

  g) $(rs)v = r(sv) = s(rv)$

  h) $1v = v$

  i) $0v = 0 = r0$

The above properties are properties of vector addition and scalar multiplication.

- In the xy-plane a vector v whose initial point is the origin and terminal point is $A(x, y)$ is represented by $v = (x, y)$ or $v = xi + yj$ where $i = (1,0)$ and $j = (0,1)$ and we have:

a) $\|v\| = \sqrt{x^2 + y^2}$ where x and y are called the first and second component of v .

b) If $p = (x_1, y_1)$ *then vectorAP* $= A - P = (x, y) - (x_1, y_1) = (x - x_1, y - y_1)$

c) $k\ v = k\ (x, y) = (kx, ky)$

d) If $u = (a, b)$ *then* $v + u = (x + a, y + b)$ *and* $v - u = (x - a, y - b)$

e) $x = \|v\| \cos\theta,\ y = \|v\| \sin\theta$ *and* $\tan\theta = y/x$

- ❖ If θ is the angle between non zero vectors u and v the u. v

$$u \cdot v = \|u\|\ \|v\| \cos\theta \text{ and } -1 \le \frac{u.v}{\|u\|\ \|v\|} \le 1$$

- ❖ Given two non zero vectors u, v then
  a) u and v are perpendicular to each other, denoted by $u \perp v$, if u. v = 0
  b) u and v are parallel, denoted by $u // v$ if there exists a scalar k such that $u = kv$.

- ❖ *Let* u, v *and* w *be vectors* $\Re^n$ *and* $c, d \in \Re$ *then*

a) $u + v$ *is a vector in* $\Re^n$

b) $u + v = v + u$

c) $(u + v) + w = u + (v + w)$

d) $u + 0 = u$

e) $u + (-u) = 0$

f) cu *is a vector in* $\Re^n$

g) $c(u + v) = cu + cv$

h) $(c + d)u = cu + du$

i) c(d u)=(c d)u=d(c u)

j) $1\ u = u$

The above properties are properties of vector addition and scalar multiplication

- ❖ *Let* $u = (x_1, x_2, \cdots, x_n)$ *and* $v = (y_1, y_2, \cdots, y_n)$ *be vectors in* $\Re^n$ *then*
  a) $\|u\| = \sqrt{x_1^2 + x_2^2 + --- + x_n^2}$
  b) $u.v = x_1\ y_1 + x_2\ y_2 + --- + x_n\ y_n$
  c) $d(u, v) = \|u - v\| = \sqrt{(x_1 - y_1)^2 + --- + (x_n - y_n)^2}$

d) $|u.v| \leq \|u\| \ \|v\|, the \ \ Cauchy\,Schwarz\,Inequality$

e) $\|u+v\| \leq \|u\| + \|v\|, \ the \ \ triangle \ inequality$

The above properties are geometric properties of the cross product

❖ Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars (numbers). By ***addition*** we mean a rule for associating with each pair of objects ***u*** and ***v*** in V an object **u + v** , called the ***sum*** of ***u*** and ***v***; by ***scalar multiplication*** we mean a rule for associating with each scalar *k* and each object ***u*** in *V* an object k**u** , called the ***scalar multiple*** of ***u*** by *k*. If the following axioms are satisfied by all objects **u,v ,w** in *V* and all scalars *k* and *m*, then we call *V* a ***vector space*** and we call the objects in *V* **vectors**.

1. If ***u*** and ***v*** are objects in *V*, then $u+v$ is in V

2. $u+v=v+u$

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

4. There is an object **0** in V, called a ***zero vector*** for V , such that

$0+\mathbf{u}=\mathbf{u}+0=\mathbf{u}$for all

u in V .

5. For each **u** in *V* , there is an object in **-u** , called a ***negative*** of **u** , such that

$\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=0$.

6. If k is any scalar and **u** is any object in *V* , then **ku** is in *V* .

7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

9. $k(m\mathbf{u})=(km)(\mathbf{u})$

10. 1.u = u

Miscellaneous Exercises

1. *Identify the following quantities as scalars or vectors*

a) Forces	*f) Speed*
b) Temperature	*g) Length*
c) Volume	*h) N 70^0 E*
d) 10k m East	*g) 9^0 c*

e) $30\,\frac{km}{hr}$

2. *(a) Find the components of the vector w if* $\|w\| = 5$ *and the direction angle for w is* $\theta = 300^0$

(b) *Let* $u = (3,1,-5),\ v = (\sqrt{2}, 1, -\sqrt{3}),$ *then find the components of* $u+v,$ $3u + 2v,\ -5u,\ u - v.$

3. *(a) Find the parametric vector equation of the line that passes through* $(1,4)$ *and*

   I. *Parallel to the line* $y = 3x - 1$
   II. *Perpendicular to the line* $x - 2y = 5$

(b) A triangle has $(-1, -2)$, $(5, -3),$ *and* $(1,\ 2)$ *as the coordinates of its vertices, then find the vector equations of its medians*

3. *(a) Let x and y be unit vectors and the angle between them is* $\pi/6$ then find $\|x+y\|$ *and* $\|x-y\|$

(b) Let u and v be vectors with $\pi/3$ *an angle between them. If* $\|u\| = 2$ *and* $\|v\| = 3$

Then find

$(i)\ u.v$ $\qquad (iii)\ (u+v)^2 \qquad (v)\ (u-2v)$

$(ii)\ u.u$ $\qquad (iv)\ (3u-2v) \qquad (vi)\ (3u-2v)\ (3v+2u)$

(c) Find a unit vector whose direction is opposite to $3i - 6j + k$.

5. *Use the given functions f and g in* $C[-1,1]$ *and the definition*

f.g = the dot product of f and g $= \int_{-1}^{1} f(x)\, g(x)\ dx$ *to find*

$(i)\ f.g \qquad (iii)\ \|g\|$

$(ii)\ \|f\| \qquad (iv)\ d(f,g)$

$(a)\ f(x) = x^2,\ g(x) = x^2 + 1 \qquad (c)\ f(x) = -x,\ g(x) = x^2 - x + 2$

$(b)\ f(x) = x,\ g(x) = e^x \qquad (d)\ f(x) = 1,\ g(x) = 3x^2 - 1$

6. *(a) Find a unit vector that is orthogonal to both* $u = i - 4j + k$ *and* $v = 2i + 3j$

(b) Find the area of a parallelogram that has $u = -3i + 4j + k$ *and* $v = -2j + 6k$ *as adjacent sides.*

7. *Find an equation of the plane passing through the given points*

a) $(1,\ -2,\ 1),\ (-1,\ -1,\ 7)$ *and* $(2,\ -1,\ 3)$

b) $(0,\ -1,\ 0),\ (1,\ 1,\ 0),\ (2,\ 1,\ 2)$

c) $(0,\ 0,\ 0),\ (1,\ -1,\ 0),\ (0,\ 1,\ -1)$

d) $(1,\ 2,\ 7),\ (4,\ 4,\ 2),\ (3,\ 3,\ 4)$

8. Prove that

a) $If\ \|u+v\| = \|u-v\|\ \ then\ \ \|u\| = \|v\|$

b) $If\, u.(v+w) = v(u-w),\ then\ \ w \perp u+v$

c) *If the cosine of the angle between the vectors* $2i + kj\ and\ ki + j\ is\ \dfrac{3}{\sqrt{10}}\ then\ k = 1\ or\ k = 2$

References

1. Bernard Kolman & David R. Hill, Elementary linear algebra, 8th ed., Prentice Hall, 2004
2. Larson/Edwards, Elementary Linear Algebra, D.C. Heath and Company, Lexington, 1988
3. H. Anton and C Rorres, Elementary linear algebra 8th ed, John Wiley &Sons, INC., 1994
4. K. Hoffman & R. Kunze, Linear Algebra, 2nd ed., Prentice Hall INC.1971
5. S. Lipschutz, Theory and problems of linear algebra, 2nd ed.McGraw-Hill1991
6. D. C. Lay, Linear algebra and its applications, Pearson Addison Wesley, 2006
7. Demissu Gemeda, An Introduction to Linear Algebra, Department of Mathematics, AAU, 2000
8. Serge Lang; Linear Algebra
9. Schaum’s Outline in Linear Algebra
10. Piage and swift: Linear Algebra
11. Beaumont: Linear Algebra
12. Halms: Finite Dimensional Vector space
13. Nomizu: Fundamentals of Linear Algebra
14. Isaak and Manougian, Basic Concepts of Linear Algebra, 1st ed., George J. McLead Limited, 1976
15. J.N. Sharma and et al, Linear Algebra, Krishna Prakashan Media(P) Ltd., 2003
16. Otto Bretscher, Linear algebra with application, 3rd ed., Prentice Hall, 2005

CHAPTER-TWO

MATRICES AND DETERMINANTS

UNIT OBJECTIVES:

At the end of this unit each student should able to:

- Learn a matrix and know properties of matrices
- Know properties of matrix operations
- Realize inverses of a matrices
- Know types of matrix
- Work with elementary row/column operations
- Understand about determinant of a matrix
- Understand how to solve system of linear equations by using Cramer's rule.
- Work with eigenvalues and eigenvectors of a matrix

Introduction

In working with a system of linear equations such as

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + x_4 &= -2 \\ 2x_1 + 3x_2 - 7x_3 - 5x_4 &= 1 \\ -x_1 + 10x_2 + x_3 - 2x_4 &= 1 \\ 8x_1 + 4x_2 - 11x_3 + 8x_4 &= 0 \end{aligned}$$

Only the coefficients and their respective positions are important. Thus these coefficients can be efficiently arranged in a rectangular array called a "matrix".

Moreover, certain abstract objects in higher mathematics can also be represented by matrices. Thus, matrices can be used to solve systems of linear equations. They have also numerous applications.

2.1. Definition of matrix and basic operations

The concept of a Matrix

Definition 2.1 : ***Definition of a Matrix***

Let m, n be natural numbers. An (read "m by n") matrix is a collection of mn numbers arranged in a rectangular array.

$$\begin{array}{cc} & \underline{\text{n columns}} \\ & \downarrow \\ \underline{\text{m rows}} \rightarrow & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \end{array}$$

in which each entry, a_{ij}, of the matrix is a number. An matrix has m rows (horizontal lines) and n columns (vertical lines).

Remarks

1. The numbers in the above matrix are called the matrix entries and are denoted by a_{ij}, *Where i, j are indices* (which are natural numbers) with $1 \le i \le m$ *and* $1 \le j \le n$.

   The index i is called the row index or row subscript because it gives the position in the horizontal lines, and j is the column index or column subscript because it gives the position in the vertical lines. So a_{ij} is the entry which appears in the i^{th} row and j^{th} column of the matrix.
2. The plural of matrix is matrices. If each entry of a matrix is a real number / complex number, then the matrix is called a real/ complex matrix.

3. An mxn matrix is said to be of order mxn. If m=n, then the matrix is square of order n. For a square matrix, the entries $a_{11}, a_{22}, a_{33}, ---$ are called the main diagonal entries.
4. We can abbreviate the above matrix by writing $A = (a_{ij})mxn.$

Example-1*:* Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ e & \sqrt{2} & -1 & -15 \\ \frac{1}{2} & -3 & 0 & 10 \end{bmatrix}$$

Since A has three rows and four columns, we say that A is a 3x4 matrix or A is a matrix of order 3x4. Moreover, $a_{11} = 0,\ a_{12} = 1,\ a_{13} = 2,\ a_{14} = 3$

$a_{21} = e,\ a_{22} = \sqrt{2}, a_{23} = -1,\ a_{24} = 5,\ a_{31} = \frac{1}{2},\ a_{32} = -3,\ a_{33} = 0,\ a_{34} = 10$

Example-2: The following matrices have the indicated orders**.**

(a) Order: 1x1

$$\begin{bmatrix} 2 \end{bmatrix}$$

(b) Order: 1x4

$$\begin{bmatrix} 0 & 2 & -1 & 3 \end{bmatrix}$$

(c) Order: 3x3

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) Order: 3x2

$$\begin{bmatrix} e & 1 \\ 2 & -1 \\ -8 & 1 \end{bmatrix}$$

The Algebra of Matrices

Equality of matrices

Definition 2.2 *: Definition of Equality of Matrices*

Two Matrices A=$\lfloor a_{ij} \rfloor$ and B= $\lfloor b_{ij} \rfloor$ are equal, denoted by A=B if and only if

I) they have the same order $(m \times n)$ and

II) $a_{ij} = b_{ij}, \forall i, j$ Such that $1 \le i \le m \ and \ 1 \le j \le n.$

That is, two matrices of the same order are said to be equal if their corresponding entries are equal.

Example-3 Equality of Matrices

Solve for x, y, z and W in the following matrix equation:

$$\begin{bmatrix} x+1 & 0 \\ 5 & z-1 \end{bmatrix} = \begin{bmatrix} 2 & y \\ w & -2 \end{bmatrix}$$

Solution:

Because two matrices are equal only if the corresponding entries are equal, we conclude that: $x+1=2,\ y=0,\ w=5 \ and \ z-1=-2$

$$\Rightarrow x=1,\ y=0,\ w=5,\ z=-1.$$

Addition and subtraction of matrices

Definition 2.3: *Definition of matrix Addition*

Let A= $\lfloor a_{ij} \rfloor$ *and* $B=\lfloor b_{ij} \rfloor$ be matrices of order mxn, then their sum is an mxn matrix given by $A+B=\lfloor a_{ij}+b_{ij} \rfloor$ That is, we add the two matrices of the same order by adding their corresponding entries and the sum of two matrices of different orders is undefined.

Example-4: Addition of Matrices

Let A = $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$, C = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then find a) A+B b) A+C.

Solution:

a) Since A and B have different orders, we have A+B is undefined.

b) A+C = $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Definition 2.4: *Definition of Matrix Subtraction*

Let A and B be two Matrices of the same order then the difference of A and B, denoted by A-B , is defined to be the matrix obtained by subtracting each element of B from the corresponding elements of A. That is if $A = (a_{ij})\ m \times n$ & $B = (b_{ij})\ m \times n$ then $A - B = (a_{ij} - b_{ij})\ m \times n$.

Example-5: Subtraction of Matrices

Let A= $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{pmatrix}$ then

$$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{pmatrix} = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

☞ **Note that**: Two matrices of the same order are said to be conformable for addition and subtraction. Only conformable matrices can be added or subtracted.

Multiplication of a matrix by scalar

Definition 2.5: *Definition of scalar multiplication*

Let $A = (a_{ij})mxn$ be a matrix and b be scalar, and then a scalar multiple of A by b is an mxn matrix given by: $bA = (ba_{ij})$. That is, bA is an mxn matrix obtained from A by multiplying each of its elements by b.

Example-6: Scalar multiplication

Let A= $\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ then find

a) 3A b) -B c) 3A-B

Solution: a) $3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-2) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

b) $-B = (-1)B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} (-1)2 & (-1)0 & (-1)0 \\ (-1)1 & (-1)-4 & (-1)3 \\ (-1)-1 & (-1)3 & (-1)2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$

c) $3A-B=\begin{bmatrix}3 & 6 & 12\\ -9 & 0 & -3\\ 6 & 3 & 6\end{bmatrix}-\begin{bmatrix}2 & 0 & 0\\ 1 & -4 & 3\\ -1 & 3 & 2\end{bmatrix}=\begin{bmatrix}1 & 6 & 12\\ -10 & 4 & -6\\ 7 & 0 & 4\end{bmatrix}$

Note:

(1) If A is any matrix and b is any scalar then bA=Ab

(2) -A represent the scalar product (-1) A.

2.2. Product of matrices and some algebraic properties; Transpose of a matrix

Matrix Multiplication

It is a basic matrix operation. At first glance the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Definition 2.6: *Definition of Matrix multiplication*

Let $A=[a_{ij}]$ be an mxn matrix and $B=(b_{ij}]$ is an nxp matrix, then the product AB is an mxp matrix given by $AB=(c_{ij}]$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

$$= a_{i1}\, b_{1j} + a_{i2}b_{2j} + a_{i3}\, b_{3j} + --- + a_{in}b_{nj}$$

That is, the ij^{th} element of AB is obtained by multiplying each element of the i^{th} row of A (denoted by A_i) in to the corresponding element of the j^{th} column of B (denoted by B^j) and adding the products or equivalently, the ij^{th} element of AB is the dot product of the i^{th} row vector of A by the j^{th} column vector of B and hence $AB=[c_{ij}]\ where\ c_{ij} = A_i.B^j$

Example-7: Finding the Product of two Matrices

Let $A=\begin{bmatrix}-1 & 3\\ 4- & 2\\ 5 & 0\end{bmatrix}$ *and* $B=\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}$ then find the product AB.

Solution:

First note that the product AB is defined because A has order 3x2, and will take the form.

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

To find c_{11} (the entry in the first row and first column of the product) we take the dot product of the first row of A and the first column of B That is,

$$c_{11} = A_1.B^1 = [-1 \quad 3] \begin{bmatrix} -3 \\ -4 \end{bmatrix} = (-1)(-3) + 3(-4) = 3 - 12 = -9$$

Similarly, to find c_{12}, we tare the dot product of the first row of A and the second column of B and hence $c_{12} = A_1 B^2$

$$= (-1 \;\; 3).\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= (-1)(2) + (3)(1) = -2 + 3 = 1$$

Continuing this pattern produces the following results.

$$c_{21} = A_2 B^1 = (4 - 2) = .\begin{pmatrix} -3 \\ -4 \end{pmatrix} .(4)\ (-3) + (-2)(-4) = -4$$

$$c_{22} = A_2 B^2 = (4 - 2) = .\begin{pmatrix} 2 \\ 1 \end{pmatrix} .(4)\ (2) + (-2)(1) = 6$$

$$c_{31} = A_3 B^1 = (5 \quad 0).\begin{pmatrix} -3 \\ -4 \end{pmatrix} = (5)\ (-3) + (0)(-4) = -15$$

$$c_{32} = A_3 B^2 = (5 \quad 0).\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (5)\ (2) + (0)(1) = 10$$

Thus the product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Remarks

1. The product of two matrices A and B is defined if the number of columns Of A is equal to the number of rows of B, say if A is an mxn matrix and B is an nxp matrix. In this case, the product AB is an mxp matrix.

2. Let A= $(a_{ij})\ mxn \ \ and \ \ B = (b_{ij})\ nxp \ \ then$

$$AB=\begin{bmatrix} A_1 B^1 & A_1.B^2 & A_1B^3 --- A_1.B^p \\ A_2.B^1 & A_2B^2 & A_3B^3 --- A_2B^p \\ A_mB^1 & A_mB^2 & A_m\mathbf{B}^3 --- A_m\mathbf{B}^p \end{bmatrix}$$

Where $A_i . B^j$ is the dot product of the i[th] row of A and j[th] column of B $\forall_i = 1, 2, ---, m$ and $j = 1, 2, ---, p$

3. If the product that AB is defined, then it is not necessary that BA must also be defined. For instance, if A is of order 3x3 and B is of order 3x2, then clearly AB is defined but BA is not defined, as the number of columns of B is not equal to the number of rows of A.

Properties of Matrix Operations

We begin by listing several properties of matrix addition and scalar multiplication

Theorem 2.7 : Properties of Matrix Addition and scalar multiplication.

Let A, B, C be an mxn matrices and c and d be scalars, then the following properties are true.

1. $A + B = B + A$ ------------------------- commutative property of Addition
2. $A + (B + C) = (A + B) + C$ $------$ Associative property of Addition
3. $(cd)A = c(dA)$
4. $1\ A = A$
5. $c(\ A + B) = cA + cB$ --------------------------Distributive property
6. $(c + d)A = cA + dA$ $------------$ *Distributive property* A

Proof:

Proof of these six properties follows directly from the definitions of matrix addition, scalar multiplication, and the corresponding properties of scalars. For example, to prove the commutative property of matrix addition, we let $A = \lfloor a_{ij} \rfloor$ *and* $B = \lfloor b_{ij} \rfloor$ *Then*, using the commutative property of scalars, we have $A + B = \lfloor a_{ij} + b_{ij} \rfloor = \lfloor b_{ij} + a_{ij} \rfloor = B + A$ similarly, to prove property 5, we use the distributive property for scalars of multiplication over addition and hence $c(A + B) = \lfloor c(a_{ij} + b_{ij}) \rfloor = \lfloor ca_{ij} + cb_{ij} \rfloor = cA + cB.$

The proofs of the remaining four properties are left as an exercise.

Note: If A is an mxn matrix and o_{mn} is the mxn matrix consisting entirely of zeros then one can observe that A+o_{mn} =A. We call o_{mn} a zero matrix or null matrix and it serves as the additive identity for the set of all mxn matrices. For example, the matrix $0_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ serves as the additive identity for the set of all 2x3 matrices.

Theorem 2.8: Properties of zero Matrices

Let A be an mxn matrix and c be a scalar then

a) $A + O_{mn} = A$

b) $A + (-A) = O_{mn}$

c) $cA = O_{mn} \Rightarrow c = o \ or \quad A = O_{mn}$

Proof: Left as an exercise.

Remarks :

1. Property b) above can be described by saying that the matrix-A is the additive inverse of A
2. The algebra of real numbers (scalars) and the algebra of matrices have many similarities.

Theorem 2.9: Properties of Matrix Multiplication

Let A,B and C be matrices with orders such that the given matrix products are defined and b is a scalar, then the following properties are true.

$a)$ $A(BC) = (AB)C$ ---------------------Associative property of multiplication

b) $A(B + C) = AB + AC$ ----------------Left distributive property

c) $(A + B)C = AC + BC$ ---------------Right distributive property

d) $b(AB) = (bA)B = A(bB)$

Proof: Exercise

Class Activity

1. Let $A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$ *then solve for X in the equation 3X+A=B*

2. *Let* $A=\begin{bmatrix}-1 & 0 & 5\\ 7 & -2 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 7 & 0\\ -3 & -1 & 0\\ 1 & 0 & 5\end{bmatrix}$ *and* $C=\begin{bmatrix}-1 & -1\\ 2 & 0\\ 0 & 4\end{bmatrix}$ *then show that*

A(BC) =(AB)C

3. *Consider* $A=\begin{bmatrix}2 & 3\\ -1 & 1\\ 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & -3\\ 3 & -4 & 5\end{bmatrix}$ *and* $C=\begin{bmatrix}-3 & -4 & 5\\ -5 & 6 & -7\end{bmatrix}$ *then show that*

   a) $A(B+C) = AB + AC$

   b) $(B+C)A = BA + CA$

4. *Let* $A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$ *and* $B=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$ *then show that* $AB \neq BA$ *and conclude that matrix multiplication is not commutative.*

Definition 2.10: ***Identity matrix***

A special type of square matrix of order n that has 1's on the main diagonal and 0's elsewhere is called the identity matrix of order n and it is denoted by I_n.

That is, $$I_n = \begin{bmatrix}1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}$$

It serves as the identity for matrix multiplication.

For instance, if n=1, 2, or 3,we have

$$I_1 = [1], \qquad I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad I_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Theorem 2.11: Properties of the identity Matrix

If A is a matrix of order mxn, then

a) $A\,I_n = A$

b) $I_m A = A$

c) If m=n then $A\ I_n = I_n A = A$

Proof: Left as an exercise

Example-8:

Let $A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ *then*

a) $AI_2 = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = A = I_2 A$

b) $I_3 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

Note: For repeated multiplication of square matrices, we use the same exponential notation used with real numbers.

That is, $A^1 = A$

$A^2 = AA \text{ and for } k \in N$

We define $A^k = AA - - - A \quad \Leftarrow K \text{ factors}$

It is convenient also to define $A^0 = I_n$, where A is a square matrix of order n

As a result one can observe that the following properties.

a) $A^j A^k = A^{j+k}$

b) $(A^j)^k = A^{jk}$

Types of Matrices: Square, identity, scalar, diagonal, triangular, symmetric, and Skew symmetric matrices

1) ***Row Matrix:*** A matrix which has only one row. It is a 1xn matrix, where n is arbitrary natural number. It is also called a row vector

Example-1: (-1 0 0 4) is a row matrix with order 1x4.

(1 -2 -3) is a 1x3 row matrix.

2) ***Column Matrix***: A matrix which has one column. It is also called a column vector.

Example-2: $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ is a 3 by 1 column matrix.

3) ***Zero matrix:*** is a matrix of arbitrary size in which all the entries are zero. It is also called null matrix and denoted by **O**.

Example-3: The following are null matrices

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\uparrow$ *order* $1x2$ $\uparrow$ *order* $2x2$ $\uparrow$ *order* $3x2$

4) ***A square matrix***: Is a matrix where the number of rows is equal to the number of columns. It is an nxn matrix; where n$\in N$.

Example 4: $\begin{bmatrix} 2 & 1 \\ 3 & 8 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 0 & -3 \\ 4 & 1 & 3 \\ 6 & 7 & \frac{1}{2} \end{bmatrix}$

Square matrix of Order $2x2$ $\qquad$ square matrix of order $3x3$

Let $A = (a_{i\,j})$ be a square matrix of order n, then

a) The entries $a_{i\,i}; 1 \le i \le n$ are called main diagonal entries of A.

b) The sum of the main diagonal entries of A is called the trace of A and it is denoted by $Tr(A)$ That is, $Tr(A) = a_{11} + a_{22} + a_{33} + --- + a_{nn} = \sum_{i=1}^{n} a_{ii}$

Example-5: Find the trace of the following matrix

$$(a)\, A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix} \quad (b)\; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: (a) $T\,r(A) = \sum_{i=1}^{3} a_{ii} = 1 + -2 + 3 = 2$

(b) $Tr(B) = \sum_{i=1}^{3} b_{ii} = b_{11} + b_{22} + b_{33} = 1 + 1 + 1 = 3$

5) ***Diagonal Matrix*:** Is a square matrix in which all entries that are not on the main diagonal are zero.

Example-6: The following are Diagonal matrices

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} a_{11} & 0 & 0 \ldots & 0 \\ 0 & a_{22} & 0 \ldots & 0 \\ 0 & 0 & a_{33} \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & a_{nn} \end{bmatrix}$

Note: If $A = (a_{ij})nxn$ *is* a diagonal matrix then $a_{ij} = O \ \forall i \neq j \ while \ a_{ii}$ can be zero or different from zero.

6) ***Scalar Matrix***: Is a diagonal matrix whose main diagonal elements are equal. That is, if $A = (a_{ij})$ is a square matrix such that

$a_{ij} = \begin{cases} 0 \ , if \ i \neq j \\ c \, , if \ i = j \end{cases}$ Where c is a scalar, then A is called scalar matrix.

Example-7: $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

7) ***Identity Matrix:*** Is a diagonal matrix in which all the main diagonal elements are equal to 1. It is also called a unit matrix.

Example-8: $[1]$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 1,2 and 3 respectively

Note

1. The nxn identity matrix is denoted by I_n

2. The identity matrix behaves like 1 in multiplication of numbers: If A is a mxn matrix, then I_m A = A and A $I_n = A$

3- $I_n = \begin{bmatrix} 1 & & 0 \\ 0 & \ddots & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & 1 \end{bmatrix}$ Are some short hand ways of drawing the matrix I_n.

8) ***Upper triangular Matrix***: Is a square matrix in which all the elements below the main diagonal are zero. That is, if $A = (a_{ij})nxn$ is a square matrix such that $a_{ij} = 0 \ for \ all \ i > j$ then A is said to be upper triangular matrix.

Example-9:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ are upper triangular matrices of order 3 and 4 respectively.

9) Lower triangular matrix: Is *a square matrix whose entries above the main* diagonal are 0.

Example-10: $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are lower triangular matrices of order 3 and 4 respectively.

Note **:**

1) We often indicate that a whole region in a matrix consists of zeros by leaving it blank or by putting in a single O.
2) We use * to indicate an arbitrary undetermined entry of a matrix.

Definition 2.12: ***The Transpose of a Matrix***

Let $A = [a_{ij}]$be an $m \times n$ matrix. Then the transpose of A, denoted by A^t (and sometimes by A'), is the matrix obtained interchanging rows and columns to produce the $n \times m$ matrix

$$A^t = [a_{ij}]^t = [a_{ji}].$$

Example-11: The Transpose of a matrix

(a) $A = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mn} \end{bmatrix}$

Solution: (a) $A^t = [3 - 4]$ (b) $B^t = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $C^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots a_{m2} \\ a_{13} & a_{23} & a_{33} & \dots a_{m3} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots a_{mn} \end{bmatrix}$

Theorem 2.13: Properties of Transposes

Let A and B be matrices with orders such that the given matrix operations are defined and c is a scalar, then the following properties are true

(a) $(A^t)^t = A$ ------Transpose of transpose

(b) $(A+B)^t = A^t + B^t$ ------Transpose of a sum

(c) $(cA)^t = c(A^t)$ ------Transpose of a scalar multiple

(d) $(AB)^t = B^t A^t$ ------Transpose of a product

(e) $(I_n)^t = I_n$

Proof: Exercise

Definition 2.14: A matrix A is said to be symmetric if it is equal to its transpose. That is, if $A^t = A$ then A is called symmetric matrix.

Note: A symmetric matrix is necessarily a square matrix

Example-12: Let $A=\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 7 \end{bmatrix}$ then since A=A^t we have A is symmetric

Definition 2.15: A square matrix a is said to be skew-symmetric if A^t = -A

Example-13: Let $A=\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}$, *then* $A^t = -A$ *and hence A is skew symmetric.*

2.3. Elementary operations and its properties

Elementary Row Operations

Consider the matrices $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $B=\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, $C=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$,

$D=\begin{bmatrix} -2 & 4 & -2 \\ 2 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$ $E=\begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$ *and* $F=\begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & -4 \\ 2 & 1 & -2 \end{bmatrix}$

Observe that:

I) B is obtained from A by interchanging the first and second rows of A

II) D is obtained from C by multiplying the first row of C by -2

III) F is obtained from matrix E by replacing the second row of E by a new row

This is made up by multiplying the first row by -3 and adding to the second row of E.

Such operations on rows of a matrix as described above in (i),(ii) and (iii) are called elementary row operations. These operations will be very useful in finding inverse of a matrix and solving systems of linear equations.

Similarly, elementary column operations can be defined. Hence we have the following definition

Definition 2.16: An elementary operation on a matrix is either elementary row operation or elementary column operation and is of the following three types.

Type I: The interchange of any two rows or (columns)

Type II: The multiplication of any row or (column) by a non zero number

Type III: The addition of a multiple of one row (or column) to another row (or column)

Notations*:* We shall use the following notations for the three types of elementary operations.

1. The interchange of i^th and j^th rows (columns) is denoted by $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$

2. The multiplication of the i^th row (column) by a none zero number k is denoted by $R_i \to kR_i \ (C_i \to kC_i)$

3. The addition of k times the j^th row (column) to i^th row (column) is denoted by $R_i \to R_i + KR_j \ (C \ \ \to C_i + kC_j)$

Note:

a) $R_i \to kR_i$ means replace the i^th row by k times the i^th row

b) $R_i \to R_i + kR_j$ Means replace the i^th row by the sum of itself and k times the j^th row.

c) Although elementary operations are simple to perform, they involve a lot of arithmetic Because it is easy to make a mistake, we suggest that you get in the habit of noting the elementary operation performed in each step so that you can go back to check your work

Definition 2.17 *: Row Equivalent and column Equivalent Matrices*

Two matrices are said to be row-equivalent or (column equivalent) if one can be obtained from the other by a finite sequence of elementary row or (column) operations. That is,

a) $A = (a_{ij})mxn$ is row equivalent to $\mathrm{B} = (b_{ij})mxn$, denoted by $A \overset{row}{\equiv} \mathbf{B}$, if B is attainable from A by successive operations of finitely many elementary row operations on A.

b) $A=(a_{ij})mxn$ is column equivalent to $B=(b_{ij})mxn,$ denoted by $A \overset{column}{\equiv} B,$ if B is attainable from A by successive operations of finitely many elementary column operations on A.

Example-1

Let $A=\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then show that

(a) $A \overset{row}{\equiv} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad$ $(b)\quad B \overset{row}{\equiv} I_3$

Solution:

(a)

$$A=\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -3 & 3 & -2 \end{bmatrix} \xrightarrow{R2\to -3R_1+R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3\to 2R_1+R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 2 \\ 0 & 7 & -2 \end{bmatrix} \xrightarrow{R_3\to R_2+R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) $$B=\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{R_2\to R_2+R_3} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{R_3\to R_3-R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-3R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1\to R_1-3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrices

Definition 2.18: ***Definition of Elementary Matrix***

Let A be an nxn matrix such that A can be Obtained from I_n by a single elementary row /column operation then A is called an elementary matrix.

Example-2: Elementary Matrices and Non elementary Matrices

Which of the following matrices are elementary? For those that are describe the corresponding elementary row operation.

(a) $A = I_n$ $\quad$ $(b)\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ $(c)\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad$ $(e)\ E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad$ $(f)\ F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad$ $(g)\ G = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

Solution:

(a) Since $A = I_n$ can be obtained from itself by multiplying any one of its rows by 1.

$(i.e.\ I_n \xrightarrow{R_1 \leftarrow 1R_1} A)$ $we've\ I_n$ is elementary matrix.

(b) Since B can be obtained from I_3 by multiplying the second row of

$I_3\ by\ -4\ \ we\ have\ \ I_3 \xrightarrow{R_2 \to -4R_2} B\ and\ hence\ B\ is\ an\ elementary\ matrix.$

(c) Since C is obtained by multiplying the fourth row of I_4 by 0 we have C is not elementary matrix. Note that, row multiplication must be anon zero constant.

(d) D is not elementary because it is not square marix.

(e) Since $I_3 \xrightarrow{R_2 \leftarrow R_3} E$ we've E is elementary.

(f) This matrix is not elementary because two elementary row operations are required to obtain it from I_3 .

(g) Since $I_2 \xrightarrow{R_2 \to 4R_1 + R_2} G$ we have G is elementary.

Note Elementary Matrices are useful because they enable us to use matrix multiplication to perform elementary row/ column operations.

Example-3: Elementary Matrices and Elementary Row Operations

Let $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\ and\ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $A_1 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{bmatrix}$ *and* $A_3 = \begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

Then

(a) $E_1\ A_1 = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$, this product matrix is a matrix obtained by interchanging the first two rows of A. And E_1 is an elementary matrix in which the first two rows of I_3 has been interchanged.

(b) $E_2\ A_2 \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

Here, E_2 is an elementary matrix in which the second row of I_3 has been multiplied by ½ and $A \xrightarrow{R_2 \to \frac{1}{2} R_2} E_2\ A_2$

(c) $E_3\ A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{bmatrix}$ Here, $I_3 \xrightarrow{R_2 \to 2R_1 + R_2} E_3$ *and* $A_3 \xrightarrow{R_2 \to 2R_1 + R_2} E_3\ A_3$

Definition 2.19: Let A, B be an *mxn* matrices. We say that A is equivalent to B, denoted by $A \equiv B$, if B can be attainable from A by successive application of finitely many elementary row or column operations.

Theorem 2.20 : (Elementary matrices are Invertible.) If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

Proof: Since elementary operations are reversible then to find the inverse of an elementary matrix E, we simply reverse the elementary row operation used to obtain E.

If E is an elementary matrix such that

(a) $I_n \xrightarrow{R_i \leftrightarrow R_j} E$ *then put* $E^{-1} = E$

(b) $I_n \xrightarrow{R_i \to cR_i} E$ *then put* $E^{-1} = E_1$ *where* E_1 *is* an elementary matrix such that

$$I_n \xrightarrow{R_i \to \frac{1}{c} R_i} E_1$$

(c) $I_n \xrightarrow{R_i \to R_i + cR_j} E$ *then put* $E^{-1} = E_2$ *where* E_2 *is* an elementary matrix *such that* $I_n \xrightarrow{R_i \to R_i - cR_j} E_2$

And to check that $E\,E^{-1} = I_n = E^{-1}\,E,$ apply theorem 3.10

Example-5:

Let $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ *and* $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ *then*

$$E_1^{-1} = E_1,\ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and } E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Row reduced echelon form of a Matrix

1. Row-Echelon Matrix

Definition 2.21 : ***Definition of Row-Echelon Form of a Matrix***

A matrix is said to be in row-echelon form if

1. All the non-zero rows, if any, precede all zero rows, if any. That is, all rows Consisting entirely of zeros occur at the bottom of the matrix.
2. In any non-zero row after the first row, the number of zeros preceding the first non-Zero elements is greater than the number of such zeros in the preceding row. That is, the first non-zero entry of row i+1 is to the right of the first non-zero entry of row i.
3. The first non-zero entry in each non-zero row is 1. This entry is called a pivot or a leading 1.

Example-1: Determine whether the given matrix is in row-echelon form or not**.**

$(a)\quad \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad (b)\quad \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 \end{bmatrix}$

Solution: The matrices shown in parts (a) and (b) are in row-echelon form, while the matrices shown in parts (c) and (d) are not in row-echelon form.

Note: One can show that every matrix is row equivalent to a matrix in row-echelon form.

Definition 2.22 : *Definition of Reduced Row-Echelon Form of a matrix*.

A matrix M in row-echelon form is in reduced row-echelon form if the first non-zero element in each non-zero row is the only non-zero element in its column.

That is, M is in reduced row-echelon form if M is in row-echelon form and if every column that has a leading 1 has zeros in every position above and below its leading 1.

Note that: It can be shown that the reduced row echelon matrix obtained from a given matrix A by elementary operations is unique. That is, that it does not depend on the particular sequence of operations used.

Example-2: The following matrices are in reduced row-echelon

(a) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Theorem 2.23: Let A be an mxn matrix, then A is row equivalent to an mxn reduced row-echelon form of a matrix.

Proof: Exercise

Example- 3 Let $A = \begin{bmatrix} 0 & 2 & -1 & 4 \\ 3 & 2 & 0 & 2 \\ 3 & 3 & 3 & 4 \end{bmatrix}$ then find the unique reduced row-echelon matrix that is row equivalent to the matrix A.

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 2 & 0 & 2 \\ 0 & 2 & -1 & 4 \\ 3 & 3 & 3 & 4 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3} R_1} \begin{bmatrix} 1 & 2/3 & 0 & 2/3 \\ 0 & 2 & -1 & 4 \\ 3 & 3 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to -3R_1 + R_3} \begin{bmatrix} 1 & 2/3 & 0 & 2/3 \\ 0 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 2/3 & 0 & 2/3 \\ 0 & 1 & -1/2 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to \frac{-2}{3} R_2 + R_1} \begin{bmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & \frac{-1}{2} & 2 \\ 0 & 0 & 7/2 & 0 \end{bmatrix} \xrightarrow{R_3 \to \frac{2}{7} R_3} \begin{bmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & \frac{-1}{2} & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_3 \to -R_2 + R_3$$

$$\xrightarrow{R_1 \to -\frac{1}{3} R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_2 \to \tfrac{1}{2} R_3 + R_2$$

Thus the reduced row-echelon matrix which is equivalent to A is $\begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

2.4. Inverse of a matrix and its properties

Definition 2.24: ***Definition of an inverse of a matrix***

An nxn matrix A is invertible (or nonsingular) if there exists an nxn matrix B such that $AB = I_n = BA$; *where* I_n is the identity matrix of order n. The matrix B is called the (multiplicative) inverse of A. A matrix that doesn't have an inverse is called noninvertible (or singular). The inverse of a matrix A is donated by A^{-1}.

Example-14:

Find the inverse of the following matrices

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Solution:

(a) To find the inverse of A. We try to solve the matrix equation $AX = I \ \ for \ X \ \ But, \ AX = I$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{11} + x_{21} & x_{12} + x_{22} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, by equation the corresponding entries, we obtain the following two systems of linear equations.

$$\left\{\begin{matrix} x_{11} + x_{21} = 1 \\ x_{21} \quad = 0 \end{matrix}\right. \quad and \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} x_{12} + x_{22} = 0 \\ x_{22} \quad = 1 \end{matrix}$$

$$\Rightarrow x_{11} = 1, \ x_{21} = 0, \quad x_{12} = -1 \ and \ x_{22} = 1$$

$$\Rightarrow x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Therefore, the inverse of A is $A^{-1} = x = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(b) Suppose $B^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{22} & b_{22} \end{bmatrix}$ be the inverse of B then $B^{-1}\mathbf{B} = I_2 = B \ B^{-1}$ But,

$B \ B^{-1} = I_2$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2b_{11} + b_{21} & 2b_{12} + b_{22} \\ 5b_{11} + 3b_{21} & 5b_{12} + 3b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{matrix} 2b_{11} + b_{21} = 1 \\ 5b_{11} + 3b_{21} = 0 \end{matrix}\right. \quad and \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} 2b_{12} + b_{22} = 0 \\ 5b_{12} + 3b_{22} = 1 \end{matrix}$$

$$\Rightarrow b_{11} = 3, \ b_{21} = -5, \ b_{12} = -1 \ and \ b_{22} = 2$$

$$\Rightarrow B^{-1} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Infact, one can check that $B\ B^{-1} = I_2 = B^{-1}B$

Remarks

1. Non square matrices do not have inverses. To see this, note that if A is of order mxn and B is of order nxm (where $m \neq n$), then the products AB and BA are of different orders and therefore could not be equal to each other.
2. Not all square matrices possess inverse. For example, the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ doesn't have inverse
3. We will see later that square matrix A is invertible if there is a square matrix B such that either one of the two relations $AB = I\ or\ BA = I$ holds, or then B is the inverse of A.

Theorem 2.25: Uniqueness of an Inverses Matrix

If A is an invertible matrix, then its inverse is unique

Proof:

Since A is invertible, we know that it has at least one inverse B such that $AB = I = BA$.
Suppose that A has another inverse C such that $AC = I = CA, then$

$$\begin{aligned} B &= I\ B = (CA)B \\ &= C(AB);\ Why? \\ &= CI \quad j\ why? \\ &= C \end{aligned}$$

$\Rightarrow B = C$ And it follows that the inverse of a matrix is unique

Theorem 2.26: Properties of Inverse Matrices

Let A be an invertible matrix, $K \in N$ and C is a scalar then the following are true

(a) $(A^{-1})^{-1} = A$

(b) $(A^k)^{-1} = A^{-1}A^{-1} --- A^{-1}\ \}K\ \ factors$

(c) $(cA)^{-1} = c^{-1}A^{-1}; c \neq 0$

(d) $(A^t)^{-1} = (A^{-1})^t$

Proof:

(a) Suppose A^{-1} is the inverse of A

$\Rightarrow A^{-1}A = I = A\ A^{-1}$

$\Rightarrow$ *the inverse of* A^{-1} *is* A ; *by definition of an inverse*

$\Rightarrow (A^{-1})^{-1} = A$

(c) Since $(cA)\ (c^{-1}A^{-1}) = (cc^{-1})\ AA^{-1}$; by property of scalar multiplication.

$$= (1)\ I$$
$$= I$$

And $(c^{-1}A^{-1})(cA) = (c^{-1}c)A^{-1}A = (1)\ I = I$ *we have* $(cA)(c^{-1}A^{-1}) = I = (c^{-1}A^{-1})(cA)$

And if follows that $c^{-1}A^{-1}$ is the inverse of CA.

$\Rightarrow (cA)^{-1} = c^{-1}A^{-1}$.

As an exercise show property b& d.

Note: For nonsingular matrices, the exponential notion used for repeated multiplication of square matrices can be extended to include exponents that are negative integers. This may be done by defying A^{-k} *to be*

$$A^{-k} = A^{-1}\ A^{-1} --- A^{-1}\ (k\ \ factors)$$

With this convention we can show that the following properties.

(a) $A^{j}A^{k} = A^{j+k}$

(b) $(A^{j})^{k} = A^{jk}$ For any integers j and K.

Theorem 2.27: The inverse of a product

Let A and B be invertible matrices of order n, then AB is invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

Proof:

Since $(AB)\left(B^{-1}A^{-1}\right) = A(BB^{-1})A^{-1}$

$$= A(I)A^{-1}$$
$$= (AI)A^{-1}$$
$$= AA^{-1}$$
$$= I$$

And in a similar way we can show that

$(B^{-1}A^{-1})\,(AB) = I$ *we have* $(AB)\;CB^{-1}A^{-1}) = I = (B^{-1}A^{-1})$ *and hence we get* $B^{-1}A^{-1}$ is the inverse of AB.

$$\Rightarrow B^{-1}A^{-1} = (AB)^{-1} \text{ and } AB \text{ is invertibe.}$$

Remarks

1. The above Theorem says that the inverse of a product of two invertible matrices is the product of their inverses taken in the reverse order. This can be generalized to include the product of several invertible matrices.

   $(A_1 A_2 --- A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} --- A_3^{-1} A_2^{-1} A_1^{-1}$

2. The inverse of AB is usually not equal to $A^{-1}B^{-1}$.

Theorem 2.28 : Cancellation Properties

Let C be an invertible matrix, then the following properties hold.

(a) If $AC = BC$, *then* $A = B$ ------------- Right cancellation property

(b) If $CA = CB$, *then* $A = B$ ------------- Left cancellation property

Proof:

(a) Suppose $AC = BC$, *then since C is invertible we have* $(AC)\,C^{-1} = (BC)C^{-1}$

$$\Rightarrow A(CC^{-1}) = B(CC^{-1})$$

$$\Rightarrow A\,I = B\,I$$

$$\Rightarrow A = B$$

(b) Similar to (a)

Note If C is not invertible then cancellation is not usually valid and hence theorem 2.28 can be applied only if C is an invertible Matrix.

Elementary Matrices and Non singularity

Recall that an elementary matrix is nonsingular as the inverse of the elementary operation $R_i \leftrightarrow R_j$, $R_i \to CR_i$ *and* $R_i \to CR_j + R_i$ *are* $R_i \leftrightarrow R_j, R_i \to \frac{1}{c}R_i$ *and* $R_i \to R_i - CR_j$ *respectively.*

Theorem 2.29: Let A be an nxn matrix then A is row-equivalent to I_n if and only if A is nonsingular

Proof: $(\Rightarrow)$ suppose $A \overset{row}{\equiv} I_n$ then by corollary 3.1 we get $I_n = PA$; where P is a product of finite number of elementary matrices.

$\Rightarrow \quad A^{-1} = P$

$\Rightarrow$ A is invertible or non singular.

$(\Leftarrow)$ Suppose A is an invertible then by theorem 3.13 we have $A \overset{row}{\equiv} B$; where B is a reduced row-echelon matrix.

Claim $B = I_n$

Let $B = (b_{ij})\ nxn$

- If $b_{ii} \neq 0\ \forall_i = 1, 2, ---, n$ then the structure of a matrix of reduced echelon form guarantes that $B = I_n$.
- If $b_{ii} = 0$ for some i and i is the first such element on the diagonal of B

Case-1 $i = n$ then the n^th^ row is a row of zeros

Case-2 $i < n$

Consider the $(i+1)^{th}$ *row* B

It has at lest $i+1$ zeros before the first non-zero entry then one of the rows after the i^{th} row must be a row of zeros, Hence, in both cases, we have at least one row of zeros in B. But B is invertible and hence B^{-1} exists. Thus $BB^{-1} = I_n$ has at least one row of zeros; which is contradiction. Therefore, $B = I_n$.

Corollary: ***A property of Invertible Matrices***
A square matrix A of order n is invertible if and only if it can be written as a product of elementary matrices.

Proof :Exercise

Corollary: Let A be an nxn matrix. If A can be reduced to I_n by a sequence of elementary row operations, then the same sequence of elementary row operations performed on I_n produces A^{-1}.

Proof: Exercise

✍ ***Remark: Finding the Inverse of a Matrix by Gauss-Jordan Elimination***

Let A be a square matrix of order n

1. Write the $nx2n$ matrix that consists of the given matrix A on the left and the nxn identity matrix I on the right to obtain $[A/I]$. Note that we separate the matrices A and I by a line. We call this process adjoining the matrices A and I
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A/I]$. The result will be the matrix $[I/A^{-1}]$. If this is not possible, then A is not invertible
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Example-5: Finding the Inverse of a matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -3 \\ -6 & 2 & 3 \end{bmatrix} \textit{ then find its inverse.}$$

Solution

We begin by adjoining the identity matrix to A to form the matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R \to -R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to 6R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to 4R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{R_1 \to R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] = [I|A^{-1}]$$

Therefore, A is invertible and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

Class Activity

1. Find the inverse of the following matrix by using Gauss-Jordan elimination

$$A=\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \qquad (b)\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \qquad (c)\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

2. *Show that the following matrices have no inverses*

$$(a)\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix} \qquad b)\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$$

Note: - Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of order $3x3$ or greater. For $2x2$ matrices, however, many people prefer to use a formula for the inverse, rather than find the inverse by Gauss-Jordan elimination. The formula is given by:

$$A^{-1}=\tfrac{1}{ab-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ where } A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } ad-bc\neq 0.$$

Thus A is invertible if and only if $ad-bc\neq 0$. . Why?

2.5. Determinant of a matrix and its properties

Consider a system of linear equations

$$ax+by=e$$
$$cx+dy=f$$

You recall that, one of the methods available to us to solve such a system was the elimination method. In this unit, using determinants, we shall exhibit a very efficient computational method to solve system of linear equations. We first define determinant and study its properties.

Definition 2.30: To every square matrix A with elements from the set of real numbers there is assigned a specific real number called the determinant of A. It is usually denoted by $det(A)$ *or* $|A|$.

Let $A = \begin{bmatrix} a_{11} & a_{12} --- a_{1n} \\ a_{21} & a_{22} --- a_{2n} \\ \vdots & \\ a_{n1} & a_{n2} --- a_{nn} \end{bmatrix}$ *then* determinant *of* A *is denoted by* $\begin{vmatrix} a_{11} & a_{12} --- a_{1n} \\ a_{21} & a_{22} --- a_{2n} \\ \vdots & \\ a_{n1} & a_{n2} --- a_{nn} \end{vmatrix}$

Remarks:

(i). Observe that determinant is a function from the set of all square matrices to the set of real numbers, and it is defined only for square matrix. In fact, if $A = (a_{ij})$ is a square matrix over a field F $\left(a_{ij} \in F\right)$ *then* $|A| \in F$. In particular for $F = \mathbb{C} =$ The set of complex numbers then $\det A \in \mathbb{C}$. From now on wards unless we have stated we consider matrices over R.

(ii). The determinant of an nxn matrix is called determinant of order n. Before we go to the definition of determinant of order $n(n \in N)$ we shall consider determinants of order 1, 2 and 3.

You recall that a single number a is considered as a 1x1 matrix. Hence if A= $[a_{11}]$ *then* determinant *of* A *is defined to be the number* itself. That is, $\det(A) = |a_{11}| = a_{11}$.

1) *Determinant of order two*

Definition 2.31: The determinant of the $2x2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be the number ad-bc. That is, det A= $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

A convenient method for remembering the formula is to note that $ad - bc$ is just the difference of the products of diagonally opposite entries.

That is, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example-1: Find the determinants of the following matrices

(a) $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix}$

Solution:

(a) $|A| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = (2)\,(2) - (-3)(1) = 4 + 3 = 7$

(b) $|B| = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = (3)\,(2) - (6)(1) = 6 - 6 = 0$

(c) $|C| = \begin{vmatrix} 1 & 2 \\ 6 & 4 \end{vmatrix} = (1)\,(4) - (6)(2) = 4 - 12 = -8$

✍ ***Remark:*** Note that the determinant of a matrix A can be positive, zero, or negative.

Definition 2.32 : ***Definition of Minors and cofactors of a matrix***

Let $A = (a_{ij})$ be a square matrix then

1-The determinant of the matrix obtained by deleting or removing the i^{th} row and j^{th} column of A is called the minor of the element a_{ij} and it is denoted by M_{ij}.

That is, $M_{ij} = |A_{ij}|$; *where* A_{ij} is a matrix in which the i^{th} row and j^{th} column of A are removed.

2- The product of the minor of the element a_{ij} and the number $(-1)^{i+j}$ is called the cofactor of a_{ij} and it is denoted by C_{ij}, *That is*, $C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} |A_{ij}|$

Example-2*:* Find all the minors and cofactors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$

Solution: To find the minor M_{ij} of a_{ij}, we delete the i^{th} row j^{th} column of A and evaluate the determinant of the resulting matrix

Minor of $a_{11} = M_{11} = |A_{11}| = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = (1(\,(5) - (2)(4) = -3$

Minor of $a_{12} = M_{12} = |A_{12}| = \begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} = (1(\,(5) - (4)(-3) = 17,$

Minor of $a_{13} = M_{13} = |A_{13}| = \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = (1(\,(5)\,(1)(2) - (1)(-3) = 5,$

Minor of $a_{21} = M_{21} = |A_{21}| = \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = (1)\,(5) - 0.2 = 5,$

$$\text{Minor of } a_{22} = M_{22} = |A_{22}| = \begin{vmatrix} 2 & 0 \\ -3 & 5 \end{vmatrix} = (2)\ (5) - 0(-3) = 10,$$

$$\text{Minor of } a_{23} = M_{23} = |A_{23}| = \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} = (2)\ (2) - (1)(-3) = 7,$$

$$\text{Minor of } a_{31} = M_{31} = |A_{31}| = \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = (1)\ (4) - (0)(1) = 4,$$

$$\text{Minor of } a_{32} = M_{32} = |A_{32}| = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = (2)\ (4) - (0)(1) = 8,$$

$$\text{Minor of } a_{33} = M_{33} = |A_{33}| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (2)\ (1) - (1)(1) = 1,$$

Now, to find the cofactor C_{ij} *of* a_{ij}, *we multiply* M_{ij} *with* $(-1)^{i+j}$

$$C_{11} = (-1)^{1+1}\ M_{11} = -3 \qquad C_{21} = (-1)^{2+1} M_{21} = -5 \qquad C_{31} = (-1)^{3+1} M_{31} = 4$$
$$C_{12} = (-1)^{1+2}\ M_{12} = -17 \qquad C_{22} = (-1)^{2+2} M_{22} = 10 \qquad C_{32} = (-1)^{3+2} M_{32} = -8$$
$$C_{13} = (-1)^{1+3}\ M_{13} = 5 \qquad C_{23} = (-1)^{2+3} M_{23} = -7 \qquad C_{33} = (-1)^{3+3} M_{33} = 1$$

✍ ***Remark***

(i). The minors and cofactors of a matrix differ at most in sign. To obtain the cofactors of a matrix, first find the minors and then apply the following checker board pattern of +'s and –'s.

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + - - - \\ - & + & - & + & - - - - \\ + & - & + & - & + - - - \\ - & + & - & + & - - - \\ + & - & + & - & + - - - \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$3x3\,marix \qquad 4x4\,matrix \qquad nxn\ \ matrix$

Note that odd positions (where i+j is odd) have negative signs, and even positions (where i+j is even) have positive signs.

ii) If A is an nxn matrix, then the order of A_{ij} *is* $(n-1)x(n-1)$)

2) *Determinant of Order Three*

Definition 2.33: The determinant of the $3x3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \textit{is defined to be the number } |A| = a_{11}\ C_{11} + a_{12}\ C_{12} + \ a_{13}\ C_{13}$$

$$= a_{11}(-1)^{1+1}\ M_{11} + a_{12}(-1)^{1+2}\ M_{12} + a_{13}(-1)^{1+3}\ M_{13}$$

$$= a_{11}\ |A_{11}| + a_{12}\ |A_{12}| + a_{13}\ |A_{13}|$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\, a_{22}\ a_{33} + a_{12}\, a_{23}\ a_{31}\ + a_{13}\, a_{21}\ a_{32}\ - a_{11}\, a_{23}\ a_{32} - a_{12}\, a_{21}\ a_{33} - a_{13}\, a_{22}\ a_{31}$$

✍ ***Remarks:*** The formula used to compute determinants is known as expansion by a row (column). Thus, if we consider the first row, then the determinant of A will be equal to the sum of the products obtained by multiplying the elements of the first row by their respective cofactors.

Example-3 Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, *then* determine $|A|$

(a) By expanding along the first row

(b) By expanding along the second row

(c) By expanding along the first column

Solution:

(a) $C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,\ C_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -(-5) = 5, \textit{and}\ \ C_{13} = +\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$

Thus, $|A| = \ a_{11}\, C_{11}\ + a_{12}\, C_{12} + a_{13}\ C_{13}$, *First row expansion*

$$= 0(-1) + 2(5) + 1(4) = 14$$

(b) Since $C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$

And $C_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} = -(-8) = 8$ *we have*

$|A| = \ a_{21}\, C_{21}\ + a_{22}\, C_{22} + a_{23}\, C_{23}$, *second row expansion*

$$= 3(-2) + (-1)(-4) + 2(8)$$

$$=14$$

(c) Since $C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2 \quad and \quad C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5 \quad we \ \ have$

$$|A| = a_{11}\,C_{11} + a_{21}C_{21} + a_{31}\,C_{31}\,, \ \ First \ \ column \ \ \exp ansion$$

$$= 0(-1) + 3(-2) + 4(5) = 14$$

As an exercise try some other possibilities to see that the determinant of A can be evaluated by expanding by any row or any column. This result is stated formally in the following theorem 4.1, called Lap lace's Expansion of a determinant, after the French Mathematician Pierre-Simon Laplace (1749-1827)

3) Determinant of order n

Definition 2.34: ***Definition of the Determinant of a Matrix***

Let A be a square Matrix of order n where n>2, then the determinant of A is the sum of the entries in the first row of A multiplied by their cofactors, That is,

$$|A| = \sum_{j=1}^{n} a_{1\,j}\ C_{1\,j} = a_{11}\,c_{11} + a_{12}\,c_{12} + - - - + a_{1n}\,c_{1n}.$$

When this definition is used to evaluate a determinant, we say that we are expanding by cofactors.

Theorem 2.35: Expansion by cofactor

Let A be a square matrix of order n, then the determinant of A is given

$$|A| = \sum_{j=1}^{n} a_{i\,j}\ c_{i\,j} = a_{i1}\,c_{i1} + a_{i2}\ c_{i2} + - - - + a_{in}\,c_{in} \quad (i^{th} \ \ row \ \ \exp ansion) \ \ or$$

$$|A| = \sum_{i=1}^{n} a_{i\,j}\ c_{i\,j} = a_{1j}\,c_{1j} + a_{2j}\ c_{2j} + - - - + a_{nj}\,c_{nj} \quad (j^{th} \ \ column \ \ \exp ansion)$$

Proof: Exercise

Note: when expanding by cofactors we do not need to evaluate the cofactors of zero entries, because a zero entry times its cofactor is zero. That is, $a_{i\,j}\ c_{i\,j} = (0)\ c_{i\,j} = 0$.

Thus the row (or column) containing the most zeros is usually the best choice for expansion by cofactors.

Example-4 Find the determinant of $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$

Solution:

Inspecting this matrix, we see that in third column of A we have more number of zeros than the other rows and columns. Thus we can eliminate some of the work in the expansion by using the third column.

Therefore,

$$|A| = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$$
$$= 3\,C_{13} + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$
$$= 3\,C_{13}$$

But

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

Again, expanding by cofactors in the second row we get

$$C_{13} = (0)\,(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)\,(-1)^4\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^5\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$
$$= 0 + 2(1)(-4) + 3(-1)\,(-7) = 13.$$

Thus we obtain $|A| = 3(13) = 39$

Theorem 2.36: Determinant of a Triangular matrix

Let $A = (a_{ij})$ be a triangular matrix of order n, and then its determinant is the product of the entries on the main diagonal.

Proof: We use mathematical induction to prove this theorem for the case in which A is an upper triangular matrix. The case in which A is lower triangular can be proven similarly. If A has order 1, then A= $[a_{11}]$ and the determinant is given by $|A| = a_{11}$.

Assuming that the theorem is true for any upper triangular matrix of order k-1, we now consider an upper triangular matrix A of order k. Expanding by the k th row, we obtain

$$|A| = o\,c_{k1} + o\,c_{k2} + --- + o\,c_{kk-1} + a_{kk}\ c_{kk}$$

$$= a_{kk}\ c_{kk}$$

$$= a_{kk}\ (-1)^{k+k} M_{kk} = a_{kk}\ (-1)^{2k}\ M_{kk}$$

$$= a_{kk}\ \ M_{kk}$$

$$= a_{kk}\ |A_{kk}|$$

$$= a_{kk}\left(a_{11}\ a_{22}\ a_{33}\ --- a_{k-1k-1}\right), Since\ A_{kk}\ is$$
$$order\ k-1\ we\ can\ apply\ induction\ assumpton\ to\ it$$

$$= a_{11}\ a_{22}\ a_{33}\ --- a_{kk}$$

This completes the proof

Example-5 (a) Let A = $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$ $and\ \ B = I_n$

Then since a diagonal matrix is both upper and lower triangular we have

$$|A| = (-1)\ (3)\ (2)\ (4)\ (-2) = 48\ and\ |B| = (1)\ (1) --- (1) = 1 = |I_n|$$

Properties of Determinant

It should be clear that calculating the determinant of a large matrix from the definition can be quite long and cumbersome. For instance, the determinant of a $6x6$ matrix involves six $5x5$ cofactors. Each of these in turn involves five $4x4$ cofactors each of which involves four $3x3$ cofactors for a total of $120x3x3$ cofactors to be calculated. Consequently other methods are often used to compute determinants. These methods depend on the properties of determinants which h are stated as follows.

Property-1: Elementary Row operations and Determinants

Let A and B be square matrices

(a) If B is obtained from A by interchanging two rows (or columns) of A, then $|B| = -|A|$

(b) If B is obtained from A by adding a multiple of a row (or a column)of A to another row (or c column) A then the determinant is unchanged. That is, $|B| = |A|$.

(c) If B is obtained from A by multiplying a row (or a column) of A by a non zero c constant c, then $|B| = c|A|$.

Proof: Exercise

Example-1: Show that the determinant of an elementary matrix E of

(i) First kind (add a multiple of one row to another) is 1

(ii) Second kind (row interchange) is -1

(iii) Third kind (multiply a row by anon zero constant c) is c

Solution: We know that $|I_n| = 1.$ *Thus, we have*

(i) $|E| = |I_n|, by \ \ property\, 1\, (b)$

$=1$

(ii) (i) $|E| = -|I_n|, by \ \ property\, 1\, (a)$

$= -1.$

(iii) $|E| = c\,|I_n|, by \ \ property\, 1\, (c)$

$=c.1=c$

Example-2: *Evaluating a determinant using elementary Row operations.*

Let $A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$ *then find determinant of A*.

Solution:

$$= -(-7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}, factor -7\, out\, of\, the\, second\, row$$

$$= 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} = 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix}, by\, R_3 \to -R_2 + R_3$$

$$= 7(1)(1)(-1) = -7, Since \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} is\ triangular\ matrix$$

$$|A| = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix}, by\, R_1 \leftrightarrow R_2$$

$$= -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}, by\, R_2 \rightarrow -2R_1 + R_2$$

Note: In the above solution we have changed the given matrix in to triangular matrix.

Property-2: Determinant of a transpose

If A is a square matrix, then $|A| = |A^t|$

Proof: Exercise (Hint: use mathematical induction and Laplace's expansion of determinant)

Example-3: *Show that* $|A| = |A^t|$ *for the matrix* $A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ -4 & -1 & 5 \end{bmatrix}$

Solution: We expand by cofactors along the second row to find $|A|$

To find the determinant of $A^t = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{bmatrix}$ We expand by cofactors down the second column to get $|A^t| = \begin{vmatrix} \overset{+}{3} & \overset{-}{2} & \overset{+}{-4} \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{vmatrix} = -2\begin{vmatrix} 1 & -1 \\ 2 & -5 \end{vmatrix} = (-2)(3) = -6$

$Thus, |A| = -6 = |A^t| \Rightarrow |A| = |A^t|$

Notation

Let A is a square matrix of order n with columns A^1, A^2, A^3… A^j…and A^n

And rows A_1, A_2, A_3…A_j…and A_n then we can also denote det (A) by

det (A^1, A^2, A^3,…A^j,…, A^n) or det (A_1, A_2, A_3…A_j…, A_n)

✍ **Remarks -** property-1 can be expressed as

(1) **a.** det $(A^1, A^2, A^3,\ldots cA^j,\ldots, A^n)$ = c det $(A^1, A^2, A^3,\ldots A^j,\ldots, A^n)$

b. det $(A_1, A_2, A_{3\ldots}cA_j\ldots, A_n)$ = c det $(A_1, A_2, A_{3\ldots}A_j\ldots, A_n)$ where c is a non zero constant

(2) a. det $(A^1, A^2, A^3,\ldots A^i,\ldots, Aj,\ldots, A^n)$ = - det $(A^1, A^2, A^3,\ldots A^j,\ldots A^i\ldots, A^n)$

b. det $(A_1, A_2, A_{3\ldots}A_i,\ldots,A_j\ldots, A_n)$ = - det $(A_1, A_2, A_{3\ldots}A_j,\ldots,A_i,\ldots, A_n)$

(3) a. det $(A^1, A^2, A^3,\ldots,cA^j+A^i,\ldots, A^j,\ldots, A^n)$

= det $(A^1, A^2, A^3,\ldots, A^i,\ldots, A^j,\ldots, A^n)$

b. det $(A_1, A_2, A_{3\ldots}\ cA_j+A_i,\ldots,A_j\ldots, A_n)$ = det $(A_1, A_2, A_{3\ldots}\ ,A_i,\ldots,A_j\ldots, A_n)$ where c is constant number.

Property-3: Linearity of determinant

Let A be a square matrix of order n then

a. the function $|A|$ is linear in the rows of the matrix. That is,

$$\det(A_1, A_2, A_{3\ldots}\ cA_i+A_{i'|},\ldots,A_j\ldots, A_n)$$

$$= c\det(A_1, A_2, A_{3\ldots}\ A_{i|},\ldots,A_j\ldots, A_n) + \det(A_1, A_2, A_{3\ldots}\ ,A_{i'|},\ldots,A_j\ldots, A_n)$$

where c is a constant number.

b. the function $|A|$ is linear in the columns of the matrix, That is,

$$\det(A^1, A^2, A^3{}_{\ldots}\ cA^i+A^{i'}{}_{|},\ldots,A^j\ldots, A^n)$$

$$= c\det(A^1, A^2, A^3{}_{\ldots}\ A^i{}_{|},\ldots,A^j\ldots, A^n) + \det(A^1, A^2, A^3{}_{\ldots}\ ,A^{i'}{}_{|},\ldots,A^j\ldots, A^n)$$

where c is a constant number.

Proof: Exercise. Hint (use mathematical induction)

Property-4: Conditions that yield a zero determinant

Let A is a square matrix of order n and any of the following conditions is true, then

$$|A| = 0$$

(a) An entire row(or an entire column) consists of zero

(b) Two rows (or columns) are equal.

(c) One row(or two columns) is a multiple of other row (column)

Proof: Let A be an nxn matrix with rows $A_1, A_2, A_{3\ldots}\ A_{i|},\ldots,A_j\ldots, A_n$

(a) Without loss of generality suppose that the first row of A is a row of zeros then

det (A)=det (0, A_2, A_3...A_j..., A_n) = det (0.0, A_2, A_3...A_j..., A_n**)**

=0 det (0, A_2, A_3...A_j..., A_n) = 0

Hence,the proof.

(b) Without loss of generality assume that A_1=A_2 then

det (A)=det (A_1, A_2, A_3...A_j..., A_n) = det (A_1-A_2, A_2, A_3...A_j..., A_n)

= det (0, A_2, A_3...A_j..., A_n), as A_1=A_2

= 0,by (a)

(c) Without loss of generality assume that A_1= cA_2 then

det (A)=det (A_1, A_2, A_3...A_j..., A_n) = det (cA_2, A_2, A_3...A_j..., A_n)

= c det (A_2, A_2, A_3...A_j..., A_n) = c.0=0, by (b)

Example- 4*:* Let $A=\begin{bmatrix}2 & 4 & -5\\ 0 & 0 & 0\\ 3 & -5 & 2\end{bmatrix}$ $B=\begin{bmatrix}1 & 2 & -4\\ 0 & 1 & 2\\ 1 & 2 & -4\end{bmatrix}$ $and\, C=\begin{bmatrix}1 & 2 & -3\\ 2 & -1 & -6\\ -2 & 0 & 6\end{bmatrix}$ then

(a) Since A_2 is a row of zeros we have $|A|=0$

(b) Since B_1=B_3 we have $|B|=0$

(c) Since C^3= -3C^1 we have $|C|=0$

Further Properties of determinant

Theorem 2.37: Determinant of a scalar multiple of a matrix

If A is an nxn matrix and c is a scalar, then the determinant of cA is given by $|cA|=c^n|A|$

Proof: $|cA|=\det(cA)=\det\left(cA^1,cA^2,cA^3,...,cA^n\right)$

$$=c\det\left(A^1,cA^2,cA^3,...,cA^n\right)$$
$$=c^2\det\left(A^1,A^2,cA^3,...,cA^n\right)$$
$$=c^3\det\left(A^1,A^2,A^3,...,cA^n\right)$$
$$\vdots$$
$$=c^n\det\left(A^1,A^2,A^3,...,A^n\right)$$
$$=c^n\det(A)=c^n|A|$$

Example- 5*:* If A is a matrix of order 3 and $\left|A^t\right| = 5$ then find determinant of 10A

Solution*:* $|10A| = 10^3|A| = 10^3\left|A^t\right| = 10^3(5) = 5000$

Theorem 2.38: Let E be an elementary matrix and A be an arbitrary square matrix then

$|EA| = |E||A|$

In general, if E_1, E_2, E_3… E_k are elementary matrix then

$|E_k \ldots E_3 E_2 E_1 A| = |E_k| \cdots |E_3||E_2||E_1||A|$

Proof: If E is an elementary matrix obtained from identity matrix I by

I- interchanging two rows then

$$|E| = -|I| = -1 \, and \, hence |EA| = -|A|, by \, property(1)$$
$$= -1|A| = |E||A|, \sin ce |E| = -1$$
$$\Rightarrow |EA| = |E||A|$$

II. By multiplying a row of I by a non zero content c then

$|E|$ = c and hence

$|EA| = C\ |A|$, by property (1)

$= |E|\ |A|$, since $|E|$ = C

III. By adding a multiple of one row of I to another row of I, then $|E|$ =1 and hence

$|EA|$ = $|A|$, by property (1)

= 1 $|A|$

= $|E|\ |A|$, since $|E|$ =1

Theorem 2.39: Determinant of a matrix product

If A and B are square matrices of order n, then $|AB| = |A|\ |B|$. In general, $|A_1\ A_2 --- A_k| = |A_1|\ |A_2| --- |A_k|$.

Theorem 2.40: Determinant of an Invertible Matrix

A square matrix A is invertible (non singular) if and only if $|A| \neq 0$

Proof: Exercise

Theorem 2.41: Determinant of an Inverse Matrix

If A is invertible, then $\left|A^{-1}\right| = \frac{1}{|A|}$.

Proof: Suppose A is invertible le then $AA^{-1} = I$ and

$$|A| \neq 0 \Rightarrow \left|AA^{-1}\right| = |I|$$

$$\Rightarrow |A|\left|A^{-1}\right| = 1$$

$$\Rightarrow \left|A^{-1}\right| = \frac{1}{|A|}$$

Example-6: uses a determinant to decide whether A is singular or non-singular. If A is invertible, then find $\left|A^{-1}\right|$.

$$\text{(a)}\quad A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad\qquad (b)\quad A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution:

(a) Because $|A| = 0$ we conclude that A has no inverse

(b) Since $|A| = 4 \neq 0$ we have A is invertible and $\left|A^{-1}\right| = \frac{1}{|A|} = 1/4$

2.6. Solving system of linear equations

Definition 2.42: ***systems of Linear equations***

A system of m linear equations in n variables is a set of m-equations, each of which is linear in the same n-variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ---+ a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ---+ a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + ---+ a_{3n}x_n = b_3$$

$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots$$

$$a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + --- + a_{mn}\,x_n = b_m$$

Remark:

The double subscript notation indicates that $a_{i\,j}$ is the coefficient of x_j *in the* i^{th} equation.

Definition 2.43: A solution of the above system of linear equations is a sequence of numbers $t_1, t_2, t_3, ---, t_n$ that is a solution of each of the linear equations in the system.

It can happen that a system of linear equations has exactly one solution, an infinite number of solutions, or no solution. A system of linear equations is called consistent if it has at least one solution and inconsistent if it has no solution.

Note: Number of solutions of a system of linear equations

For a system of linear equations in n-variables, precisely one of the following is true.

(a) The system has exactly one solution (consistent system)

(b) The system has an infinite number of solutions (consistent system)

(c) The system has no solution (inconsistent system)

Notation*:* Matrix Notation of system of linear equations

One very common use of matrices is to represent a system of linear equations. Matrix notation was introduced in the nineteenth century to provide a short-hand way of writing linear equations. The system of m linear equations in n variables

$$a_{11}\,x_1 + a_{12}\,x_2 + --- + a_{1n}\,x_n = b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + --- + a_{2n}\,x_n = b_2$$

$$\vdots \qquad \vdots \qquad\qquad \vdots$$

$$a_{m1}\,x_1 + a_{m2}\,x_2 + --- + a_{mn}\,x_n = b_m$$

Can be written in matrix notation as $AX = B$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

And AX is the matrix product

Here, in matrix notation $AX = B,$ we have the following definitions.

(1) $A = (a_{ij})\ mxn$ is called the coefficient matrix of the system.

(2) A and B are column vectors in R^n and R^m respectively. $is\ unknown\ and$ B is a given column vector.

(3) The matrix derived from the coefficients and constant terms of a system of linear m equations is called the augmented matrix of the system. That is, the matrix $[A|B]$ is called the augmented matrix of the system.

Example-1: Use matrix notation to denote the following system of linear equations. Also find the coefficient matrix and the augmented matrix of the system.

$$-x_2 + 2x_3 = 2$$
$$3x_1 + 4x_2 - 6x_3 = 1$$

Solution:

$$\begin{cases} -x_2 + 2x_3 = 2 \\ 3x_1 + 4x_2 - 6x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 0x_1 + -1x_2 + 2x_3 = 2 \\ 3x_1 + 4x_2 - 6x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 2 \\ 3 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow (i)\ \ Ax = B \ \ where\ \ A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 4 & -6 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \ and\ B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the matrix notation for the given system.

$$\Rightarrow \quad (ii) \quad A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 4 & -6 \end{bmatrix}$$

is the coefficient matrix of the given system and

$$\Rightarrow (iii) \; [A|B] = \begin{bmatrix} 0 & -1 & 2 & 2 \\ 3 & 4 & -6 & 1 \end{bmatrix}$$

is the associated augmented matrix of the system.

2.6.1 Gaussian's Method:

Gaussian Elimination and Gauss-Jordan Elimination

(I). Gausian Elimination

It is a procedure for solving a system of linear equations.

Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called Gaussian elimination, after the German mathematician Carl Friedrich Gauss (1777-1855)

Theorem 2.44: Let $AX = B$ be represent the system of linear equations and $M = [A|B]$ be the block matrix.

If $m' = [A'|B']$ is row equivalent to

m then the solutions of $A^{'} x = B^{'}$ *are the same as those of* $Ax = B$.

Proof : Exercise

(II). GausIan Elimination with Back Substitution

The general procedure for using Gaussian elimination with back substitution to solve a system of linear equations is summarized as follows:

Step-1 Write the augmented matrix of the system

Step- 2 Use elementary row operations to rewrite the augmented matrix in Row-echelon form

Step- 3 Write the system of linear equations corresponding to the matrix in Row-echelon form, and use back-substitution to find the solution.

Example-2

Use Gauss Ian elimination with back-substitution to solve the following system.

$$x_2 + x_3 - 2x_4 = -3$$

$$x_1 + 2x_2 - x_3 = 2$$

$$2x_1 + 4x_2 + x_3 - 3x_4 = -2$$

$$x_1 - 4x_2 - 7x_3 - x_4 = -19$$

Solution:

Step-1: The augment matrix for this system is $M = [A|B]$ where,

$$M = [A|B] = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

Step-2: using the elementary operations

$R_1 \leftrightarrow R_2, R_3 \to -2R_1 + R_3, R_4 \to -R_1 + R_4, R_4 \to 6R_2 + R_4, R_3 \to \frac{1}{3}R_3$ *and* $R_4 \to \frac{-1}{13}R_4$ respectively we get the row-echelon form of the matrix M,

$$\mathrm{M} = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Which is $M' = [A'|B']$

Step-3*:* The corresponding system of linear equations to the matrix M' is

$$x_1 + 2x_2 - x_3 = 2$$

$$x_2 + x_3 - 2x_4 = -1$$

$$x_3 - x_4 = -2$$

$$x_4 = 3$$

and using back substitution, we can determine that the solution is

$$x_1 = -1,\ x_2 = 2,\ x_3 = 1,\ x_4 = 3$$

(III). Gauss-Jordan Elimination

With Gaussian elimination, we apply elementary row operations to a matrix to obtain a row equivalent row-echelon form. A second method of elimination called Gauss Jordan elimination after Carl Gauss and Wilhelm Jordan (1842-1899), Continues the reduction process until a reduced row-echelon form is obtained.

Example-3: use Gauss-Jordan elimination method to solve the system

$$\begin{aligned} &x-2y+3z=9 \\ &-x+3y=-4 \\ &2x-5y+5z=17 \end{aligned}$$

Solution

Step-1: The associated augmented matrix of the system is

$$M=\left[A|B\right]=\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Step-2: You can show that the reduced row-echelon form of the matrix M is given by

$$M'=\left[A'|B'\right]=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step-3: The corresponding system of linear equations to the matrix M' is

$$x=1,\ y=-1 \ \ and \ \ z=2$$

Note:

(1) If in the elimination process, you obtain a row with zeros except for the last entry, it is unnecessary to continue the elimination process. You can simply conclude that the system is inconsistent.

(2) If in the row-echelon form system the number of equations and the number of unknowns are equal then the system will have a unique solution.

(3) Observe that if in the row-echelon form system the number of equations is less than the number of unknown, then the system will have infinitely many solutions.

Definition 2.45: ***Homogeneous system of linear equations***

A system of linear equations in which each linear equation is equal to zero is called a homogeneous system and it is denoted by $Ax=0$.

It is easy to see that a homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called trivial or obvious.

2.6.2 Cramer's rule

Cramer's Rule, named after Gabriel Cramer (17 04- 17 52), is a formula that uses determinants to solve a system of n linear equations in n-variables. This rule can be applied only to systems of linear equations that have unique solutions.

Theorem 2.46: Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a non-zero determinant $|A|$, then the solution to the system $Ax = B$ is given by $x_j = \dfrac{\det\,(A^1, A^2, A^{j-1}, B, A^{j+1}, ---, A^n)}{|A|}$ $\forall_j = 1,\ 2, ---, n.$

Proof: Exercise

Example: Use Cramer's Rule to solve the following system of linear equations

for x, y &z

$$\begin{cases} -x + 2y - 3z = 1 \\ 2x + z = 0 \\ 3x - 4y + 4z = 2 \end{cases}$$

Solution: Let $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, then the system can be represented by $Ax = B$. And since

$$|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \neq 0$$

Then by Cramer's rule the system $Ax = B$ has a unique solution given by

$$x = \det \frac{(B, A^2, A^3)}{|A|},\quad y = \det \frac{(, A^1, B,, A^2)}{|A|}\ \ and\ \ z = \det \frac{(, A^1, A^2\ B)}{|A|}$$

$$\Rightarrow x=\frac{\begin{vmatrix}1 & 2 & -3\\ 0 & 0 & 1\\ 2 & -4 & 4\end{vmatrix}}{10},\quad y=\frac{\begin{vmatrix}-1 & 1 & -3\\ 2 & 0 & 1\\ 3 & 2 & 4\end{vmatrix}}{10},\ and\ \ z=\frac{\begin{vmatrix}-1 & 2 & 1\\ 2 & 0 & 0\\ 3 & -4 & 2\end{vmatrix}}{10}$$

$$\Rightarrow x = 4/5,\quad y = \tfrac{-3}{2}\ \ and\ \ z = -8/5$$

Thus the solution set for $Ax = B$ is given by

$$S.S. = \{(x,y,z) : x = 4/5,\ y = -3/2\ \ and\ \ z = -8/5\}$$

Exercises:

Use Cramer's Rule to solve the given system of linear equations

$(a)\ 3x - 5y = 13$
$\quad 2x + 7y = 81$

$(b)\ x + y + Z = 3$
$\quad 2x - y + 4z = 5$
$\quad x - 3y - 9z = -11$

$(c)\ 4x_1 - x_2 - x_3 = 1$
$\quad 2x_1 + 2x_2 + 3x_3 = 10$
$\quad 5x_1 - 2x_2 - 2x_3 = -1$

$(d)\ 4x_1 - 2x_2 + 3x_3 = -2$
$\quad 2x_1 + 2x_2 + 5x_3 = 16$
$\quad 8x_1 - 5x_2 - 2x_3 = 4$

2.6.3 Inverse matrix Method

Thus far, we have studied three methods for solving linear systems: Gaussian elimination , Gauss–Jordan elimination and Cramer's rule. The following theorem provides a new method for solving certain linear systems.

Theorem 2.47: If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix **b,** the system of equations $\boldsymbol{A}x = \boldsymbol{b}$ has exactly one solution, namely, $x = A^{-1}b$

Proof Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. To show that this is the only solution, we will assume that $\mathbf{x}_0$ is an arbitrary solution and then show that $\mathbf{x}_0$ must be the solution $A^{-1}\mathbf{b}$.
If $\mathbf{x}_0$ is any solution, then $A\mathbf{x}_0 = \mathbf{b}$. Multiplying both sides by A^{-1}, we obtain $\mathbf{x}_0 = A^{-1}\mathbf{b}$.

Example: solve the following system of linear equation using inverse method

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

2.7. Eigenvalues and Eigenvectors

Definition 2.48: ***Eigenvalue and Eigenvector***

Let A be an *nxn* matrix over a field F, then the scalar is called an Eigenvalue of A if there is a non zero vector $v \in F^n$ such that The vector is called an Eigen vector of A corresponding to .

Remarks:

1. The term Eigen value is derived from the German word eigenwerte, meaning "proper value"
2. An eigenvector v cannot be zero. Allowing v to be the zero vector would render the definition meaningless because $A0 = \lambda 0$ is true for all values of λ. An Eigen value of $\lambda = 0$, however, is possible.
3. A matrix can have more than one Eigen value.

Example-1:

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ then verify that $v_1 = (1,\ 0)$ *and* $v_2 = (0, 1)$ are eigenvectors of A corresponding to the Eigen values $\lambda_1 = 2$ *and* $\lambda_2 = -1$ respectively.

Solution:

Since $Av_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2v_1$, we have $v_1 = (1,\ 0)$ is an eigenvector of A corresponding to the Eigen values $\lambda_1 = 2$.

Also, since $Av_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1v_2$

We have $v_2 = (0,\ 1)$ is an eigenvector of A corresponding to the Eigen value $\lambda_2 = -1$.

Note

1. The sum of two Eigenvectors with the same Eigen value λ is also an Eigenvector with Eigen value λ.
2. A nonzero multiple of an Eigenvector with Eigen value λ is also an Eigenvector with Eigen value λ

Example:-2

Find the Eigen values and corresponding Eigen spaces of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution

i. Let $v_1 = (x,0)$ *then* $Av_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix} = -1\, v_1$

$\Rightarrow$The eigenvectors corresponding to $\lambda_1 = -1$ are the nonzero vectors on the x-axis and hence

$A_{-1} = \{(x,0):\ x \in \Re|\{0\}\}$ is eigenspace of A belonging to eigen value $\lambda_1 = -1$.

ii. Let $v_2 = (0, y)$ *then* $Av_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix} = 1\, v_2$

$\Rightarrow$The eigenvectors on the corresponding to $\lambda_2 = 1$ are the non zero vectors on the y-axis and hence

$A_1 = \{\ (0,\ y): y \in \Re|\{0\}\}$ is eigenspace of A belonging to eigenvalue $\lambda_2 = 1$.

iii. For $v = (x, y) \in \Re^2$, we have

$$Av = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$, which is a reflection in the y-axis.

Thus, the eigenspace corresponding to $\lambda_1 = -1$ is the x-axis and the eigenspace corresponding to $\lambda_2 = 1$ is the y-axis

***Question*: How do we determine eigenvalues and the corresponding eigenvectors of a matrix?**

Theorem 2.49: Eigenvalues and eigenvectors of a matrix

Let A be an nxn matrix over a field F

1-An eigenvalue of A is a scalar λ such that $\det(\lambda I_n - A) = 0$

2-The eigenvectors of A corresponding to λ are the non zero solutions x of $(\lambda I_n - A)x = 0$

Proof*:*

Let $x \in F^n$ be an eigen vector of A with eigen value λ then

$$\begin{aligned} & Ax = \lambda x \\ \Leftrightarrow\ & Ax = \lambda I_n x, \text{ since } I_n x = x \ \forall_x \in F^n \\ \Leftrightarrow\ & (A - \lambda I_n)x = 0 \\ \Leftrightarrow\ & A - \lambda I_n \text{ is singular, why?} \\ \Leftrightarrow\ & |A - \lambda I_n| = \det(A - \lambda I_n) = 0 \end{aligned}$$

This completes the proof.

Note:

1. The equation $|A - \lambda I_n| = 0$ is called the characteristic equation of A. And the matrix $A - \lambda I_n$ *is called characterstic matrix of A*
2. The determinantal polynomial $|A - \lambda I_n| = \lambda^n + c_{n-1}\lambda^{n-1} + --- + c_1\lambda + c_0$ is called the characteristic polynomial of A and it is denoted χ_A. This definition tells us that the eigenvalues of an nxn matrix A corresponds to the roots of the characteristic polynomial of A. Because the characteristic polynomial of A is of degree n, A can have at most n-distinct eigenvalues.

3. For each eigenvalue λ, the corresponding eigenvector is found by substituting λ back in to the characteristic equation $|A-\lambda I_n| = 0$ *of A.*

Example 3: Find the eigenvalues and corresponding eigenvectors of

$$(a)\quad A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \qquad (b)\quad B = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Solution:

(a) The characteristic equation of A is

$$\begin{aligned}
&|A-\lambda I_2| = 0 \\
&\Leftrightarrow \left| \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \\
&\Leftrightarrow \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = 0 \\
&\Leftrightarrow \lambda^2 - 3\lambda - 28 = 0 \\
&\Leftrightarrow (\lambda - 7)\ (\lambda + 4) = 0 \\
&\Leftrightarrow \lambda = 7 \ or\ \lambda = -4.
\end{aligned}$$

The corresponding eigenvectors are now found

Let $\lambda = 7$ then $(A - 7I_2)x = 0$

$$\begin{aligned}
&\Leftrightarrow \left[\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\Leftrightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -6x + 6y = 0 \\ 5x - 5y = 0 \end{cases} \\
&\Leftrightarrow y = x
\end{aligned}$$

Hence, any vector of type β (1, 1), where β is a non zero scalar, is an eigenvector corresponding to eigenvalue $\lambda = 7$.

Let $\lambda = -4$ then $(A + 4I_2)x = 0$

$$\begin{aligned}
&\Leftrightarrow \left[\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\Leftrightarrow \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 5x + 6y = 0 \\ 5x + 6y = 0 \end{cases} \\
&\Leftrightarrow y = {}^{-5}\!/_{6}\, x
\end{aligned}$$

Hence, any vector of type $\beta(1, {}^{-5}\!/_{6})$; *where* β is a non zero scalar, is an eigenvector corresponding to eigenvalue $\lambda = -4$

Try checking that $Ax = \lambda_i x$ for the eigenvalues and eigenvectors found in this example.

(b) Exercise

Remark:

Steps in Finding Eigenvalues and Eigenvectors

Let A be an nxn matrix over a field $\Re$

Step-1: Form the characteristic equation $|A - \lambda I_n| = 0$. It will be a polynomial equation of degree n in the variable λ.

Step-2: Find the real roots of the characteristic equation. These are the eigen values of A.

Step-3: For each eigen value λ_i, find the eigen vectors corresponding to λ_i by solving the homogeneous system

$(A - \lambda_i I_n)\, x = 0$. This may requires row-reducing an nxn matrix. The resulting reduced row-echelon form must have at least one row of zeros.

SUMMARY OF CHAPTER TWO

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.
- Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal. In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then A =B if and only if $(A)_{ij} = (B)_{ij}$, or, equivalently, $a_{ij} = b_{ij}$ for all i and j.
- If A and B are matrices of the same size, then the **sum** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the **difference** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

- If A is any matrix and c is any scalar, then the **product** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a **scalar multiple** of A.
- If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the **product AB** is the $m \times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of AB , single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.
- If A is any $m \times n$ matrix, then the **transpose of A**, denoted by A^t , is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A; that is, the first column of A^t is the first row of A, the second column of A^t is the second row of A, and so forth.
- If A is a square matrix, then the **trace of A**, denoted by **tr(A)** , is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.
- If R is the reduced row-echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n .

Miscellaneous Exercises

1. Find the determinant of the following matrices:

$$(a)\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{bmatrix} \qquad (b)\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (c)\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 1 & 3 \\ 1 & 2 & -2 & 3 \end{bmatrix}$$

$$(d)\begin{bmatrix} -1 & 5 & 20 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix} \qquad (e)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix}$$

2. Let A and B be $3x3$ *matrices such that*

$\det(A) = 4$ *and* $\det B = 5$ *then find the value of* :

$$(a)\ |AB| \qquad (c)\ |2AB|$$
$$(b)\ |3A| \qquad (d)\ |A^{-1}B|$$

3. For each of the following compute A^{-1}

$$(a)\ A=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix} \qquad (b)\begin{bmatrix}1 & 3 & 1\\ 2 & 1 & 1\\ -2 & 2 & -1\end{bmatrix} \qquad (c)\ A=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

4. Use Cramer's Rule to solve for each of the following systems

$$(a)\ \begin{aligned} x+2y+3z&=8\\ 2x-y+z&=7\\ -y+z&=1\end{aligned} \qquad (b)\ \begin{aligned} x+y+2z&=0\\ 3x-y-z&=3\\ 2x+5y-3z&=4\end{aligned}$$

5. *Evaluate* $\begin{vmatrix}a & 1 & 1 & 1 & 1\\ 1 & a & 1 & 1 & 1\\ 1 & 1 & a & 1 & 1\\ 1 & 1 & 1 & a & 1\\ 1 & 1 & 1 & 1 & a\end{vmatrix}$

6. *a) Find the ad joint of* $\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}$

b) If A is a triangular matrix, then show that $adj\ (A)\ is\ also\ triangular$

7. (a) Let $A=\begin{bmatrix}a & b & 0 & 0\\ c & d & 0 & 0\\ e & f & g & h\\ x & y & z & w\end{bmatrix}$$,then\ \ find\ rank\ (A)$

(b) Let A be a square matrix of order n, then show that A is invertible if rank (A) = n

8. Let $A=(a_1,a_2,a_3)$ and $B=(b_1,b_2,b_3)$ be two vectors in $\Re^n$ then show that their cross product AxB can be restated using determinants as

$$AxB=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad where \quad i=(1,\ 0,\ 0)$$

$$j=(0,\ 1\ \ 0)$$

$$k=(0,\ 0,\ 1) \text{ are the standard unit vectors in } \Re^n$$

9. Let $A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & c & d \\ e & f & g & h \end{bmatrix}$ $B=\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Then prove that

a) $\det\ A=\begin{vmatrix} c & d \\ g & h \end{vmatrix}$ b) $\det(B)=\begin{vmatrix} a & b & d \\ e & f & h \end{vmatrix}$

10. Let A and B square matrices of order n then prove or disprove the following

a) $\det\ (A+B)=\ \det(A)+\det(B)$

b) $\det\ \left[(A+B)^2\right]=\ [\det\ (A+B)]^2$

References

1. James Stewart, Calculus early trancedental, 6th ed., Prentice Hall, 2008
2. Howard Anton,Calculus a new horizon, 6th ed., John wiley and Sons Inc
3. Robert Ellis and Denny Gulick, Calculus with analytic geometry, 6th ed, Harcourt Brace Jovanovich, Publishers, 5th ed, 1993
4. Adams, Calculus: A complete course, 5th ed, Addison Wesley, 2003

5. R. Wrede and M. R. Spiegel, Theory of advanced calculus, 2nd ed., McGraw-Hill, 2002.
6. A. E. Taylor and W. R. Mann, Advanced calculus, 3rd ed, John-Wiley and Son, INC, 1995.
7. R. T. Smith and R. B. Minton, Calculus concepts and connections, McGram-Hill book company, 2006
8. D. V. Widder, Advanced calculus, Prentice-Hall, 1979
9. Ross L. Finney et al, Calculus, Addison Wesley, 1995
10. E. J. Purcell and D. Varberg, Calculus with analytic geometry, Prentice-Hall INC., 1987
11. Lynne, Garner: Calculus and Analytic Geometry, Dellen Publishing Company
12. John A. Tierney: Calculus and Analytic, 4th edition, Allyn and Bacon, Inc. – Boston.
13. Earl W. Swokowski: Calculus with Analytic Geometry, 2nd edition, Prindle, Weber and Schmidt.
14. Goldstein, L. J., Lay D. C. , and Schneider D. I (1987). Calculus and its applications, 4th, edn, Prentice-Hall, Inc, London
15. Kresying, Advanced Engineering Mathematics.
16. Alan Jeffry , Advanced Engineering Mathematics

CHAPTER 3
LIMITS AND CONTINUITY

Objectives:

By the end of this chapter, students will be able to:

- Understand the formal definition of limit and continuity;

- State some limit theorems;
- Evaluate limits of functions;
- Determine points of discontinuity of functions;
- Apply Intermediate Value Theorem;

Introduction

The topic that we will be examining in this chapter is that of Limits. Limits are very important in the study of calculus. We will be see limits in a variety of places once we move out of this chapter. In particular we will see that limits are part of the formal definition of the other two major topics. Here is a quick listing of the material that will be covered in this chapter.

3.1 DEFINITION OF LIMIT

Objectives

At the end of this section, students should able to:

- ✓ Define informal definition of limit;
- ✓ Define formal definition of limit;
- ✓ Use $\varepsilon - \delta$ definition to show that the limit of f at a is L;
- ✓ Find a non-existence of limit.

Definition 3.1: Let $f(x)$ be defined on an open interval about a except possibly at a itself. If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to a we say that f approaches the limit L as x approaches a and we write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L".

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to a (on either side of a). This definition is "informal" because phrases like arbitrarily close and sufficiently close are imprecise: Their meaning depends on the context.

Example 1: ***The behavior of a function near a point.***

How does the function $f(x) = \frac{x^2-1}{x-1}$ behave near x = 1?

Solution: The given formula defines f for all real numbers x except $x \neq 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x + 1 \quad for \quad x \neq 1$$

The graph of f is thus the line with the point (1, 2) removed. This removed point is shown as a "hole" in Figure 3.1. Even though $f(1)$ is not defined, it is clear that we can make the value of f(x) as close as we want to 2 by choosing x close enough to 1 and we say that $f(x)$ approaches the limit 2 as x approaches 1 and we write,

$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{x^2-1}{x-1} = 2$$

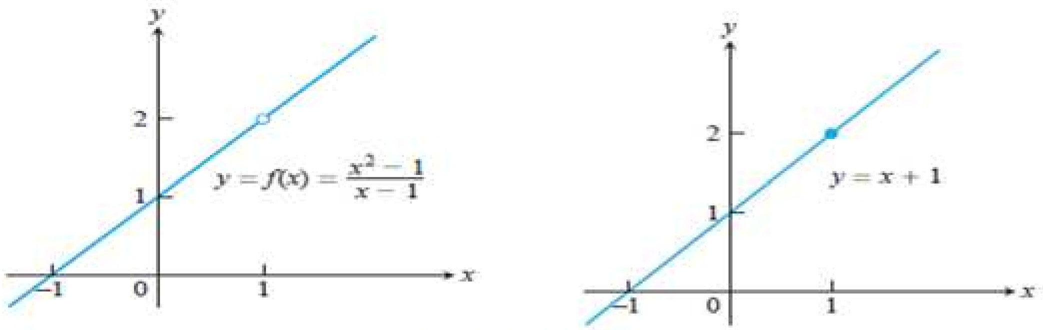


Figure 3.1

The graph of f is identical with the line $y = x + 1$ except at $x = 1$ where f is not defined.

Example 2: The identity and constant functions have limits at every point. If f is the identity function $f(x) = x$, then for any value of a (Figure 3. 2).

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x = x_0$$

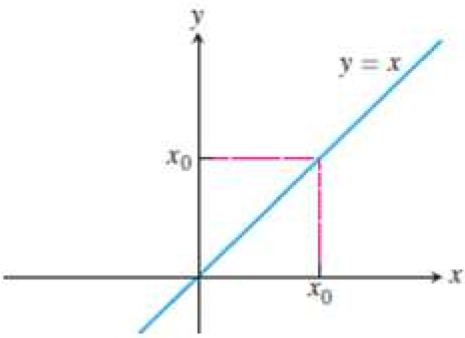


Figure 3.2 Identity function

If f is the constant function $f(x) = k$ (function with the constant value k), then for any value of x_o (Figure 3.3)

$$\lim_{x \to x_o} f(x) = \lim_{x \to x_o} k = k.$$

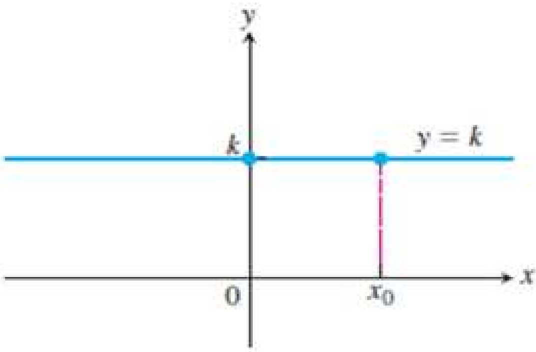


Figure 3.3 constant function

Now that we have gained some insight into the limit concept, working intuitively with the informal definition, we turn our attention to its precise definition. We replace vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example. With a precise definition we will be able to prove conclusively the limit properties given in the preceding section, and we can establish other particular limits important to the study of calculus.

To show that the limit of $f(x)$ as equals the number L, we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to arbitrary number.

Definition 3.2 (***Formal Definition of Limits***)

Let f be a function defined at each point of some open interval containing a, possibl at a itself,then a number L is the limit of f(x) as x approaches a (or is the limit of f at a) if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{If } 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

If L is the limit of f(x) as x approaches a then we write

$$\lim_{x \to a} f(x) = L$$

If such an L can be found we say that the limit of f at a exists or that f has a limit at a or that $\lim_{x \to a} f(x)$ exists.

Let us look at the following graph and let us also assume that the limit does exist

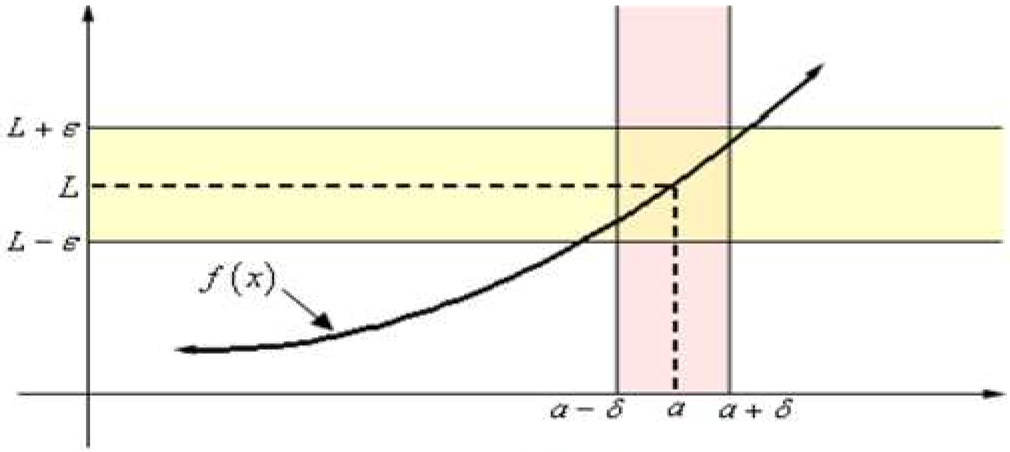


Figure 3.4

What the definition is telling us is for any number $\varepsilon > 0$ that we pick we can go to our graph and sketch two horizontal lines at $L+\varepsilon$ and $L-\varepsilon$ as shown on the graph above. Then somewhere out here in the world is another number $\delta > 0$. Which we will need to determine that will allow us to add in two vertical lines to our graph at $a+\delta$ and $a-\delta$. Now, if we take any x in the vertical strip, i.e, between $a+\delta$ and $a-\delta$, then this x will be close to a than either of $a+\delta$ and $a-\delta$ or $|x-a|<\delta$. If we now identify the point on the graph that our choice of x gives, then this point on the graph will lie in the intersection of the horizontal and vertical strip. This means that, this functional value of $f(x)$ will be close to L than either of $L+\varepsilon$ and $L-\varepsilon$, or $|f(x)-l|<\varepsilon$. So, if we take value of x in the horizontal strip then the graph for those values of x will lie in the vertical strip.

Notice that there are actually an infinite number of δ's that we can choose. In fact, if we go back and look at the graph above it looks like we could have taken slightly large δ and still gotten the graph from that horizontal strip to be completely contained in the vertical strip.

Example 1: Use the definition of limit to prove the following limit

$$\lim_{x\to 2} 5x-7=3$$

Solution: We need to show that given $\varepsilon>0$ then there exists $\delta>0$ such that

$$if\ \ 0<|x-2|<\delta\ \ then\ |\ (5x-7)-3|\ <\varepsilon$$

To choose an appropriate δ we start with $|\ (5x-7)-3|<\varepsilon$ then we have

$$|5x\text{-}10|<\varepsilon \quad\Rightarrow\quad 5|x-2|<\varepsilon$$

Now we can Choice $\delta=\frac{\epsilon}{5}$

Assume $0 < |x - 2| < \delta = \frac{\varepsilon}{5}$ and we have

$$|(5x - 7) - 3| = |5x - 10| = 5|x - 2| < 5\delta = \frac{5\varepsilon}{5} = \varepsilon$$

$$|(5x - 7) - 3| < \varepsilon$$

Therefore, $\lim_{x\to 2} 5x - 7 = 3$. ■

Example 2: Show that $\lim_{x\to 0} x^2 = 0$

Solution: Let $\varepsilon > 0$ be given we need tofind a $\delta > 0$ such that

$$\text{if } 0 < |x - 0| < \delta, \quad \text{then} \quad |x^2 - 0| < \varepsilon$$

Equivalently,

$$\text{if } 0 < |x| < \delta \text{ then } |x^2| < \varepsilon$$

It is enough to let $\delta = \varepsilon^{1/2}$ because

$$\text{if } 0 < |x| < \varepsilon^{1/2} \quad \text{then } |x^2| = |x|^2 < (\varepsilon^{1/2})^2 = \varepsilon$$

This proves that $\lim_{x\to 0} x^2 = 0$ ■

Example 3: Prove that $\lim_{x\to 3} x^2 = 9$

Solution:

I. Guessing a value for δ.

Let $\varepsilon > 0$ be given, we have to find a number a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |x^2 - 9| < \varepsilon$$

To connect $|x^2 - 9|$ with $|x - 3|$ we write $|x^2 - 9| = |(x - 3)(x + 3)|$. Then we want

$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |x + 3|\,|x - 3| < \varepsilon.$$

Notice that if we can find a positive constant C such that $|x + 3| < C$ then

$$|x + 3||x - 3| < C|x - 3|$$

And we can make $C|x - 3| < \varepsilon$ by taking $|x - 3| < \frac{\varepsilon}{C} = \delta$. We can find such a number C if we restrict x to lie in some interval centered at 3. Infact, since we are interested only in values of x that are close to 3, it is reasonable to assume that x is within a distance 1 from 3, that is , $|x - 3| < 1$. then $2 < x < 4$, so $5 < x + 3 < 7$. Thus we have $|x + 3| < 7$, and so $C = 7$ is a suitable choice for the constant . But now there are two restrictions on $|x + 3|$, namely

$$|x - 3| < 1 \quad \text{and} \quad |x - 3| < \frac{\varepsilon}{C} = \frac{\varepsilon}{7}$$

To make sure that both of these inequalities are satisfied , we take δ to be the small of the two numbers 1 and $\frac{\varepsilon}{7}$. The notation for this is $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$.

II. Showing that this δ works .

Given $\varepsilon > 0$, let $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$. If $0 < |x-3| < \delta$,

Then $|x-3| < 1 \implies 2 < x < 4 \Rightarrow |x+3| < 7$ (as in part I). We also have $|x-3| < \frac{\varepsilon}{7}$, so

$$|x^2-9| = |x+3|\,|x-3| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon$$

This shows that $\lim_{x\to 3} x^2 = 9$. ■

Example 4: Show that $\lim_{\theta\to 0} \sin\theta = 0$ and $\lim_{\theta\to 0} \cos\theta = 1$

Solution: Referring to the quarter of the unit circle in figure 3.5, along with the accompanying line segments, we see that $\leq \sin\theta \leq |PA| \leq \theta$ for $0 \leq \theta < \frac{\pi}{2}$. So that

$$|sin\theta| \leq |\theta| \text{ for } 0 \leq \theta < \frac{\pi}{2}.$$

Since $sin(-\theta) = -sin\theta$, and hence $|sin(-\theta)| = |sin\theta|$, it follows that $|\sin\theta| \leq |\theta|$ for $\frac{-\pi}{2} < \theta \leq 0$.Consequently

$$|sin\theta| \leq |\theta| \;\; for \quad (-\pi)/2 \leq \theta < \pi/2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Now let $\varepsilon > 0$, let δ be any positive number less than both ε and $\frac{\pi}{2}$. From (1), it follows that if $0 < |\theta - a| < \delta$, then $|sin\,\theta - 0| \leq |\theta| < \delta < \varepsilon$.

We conclude that $\lim_{\theta\to 0} sin\,\theta = 0$

To show that $\lim_{\theta\to 0} cos\,\theta = 1$, observe form figure 3.5 that

$|cos\theta - 1| = |AQ| \leq |AP| \leq |\theta|$ For $0 \leq \theta < \frac{\pi}{2}$. Since $cos(-\theta) = cos\theta$, it follows that

$$|cos\theta - 1| \leq |\theta| \text{ for } \frac{-\pi}{2} \leq \theta < \frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Now let $\varepsilon > 0$, let δ be any positive number less than both ε and $\frac{\pi}{2}$. From (2) we find,

if $0 < |\theta - a| < \delta$, then $|cos\theta - 1| \leq |\theta| < \varepsilon$. We conclude that $\lim_{\theta\to 0} cos\,\theta = 1$

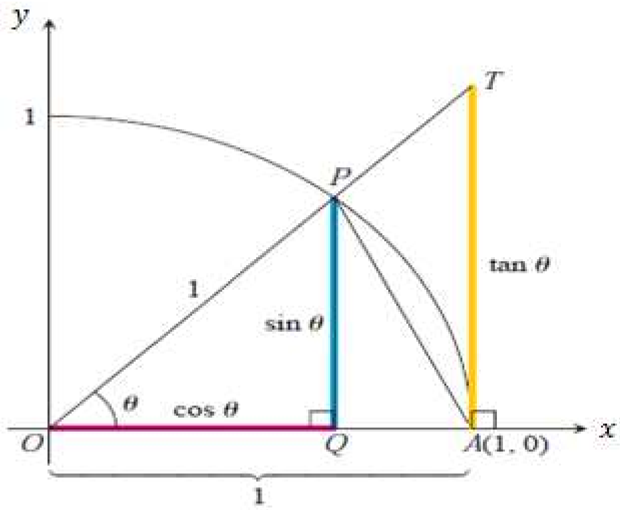


Figure 3.5

■

Next, we present an example of a function that does not have a limit at a certain point. For a function f not to have a limit at a means that for every real number L, the statement "L is the limit of f at a" is false. What does it mean for that statement to be false?

Definition 3.3 (***Negation of the Existence of a Limit***)

"L is the limit of f at a" means that for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta, \qquad \text{then} \quad |f(x) - L| < \varepsilon$$

For this statement to be false there must be some $\varepsilon > 0$ such that for every $\delta > 0$ it is false that

$$\text{if} \quad 0 < |x - a| < \delta, \qquad \text{then} \quad |f(x) - L| < \varepsilon \qquad (*)$$

But to say that $(*)$ is false is the same, as to say that there must be a number x such that

$$0 < |x - a| < \delta, \qquad \text{then} \quad |f(x) - L| \geq \varepsilon$$

Thus to say that the statement $\lim_{x\to a} f(x) = L$ is false is the same as to that there is some $\varepsilon > 0$ such that for every $\delta > 0$ there is a number x satisfying

$$0 < |x - a| < \delta, \qquad \text{then} \quad |f(x) - L| \geq \varepsilon$$

Example 5: Let f be defined by

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases}$$

has no limit at 0.

Solution: Let L be any number. We will prove that the statement "L is the limit of f at 0" is false by letting $\varepsilon = 1/2$ and showing that for any $\delta > 0$ there is an x satisfying

$$0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \frac{1}{2} = \varepsilon$$

Let δ be any positive number. If $\leq -\frac{1}{2}$, then we let $x = \delta/2$ and note that $f(x) = x^2$ so that

$$|f(x) - L| = \left|\frac{\delta^2}{4} - L\right| \geq \left|\frac{\delta^2}{4} + \frac{1}{2}\right| > \frac{1}{2} = \varepsilon$$

If $L \geq -\frac{1}{2}$ then we let $x = -\delta/2$ and note that $f(x) = -1$, so that

$$|f(x) - L| = |-1 - L| = |-1||1 + L| \geq \left|1 - \frac{1}{2}\right| = \frac{1}{2} = \varepsilon$$

In either case we have shown that for any $\delta > 0$ there is an x satisfying

$$0 < |x - a| < \delta \quad \text{and} \quad |f(x) - L| \geq \tfrac{1}{2} = \varepsilon$$

Therefore f has no limit at 0. ■

Exercise

1. Use $\varepsilon - \delta$ definition proves the following limits.

a) $\lim_{x\to 2} 2x + 3 = 7$

b) $\lim_{x\to 10} \sqrt{x-1} = 3$

c) $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ *If* $a > 0$

d). $\lim\limits_{x\to 2} \frac{1}{x} = \frac{1}{2}$

e) $\lim_{x\to 4} x^2 = 16$

f) $\lim_{x\to k} k = k$

g) $\lim_{x\to 4}(x-4) = 0$

h) $\lim_{x\to 0} 3x \sin\frac{1}{x}$

i) $\lim_{x\to 4} x^2 + x - 11 = 9 \left[Hint: Use \left|\sqrt{x} - \sqrt{a}\right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}.\right]$

2. Find the number δ such that

a) If $|x-2| < \delta$ *, then* $|4x-8| < \varepsilon$, *where* $\varepsilon = 0.1$

b) Repeat part (a) with $\varepsilon = 0.01$

3.Let $f(x) = \begin{cases} 0 & if\ \ is\ \ rational \\ 1 & if\ x\ is\ irrational \end{cases}$. *Prove that* $\lim_{x\to 0} f(x)$ *does not exist.*

3.2 Basic Limit Theorems

Objectives

By the end of this section, students will able to:

- ✓ State some limit theorems;
- ✓ Find the limit of function by using limit theorems;
- ✓ Apply Squeezing Theorem to find limits of some functions.

Even if we have developed important techniques of solving limit problems by using the formal definition, by now we have realized that it is not that easy to use this definition to solve each and every problem. Nevertheless, the student had encountered in his or her earlier studies of calculus rather easy ways of evaluating limits by the help of different rules. Here we state and prove some of them by using definition 3.2 and use those to evaluate more complex limit cases.

Theorem 3.4:

i. If the limit of a function f at a exists, then this limit is unique.

ii. If L and M are both limits of f at a then $L = M$.

Theorem 3.5: Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and c is any constant. Then

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
3. $\lim_{x\to a} cf(x) = \quad c\lim_{x\to a} f(X)$.
4. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$.
5. If $\lim_{x\to a} g(x) \neq 0$ *and* $\lim_{x\to a}\frac{f(x)}{g(x)}$ exist then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a positive integer.
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$ where n is positive integer (if n is even. We assume that a>0)
11. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is positive integer (if n is even. We assume that $a > 0$)

Proof: We are going to proof only the first one the other properties are left as an exercise to the reader.

Let

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M \text{ then } \lim_{x\to a}[f(x) + g(x)] = L + M$$

Le t $\varepsilon > 0$ *be* given. We want to find a positive number δ such that for all x

$$0 < |x - a| < \delta \qquad \Rightarrow \qquad |[f(x) + g(x) - (L + M)]| < \varepsilon.$$

Regrouping terms, we get

$$|[f(x)+g(x)]-(L+M)| = |(f(x)-L)+(g(x)-M)|$$
$$\leq |f(x)-L|+|g(x)-M|$$

By triangle inequality. Since $\lim_{x\to a} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x

$$0 < |x-a| < \delta_1 \text{ then } |f(x)-L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x\to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x-a| < \delta_2 \quad \text{then } |g(x)-M| < \frac{\varepsilon}{2}.$$

$Let\ \delta = \min\{\delta_1, \delta_2\}$ the smaller of $\delta_1\ and\ \delta_2$ if $|x-a| < \delta_1\ \ then\ |\ x-a| < \delta.$

So $|f(x)-L| < \frac{\varepsilon}{2}$ and $|x-a| < \delta_2,$ hence $|g(x)-M| < \frac{\varepsilon}{2}$. Therefore,

$$|f(x)+g(x)-(L+M)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L+M.$

Example 1: Evaluate the following limits and justify each step.

a. $\lim_{x\to 5}(2x^2-3x+4)$ b. $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x}$

Solution:

a. $\lim_{x\to 5}(2x^2-3x+4) = \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4$ By law 1 and 2

$=2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4$ by 3

$= 2(5^2)-3(5)+4$ by 9,7 and 7

$= 39$ ■

b. $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2} 5-3x}$ (by law 5)

$= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}$ (By 1, 2 and 3)

$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \frac{-1}{11}$ (By 9, 8 and 7 ■

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f. Then

$$\lim_{x\to a} f(x) = f(x = f(a)$$

A function with the direct substitution property is called continuous at a. However, not all limits can be evaluated by direct substitution as the following examples show.

Example 2: Find $\lim_{x\to 1}\frac{x^2-1}{x-1}$

Solution: Let $f(x) = (x^2-1)/x-1$. We can't find the limit by substitution $x = 1$ because $f(1)$ isn't defined nor can we apply the quotient rule because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of square:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$$

The numerator and denominator have a common factor of $x-1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x-1 \neq 0$. Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1}(x+1)$$

$$= \quad 1+1 = 2 \qquad \blacksquare$$

Example 3: Find $\lim_{x\to 1} g(x)$ where

$$g = \begin{cases} x+1 & if \;\; x \neq 1 \\ \pi & if \;\; x = 1 \end{cases}$$

Solution: Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1, since $g(x) = x+1$ for $x \neq 1$, we have

$$\lim_{x\to 1} g(x) = \lim_{x\to}(x+1) = 2 \qquad \blacksquare$$

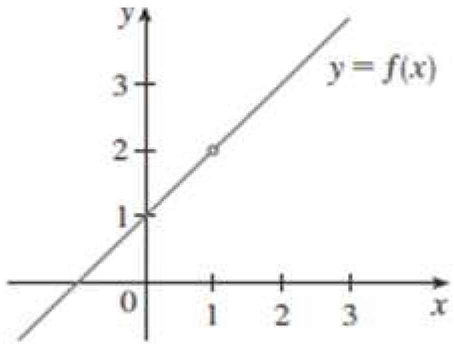


Figure 3.6

Example 4: Evaluate the following limits**.**

a. $\lim_{x\to 0}(x^2+\cos x)$ b. $\lim_{x\to 9}\frac{3}{4}\sqrt{x}$

c. $\lim_{x\to 0} x\cos x$ d. $\lim_{x\to 0}\frac{x\cos x}{x^2+\cos x}$

Solution: a. by the sum theorem, we have

$$\lim_{x\to 0}(x^2+\cos x)=\lim_{x\to 0}x^2+\lim_{x\to 0}\cos x=0+1=1$$

b. From the constant multiple theorems we find that.

$$\lim_{x\to 9}\frac{3}{4}\sqrt{x}=\frac{3}{4}\lim_{x\to 9}\sqrt{x}=\frac{3}{4}\sqrt{9}=\frac{9}{4}$$

c. By the product theorem and we conclude that

$$\lim_{x\to 0}x\cos x=\lim_{x\to 0}x\lim_{x\to 0}\cos x=0\cdot 1=0$$

d. By the quotient Theorem along the results of part (a)and(c), we find that

$$\lim_{x\to 0}\frac{x\cos x}{x^2+\cos x}=\frac{\lim_{x\to 0}x\cos x}{\lim_{x\to 0}(x^2+\cos x)}=\frac{0}{0+1}=0 \qquad ■$$

Activitiy: *Evaluate the following limits.*

a) $lim_{x\to 0}\frac{sin(2x)}{x}$ b) $lim_{x\to 0}\frac{cos\,x-1}{x}$ c) $lim_{x\to 0}\frac{sin^2x}{1-cos\,x}$ d) $lim_{x\to 0}\frac{e^{3x}-1}{e^x-1}$

Note:

Another widely used form equivalent to $\lim_{x\to a}f(x)=L$ is

$$lim_{h\to 0}f(x+h)=L \qquad (1)$$

and this is obtained by replacing x by $a+h$.

Using the sum and constant multiple theorems and (1) we can show that the sine cosine functions have limits at any number.

Example 5: Show that for any number a, $lim_{x\to a}sinx=sina$ and $lim_{x\to a}cosx=cosa$

Solution: Using the trigonometric identities

$$sin(a+h)=sin\,a\,cos\,h+sin\,h\,cos\,a \qquad (2)$$

$$cos(a+h)=cos\,a\,cos\,h-sin\,a\,sin\,h \qquad (3)$$

We will prove that $\lim_{x\to a}sinx=sina$ and $\lim_{x\to a}cosx=cosa$ beginning with the sine function, from (2) it follows that

$$\lim_{h\to 0}sin(a+h)=\lim_{h\to 0}(sin\,a\,cos\,h+sin\,h\,cos\,a)$$

$$=\sin a\lim_{h\to 0}\cos h+\cos a\lim_{h\to 0}\sin h$$

$$=\sin a\,.\,1+\cos a\,.\,0=\sin a \qquad ■$$

For the cosine function we use (3) and find that

$$\lim_{h\to 0}\cos(a+h) = \lim_{h\to 0}(cosacosh - sinasinh)$$

$$= \cos a \lim_{h\to 0}\cos h - \sin a \lim_{h\to 0}\sin h = \cos a\,.\,1 - \sin a\,.\,0 = \cos a$$

The other trigonometric functions also have the property that the limit at a point a in the domain by evaluating the function at a. Thus

$$\lim_{x\to\frac{\pi}{4}}\tan x = \lim_{x\to\frac{\pi}{4}}\frac{\sin x}{\cos x} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \tan\frac{\pi}{4} = 1$$

Although the quotient theorem does not guarantee the existence of $\lim_{x\to a}\frac{f(x)}{g(x)}$, when $\lim_{x\to a} g(a) = 0$, sometimes it is still possible to evaluate such limits.

Theorem 3.6 (*Substitution Theorem for limits*)

Suppose $lim_{x\to a} f(x) = c,$ and $f(x) \neq c$ for all x in some open interval about a with the possible exception of a itself. Suppose also that $lim_{y\to c} g(y)$ exists. Then

$$lim_{x\to a} g\big(f(x)\big) = lim_{y\to c} g(y).$$

Proof: Let $\varepsilon > 0$ Suppose $lim_{y\to c} g(y)=L$, then there is a $\delta_1 > 0$ such that

$$\text{if } 0 < |y-c| < \delta_1. \text{ Then } |g(y) - L| < \varepsilon \qquad (1)$$

Since $lim_{x\to a} f(x) = c$ and $\delta_1 > 0$, there is a $\delta > 0$ such that

$$\text{if } 0 < |y-a| < \delta \text{, then } |f(x) - c| < \delta_1$$

By hypothesis, δ may be chosen so small that if $0 < |x-a| < \delta$, then $f(x) \neq c$.

Thus,

$$\text{if } 0 < |x-a| < \delta, \text{ then } 0 < |f(x) - c| < \delta_1.$$

Hence by (1) $\quad \big|g\big(f(x)\big) - L\big| < \varepsilon$

Consequently $\quad lim_{x\to a} g\big(f(x)\big) = L = lim_{y\to c} g(y).$

Example 6: Find $\lim_{x\to 0}\sqrt{1-x^2}$

Solution: Let y is the expression $1 - x^2$, since $1 - x^{\,2}$ approaches 1 as x approaches 0, it follows those y approaches 1, and we have

$$\lim_{x\to 0}\sqrt{1-x^2} = \lim_{y\to 1}\sqrt{y} = 1. \qquad ■$$

Example 7:

Find $\lim_{x\to\frac{\pi}{3}}\cos(x+\frac{\pi}{6})$.

Solution: Let y is the expression $x+\frac{\pi}{6}$. Since $x+\pi/6$ approaches $\frac{\pi}{3}+\frac{\pi}{6}$ as x approaches $\frac{\pi}{3}$. It follows that y approach $\frac{\pi}{2}$, so we have

$$\lim_{x\to\frac{\pi}{3}}\cos(x+{}^{\pi}/_{6})=\lim_{y\to\pi/_2}\cos y=\cos{}^{\pi}/_{2}=0$$ ■

Activities :

Find the following limits

a) $\lim_{x\to-\pi/_4}\tan x$ *b)* $\lim_{x\to\pi/_{12}}\sqrt{\sin 2x}$ *c)* $\lim_{x\to 2}\sqrt{x^2+5}$

Theorem 1.7 (*squeezing theorem*)

Assuming that $f(x)\ \le g(x)\ \le h(x)$ for all x in the some open interval I about a except possible a itself. If $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$, Then $\lim_{x\to a}g(x)$ exists $\lim_{x\to a}g(x)=L$

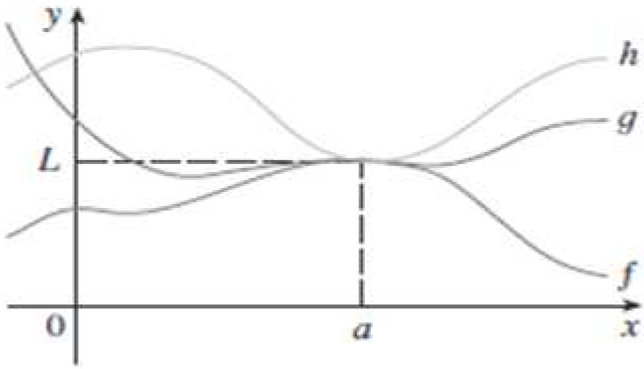


Figure 3.7

Example 8: Show that $\lim_{x\to 0}x^2\sin\frac{1}{x}=0$.

Solution: We can consider $x^2\sin\frac{1}{x}$ as product of x^2 and $\sin\frac{1}{x}$, but we cannot use product rule because $\lim_{x\to 0}\sin\frac{1}{x}$ does not exists, since the value of the $\sin x$ function lies in the interval $[-1,1]$ it follows that

$$-1\le\sin\frac{1}{x}\le 1 \text{ for } \ x\ne 0$$

Multiplying by non negative number x^2 yeild

$$-x^2\le x^2\sin\frac{1}{x}\le x^2$$

Let $f(x)=-x^2$, $g(x)=x^2\sin\frac{1}{x}$ and $h(x)=x^2$. Now,

$$\lim_{x\to 0} x^2 = 0 \text{ and } \lim_{x\to 0}(-x^2) = 0.$$

That is

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x).$$

Hence by Squeezing Theorem the limit of $g(x)$ exists and $\lim_{x\to 0} g(x) = 0$

Therefore,

$$lim_{x\to 0}\, x^2\, sin\frac{1}{x} = 0. \qquad \blacksquare$$

Example 9: Show that

$$\lim_{x\to a}\frac{\sin\theta}{\theta}=1$$

Solution: Using figure 3.8 we obtain the following equations which are valid for $0 < \theta < \frac{\pi}{2}$.

$$\text{Area of triangle OPA}= \frac{1}{2}|OA||QP| = \frac{1}{2}\times 1\times \sin\theta = \frac{\sin\theta}{2}.$$

$$\text{Area of sector OPA} = \frac{\theta}{2\pi}(\text{area of circle})= \frac{\theta}{2\pi}\pi = \frac{\theta}{2}.$$

$$\text{Area of triangle OAT}= \frac{1}{2}|OA||AT| = \frac{1}{2}\times 1\times \tan\theta = \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

It is geometrically clear that

Area of triangle OAP $\leq$ area of sector OAP $\leq$ area of triangle OAT

Thus

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

so that

$$\frac{\sin\theta}{2} \leq \frac{1}{2}\frac{\sin\theta}{\cos\theta} \text{ for } 0 < \theta < \frac{\pi}{2}$$

Separately, the first and the second inequalities yield

$$\frac{\sin\theta}{\theta} \leq 1 \text{ and } \cos\theta \leq \frac{\sin\theta}{\theta} \qquad (1)$$

Combining the inequality in (1) and use the fact that

$$\cos(-\theta) = \cos\theta \text{ and } \frac{\sin(-\theta)}{-\theta} = \frac{\sin\theta}{\theta}$$

we obtain

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1, \text{ for } 0 < |\theta| < \frac{\pi}{2}$$

Since $\lim_{\theta\to}\cos\theta = 1 = \lim_{\theta\to 0} 1$ it follows from the squeezing theorem that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad \text{exists} \quad \text{and} \quad \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$ ■

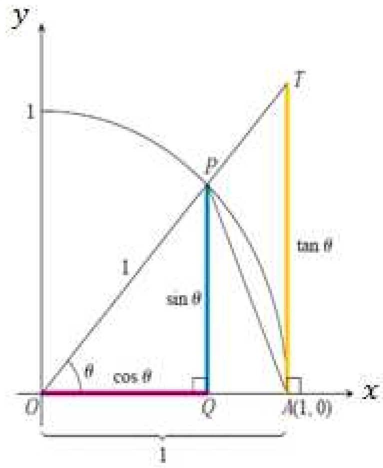


Figure 3.8

Exercises

1. *Given that* $lim_{x\to a} f(x) = -3$ $\quad lim_{x\to a} g(x) = 0$ $\quad lim_{x\to a} h(x) = 8$

Find the limits if exist.If the limit does not exist, explain why.

a) $lim_{x\to a}[f(x) + g(x)]$

b) $lim_{x\to a}(f(x))^2$

c) $lim_{x\to a}\sqrt{h(x)}$

d) $lim_{x\to a}\frac{2f(x)}{h(x)-f(x)}$

e) $lim_{x\to a}\sqrt{f(x)}$

f) $lim_{x\to a}\frac{f(x)}{g(x)}$

2. *Evaluate the limit and justify each step by indicating the appropriate limit laws.*

a) $lim_{x\to -2}(2x^4 + 5x^2 - 2x + 9)$

b) $lim_{x\to 2}\frac{2x^2+1}{x^2-3x+4}$

c) $lim_{t\to -1}(t^2+1)^3(t+3)^5$

d) $lim_{x\to 8}\left(1+\sqrt[3]{x}\right)(2-3x^2+x^3)$

e) $lim_{x\to 3}\frac{\sqrt{x^2+1}}{x+4} + \sqrt{16-x^2}$

f) $lim_{x\to 2}(1+x^2)(2x-1)$

g) $lim_{x\to 4^-}\sqrt{16-x^2}$

3. *Evaluate the limit if it exists.*

a) $lim_{x\to 2}\frac{x^2+x-6}{x-2}$

b) $lim_{x\to 1}\frac{x^3-1}{x^2-1}$

c) $lim_{h\to 0}\frac{(4+h)^2-16}{h}$

e) $lim_{t\to -3}\frac{t^2-9}{2t^{2+7t+3}}$

f) $lim_{u\to -2}\sqrt{u^4+3u+6}$

g) $lim_{x\to 1}\left(\frac{1+3x}{1+4x^2+3x^4}\right)^3$

d) $\lim\limits_{h\to 0}\frac{\sqrt{1+h}-1}{h}$ *h)* $lim_{t\to -1}(t^2+1)^3(t+1)^5$

4. *Find the limit, if it exists. If the limit does not exist, explain why.*

a) $lim_{x\to 3}(2x+|x-3|)$ *e)* $lim_{x\to o^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

b) $lim_{x\to -6}\frac{2x+12}{|x+6|}$ *f)* $lim_{x\to 0.5^-}\frac{2x-1}{|3x^3-x^2|}$

c) $lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ *g)* $lim_{x\to 0^-}\frac{x}{|x|}$ *and* $lim_{x\to 0^+}\frac{x}{|x|}$

d) $lim_{x\to -2}\frac{2-|x|}{2+x}$ *h)* $lim_{x\to 0}\frac{x}{|x|}$

5. *Let* $f(x)=\begin{cases}4-x^2 & if\ x\le 2\\ x-1 & if\ \ x>2\end{cases}$

a) *Find* $lim_{x\to 2^-}f(x)$ *and* $lim_{x\to 2^+}f(x)$

b) *Does* $lim_{x\to 2}f(x)$ *exist?*

6. *Let* $f(x)=\frac{x^2-1}{|x-1|}$. *then find* $lim_{x\to 1^+}f(x)$ *and* $lim_{x\to 1^+}f(x)$

7. *Use the squeeze Theorem to show that*

$a)$ $lim_{x\to 0}\sqrt{x^3+x^2}\,sin\frac{\pi}{x}=0$ *b)* $lim_{x\to 0}x^4\,cos\frac{2}{x}=0$ *c)* $lim_{x\to 0}x^2\,cos\frac{1}{x}$

3.3. One Sided Limits

Objectives

At the end of this section, students will be able to:

- ✓ Find one sided limit;
- ✓ Find the right hand side limit;
- ✓ Find the right hand side limit.

To have a limit L as x approaches a, a function f must be defined on both sides of a and its values f(x) must approaches L as x approaches a from either side. Because of this,ordinary, limits are called two-sided.

If fails to have a two-sided limit at a, it may still have a one sided limit, that is ,a limit if the approaches is only from one side. If the approaches is from the right, the limit is right-hand limit. From the left, it is a left-hand limit.

Example 1: one-side limit for simecircle

The domain of $\mathrm{f(x)}=\sqrt{4-\mathrm{x}^2}$ is [-2,2], its graph is the semicircle in figure 1.9, we have

$$\lim_{x\to 2^+}\sqrt{4-x^2}=0 \text{ and } \lim_{x\to 2^-}\sqrt{4-x^2}=0$$

The function does not have a left hand limit at $x = -2$ or a right hand limit at $x= 2$. It does not have ordinary two sided limits at either -2 or 2.

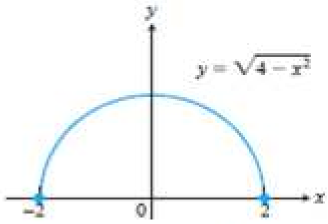


Figure 3.9

Definition 3.8:

a) Let f be defined on some open interval (a, c). A number L is the limit of $f(x)$ as x approaches a from the right (or the right-hand limit of f at a) if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < x - a < \delta, \text{then } |f(x) - L| < \varepsilon.$$

In this case, we write $lim_{x\to a^+} f(x) = L$.

b) Let f be defined on some open interval (c, a). A number L is the limit of $f(x)$ as x approaches a from the left (or the left-hand limit of f at a) if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } -\delta < x - a < 0. \text{ then } |f(x) - L| < \varepsilon$$

In this case we write $lim_{x\to a^-} f(x) = L$.

Thus the symbol $x \to a^+$ and "$x \to a^-$ means that we consider only $x > a$ *and* $x < a$ respectively.

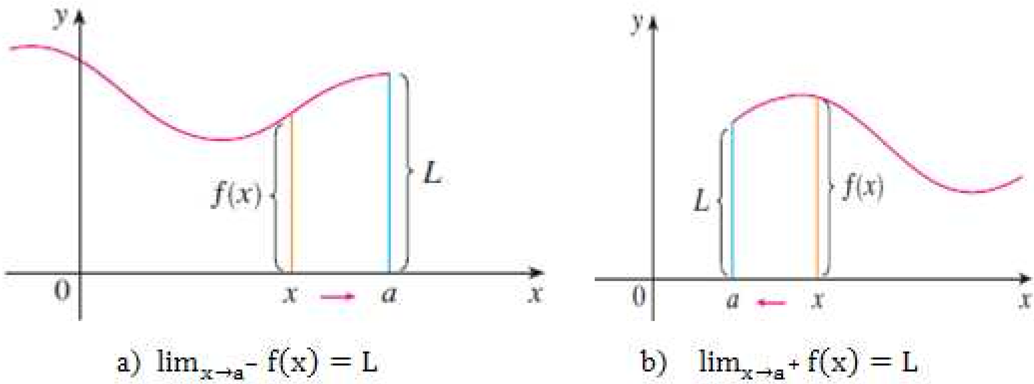


Figure 3.10

Example 2: Prove that $lim_{x\to o^+} \sqrt{x}$=0

Solution: Let $\varepsilon > 0$ be given, here x_o=0 and L=0, so we want to find a $\delta > 0$ such that for all x $0 < x < \delta$ implies $|\sqrt{x} - 0|<\varepsilon$ or

$$0 < x < \varepsilon \quad \text{then } \sqrt{x} < \varepsilon$$

Squaring both sides of this last inequality gives $x < \varepsilon^2$ if $0 < x < \delta$.

If we choose $\delta = \varepsilon^2$ we have $0 < x < \delta = \varepsilon^2$ implies $\sqrt{x} < \varepsilon$, or $0 < x < \varepsilon^2$ implies that $\left|\sqrt{x} - 0\right| < \varepsilon$

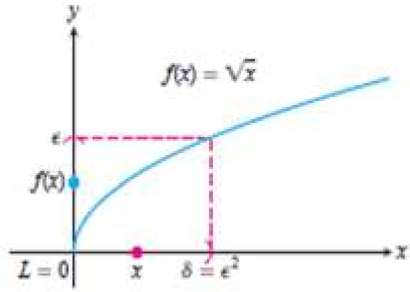


Figure 3.11

According to definition, this shows that $\lim_{x\to 0^+} \sqrt{0}$=0 ■

Example 3: Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side.

Solution: As x approaches zero its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatdly from -1 to 1. There is no single number L such that function`s values stay increasing close to x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

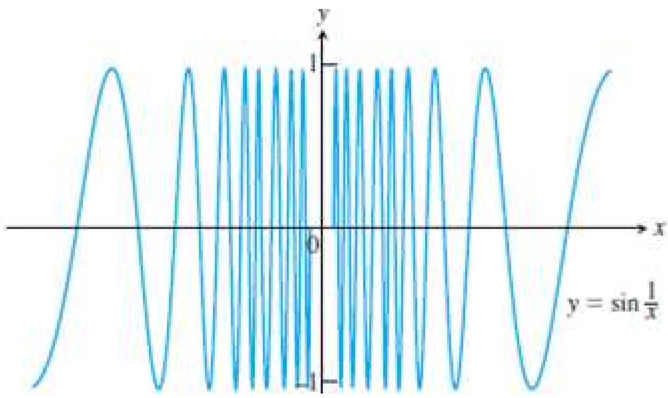


Figure 3. 12

From figure 1.12 we see that $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero. ■

Theorem 3.9: Let f be defined on an open interval about a except possible at a it self. Then $\lim_{x\to a} f(x)$ exists if and only if both one sided limits, $lim_{x\to a^+} f(x)$ *and* $lim_{x\to a^-} f(x)$ exists and $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

In this case

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

Example 1: Show that $\lim_{x\to 0}|x| = 0$

Solution: Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, *we have* $\lim_{x\to 0^+}|x| = 0$

For $x < 0$, we have $|x| = -x$ and so $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = 0$

Therefore, by Theorem 3.9

$$\lim_{x\to 0}|x| = 0$$ ■

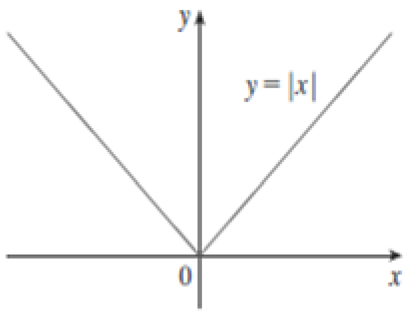


Figure3.13

Example 2: Let $f(x) = \begin{cases} 2x+1 & \text{for } x < 2 \\ x+3 & \text{for } x > 2 \end{cases}$

find $\lim_{x\to 2} f(x)$.

Solution: since $\lim_{x\to 2}(2x+1) = 5$ and $\lim_{x\to 2}(x+3) = 5$ by theorem 3.9 we have

$$\lim_{x\to 2^-}(2x+1) = 5 \text{ and } \lim_{x\to 2^+}(x+3) = 5$$

Hence $\lim_{x\to 2} f(x) = 5$ ■

Example 3: Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Solution : $\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+}\frac{1}{1} = 1$. And $\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = \lim_{x\to 0^-}(-1) = -1$

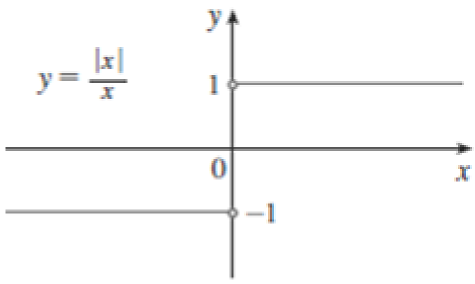


Figure 3.14

Since the right and the left limits are different, it follows that from theorem 1.9 that $\lim_{x\to 0}\frac{|x|}{x} = 0$ does not exist.The graph of the function $f(x) = |x|/x$ is shown in Figure 3.14 supports the one-sided limits that we found. ■

Example 4: The greatest integer function is defined by $[\![x]\!]$is equal to the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$). Show that $\lim_{x\to 3}[\![x]\!]$ does not exist.

Solution: The graph of the greatest integer function is shown in Figure 3.15

For $3 \le x < 4$. we have.

$$\lim_{x\to 3^+}[\![x]\!] = \lim_{x\to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \le x < 3$, we have $\lim_{x\to 3^-}[\![x]\!] = \lim_{x\to 3^-} 2 = 2$ ■

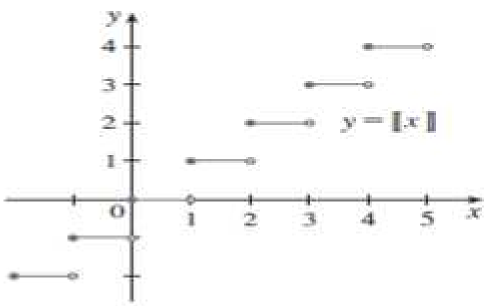


Figure3.15

Because of one side limits are not equal. $lim_{x\to 3}[\![x]\!]$ does not exist by Theorem 3.9.

Exercises

1. *Let* $f(x) = \begin{cases} -x & if\ x \leq -1 \\ 1 - x^2 & if\ -1 < x < 1 \\ x - 1 & if x > 1 \end{cases}$

Then evaluate the limit,if it exists.

a) $\lim_{x\to 1^+} f(x)$ *b)* $\lim_{x\to 1} f(x)$ *c)* $\lim_{x\to 0} f(x)$

d) $\lim_{x\to -1^-} f(x)$ *e)* $\lim_{x\to -1^+} f(x)$ *f)* $\lim_{x\to -1} f(x)$

2. If n is an integer ,evaluate.

a) $\lim_{x\to n^+} [\![x]\!]$ *b)* $\lim_{x\to n^-} [\![x]\!]$ *c)* $\lim_{x\to n} [\![x]\!]$

3**.** *Which of the following statements about the function* $y = f(x)$ *sketched in figure 3.16 are true, and which are false?*

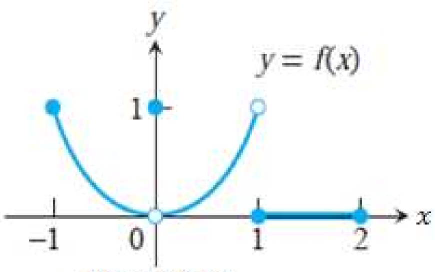


Figure 3.16

a) $\lim_{x\to -1^+} f(x) = 1$ *e)* $\lim_{x\to 0} f(x) = 1$ *i)* $\lim_{x\to 2^-} f(x) = 0$

b) $\lim_{x\to 0^-} f(x) = 1$ *f)* $\lim_{x\to 1} f(x) = 1$ *j)* $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

c) $\lim_{x\to 0} f(x)$ *exists* *g)* $\lim_{x\to 1} f(x) = 0$ *k)* $\lim_{x\to -1^-} f(x)$ *does not exist.*

d) $\lim_{x\to 0^-} f(x) = 0$ *h)* $\lim_{x\to 2^+} f(x) = 0$ *l)* $\lim_{x\to 0^-} f(x) = 0$

3.4- Infinite limits, limit at infinity and asymptotes

Objectives

At the end of this section, students will be able to:

- ✓ Find infinite limits;
- ✓ Find limits at infinity;
- ✓ Find infinite limits at infinity;
- ✓ Use the limit to find asympotes of function.

3.4.1 Limit at Infinity

Now we consider the limit of f as x becomes larger and larger in absolute value. Here we see precise definition of limit.

Definition 3.10

a) $\lim_{x\to\infty} f(x) = L$ if for every ε > there is a number M such that

$$\text{if } x > M, \text{ then } |f(x) - L| < \varepsilon$$

b) $\lim_{x\to-\infty} f(x) = L$ if for every ε >0 there is a number N such that

$$\text{if } x < N, \text{then } |f(x) - L| < \varepsilon$$

The symbol for infinity (∞) does not reprsent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range out grow all finite bounds. For example, the function $f(x) = 1/x$ is define for all $x \neq 0$ (Figure 3.17). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large. $1/x$ again becomes small. We summarize these observation by saying $f(x) = 1/x$ at infinity and negatve infinity.

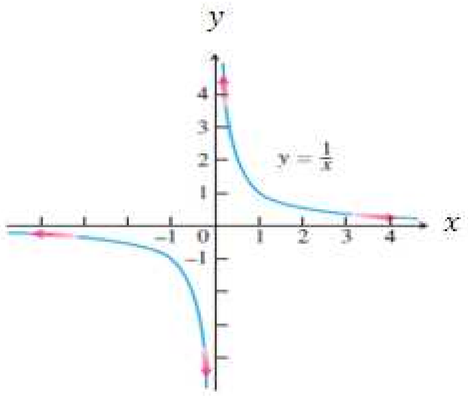


Figure 3.17 Limit at infinity for $f(x) = 1/x$

Example 1: Show that a. $\lim_{x\to\infty} 1/x = 0$ b. $\lim_{x\to-\infty} 1/x = 0$

Solution: a. Let ε > 0 be given. We must find a number M such that for all x,

$$x > M \text{ implies } \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$$

The implication will hold if $M = 1/\varepsilon$ or any large positive number.This proves

$$lim_{x\to\infty} 1/x = 0$$

b. Let $\varepsilon > 0$ be given. We must find a number N such that for all x, $x < N$ implies $\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon$. This implication will hold if $N = {-1}/_{\varepsilon}$ or any less than ${-1}/_{\varepsilon}$. This prove $lim_{x\to-\infty} {1}/_{x} = 0$ ■

Example 2: Show that $\lim_{x\to\infty} \frac{1}{x^2} = 0$ and $\lim_{x\to-\infty} \frac{1}{x^2} = 0$.

Solution: Let $\varepsilon > 0$ To show that $\lim_{x\to\infty} \frac{1}{x^2} = 0$ we must find an M such that

If $x > M$, then $\left|\frac{1}{x^2} - 0\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} < \varepsilon$, but then if $x > \frac{1}{\sqrt{\varepsilon}}$ then $\frac{1}{x^2} < \varepsilon$

Therefore we let $M = \frac{1}{\sqrt{\varepsilon}}$ and conclude that $lim_{x\to\infty} \frac{1}{x^2} = 0$

To show that $lim_{x\to-\infty} \frac{1}{x^2} = 0$ we simple choose $N = -\frac{1}{\sqrt{\varepsilon}}$ then $N < 0$ and thus

$$\text{if } x < N, \quad \text{then } \left|\frac{1}{x^2} - 0\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} < \frac{1}{N^2} = \varepsilon$$

This proves that $lim_{x\to-\infty} \frac{1}{x^2} = 0$. ■

Definition 3.11: Horizontal Asymptote

A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$lim_{x\to\infty} f(x) = b \quad \text{or} \quad lim_{x\to-\infty} f(x) = b.$$

In the above examples that $y = 0$ is the horizontal asymptote of the graphs of ${1}/_{x}$ and ${1}/_{x^2}$.

Example 3:

Let's begin by investigating the behavior of the function f defined by $f(x) = \frac{x^2-1}{x^2+1}$.

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1 (figure 3.18). In fact ,it seems that we can make the values of $f(x)$ as close we like 1 by taking x sufficient large.This situation is expressed symolically by writing

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$$

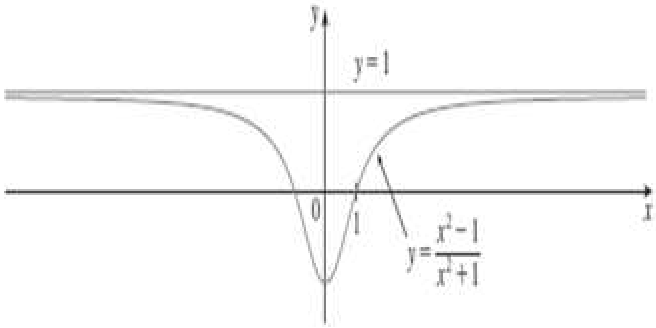


Figure 3.18

The curve illustrated in Figure 18 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$$ ■

Theorem 3.12 If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

if $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator.Then depends on the degrees of the polynomails involved. Example 4 Numerator and denominator of the same degree.

Example 4: Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

and indicate which properties of limits are used at each stage.

Solution:

To evaluate the limit at infinity of any rational function ,we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x).

In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2 - x - 2}{x^2}}{\frac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

$$= \frac{lim_{x\to\infty}\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{lim_{x\to\infty}\left(5+\frac{4}{x}+\frac{1}{x^2}\right)} \qquad \text{(by limit law 5)}$$

$$= \frac{lim_{x\to\infty}3-lim_{x\to\infty}\frac{1}{x}-2\,lim_{x\to\infty}\frac{1}{x^2}}{lim_{x\to\infty}5+4\,lim_{x\to\infty}\frac{1}{x}+lim_{x\to\infty}\frac{1}{x^2}} \qquad \text{(by limit laws1,2 and 3)}$$

$$= \frac{3-0-0}{5+0+0} = \frac{3}{5}$$ ■

Example 5: Evaluate

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}\ -x\right)$$

Solution: Because both $\sqrt{x^2+1}$ and x are large when x is large, it is difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical.

$$lim_{x\to\infty}\left(\sqrt{x^2+1}\ -x\right) = lim_{x\to\infty}\left(\sqrt{x^2+1}\ -x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$\lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

now divide numerator and denominator by $x = \sqrt{x^2}$ for $x > 0$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}+x}{x}}$$

$$lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{0}{\sqrt{1+0}+1} = 0$$ ■

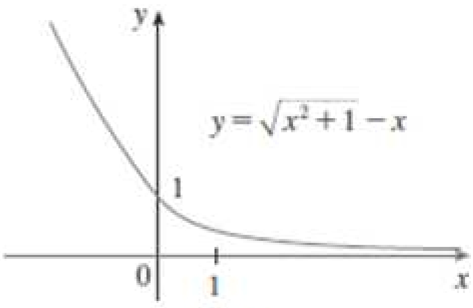


Figure 3.19

Theorem 3.13

If L, M and K are real number and

$lim_{x\to\pm\infty} f(x) = L$ and $lim_{x\to\pm\infty} g(x) = M$, then

1. Sum Rule: $lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. Difference Rule: $lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. Product Rule: $lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. Constant Multiple Rule: $lim_{x\to\pm\infty}(K \cdot f(x)) = K \cdot L$
5. Quotient Rule: $lim_{x\to\pm} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. Power Rule: if r and s are integers with no common factors,s$\neq$ 0.then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is real number.(If s is even we assume that $L > 0$.)

Example 6: Using theorem 3.13

a. $lim_{x\to\infty}(5 + \frac{1}{x}) = lim_{x\to\infty} 5 + lim_{x\to\infty} \frac{1}{x}$ Sum rule

$= 5 + 0 = 5$ known limits

b. $lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2} = lim_{x\to-\infty} \pi\sqrt{3}\, x \frac{1}{x} \frac{1}{x}$

$= lim_{x\to-\infty} \pi\sqrt{3}\, lim_{x\to-\infty} \frac{1}{x} lim_{x\to-\infty} \frac{1}{x}$ product rule

$= \pi\sqrt{3} \cdot 0 \cdot 0 = 0$ ■

3.4.2 Infinite Limits at Infinity

The notation $lim_{x\to\infty} f(x) = \infty$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols.

$$\lim_{x\to-\infty} f(x) = \infty\ , \lim_{x\to\infty} f(x) = -\infty\ , \quad \lim_{x\to-\infty} f(x) = -\infty$$

Example 7: Find $lim_{x\to\infty} x^3$ and $lim_{x\to-\infty} x^3$.

Solution: When x becomes large, x^3 also becomes large, For instance.

$$10^3 = 1000 \qquad 100^3 = 1{,}0000{,}000 \quad 1000^3 = 1{,}000{,}000{,}000$$

In fact ,we can make x^3 as big as we like by taking x large enough. x y

Therefore we can write

$$\lim_{x\to\infty} x^3 = \infty$$

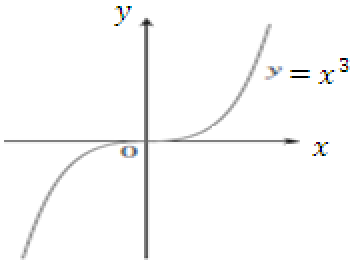


Figure 3 20

Similarly, when x is large negative, so is x^3. Thus $lim_{x\to-\infty} x^3 = -\infty$ ■

Definition 3.14. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

Means that for every positve number M there is a corresponding positve number N such that

$$\text{If} \quad \mathrm{x} > N \qquad then \quad f(\mathrm{x}) > M$$

Similar definition apply when the symbol ∞ is replace by $-\infty$.

3.4.3 Infinite Limits

Infinite limits can also be defined in a precise way. Let us look again at the function $\mathrm{f(x)} = \frac{1}{\mathrm{x}}$ as $\mathrm{x} \to \mathrm{o}^+$ the values of f grow with out bound, that is

$$\lim_{\mathrm{x}\to0^+} \mathrm{f(x)} = \lim_{\mathrm{x}\to0^+} \frac{1}{\mathrm{x}} = \infty$$

And as $\mathrm{x} \to 0^-$, the values of $\mathrm{f(x)} = \frac{1}{\mathrm{x}}$ become negative, that is

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{1}{x} = -\infty$$

Example 8: Find $lim_{x\to1^+} \frac{1}{x-1}$ and $lim_{x\to1^-} \frac{1}{x-1}$

Geomeric solution: The graph of $f(x) = \frac{1}{x-1}$ *is* the graph of $f(x) = \frac{1}{x}$ shifted 1 unit to the right (see Figure 3.21) therefore $y = \frac{1}{x-1}$ be haves near 1 exactly the way $y = \frac{1}{x}$ behaves near 0.

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty \qquad \blacksquare$$

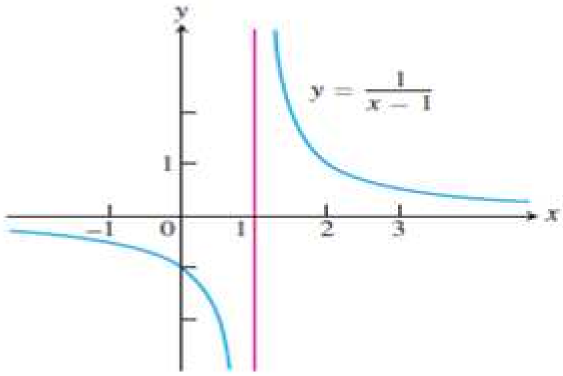


Figure 3.21

Example 9: discuss the behavior of th functions

a. $f(x) = \frac{1}{x^2}$ near $x = o$

b. $g(x) = \frac{1}{(x+3)^2}$ near $x = -3$

Solution:

a. As x appraoches zero from either side,the values of $f(x) = \frac{1}{x^2}$ are positive and become arbitrarily large.We write

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty$$

b. The graph of $g(x) = \frac{1}{(x+3)^2}$ is the graph of $f(x) = \frac{1}{x^2}$ shifted 3 units to the left (Figure 3.22).Therefore g behaves near -3 exactly the way f behaves near 0.

$$\lim_{x\to -3} g(x) = \lim_{x\to -3} \frac{1}{(x+3)^2} = \infty \qquad \blacksquare$$

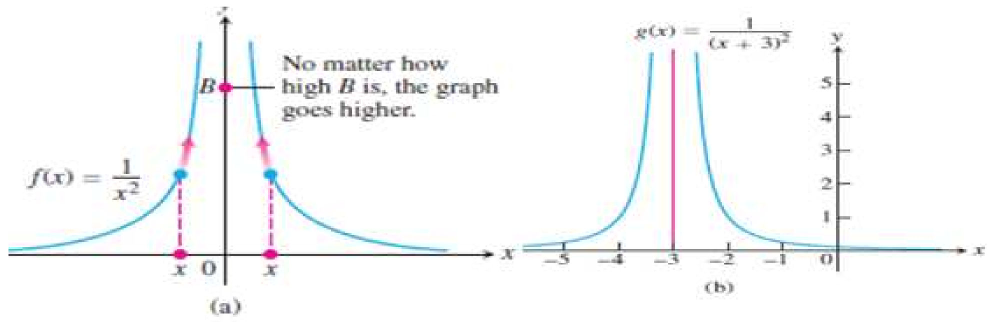


Figure 3.22

Instead of requiring $f(x)$ to lie arbitrarily close to a finite L for all sufficiently close to a, the definition of infinte limits require $f(x)$ to arbitrarily far from the origin. Except for this change, the langauge is identical with what we have seen before.

Defination 3.15 (*Precise definition of infinite limits*).

Let f be a function defined on some open interval that contains the number a except possible at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for vevery positve number M there is positve number δ such that $f(x) > M$ whenver $0 < |x - a| < \delta$

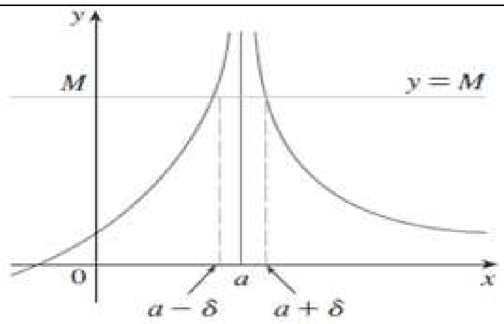


Figure 3.23

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given Number M) by taking x close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 3.23.

Given any horizontal line M, we can find a number $\delta > 0$ such that if we restrict x to lie in the interval$(a - \delta, a + \delta)$ but $x \neq$ a , then the curve $y = f(x)$ lies above the line $y = M$. You can see that if a larger M is chosen, then a smaller δ may be required.

Example 10: Use the definition to prove that $lim_{x \to 0} \frac{1}{x^2} = \infty$

Solution: 1. guessing a value for δ. Given that $M > 0$, we want to find $\delta > 0$ such that

$\frac{1}{x^2} > M$ Whenever $0 < |x - 0| < \delta$

That is $x^2 < \frac{1}{M}$ whenever $0 < |x| < \delta$

or $|x| < \frac{1}{\sqrt{M}}$ whenever $0 < |x| < \delta$ work

This suggests that we should take $\delta = 1/\sqrt{M}$.

2. Showing that this δ works. If $M > 0$ is given. Let $\delta = 1/\sqrt{M}$. If $0 < |x - c| < \delta$

Then $|x| < \delta \Rightarrow x^2 < \delta^2$

$\Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = M$

Thus $\frac{1}{x^2} > M$ Whenever $0 < |x - 0| < \delta$

Therefore, $lim_{x\to 0}\frac{1}{x^2} = \infty$. ■

Definition 3.16: Let f be a function defined on some open interval that contains the number a except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that $f(x) < N$.

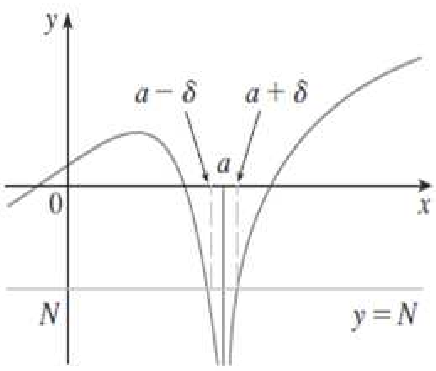


Figure 3.24

Notice that the distance between a point on the graph of $y = 1/x$ and the y-axis approaches zero as the point moves vertically along the graph and away from the origin

$$lim_{x\to 0^+}\frac{1}{x} = \infty \text{ and } \lim_{x\to 0^-}\frac{1}{x} = -\infty$$

We say the line $x = 0$ is the vertical asympote of the graph of $y = 1/x$.

Definition 3.17: A line $y = a$ is a vertical asymptote of the graph of the function $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

Remark: If a function f has a vertical asymptote at $x = a$ then f is not continuous at $x = a$.

Exercises

1. *Determine the infinite limit.*

   a) $\lim\limits_{x\to\infty} \frac{1}{2x+3}$ b) $\lim_{y\to-\infty} \frac{2-3y^2}{5y^2+4y}$ c) $\lim_{x\to 0} \frac{5}{x^5}$

   d) $\lim\limits_{x\to-\infty} \frac{1-x-x^2}{2x^3-x^2+3}$ e) $\lim\limits_{x\to\infty} \frac{3x+5}{x-4}$ f) $\lim_{x\to 0^+} \frac{2}{x^{1/4}}$

   g) $\lim\limits_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1}$ h) $\lim\limits_{x\to-\infty} \frac{x+2}{\sqrt{9x+1}}$ i) $\lim\limits_{x\to-\infty} (x^4 + x^5)$

   j) $\lim\limits_{u\to-\infty} \frac{-u^4+u}{(u^2-2)(2u^2-1)}$ k) $\lim\limits_{x\to-\infty} \left(x + \sqrt{x^2 + 2x}\right)$ l) $\lim\limits_{x\to-\infty} \frac{\sqrt{9x^{6-x}}}{x^3+1}$

   m) $\lim\limits_{x\to-\infty} \frac{\sqrt{9x^{6-x}}}{x^3+1}$

2. *Let P and Q be polynomials. Find* $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$

   a) *If the degree of P is less than the degree of Q*

   b) *If the degree of P is greater than the degreeof Q*

3. *Use the definition to prove that* $\lim_{x\to-\infty} \frac{1}{x} = 0.$

4. *Find all vertical asymptotes .*

   a) $f(x) = \frac{1}{x+4}$ b) $f(x) = \frac{x-2}{x-3}$ c) $f(x) = \frac{2x-9}{x+\sqrt{2}}$

   d) $f(x) = \frac{x^2-1}{x^2-4}$ e) $f(x) = \frac{-3x}{x^2+9}$ f) $f(x) = \frac{\sin x}{x(x^2-1)}$

3.5. Continuity; one- sided continuity

Objectives

At the end of this section, students will be able to:

- ✓ Define continuity of a function;

- ✓ Define one-sided continuity of a function;
- ✓ Distinguish continuous function on any given interval.

We noticed that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called continuous at a. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word continuity in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change).

Definition 3.18 A function f is continuous at a number a if

$$\lim_{x\to a} f(x) = f(a)$$

Notice that definition 3.18 requires three properties if f is continuous at a

1. $f(a)$ is defined (that is ,a is in the domain of)
2. $lim_{x\to a} f(x)$ exists.
3. $lim_{x\to a} f(x) = f(a)$

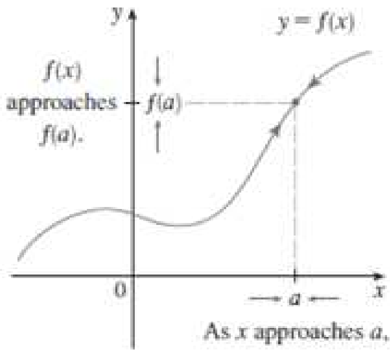


Figure 3.25

If a function f is not continuous at a point c, we say that f is discontinuous at c and c is a point of discontinuos of f.

A function f is right-continuous (continuous from the right) at a point $x = c$ in its domain if $lim_{x\to c^+} f(x) = f(c)$. It is left-continuous(continuous from the left) at c if $lim_{x\to c^-} f(x) = f(c)$. Thus a function is continuous at a left end point a of its domain if it is right -continuous at a and continuous at a right end point b of it is domain if it is left continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c.

Example 1:

The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain ,$[-2,2]$,including $x = -2$. Where f is right-continuous, and $x = 2$ where f is left-continuous.

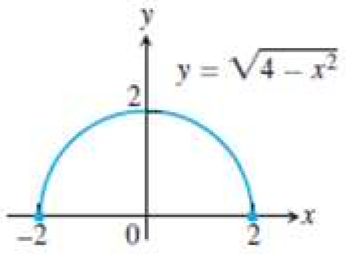


Figure 3.26

Figure 3.26 shows an example of function that is continuous at every point on its domain

Example 2: Figure 3.27 shows the graph of a function f. At which number is f discontinuous?

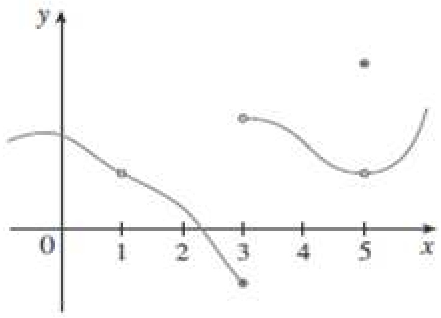


Figure 3.27

Solution: It looks as if there is a discontinuity where $a = 1$ because the graph has a break there. The main reason that f is discontinuous at 1 is that $f(1)$ is not defined.The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined , but $lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuouts at 3. What about $a = 5$? Here, $f(5)$ is defined and $lim_{x\to 5} f(x)$ exist (because the left and right limits are the same). But, $lim_{x\to 5} f(x) \neq f(5)$ So f is discontinuous at 5

Class activity

Where are each of the following functions discontinuous?

$a.\ f(x) = \frac{x^2-x-2}{x-2}$ $\quad b\ .f(x) = \begin{cases} \frac{1}{x^2} & if\, x \neq 0 \\ 1 & \end{cases}$ $\quad c.\ f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if x \neq 2 \\ 1 \quad if & x = 2 \end{cases}$

Definition 3.19 (***One Sided Continuity***)

i. A function f is continuous from the right at a number a if $\lim_{x\to a^+} f(x) = f(a)$.

ii. A function f is continuous from the right at a number a if $\lim_{x\to a^+} f(x) = f(a)$.

Example 3: At each integer n, the function $f(x) = [\![x]\!]$ is continuous from the right but

Definition 3.20 A functions f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an end point of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left).

A continuous functions need not be continuous on every interval. For example $y = 1/x$ is not continuous on [-1,1] (Figure 3.28), but it is continuous at its domain $(-\infty, 0)U(0, \infty)$.

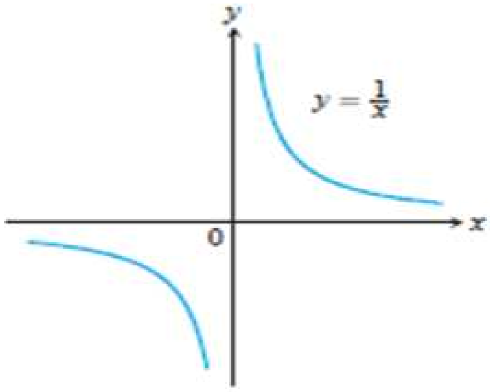


Figure 3.28

The function $y = {}^{1}/_{x}$ is continuous at every values of x except $x = 0$. It has a point of discontiniuty at $x = 0$.

Theorem 3.21: ***properties of continuous function at $x = c$***

If the functions f and g are continuous at $x = c$ then the following combinations are continuous at $x = c$

1. Sum: $f + g$
2. Differences: $f - g$
3. Product: fg
4. Constant multiple: kf for any number k
5. Quotients : f/g provided $g(c) \neq 0$

6. Power: $f^{r/s}$, provided it is defined on an open interval containig c, where r and s are integer.

Proof: by the limit rules

To proof the sum property we have

$$\lim_{x\to c}(f+g)(x) = \lim_{x\to c}(f(x)+g(x))$$
$$= \lim_{x\to c} f(x) + \lim_{x\to c} g(x), \text{ sum rule theorem.}$$
$$= f(c) + g(c) \qquad \text{continuity of } f, g \text{ at } c$$
$$= (f+g)(c)$$

This show that $f+g$ is continuous.

It follows from theorem 3.21 and definition 3. 20 that if f and g are continuous on an intervals, then so are the functions $\mathrm{f+g, f-g, cf, fg}$ and (if g is never zero 0) $\mathrm{f/g}$.

Theorem 3.22

a. Any polynomial is continuous every where, that is continuous on $\mathbb{R} = (-\infty, \infty)$.
b. Any rational function is continuous wherever it is defined, that is continuous on its domain.

Proof:

a. A polynomail is a function of the form:

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x^1 + c_o$$

Where $\mathrm{c_o, c_1, \ldots, c_n}$ are constants. We know that

$$lim_{x\to a}\, c_o = c_o \qquad \text{(by law 7)}$$

And

$$lim_{x\to a}\, x^m = a^m \qquad m = 1,2,3\ldots,n \qquad \text{(by law 9)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus by part 3 of theorem 3.21 the function $g(x) = cx^m$ is continuous.Since p is a sum of function of this form and a constant funcition. It follow from part 1 of the theorem 3.21 that p is continuous.

b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomails.The domain of f is $D = \{x \epsilon R | Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous every where. Thus, by part 5 of Theorem 3.21 f is continuous at every number in *D*.

Example 4: Find $lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$

Solution: The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$

is rational, so by theorem 3.22 it is continuos on it domain, which is $\left\{x | x \neq \frac{5}{3}\right\}$.

$$lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = lim_{x\to -2} f(x) = f(-2) = \frac{(-2)^3+2(-2)^{2-1}}{5-3(-2)} = -\frac{1}{11} \quad \blacksquare$$

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function fog this fact is a consequence of the following theorem.

Theorem 3.23 If f is continuous at b and $\lim_{x\to a} g(x) = b$ *then*

$$\lim_{x\to a} f(g(x)) = f(b) = f(\lim_{x\to a} g(x))$$

Theorem 3.24 tells us that the composite of two continuous functions at a given number is continuous.

Theorem 3.24

If g is continuous at a and f is continuous at $g(a)$, then the composite fog given by $(fog)(x) = f(g(x))$ is continuous at a.

Example 5: where are the following functions continuous?

a) $h(x) = sin(x^2)$ b) $F(x) = ln(1 + cos\, x)$

Solution:

a. Let $g(x) = x^2$ and $f(x) = sin\, x$, we have $h(x) = f(g(x))$

Now g is continuous on $\mathbb{R}$ since it is polynomial and f is continuous everywhere. Thus $h = fog$ is continuous on $\mathbb{R}$ by theorem 3.24.

b. We know from theorem 3.22 that $f(x) = ln\, x$ is continuous and $g(x) = 1 + cos\, x$ (because $y = 1$ and $y = cos\, x$ continuous). Therefore by Theorem 3.24 $F(x) = f(g(x))$ is continuous, where it is defined. Now $ln(1 + cos\, x)$ is defined when $1 + cos\, x > 0$. So it is undefined when $cos\, x = -1$ and this is happened when $x = \pm\pi, \pm 3\pi, ...$ Thus F is discontinuous when x is an odd multiple of π and is continuous on the intervals between these values. ■

Definition 3.25

a) A function f is continuous on (a, b),if it is continuous at every point in (a, b).

b) A function f is continuous on $[\mathrm{a}, \mathrm{b}]$, if it is continuous on (a, b), and is also continuous from the right at a and continuous from the left at b.

Exercises

1.Write an equation that expresses the fact that a function f if continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$,what can you say about its graph?

3. a) From figure 3.29. State the number at which f is discontinuous and explain why.

b) For each of the numbers stated in part (a) determine whether f is continuous from the right, or from the left ,or neither.

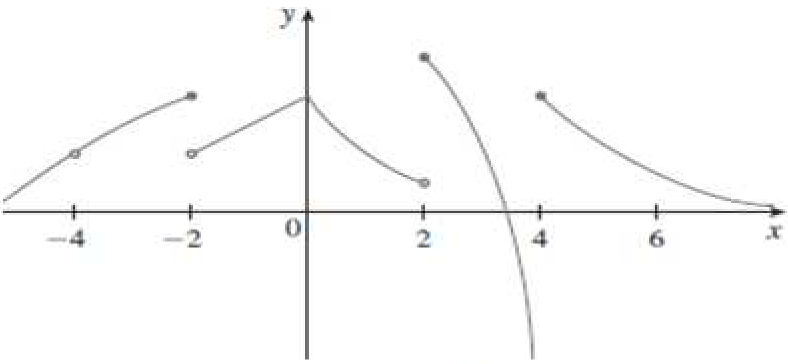


Figure 3.29

4. Determine whether f is continuous or discontinuous at a.if f is discontinuous, determine whether f is continuous from the left at a, is continuous from the right at a.

$$a)\ f(x) = \sqrt{x^2 - 13} : a = -\sqrt{13}$$

$$b)\quad f(x) = \begin{cases} 3x - 4 & for\ x < 2 \\ 3x + 4 & for\ x > 2 \end{cases} \quad a = 2$$

$$c)\ \ f(x) = \begin{cases} \dfrac{|x - 4|}{x - 4} & for\ x \neq 4 \\ 1 & for = 4 \end{cases}$$

5. Find the values for the constant k, that makesthe following functions are continuous at $x = 0$ or continuous every where.

a) $f(x) = \begin{cases} \frac{\sin 3x}{x}, & x \neq 0 \\ k \quad, & x = 0 \end{cases}$ b) $g(x) = \begin{cases} \frac{\tan kx}{x} \quad, & x < 0 \\ 3x + 2k^2, & x > 0 \end{cases}$

c) $f(x) = \begin{cases} 7x - 2, x < 1 \\ kx^2 \ , x > 1 \end{cases}$ d) $f(x) = \begin{cases} kx^2 \, , x \leq 2 \\ 2x + k, x > 2 \end{cases}$

e) $f(x) = \begin{cases} k_1x - k, & x \leq -1 \\ 2x^2 + 3k_1x + k_2 \, , & -1 \leq x \leq 1 \\ 4 \quad , & x > 1 \end{cases}$

6. If f and g are continuous functions with $f(3) = 5$ and $lim_{x\to 3}[2f(x) - g(x)] = 4$ then find $g(3)$.

3.6- Intermediate value theorem

Objectives

At the end of this section, students will be able to:

- ✓ State Intermediate value Theorem;
- ✓ Use the intermedaite value Theorem to show that there is a root of the given equation in the specified interval.

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the Intermediate Value Property. A function is said to have the Intermediate Value Property if whenever it takes on two values, it also takes on all the values in between.

Theorem 3.26 (*Intermediate Value Theorem*)

Suppose that f is continuous on the closed interval $[a \, , b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$

The intermediate value Theorem states that a continuous function takes on every intermediate value between the functional values $f(a)$ and $f(b)$. It is illustrated by Figure 3.30 below. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the intermediate value theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 1.32, then the graph of f cannot jump over the line.it must intersect y = N somewhere.

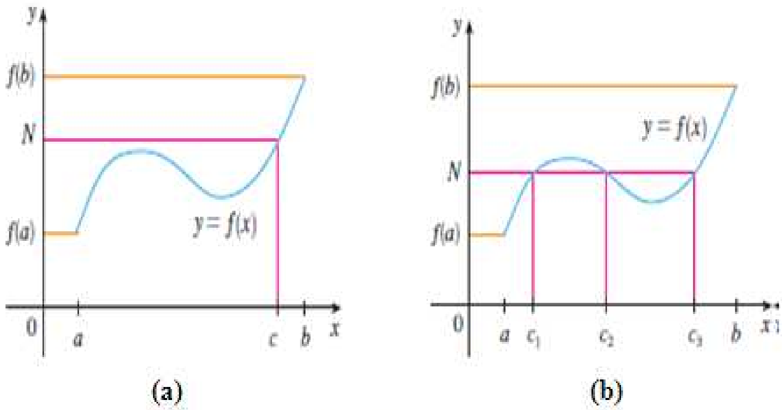


Figure 3.30 Illustration of IVT

The intermediatevalue theorem is not true in general for discontinuous functions.

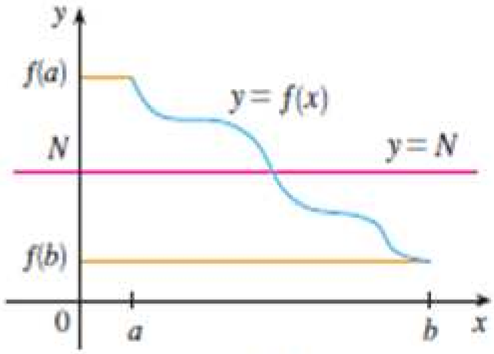


Figure 3.31

One use of the intermediate value theorem is in locating roots of equations as in the following example.

Example 1: Show that the expression $4x^3 - 6x^2 + 3x - 2$ has at least one root between 1 and 2.

Solution: Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. we are lookig for a solution of the given equation , that is a number c between 1 and 2 such that $f(c) = 0$. Therefore ,we take $a = 1, b = 2$ and $N = 0$ by the theorem, we have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < f(2)$; that is $N = 0$ is a number between $f(1)$ $and\, f(2)$.Now f is continuous since it is polynomial, so the intermediate value theorem says there is a number c between 1 and 2 such that $f(c) = 0$. ■

In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1,2)$,

Summary

I. Let f be a function defined at each point of some open interval containing a, possibly at a itself, then a number L is the limit of $f(x)$ as x approaches a (or is the limit of f at a) if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$If\ 0 < |x - a| < \delta, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

If L is the limit of $f(x)$ as x approaches a then we write

$$\lim_{x \to a} f(x) = L$$

If such an L can be found we say that the limit of f at a exists or that f has a limit at a or that $\lim_{x \to a} f(x)$ exists.

II. For real number a and L and function f

- $\lim_{x \to a^+} f(x) = L$, right-hand limit
- $\lim_{x \to a^-} f(x) = L$, left-hand limit
- If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$, then it is said that f has limit at $x = a$ or the limit of $f(x)$ exists at $x = a$ and is expressed as

$$\lim_{x \to a} f(x) = L$$

- If the limit of a function $f(x)$ exist, then it unique.

III. Limit at $\pm\infty$ and infinite limit

- $\lim_{x \to \pm\infty} f(x) = L$ limit at infinity

  In this case, the line $y = L$ is a horizontal asymptote of the graph of f.
- $\lim_{x \to a} f(x) = \pm\infty$, infinite limit

In this case, the line $x = a$ is a vertical asymptote of the graph of f. If $a = \pm\infty$, we have

- $\lim\limits_{x \to \infty} f(x) = \pm\infty$, and

- $\lim_{x\to-\infty} f(x) = \pm\infty$

IV. A function f is continuous at a point $x = a$ if:

$$\lim_{x\to a} f(x) = f(a)$$

V. If f is continuous on$[a,b]$, $f(a)$ and $f(b)$ have opposite signs, then there is at least one number $c\epsilon(a,b)$ such that $f(c) = 0$

Review Exercises

1. Using $\varepsilon - \delta$ Prove the following limits

a) $lim_{x\to\frac{-3}{2}}(1-4x) = 7$ *b)* $lim_{x\to-1}(x^2+3) = 4$ *c)* $lim_{x\to2}(x^2+3x-1) = 9$

d) If $lim_{x\to5} f(x) = 4$ *then* $lim_{x\to5} f(x) \neq 2$ *e)* $lim_{x\to9}\left(2+\sqrt{x}\right) = 5$

f) $lim_{x\to1}\frac{(x+3)}{1+\sqrt{x}} = 2$

2. Find the limits of the following.

a) $lim_{x\to+\infty}\frac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}$ *Where* $c_n \neq 0$ *and* $d_m \neq 0$ *and where* $c_{0,}c_{1,\ldots,}c_{n,}d_0, d_{1,\ldots,} d_m\epsilon\mathbb{R}$ *b)* $\lim_{x\to+\infty}\frac{1-\cos x}{x+2}$ *c)* $\lim_{x\to+\infty}\frac{\cos^2(2x)}{3-2x}$ *d)* $lim_{x\to\infty}\frac{5x^2-\sin(3x)}{x^2+10}$ *e)* $lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$

f) $lim_{x\to\infty}\frac{\sqrt{9x^6-x}}{x^3+1}$ *g)* $lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$

h) $lim_{x\to\infty}\cos x$

3. Find two function f and g such that $lim_{x\to0} f(x)$ *and* $\lim_{x\to0} g(x)$ *does not exist but* $lim_{x\to0}(f(x)g(x))$ *exists.*

4. $lim_{x\to a}[f(x)+g(x)] = 2$ *and* $lim_{x\to a}[f(x)-g(x)] = 1$ *Then find* $lim_{x\to a}[f(x)g(x)]$, *provided that the limits exist at a.*

5. Explain why the function is discontinuous at the given number a

a) $f(x) = ln|x-2|$ $a = 2$

b) $f(x) = \begin{cases} \frac{1}{x-1} & if\ \ x \neq 1 \\ 2 & if\ \ x = 1 \end{cases}$ $a = 1$

c $f(x) = \begin{cases} e^x & if\ \ x < 0 \\ x^{2v} & if\ \ x \geq 0 \end{cases}$ $a = 0$

6. Show that the number at which the function

$$f(x)=\begin{cases} x+2 & if \;\; x<0 \\ e^{x} & if \;\; 0\le x\le 1 \\ 2-x & if \;\; x>1 \end{cases}$$

is discontinuous?At which of these points is f continuous from the right, from the left.

7. *For what value of the constant c is the function f continuous on $(-\infty,\infty)$?*

$$f(x)=\begin{cases} cx^2+2x & if \;\; x<2 \\ x^3-cx & if \; x\ge 2 \end{cases}$$

8. *Find the values of a and b that make f continuous every where.*

$$f(x)=\begin{cases} \dfrac{x^2-2}{x-2} & if \;\; x<2 \\ ax^2-bx+3 & if \; 2<x<3 \\ 2x-a+b & if \;\; x>3 \end{cases}$$

9. *Show that f is continuous on $(-\infty,\infty)$.*

a) $f(x)=\begin{cases} x^2 & if \;\; x<1 \\ \sqrt{x} & if \; x\ge 1 \end{cases}$

b) $f(x)=\begin{cases} \sin x & if \;\; x<\frac{\pi}{4} \\ \cos x & if \; x\ge \frac{\pi}{4} \end{cases}$

10.Use the intermedaite value Theorem to show that there is a root of the given equation in the specified interval.

a) $x^4+x-3=0,\; (1,2)$ *b)* $\cos x = x,\; (0,1)$ *c)* $\ln x = e^{-x},\; (1,2)$

d) $\sqrt[3]{x}=1-x\,,\;\; (0,1)$ *e)* $x^3+2x-1\,,\;\; (0,1)$

References

1. James Stewart, Calculus early trancedental, 6th ed., Prentice Hall, 2008
2. Howard Anton,Calculus a new horizon, 6th ed., John wiley and Sons Inc
3. Robert Ellis and Denny Gulick, Calculus with analytic geometry, 6th ed, Harcourt Brace Jovanovich, Publishers, 5th ed, 1993
4. Adams, Calculus: A complete course, 5th ed, Addison Wesley, 2003

5. R. Wrede and M. R. Spiegel, Theory of advanced calculus, 2nd ed., McGraw-Hill, 2002.
6. A. E. Taylor and W. R. Mann, Advanced calculus, 3rd ed, John-Wiley and Son, INC, 1995.
7. R. T. Smith and R. B. Minton, Calculus concepts and connections, McGram-Hill book company, 2006
8. D. V. Widder, Advanced calculus, Prentice-Hall, 1979
9. Ross L. Finney et al, Calculus, Addison Wesley, 1995
10. E. J. Purcell and D. Varberg, Calculus with analytic geometry, Prentice-Hall INC., 1987
11. Lynne, Garner: Calculus and Analytic Geometry, Dellen Publishing Company
12. John A. Tierney: Calculus and Analytic, 4th edition, Allyn and Bacon, Inc. –Boston.
13. Earl W. Swokowski: Calculus with Analytic Geometry, 2nd edition, Prindle, Weber and Schmidt.
14. Goldstein, L. J., Lay D. C. , and Schneider D. I (1987). Calculus and its applications, 4th, edn, Prentice-Hall, Inc, London
15. Kresying, Advanced Engineering Mathematics.
16. Alan Jeffry , Advanced Engineering Mathematics

CHAPTER-FOUR
DERIVATIVES AND APPLICATION OF DERIVATIVES

Objectives

At the end of this chapter, students will be able to:

- Define the derivative and differntiability of a function;

- Find the slope and the equation of the tangent line to the graph of a given function at a given point;
- State and prove some techniques of differntiation;
- Compute the derivative of a given function;
- To find higher order of derivative of functions.

Introduction

In this chapter,we will discuss the concept of tangent line and normal line to graph of a function at a given point. With the concept of geometric interpretation of tangent line to a curve at a point,we will see the formal definition of the derivative of a function at point ,which will be followed by the derivative of a function at any point along with the usual notations for the derivative. We will also duscuss the concept of differentiablility and its relationship to continuity. We will also develop the Chain Rule to find the derivative of composition of functions. we will also emphasize the application of the derivative to graphing functions. We will learn how to determine where the graph of a differentiable function rise and where it falls: where it has peak and where it has valleys: where it curves upward and where it curves downward. The concepts we will introduce have application not only to graphing functions but also to problems in such widely varying areas.

4.1- Definition of derivatives; basic rules

Objectives:

At the end of this section, students will be able to:

- Develop formulas to differentiate different types of functions;
- Find the derivatives of functions using an appropriate method.

Definition 4.1: Let a be a number in the domain of a function f. If

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad (1)$$

exists, we call this limit the derivative of f at a and write it $f'(a)$, so that

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad (2)$$

If the limit in (2) exists, we say that f has a derivative at a, and f is defferentiable at a or that $f'(a)$ exists.

The derivative of a function f at a point a can also be defined as

$$f'(a) = \ lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad (3)$$

if this limit exists.

Formula (3) is obtained by replacing x by $a+h$ and $x-a$ by h in formula (2).

Example 1: Find the derivative of the function $f(x) = x^2 - 8x + 9$ at a number a

Solution: From definition 2.1 we have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h\to 0} \frac{(2a + h - 8)}{h}$$

$$= 2a - 8 \qquad \blacksquare$$

Differentaible functions

The derivative of function may or may not exist at particular point x. If the limit in (3) fails to exist, (say at a) we say f is not differentaible at a.

Definition 4.2: Let a function f be defined in open interval containig the point c. The function is **differentaible** at c if and only if the derivative f′(c) exists. If f is differentaile at every point of its domain we say simply that f is **differentaible**.

Example 2: Let $f(x) = x^2$. Then determine the set of values of x for which f is differentaible.

Solution:

Let x ϵ ℝ

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} x^2$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x$$

Since this expression is defined for every real number x, f is defferentaible in the whole real line. ■

Theorem 4.3: If f differentaible at point a, then it is continuous at a.

Proof: To prove that f is continuous at a, we have to show that $\lim_{x\to a} f(x) = f(a)$.

We do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. Now divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$)

Taking the limit on both side of equation, we get

$$\lim_{x\to a}(\ f(x) - f(a)) = \lim_{x\to a} \frac{[f(x) - f(a)](x - a)}{x - a}$$

$$= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x - a)$$

$$= f'(a)(0)$$

$$\Rightarrow \quad \lim_{x\to a} f(x) = f(a)$$

Therefore, f is continuous at a.

The converse of this theorem is not true ; that is ,there are functions that are continuous at a point but not differentaible at that point.

Example 3: Show that the function $f(x) = |x|$ is continuous at 0, but not differentiable at 0.

Solution: The function $f(x) = |x|$ continuous at 0, since

$$\lim_{x\to o^-} |x| = \lim_{x\to 0^+} |x| = 0 = f(0)$$

If we compute the derivetive, we obtain

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}$$

For $x > 0$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

And for $x < 0$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

Thus $lim_{x\to 0}\frac{|x|}{x}$ does not exists, because the right and the left hand side limits are not equal. Therefore, f is not differentaible at 0. ■

Notation: Alternative notations for $f'(x)$ of a function $y = f(x)$ are y',$\frac{dy}{dx}$,$\frac{df(x)}{dx}$,$D_x f$, $D_x y$ f', f_x etc.

Exircses

1. Find $f'(x)$.

a) $f(x) = 3 - 2x + 4x^2$ *b)* $f(t) = t^4 - 4t$ *c)* $f(x) = 5\pi^4 + 6$

d) $f(t) = t^{-2} + 4t$ *e)* $f(t) = \frac{2t+1}{t+3}$ *f)* $f(x) = \frac{x^2+1}{x-2}$

g) $f(x) = \frac{1}{\sqrt{x+2}}$ *h)* $f(x) = \sqrt{3x+1}$

2. Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

a) $lim_{h\to 0}\frac{(1+h)^{10}-1}{h}$ *b)* $lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h}$ *c)* $lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{x-\pi/4}$

d) $lim_{x\to 5}\frac{2^x-32}{x-5}$ *e)* $lim_{h\to 0}\frac{\cos(\pi+h)+1}{h}$ *f)* $lim_{t\to 1}\frac{t^2+t-2}{t-1}$

3. Determine whether $f'(0)$ exists.

a) $f(x) = \begin{cases} x \sin\frac{1}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

b) $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

4. *Given* $f(x) = \begin{cases} x^2 \cos\frac{1}{x} & if\ \ x \neq 0 \\ 0 & if\ \ x = 0 \end{cases}$

Show that :

a) f *is continuous for all values of* x.

b) f *is differentiable for all value of* x.

c) f' *is not continuous at* $x = 0$.

Tangent Lines

Activity

Q1.What is a tangent line? a normal line?

Q2. Is any line that touches a curve at one point a tangent line?

For circle, tangency is striaght forward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the raduis at P (Figure 4.1). Such a line just touches the circle. But what does it mean to say that a line L is tangent to some other curve C at a Point P? Generalizing from the geometry of the circle, we might say that it means one of the following:

1. *L* passes through *P* perpendicular to the line from *P* to the center of *C*.

2. *L* passes through only one point of *C*, namely *P*.

3. *L* passes through *P* and lies on one side of *C* only.

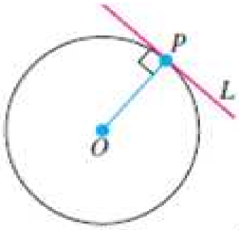


Figure 4.1 Tangent line

L Passes through P perpendicular to radius OP.

Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at point of tangency (Figure 4.2).

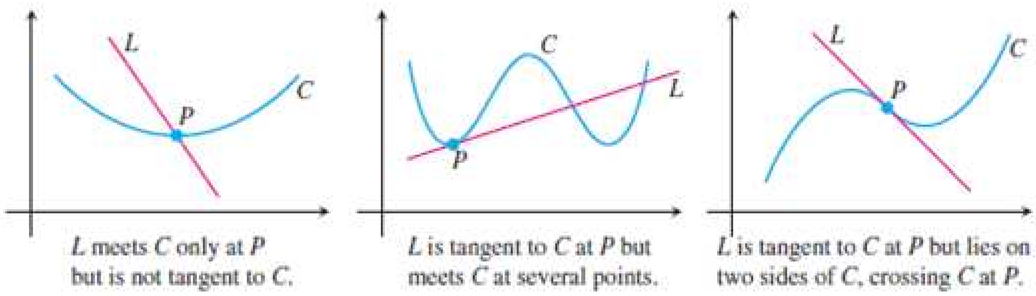


Figure 4.2

In this section, we make use of limit concept to find the equation of a line tangent to the graph of a function at a given point.

Definition 4.4: The tangent line to the curve $y = f(x)$ at the point $p(a, f(x))$ is the line through p with slope

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \tag{4}$$

provided that this limit exists.

Example 1*:*

Find an equation of the tangent line to the parabola $y = x^2$ at the point $p(1,1)$.

Solution: Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x\to 1}\frac{x^2-1}{x-1}$$
$$= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$
$$= \lim_{x\to 1}(x+1) = 1+1 = 2$$

Using the point slope form of the equation of the line, we find that an equation of the tangent line at $p(1,1)$ is $y-1 = 2(x-1)$ or $y = 2x-1$ ■

There is another expression for the slope of tangent line that is sometimes easier to use.

Let $h = x - a$ so that $x = a + h$.

So the slope of the secant line PQ is

$$M_{PQ} = \frac{f(a+h)-f(a)}{h} \qquad (5)$$

See Figure 4.3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$. However, Q would be to the left of P.

Notice that as x approaches a, h approaches 0 (because $h = x - a$) so that the expression for the slope of the tangent line in definition 4.4 becomes

$$M = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad (6)$$

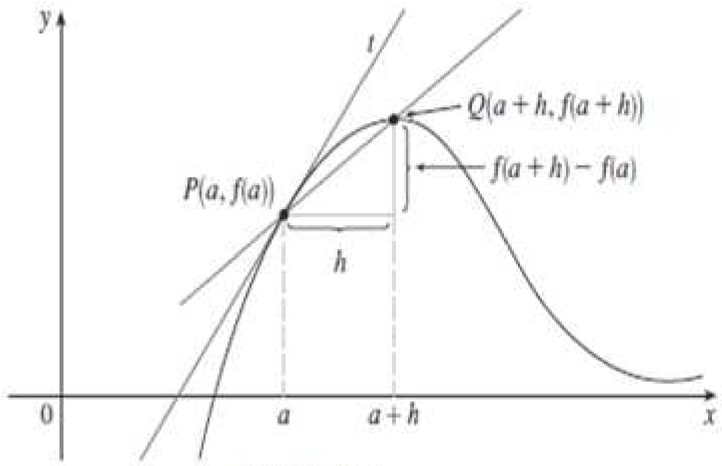


Figure 4.3

Example 2:

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3,1).

Solution: Let $f(x) = 3/x$. Then the slope of tangent at (3,1) is

$$m = \lim_{h \to 0} \frac{f(3+h)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)}$$

$$= \lim_{h \to 0} -\frac{1}{3+h} = \frac{-1}{3}$$

Therefore, the equation of the tangent at the point (3,1) is

$$y - 1 = -\frac{1}{3}(x - 3)$$

which simplifies to $x + 3y - 6 = 0$ ■

Definition 4.5: The slope of the line tangent to the graph of the function y = f(x) at $(a, f(a))$ is equal to

$f'(a)$, the derivative of f at a.

The geometric interpretion of a derivative is shown in Figure 4.4.

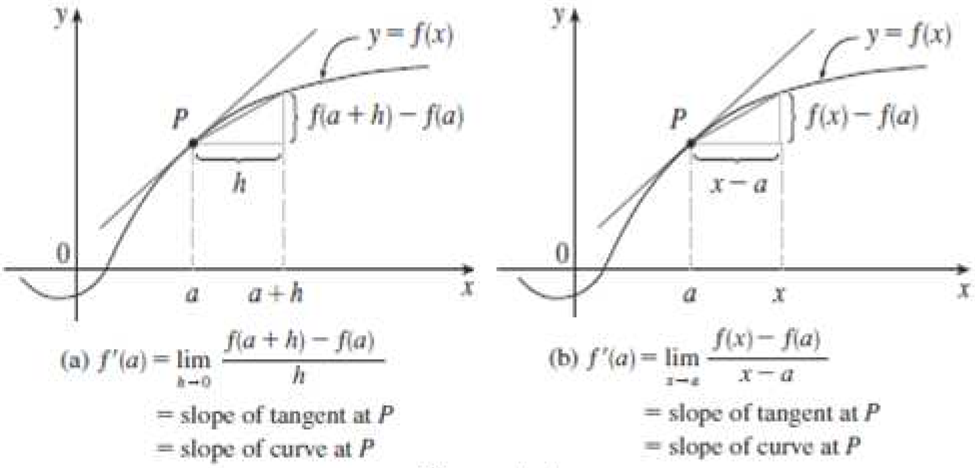


Figure 4.4

If we use the point - slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Example 3: Find the equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point (3,-6).

Solution: We know that the derivetive of $\mathrm{y} = \mathrm{x}^2 - 8\mathrm{x} + +9$ at the number a is $f'(a) = 2a - 8$. Therefore, the slope of the tangent line at (3,-6) is

$$f'(3) = 2(3) - 8 = -2$$

Thus, the equation of the tangent line to the parabola is:

$$y - (-6) = (-2)(x - 3) \quad or \; y = -2x$$ ■

Definition 4.6: Let f be continuous at a. If

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \infty \qquad or \qquad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = -\infty$$

Then we say that the graph of f has a vertical tangent line at $(a, f(a))$. In that case the vertical line $x = a$ is called the line tangent to the graph of f at a.

Example 4*:*

Let $f(x) = x^{1/3}$. Show that the graph of f has vertical tangent line at (0,0) and find an equation for it.

Solution: We observe that f is continuous at 0 an d that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0} x^{-1/3} = \infty.$$

By definition 2.6 the graph has a vertical tangent line at $(0,0)$ and an equation of the tangent is $x = 0$.

Normal Lines

Definition 4.7: A line is said to be *normal* to a curve at point P, if it is perpendicular to the tangent line at P.

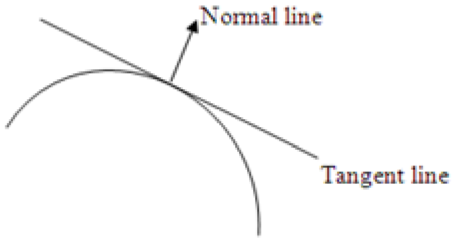


Figure 4.5 Illustration of normal line

If the tangent line has slope $\mathrm{m}_{\tan}$ then the normal line will have slope $\frac{-1}{m_{tan}}$, and the equation of the normal line to the curve at $(a, f(a))$ is given by

$$y - f(a) = \frac{-1}{m_{tan}}(x - a).$$

It is clear that if the tangent line is horizontal then the normal lne will be vertical, and vice varsa.

Example 5: Find the equation of the normal line to the function $\mathrm{f(x) = x^2}$ at $(2,4)$

Solution: Let us compute the slope of the tangent line to the curve at $(2,4)$. This is done in the following ways.

$$m = \lim_{h\to 0}\frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0}\frac{(2+h)^2 - 4}{h}$$

$$= \lim_{h\to 0}\frac{4 + 4h + h^2 - 4}{h}$$

$$= lim_{h\to 0}\frac{h(4+h)}{h} = \lim_{h\to 0} 4 + h = 4.$$

That is $m_{tan} = 4$, so the normal line is will have slope $m = \frac{-1}{m_{tan}} = \frac{-1}{4}$, hence the equation of normal line is

$$y - f(2) = \frac{-1}{m_{tan}}(x - 2)$$

$$\Leftrightarrow y-4=\frac{-1}{4}(x-2)=4y-16=-x+2$$

Or

$$4y+x+18=0 \qquad \blacksquare$$

Execises

1. Find the slope of the tangent line to the curve $y=x^3$ *at the point* $(-1,-1)$

a) using definition 1

b) using equation 3

2. Find an equation of the tangent line to the curve given point.

a) $y=1+2x-x^3, \quad (1,2)$

b) $y=\sqrt{2x+1}\,, \quad (4,3)$

c) $y=(x-1)/x-2, \quad (3,2)$

d) $y=2x/(x+1)^2, \quad (0,0)$

3. a) Find the slope of the tangent to curve $y=2/(x+3)$ *at the point* $x=a$.

b) Find the slope of the tangent lines at the point whose x-coordinates ar $x=-1,\ x=0$ *and* $x=1$

4. Compute tangent line if it exists for each curve at the given value of a.

a) $y=3x, \quad a=2$ *b)* $y=x^{2/3} \quad a=0$ *c)* $y=mx+b,$ *any a*

d) $y=ax^2+bx+k,$ *any a* *e)* $y=\sin x \quad a=0$ *f)* $y=\frac{1}{x+1}$

5. Find the equation of tangent line if it exists to the given curve at given point .

a) $y=x^2-x\,, \quad (1,0)$ *b)* $y=\frac{1}{x}\,, \quad (1,1)$ *c)* $y=\frac{1}{1-x}\,, \quad (2,1)$

d) $y=\sqrt{x+1}\,, \quad (3,2)$ *e)* $y=|x|\,, \quad (0,0)$ *f)* $y=\frac{1}{\sqrt{x}}, \quad (1,1)$

g) $y=\cos x\,, \left(\frac{1}{4\pi},\frac{1}{2\pi}\right)$

6. Explian why there is no tangent line to the given curve at the given point.

a) $y=\frac{|x-2|}{x-2}, \quad a=2$ *b)* $y=\sqrt{x}\,, \quad a=0$ *c)* $y=x^{2/3}, \quad a=0$

d) $y=|1-x^2|, \quad a=-1$ *e)* $y=|x-1|, \quad a=1$ *f)* $y=\frac{1}{x}, \quad a=0$

7. Find the equation for the normal line to the given curve at the given point.

a) $y = 1 - x^2,\ \left({}^{1}/_{2}, {}^{-1}/_{4}\right)$ *b)* $y = x^{2/3},\ (8,4)$ *c)* $y = \sqrt{4 - x^2},\ (0,2)$

d) $y = \sin x,\ (0,0)$ *e)* $y = x^2,\ (2,4)$ *f)* $y = \sqrt{4 - x^2}\ ,\ \left(\sqrt{2}, \sqrt{2}\right)$

Basic Rules of Derivatives

Here, we will discuss rules stated as theorems that help us differentiate combinations of functions. The proofs of these theorems depend mainly on the appropriate limit theorems.

Rule 1 :Derivative of constant function.

If f has the constant value $f(x) = c$, then $\frac{d}{dx} f(x) = \frac{d}{dx}(c) = c$.

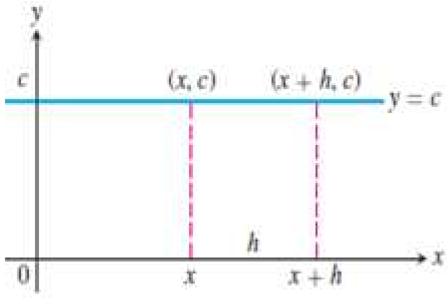


Figure 4.6 $\frac{d}{dx}(c) = 0$

Example 1: Let $f(x) = 8$ then

$$\frac{d}{dx} f(x) = \frac{d}{dx} 8 = 0$$

Similarly, $\frac{d}{dx} \frac{(-\pi)}{4} = 0$

Rule 2: Power rule

If n is nonnegative integer and $f(x) = x^n$, then f is defferentaible on the set of real numbers $\mathbb{R}$ and is given by $f'(x) = nx^{n-1}$

Proof: when $n = 0$, then $f(x) = 1$ thus f is constant function , and

$$f'(x) = 0.$$

When n is any positive integer ,then we have form the definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

By the binomial expansion theorem, we have

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ ... + h^n$$

so that $$(x+h)^n - x^n = h[nx^{n-1} + \tfrac{n(n-1)}{2}x^{n-2}h + \ ... + h^{n-1}]$$

$$\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{h[nx^{n-1} + n(n-1)^{x-2}h + \cdots + h^{n-1}]}{h}$$

$$= \lim_{h\to 0}[nx^{n-1} + n(n-1)x^{n-2}h + \cdots + h^{n-1}]$$

$$= nx^{n-1}.$$

Therefore, $f'(x) = nx^{n-1}$

Example 1: Let $y = x^{100}$. Then find $\frac{dy}{dx}$.

Solution: By the power rule we obtain $\frac{dy}{dx} = 100x^{100-1} = 100x^{99}$. ■

The power rule (general version)

If n is any real number , then $\frac{dy}{dx}(x^n) = nx^{n-1}$.

Activities: Differentiate

a. $f(x) = \frac{1}{x^2}$ *b.* $y = \sqrt[3]{x^2}$ *c.* $g(x) = \frac{1}{\sqrt{x+2}}$ *d)* $f(x) = 8\pi^{15}$

Rule 3: The constant multiple

If c is a constant and f is differentaible function, then $\frac{dy}{dx}[cf(x)] = c\frac{dy}{dx}f(x)$.

Proof: Let $g(c) = cf(x)$, then

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h\to 0}\frac{c[f(x+h) - f(x)]}{h}$$

$$= c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = c\frac{dy}{dx}f(x).$$

Example 2: $\frac{dy}{dx}3x^4 = 3\frac{dy}{dx}x^4 = 3(4x^3) = 12x^3$

Rule 4 : Sum rule

If the functions f and g are differentiate at a point a, then so are $f+g$ and $f-g$ and

1. $(f+g)'(a) = f'(a) + g'(a)$ sum rule
2. $(f-g)'(a) = f'(a) - g'(a)$ difference rule

Proof

1. Using the limit theorem , we find that

$$(f+g)'(a) = \lim_{x\to a}\frac{(f+g(x)-(f+g)(a)}{x-a} = \lim_{x\to a}\frac{f(x)+g(x)-f(a)-g(a)}{x-a}$$

$$= \lim_{x\to a}\frac{(f(x)-f(a)+g(x)-g(a))}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$

$$= f'(a)+g'(a)$$

Proof of 2 is similar with that of 1.

This can also be extended to finite number differetaible function as:

$$(f_1+f_2+f_3+\cdots+f_n)' = {f_1}'+{f_2}'+{f_3}'+\cdots+{f_n}'.$$

Class activities*: Find the derivative of*

a. $f(x) = x^2(5x^2-3x)$ *b.* $g(x) = (x+1)^3$ *c.* $y = (3x^2-2)^2$

Rule 5 :The product rule

If f and g are differentaible, then the product fg is differentailble, and

$$(fg)'(x) = f'(x)g(x)+f(x)g'(x)$$

Proof: Let $F(x) = f(x)g(x)$. We need to show that F is differentaible by finding its derivative.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x)\ \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

Now, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$ and $\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = g'(x)$

Since g is differentaible it is continuous so $\lim\limits_{h\to 0} g(x+h) = g(x)$.

Therefore by limit theorem ,we have $(fg)'(x) = f'(x)g(x)+f(x)g'(x)$

Example 3*:*

a) Let $f(x) = (x^2+1)(x^4-1)$ then find $f'(x)$

b) $f(x) = xe^x$ then find $f'(x)$,

Solution: a) Let $g(x) = x^2+1$ so that $g'(x) = 2x$ and $h(x) = x^4-1$ so that $h'(x) = 4x^3$. By product rule, we have

$$f'(x) = h'(x)g(x)+h(x)g'(x)= 4x^3(x^2+1)+(x^4-1)2x$$
$$=4x^5+4x^3+2x^5-2x=6x^5+4x^3-2x$$

b) By product rule we have

$$f'(x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$
$$= xe^x + e^x \cdot 1 = (x+1)e^x$$ ■

Rule 6: The quotient rule

If f and g are differentaibe at a, and $g(a) \neq 0$ then f/g is differentaible at a and

$$(f/g)'(a) = \frac{f'(a)\,g(a) - f(a)g'(a)}{[g(a)]^2}$$

Proof : Since $g'(a)$ exists by hypothesis, it follows that g is continuous at a so that $lim_{x\to a}\, g(x) = g(a)$, because $g(a) \neq 0$ by hypothesis.

Therefore, f/g is defined throught some open interval about a , and the following limits exist:

$$(f/g)'(a) = \lim_{x\to a}\frac{f/g(x) - f/g(a)}{x-a}$$
$$= \lim_{x\to a}\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x\to a}\frac{f(x)g(a) - f(a)g(x)}{(x-a)g(x)g(a)}$$
$$= \lim_{x\to a}\frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{(x-a)g(x)g(a)}$$
$$= \lim_{x\to a}\frac{f(x)g(a) - f(a)g(a)}{(x-a)g(x)g(a)} + \lim_{x\to a}\frac{f(a)g(a) - f(a)g(x)}{(x-a)g(x)g(a)}$$
$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{(x-a)}\frac{g(a)}{g(a)g(x)}\right) - \lim_{x\to a}\left(\frac{f(a)\cdot g(a) -}{(x-a)}\frac{g(a)}{g(a)g(x)}\right)$$
$$= \frac{f'(a)\,g(a)}{[g(x)]^2} - \frac{g'(a)\,f(a)}{[g(x)]^2} = \frac{f'(a)\,g(a) - f(a)g'(a)}{[g(x)]^2}$$

Therefore,

$$(f/g)'(a) = \frac{f'(a)\,g(a) - f(a)g'(a)}{[g(a)]^2}$$

Example 4: $y = \frac{x^2+x-2}{x^3+6}$. Then find y'.

Solution:

$$y' = \frac{(x^3+6)\frac{dy}{dx}\left(x^2+x-2) - (x^2+x-2)\frac{dy}{dx}(x^3+6\right)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1)-(x^2+x-2)(\ 3x^2)}{(x^3+6)^2}$$

$$= \frac{(2x^4+x^3+12x+6)-(3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2 12x+6}{(x^3+6)^2} \qquad \blacksquare$$

Rule 7: Reciprocal rule

If g is differentaible function, then $\frac{1}{g}$ is differentaible whenever it is defined, and

$$\frac{dy}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{(g(x))^2}$$

Activities.

Differentaite the following function.

a. $f(x) = \frac{3x}{x+2}$ b. $y = 6x^{-2}$ c. $\frac{2x^3-x+1}{x^2+1}$

We smmarize the differentiation formulas we have learned so far as follows.

1. $\frac{d}{dx}(c) = 0$
2. $(cf)' = cf'$
3. $(fg)' = f'g + g'f$
4. $(f+g)' = f' + g'$
5. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
6. $(f-g)' = f' - g'$
7. $\frac{d}{dx}(x^n) = nx^{n-1}$

Exercises

1. Find the derivative of $y = (x^2+1)(x^3+1)$ *in two ways: by using the product Rule and performing the multiplication first. Do your ansewrs agree?*

2. Find the derivative of the function

$$f(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$

In two ways:by using the Quotient Rule and by simplifing first. Show that your answers are equivalent.Which method do you prefer?

3. *Differentiate.*

a) $f(x) = (x^3 + 2x)e^x$

b) $y = \frac{e^x}{x^2}$

c) $g(x) = \frac{e^x}{1+x}$

d) $y = \sqrt{x}e^x$

e) $g(x) = \frac{3x-1}{2x+1}$

f) $y = (2x^3)(x^4 - 2x)$

g) $y(u) = (u^{-2} + u^{-3})(u^5 - 2u)$

h) $f(y) = \left(\frac{1}{y^2} - \frac{4}{y^4}\right)(y + 5y^3)$

i) $r(t) = (1 + e^{2t})\left(3 - \sqrt{t}\right)$

j) $f(t) = \frac{2t}{t^2+4}$

4. *Suppose that* $f(5) = 1, f'(5) = 6,$ $g(5) = -3$ *and* $, g'(5) = 2$ *then find the following values.*

a) $(fg)'(5)$ b) $(g/f)'(5)$ c) $(f/g)'(5)$

5. *Suppose that* $f(2) = -3,$ $g(2) = 4.$ $f'(2 = -2) and$ $g'(2) = 7.$ *Find* $h'(2), if$

a) $h(x) = 5f(x) - 4g(x)$

b) $h(x) = \frac{f(x)}{g(x)}$

c) $h(x) = f(x)g(x)$

d) $h(x) = \frac{g(x)}{1+f(x)}$

6. *If* $f(x) = e^x g(x)$, *where* $g(0) = 2$ $g'(0) = 5$,*Find* $f'(0)$.

7. *If* $h(2) = 4$ *and* $h'(2) = -3$ *find* $\left.\frac{d}{dx}\left(\frac{h(x)}{x}\right)\right|_{x=2}$

8. *Find equation of the tangent line and normal line to the given curve at the specified point.*

a) $y = 2xe^x$, (0,0) b) $y = \frac{\sqrt{x}}{x+4}$, (4,0.4)

9. *If f is a differentiable function, find an expression for the derivative of each of the following functions.*

a) $y = x^2 f(x)$ c. $y = \frac{f(x)}{x^2}$ b) $y = \frac{x^2}{f(x)}$ d) $y = \frac{1+xf(x)}{\sqrt{x}}$

10. *Let* $P(x) = F(x)G(x)$ *and* $Q(x) = F(x)/G(x)$ *,where F and G are the functions whose graphs are shown.*

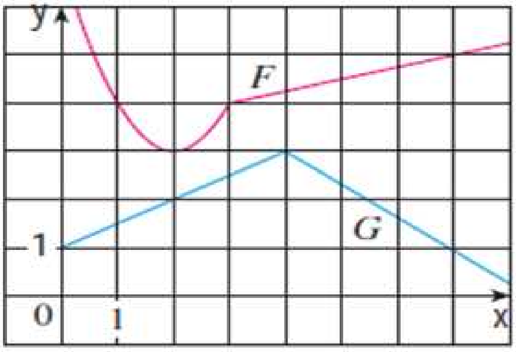


Figure 4.7

Then find;

a) $p'(2)$ b) $Q'(7)$

11. *Find equations of the tangent lines to the curve*

$$y = \frac{x-1}{x+1}$$

That are parallel to the line $x - 2y = 2$.

12. *a)Use the poduct Rule twice to prove that if* f, g *and* h *are differentiable,then*

$$(fgh)' = f'gh + fg'h + fgh'$$

b) *Taking* $f = g = h$ *in part (a), show that*

$$\frac{dy}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

Derivatives of Polynomial Functions

The constant multiple rule, sum rule the difference rule can be combined with the power rule to find any polynomial, as the following example.

Example 1: Given the polynomial function $f(x) = x^8 + 12x^5 + 10x^3 - 6x + 5$, its derivative is given as

$$f'(x) = \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$
$$= 8x^7 + 12(5x^4) + 10(3x^2) - 6(1) + 0$$
$$= 8x^7 + 60x^4 + 30x^2 - 6$$

Example 2: Find the horizontal tangents of the curve $y = x^4 - 2x^2 + 2$

Solution: The horizontal tangent, if any, occurs at the points where $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$$

Now solve the equation $\frac{dy}{dx} = 0$ for x.

$$4x^3 - 4x = 4x(x^2 - 1) = 0$$

$$\Rightarrow x = 0, x = 1, -1$$

Thus, $y = x^4 - 2x^2 + 2$ has any horizontal tangent at $x = 0, x = 1, x = -1$. ■

the corresponding points on the curve are (0.2),(1,1) and (-1,1) see Figure 2.8.

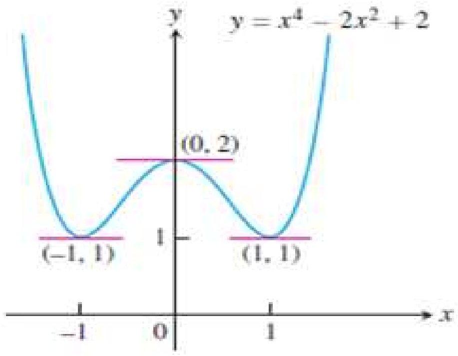


Figure 4.8 $y = x^4 - 2x^2 + 2$

Derivatives of Rational Functions

The Quotient Rule and the other differentiation formulas enable us to compute the derivative of any rational function, as the next example illustrates.

Example 3: Let $y = \frac{x^2+x-2}{x^3+6}$. Then

$$y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6)x^4 + x^3 + 12x + -(3x^4 + 3x^3 - x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Derivatives of Trigonometric Functions

Activity

1. *State and describe the basics of derivatives of trignometric functons.*
2. *Show that the derivative of* $tan\, x$ *is* $sec^2 x$.

In this discussion, we will find the derivatives of all the trignometric functions. There are six important formulas for differentiating trignometric functons.

Theorem 4.8:

The trignometric functons are differentiable whenever they are defined and

1. $(\sin x)' = \ cos\ x$
2. $(\cos x)' = \ -sin\, x$
3. $(\tan x)' = sec^2 x$
4. $(\cot x)' = -csc^2 x$
5. $(\sec x)' = \mathrm{sec\, x\, tan\, x}$
6. $(\csc x)' = -\mathrm{csc\, x\, cot\, x}$

Proof:

1. Using the definition of derivative, we have ,

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x(\cos x-1)+\sin h\cos x}{h}$$

$$=\sin x\lim_{h\to o}\frac{\cos h-1}{h}\ \cos x\lim_{h\to 0}\frac{\sin x}{h}$$

Notice that $\lim_{h\to o}\frac{\cos h-1}{h}=0$ and $\lim_{h\to 0}\frac{\sin x}{h}=1$

Therefore,

$$\frac{d}{dx}\sin x = \sin x\cdot 0+\cos x\cdot 1=\cos x$$

2. To show (3), we make use of the definition $\tan x = \frac{\sin x}{\cos x}$ and the quotient rule.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\frac{d}{dx}(\sin x)\cos x - \sin x\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$=\frac{\cos x\cos x+\sin x\sin x}{(\cos x)^2}$$

$$= \frac{sin^2x + cos^2x}{(\cos x)^2} = \frac{1}{(\cos x)^2} \qquad \text{why?}$$

$$= sec^2x \qquad \text{why?}$$

3. similar with the proof of (3).
4. To prove (5),we use the fact that $\sec x = \frac{1}{\cos x}$, and along with the reciprocal rule

$$(\cos x)^2 = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = -\frac{(\cos x)'}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \sec x \tan x$$

5. This can be shown in a similar manner as that of (5).

Example 4:

Differentiate the following functions.

a. $f(x) = x^2 \cos x + x \tan x$ b. $g(x) = \frac{x \cot x}{\sin x}$

Solution:

a. Using the sum and product rule,we get,

f'(x)=2xcos $x - x^2 \sin x + \tan x + x\, sec^2x$

b. Use the product and the quotient rule , we have,

$$g'(x) = \frac{d}{dx}\frac{(x \cot x) \sin x - x \cot x \frac{d}{dx} \sin x}{(\sin x)^2}$$

$$= \frac{(\cot x - x csc^2x) \sin x - x \cot x \cos x}{(\sin x)^2} \qquad \blacksquare$$

Exercises

1-24 Differentiate the given functions.

1. $f(x) = 3x^2 - 2\, cos\, x$ 2. $h(x) = csc\, x + e^x\, cot\, x$ 3. $y = sec^2x$
4. $f(x) = sin\, x + \frac{1}{2} cot\, x$ 5. $y = e^u(cos\, u + cu)$ 6. $y = \frac{1}{x} - \frac{1}{sin\, x}$
7. $f(t) = t^3\, cos\, t$ 8. $y = \frac{x}{4 - tan\, x}$ 9. $f(x) = (x^2 + 3x + 1)\, sin\, x$
10. $f(t) = \sqrt{t}\, sin\, t$ 11. $y = \frac{1 + sin\, x}{x + cos\, x}$ 12. $y = 2\, sin\, x\, cos\, x$
13. $y = 2\, csc\, x + 4tan\, x$ 14 . $f(\theta) = \frac{sec\, \theta}{1 + sec\, \theta}$ 15. $y = cos^2(x - 1)$
16. $y = \frac{1 - sec\, x}{tan\, x}$ 17. $y = csc\, \theta\, (\theta + cot\, \vartheta)$ 18. $y = \frac{x\, sin\, x}{1 - cos\, x}$

19. $y = \frac{\sin x}{x^2}$ 20. $y = x^2 \sin x \tan x$ 21. $f(x) = \frac{4x+3}{2x \sin x}$

22. f(x)=$xe^r \csc x$ 23. $y = \sec x \tan x$ 24. $y = \sqrt{\sin 2x}$

25. *prove that* $\frac{d}{dx}(\csc x) = -\csc x \cot x$. 26. *prove that* $\frac{d}{dx}(\sec x) = \sec x \tan x$.

27. *prove that* $\frac{d}{dx}(\cot x) = -\csc^2 x$.

28. *prove ,using the definitionof derivative,that if* $f(x) = \cos x$. *Then* $f'(x) = -\sin x$.

29-32 Find an equation of the tangent line to the curve at the given point.

29. $y = \sec x, (\pi/3, 3)$ 30. $y = e^x \cos x, \quad (0,1)$ 31. $y = x + \cos x, \quad (0,1)$

32. $y = \frac{1}{\sin x + \cos x}, \quad (0,1)$

33. *Let* $f(x) = \frac{\tan x - 1}{\sec x}$

a) *Use the Quotient Rule to differentiate the function*

b) *Simplify the expression for f(x) by write it in terms of* $\sin x$ *and* $\cos x$ *and then find* $f'(x)$.

c) *Show that your answers to parts (a) and (b) are equivalent.*

34. *Suppose* $f(\pi/3) = 4$ *and* $f'(\pi/3) = -2$ *and. Let* $g(x) = f(x) \sin x$ *and* $h(x) = \frac{\cos x}{f(x)}$. *Then find* $g'(\pi/3)$ *and* $h'(\pi/3)$.

35. *For what values of x does the graph of* $f(x) = x + \sin x$ *have a horizontal tangent?*

Derivatives of Exponential Functions

Let us try to find the derivative of the exponential function $y = a^x$ using the definition of a derivative .

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - f'(0)}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

The factor a^x does not depend on h, so we take it infront of the limit :

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Notice that the limit is the value of derivative of f at 0 . That is,

$$\lim_{h\to 0}\frac{a^h-1}{h}=f'(0)$$

Therefore, we have shown that the exponential function $y=a^x$ is differentiable at 0, and it is differntaible every where and

$$f'(x)=f'(0)a^x.$$

Example 5: Differentiate $y=e^{\sin x}$

Solution: Here the inner function is $g(x)=\sin x$ and the outer function is the exponential function $f(x)=e^x$. So by Chain Rule ,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin x}\right)=e^{\sin x}\frac{d}{dx}(\sin x)=e^{\sin x}\cos x \qquad ■$$

We can use chain Rule to differentiate an exponential function with any base $a>0$.

let $a=e^{\ln a}$ then $a^x=e^{(\ln a)x}$ and Chain Rule gives

$$\frac{d}{dx}(a^x)=\frac{d}{dx}\left(e^{(\ln a)x}\right)=e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$

$$=e^{(\ln a)x}\cdot\ln a=a^x\ln a$$

Because $\ln a$ is a constant, we have the formula

$$\frac{d}{dx}(a^x)=a^x\ln a \qquad (1)$$

Derivatives of Logarithmic Functions

In this section, we use implicit differentaition to find the derivatives of the logarithmic function $y=\log_a x$ and in particular, the natural logarithmic function $y=\ln x$.

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\ln x} \qquad (2)$$

Proof: Let $y=\log_a x$ then $a^y=x$. Differentiating this equation implicity with respect to x and using formula (1) we get

$$a^x(\ln a)\frac{dy}{dx}=1$$

and so

$$\frac{dy}{dx}=\frac{1}{a^x\ln a}=\frac{1}{x\ln a}$$

If we put $a = e$ in formula 1, then the factor $\ln a$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$.

$$\frac{dy}{dx}(\ln x) = \frac{1}{x} \qquad (3)$$

By comparing formula 1 and 2 ,we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus : The differentiation formula is simplest when $a = e$ become $\ln e = 1.$ In general , if we combine formula 2 with the chain rule we get

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)} \qquad (4)$$

Example 6: Differentiate $y = \ln(x^3 + 1)$.

Solution: let $g(x) = x^3 + 1$ and $g'(x) = 3x^2$ now by formula (3) we have

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(x^3 + 1)] = \frac{1}{x^3 + 1}\frac{d}{dx}\ (x^3 + 1) = \frac{3x^2}{x^3 + 1} \qquad \blacksquare$$

Example 7: Find $\frac{d}{dx}\ln(\sin x)$.

Solution: Using (3),we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x \qquad \blacksquare$$

Example 8: Differentiate $f(x) = \log_{10}(2 + \sin x)$.

Solution: Using formula 1 with $a = 10$, we have

$$f'(x) = \frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) = \frac{\cos x}{(2 + \sin x)\ln 10} \qquad \blacksquare$$

Example 9: Find f'(x) if $f(x) = \ln|x|$.

Solution : Since $f(x) = \begin{cases} \ln x & if\ x > o \\ \ln(-x) & if\ x < 0 \end{cases}$

It follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & if\ x > 0 \\ \dfrac{1}{-x}(-1) \ = \dfrac{1}{x} & if\ x < 0 \end{cases}$$

Thus f'(x)= $\frac{1}{x}$ for all $x \neq 0$.

The result is

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \blacksquare$$

Logarithmic differentiation

The calculation of deravative of complicated functions involving product, quotient, or power can often be simplified by taking logarithms. The method used in the following example is called logarithmic differntiation.

Example 10: differentiate $y=\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution : We take logarithms of both sides of the equation and use the law of logarithms to simplify:

$$\ln y=\frac{3}{4}\ln x+\frac{1}{2}\ln(x^2+1)-5\ln(3x+2)$$

Diffentiating implicity with respect to x gives

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{4}\frac{1}{x}+\frac{1}{2}\frac{2x}{x^2+1}-5\frac{3}{3x+2}$$

Solving for dy/dx , we get

$$\frac{dy}{dx}=y\left(\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx}=\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right) \qquad \blacksquare$$

Steps in logarithmic differentiation:

1. Take natural logarithms of both sides of an equation $y=f(x)$ and use the laws of logarithms to simplify;
2. Differentiate with respect to x;
3. Solve the resulting equation for y'.

The Power Rule:

If n is any real number and $f(x)=x^n$, then

$$f'(\text{x})=nx^{n-1}$$

Proof: Let $y=x^n$ and use logarithmic differntiation:

$$\ln|y|=\ln|x|^n=n\ln|x| \qquad x\neq 0$$

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence $y' = n\frac{y}{x} = nx^{n-1}$

Remark:

You should distinguish carefully between the Power Rule $[(x^n)' = nx^{n-1}]$ where the base is variavble and the exponent is constant, and the rule for differentiating exponential functions $[(a^x)' = a^x \ln a]$, where the base is constant and the exponent is variable.

In general there are four cases for exponent and bases:

1. $\frac{d}{dx}(a^b) = 0$ (a and b are constants)
2. $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$
3. $\frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)}(\ln a)g'(x)$
4. To find $\frac{dy}{dx}[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

Example 11: Differenttiate $y = x^{\sqrt{x}}$.

Solution: Using logarithmic differentiation , we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}\right) \quad \blacksquare$$

Another method is to write $x^{\sqrt{x}} = \left(e^{\ln x}\right)^{\sqrt{x}}$

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$

Activity:

Show that $lim_{x\to 0}(1+x)^{1/x} = e.$

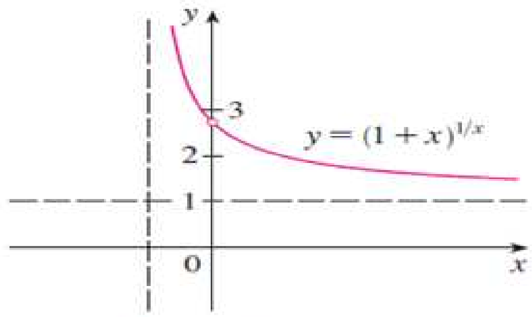


Figure 4.9

Note: $e \approx 2.7182818$

If we put $n = 1/x$ in the above limit, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \cos 3x$$

The differentiation formulas that we have learned so far do not enable us to calculate $F'(x)$. Observe that F is a composite function. In fact if we let $f(x) = 3x$ and $g(x) = \cos x$ and $F(x) = g \circ f$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F(x) = g \circ f$ in terms of the derivatives of f and g.

Theorem 4.9 (The chain rule)

If f is differentiable at a and g is differentiable at $f(a)$. Then $g°f$ is differentiable at a, and

$$(g \circ f)'(a) = g'\big(f(a)\big)f'(a)$$

The chain rule can be written in the form

$$\frac{d}{dx}\Big(g\big(f(x)\big)\Big) = g'\big(f(x)\big)f'(x)$$

Example 1: Let $(x) = \cos 3x$. Find a formula for $F'(x)$.

Solution: Let $f(x) = 3x$ and $g(x) = cos\ x$. Then $F = g \circ f$, since $f'(x) = 3$ and $g'(x) = -\sin x$ we conclude that

$$\begin{aligned} F'(x) = (g \circ f)'(x) &= g'(f(x))f'(x) \\ &= -\sin(3x)3 \\ &= -3\sin 3x \end{aligned}$$ ■

Example 2 Find a formula for $\frac{d}{dx}(\sqrt{x^2+1})$.

Solution: Let $F(x) = \sqrt{x^2+1}$, $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, so $F(x) = f(g(x))$ and $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = 2$. By chain rule

$$\frac{d}{dx}(\sqrt{x^2+1}) = \frac{d}{dx}(f{\circ}g)(x) = f'\big(g(x)\big)g'(x) = \frac{1}{2\sqrt{x^2+1}}\,2x = \frac{x}{\sqrt{x^2+1}}$$ ■

Activity 3.10

Differentiate a) $\boldsymbol{ln\sqrt{x}}$ b) $\boldsymbol{\left(\frac{x-2}{2x+1}\right)^5}$ c) $\boldsymbol{cos^2 x}$

The chain rule assumes a very suggestive form in the Leibniz notation.

Suppose the function f and g in the chain rule are already given and let

$u = f(x)$and $y = g(u)$. Then $y = g\big(f(x)\big)$, $\frac{du}{dx} = f'(x)$ and $\frac{dy}{du} = g'(u)$.

Therefore

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\Big(g\big(f(x)\big)\Big) &= g'(f(x))f'(x) \\ &= g'(u)f'(x) \\ &= \frac{dy}{du}\,\frac{du}{dx} \end{aligned}$$

Or more concisely,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}} \qquad (8)$$

If $y = [g(x)]^n$ then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the chain rule and the power rule we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\ \frac{du}{dx}$$

Therefore $\frac{dy}{dx} = nu^{n-1}\ \frac{du}{dx} = n[g(x)^{n-1}]\ \ g'(x)$

If n is any real number and $u = g(x)$ is differentiable then

$$\boxed{\frac{d}{dx}(u^n) = nu^{n-1}\ \frac{du}{dx}} \qquad (9)$$

Example 3: Let $y = cos^4 x$. Find $\frac{dy}{dx}$

Solution: Put $u = \cos x$ and $y = u^4$. Then from (8), it follows that

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx} = \frac{d}{du}(u^4)\frac{d}{dx}(\cos x)$$

$$= 4u^3\ \ (-\sin x)$$

$$= -4\ cos^3 x \sin x. \qquad \blacksquare$$

Example 4: Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$.

Solution: Let $u = \frac{1}{x^2+x+1}$ and $y = \sqrt[3]{x}$. So, $\frac{du}{dx} = \frac{-(2x+1)}{(x^2+x+1)^2}$ and $\frac{dy}{dx} = \frac{1}{3}u^{-2/3}$

Then from (8) it follows that

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx} = \frac{d}{du}\left(\frac{1}{x^2+x+1}\right)\frac{d}{dx}(\sqrt[3]{x})$$

$$= \left(\frac{-(2x+1)}{(x^2+x+1)^2}\right)\frac{1}{3}u^{-2/3}$$

$$= \frac{1}{3}\left(\frac{1}{x^2+x+1}\right)^{-2/3}\frac{-(2x+1)}{x^2+x+1}$$

$$= -\frac{1}{3}(x^2 + x + 1)^{-4/3}\ (2x + 1) \qquad \blacksquare$$

Example 5: Suppose the radius r of a ballon varies with respect to time according to the equation $r = 1 + 2t$. Find the rate of change of the ballon's volume with respect to time.

Solution: Let V be the volume, then $V = \frac{4}{3}\pi r^3$, while by assumption $r = 1 + 2t$

Therefore (8) tells us that

$$\frac{dv}{dt} = \frac{dv}{dr}\ \frac{dr}{dt}$$

$$= 4\pi r^2\ 2$$

$$= 8\pi r^2 = 8\pi(1+2t)^2 \qquad \blacksquare$$

The compound chain rule

Activity:

Find $k'(x)$ where $k(x) = \sqrt[4]{sec(\tan x\,)}$

Let $k(x) = (h \circ g \circ f)(x) = h(g(f(x)))$ and f is differentiable at x, g differentiable at $f(x)$ and h differentiable at $g(f(x))$. Since

$$k(x) = h((g \circ f)(x))$$

first application of the chain rule yields

$$k'(x) = h'((g \circ f)(x)(g \circ f)'(x)$$

But

$$(g \circ f)'(x) = g'(f(x))\, f'(x)$$

So, $k'(x) = h'(g(f(x)))\, g'(f(x))\, f'(x)$

Therefore,

$$\boxed{k'(x) = h'(g(f(x)))\, g'(f(x))\, f'(x)} \qquad (10)$$

In the formula, the derivative of h at the number $g(f(x))$ appears first, then the derivative of g at the number $f(x)$ and finally the derivative of f at the number x.

Example 6: Let $k(x) = cos^3 4x$. Find $k'(x)$ and calculate $k'(\pi/_6)$

Solution: Let $h(x) = x^3$, $g(x) = \cos x$ and $f(x) = 4x$

Then $k(x) = h(g(f(x)))$, $h'(x) = 3x^2$, $g'(x) = -\sin x$, and $f'(x) = 4$

From (9) we have

$$k'(x) = h'((g^\circ f)(x)g'(f(x))f'(x)$$

$$= 3(\cos 4x)^2\, (-\sin 4x).4$$

$$= -12\, cos^2 4x\ \sin 4x$$

In particular,

$$k'(\pi/_6) = -12\, cos^2 4(\pi/_6) \sin 4(\pi/_6)$$

$$= -12(-{}^1/_2)^2 \left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{-3\sqrt{3}}{2} \qquad \blacksquare$$

Exercises

1. *Find the derivative of the function*

   a. $f(x) = x^{7/6} - x^{-7/6}$

   b. $f(x) = \sqrt{2x^2 + 3x - 1}$

   c. $f(x) = \frac{\sqrt{x^3-1}}{x^2}$

   d) $f(x) = \sqrt[3]{x + \tan x}$

   e) $y = a^3 + \cos^3 x$

   f) $g(t) = (6t^2 - 5)^3(t^2 - 2)^4$

2. *Write the composite function in the form* $f(g(x))$. *Then find* $\frac{dy}{dx}$

   a. $y = \sin\frac{1}{2}x$

   b. $y = e^{\sqrt{x}}$

   c. $y = \sqrt{3x + 1}$

   d) $y = (\sin x - \cos x)^{-3/2}$

   e) $y = \sqrt{2x + 1}$

3. *Write the composite in the form of* $h(f(g(x)))$. *Then find* $\frac{dy}{dx}$

   a. $y = (\cos\,(4x))^{1/2}$

   b. $y = \cos^2(3x^6)$

   c. $y = \sin\sqrt{2x + 1}$

4. *Find an equation of the tangent line to the graph of f at the given point*

   a. $f(x) = \frac{2}{1+e^{-x}}$ *at the point (0,1)*

   b. $f(x) = x\sqrt{2 - x^2}$ *, find* $f'(x)$

5. *If* $F(x) = f(g(x)$, *where* $f(-2) = 8, f'(-2) = 4$, *f'(5)=32 and g'(5)=6,* $g(5) = -2$ *find* $F'(5)$

6. *If* $F(x) = f\big(g(x)\big)$, *where* $f(-2) = 8, f'(-2) = 4, f'(5) = 3$ $g(5) = -2$ *and* $g'(5)$ *=6 find* $F'(5)$.

7. *If* $g(x) = \sqrt{4 + 3f(x)}$ *where* $f(1) = 7$ *and*$f'(1) = 4$.*find* $g'(1)$

4.2. Derivatives of Inverse Functions

4.2.1. *Inverse Trigonometric Functions*

Let's start with inverse sine. Here is the definition of the inverse sine.

$$y = \sin^{-1} x \qquad \Leftrightarrow \qquad \sin y = x \qquad \text{for} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

So, evaluating an inverse trig function is the same as asking what angle (*i.e.* y) did we plug into the sine function to get x. The restrictions on y given above are there to make sure that we get a consistent answer out of the inverse sine. We know that there are in fact an infinite number of angles that will work and we want a consistent value when we work with inverse sine. When using the range of angles above gives all possible values of the sine function exactly once. If you're not sure of that sketch out a unit circle and you'll see that that range of angles (the y's) will cover all possible values of sine.

s

Note as well that since $-1 \le \sin(y) \le 1$ we also have $-1 \le x \le 1$.

Let's work a quick example.

Example 1 Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$

Solution

So we are really asking what angle y solves the following equation.

$$\sin(y) = \frac{1}{2}$$

and we are restricted to the values of y above.

From a unit circle we can quickly see that $y = \frac{\pi}{6}$.

Using the first part of this definition the denominator in the derivative becomes,

$$\cos\left(\sin^{-1} x\right) = \cos(y)$$

Now, recall that

$$\cos^2 y + \sin^2 y = 1 \qquad \Rightarrow \qquad \cos y = \sqrt{1 - \sin^2 y}$$

Using this, the denominator is now,

$$\cos\left(\sin^{-1} x\right) = \cos(y) = \sqrt{1 - \sin^2 y}$$

Now, use the second part of the definition of the inverse sine function. The denominator is then,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Putting all of this together gives the following derivative.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$$

Inverse Cosine

Now let's take a look at the inverse cosine. Here is the definition for the inverse cosine.

$$y = \cos^{-1} x \qquad \Leftrightarrow \qquad \cos y = x \quad \text{for} \quad 0 \le y \le \pi$$

As with the inverse since we've got a restriction on the angles, y, that we get out of the inverse cosine function. Again, if you'd like to verify this a quick sketch of a unit circle should convince you that this range will cover all possible values of cosine exactly once. Also, we also have $-1 \le x \le 1$ because $-1 \le \cos(y) \le 1$.

Example 2 Evaluate $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution

As with the inverse sine we are really just asking the following.

$$\cos y = -\frac{\sqrt{2}}{2}$$

where y must meet the requirements given above. From a unit circle we can see that we must have $y = \frac{3\pi}{4}$.

The inverse cosine and cosine functions are also inverses of each other and so we have,

$$\cos\left(\cos^{-1} x\right) = x \qquad \cos^{-1}(\cos x) = x$$

To find the derivative we'll do the same kind of work that we did with the inverse sine above. If we start with

$$f(x) = \cos x \qquad\qquad g(x) = \cos^{-1} x$$

then,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{-\sin(\cos^{-1} x)}$$

Simplifying the denominator here is almost identical to the work we did for the inverse sine and so isn't shown here. Upon simplifying we get the following derivative.

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

So, the derivative of the inverse cosine is nearly identical to the derivative of the inverse sine. The only difference is the negative sign.

Inverse Tangent

Here is the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Leftrightarrow \qquad \tan y = x \qquad \text{for} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Again, we have a restriction on y, but notice that we can't let y be either of the two endpoints in the restriction above since tangent isn't even defined at those two points. To convince yourself that this range will cover all possible values of tangent do a quick sketch of the tangent function and we can see that in this range we do indeed cover all possible values of tangent. Also, in this case there are no restrictions on x because tangent can take on all possible values.

There are three more inverse trig functions but the three shown here the most common ones. Formulas for the remaining three could be derived by a similar process as we did those above. Here are the derivatives of all six inverse trig functions.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

We should probably now do a couple of quick derivatives here before moving on to the next section.

Example 4 Differentiate the following functions.

(a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$

(b) $y = \sqrt{z}\sin^{-1}(z)$

Solution

(a) Not much to do with this one other than differentiate each term.

$$f'(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b) Don't forget to convert the radical to fractional exponents before using the product rule.

$$y' = \frac{1}{2}z^{-\frac{1}{2}}\sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

$$\sin^{-1} x = \arcsin x \qquad \cos^{-1} x = \arccos x$$

$$\tan^{-1} x = \arctan x \qquad \cot^{-1} x = \operatorname{arccot} x$$

$$\sec^{-1} x = \operatorname{arcsec} x \qquad \csc^{-1} x = \operatorname{arccsc} x$$

4.2.2. Hyperbolic and Inverse hyperbolic Functions

Derivatives of Hyperbolic Functions

Certain combinations of exponential functions e^x and e^{-x} occur in advanced application of calculus. Their properties are similar in many ways to those of $\sin x$ and $\cos x$, they have the same relationship to the hyperbola that the trigonometric function have to the circle. For this reason they are collectivelly called **hyperbolic functions** and individually called **hyperbolic sine** and **hyperbolic cosine** and so on. We also define the rest of the hyperbolic functions interms of these functions.

Definition 4.10:

$$\sin hx = \frac{e^{x} - e^{-x}}{2} \qquad \csc hx = \frac{1}{\sin hx}$$

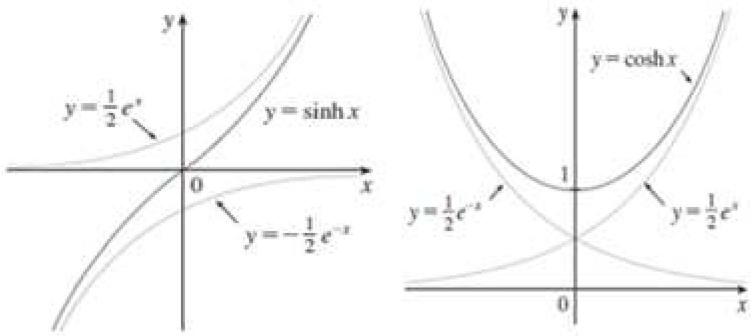
$$\cos hx = \frac{e^{x} + e^{-x}}{2} \qquad \sec hx = \frac{1}{\cos hx}$$

$$\tan hx = \frac{\sin hx}{\cos hx} \qquad \cot hx = \frac{1}{\tan hx}$$

The graphs of **hyperbolic sine** and **hyperbolic cosine** can be sketched using graphical addition as in the Figure belows.

Note that $\sin hx$ has domain $\mathbb{R}$ and range $\mathbb{R}$. While $\cos hx$ has domain $\mathbb{R}$ and range $[1, \infty]$

The hyperbolic functions satisfy a number of identities that are similar to well - known trigonometric identities. We list some of them here and leave most of the proof as an exercises.



a) $y = \sin hx = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$ b) $y = \cos hx = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

Figure 4.10

Hyperbolic Identities

1. $\sin h(-x) = -\sin hx$
2. $\cos h^2 x - \sin h^2 x = 1$
3. $\cos h(-x) = \cos hx$
4. $1 - \tan h^2 x = \sec h^2$
5. $\sin h(x \pm y) = \sin hx \cos hy \pm \cos hx \sin hy$
6. $\cos h(x \pm y) = \cos hx \cos hy \pm \sin hx \sin hy$

Proof:

1. Trivial!
2. By definition, we have

$$\cos h^2 x - \sin h^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

3. Trivial!
4. Left as an exercise

Derivatives of hyperbolic functions

We list the differentiation formula for the hyperbolic functions below. We prove (1) and (2). The remaining proof are left as exercise.Obserbe that the analogy with the differentiation formulas for trigonometric functions, but be aware that the signs are different in some cases.

1. $\frac{d}{dx}(\sin hx) = \cos \mathrm{hx}$
2. $\frac{d}{dx}(\cos \mathrm{hx}) = \sin hx$
3. $\frac{d}{dx}(\tan hx) = \sec \mathrm{h}^2 \mathrm{x}$
4. $\frac{d}{dx}(\csc hx) = -\csc hx \cot hx$
5. $\frac{d}{dx}(\sec hx) = \sec hx \tan hx$
8. $\frac{dy}{dx}(\cot hx) = -\csc h^2 x$

Proof:

$$\frac{d}{dx}(\sin hx) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cos \mathrm{hx}$$

$$\frac{d}{dx}(\cos \mathrm{hx}) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sin hx$$

Inverse of hyperbolic functions

It can be seen from their graphs that $\sin hx$ and $\tan hx$ are one-to-one functions and so they have inverse functions denoted by $\sin h^{-1}$ and $\tan h^{-1}$. But, $\cos h$ is not one –to-one. However, when restricted to the domain $[0, \infty]$ it become one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

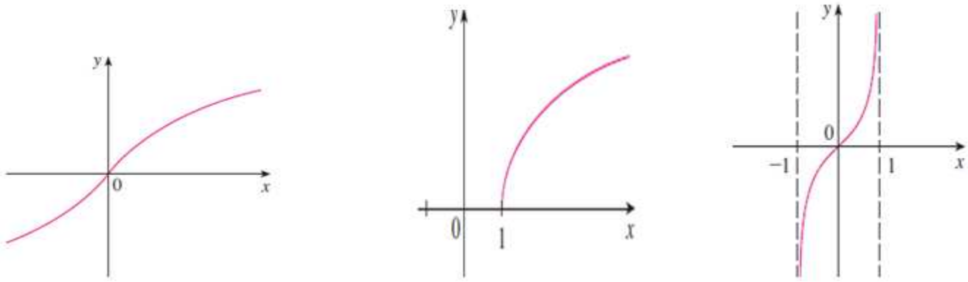
$$y = \sin h^{-1} x \quad \text{iff} \quad \sin hy = x$$

$$y = \cos h^{-1} x \quad \text{iff} \quad \cos hy = x$$

$$y = \tan h^{-1} x \quad \text{iff} \quad \tan hy = x$$

The remaining invese hyperbolic functions are defined similarly.

We can sketch the graphs of $\sin h^{-1}$, $\cos h^{-1} x$ *and* $\tan h^{-1} x$



a) $y = \sin h^{-1} x$ Domain= $\mathbb{R}$ Range = $\mathbb{R}$

b) $y = \cos h^{-1} x$ Domain = $[1, \infty)$ Range = $[0, \infty)$

c) $y = \tan h^{-1} x$ Domain = $(-1,1)$ Range = $\mathbb{R}$

Figure 4 .11

Since the hyperbolic functions are definded in terms of exponential funfions ,it is not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular,we have :

Theorem 4.11:

1. $\sin h^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\,\right)$
2. $\cos h^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\,\right)$ $x \geq 1$
3. $\tan h^{-1} = \frac{1}{2}\ln\frac{1+x}{1-x}$, $|x| < 1$
4. $\sec h^{-1} x = \ln\frac{1+\sqrt{x^2+1}}{x}$, $0 < x \leq 1$

Proof: To pove (1):

In order to derive a formula for $\sin h^{-1} x$ we note that $y = \sin h^{-1} x$,then by definition

$$x = \sin hy = \frac{e^y - e^{-y}}{2} = \frac{e^{2y} - 1}{2e^y}$$

$$\Rightarrow e^y x = e^{2y} x - 1, \qquad \text{or } (e^y)^2 - 2xe^y - 1 = 0$$

Our aim now is to write y as a function of x. Using the quadratic formula and the fact that $e^y > 0$,we deduce that

$$e^y = \frac{2x + \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2 + 1} \qquad (6)$$

We can find y by taking natural logarithms in (6) this yeilds

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Thus our formula for $\sin h^{-1} x$ is

$$\sin h^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

The remaining proofs are left as an exercise.

Exercises

1. Differentiate the following function.

a) $f(x) = ln(x^2 + 1)$ *b)* $f(x) = cos(ln\, x)$ *c)* $f(x) = ln(cos\, x)$

d) $f(x) = log_2(1 - 3x)$ *e)* $f(x) = log_{10}\left(\frac{x}{x-1}\right)$ *f)* $f(x) = \sqrt[3]{ln\, x}$

g) $y = ln(x^4 sin^2 x)$ *h)* $G(u) = \sqrt{\frac{3u+2}{3u-2}}$ *i)* $y = [ln(1 + e^x)]^2$

2. Find an equation of the tangent line to the curve at the given point.

a) $y = ln(ln\, x),\quad (e, 0)$ *b)* $y = ln(x^3 - 1),\quad (2.0)$

3. find the numerical value of the expression.

a) $sin\, h0$ *b)* $cos\, h0$ *c).* $tan\, h0$ *d)* $tan\, h1$

e) $cot\, h(-1)$ *f)* $sinh(ln\, 2)$ *g)* $sech\left(ln\sqrt{2}\right)$ $cos\, h(ln\, 3)$

4.Prove following function:

(a). $\frac{d}{dx} tan\, hx = sec\, h^2\, x$ (b) $\frac{d}{dx} sec\, hx = -\, sec\, hx\, tan\, hx$

(c). $\frac{d}{dx} cot\, hx = -\, csc\, h^2\, x$ (d) $\frac{d}{dx} csc\, hx = -csc\, hx\, cot\, hx$

4.3. Higher Order Derivatives

Objectives:

After completion of this section, students will be able to:

- ✓ Define the higher order derivative of a function
- ✓ Understand the method of finding the higher order derivative.

If the derivative f' of a function f is itself differentiable, then the derivative of f' is denoted by f'' and is called the **second derivative** of f. As long as we have differentiability, we can continue the process of differentiating derivatives to obtain third, fourth, fifth and even higher derivatives of f . The successive derivatives of f are denoted by

$$f', f'' = (f')', f''' = (f'')', f^{(4)} = (f''')' \ldots$$

These are called the first derivative, the second derivative, the third derivative and so forth. The notation of a derivative of arbitrary order is

$f^{(n)}(x)$ and is read as n^{th} order derivative.

Activity : Find

a) $y'''(0)$, *where* $y = 4x^4 + 2x^3 + 3$

b) $\left.\frac{d^4y}{dx^4}\right|_{x=1}$, *where* $y = \frac{6}{x^4}$

Example 1: If $f(x) = 3x^5 - 2x^4 + x^3 - 4x^2 + 2x + 4$. Find the successive derivatives of f.

Solution:

$$f'(x) = 15x^4 - 8x^3 + 3x^2 - 8x + 2$$

$$f''(x) = 60x^3 - 24x^2 + 6x - 8$$

$$f'''(x) = 180x^2 - 48x + 6$$

$$f^{(4)}(x) = 360x - 48$$

$$f^{(5)}(x) = 360$$

$$f^{(6)}(x) = 0$$

.

.

.

$$f^{(n)}(x) = 0 \qquad (n \geq 6) \qquad \blacksquare$$

Example 2: Find a general formula for $F''(x)$ if $F(x) = xf(x)$ and f and $f'(x)$ are differentiable at x.

Solution: using the product rule differentiate F

$$F(x) = xf(x)$$

$$F'(x) = (xf(x))' = f(x) + xf'(x)$$

$$F''(x) = (f(x) + xf'(x))' = \big(f(x)\big)' + (xf'(x))'$$

$$= f'(x) + f'(x) + xf''(x) = 2f'(x) + xf''(x)$$

Therefore,

$$F''(x) = 2f'(x) + xf''(x) \qquad \blacksquare$$

Successive derivatives can also be denoted as follows:

$$f'(x) = \frac{d}{dx}[f(x)]$$

$$f''(x) = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)]$$

$$f'''(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)]$$

.
.
.

In general, we write

$$\boxed{f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]} \qquad (11)$$

which is read "the n^th derivative of f with respect to x. "

When a dependent variable is involved, say $y = f(x)$. Then the successive derivatives can be denoted by writing

$$\frac{dy}{dx}, \frac{d^2y}{dy^2}, \frac{d^3y}{dy^3}, \frac{d^4y}{dy^4}, \dots, \frac{d^ny}{dy^n}, \cdots$$

or more briefly

$$y', y'', y''', \dots, y^{(n)}, \dots$$

Example 3: Find the 27^th derivative of $cosy$

Solution: the first few derivative of $f(y) = cos\ y$ are as follows:

$$f'(y) = -\sin y$$

$$f''(y) = -\cos y$$

$$f'''(y) = \sin y$$

$$f^{(4)}(y) = \cos y$$

$$f^{(5)}(y) = -\sin y = f'(y)$$

.
.
.

We see that the successive derivatives occur in a cycle of length 4 and, in particular $f^{(n)}(y) = \cos y$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(y) = \cos y$$

and , differentiating three more times, we have

$$f^{(27)}(y) = \sin y \qquad \blacksquare$$

Exercises

1. *Show that* $y = x^3 + 3x + 1$ *satisfies* $y''' + xy'' + y' = 0$.
2. *Show that if* $x \neq 0$, *then* $y = \frac{1}{x}$ *satisfies the equation*

$$x^3y''' + x^2y''' = xy = 0$$

3. *find* $f'(x)$, $f''(x)$and $f'''(x)$

   a) $f(x) = x^{-3} + \frac{1}{x^5}$

   b) $f(x) = (3x^2 + 6)(2x - 1)$

   c) $f(x) = (x^3 + 2x^2 - 6x + 8)(2x^{-1} + x^{-2})$

   d) $f(x) = (3x^2 + 4x + 1)^2$

4. a) *Use the Quotient Rule to differentiate the function*

   $f(x) = \frac{\tan x - 1}{\sec x}$

   b) *Simplify the expression for* $f(x)$ *by writing it in terms of* sin *x and* cos *x and then find*$f'(x)$

   c) *show that your answers to part (a) and (b) are equivalent.*

5. *Suppose* $f(\pi/3) = 4$ *and* $f'(\pi/3) = -2$ *and let* $g(x) = f(x)\sin x$ *and* $h(x) = \frac{\cos x}{f(x)}$ *then find*

   a) $g'(\pi/3)$ b) $h'(\pi/3)$.

4.4. Implicit Differentiation

Objectives:

At the end of this section, students will be able to:

- ✓ Understand the implicit differentiation
- ✓ Differentiate the functions using implicit differentiation

Up to now, we have been concerned with differentiating functions that are expressed in the form $y = f(x)$, that is, functions that can be expressed in one variable explicitly. Because the variable y appears alone on one side of the equation. However, sometimes functions are defined by equations in which y is not alone on one side: For example, the equation

$$yx + y + 1 = x$$

is not of the form $y = f(x)$. However, this equation still defined y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1} \qquad (1)$$

Thus, we say that (1) defines y implicitly as a function of $x,$ the function being

$$f(x) = \frac{x-1}{x+1}$$

The method of implicit differentiation consists of differentating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentable function of x so that the method of implicit differentation can be applied.

Example 1:

a) If $x^2 + y^2 = 25$ find $\frac{dy}{dx}$

b) Find an equation of the tangent to the circle$x^2 + y^2 = 25$ at the point (3,4)

Solution:

a) Differentiate both sides of the equation $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the chain rule, we have

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

Thus, $2x + 2y\frac{dy}{dx} = 0$

Now we solve this equation for $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{x}{y}$$

b) At the point (3,4) we have $x = 3$ and $y = 4$, so

$$\frac{dy}{dx} = -\frac{3}{4}$$

An equation of the tangent to the circle at (3,4) is therefore

$$y - 4 = -\frac{3}{4}(x - 3) \text{ or } 3x + 4y = 25$$

Alternatively, solving the equation $x^2 + y^2 = 25$, we get $y = \pm\sqrt{25 - x^2}$. The point (3,4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$f'(x) = \frac{1}{2}(25 - x^2)^{-1/2}\frac{d}{dx}(25 - x^2)$$

$$= \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

So,

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and an equation of the tangent is $3x + 4y = 25$. ■

Example 2: Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$

Solution:

$$\frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}(x^2)$$

$$5\frac{d}{dx}(y^2) + \frac{d}{dx}(\sin y) = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (cos\ y)\frac{dy}{dx} = 2x$$

$$10y\frac{dy}{dx} + (cos\ y)\frac{dy}{dx} = 2x$$

Solving for $\frac{dy}{dx}$ we obtain

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$ ■

Activity.

a) *Use implicit differentiation to find* $\frac{dy}{dx}$ *for the equation* $x^3 + y^3 = 3xy$

b) *Find an equation for the tangent line to the equation* $x^3 + y^3 = 3xy$ *at the point* $(\frac{3}{2}, \frac{3}{2})$.

Example 3: Using implicit differentiation, find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$

Solution: Differentiating both sides of $x^2 + y^2 = 1$ with respect to x implicitly yields

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Longrightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Again applying implicit differentiation for the second time, we get

$$\frac{d^2y}{dx^2} = \frac{(-1)y-(-x)y\prime}{y^2}$$

$$= \frac{-y+x\,\frac{-x}{y}}{y^2} = -\frac{1}{y^3}$$ ■

Example 4: Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$

Solution: Differentiating both sides of $4x^2 - 2y^2 = 9$ implicitly yields

$$8x - 4y\frac{dy}{dx} = 0$$

from which we obtain

$$\frac{dy}{dx} = \frac{2x}{y} \qquad (1)$$

Differentiating both sides of (1) implicitly yields

$$\frac{d^2y}{dx^2} = \frac{(y)(2)-(2x)(dy/dx)}{y^2} \qquad (2)$$

Substituting (1) into (2) and simplifying using the original equation, we obtain

$$\frac{d^2y}{dx^2} = \frac{2y-2x(2x/y)}{y^2} = \frac{2y^2-4x^2}{y^3} = \frac{9}{y^3}$$ ■

Example 5: Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2 \cos x$

Solution: Differentiating implicitly we get

$$\frac{d}{dx}(\sin(x + y)) = \frac{d}{dx}(y^2 \cos x)$$

$$\cos(x + y)\ \frac{d}{dx}(x + y) = y^2\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(y^2)$$

$$\cos(x+y)(1+\frac{dy}{dx}) = y^2\sin x + \cos x\,(2y\frac{dy}{dx})$$

$$\cos(x+y) + cos(x+y)\frac{dy}{dx} = y^2\sin x + 2y\cos x\,\frac{dy}{dx}$$

$$\cos(x+y) - y^2\sin x = (2y\cos x\ - cos(x+y))\frac{dy}{dx}$$

$\frac{dy}{dx} = \frac{cos(x+y)-y^2\sin x}{2y\cos x\ -cos(x+y)}$ ∎

Exercises

1. *Find* $\frac{dy}{dx}$

   a) $xy + 2x + 3x^2 = 4$

   b) $\frac{1}{x}+\frac{1}{y} = 1$

   c) $y = \sqrt[3]{4x-5}$

   d) $4x^2 + 9y^2 = 36$

   e) $\cos x + \sqrt{y} = 5$

   f) $y = \sqrt{\frac{x^2+1}{x^2-5}}$

2. *Find* $\frac{dy}{dx}$ *by implicit differentiation.*

   a) $x^2 + y^2 = 68$

   b) $x^3y^2 - 5x^2y + x = 1$

   c) $x^2 = \frac{x+y}{x-y}$

   d) $\sqrt{xy} = 1 + x^2y$

   e) $e^y cosx = 1 + \sin(xy)$

   f) $x^2y + 3xy^3 - x = 3$

   g) $e^{x/y} = x - y$

   h) $ysin(x^2) = xsin\,(y^2)$

   i) $\sqrt{x+y} = \frac{y}{1+x^2}$

3. *Find the slope of the tangent line to the curve at the given points in two ways: first by solving for y interns of x and differentiating and then by implicit differentiation.*

   a) $x^2 + y^2 = 1: \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   b) $y^2 - x + 1 = 0: (10,3), (10,-3)$

4. *Find an equation of the tangent line to the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

   at the point (x_0, y_0)

5. *Find the value of a and b for the curve* $x^2y + ay^2 = b$ *if the point (1,1) is on its graph and the tangent line at (1,1) has the equation* $4x + 3y = 7$.

4.5. Applications of Derivatives

Objectives:

On completion of the section, students will be able to:

- Define the extreme value of a function
- Differentiate the relative and absolute extreme value of a function
- Calculate the relative maximum and minimum value of a function
- Calculate the absolute maximum and minimum value of a function
- Understand the Mean Value Theorem;
- Solve problems using Mean Value Theorem.
- Apply derivatives to determine the intervals on which a given function is increasing and decreasing.
- Apply derivatives to determine relative extremes
- Define concavity of functions;
- Identify where a graph of a function is concave upward and concave downward;
- Define inflection points.
- Sketch the graph of a given function.

In this section , we will give emphasize the application of the derivative to graphing functions. We will learn how to determine where the graph of a differentiable function rise and where it falls: where it has peak and where it has valleys: where it curves upward and where it curves downward. The concepts we will introduce have application not only to graphing functions but also to problems in such widely varying areas.

4.5.1. Extreme Values of Functions

Some of the most important applications of differential calculus are optimization problems, on which we are required to find the optima (best) way of doing something.

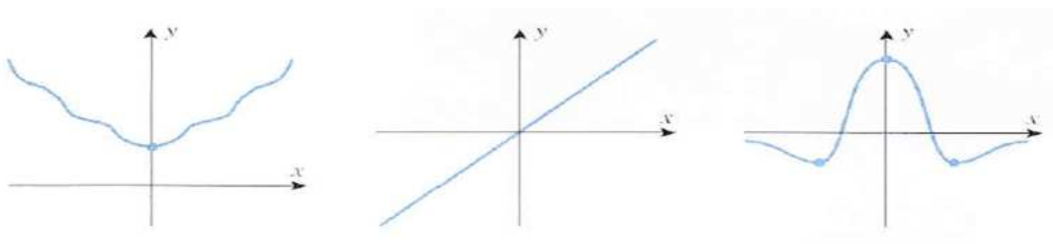
Definition 4.12: A function f is said to have an

a. *Absolute maximum (or global max*imum) at the point x_0 on an interval I if $f(x_0)$ is the largest value of f on I: that is $f(x_0) \geq f(x)$. The number $f(x_0)$ is called the maximum value of f on I.

b. Absolute minimum (or global minimum) at the point x_0 on I if $f(x_0)$ is the smallest value of f on I: that is $f(x_0) \leq f(x)$ for all x in I. The number $f(x_0)$ is called the minimum value of f on I.

If f has either an absolute maximum or absolute minimum on I at x_0, then f is said to have an *absolute extreme* on I at x_0.

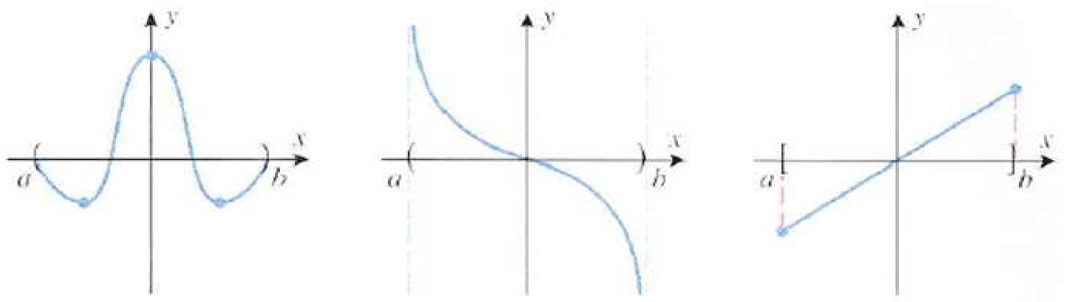
As it is illustrated in figure 3.1, there is no guarantee that a function f will have absolute extrema on given interval.



f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$
(a)

f has no absolute extrema on $(-\infty, +\infty)$
(b)

f has an absolute maximum and minimum on $(-\infty, +\infty)$
(c)



f has absolute maximum and minimum on (a, b) (d)

f has no absolute extrema on (a, b) (e)

f has an absolute maximum and minimum on $[a, b]$ (f)

Figure 4.12

Part (a)-(e) of figure 4.12 show that a continous function may or may not have relative maximum or minimum on an infinite interval or on a finite open interval. However, theorem 4.14 will show us that a continous function must have both an absolute maximum and an absolute minimum on every finite closed interval(see part (f) of figure 4.12)

> **Definition 4.13** A function f has a local maximun (or relative maximum) at x_0 if
>
> $f(x_0) \geq f(x)$ when x is near x_0 [this means that $f(x_0) \geq f(x)$

Example 1. If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x. therefore $f(0) = 0$ is the absolute (and local) minimum value of f. this corresponding to the fact that the origin is the lowest point on the parabola $y = x^2$. (see figure 4.13) however, there is no highest point on the parabola and so this function has no maximum value.

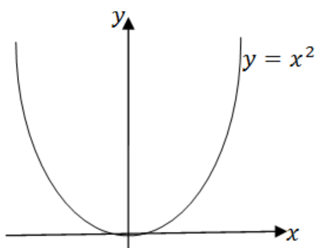


Figure 4.13

Figure 4.13 shows an example of a function which has minimum value and no maximum value.

Example 2. the function $f(x) = 2x + 1$ is continuous, and hence is guaranteed to have both an absolute maximum and an absolute minimum on every finite closed interval and, in particular, on the interval [0,3] the absolute minimum occures at $x = 0$ and the absolute maximum occurs at $x = 3$, at which points the absolute minimum and maximum values are $f(0) = 1$ and $f(3) = 7$.

Finding absolute extrema on finite closed intervals:

The Extreme Value Theorem is an example of what mathematicians call an existence theorem. Such theorems state conditions under which something exists, in this case absolute extrema. However, knowing that something exists and finding it are two separate things, so we will now address the problem of finding the absolute extrema.

Theorem 4.14 (***The Extreme Value Theorem***)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Definition 4.15: A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ doesn't exist.

Example 3: Find the critical number of $f(x) = x^3$.

Solution: $f'(x) = 3x^2$. Now,

$$f'(x) = 0 \Leftrightarrow 3x^2 = 0 \Leftrightarrow x = 0.$$

Thus, $x = 0$ is the only critical number of the given function. ■

Theorem 4.16: If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f.

Steps to find the extreme value of a function on an interval (a, b):

1. Compute the value of f at all the critical points in (a, b)
2. Compute the value of f at the end points a snd b.

3. Compare the results in (1) and (2). The largest of those values is the maximum value of f on $[a, b]$, the smallest of those values is the minimum value of f on $[a, b]$.

Example 4: Let $f(x) = x - x^3$. Find the extreme value of f on [0,1] and determine at which numbers in [0,1] they occur.

Solution: First we find the critical point f has extreme values because it is continuous on [0,1], since f is differentiable,

$$f'(c) = 1 - 3c^2 = 0$$
$$c = \pm\frac{\sqrt{3}}{3}.$$

Since $-\frac{\sqrt{3}}{3}$ is not in the interval [0,1], the only critical number in our focus is $\frac{\sqrt{3}}{3}$.

We compute the corresponding value of f,

$$f(0) = 0, f(1) = 0, f\left(\frac{\sqrt{3}}{3}\right) = \frac{2}{9}\sqrt{3}$$

Consequently, the minimum value of f on [0,1] is 0 and it occurs at 0 and 1. The maximum value of f on [0,1] is $\frac{2}{9}\sqrt{3}$ and it occurs at $\frac{\sqrt{3}}{3}$. ■

Example 5: Let $f(x) = x - x^3$. Find the extreme value of f on [2,4] and determine at which numbers in [2,4] they occur.

Solution: From example 4, the critical point is $c = \frac{\sqrt{3}}{3}$, therefore $f'(x) \neq 0$ for all x in $(2,4)$. Hence the extreme value of f on [2,4] must occur at the end points of the interval. Since $f(2) = -6, f(4) = -60$, we conclude that -6 is the maximum value and occurs at 2, whereas -60 is the minimum value and occur at 4. ■

Example 6: Find the extreme value of f, $f(x) = x^2(4 - x)$on, $[-1,2]$

Solution: $f'(x) = 2x(4 - x) + x^2(-1)$

$$= 8x - 2x^2 - x^2 = -3x^2 + 8x = x(-3x + 8)$$

$$f'(x) = 0 => x(-3x + 8) = 0$$

$$=> x = 0 \text{ or } -3x + 8 = 0$$

$$=> x = 0 \text{ or } x = \frac{8}{3}$$

But $x = \frac{8}{3}$ is not in $[-1,2]$ so, $c = 0$

We compute the corresponding value of f

$$f(-1) = 5, f(0) = 0, f(2) = 8$$

So, the maximum value of f on $[-1,2]$ is 8, and it occurs at 2 and the minimum value of f on $[-1,2]$ is 0, and it occurs at 0. ■

Activity

a) *Find the critical number of the function:* $f(x) = x^{4/5}(x - 4)^2$

b) *Find the absolute maximum and absolute minimum value of f on the given interval* $f(x) = x^3 - 6x^2 + 9x + 2$, $[-1,4]$

Example 7: A landowner wishes to use 2 mile of fencing to enclose a rectangular region of maximum area, what should the lengths of the sides be?

Solution: Any rectangular region the landowner could enclose must have a length x and width y (figure 4.14)

since the perimeter is to be 2(miles), we have

$2x + 2y = 2$ for $0 \le x \le 1$ and $0 \le y \le 1$.

Rectangle with 2 miles perimeter

Figure 4.14

Therefore, $y = 1 - x$, so the area xy of the rectangle can be written as a function of x alone.

$$A(x) = x(1 - x) \text{ for } 0 \le x \le 1$$

The problem has been reduced to finding the maximum value of A on [0,1], since $A'(x) = 1 - 2x$ and thus $A'(x) = 0$ only when $x = \frac{1}{2}$, A can have its maximum value on [0,1] only at 0, $\frac{1}{2}$ or 1. But

$A(0) = 0, A\left(\frac{1}{2}\right) = \frac{1}{4}$ and $A(1) = 0$

Thus the maximum value of A occurs for $x = \frac{1}{2}$(mile) (see figure 3.4); since $y = 1 - x$, the value of y corresponding to $x = \frac{1}{2}$ is also $\frac{1}{2}$ (mile). Consequently, the fence should enclose a square region of area $A\left(\frac{1}{2}\right) = \frac{1}{4}$(square miles). ■

Exercise .

1. *Find all critical points (if any) of the given function*
   - *a)* $f(x) = 5x^2 + 4x$
   - *b)* $f(x) = x^2 + 4x + 6$
   - *c)* $f(x) = x^5 - 5x^3 + 10x - 3$
   - *d)* $f(x) = x^3 + x^2 - x$
   - *e)* $\frac{x+1}{x^2+x+1}$
   - *f)* $f(x) = \frac{1}{x^2+1}$
2. *Find all extreme values (if any) of the given function on the given interval. Determine at which number in the interval these value occur*
   - *a)* $f(x) = x^2 - 5x - 6, [2.5,6]$
   - *b)* $f(x) = x^3 - 3x + 1 \;, [0,3]$
   - *c)* $f(x) = sin\, x + cos\, x \quad , [0, \frac{\pi}{3}]$
   - *d)* $f(x) = \frac{1}{x}, \quad [-1, 3]$
   - *e)* $f(x) = xe^{-x^2/8}, [-1,4]$
3. *If a and b are positive number , find the maximum value of*

   $f(x) = x^a(1-x)^b \; 0 \le x1$
4. *Assume that* f *is defined on* $[a, b]$ *and that* $g = -f$. *prove that* $f(x_0)$ *is the maximum value of* f *on* $[a, b]$ *if and only if* $g(x_0)$ *is the minimum value of* g *on* $[a, b]$.

4.5.2. Mean - Value Theorem

Theorem 4.17 (*Rolle's Theorem*)

Let f be continuous on [a, b], and differentiable on (a, b). If $f(a) = f(b)$ then there is a number c in (a, b) such that $f'(c) = 0$.

Theorem 4.18 (*The Mean Value Theorem*)

Let f be a function that satisfies the following:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad (1)$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \qquad (2)$$

Proof: We introduce an auxiliary function g that allows us to simplify the proof by using Rolle's Theorem. The function g is defined by

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x - a)\right]$$

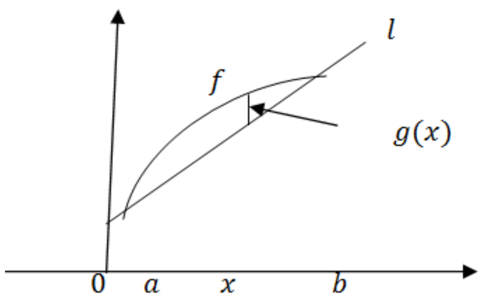


Figure 4.15

(see figure 4.15), now g is continuous on $[a, b]$ and differentiable (a, b), since g is a simple combination of f, constant functions and a linear functions. Substituting in the equation above, we find that

$$g\ (a) = g(b) = 0$$

So, that by Roll's Theorem there is a number c in (a, b) such that $g'(c) = 0$

However, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ for $a \leq x \leq b$ and thus

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

Solving for $f'(c)$, we obtain

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1. Let $f(x) = x - 3x^2$. Find a number c in (-1,3) such that

$$f'(c) = \frac{f(3)-f(-1)}{3-(-1)}$$

Solution: since $\frac{f(3)-f(-1)}{3-(-1)} = \frac{24-(-4)}{4} = 7$

We see a number c in (-1,3) such that $f'(c) = 7$. But $f'(x) = 1 - 6x$

So that, $f'(c) = 1 - 6c = 7$

$$c = 1$$

Since $1\epsilon(-1,3)$, we conculude that $c = 1$. ■

Applications of the Mean Value Theorem

The Mean Value Theorem is one of the most important results in calculus. We will use it now to prove two very different theorems; the first implies that if two functions have identical slopes at each number in an interval; then the functions differ by a constant on that interval.

Theorem 4.19

a) Let f be contionous on an interval I. If $f'(x) = 0$ for each interior point x of I, then f is constant on I.

b) Let f and g be contionous on an interval I. If $f'(x) = g'(x)$ for each interior point x of I, then $f - g$ is constant on I. In other words, there is a constant c such that $f(x) = g(x) + c$ for all x in I.

Proof:

To prove (a), let x and z be arbitrary numbers in I with $x < z$. By the Mean Value Theorem there is a number c in (x, z) such that

$$\frac{f(z)-f(x)}{z-x} = f'(c) \qquad (1)$$

by assumption, $f'(c) = 0$ and thus (1) reduce to

$$f(z) - f(x) = 0$$

Therefore, $f(z) = f(x)$.

It follows that f assigns the same value at any two points in I, so f is constant on I.

To prove (b), notice that

$$(f - g)'(x) = f'(x) - g'(x) = 0$$

So $f - g$ satisfies the conditons of point (a). consequently $f - g$ is constant on I. in other words, there is a constant c such that $f(x) = g(x) + c$ for all x in I.

If f is a function defined on an interval I, then any function F such that $F'(x) = f(x)$ for each x in I is called an antiderivative of f(since f is the derivative of F on I)

Example 2: Find the antiderivatives of the function $f(x) = sec^2x$.

Solution: tan x is an antiderivative of sec^2x, so the antiderivatives of sec^2x have the form $\tan x + c$, where c is any constant. ■

Example 3. Let f be suchthat $f'(x) = \cos x$ and $f(\frac{\pi}{2}) = -1$. Determine the function f.

Solution: Since f and $sin\ x$ are both antiderivatives of $cos\ x$, by theorem () (b) there is a constant c such that

$$f(x) = \sin x + c$$

For the appropriate constant c. to determine c, we use the assumption that $f(\frac{\pi}{2}) = -1$, which yhields

$$-1 = f(\frac{\pi}{2}) = \sin\frac{\pi}{2} + c = 1 + c$$

therefore $c = -2$ and hence

$$f(x) = \sin x - 2.$$ ■

Exercises:

1. *Let $f(x) = Ax^2 + Bx + C$, where A, B, and C are costants with $A \neq 0$. Show that for any interval [a, b], the number c guaranteed by the Mean Value Theorem is the mid point of [a, b].*
2. *Let $f(x) = |x|$. Show that $f(-2) = f(2)$, but there is no number c in (-2, 2) such that $f'(c) = 0$. Does this result contradict Rolle's Theorem? Explain.*

Increasing and Decreasing Functions

The main singificance of the Mean Value Theorem is that it enables us to analyze the nature of graphs of functions. Our immediate use of this principle is to prove the basic fact concerning increasing and decreasing functinos. But, before stating the theorem let us see the following definitions.

Definition 4.20: A function f is said to be increasing on an interval I provided that $f(x) \leq f(z)$ whenever x and z are in I and $x < z$

The function f is strictly increasing on I provided that $f(x) < f(z)$ whenever x and z are in I and $x < z$.

Definition 4.21: A function f is said to be decreasing on an interval I provided that $f(x) \geq f(z)$ whenever x and z are in I and $x < z$

The function f is strictly increasing on I provided that $f(x) > f(z)$ whenever x and z are in I and $x < z$.

Theorem 4.22: Let f be continuous on an interval $I$and differentiable at each interior point of I

a. If $f'(x) \geq 0$ at each interior point of I then f is increasing on I. Moreover, f is strictly increasing on I if $f'(x) = 0$ for at most a finite number of points in I.
b. If $f'(x) \leq 0$ at each interior point of I then f is decreasing on I. Moreover, f is strictly decreasing on I if $f'(x) = 0$ for at most a finite number of points in I

Definition 4.23: A function f *which* is either increasing or decreasing is called monotonic function.

Example 1: Let $f(x) = x^3 + 3x^2 + 3x + 4$. On which interals is f strictly increasing and strictly decreasing.

Solution: For all x, we have

$$f'(x) = 3x^2 + 6x + 3. = 3(x+1)^2 \geq 0$$

Thus, $f'(x)$ is positive except for $x = -1$. It follows that f is strictly increasing on $(-\infty, \infty)$. ■

Example 2: Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. On which interval is f incerasing and on which it is decerasing.

Solution: For all x, we have

$$f'(x) = 12x^3 - 12\mathrm{x}^2 - 24\mathrm{x} = 12\mathrm{x}(\mathrm{x}-2)(\mathrm{x}+1)$$

Using sign chart Method to know where $f'(x) > 0$ and where $f'(x) < 0$.this depends on the sign of the three factors of $f'(x)$, namely, $12x, x-2, x+1$.

From figure 3.6, we conclude that f decreasing on $(-\infty, -1)$ and $(0, \infty)$ and increasing on $(-1,0)$ *and* $(2, \infty)$. ■

		-1	0	2
$x+1$	- - -	+ + +	+ + +	+ + +
$12x$	- - - -	- - - -	+ + +	+ + +
$x-2$	- - - - -	- - -	- - -	+ + +
$f'(x)$	- - - -	+ + +	- - -	+ + +

Figure 4.15

Exercises:

Find the intervals on which the given function is strictly increasing and those on which it is strictly decreasing.

1. $f(x) = x^2 + x + 1$
2. $g(x) = x^3 - 12x + 4$
3. $h(x) = x^3 - x^2 + x - 1$
4. $l(x) = \frac{x-2}{x-1}$

4.5.3. The First and Second Derivative Tests

i. If a continuous f is increasing on the portion of an interval I to the left of c and decreasing on the portion to the right of c, then $f(c)$ is the maximum value of f on I.

ii. If a continuous f is decreasing on the portion of an interval I to the left of c and increasing on the portion to the right of c, then $f(c)$ is the minimum value of f on I.

First Derivative Test

The derivative f' changes from **positive to negative** at c if there exists some number $\delta > 0$ such that $f'(x) > 0$ for all x in $(c - \delta, c)$ and $f'(x) < 0$ for all x in $(c, c + \delta)$.

The definition of f' changes from **negative to positive** at c, results from replacing $f'(x) > 0$ by $f'(x) < 0$ and vice versa.

Example 1: Let $f(x) = x^{2/3}(6-\mathrm{x})^{1/3}$. Determine where f' changes from positive to negative and where it changes from negative to positive.

Solution: First we find the derivative of f

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ doesn't exist when $x = 0$ or $x = 6$. To determine where f' changes sign, we assemble the chart

		0	4	6
$x^{1/3}$	- - -	+ + +	+ + +	+ + +
$4-x$	+ + +	+ + +	- - -	- - -
$(6-x)^{2/3}$	+ + +	+ + +	+ + +	+ + +
$f'(x)$	- - -	+ + +	- - -	- - -

Figure 4.16

Consequently, f' changes from negative to positive at 0 and from positive to negative at 4. But at 6 the sign f' doesn't change (negative to negative) ■

Theorem 4.24 (*First Derivative Test*)

Suppose that c is a critical number of a continuous function f on an interval I

a. if f' changes from positive to negative at c, then f has a relative maximum at c.
b. If f' changes from negative to positive at c, then f has a relative minimum value at c.
c. If f' doesn't change sign at c, (that is f' is positive on both sides or negative on both sides), then f has no relative maximum or minimum at c.

Example 2: Let $f(x) = 4x^3 + 9x^2 - 12x + 3$. Show that f has a relative maximum at -2 and a relative minimum value at $\frac{1}{2}$.

Solution: First we find the derivative of f

$$f'(x) = 12x^2 + 18x - 12 = 12\left(x^2 + \frac{3}{2}x - 1\right) = 12(x+2)(x - \frac{1}{2})$$

To determine where f' changes the sign

		-2	$^1/_2$
$x+2$	- -	+ +	+ +
$x - {}^1/_2$	- -	- -	+ +
$f'(x)$	+ +	- -	+ +

Figure 4.17

Consequently, f' changes frome positive to negative at -2 and frome negative to positive at $^1/_2$. Thus the first derivative test implies that f has a relative maximum value at -2, similarly, f has a relative minimum value at $^1/_2$. ■

Together, the First Derivative Test and theorem (4.22) can help us sketch the graph of a function f. The procedure is as follows. We compute the derivative of f and examine it:

1. From theorem 4.24, if $f'(x) \geq 0$ for all x in an interval I, then f is increasing on I, whereas if $f'(x) \leq 0$ for all x in an interval I then f is decreasing on the interval I (figure 4.18(a))
2. From the First Derivative Test, if f' changes from positive to negative at c, then f has a relative maximum value at c, if f' changes from negative to positive at c, then f has a relative minimum value at c. (figure 4.18(b)).

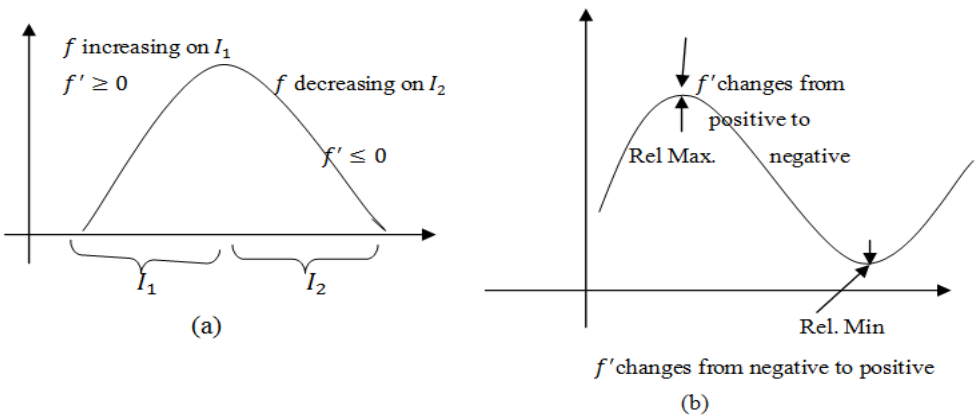


Figure 4.18

Example 3: Let $f(x) = (x-1)^2(x-3)^2$. Sketch the graph of f.

Solution: First we find the derivative of f

$$\begin{aligned} f'(x) &= 2(x-1)(x-3)^2 + 2(x-3)(x-1)^2 \\ &= 2(x-1)(x-3)[(x-3)+(x-1)] \\ &= 2(x-1)(x-3)(2x-4) \\ &= 4(x-1)(x-3)(x-2) \end{aligned}$$

Since $f'(x) = 0$ when $x = 1, x = 3$ or $x = 2$, we determine the sign change at each of these points.

		1	2	3
$x-1$	- -	+ +	+ +	+ +
$x-2$	- -	- -	+ +	+ +
$x-3$	- -	- -	- -	+ +
$f'(x)$	- -	+ +	- -	+ +

Figure 4.19

From figure 4.19, f is strictly increasing on the interval $[1,2]$ and $[3, \infty)$ and strictly decreasing on the intervals $(-\infty, 1]$ and $[2,3]$. We also find that f' changes from negative to positive at 1 and 3 and from positive to negative at 2. As a result, the first derivative test implies $f(1) = 0$ and $f(3) = 0$ are relative minimum value and $f(2) = 1$ is relative maximum value. Figure 4.20 shows the graph of f. ■

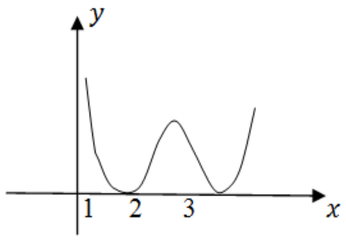


Figure 4.20

Theorem 4.25 (*Second Derivate Test*)

Assume that $f'(c) = 0$

a. If $f''(c) < 0$ then $f(c)$ is a relative maximum value of f.

b. If $f''(c) > 0$ then $f(c)$ is a relative minimum value of f .

If $f''(c) = 0$ then from this test alone we cannot draw any conclusion about a relative exterme value f at c.

Example 4: Let $f(x) = x^3 - 3x - 2$. Using the Second Derivative Test, find the relative exteme value of f.

Solution: By differentiation we obtain

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1) \text{ and } f''(x) = 6x$$

Therfore $f'(x) = 0$ when $x = 1$ or $x = -1$. Since

$$f''(-1) = -6 < 0 \text{ and } f''(1) > 0$$

We know from the second derivative test that $f(-1) = 0$ is a relative maximum value of f, whereas $f(1) = -4$ is relative minimum value of f. ■

Example 5: Use the Second Derivative Test to determine the relative extreme values of the function $f(x) = -4x^2 + 16x - 1$

Solution: Differentiating f with respect to x we get

$$f'(x) = -8x + 16 \text{ and } \quad f''(x) = -8$$

Now,

$$f'(x) = 0 \Leftrightarrow -8x + 16 = 0 \Leftrightarrow x = 2$$

Thus $x = 2$ is the only critical number of f. More over, $f''(2) = -8 < 0$. Therefore, by the Second Derivative Test $f(2) = 15$ relative maximum value. ■

Example 6: Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$

Solution: $f'(x)$ changes from negative to positive at 0, $f(0) = 0$ is a relastive minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a relative maximum. The sign of f' doesn't change at 6, so there is no minimum or maximum there.

And $f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$, $x^{4/3} \geq 0$ for all x,we have $f'(x) < 0$for $0 < x < 6$ and $f''(x) > 0$for all $x > 6$. ■

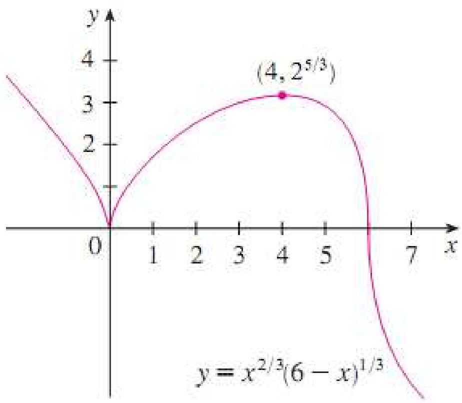


Figure 4.21

Exercise

1. *Determine the value of c at which f'changes from positive to negatiove or frome negative to positive.*
   a. $f(x) = x^2 + 6x - 11$
   b. $f(x) = 2x^3 - 4x^2 + 3$
   c. $f(x) = \frac{x}{x^3-2}$
   d. $f(x) = \sin x - \cos x$
2. *Use the first derivative test to determine the relative extreme values(if any) of the function*
   a. $f(x) = x^3 - 12x + 2$
   b. $f(x) = x^4 - 8x^2 + 1$
   c. $f(x) = \frac{x}{16+x^3}$
   e. $f(x) = \frac{\cos x}{1+\sin x}$
3. *Use the second derivative test to determine the relative extreme value of the function*
   a. $f(x) = 4x^2 + 3x - 1$
   b. $f(x) = x^4 + \frac{1}{2}x$

d. $f(x) = (x^2 + 2)^6$

c. $f(x) = x^3 - \frac{48}{x^2}$

4. *Suppose* f'' *is contionous on* $(-\infty, \infty)$

   a. *If* $f'(2) = 0$ *and* $f''(2) = 5$. *What cann you say about* f?

   b. *If* $f'(6) = 0$ *and* $f''(6) = 0$. *What cann you say about* f?

Applications of Extreme Values

To solve many applied problems, one needs to find a maximum or minimum value of a suitable function on an interval I. Recall from the Extreme Value Theorem that if f is continuous on a closed, bounded interval $[a, b]$, f assumes a maximum and a minimum value. Moreover thes values can be assumed only at the end points a and bof the interval or at critical points in (a, b). The following examples illustrate this idea.

Example 1: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution: In order to get a feeling for what is happening in this problem, let's experiment with some special cases.

We see that when we try shallow, wild fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area. Figure 4.22 shows the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle in feet. Then we express A in terms of x and y:

$$A = xy$$

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation, we have $y = 2400 - 2x$, which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

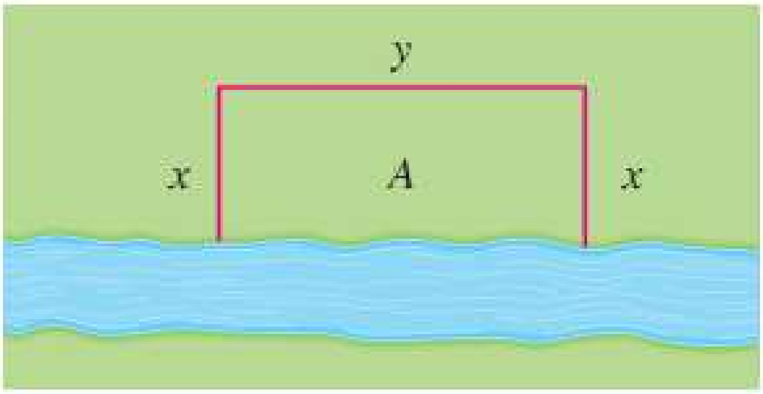


Figure 4.22

Note that $x \geq 0$ and $x \leq 1200$ (otherwise A < 0). So the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

Which gives $x = 600$. The maximum value of A must occur either at this critical number or at an end point of the interval. Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, the closed interval method gives the maximum value as $A(600) = 720{,}000$.

Therefore, the rectangular field should be 600 ft deep and 1200 ft wide. ■

Example 2: Find the two nonnegative numbers whose sum is 18 and whose product is as large as possible.

Solution: Let x and y be positive numbers such that $x + y = 18 \implies y = 18 - x$ with $0 \leq x \leq 18$. Let the product be $A = xy$. Then,

$$A(x) = x(18 - x) = 18x - x^2$$
$$\implies A'(x) = 18 - 2x.$$

Now, $A'(x) = 0 \implies 18 - 2x = 0 \implies x = 9$ is the critical number of A. From this we have $y = 9$. $A(0) = 0 = A(18)$ and $A(9) = 81$

Thus, by the closed interval method, we have that $A(9) = 81$ is the largest possible product of the numbers $x = 9$ and $y = 9$. ■

Related Rates

In this subtopic, we are going to study problems involving variables that are changing with respect to time. If two or more such variables are related to each other, then their rates of change with respect to time are also related.

For instance, suppose that x and y are related by the equation $y = 2x$. If both variables are changing with respect to time then their rates of change will also be related by the equation

$$\frac{dy}{dt} = 2\frac{dx}{dt}$$

Example 3: Given $x + xy = 6$.Find

a. The rate of change of x with respect to y.
b. The rate of change of y with respect to x.

Solution:

a. In this case we assume x is differentiable with respect to y.

   Thus,

$$\frac{d}{dy}(x + xy) = \frac{d}{dy}(6)$$

$$\Rightarrow x + y\frac{dx}{dy} = 0(By\ Chain\ Rule)$$

$$\Rightarrow \frac{dx}{dy} = \frac{-x}{y}$$

b. Similarly

$$\frac{d}{dx}(x + xy) = \frac{d}{dx}(6)$$

$$\Rightarrow 1 + y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1-y}{x}$$ ■

Example 4: A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution:

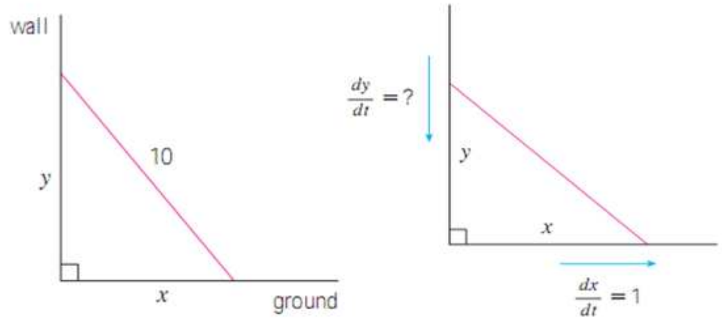


Figure 4.23

We first draw a diagram and label it as in the figure. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of time t measured in seconds.

We are given that $\frac{dx}{dt} = 1\,{}^{ft}/_{s}$ and we are asked to find $\frac{dy}{dt}$ when $x = 6ft$. In this problem,

the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the chain rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}\,.$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values we have

$$\frac{dy}{dt} = -\frac{3}{4}(1\,ft/s) = -\frac{3}{4}ft/s.$$

The fact that $\frac{dy}{dt}$ is negative means that the distance from the top of the ladder to the ground is decreasing. ■

Example 5: A water tank has the shape of an inverted circular cone with base radius 2m and height 4m . If water is being pumped in to the tank at a rate of 2 m^3/min, find the rate at which the water level is raising when the water is 3m deep.

Solution:

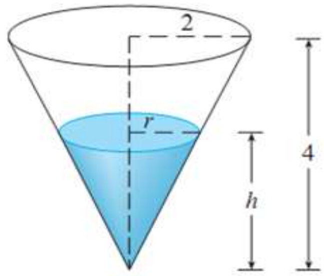


Figure 4.24

Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at a time t, where t is measured in minutes.

We are given that $\frac{dV}{dt}$ =2 m^3/min and we are asked to find $\frac{dh}{dt}$ when h is 3m. The given quantities are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

Now, $\frac{r}{h} = \frac{2}{4}$ implies that $r = \frac{h}{2}$. Thus,

$$V=\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting h = 3 and $\frac{dV}{dt}$ =2 m^3/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi 3^2}\cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $\frac{8}{9\pi}$ m/min. ■

Example 6: Air is being pumped into a spherical balloon at the rate of 4.5cm^3/min. Find the rate of change of the radius when the radius is 2cm.

Solution: Let r be the radius of the sphere, then the volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Longrightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi(2)^2}\times 4.5 \approx 0.09\ cm/\min \quad ■.$$

Example 7: The radius r of a circle is increasing at a rate of 3cm/min. Find the rate of change of the area when r = 8cm.

Solution: If A is the area of the circle and r is its radius, then

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi \times 8cm \times \frac{3cm}{min} = 48\pi cm/\min \qquad \blacksquare$$

Exercises

1. *Find two real numbers whose difference is 16 and whose product is as small as possible.*
2. *Find the area of the largest rectangle that can be inscribed in a semicircle.*
3. *A metal box (without top) is to be constructed from a square sheet of metal that is 10m on a side by first cutting square pieces of the same size from each corner of the sheet and then folding the sides. Find the dimensions of the box with largest volume.*
4. *A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.*
5. *A ladder 5m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 0.25m/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3m from the wall?*

4.5.4. Concavity and Inflection Points

Definition 4.26: Let f be differentiable at c and let l_c be the line tangent to the graph of f at $(c, f(c))$.

a. The graph of f is **concave upward** at $(c, f(c))$ if there is an open interval I_c about c such that if x is in I_c and $x \neq c$ then $(x, f(x))$ lies above l_c.
b. The graph of f is **concave downward** at $(c, f(c))$ if there is an open interval I_c about c such that if x is in I_c and $x \neq c$, then $(x , f(x))$ lies below l_c.

c. The graph of a function f is **concave upward** (respectively **concave downward**) on an open interval I if it is concave upward (respectively concave downward) at $(x, f(x))$ for each x in I.

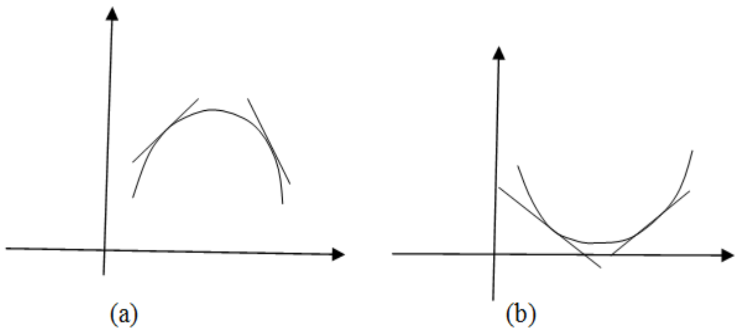


Figure 4.25

Notice from figure 4.25 (a) that the graph of f lies below all its tangents and the slopes of the tangents decrease from left to right. Thus, the graph is concave down ward. Similarly, the graph of f in figure 4.25 (b) lies above all its tangents. Thus it is concave upward.

A point where a curve changes its direction of concavity is called an **inflection point**.

Theorem 4.27 (*concavity test*)

Assume that f'' exist on an open interval I

a. If $f'' > 0$ for all x in I, then the graph of f is concave upward on I.
b. If $f'' < 0$ for all x in I, then the graph of f is concave downward on I.

Example 1: Let $f(x) = 3x^4 - 4x^3$. Find the interval on which the graph of f is concave upward and those on which it is concave downward. Then sketch the graph of f.

Solution:

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$$

$$f''(x) = 36x^2 - 24x = 36x(x - \frac{2}{3})$$

and that $f(1) = -1$ is a relative minimum value of f. Now we determine the sign of $f''(x)$.

	0		$\frac{2}{3}$
x	$- - -$	$+ + +$	$+ + +$
$x - \frac{2}{3}$	$- -$	$- - -$	$+ + +$
$f''(x)$	$+ + +$	$- - -$	$+ + +$

Figure 4.26

From figure 4.26 and theorem 4.27 we deduce that the graph of f is concave upward on $(-\infty, 0)$and $({}^2/_3, 0)$ and is concave downward on $(0, {}^2/_3)$. From this information we conclude the graph of f is as shown in figure 4.27. ■

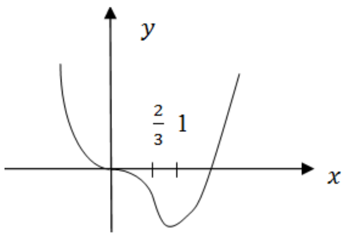


Figure 4.27

Activity

Discuss the curve $y = 4x^3 - 6x^2 - 9x$ with respect to concavity, inflection point and relative maximum and minimum. Use this information to sketch the curve.

Assume that f'' exists and is continuous on an interval containing c. Assume also that the graph of f has an inflection point at $(c, f(c))$, there is a change of concavity at $(c, f(c))$. Because of the continuity of f'' it is possible to show that $f''(c) = 0$. Thus we are led to the following method of finding inflection points:

1. Find the values of c for which $f''(c) = 0$.
2. For each value of c found in step 1, determine whether f'' changes sign at c.

3. If f'' changes sign at c, the point $\left(c, f(c)\right)$ is an inflection point.

Example 2: Let $f(x) = x^3$. Find the inflection points of the graph of f.

Solution: The derivatives of f are

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

Since $f''(0) = 0$ and f'' changes sign at 0, the point $\left(0, f(0)\right) = (0, 0)$ is the only inflection point of the graph of f. ■

Exercises

1. *Find the interval on which the graph of the function is concave upward and those on which it concave downward. Then sketch the graph of the function.*
   a. $f(x) = \frac{3}{2}x^2 + x$
   b. $f(x) = x^3 - 6x^2 + 9x + 2$
   c. $f(x) = 3x^5 - 5x^3$
   d. $g(x) = x\sqrt{x^2 - 4}$
   e. $g(x) = x^4 - 6x^2 + 8$
2. *Find the interval of concavity and the inflection points*
   a. $f(x) = x^3 - 12x + 1$
   b. $f(x) = x^4 - 4x - 1$
   c. $f(x) = \frac{x^2}{x^2+3}$
3. *Let* $f(x) = ax^3 + bx^2 + cx + d$ *with* $a \neq 0$. *Show that the graph of f has exactly one inflection point and find it.*
4. *Suppose* f'' *is continuous on* $(-\infty, \infty)$
   a. *If* $f'(2) = 0$ *and* $f''(2) = -5$. *What can you say about* f*?*
   b. *If* $f'(6) = 0$ *and* $f''(6) = 0$. *What can you say about* f*?*

5. *Suppose the derivative of function f is*

$$f'(x) = (x+1)^2(x-3)^5(x-6)^4.$$

On which interval is f increasing?

Curve sketching

As we have seen throughout chapter three, a knowledge of derivatives is very important in sketching graphs of functions. In this section we collect the method we have encountered for sketching graphs and we illustrate their uses.

Table 4.1 lists the items that are must important in graphing a function.

Table 4.1

Property	Test
f has y intercept c	$f(0) = c$
f has x intercept c	$f(c) = 0$
Graph of f is symmetric with respect to $\begin{cases} y\ axis \\ origin \end{cases}$	$f(-x) = f(x)$ $f(-x) = -f(x)$
f has a relative maximum value at c	$\begin{cases} f'(c) = 0\ and\ f'\ changes\ from + ve\ to - ve \\ f'(c) = 0\ and\ f''(c) < 0 \end{cases}$
f has a relative minimum value at c	$\begin{cases} f'(c) = 0\ and\ f'\ changes\ from - ve\ to\ + ve \\ f'(c) = 0\ and\ f''(c) > 0 \end{cases}$
f is strictly increasing on an open interval I	$f'(x) > 0$ for all except finitely many x in I
f is strictly decreasing on an open interval I	$f'(x) < 0$ for all except finitely many x in I
Graph f is concave upward on an open interval I	$f''(x) > 0$ for all x in I
Graph f is concave downward on an open interval I	$f''(x) < 0$ for all x in I
$(c, f(c))$ is an inflection point of the graph of f	f'' changes sign at c(and usually $f''(c) = 0$)
f has a vertical asymptote $x = c$	$\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$
f has a horizontal asymptote $y = d$	$\lim_{x\to d^+} f(x) = d$ or $\lim_{x\to d^+} f(x) = d$

Example 1: Let $f(x) = \frac{2}{1+x^2}$. Sketch the graph of f

Solution:

- y −intercept

  $f(0) = 2$, so the y –intercept is 2.

- x −intercept

  $f(x) = 0 => 2 = 0$ it is false for all x, so f has no x −intercept.

- Symmetry: $f(-x) = \frac{2}{1+(-x)^2} = \frac{2}{1+x^2} = f(x)$

  so the graph of f is symmetric with respect to y−axis.

- Extreme

  $f'(x) = -\frac{4x}{(1+x^2)^2}$ and $f''(x) = \frac{4(3x^2-1)}{(1+x^2)^3}$

  Since $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, it follows that f is strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$, so that $f(0) = 2$ is the maximum value of f.

- Concavity

  Now we display the sign of f''

		$\frac{-1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
$\sqrt{3x}+1$	− − −	+ + +	+ + +
$\sqrt{3x}-1$	− − −	− − −	+ + +
$f''(x)$	+ + +	− − −	+ + +

Figure 4.28

From figure 4.28 , the graph is concave upward on $(-\infty, \frac{-1}{3}\sqrt{3})$ and on $(\frac{1}{3}\sqrt{3}, \infty)$ and it is concave downward on $\left(\frac{-1}{3}\sqrt{3}, \frac{3}{2}\right)$ and $(\frac{1}{3}\sqrt{3}, \frac{3}{2})$.

- Asymptote

  $\lim_{x\to\infty} \frac{2}{1+x^2} = \lim_{x\to-\infty} \frac{2}{1+x^2} = 0$

Which means that x - axis is a horizontal asymptote of the graph.

We are now ready to sketch the graph of f shown in figure 3.20

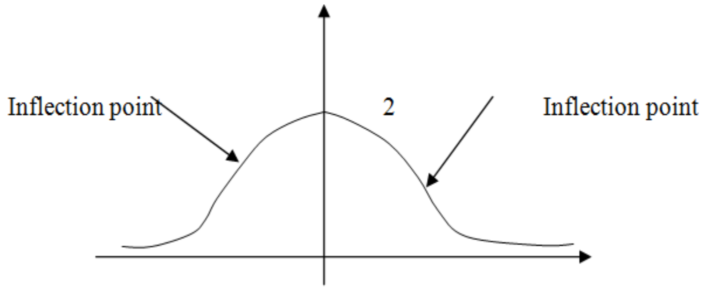


Figure 4.29 ■

> ***Activity***
>
> *Skecth the graph of* $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Exercises

1. *Skecth the graph of the given function noting all relevant proberties listed in table3.1.*
   a. $f(x) = x^3 - 8x^2 - 16x - 3$
   b. $f(x) = x^4 - 6x^2$
   c. $f(x) = x + \frac{4}{4}$
   d. $f(x) = \frac{(x+1)^2}{x-1}$
2. *Skecth the graph of the given equation*
   a. $\frac{x^2}{9} + \frac{y^2}{4} = 1$
   b. $x^2 + y^2 = 9$ c. $6x^2 + 24y^2 = 9$

Summary

- The function f is said to have a derivative at point a if

  $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ or $\lim_{h \to o} \frac{f(a+h)-f(a)}{h}$ exists.
- The slope of a function means the derivative of a function.
- The equation of a tangent line to the curve $y = f(x)$ at a point $(a, f(a))$ is $y = m(x - a) + f(a)$ where m is the slop.

- ❖ If the derivative of a function f is exist at each point in its domain then we say that a function f is differentiable.
- ❖ If f is differentiable at a then f is contionous.
- ❖ A function f is differentiable on $[a, b]$ if it is differentiable on (a, b) and if the left and right sided limit are exist.
- ❖ Differentiation rule: if f and g are differentiable function and c be any constant number then:
- ❖ $\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$
- ❖ $\frac{d}{dx}(cf) = c\frac{d}{dx}(f)$
- ❖ $\frac{d}{dx}(f-g) = \frac{d}{dx}(f) - \frac{d}{dx}(g)$
- ❖ $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f\frac{d}{dx}(g)}{g^2}$
- ❖ $\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g)$
- ❖ $\frac{d}{dx}(fog)(a) = g'(f(a))f'(a)$
- ❖ A function f is said to have an absolute maximum on an interval I at the point x_0 if $f(x_0) \geq f(x)$, $f(x_0)$ is called the maximum value of f on I.
- ❖ A function f is said to have an absolute minimum on an interval I at the point x_0 if $f(x_0) \leq f(x)$, $f(x_0)$ is called the minimum value of f on I.
- ❖ If f has either an absolute maximum or minimum at x_0 then f is said to have an absolute extremum on I at x_0 .
- ❖ A function f is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I
- ❖ A function f is called decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I
- ❖ A function is monotonic if it is either increasing or decreasing function.
- ❖ A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ doesn't exist.
- ❖ If $f'(x) > 0$ on an interval I then f is increasing on I.

- If $f'(x) < 0$ on an interval I then f is decreasing on I.
- If f' changes from positive to negative at c, then f has a relative maximum at c. Were c is a critical number.
- If f' changes from negative to positive at c, then f has a relative minimum value at c. Where c is a critical number.
- If $f''(c) < 0$ then $f(c)$ is a relative maximum value of f. Where c is a critical number.
- If $f''(c) > 0$ then $f(c)$ is a relative minimum value of f . Where c is a critical number.
- If $f'' > 0$ for all x in I, then the graph of f is concave upward on I.
- If $f'' < 0$ for all x in I, then the graph of f is concave downward on I.

Review exercises

1. *Find the derivative of the following function at x*

   a. $f(x) = \sqrt{x}(x^2 - 3)^{4/7}$ *c.* $f(x) = \frac{1}{(4-x^2)^{3/2}}$

   b. $f(x) = \frac{4x^2+2}{3x-8}$ *d.* $f(t) = t^2 \sin\frac{1}{t}$

2. *Find* $\frac{dy}{dx}$

   a. $y = 4x^3 - \sqrt{3}x + \frac{2}{5x}$ *c.* $y = x^2 tan^2 x$

   b. $y = 3\sin 2x - \sqrt{x}\cos x$ *d.* $y = \frac{x^2-x+1}{x^2+x+1}$

3. *Find the equation of the tangent line at the given point.*

   a. $f(x) = 3x^3 - 2x^2 + 4; (1,5)$ *c.* $f(x) = (x-2)^{1/7}$

   b. $f(x) = \sin x - 3\cos 2x; (\frac{\pi}{6}, -1)$ *d.*

   $f(x) = \begin{cases} 2\sin x & for\ x < 0 \\ 3x^2 + 2x & for\ x \geq o \end{cases}$ (0,0)

4. *Find* $\frac{dy}{dx}$ *by implicit differentiation*

   a. $y(\sqrt{x} + 1) = x$

   b. $x^2 + y^2 = \frac{x^2}{y^2}$

   c. $xy = \sqrt{x} + \sqrt{y}$

5. *Assume that* x *and* y *are differentiable function of* t. *find* $\frac{dy}{dt}$ *in terms of* x, y *and* $\frac{dx}{dt}$

   a. $xy = 3$ b. $y = \sin xy^2$

6. *Let* f *be differentiable at* 0 *and let* $g(x) = f(x^2)$. *Show that* $g'(0) = 0$
7. *What is the equation of a tangent line to the parabola* $y = x^2$ *at* $(-2,4)$
8. *If* $f(t) = \sqrt{4t+1}$, *find* $f''(2)$
9. *If* $g(x) = x\sin x$, *find* $g''(\frac{\pi}{6})$
10. *If* $f(x) = 2^x$ *find* $f^n(x)$
11. *Find the relative extreme value of the function on the given interval. Determine at which numbers in the interval they are assumed.*
    a. $f(x) = x^2 + x + 1, [-2,2]$
    b. $f(x) = x - \sqrt{x}$, $[0,4]$
    c. $f(x) = x^{2/3} - x$, $[-\frac{1}{8}, \frac{1}{8}]$
    d. $f(x) = \frac{\ln x}{x^2}$ $[1,3]$
12. *Find the critical points of the given function*
    a. $f(x) = x^2\sqrt{2-x}$
    b. $f(x) = \cos x^{1/3}$
13. *Use the first derivative test or the second derivative test to determine the relative extreme value of the function*
    a. $f(x) = 3x^4 - 10x^3 + 6x^2 + 3$
    b. $f(x) = \frac{x-1}{x^2+3}$
    c. $f(x) = 2\sqrt{x+1} - \sqrt{x-1}$
    d. $f(x) = x + \sqrt{1-x}$
14. *For what value of the constant* a *and* b *is* $(1,6)$ *a point of inflection of the curve* $y = x^3 + ax^2 + bx + 1$?
15. *Determine the intervals on which f is increasing and those on which f is decreasing,*
    a. $f(x) = \frac{1}{3}x^3 - x^2 + x - 2$

b. $f(x) = x^{1/3} - x$

c. $f(x) = x^4 + x^3 + x^2 + x$

16. Let

$$f(x) = \begin{cases} x + 1, for\ x \leq 0 \\ x - 1\ for\ x > 0 \end{cases}$$

a. *Show that* $f(-1) = f(1) = 0$, *but that there is no number c in* $(-1,1)$ *such that* $f'(c) = 0$

b. *Why does this not contradict Rolle's Theorem*

17. *Determine the intervals on which the graph of f is concave upward and the intervals on which the graph is concave downward*

a. $f(x) = \sqrt{x} + \frac{1}{x}$

b. $f(x) = \frac{1}{2}x^4 + x^3 - 6x^2$

c. $f(x) = sin\, x + \frac{1}{4} sin\, 2x$

18. *Sketch the graph of the function, indicating all relevant properties listed in table3.1*

a. $f(x) = x^3 - 6x - 1$

b. $f(x) = x^4 + 2x^3 + 1$

c. $f(x) = x + \sqrt{1 - x}$

d. $f(x) = x^4 + 4x^3$

e. $f(x) = \frac{1}{1-x^2}$

19. *Suppose the distance* $D(v)$ *a car can travel on one tank of gas at a velocity of v miles per hour is given by*

$$D(v) = \frac{\sqrt{3}}{48}(80v^{3/2} - v^{5/2})\ for\ 0 \leq v \leq 75$$

What velocity maximizes D(and hence minimizes fue)

20. *Evaluate the limit*

a. $lim_{x\to 0} \frac{tan\, \pi x}{ln\,(1+x)}$

b. $lim_{x\to \infty} \frac{e^{4x}-1-4x}{x^2}$

c. $lim_{x\to 0^+} x^2\, ln\, x$

21. *A 10 cm ladder is leaning against a house. The base of the ladder is pulled away from the house at a rate of 0.25m/sec. How fast is the top of the ladder moving down the wall when the base is*

a. 6 m from the house

b. 8 m from the house

c. 9 m from the house

22. *A water tank is in the shape of an inverted circular cone with base radius 3m and height 5m. If water is being pumped in to the tank at a rate of 2 m^3/min, find the rate at which the water level is raising when the water is 3m deep.*

23. *Find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ assuming that y is differentiable wrt x and x is differentiable wrt y.*

$$x^2 + y^2 = 25$$

24. *The radius r of a sphere is increasing at a rate of 3cm/min. Find the rate of change of the volume when*

a) r = 2 cm *b) r = 3 cm.*

References

1. James Stewart, Calculus early trancedental, 6th ed., Prentice Hall, 2008
2. Howard Anton,Calculus a new horizon, 6th ed., John wiley and Sons Inc
3. Robert Ellis and Denny Gulick, Calculus with analytic geometry, 6th ed, Harcourt Brace Jovanovich, Publishers, 5th ed, 1993
4. Adams, Calculus: A complete course, 5th ed, Addison Wesley, 2003
5. R. Wrede and M. R. Spiegel, Theory of advanced calculus, 2nd ed., McGraw-Hill, 2002.
6. A. E. Taylor and W. R. Mann, Advanced calculus, 3rd ed, John-Wiley and Son, INC, 1995.
7. R. T. Smith and R. B. Minton, Calculus concepts and connections, McGram-Hill book company, 2006
8. D. V. Widder, Advanced calculus, Prentice-Hall, 1979
9. Ross L. Finney et al, Calculus, Addison Wesley, 1995
10. E. J. Purcell and D. Varberg, Calculus with analytic geometry, Prentice-Hall INC., 1987
11. Lynne, Garner: Calculus and Analytic Geometry, Dellen Publishing Company
12. John A. Tierney: Calculus and Analytic, 4th edition, Allyn and Bacon, Inc. –Boston.
13. Earl W. Swokowski: Calculus with Analytic Geometry, 2nd edition, Prindle, Weber and Schmidt.
14. Goldstein, L. J., Lay D. C. , and Schneider D. I (1987). Calculus and its applications, 4th, edn, Prentice-Hall, Inc, London
15. Kresying, Advanced Engineering Mathematics.
16. Alan Jeffry , Advanced Engineering Mathematics

CHAPTER-FIVE:

INTEGRATION

UNIT OBJECTIVES:

At the end of this unit each student will able to:

- Know antidifferentiation
- Understand indefinite and definite integrals.
- Realize and apply techniques of integration.
- Understand the **Fundamental Theorem of Calculus**.
- Know properties of indefinite and definite integrals
- Learn about **improper integrals**.

Introduction

In this chapter we will look integrals. As with derivatives, this chapter will be devoted almost exclusively to finding and computing integrals. Applications will be given in the following chapter. There are really two types of integrals that we'll be looking at in this chapter : Indefinite Integrals and Definite Integrals. The first half of this chapter is devoted to indefinite integrals and the second half is devoted to definite integrals. As we will see in the second half of the chapter if we don't know indefinite integrals we will not be able to do definite integrals.

5.1- Antiderivatives; indefinite integrals

In the past chapter we've been given a function, , and asking what the derivative of this function was. Starting with this section we are not going to turn things around. We now want to ask what function we differentiated to get the function .

Definition 5.1

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

Question: What function did we differentiate to get the following function.

$$f(x) = x^4 + 3x - 9$$

Solution :

Let's actually start by getting the derivative of this function to help us see how we're going to have to approach this problem. The derivative of this function is,

$$f'(x) = 4x^3 + 3$$

The point of this was to remind us of how differentiation works. When differentiating powers of x we multiply the term by the original exponent and then drop the exponent by one.

Now, let's go back and work the problem. In fact let's just start with the first term. We got x^4 by differentiating a function and since we drop the exponent by one it looks like we must have differentiated x^5. However, if we had differentiated x^5 we would have $5x^4$ and we don't have a 5 in front our first term, so the 5 needs to cancel out after we've differentiated. It looks then like we would have to differentiate $\frac{1}{5}x^5$ in order to get x^4.

Likewise for the second term, in order to get $3x$ after differentiating we would have to differentiate $\frac{3}{2}x^2$. Again, the fraction is there to cancel out the 2 we pick up in the differentiation.

The third term is just a constant and we know that if we differentiate x we get 1. So, it looks like we had to differentiate $-9x$ to get the last term.

Putting all of this together gives the following function,

$$F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x$$

Our answer is easy enough to check. Simply differentiate $F(x)$.

$$F'(x) = x^4 + 3x - 9 = f(x)$$

So, it looks like we got the correct function. Or did we? We know that the derivative of a constant is zero and so any of the following will also give $f(x)$ upon differentiating.

$$F(x)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x+10$$
$$F(x)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x-1954$$
$$F(x)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x+\frac{3469}{123}$$
$$etc.$$

In fact, any function of the form,

$$F(x)=\frac{1}{5}x^5+\frac{3}{2}x^2-9x+c, \quad c \text{ is a constant}$$

will give $f(x)$ upon differentiating.

EXAMPLE 1 Finding Antiderivatives

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$
(b) $g(x) = \cos x$
(c) $h(x) = 2x + \cos x$

Solution

(a) $F(x) = x^2$
(b) $G(x) = \sin x$
(c) $H(x) = x^2 + \sin x$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$ and the derivative of $H(x) = x^2 + \sin x$ is $2x + \cos x$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.2 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

Theorem 5.2 : If F(x) is an antiderivative of f(x) on an interval I, then for any constant C the function F(x) + C is also an antiderivative of f(x) on that interval. Moreover, each antiderivative of f(x) on the interval I can be expressed in the form F(x) + C by choosing the constant C appropriately.

Proof: Exercise

The process of finding antiderivatives is called antidifferentiation or integration

EXAMPLE 2 Finding a Particular Antiderivative

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Solution Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative

$$F(x) = -\cos x + C$$

gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ into $F(x) = -\cos x + C$ gives

$$F(0) = -\cos 0 + C = -1 + C.$$

Since $F(0) = 3$, solving for C gives $C = 4$. So

$$F(x) = -\cos x + 4$$

is the antiderivative satisfying $F(0) = 3$. ■

EXAMPLE 3: Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$

Solution

(a) $F(x) = \dfrac{x^6}{6} + C$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

(c) $H(x) = \dfrac{-\cos 2x}{2} + C$

(d) $I(x) = \dfrac{\sin (x/2)}{1/2} + C = 2 \sin \dfrac{x}{2} + C$

Definition 5.3

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\,dx = F(x) + c, \quad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, x is called the **integration variable** and the "c" is called the **constant of integration**.

Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the context of the problem that we are talking about an indefinite integral (or definite integral).
The process of finding the indefinite integral is called **integration** or **integrating *f(x)***. If we need to be specific about the integration variable we will say that we are **integrating *f(x)* with respect to *x***.

EXAMPLE 4*:* Evaluate the following indefinite integral.

$$\int x^4 + 3x - 9\,dx$$

Solution

Since this is really asking for the most general anti-derivative we just need to reuse the final answer from the first example.

The indefinite integral is,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

A couple of warnings are now in order. One of the more common mistakes that students make with integrals (both indefinite and definite) is to drop the *dx* at the end of the integral. This is required! Think of the integral sign and the *dx* as a set of parenthesis. You already know and are probably quite comfortable with the idea that every time you open a parenthesis you must close it. With integrals, think of the integral sign as an "open parenthesis" and the *dx* as a "close parenthesis".
If you drop the *dx* it won't be clear where the integrand ends. Consider the following variations of the above example.

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

$$\int x^4 + 3x\,dx - 9 = \frac{1}{5}x^5 + \frac{3}{2}x^2 + c - 9$$

$$\int x^4\,dx + 3x - 9 = \frac{1}{5}x^5 + c + 3x - 9$$

You only integrate what is between the integral sign and the *dx*.
Knowing which terms to integrate is not the only reason for writing the *dx* down. In the Substitution Rule section we will actually be working with the *dx* in the problem and if we aren't in the habit of writing it down it will be easy to forget about it and then we will get

the wrong answer at that stage. The moral of this is to make sure and put in the *dx*! At this stage it may seem like a silly thing to do, but it just needs to be there, if for no other reason than knowing where the integral stops.

On a side note, the *dx* notation should seem a little familiar to you. We saw things like this a couple of sections ago. We called the *dx* a differential in that section and yes that is exactly what it is. The *dx* that ends the integral is nothing more than a differential. The next topic that we should discuss here is the integration variable used in the integral. Actually there isn't really a lot to discuss here other than to note that the integration variable doesn't really matter. For instance,

$$\int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

$$\int t^4 + 3t - 9\,dt = \frac{1}{5}t^5 + \frac{3}{2}t^2 - 9t + c$$

$$\int w^4 + 3w - 9\,dw = \frac{1}{5}w^5 + \frac{3}{2}w^2 - 9w + c$$

Changing the integration variable in the integral simply changes the variable in the answer. It is important to notice however that when we change the integration variable in the integral we also changed the differential (*dx*, *dt*, or *dw*) to match the new variable. This is more important that we might realize at this point.

Another use of the differential at the end of integral is to tell us what variable we are integrating with respect to. To see why this is important take a look at the following two integrals.

$$\int 2x\,dx \qquad\qquad \int 2t\,dx$$

The first integral is simple enough.

$$\int 2x\,dx = x^2 + c$$

The second integral is also fairly simple, but we need to be careful. The *dx* tells us that we are integrating *x*'s. That means that we only integrate *x*'s that are in the integrand and all other variables in the integrand are considered to be constants. The second integral is then,

$$\int 2t\,dx = 2tx + c$$

So, it may seem silly to always put in the *dx*, but it is a vital bit of notation that can cause us to get the incorrect answer if we neglect to put it in.

Now, there are some important properties of integrals that we should take a look at.

Properties of the Indefinite Integral

1. $\int k\, f(x)\,dx = k\int f(x)\,dx$ where k is any number. So, we can factor multiplicative constants out of indefinite integrals.

2. $\int -f(x)\,dx = -\int f(x)\,dx$. This is really the first property with $k=-1$ and so no proof of this property will be given.

3. $\int f(x)\pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$. In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.

Proof of : $\int k\, f(x)\,dx = k\int f(x)\,dx$ where k is any number.

This is a very simple proof. Suppose that $F(x)$ is an anti-derivative of $f(x)$, *i.e.* $F'(x) = f(x)$. Then by the basic properties of derivatives we also have that,

$$(k\,F(x))' = k\,F'(x) = k\,f(x)$$

and so $k\,F(x)$ is an anti-derivative of $k\,f(x)$, *i.e.* $(k\,F(x))' = k\,f(x)$. In other words,

$$\int k\,f(x)\,dx = k\,F(x) + c = k\int f(x)\,dx$$

Proof of : $\int f(x)\pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

This is also a very simple proof Suppose that $F(x)$ is an anti-derivative of $f(x)$ and that $G(x)$ is an anti-derivative of $g(x)$. So we have that $F'(x) = f(x)$ and $G'(x) = g(x)$. Basic properties of derivatives also tell us that

$$(F(x)\pm G(x))' = F'(x)\pm G'(x) = f(x)\pm g(x)$$

and so $F(x)+G(x)$ is an anti-derivative of $f(x)+g(x)$ and $F(x)-G(x)$ is an anti-derivative of $f(x)-g(x)$. In other words,

$$\int f(x)\pm g(x)\,dx = F(x)\pm G(x) + c = \int f(x)\,dx \pm \int g(x)\,dx$$

Not listed in the properties above were integrals of products and quotients. The reason for this is simple. Just like with derivatives each of the following will NOT work.

$$\int f(x)\,g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx \qquad\qquad \int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

With derivatives we had a product rule and a quotient rule to deal with these cases. However, with integrals there are no such rules. When faced with a product and quotient in an integral we will have a variety of ways of dealing with it depending on just what the integrand is.

We can now answer this question easily with an indefinite integral.

$$f(x) = \int f'(x)\,dx$$

EXAMPLE 5: *If* $f'(x) = x^4 + 3x - 9$ *then find* $f(x)$

Solution

By this point in this section this is a simple question to answer.

$$f(x) = \int f'(x)\,dx = \int x^4 + 3x - 9\,dx = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x + c$$

Computing Indefinite Integrals

The table below shows list of indefinite integrals of some functions

1 TABLE OF INDEFINITE INTEGRALS

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x}\,dx = \ln\|x\| + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$\int \sinh x\,dx = \cosh x + C$	$\int \cosh x\,dx = \sinh x + C$

EXAMPLE 6: Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

Solution:

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$$

You should check this answer by differentiating it.

EXAMPLE 7: Evaluate $\int \frac{\cos\theta}{\sin^2\theta}d\theta$

Solution:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C$$

EXAMPLE 7: Evaluate $\int \frac{t^2 - 2t^4}{t^4}dt$

Solution:

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

Activity:

Evaluate each of the following indefinite integrals

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$ **(b)** $\int x^8 + x^{-8}\,dx$ **(c)** $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$ **(e)** $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$ **(f)** $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Evaluate each of following integrals

(a) $\int 3e^x + 5\cos x - 10\sec^2 x\,dx$ **(b)** $\int 2\sec w\tan w + \frac{1}{6w}\,dw$

5.2- *Techniques of integration*

Objectives:

By the end of this section, students will be able to:

- Evaluate indefinite integrals using integration by substitution
- Find indefinite integrals using integration by parts & by partial fraction
- Determine the value of trigonometric integrals
- Apply integration by trigonometric substitution to calculate integrals

Over view: In this section, we are going to discuss Integration by substitution, by parts and by partial fraction, Trigonometric integrals, Integration by trigonometric substitution.That is we develop techniques for using the basic integration formulas

To obtain indefinite integrals of more complicated functions. Integration is not as Straightforward as differentiation; there are no rules that absolutely guarantee obtaining an indefinite integral of a function. Therefore we discuss a strategy for **integration.**

5.2.1: Integration by substitution, by parts and by partial fraction

Integration by substitution method

Our antidifferentiation formulas don't tell us how to evaluate integrals such as

$\int 2x\sqrt{x^2+1}\,dx$, $\int \sin 3x\,dx$ but powerful method for changing the variable

of integration so that these integrals (and many others) can be evaluated by using the basic integration formulas which are given in section 5.1.

Theorem 5.4 ***: The substitution Rule***

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof The rule is true because, by the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f:

$$\begin{aligned}\frac{d}{dx}F(g(x)) &= F'(g(x)) \cdot g'(x) && \text{Chain Rule}\\ &= f(g(x)) \cdot g'(x). && \text{Because } F' = f\end{aligned}$$

If we make the substitution $u = g(x)$ then

$$\begin{aligned}\int f(g(x))g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx\\ &= F(g(x)) + C && \text{Fundamental Theorem}\\ &= F(u) + C && u = g(x)\\ &= \int F'(u)\,du && \text{Fundamental Theorem}\\ &= \int f(u)\,du && F' = f\end{aligned}$$

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

EXAMPLE 1 Using the Power Rule

$$\begin{aligned}\int \sqrt{1+y^2}\cdot 2y\,dy &= \int \sqrt{u}\cdot\left(\frac{du}{dy}\right)dy && \text{Let } u = 1 + y^2,\ du/dy = 2y\\ &= \int u^{1/2}\,du\\ &= \frac{u^{(1/2)+1}}{(1/2)+1} + C && \text{Integrate, using Eq. (1) with } n = 1/2.\\ &= \frac{2}{3}u^{3/2} + C && \text{Simpler form}\\ &= \frac{2}{3}(1+y^2)^{3/2} + C && \text{Replace } u \text{ by } 1 + y^2.\end{aligned}$$

EXAMPLE 2 Adjusting the Integrand by a Constant

$$\int \sqrt{4t-1}\,dt = \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt$$

$$= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt \qquad \text{Let } u = 4t-1,\ du/dt = 4.$$

$$= \frac{1}{4}\int u^{1/2}\,du \qquad \text{With the } 1/4 \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{1}{6}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{1}{6}(4t-1)^{3/2} + C \qquad \text{Replace } u \text{ by } 4t-1.$$

EXAMPLE 3 Using Substitution

$$\int \cos(7\theta + 5)\,d\theta = \int \cos u \cdot \frac{1}{7}\,du \qquad \text{Let } u = 7\theta + 5,\ du = 7\,d\theta,\ (1/7)\,du = d\theta.$$

$$= \frac{1}{7}\int \cos u\,du \qquad \text{With the } (1/7) \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate with respect to } u, \text{ Table 4.2.}$$

$$= \frac{1}{7}\sin(7\theta + 5) + C \qquad \text{Replace } u \text{ by } 7\theta + 5.$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 5)$.

EXAMPLE 4 Using Substitution

$$\int x^2 \sin(x^3)\,dx = \int \sin(x^3)\cdot x^2\,dx$$

$$= \int \sin u \cdot \frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate with respect to } u.$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

EXAMPLE 5 Using Identities and Substitution

$$\begin{aligned}
\int \frac{1}{\cos^2 2x}\,dx &= \int \sec^2 2x\,dx && \frac{1}{\cos 2x} = \sec 2x \\
&= \int \sec^2 u \cdot \frac{1}{2}\,du && u = 2x,\ du = 2\,dx,\ dx = (1/2)\,du \\
&= \frac{1}{2}\int \sec^2 u\,du \\
&= \frac{1}{2}\tan u + C && \frac{d}{du}\tan u = \sec^2 u \\
&= \frac{1}{2}\tan 2x + C && u = 2x
\end{aligned}$$

The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. If the first substitution fails,try to simplify the integrand further with an additional substitution or two. Alternatively, we can start fresh. There can be more than one good way to start,as in the next example.

EXAMPLE 6 Using Different Substitutions

Evaluate $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Solution:

Solution 1: Substitute $u = z^2 + 1$.

$$\begin{aligned}
\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2+1,\ du = 2z\,dz. \\
&= \int u^{-1/3}\,du && \text{In the form } \int u^n\,du \\
&= \frac{u^{2/3}}{2/3} + C && \text{Integrate with respect to } u. \\
&= \frac{3}{2}u^{2/3} + C \\
&= \frac{3}{2}(z^2+1)^{2/3} + C && \text{Replace } u \text{ by } z^2+1.
\end{aligned}$$

Solution 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2+1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

Integration by Parts

If we try to evaluate integrals of the type $\int xe^x dx$, and $\int \ln x\, dx$ by using the method of substitution we obviously fail. But don't worry the next theorem will enable us to evaluate not only these, but also many other types of integrals.

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

Integration by parts *is* a technique for simplifying integrals of the form $\int f(x)g(x)dx$ It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty.

Theorem 5.5 ***: Integration by parts***

If f and g are differentiable and f' and g' are continous then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Proof:

If f and g are differentiable functions, then by the rule for differentiating products

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integrating both sides we obtain

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

or

$$f(x)g(x) + C = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

or

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx + C$$

Since the integral on the right will produce another constant of integration, there is no need to keep the C in this last equation; thus, we obtain

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \tag{1}$$

which is called the formula for ***integration by parts***. By using this formula we can sometimes reduce a hard integration problem to an easier one.

In practice, it is usual to rewrite (1) by letting

$u = f(x), \quad du = f'(x)\,dx$

$v = g(x), \quad dv = g'(x)\,dx$

This yields the following alternative form for (1):

$$\int u\,dv = uv - \int v\,du \tag{2}$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 **Using Integration by Parts**

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C. \quad ■$$

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x \cos x \, dx = \int u \, dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x \, dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x)\, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v \, du = \int (x \cos x - x^2 \sin x)\, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x \, dx, \\ du &= -\sin x \, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v \, du = -\int \frac{x^2}{2} \sin x \, dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u \, dv$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\quad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\quad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

Sometimes we have to use integration by parts more than once.

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2\int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

The technique of Example 4 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x\, dx.$$

Solution Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\, dx, \qquad v = -\cos x, \qquad du = e^x\, dx.$$

Then

$$\begin{aligned}\int e^x \cos x\, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right)\\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Example 6 : ***A Reduction Formula***

Obtain a "reduction" formula that expresses the integral

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$

Hence

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx, \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\, dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

This allows us to reduce the exponent on $\cos x$ by 2 and is a very useful formula. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is either

$$\int \cos x\, dx = \sin x + C \qquad \text{or} \qquad \int \cos^0 x\, dx = \int dx = x + C. \qquad \blacksquare$$

Example 7 : Using a reduction formula evaluate

$$\int \cos^3 x\, dx.$$

Solution: **From the result in example 6**

$$\begin{aligned}\int \cos^3 x\, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\, dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C. \qquad \blacksquare\end{aligned}$$

Summary

When substitution doesn't work, try integration by parts. Start with an integral in which the integrand is the product of two functions,

$$\int f(x)g(x)\,dx.$$

(Remember that g may be the constant function 1, as in Example 3.) Match the integral with the form

$$\int u\,dv$$

by choosing dv to be part of the integrand including dx and either $f(x)$ or $g(x)$. Remember that we must be able to readily integrate dv to get v in order to obtain the right side of the formula

$$\int u\,dv = uv - \int v\,du.$$

If the new integral on the right side is more complex than the original one, try a different choice for u and dv.

Integration by Partial fraction

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3},$$

which can be verified algebraically by placing the fractions on the right side over a common denominator $(x + 1)(x - 3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x + 1)\,(x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the above example, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \tag{1}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x+1)$ and $B/(x-3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until proper values for them have been found.

To find A and B, we first clear Equation (1) of fractions, obtaining

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. See Example 3 of this section.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Distinct Linear Factors

Evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx$$

using partial fractions.

Solution The partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C we clear fractions and get

$$\begin{aligned} x^2+4x+1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\begin{array}{ll} \text{Coefficient of } x^2: & A + B + C = 1 \\ \text{Coefficient of } x^1: & 4A + 2B \quad = 4 \\ \text{Coefficient of } x^0: & 3A - 3B - C = 1 \end{array}$$

There are several ways for solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables, or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K, \end{aligned}$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 A Repeated Linear Factor

Evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x+7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$

EXAMPLE 3 Integrating an Improper Fraction

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{\smash{)}\, 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \int 2x\,dx + \int \frac{5x-3}{x^2 - 2x - 3}\,dx$$

$$= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$
$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

EXAMPLE 4 Integrating with an Irreducible Quadratic Factor in the Denominator

Evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

using partial fractions.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \qquad (2)$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 \\ &\quad + (A-2B+C)x + (B-C+D). \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned} &\text{Coefficients of } x^3: && 0 = A + C \\ &\text{Coefficients of } x^2: && 0 = -2A + B - C + D \\ &\text{Coefficients of } x^1: && -2 = A - 2B + C \\ &\text{Coefficients of } x^0: && 4 = B - C + D \end{aligned}$$

We solve these equations simultaneously to find the values of A, B, C, and D:

$$\begin{aligned} &-4 = -2A, \quad A = 2 && \text{Subtract fourth equation from second.} \\ &C = -A = -2 && \text{From the first equation} \\ &B = 1 && A = 2 \text{ and } C = -2 \text{ in third equation.} \\ &D = 4 - B + C = 1. && \text{From the fourth equation} \end{aligned}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C.\end{aligned}$$

EXAMPLE 5 A Repeated Irreducible Quadratic Factor

Evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \quad C=0, \quad 2A+B+D=0, \quad C+E=0, \quad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad u = x^2+1,\ du = 2x\,dx \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.\end{aligned}$$

Note : we can determine the constants that appear in partial fractions using different methods such as differentiation method and assigning numerical Values to *x*

Example 6 : Find *A*, *B*, and *C* in the equation using differentiation method

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$ ■

In some problems, assigning small values to x such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

Example 7: Find *A*, *B*, and *C* in the equation using assigning numerical value for x

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \quad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\
2 &= 2A \\
A &= 1 \\
x = 2: \quad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\
5 &= -B \\
B &= -5 \\
x = 3: \quad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\
10 &= 2C \\
C &= 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$ ■

Exercise

I- *Evaluate the following integrals using integration by substitution*

1. $\int \sin^2 x dx$
2. $\int \csc x dx$
3. $\int \frac{x}{(x^2+5)^3} dx$
4. $\int \frac{x}{\sqrt[3]{1-2x^2}} dx$
5. $\int \frac{1}{x}(1+\ln x)^4 dx$
6. $\int \tan x \, dx$
7. $\int \sec x \, dx$

II- *Evaluate the following integrals using integration by Parts*

1. $\int xe^{-x} dx$
2. $\int x \ln x dx$
3. $\int \sec^3 x dx$
4. $\int x2^x dx$
5. $\int x \tan x \sec x dx$
6. $\int_0^{\pi/2} 2t \sin 2t dt$
7. $\int (x+1)^{10}(x+2) dx$
8. $\int \sin(\ln x) dx$ (Hint: Let $u = \sin(\ln x)$)
9. $\int \tan^{-1} x dx$
10. $\int \cos\sqrt{x} dx$

III- *Evaluate the following integrals using integration by Partial fractions*

1. $\int \frac{x^2}{x^2-1} dx$
2. $\int \frac{2x^2-12x+4}{x^3-4x^2} dx$
3. $\int_{-1}^{0} \frac{x^2+x+1}{x^2+1} dx$
4. $\int \frac{-x^3+x^2+x+3}{(x+1)(x^2+1)^2} dx$
5. $\int \frac{x^2-1}{x^3+3x+4} dx$
6. $\int \frac{\sqrt{x}+1}{x+1} dx$; ($H$ int : *Substitute* $u = \sqrt{x}$)
7. $\int \frac{\sin^2 x \cos x}{\sin^2 x + 1} dx$
8. $\int_0^{\pi/4} \tan^3 x dx$; ($H$ int : *Substitute* $u = \tan x$)
9. $\int \frac{e^x}{1-e^{3x}} dx$
10. $\int \frac{dx}{1+3e^x+2e^{2x}}$

5.2.2 Trigonometric Intergals

In this subsection we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines,but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

STRATEGY FOR EVALUATING $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Here are some examples illustrating each case.

EXAMPLE 1 ***m* is Odd**

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x))$$

$$= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x$$

$$= \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x\, dx.$$

Solution

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x) \qquad m = 0$$

$$= \int (1 - u^2)^2\, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x\, dx.$$

Solution

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x\, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx$$

$u = \sin 2x,$
$du = 2\cos 2x\,dx$

$$= \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x\cos^4 x\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Integrals of Powers of tan x and sec x

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

STRATEGY FOR EVALUATING $\int \tan^m x\sec^n x\,dx$

(a) If the power of secant is even ($n = 2k,\ k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x\sec^{2k} x\,dx = \int \tan^m x(\sec^2 x)^{k-1}\sec^2 x\,dx$$

$$= \int \tan^m x(1 + \tan^2 x)^{k-1}\sec^2 x\,dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x\tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x\sec^n x\,dx = \int (\tan^2 x)^k\sec^{n-1} x\ \sec x\ \tan x\,dx$$

$$= \int (\sec^2 x - 1)^k\sec^{n-1} x\ \sec x\ \tan x\,dx$$

Then substitute $u = \sec x$.

Example 5 Evaluate

$$\int tan^4 x dx$$

Solution

$$\begin{aligned}\int \tan^4 x\,dx &= \int \tan^2 x \cdot \tan^2 x\,dx = \int \tan^2 x \cdot (\sec^2 x - 1)\,dx \\ &= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx \\ &= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx \\ &= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx.\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\,dx$$

and have

$$\int u^2\,du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$ ■

EXAMPLE 6 Evaluate

$$\int \sec^3 x\,dx.$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x\,dx, \qquad v = \tan x, \qquad du = \sec x \tan x\,dx.$$

Then

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\,dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$ ■

EXAMPLE 7 Evaluate

$$\int tan^6 x sec^4 x dx$$

SOLUTION If we separate one $\sec^2 x$ factor, we can express the remaining $\sec^2 x$ factor in terms of tangent using the identity $\sec^2 x = 1 + \tan^2 x$. We can then evaluate the integral by substituting $u = \tan x$ so that $du = \sec^2 x\, dx$:

$$\begin{aligned}
\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x \sec^2 x \sec^2 x\, dx \\
&= \int \tan^6 x (1 + \tan^2 x) \sec^2 x\, dx \\
&= \int u^6 (1 + u^2)\, du = \int (u^6 + u^8)\, du \\
&= \frac{u^7}{7} + \frac{u^9}{9} + C \\
&= \tfrac{1}{7} \tan^7 x + \tfrac{1}{9} \tan^9 x + C
\end{aligned}$$

□

EXAMPLE 8 Evaluate

$$\int tan^5 \theta sec^7 \theta d\theta$$

SOLUTION If we separate a $\sec^2\theta$ factor, as in the preceding example, we are left with a $\sec^5\theta$ factor, which isn't easily converted to tangent. However, if we separate a $\sec\theta\ \tan\theta$ factor, we can convert the remaining power of tangent to an expression involving only secant using the identity $\tan^2\theta = \sec^2\theta - 1$. We can then evaluate the integral by substituting $u = \sec\theta$, so $du = \sec\theta\ \tan\theta\, d\theta$:

$$\begin{aligned}
\int \tan^5\theta\ \sec^7\theta\, d\theta &= \int \tan^4\theta\ \sec^6\theta\ \sec\theta \tan\theta\, d\theta \\
&= \int (\sec^2\theta - 1)^2 \sec^6\theta\ \sec\theta\ \tan\theta\, d\theta \\
&= \int (u^2 - 1)^2 u^6\, du \\
&= \int (u^{10} - 2u^8 + u^6)\, du \\
&= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C \\
&= \tfrac{1}{11} \sec^{11}\theta - \tfrac{2}{9} \sec^9\theta + \tfrac{1}{7} \sec^7\theta + C
\end{aligned}$$

□

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

These come from the angle sum formulas for the sine and cosine functions (Section 1.6). They give functions whose antiderivatives are easily found.

EXAMPLE 9 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$

Exercise:

Evaluate the following integrals

1. $\int \sin^3 x \cos^4 x dx$
2. $\int_0^{\pi/2} \sin^2 x \cos^5 x dx$
3. $\int \sqrt{\sin x} \cos^3 x dx$
4. $\int (\tan x + \cot x)^2 dx$
5. $\int \tan^3 x \csc^4 x dx$
6. $\int \cot^3 x \csc^3 x dx$
7. $\int \sin 5x \sin 3x dx$
8. $\int_0^{\pi/4} \cos x \cos 5x dx$
9. $\int_0^{\pi/3} \tan x \sec^{3/2} x dx$
10. $\int \tan^6 x dx$

5.2.3. Integration by Trigonometric Substitution

In this section we will discuss a method for evaluating integrals containing radicals by making substitutions involving trigonometric functions. We will also show how integrals containing quadratic polynomials can sometimes be evaluated by completing the square.

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a \tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$.

With $x = a\tan\theta$,

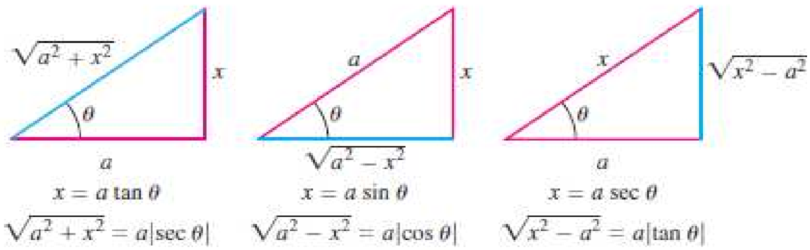
$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$



We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

The functions in these substitutions have inverses only for selected values of θ. For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\ \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

TABLE OF TRIGONOMETRIC SUBSTITUTIONS

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

EXAMPLE 1 Using the Substitution $x = a\tan\theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

Then

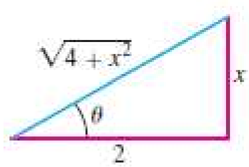
$$\int \frac{dx}{\sqrt{4 + x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4 + x^2}}{2} + \frac{x}{2}\right| + C \qquad \text{From Fig. 8.4}$$

$$= \ln|\sqrt{4 + x^2} + x| + C'. \qquad \text{Taking } C' = C - \ln 2$$



EXAMPLE 2 Using the Substitution $x = a\sin\theta$

Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$
$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \mathbf{9}\int \frac{\mathbf{1-cos2\theta}}{\mathbf{2}}\,\boldsymbol{d\theta}$$

$$= \frac{9}{2}\left(\theta - \frac{sin2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - sin\theta cos\theta) + C \quad , since\ sin2\theta = 2sin\theta cos\theta$$

$$= \frac{9}{2}(sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}.\frac{\sqrt{9-x^2}}{3}) + C$$

$$= \frac{9}{2}sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + C$$

EXAMPLE 3 Using the Substitution $x = a\sec\theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$
$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

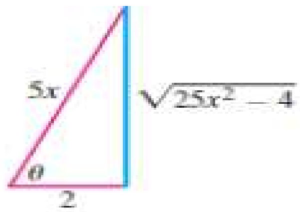
$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$
$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$
$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for $0 < \theta < \pi/2$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$
$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$



EXAMPLE 4 :Evaluate

$$\int \frac{1}{\sqrt{x^2 + 8x + 25}}\, dx$$

Solution: We complete the square for the quadratic expression as follows:

$$x^2+8x+25=(x^2+8x\qquad)+25$$
$$=(x^2+8x+16)+25-16$$
$$=(x+4)^2+9$$

Thus,

$$\int\frac{1}{\sqrt{x^2+8x+25}}dx=\int\frac{1}{\sqrt{(x+4)^2+9}}dx.$$

If we make the trigonometric substitution

$$x+4=3\tan\theta,\quad dx=3\sec^2\theta\,d\theta$$

then

$$\sqrt{(x+4)^2+9}=\sqrt{9\tan^2\theta+9}=3\sqrt{\tan^2\theta+1}=3\sec\theta$$

and

$$\int\frac{1}{\sqrt{x^2+8x+25}}dx=\int\frac{1}{3\sec\theta}3\sec^2\theta\,d\theta$$
$$=\int\sec\theta\,d\theta$$
$$=\ln\left|\sec\theta+\tan\theta\right|+C.$$

Using our formulas for $\tan\theta$ *and* $\sec\theta$, we conclude that

$$\int\frac{1}{\sqrt{x^2+8x+25}}dx=\ln\left|\frac{\sqrt{x^2+8x+25}}{3}+\frac{x+4}{3}\right|+C.$$

Exercise

Evaluate the following Integrals

Integrals containing $\sqrt{bx^2+cx+d}$

By completing the square in bx^2+cx+d we can express $\sqrt{bx^2+cx+d}$ in terms of $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, *or* $\sqrt{x^2-a^2}$ for suitable $a>0$. Then a trigonometric substitution eliminates the square root as before.

1. $\int \frac{1}{x^2\sqrt{9-x^2}}dx$
2. $\int \frac{(1-x^2)^{3/2}}{x^6}dx.$
3. $\int_0^1 \frac{1}{(3x^2+2)^{5/2}}dx$
4. $\int_1^{\sqrt{2}} \frac{1}{\sqrt{2x^2-1}}dx$
5. $\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}dx$
6. $\int_{\sqrt{2}}^{2} arc\sec dx$
7. $\int \frac{e^{3x}}{\sqrt{1-e^{2x}}}dx$
8. $\int \frac{1}{\sqrt{4x-x^2}}dx$
9. $\int \frac{1}{x^2-2x+2}dx$
10. $\int \frac{x+5}{9x^2+6x+17}dx$

5.3- Definite Integrals; Fundamental Theorem of Calculus

Objectives:

By the end of this section, students will be able to:

- Define Definite Integral
- Evaluate definite integral
- Apply properties of definite integrals to calculate integrals
- State fundamental theorem of calculus and apply it.

Over view: In this section we will formally define the definite integral and give many of the properties of definite integrals ,then we will present Fundamental Theorem of Calculus ,which is central theorem of integral calculus. Let's start off with the definition of a definite integral.

Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a,b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of *f(x)* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

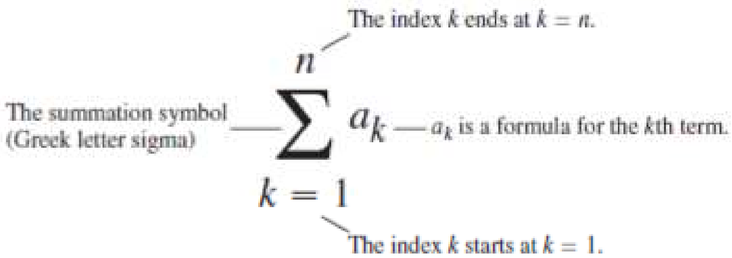
The number "*a*" that is at the bottom of the integral sign is called the **lower limit** of the integral and the number "*b*" at the top of the integral sign is called the **upper limit** of the integral. Also, despite the fact that *a* and *b* were given as an interval the lower limit does not necessarily need to be smaller than the upper limit. Collectively we'll often call *a* and *b* the **interval of integration**.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.



Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The sigma notation used on the right side of these equations is much more compact than the summation expressions on the left side.

EXAMPLE 1 Using Sigma Notation

The sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

The lower limit of summation does not have to be 1; it can be any integer.

EXAMPLE 2 Using Different Index Starting Values

Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k = 0$ or $k = 1$.

Starting with $k = 0$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^{4}(2k + 1)$

Starting with $k = 1$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^{5}(2k - 1)$

Starting with $k = 2$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^{6}(2k - 3)$

Starting with $k = -3$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^{1}(2k + 7)$ ■

When we have a sum such as

$$\sum_{k=1}^{3}(k + k^2)$$

we can rearrange its terms,

$$\begin{aligned}\sum_{k=1}^{3}(k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2)\\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \qquad \text{Regroup terms.}\\ &= \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2\end{aligned}$$

This illustrates a general rule for finite sums:

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Algebra Rules for Finite Sums

1. *Sum Rule*: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule*: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule*: $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule*: $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

EXAMPLE 3 Using the Finite Sum Algebra Rules

(a) $\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$ — Difference Rule and Constant Multiple Rule

(b) $\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1\cdot\sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$ — Constant Multiple Rule

(c) $\sum_{k=1}^{3}(k + 4) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} 4$ — Sum Rule

$= (1 + 2 + 3) + (3\cdot 4)$ — Constant Value Rule

$= 6 + 12 = 18$

(d) $\sum_{k=1}^{n}\frac{1}{n} = n\cdot\frac{1}{n} = 1$ — Constant Value Rule ($1/n$ is constant)

Formulas

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 4: Using the formulas and properties from above determine the value of the following summation.

$$\sum_{i=1}^{100}(3-2i)^2$$

Solution:

The first thing that we need to do is square out the stuff being summed and then break up the summation using the properties as follows,

$$\begin{aligned}\sum_{i=1}^{100}(3-2i)^2 &= \sum_{i=1}^{100} 9 - 12i + 4i^2\\ &= \sum_{i=1}^{100} 9 - \sum_{i=1}^{100} 12i + \sum_{i=1}^{100} 4i^2\\ &= \sum_{i=1}^{100} 9 - 12\sum_{i=1}^{100} i + 4\sum_{i=1}^{100} i^2\end{aligned}$$

Now, using the formulas, this is easy to compute,

$$\begin{aligned}\sum_{i=1}^{100}(3-2i)^2 &= 9(100) - 12\left(\frac{100(101)}{2}\right) + 4\left(\frac{100(101)(201)}{6}\right)\\ &= 1293700\end{aligned}$$

EXAMPLE 5: Using the definition of the definite integral compute the following.

$$\int_0^2 x^2 + 1\,dx$$

Solution

First, we can't actually use the definition unless we determine which points in each interval that well use for x_i^*. In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general n the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$\left[0,\frac{2}{n}\right],\ \left[\frac{2}{n},\frac{4}{n}\right],\ \left[\frac{4}{n},\frac{6}{n}\right],\ldots,\left[\frac{2(i-1)}{n},\frac{2i}{n}\right],\ldots,\left[\frac{2(n-1)}{n},2\right]$$

As we can see the right endpoint of the i^{th} subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^2+1\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3}+\frac{2}{n}\right)$$

$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2$$

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{1}{n}(2n)$$

$$= \frac{4(n+1)(2n+1)}{3n^2}+2$$

$$= \frac{14n^2+12n+4}{3n^2}$$

We can now compute the definite integral.

$$\int_0^2 x^2+1\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

$$= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2}$$

$$= \frac{14}{3}$$

Properties of Definite Integral

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, $[a,c]$ and $[c,b]$. Note however that c doesn't need to be between a and b.

6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

Proof of : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

From the definition of the definite integral we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

and we also have,

$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{a-b}{n}$$

Therefore,

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{-(a-b)}{n}\\ &= \lim_{n\to\infty}\left(-\sum_{i=1}^{n} f(x_i^*)\frac{a-b}{n}\right)\\ &= -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\frac{a-b}{n} = -\int_b^a f(x)\,dx\end{aligned}$$

■

Proof of : $\int_a^a f(x)\,dx = 0$

From the definition of the definite integral we have,

$$\int_a^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x = \frac{a-a}{n} = 0$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(0)$$

$$= \lim_{n\to\infty} 0$$

$$= 0$$

■

EXAMPLE 6: Evaluate each of the following.

(a) $\int_2^0 x^2 + 1\,dx$

(b) $\int_0^2 10x^2 + 10\,dx$

(c) $\int_0^2 t^2 + 1\,dt$

Solution

All of the solutions to these problems will rely on the fact we proved in the first example. Namely that,

$$\int_0^2 x^2 + 1\,dx = \frac{14}{3}$$

(a) $\int_{2}^{0} x^2+1\,dx$

In this case the only difference between the two is that the limits have interchanged. So, using the first property gives,

$$\int_{2}^{0} x^2+1\,dx=-\int_{0}^{2} x^2+1\,dx$$
$$=-\frac{14}{3}$$

(b) $\int_{0}^{2} 10x^2+10\,dx$

For this part notice that we can factor a 10 out of both terms and then out of the integral using the third property.

$$\int_{0}^{2} 10x^2+10\,dx=\int_{0}^{2} 10\left(x^2+1\right)dx$$
$$=10\int_{0}^{2} x^2+1\,dx$$
$$=10\left(\frac{14}{3}\right)$$
$$=\frac{140}{3}$$

(c) $\int_{0}^{2} t^2+1\,dt$

In this case the only difference is the letter used and so this is just going to use property 6.

$$\int_{0}^{2} t^2+1\,dt=\int_{0}^{2} x^2+1\,dx=\frac{14}{3}$$

EXAMPLE 7: Evaluate the definite integral.

$$\int_{130}^{130} \frac{x^3-x\sin(x)+\cos(x)}{x^2+1}\,dx$$

Solution

There really isn't anything to do with this integral once we notice that the limits are the same. Using the second property this is,

$$\int_{130}^{130} \frac{x^3-x\sin(x)+\cos(x)}{x^2+1}\,dx=0$$

EXAMPLE 8:

Given that $\int_{6}^{-10} f(x)\,dx=23$ and $\int_{-10}^{6} g(x)\,dx=-9$ determine the value of

$$\int_{-10}^{6} 2f(x)-10g(x)\,dx$$

Solution

We will first need to use the fourth property to break up the integral and the third property to factor out the constants.

$$\int_{-10}^{6} 2f(x)-10g(x)\,dx = \int_{-10}^{6} 2f(x)\,dx - \int_{-10}^{6} 10g(x)\,dx$$
$$= 2\int_{-10}^{6} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$

Now notice that the limits on the first integral are interchanged with the limits on the given integral so switch them using the first property above (and adding a minus sign of course). Once this is done we can plug in the known values of the integrals.

$$\int_{-10}^{6} 2f(x)-10g(x)\,dx = -2\int_{6}^{-10} f(x)\,dx - 10\int_{-10}^{6} g(x)\,dx$$
$$= -2(23)-10(-9)$$
$$= 44$$

EXAMPLE 9

Given that $\int_{12}^{-10} f(x)\,dx = 6$, $\int_{100}^{-10} f(x)\,dx = -2$, and $\int_{100}^{-5} f(x)\,dx = 4$ determine the value of $\int_{-5}^{12} f(x)\,dx$ ..

Solution

$$\int_{-5}^{12} f(x)\,dx = \int_{-5}^{100} f(x)\,dx + \int_{100}^{12} f(x)\,dx$$
$$= \int_{-5}^{100} f(x)\,dx + \int_{100}^{-10} f(x)\,dx + \int_{-10}^{12} f(x)\,dx$$
$$= -\int_{100}^{-5} f(x)\,dx + \int_{100}^{-10} f(x)\,dx - \int_{12}^{-10} f(x)\,dx$$
$$= -4-2-6$$
$$= -12$$

More Properties

7. $\int_a^b c\,dx = c(b-a)$, c is any number.
8. If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$.
9. If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
10. If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,d \le M(b-a)$.
11. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Proof of : $\int_a^b c\,dx = c(b-a)$, c is any number.

If we define $f(x)=c$ then from the definition of the definite integral we have,

$$\begin{aligned}\int_a^b c\,dx &= \int_a^b f(x)\,dx \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad\qquad \Delta x=\frac{b-a}{n} \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} c\right)\frac{b-a}{n} \\ &= \lim_{n\to\infty}(cn)\frac{b-a}{n} \\ &= \lim_{n\to\infty} c(b-a) \\ &= c(b-a)\end{aligned}$$

■

Proof of : If $f(x)\ge 0$ for $a\le x\le b$ then $\int_a^b f(x)\,dx \ge 0$.

From the definition of the definite integral we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \Delta x=\frac{b-a}{n}$$

Now, by assumption $f(x)\ge 0$ and we also have $\Delta x>0$ and so we know that

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \ge 0$$

So, from the basic properties of limits we then have,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \ge \lim_{n\to\infty} 0 = 0$$

But the left side is exactly the definition of the integral and so we have,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \ge 0$$

■

Proof of : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

First let's note that we can say the following about the function and the absolute value,

$$-|f(x)| \le f(x) \le |f(x)|$$

If we now use Property 9 on each inequality we get,

$$\int_a^b -|f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

We know that we can factor the minus sign out of the left integral to get,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

Finally, recall that if $|p| \le b$ then $-b \le p \le b$ and of course this works in reverse as well so we then must have,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

■

Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a,b]$ then,

$$g(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and it is differentiable on (a,b) and that,

$$g'(x) = f(x)$$

An alternate notation for the derivative portion of this is,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

EXAMPLE -10 : Differentiate each of the following

(a) $g(x)=\int_{-4}^{x}\mathbf{e}^{2t}\cos^2(1-5t)\,dt$

(b) $\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt$

Solution

(a) $g(x)=\int_{-4}^{x}\mathbf{e}^{2t}\cos^2(1-5t)\,dt$

This one is nothing more than a quick application of the Fundamental Theorem of Calculus.

$$g'(x)=\mathbf{e}^{2x}\cos^2(1-5x)$$

(b) $\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt$

This one needs a little work before we can use the Fundamental Theorem of Calculus. The first thing to notice is that the FToC requires the lower limit to be a constant and the upper limit to be the variable. So, using a property of definite integrals we can interchange the limits of the integral we just need to remember to add in a minus sign after we do that. Doing this gives,

$$\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}dt=\frac{d}{dx}\left(-\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt\right)=-\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}dt$$

The next thing to notice is that the FToC also requires an x in the upper limit of integration and we've got x^2. To do this derivative we're going to need the following version of the chain rule.

$$\frac{d}{dx}\left(g(u)\right)=\frac{d}{du}\left(g(u)\right)\frac{du}{dx} \qquad \text{where } u=f(x)$$

So, if we let $u = x^2$ we use the chain rule to get,

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -\frac{d}{dx}\int_{1}^{x^2}\frac{t^4+1}{t^2+1}\,dt\\ &= -\frac{d}{du}\int_{1}^{u}\frac{t^4+1}{t^2+1}\,dt\;\frac{du}{dx} \qquad \text{where } u = x^2\\ &= -\frac{u^4+1}{u^2+1}(2x)\\ &= -2x\frac{u^4+1}{u^2+1}\end{aligned}$$

The final step is to get everything back in terms of x.

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{1}\frac{t^4+1}{t^2+1}\,dt &= -2x\frac{\left(x^2\right)^4+1}{\left(x^2\right)^2+1}\\ &= -2x\frac{x^8+1}{x^4+1}\end{aligned}$$

Using the chain rule as we did in the last part of this example we can derive some general formulas for some more complicated problems.

First,

$$\frac{d}{dx}\int_{a}^{u(x)} f(t)\,dt = u'(x)\,f\left(u(x)\right)$$

This is simply the chain rule for these kinds of problems.

Next, we can get a formula for integrals in which the upper limit is a constant and the lower limit is a function of *x*. All we need to do here is interchange the limits on the integral (adding in a minus sign of course) and then using the formula above to get,

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -\frac{d}{dx}\int_{b}^{v(x)} f(t)\,dt = -v'(x)\,f\left(v(x)\right)$$

Finally, we can also get a version for both limits being functions of *x*. In this case we'll need to use Property 5 above to break up the integral as follows,

$$\int_{v(x)}^{u(x)} f(t)\,dt = \int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt$$

We can use pretty much any value of a when we break up the integral. The only thing that we need to avoid is to make sure that $f(a)$ exists. So, assuming that $f(a)$ exists after we break up the integral we can then differentiate and use the two formulas above to get,

$$\begin{aligned}\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt &= \frac{d}{dx}\left(\int_{v(x)}^{a} f(t)\,dt + \int_{a}^{u(x)} f(t)\,dt\right) \\ &= -v'(x)\,f(v(x)) + u'(x)\,f(u(x))\end{aligned}$$

EXAMPLE -11 : Differentiate the following integral

$$\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)dt$$

Solution

This will use the final formula that we derived above.

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{3x} t^2 \sin\left(1+t^2\right)dt &= -\frac{1}{2}x^{-\frac{1}{2}}\left(\sqrt{x}\right)^2 \sin\left(1+\left(\sqrt{x}\right)^2\right) + (3)(3x)^2 \sin\left(1+(3x)^2\right) \\ &= -\frac{1}{2}\sqrt{x}\sin(1+x) + 27x^2\sin\left(1+9x^2\right)\end{aligned}$$

Computing Definite Integrals

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ is a continuous function on $[a,b]$ and also suppose that $F(x)$ is any anti-derivative for $f(x)$. Then,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

EXAMPLE -12 : Evaluate each of the following

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

(b) $\int_{4}^{0} \sqrt{t}\,(t-2)\,dt$

(c) $\int_{1}^{2} \frac{2w^5 - w + 3}{w^2}\,dw$

(d) $\int_{25}^{-10} dR$

Solution:

(a) $\int_{-3}^{1} 6x^2 - 5x + 2\,dx$

There isn't a lot to this one other than simply doing the work.

$$\begin{aligned}\int_{-3}^{1} 6x^2 - 5x + 2\,dx &= \left(2x^3 - \frac{5}{2}x^2 + 2x\right)\Big|_{-3}^{1} \\ &= \left(2 - \frac{5}{2} + 2\right) - \left(-54 - \frac{45}{2} - 6\right) \\ &= 84\end{aligned}$$

(b) $\int_{4}^{0} \sqrt{t}\,(t-2)\,dt$

Recall that we can't integrate products as a product of integrals and so we first need to multiply the integrand out before integrating, just as we did in the indefinite integral case.

$$\begin{aligned}\int_{4}^{0} \sqrt{t}\,(t-2)\,dt &= \int_{4}^{0} t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\,dt \\ &= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Big|_{4}^{0} \\ &= 0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(2)^{\frac{3}{2}}\right) \\ &= -\frac{32}{15}\end{aligned}$$

In the evaluation process recall that,

$$(4)^{\frac{5}{2}} = \left((4)^{\frac{1}{2}}\right)^5 = (2)^5 = 32$$

$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = (2)^3 = 8$$

Also, don't get excited about the fact that the lower limit of integration is larger than the upper limit of integration. That will happen on occasion and there is absolutely nothing wrong with this.

(c) $\displaystyle\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw$

First, notice that we will have a division by zero issue at $w = 0$, but since this isn't in the interval of integration we won't have to worry about it.

Next again recall that we can't integrate quotients as a quotient of integrals and so the first step that we'll need to do is break up the quotient so we can integrate the function.

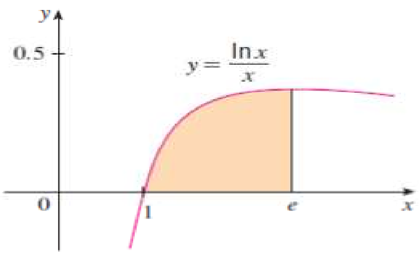
$$\begin{aligned}\int_1^2 \frac{2w^5 - w + 3}{w^2}\,dw &= \int_1^2 2w^3 - \frac{1}{w} + 3w^{-2}\,dw \\ &= \left(\frac{1}{2}w^4 - \ln|w| - \frac{3}{w}\right)\Big|_1^2 \\ &= \left(8 - \ln 2 - \frac{3}{2}\right) - \left(\frac{1}{2} - \ln 1 - 3\right) \\ &= 9 - \ln 2\end{aligned}$$

EXAMPLE -13 : Evaluate the following definite integral

$$\int_1^e \frac{\ln x}{x}\,dx.$$

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$ □



EXAMPLE -13 : Evaluate the following definite integral

$$\int_0^1 \tan^{-1} x\,dx.$$

SOLUTION Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then
$$du = \frac{dx}{1+x^2} \qquad v = x$$

So Formula 6 gives

$$\begin{aligned}\int_0^1 \tan^{-1}x\,dx &= x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= 1\cdot\tan^{-1}1 - 0\cdot\tan^{-1}0 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx\end{aligned}$$

To evaluate this integral we use the substitution $t = 1 + x^2$ (since u has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{dt}{t} &= \tfrac{1}{2}\ln|t|\Big]_1^2 \\ &= \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2\end{aligned}$$

Therefore
$$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$ □

Exercise

1. *In each part, evaluate the integral, given that*

$$f(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$$

(a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^1 f(x)\,dx$

(c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^5 f(x)\,dx$

2. Let $F(x) = \int_2^x \sqrt{3t^2+1}\,dt$. Find

(a) $F(2)$ (b) $F'(2)$ (c) $F''(2)$

5.4- *Improper Integrals*

Objectives:

By the end of this section, students will be able to:

- ❖ Define Improper Integral
- ❖ Evaluate Improper integral

Over view: In this section we will formally define the improper integral and evaluate improper integrals.

In defining a definite integral $\int_a^b f(x)\,dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity (see Section 5.2). In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$.

TYPE I: INFINITE INTERVALS

we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

1 DEFINITION OF AN IMPROPER INTEGRAL OF TYPE I

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}(\ln t - \ln 1) = \lim_{t\to\infty}\ln t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^\infty (1/x)\,dx$ is divergent. □

EXAMPLE 2 Evaluate $\int_{-\infty}^0 xe^x\,dx$.

SOLUTION Using part (b) of Definition 1, we have

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = xe^x\Big]_t^0 - \int_t^0 e^x\,dx$$

$$= -te^t - 1 + e^t$$

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hospital's Rule we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty}(-e^t) = 0$$

Therefore

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}(-te^t - 1 + e^t)$$

$$= -0 - 1 + 0 = -1$$ □

EXAMPLE 3 Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

SOLUTION It's convenient to choose $a = 0$ in Definition 1(c):

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

We must now evaluate the integrals on the right side separately:

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1}x\Big]_0^t$$

$$= \lim_{t\to\infty}(\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty}\tan^{-1}t = \frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1}x\Big]_t^0$$

$$= \lim_{t\to-\infty}(\tan^{-1}0 - \tan^{-1}t)$$

$$= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

EXAMPLE 4 For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

SOLUTION We know from Example 1 that if $p = 1$, then the integral is divergent, so let's assume that $p \neq 1$. Then

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$. Therefore

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \qquad \text{if } p > 1$$

and so the integral converges. But if $p < 1$, then $p - 1 < 0$ and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \qquad \text{as } t \to \infty$$

and the integral diverges. □

Therefore,

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

TYPE 2: DISCONTINUOUS INTEGRANDS

3 DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

EXAMPLE 5 Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

SOLUTION We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left endpoint of $[2, 5]$, we use part (b) of Definition 3:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t \to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \\ &= \lim_{t \to 2^+} 2\sqrt{x-2}\,\Big]_t^5 \\ &= \lim_{t \to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) \\ &= 2\sqrt{3}\end{aligned}$$

EXAMPLE 6 Determine whether $\displaystyle\int_0^{\pi/2} \sec x\,dx$ converges or diverges.

SOLUTION Note that the given integral is improper because $\lim_{x \to (\pi/2)^-} \sec x = \infty$. Using part (a) of Definition 3 and Formula 14 from the Table of Integrals, we have

$$\begin{aligned}\int_0^{\pi/2} \sec x\,dx &= \lim_{t \to (\pi/2)^-} \int_0^t \sec x\,dx = \lim_{t \to (\pi/2)^-} \ln|\sec x + \tan x|\Big]_0^t \\ &= \lim_{t \to (\pi/2)^-} [\ln(\sec t + \tan t) - \ln 1] = \infty\end{aligned}$$

because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$. Thus the given improper integral is divergent. □

EXAMPLE 7 Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

SOLUTION Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of Definition 3 with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where
$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|x-1|\Big]_0^t$$
$$= \lim_{t\to 1^-} \left(\ln|t-1| - \ln|-1|\right)$$
$$= \lim_{t\to 1^-} \ln(1-t) = -\infty$$

because $1 - t \to 0^+$ as $t \to 1^-$. Thus $\int_0^1 dx/(x-1)$ is divergent. This implies that $\int_0^3 dx/(x-1)$ is divergent. [We do not need to evaluate $\int_1^3 dx/(x-1)$.] □

EXAMPLE 8 Evaluate $\int_0^1 \ln x\,dx$.

SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ln x = -\infty$. Thus the given integral is improper and we have

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x\,dx = x\ln x\Big]_t^1 - \int_t^1 dx$$
$$= 1\ln 1 - t\ln t - (1-t)$$
$$= -t\ln t - 1 + t$$

To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} (-t) = 0$$

Therefore
$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} (-t\ln t - 1 + t) = -0 - 1 + 0 = -1$$

A COMPARISON TEST FOR IMPROPER INTEGRALS

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

COMPARISON THEOREM Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

EXAMPLE 9 Show that $\int_0^\infty e^{-x^2}dx$ is convergent.

SOLUTION We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function (as explained in Section 7.5). We write

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

and observe that the first integral on the right-hand side is just an ordinary definite integral. In the second integral we use the fact that for $x \geqslant 1$ we have $x^2 \geqslant x$, so $-x^2 \leqslant -x$ and therefore $e^{-x^2} \leqslant e^{-x}$. (See Figure 13.) The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}(e^{-1} - e^{-t}) = e^{-1}$$

Thus, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}\,dx$ is convergent. It follows that $\int_0^\infty e^{-x^2}\,dx$ is convergent. □

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. □

Exercise

Determine whether the integral converges or diverges, and if it converges, find its value.

1. $\int_{0}^{\infty} \frac{x}{1+x^2} dx$
2. $\int_{-\infty}^{0} \frac{1}{(x+3)^2} dx$
3. $\int_{1}^{\infty} \frac{1}{\sqrt{x^2-1}} dx$
4. $\int_{-\infty}^{\infty} xe^{-x^2} dx$
5. $\int_{0}^{9} \frac{1}{\sqrt{x}} dx$
6. $\int_{0}^{\pi/2} \sec^2 x dx$
7. $\int_{-2}^{0} \frac{1}{\sqrt{4-x^2}} dx$
8. $\int_{0}^{\pi} \sec x dx$
9. $\int_{-2}^{7} \frac{1}{(x+1)^{2/3}} dx$
10. $\int_{0}^{1} \frac{3x^2-1}{x^3-x} dx$

SUMMARY OF CHAPTER-FIVE

1.The table below shows list of Basic Integration formulas

Derivative	Indefinite integral
$D_x(x) = 1$	1. $\int 1dx = \int dx = x + c$
$D_x\left(\frac{x^{r+1}}{r+1}\right) = x^r \ (r \neq -1)$	2. $\int x^r dx = \frac{x^{r+1}}{r+1} + c \ \ (r \neq -1)$
$D_x(\sin x) = \cos x$	3. $\int \cos x dx = \sin x + c$
$D_x(-\cos x) = \sin x$	4. $\int \sin x dx = -\cos x + c$
$D_x(\tan x) = \sec^2 x$	5. $\int \sec^2 x dx = \tan x + c$
$D_x(-\cot x) = \csc^2 x$	6. $\int \csc^2 x dx = -\cot x + c$
$D_x(\sec x) = \sec x \tan x$	7. $\int \sec x \tan x dx = \sec x + c$
$D_x(-\csc x) = \csc x \cot x$	8. $\int \csc x \cot x dx = -\csc x + c$
$D_x(e^x) = e^x$	9. $\int e^x dx = e^x + c$
$D_x\left(\frac{a^x}{\ln a}\right) = a^x$	10. $\int a^x dx = \frac{a^x}{\ln a} + c$
$D_x(\ln\|x\|) = \frac{1}{x}$	11. $\int \frac{1}{x} dx = \ln\|x\| + c$

2. To find an integral of a function we can follow the following strategies
   - ✓ Simplify the Integrand if Possible

- ✓ Look for an Obvious Substitution
- ✓ Classify the Integrand According to Its Form (Trigonometric functions, Rational functions, Integration by parts., Radicals.)

Miscellaneous Exercises

Evaluate the integral

1. $\int_0^5 \frac{x}{x+10}\,dx$
2. $\int_0^5 ye^{-0.6y}\,dy$
3. $\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta$
4. $\int_1^4 \frac{dt}{(2t+1)^3}$
5. $\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$
6. $\int \frac{1}{y^2-4y-12}\,dy$
7. $\int \frac{\sin(\ln t)}{t}\,dt$
8. $\int \frac{dx}{\sqrt{e^x-1}}$
9. $\int_1^4 x^{3/2}\ln x\,dx$
10. $\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx$
11. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$
12. $\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx$
13. $\int e^{\sqrt[3]{x}}\,dx$
14. $\int \frac{x^2+2}{x+2}\,dx$
15. $\int \frac{x-1}{x^2+2x}\,dx$
16. $\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$
17. $\int x\sec x\tan x\,dx$
18. $\int \frac{x^2+8x-3}{x^3+3x^2}\,dx$
19. $\int \frac{x+1}{9x^2+6x+5}\,dx$
20. $\int \tan^5\theta\sec^3\theta\,d\theta$
21. $\int \frac{dx}{\sqrt{x^2-4x}}$
22. $\int te^{\sqrt{t}}\,dt$

References

1. James Stewart, Calculus early trancedental, 6th ed., Prentice Hall, 2008

2. Howard Anton,Calculus a new horizon, 6th ed., John wiley and Sons Inc
3. Robert Ellis and Denny Gulick, Calculus with analytic geometry, 6th ed, Harcourt Brace Jovanovich, Publishers, 5th ed, 1993
4. Adams, Calculus: A complete course, 5th ed, Addison Wesley, 2003
5. R. Wrede and M. R. Spiegel, Theory of advanced calculus, 2nd ed., McGraw-Hill, 2002.
6. A. E. Taylor and W. R. Mann, Advanced calculus, 3rd ed, John-Wiley and Son, INC, 1995.
7. R. T. Smith and R. B. Minton, Calculus concepts and connections, McGram-Hill book company, 2006
8. D. V. Widder, Advanced calculus, Prentice-Hall, 1979
9. Ross L. Finney et al, Calculus, Addison Wesley, 1995
10. E. J. Purcell and D. Varberg, Calculus with analytic geometry, Prentice-Hall INC., 1987
11. Lynne, Garner: Calculus and Analytic Geometry, Dellen Publishing Company
12. John A. Tierney: Calculus and Analytic, 4th edition, Allyn and Bacon, Inc. – Boston.
13. Earl W. Swokowski: Calculus with Analytic Geometry, 2nd edition, Prindle, Weber and Schmidt.
14. Goldstein, L. J., Lay D. C. , and Schneider D. I (1987). Calculus and its applications, 4th, edn, Prentice-Hall, Inc, London
15. Kresying, Advanced Engineering Mathematics.
16. Alan Jeffry , Advanced Engineering Mathematics

CHAPTER-SIX
APPLICATION OF INTEGRATION

UNIT OBJECTIVES:

At the end of this unit each student should able to:

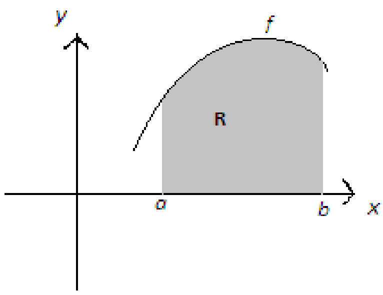
- Apply integration to find area
- Apply integration to find volume

In this chapter, we will see some applications of integration such as: Area of a region bounded by curves of continuous defined on a closed interval $[a, b]$ and volume of a solid of revolutions.

6.1. Area

Let f be a non-negative continuous function on $[a, b]$, then the area of the region R bounded by f and the x-axis between $x = a$ & $x = b$ (as shown in the fig. below) is given by:

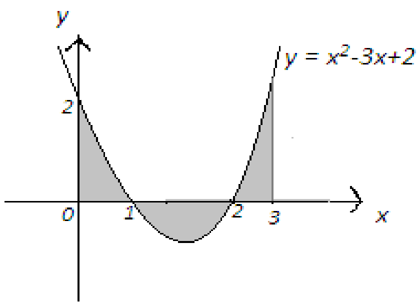
$$A(R) = \int_a^b f(x)dx \quad \text{………………………………(*)}$$



Example:

1) Find the area of the region bounded by the graph of the function $f(x) = x^2 - 3x + 2$ and the *x-axis* between *x=0* & *x=3.*

Solution*:* The region R is the shaded region as shown below.



Now let R_1, R_2 and R_3 be the sub-regions b/n x=0 and x=1, x=1 and x=2, and x=2 and x=3, respectively. Then, using (*):

$$A(R) = A(R_1) + A(R_2) + A(R_3)$$

Where,

$$A(R_1) = \int_0^1 f(x)dx = \int_0^1 (x^2 - 3x + 2)dx = (\frac{x^3}{3} - 3\frac{x^2}{2} + 2x)\Big|_0^1 = \frac{5}{6}$$

$$A(R_2) = -\int_1^2 (x^2 - 3x + 2)dx = -(\frac{x^3}{3} - 3\frac{x^2}{2} + 2x)\Big|_1^2 = \frac{1}{6}$$

Since the sub-region R_2 is below the *x- axis*, the negative sign is important. Similarly,

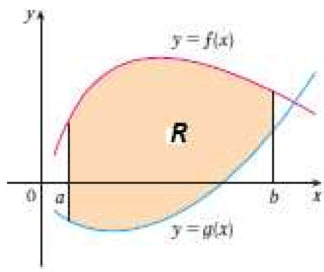
$$A(R_3) = \int_2^3 (x^2 - 3x + 2)dx = (\frac{x^3}{3} - 3\frac{x^2}{2} + 2x)\Big|_2^3 = \frac{5}{6}.$$

Thus, area of the region is:

$$A(R) = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}.$$

We can see that the area of a region bounded by a continuous curves *y = f(x), y = g(x)* and the lines *x = a, x = b* such that $f(x) \geq g(x)$ for all *x* in [*a*, *b*] is:

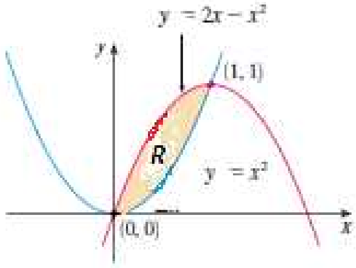
$$A(R) = \int_a^b [f(x) - g(x)]dx \text{ ……………………(*)}$$



$R = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$

Example 2: Find area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution: The sketch of the region is shown below. So, using (**):



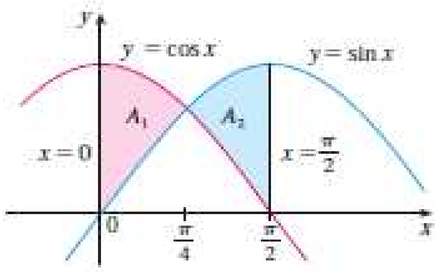
$$A(R) = \int_0^1 [f(x) - g(x)]dx = \int_0^1 [(2x - x^2) - x^2]dx = \int_0^1 (2x - 2x^2)dx$$

$$= (x^2 - 2\frac{x^3}{3})\Big|_0^1 = \frac{1}{3}.$$

Note that the intersection points can be obtained as: $y = 2x - x^2 = x^2 \Rightarrow 2x^2 - 2x = 0$. Solving this equation for *x*, we get $x = 0$ *or* $x = 1$, then $y = 0$ *or* $y = 1$, as indicated in the graph above.

Example 3: Find the area of the region bounded by the curves *y = sin x* and *y = cos x*, between *x = 0* & $x = \frac{\pi}{2}$.

Solution: The region R with its parts is as shown below:



The points of intersection occur when *sin x* = *cos x*, that is, when $x = \frac{\pi}{4}$(since $0 \leq x \leq \frac{\pi}{2}$). Observe that *cos x* ≥ *sin x* when $0 \leq x \leq \frac{\pi}{4}$ but *cos x* ≤ *sin x* when $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$. Therefore, area of the region is:

$$A(R) = A_1 + A_2 = \int_0^{\frac{\pi}{4}} [cosx - sinx]dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [sinx - cosx]dx$$

$$= [sinx + cosx]\Big|_0^{\frac{\pi}{4}} + [-cosx - sinx]\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2\sqrt{2}.$$

Exercise:

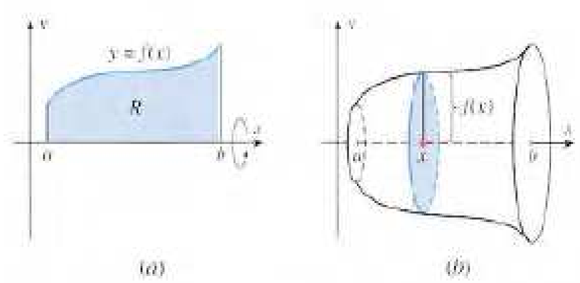
1. Find the area of the region enclosed by the graph of *f(x) = sin x* and the *x-axis* between $x = -\frac{\pi}{2}$ and $x = 2\pi$.
2. Find the area of the region in the first quadrant which is enclosed by the *y-axis* and the curves of *f(x) = cos x* and *g(x) = sinx.*
3. Determine the area of the region enclosed by the graphs of $x = -y^2$ and $x = 9 - 2y^2$
4. Find the area of the region enclosed by the graphs $y = x^2 + 1$ and the line $y = 5$.

6.2. Volume

In section, we will apply integral calculus to determine the volume of a solid region by considering cross sections.

Suppose a region rotates about a straight line, then a solid figure called a *solid of revolution,* is formed. The volume of such a solid is said to be a *volume of revolution* and the line about which the region rotates is an *axis of symmetry*.

Now consider the solid of revolution generated by revolving the region between the curve $y = f(x)$ and the *x-axis* from $x = a$ to $x = b$ as shown below.



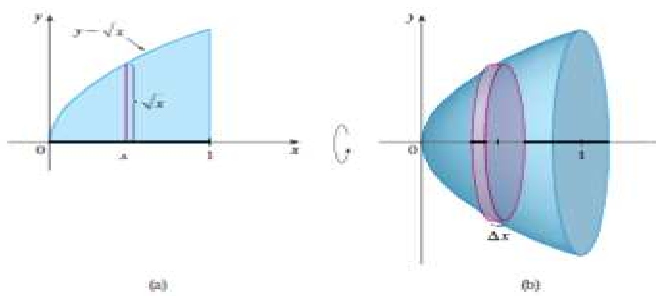
Every cross section which is perpendicular to the *x-axis* at *x* a circular region with radius, r = *f(x).* Thus, the area of the cross section *A(x)* is $A(x) = \pi r^2 = \pi(f(x))^2$.

Thus, we define the volume of revolution V as by:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi(f(x))^2\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(\Delta)$$

Example 1: Find the volume of the solid obtained by rotating about the *x-axis* the region under the curve $y = \sqrt{x}$ from *0* to *1*.

Solution: The region and the solid figure rotated about the *x-axis* from 0 to 1 are shown in fig.(a) and fig.(b) below.

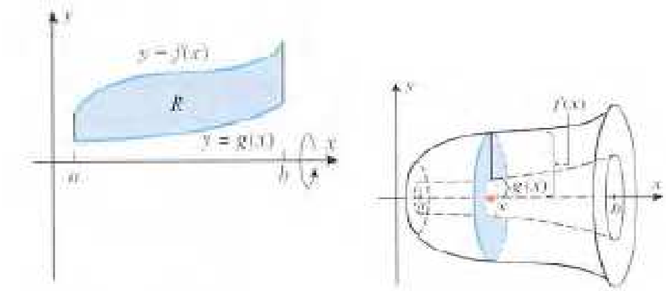


Now, the area of the cross section is $A(x) = \pi(f(x))^2 = \pi(\sqrt{x})^2 = \pi x$. So, using (Δ), the volume of revolution V is given by:

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}.$$

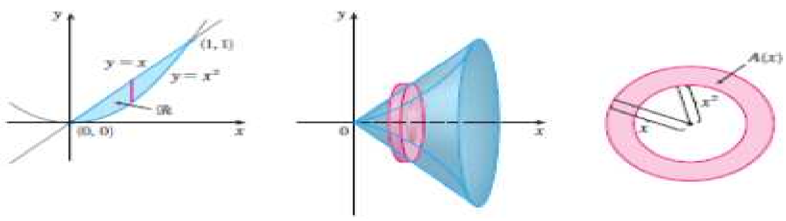
Note that if f and g are continuous, non-negative on $[a, b]$ such that $f(x) \geq g(x)$ for all x *in* $[a, b]$. let R be a region bounded by *f(x)* and *g(x),* on $[a, b]$ as in the following figure. Then, the volume of the solid of revolution generated by revolving the region R about the *x-axis* is given by:

$$V = \int_a^b A(x)\,dx, \text{ where } A(x) = \pi[\left(f(x)\right)^2 - \left(g(x)\right)^2].$$



Example 2: The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the *x-axis*. Find the volume of the resulting solid.

Solution:

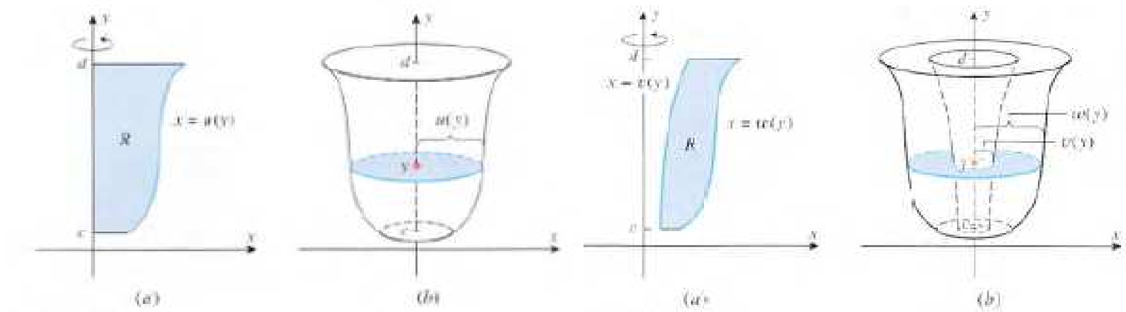


The curves $y = x$ and $y = x^2$ intersect at the points $(0,0)$ and $(1, 1)$. The region and solid of rotation are shown in the fig. above. The cross-section in the plane has the shape of annular ring with inner radius x^2 and outer radius x. So, $A(x) = \pi[x^2 - (x^2)^2] = \pi(x^2 - x^4)$.

Therefore, we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi[\frac{x^3}{3} - \frac{x^5}{5}]\Big|_0^1 = \frac{2\pi}{15}$$

Consider again the case where a region R is rotated about the *y-axis*: where, either R is a region bounded by the curve $x = u(y)$ and the *y-axis,* between the lines $y = c$ & $y = d$ or R is a region between two curves *v(y)* and *w(y)* on $[c, d]$ as in following figures:

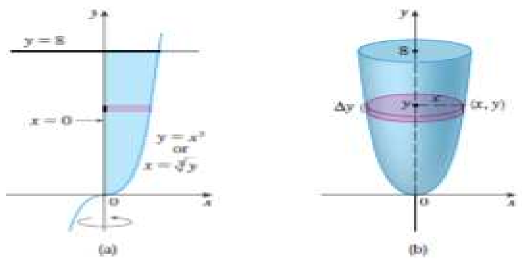


Then the corresponding volume of revolution is given by:

$V = \int_c^d \pi[u(y)]^2\,dy$ and $V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$ respectively.

Example 3: Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the *y-axis*.

Solution: The rotation of the region is shown as:



Here, $u(y) = x = \sqrt[3]{y}$. So, the area of the cross-section is $A(y) = \pi[u(y)]^2 = \pi y^{\frac{2}{3}}$. Thus, the volume V of revolution is:

$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^8 \pi y^{\frac{2}{3}}\,dy = \frac{96\pi}{5}.$$

Exercises:

1. Find the volume of solid of revolution about the *x-axis* generated by revolving the area between the lines $y = x$ and $y = 4$ from $x = 1$ to $x = 3$.
2. Find the volume of the solid of revolution about the *y-axis* generated by revolving the region enclosed by the curve $x = \sqrt{y}$ and the *y-axis* from $y = 0$ to $y = 4$.
3. The area bounded by the graph of $y = x^2+1$ and the line $y = 4$ rotates about the *y – axis*, find the volume of the solid generated.
4. Find the volume of the solid obtained by rotating the region bounded by the curves $y^2 = x$ and $x = 2y$ about *y-axis*.

References

1. James Stewart, Calculus early trancedental, 6^{th} ed., Prentice Hall, 2008
2. Howard Anton,Calculus a new horizon, 6^{th} ed., John wiley and Sons Inc
3. Robert Ellis and Denny Gulick, Calculus with analytic geometry, 6^{th} ed, Harcourt Brace Jovanovich, Publishers, 5th ed, 1993
4. Adams, Calculus: A complete course, 5th ed, Addison Wesley, 2003
5. R. Wrede and M. R. Spiegel, Theory of advanced calculus, 2nd ed., McGraw-Hill, 2002.
6. A. E. Taylor and W. R. Mann, Advanced calculus, 3rd ed, John-Wiley and Son, INC, 1995.
7. R. T. Smith and R. B. Minton, Calculus concepts and connections, McGram-Hill book company, 2006
8. D. V. Widder, Advanced calculus, Prentice-Hall, 1979
9. Ross L. Finney et al, Calculus, Addison Wesley, 1995
10. E. J. Purcell and D. Varberg, Calculus with analytic geometry, Prentice-Hall INC., 1987
11. Lynne, Garner: Calculus and Analytic Geometry, Dellen Publishing Company
12. John A. Tierney: Calculus and Analytic, 4th edition, Allyn and Bacon, Inc. – Boston.
13. Earl W. Swokowski: Calculus with Analytic Geometry, 2nd edition, Prindle, Weber and Schmidt.
14. Goldstein, L. J., Lay D. C. , and Schneider D. I (1987). Calculus and its applications, 4th, edn, Prentice-Hall, Inc, London
15. Kresying, Advanced Engineering Mathematics.
16. Alan Jeffry , Advanced Engineering Mathematics