Chapter One :Vectors

1.1 What is a Vector?

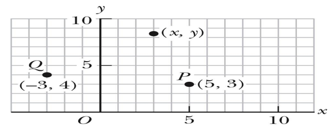
- Quantities that need to be represented by magnitude (size) *and* direction are called "vectors".
- Examples
  - Wind
  - Boat or aircraft travel
  - Forces in physics

1.2. Coordinate Systems

- Used to describe the position of a point in space
- Common coordinate systems are:
  - Cartesian (x,y,z)
  - Polar (r,θ)

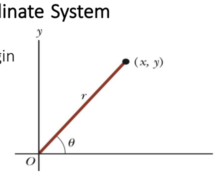
1.2.1. Cartesian Coordinate System

- Also called rectangular coordinate system, *x*- and *y*- axes intersect at the origin and Points are labeled (*x*,*y*)



1.2.2. Polar Coordinate System

A Point is at a distance *r* from the origin in the direction of angle *θ and* the reference line is often the x-axis and Points are labeled (*r*,*θ*)



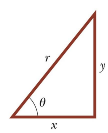
- Based on forming a right triangle from *r* and *θ*

$$x = r\cos\theta$$
$$y = r\sin\theta$$

- If the Cartesian coordinates are known

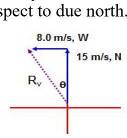
$$\tan\theta = \frac{y}{x}$$
$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$



Example

A boat moves with a velocity of 15 m/s, N in a river which flows with a velocity of 8.0 m/s, west. Calculate the boat's resultant velocity with respect to due north.



$$R_v = \sqrt{8^2 + 15^2} = 17\ m/s$$
$$Tan\ \theta = \frac{8}{15} = 0.5333$$
$$\theta = Tan^{-1}(0.5333) = 28.1^\circ$$

1.3. Vectors and Scalars

- A ***scalar quantity*** is any quantity in physics that has magnitude, but not a direction associated with it.
- Magnitude – A numerical value with units .

Ex: Length, temperature, mass, speed

- A vector quantity is ANY quantity in physics that has both magnitude and direction.
- Ex. Force, velocity, displacement, acceleration

•Also used for printing is simple bold print: **A**

•When dealing with just the magnitude of a vector in print, an italic letter will be used:

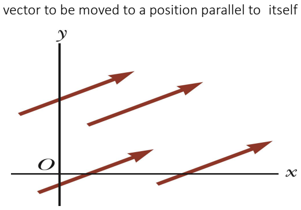
A or $|\vec{\mathbf{A}}|$

- The magnitude of the vector has physical units.
- The magnitude of a vector is always a positive number.

•When handwritten, use an arrow : $\vec{A}$

1.4. Equality of Two Vectors

- Two vectors are ***equal,*** if they have the same magnitude and the same direction.
- if *A* = *B* and they point along parallel lines
- All of the vectors shown are equal.
- Allows a vector to be moved to a position parallel to itself



1.5. Adding Vectors

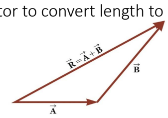
- Vector addition is very different from adding scalar quantities.
- When adding vectors, their directions must be taken into account.
- Units must be the same
- Graphical Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient

➢ Adding Vectors Graphically

- Draw the first vector, $\vec{\mathbf{A}}$ with the appropriate length and in the direction specified, with respect to a coordinate system.
- Draw the next vector with the appropriate length and in the direction specified, with respect to a coordinate system whose origin is the end of vector $\vec{\mathbf{A}}$ and parallel to the coordinate system used for $\vec{\mathbf{A}}$.

(A). Triangle method of vector addition

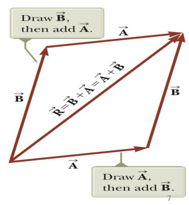
- Continue drawing the vectors "tip-to-tail" or "head-to-tail".
- The resultant is drawn from the origin of the first vector to the end of the last vector.
- Measure the length of the resultant and its angle.
  - Use the scale factor to convert length to actual magnitude.



(B).Parallelogram method of vector addition

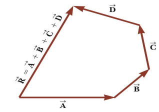
When two vectors are added, the sum is independent of the order of the addition. This is the *Commutative Law of Addition*.

$\vec{A}+\vec{B}=\vec{B}+\vec{A}$



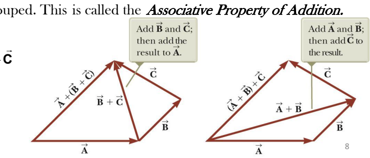
(C). Polygon method of vector addition

- When you have many vectors, just keep repeating the process until all are included.
- The resultant is still drawn from the tail of the first vector to the tip of the last vector.



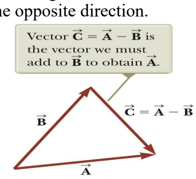
- When adding three or more vectors, their sum is independent of the way in which the individual vectors are grouped. This is called the ***Associative Property of Addition.***

$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$



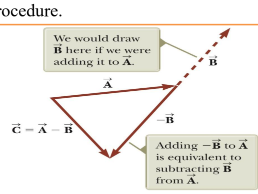
1.6. Negative of a Vector

- The negative of a vector is defined as the vector that, when added to the original vector, gives a resultant of zero.
  - Represented as $-\vec{A}$
  - $\vec{A}+(-\vec{A})=0$
- The negative of the vector will have the same magnitude, but point in the opposite direction.



b. Added to the second vector gives you the first vector.

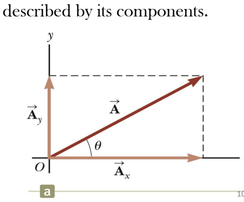
a. If $\vec{A}-\vec{B}$, then use $\vec{A}+(-\vec{B})$ Continue with standard vector addition procedure.



1.7. Component Method of Adding Vectors

- Graphical addition is not recommended when:
  - High accuracy is required
  - If you have a three-dimensional problem
- Component method is an alternative method
  - It uses projections of vectors along coordinate axe
- A **component** is a projection of a vector along an axis.
  - Any vector can be completely described by its components.

- It is useful to use **rectangular components.** These are the projections of the vector along the x- and y-axes.

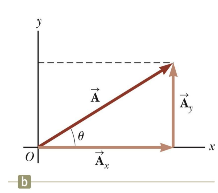


- $\vec{A}_x$ and $\vec{A}_y$ are the *component vectors* of $\vec{A}$
  - They are vectors and follow all the rules for vectors.
- A_x and A_y are scalars, and will be referred to as the *components* of $\vec{A}$
- Assume you are given a vector $\vec{A}$
- It can be expressed in terms of two other vectors, $\vec{A}_x$ and $\vec{A}_y$

$$\vec{A}=\vec{A}_x+\vec{A}_y$$

- These three vectors form a right triangle.

- The y- component is moved to the end of the x- component.
- This is due to the fact that any vector can be moved parallel to itself without being affected.
- This completes the triangle.



- The x-component of a vector is the projection along the x-axis.

$$A_x = A\cos\theta$$

- The y-component of a vector is the projection along the y-axis.

$$A_y = A\sin\theta$$

- This assumes the angle θ is measured with respect to the x-axis.
  - If not, do not use these equations, use the sides of the triangle directly.
- The components are the legs of the right triangle whose hypotenuse is the length of A.

$$A=\sqrt{A_x^2+A_y^2} \quad \text{and} \quad \theta=\tan^{-1}\frac{A_y}{A_x}$$

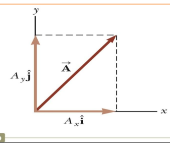
- May still have to find θ with respect to the positive x-axis
- In a problem, a vector may be specified by its components or its magnitude and direction.

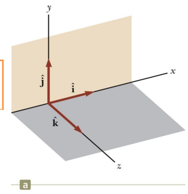
1.8. Unit Vectors

- A ***unit vector*** is a dimensionless vector with a magnitude of exactly 1.
- Unit vectors are used to specify a direction and have no other physical significance.
- The symbols $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ represent unit vectors
- They form a set of mutually perpendicular vectors in a right-handed coordinate system
- The magnitude of each unit vector is 1

$\mathbf{A_x}$ is the same as A_x and $\mathbf{A_y}$ is the same as A_y etc.The complete vector can be expressed as:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$





- A point lies in the *xy* plane and has Cartesian coordinates of (x, y).
- The point can be specified by the position vector.

$$\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$$

- This gives the components of the vector and its coordinates.
- Using $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$

Then

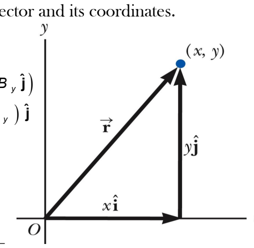
$$\vec{\mathbf{R}} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right)$$

$$\vec{\mathbf{R}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

So $R_x = A_x + B_x$ and $R_y = A_y + B_y$

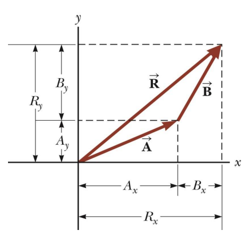
$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$



- Note the relationships among the components of the resultant and the components of the original vectors.

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$



- for Three-Dimensional Extension
- Using $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$
- Then

$$\vec{\mathbf{R}} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$$

$$\vec{\mathbf{R}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} + \left(A_z + B_z\right)\hat{\mathbf{k}}$$

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad \theta_x = \cos^{-1}\frac{R_x}{R}, \text{etc.}$$

- So $R_x = A_x + B_x$, $R_y = A_y + B_y$, and $R_z = A_z + B_z$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad \theta_x = \cos^{-1}\frac{R_x}{R}, \text{etc.}$$

1.9. Product of Vector

i). Dot(Scalar) product:

- If we take two vectors **A** and **B,** their dot product is

$$A \cdot B = a_1 a_2 + b_1 b_2$$

If A and B are two nonzero vectors, the angle $\theta, 0 \le \theta < \pi$, between A and B is determined by the formula

$$\cos\theta = \frac{A \cdot B}{\|A\|\|B\|}$$

$i \cdot j = 0$	$i \cdot k = 0$	$j \cdot k = 0$
$i \cdot i = 1$	$j \cdot j = 1$	$k \cdot k = 1$

Find the angle θ between $A = 2\mathbf{i} - \mathbf{j}$ and $B = 4\mathbf{i} + 3\mathbf{j}$

$$A \cdot B = (2)4 + (-1)3 = 8 - 3 = 5$$

$$\|A\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\|B\| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\cos\theta = \frac{A \cdot B}{\|A\|\|B\|} = \frac{5}{\sqrt{5} \cdot 5} = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{5}} \approx 63.4°$$

ii). Cross(Vector) Product:

- The cross product between A and B produces a vector quantity. The magnitude of the vector product is defined as:

$$|AxB| = |A||B|\sin\phi$$

- Where ϕ is the angle between the two vectors.

Consider:

$$\mathbf{A} = \mathbf{A_x i} + \mathbf{A_y j} + \mathbf{A_z k}$$

$$\mathbf{B} = \mathbf{B_x i} + \mathbf{B_y j} + \mathbf{B_z k}$$

$$B \otimes A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

$\mathbf{i} \times \mathbf{i} = 0$	$\mathbf{j} \times \mathbf{j} = 0$	$\mathbf{k} \times \mathbf{k} = 0$
$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Example: Let ***A*** = *3i - 4j + 5k* and ***B*** = *-2i - 4j - 6k*. What is ***B*** *x* ***A***?

$$\mathbf{B \times A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -4 & -6 \\ 3 & -4 & 5 \end{vmatrix} \qquad BxA = -44\hat{i} - 8\hat{j} + 20\hat{k}$$

iii). Scalar and Vector Triple product

- The **scalar triple product** is defined as the dot product of one of the vectors with the cross product of the other two.

A.(B x C) = C.(A x B) = B.(C x A)

A.(A x B) = B.(A x B) = 0

- Product of three vectors. Given three vectors A, B, and C. We list three products with formula

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C});$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B});$$

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$