European Journal of Operational Research 196 (2009) 516–525

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Production, Manufacturing and Logistics

The variable radius covering problem

Oded Berman [a], Zvi Drezner [b,*], Dmitry Krass [a], George O. Wesolowsky [c]

[a] Joseph L. Rotman School of Management, University of Toronto, 105 St. George Street, Toronto, Ontario, Canada M5S 3E6
[b] Steven G. Mihaylo College of Business and Economics, California State University-Fullerton, Fullerton, CA 92834, United States
[c] Faculty of Business, McMaster University, Hamilton, Ont., Canada L8S 4M4

ARTICLE INFO

Article history:
Received 10 February 2007
Accepted 21 March 2008
Available online 9 April 2008

Keywords:
Location
Heuristics
Cover
Planar
Network
Discrete

ABSTRACT

In this paper we propose a covering problem where the covering radius of a facility is controlled by the decision-maker; the cost of achieving a certain covering distance is assumed to be a monotonically increasing function of the distance (i.e., it costs more to establish a facility with a greater covering radius). The problem is to cover all demand points at a minimum cost by finding optimal number, locations and coverage radii for the facilities. Both, the planar and discrete versions of the model are considered. Heuristic approaches are suggested for solving large problems in the plane. These methods were tested on a set of planar problems. Mathematical programming formulations are proposed for the discrete problem, and a solution approach is suggested and tested.

1. Introduction

The location set covering problem (LSCP), introduced by ReVelle et al. (1976), is one of the classic models in the location literature. The problem is to cover the customer demand, assumed to be concentrated at a discrete set of points, with the minimum number of facilities. Each facility has a constant coverage radius r, a demand point is assumed to be "covered" if it is within distance r of a facility. This model has a wide range of applications, including emergency response system design, location of retail facilities, design of computer networks, etc. (for reviews see Schilling et al., 1993; Daskin, 1995; Current et al., 2002; Plastria, 2002). A close relative of LSCP is the maximal cover location problem MCLP, where the goal is to cover the maximum amount of demand with a certain number of facilities; for network formulations see Church and ReVelle (1974), Megiddo et al. (1983), ReVelle (1986) and Berman (1994), and for planar problems see Drezner (1981), Watson-Gandy (1982), Drezner (1986) and Canovas and Pelegrin (1992).

The covering radius r in LSCP (and in MCLP) is assumed to be an exogenous parameter, outside of the control of the decision-maker. However, in many (if not most) applications, the coverage radius of a facility is one of the design parameters: a larger or smaller coverage radius can be achieved by increasing or decreasing the "size" of the facilities (here by "size" we refer to the physical characteristic of the facility that is related to the coverage radius). For example, locating warning sirens (Current and O'Kelly, 1992) is modeled as a p-center problem but the coverage radius depends on the intensity of the siren. When locating light posts to illuminate a certain area, the coverage radius of each individual light depends on the intensity of the light bulb and the height of the post. When locating detectors to warn of fire or other hazards (Drezner and Wesolowsky, 1997), the distance at which the detector discovers the hazard depends on the sensitivity of the detector. The signal strength of a radio station determines the coverage area. In the design of cellular telephone networks, the height of the tower and the signal strength of the transmitter affect the coverage radius. Retailers generally expect larger stores to have larger trading areas. In design of public service facilities (such as schools or hospitals), larger facilities serve more patients (or students) leading to a larger covering distance. Longer runways at an airport allow it to service larger planes, thus allowing arrivals and departures of airplanes from larger distances.

The common feature of the examples above is that, at a certain cost, it is possible to adjust the coverage radius of a facility, with larger coverage radii requiring larger capital investments. In this paper we propose a Variable Radius Covering Problem where a decision-maker has to determine the optimal number, locations and coverage radii for the facilities to cover a discrete number of demand points at a minimal cost. We assume that the cost of constructing a facility with coverage radius r is given by a non-decreasing cost function $\phi(r)$. Two versions of the model are analyzed: in the planar version, the facilities can be located anywhere within a certain region of a plane; in the discrete version, the demand is assumed to come from a finite set of points and the facilities locations set is also finite.

* Corresponding author. Tel.: +1 714 278 2712; fax: +1 714 278 5940.
E-mail address: zdrezner@fullerton.edu (Z. Drezner).

0377-2217/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.ejor.2008.03.046

To the best of our knowledge, the direct trade-off between the coverage radius and the facility cost is novel to our model. However, there has been some prior work on relaxing the "fixed coverage radius" assumption of the LSCP. One stream of research investigated the hierarchical covering problem where a set of different possible coverage radii is given and two possible objectives apply (Daskin and Stern, 1981; Church and Eaton, 1987). The maximal expected covering problem (where there is some uncertainty about whether a demand point will be covered) is investigated in Daskin (1983) and Batta et al. (1989). Another stream of research, in the MCLP context, is gradual covering problems, where demand points that are too far away from the facility are assumed to be only partially covered. In one version of the problem (Berman and Krass, 2002) the proportion of the demand covered is a decreasing stepwise function of the coverage radius. In another version there is a minimum and maximum covering distance. The demand is fully covered within the minimum distance and is not covered at all beyond the maximum distance. Between these two distances the coverage is gradually declining either linearly or otherwise (Berman et al., 2003; Drezner et al., 2004). In Drezner et al. (2004) the single facility case in the plane using Euclidean distances is optimally solved.

Carrizosa and Plastria (1998) and Plastria and Carrizosa (1999) studied covering problems with varying radii. In their models the coverage radius is the same for all facilities. They develop the efficient frontier between the covering radius and the maximal cover with that radius. We note that in our model, each facility will, in general, have its own coverage radius.

The idea of selecting the attractiveness of a facility as a variable in competitive location models was suggested in Drezner (1998), Plastria and Carrizosa (2004), Fernandez et al. (2007) and Aboolian et al. (2008). This is similar, in spirit, to our model in which the cost of building a facility is a function of its coverage radius (attractiveness). However, the objective of our model and of the competitive location models mentioned above are obviously quite different: in our mode, it is to cover all demand point at the minimal total cost; in competitive location models it is generally to select the most profitable strategy, given certain actions of the competitor(s).

This paper is organized as follows. In Section 2 we develop the model in the plane, establish certain structural results, and construct three heuristic approaches. In Section 3 we report computational experience with these heuristic algorithms; two of the heuristics appear to be quite efficient in obtaining accurate solutions for problems with up to 10,000 demand nodes. The experiments are performed on both, randomly generated and real-life problem sets. In Section 4 we formulate the discrete model, and develop an exact solution algorithm. In Section 5 we report computational experiments with a set of test problems on a network; the algorithm appears to be capable of handling instances with up to 1000 demand points in a few seconds. We conclude in Section 6 and suggest topics for future research.

2. The planar Variable Radius Covering Problem

In the plane, each facility covers all the demand points within a circle of a given radius. While the model could be developed with any distance metric, we will use Euclidean distances, unless stated otherwise.

We introduce the following notation:

Decision variables

p the unknown number of facilities
$X_j = (x_j, y_j)$ be the unknown location of facility j for $j = 1, \ldots, p$
X the vector $\{X_j\}$ for $j = 1, \ldots, p$
Y_{ij} a binary variable. $Y_{ij} = 1$ if demand point i is assigned to facility j, and $Y_{ij} = 0$ otherwise
Y the matrix $\{Y_{ij}\}$
r_j the unknown coverage radius for facility $j = 1, \ldots, p$
$P = \{p, X, (r_1, \ldots, r_p)\}$ the facility set describing the number, locations, and coverage radii of the facilities

Problem parameters

n the number of demand points
(a_i, b_i) the location of demand point i for $i = 1, \ldots, n$
$d_i(X_j)$ the Euclidean distance between demand point i and facility j
F the non-negative fixed cost of locating one facility
$\phi(r)$ the variable cost of building a facility of radius r. By definition $\phi(0) = 0$

The total cost of building a facility with coverage radius $r > 0$ is assumed to be $F * I\{r > 0\} + \phi(r)$, where $I\{\cdot\}$ is the indicator function (for $r = 0$ the total cost is zero and the fixed cost is not charged). The variable cost function $\phi(r)$ is assumed to be non-decreasing in r. Note that to minimize the overall cost of coverage, for a given variable cost expenditure, it is always beneficial to construct a facility with the largest possible coverage radius. Thus, even if a facility with the larger radius has a lower cost than one with a smaller radius, the larger radius option will always be chosen, thus the assumption above is made without loss of generality. We do not assume continuity of $\phi(r)$ as construction costs may jump in value at some specific radii.

An example of the cost function applicable in several settings is $\phi(r) = Cr^2$ for $C > 0$: when coverage is defined as a physical intensity of the transmitter/receiver, such intensity is often inversely proportional to the square of the distance. Note that if the total number of facilities p is fixed, rather than being a decision variable, we can take $F = 0$ for the fixed cost component (this is the case in many of the competitive location models mentioned in Section 1). Several different forms of the coverage cost function have been suggested by other authors. For example, in Drezner (1998) four cost structures were suggested: a decreasing marginal return curve $\phi(r) = C\sqrt{r}$, a fixed marginal return curve $\phi(r) = Cr$, and two increasing marginal return curves, $\phi(r) = Cr^2$ for rapidly increasing, and $\phi(r) = Cr(1 + \alpha r)$ for mildly increasing ($\alpha > 0$). In Plastria and Carrizosa (2004) a list of possible radii each with a corresponding cost is suggested. In Fernandez et al. (2007) it is required that $\phi(r)$ is differentiable and the function may be different for different demand points. They suggest $\phi(r) = Cr^k$ for $k \geqslant 1$ or $\phi(r) = C(e^{kr} - 1)$ for $k > 0$.

Since the variable cost is non-decreasing in the coverage radius, the latter is uniquely determined by the assignment variables through the following relationship:

$$r_j = \max_{1 \leqslant i \leqslant n} \{Y_{ij} d_i(X_j)\}$$

(i.e., each facility must cover all demand assigned to it and the coverage radius is determined by the furthest assigned demand point). It follows that the cost corresponding to given values of the decision variables X, Y and p is

$$F(X, Y, p) = pF + \sum_{j=1}^{p} \phi\left(\max_{1 \leqslant i \leqslant n} \{Y_{ij} d_i(X_j)\}\right),$$

and the problem can be stated as follows:

$$\min_{X,Y,p} \left\{ F(X, Y, p) | Y_{ij} \in \{0, 1\}, i = 1, \ldots, n, j = 1, \ldots, p; \sum_{j=1}^{p} Y_{ij} = 1 \right\}. \tag{1}$$

We note that the formulation (1) above cannot be directly solved by standard mathematical programming solvers since the number of decision variables depends on p, which is itself as a variable. For this reason we develop several heuristic algorithms developed for planar problems. For discrete location problems a formulation similar to (1), yet suitable for mathematical programming solvers will be presented in Section 4.

We also note that the coverage radius r_j for facility j does not appear explicitly in cost function $F(X, Y, p)$ or in the formulation (1) since it was represented through the assignment variables Y_{ij}. In the development of the heuristics that follows it will be more useful to represent the total cost as a function of the facility set $P = \{p, X, (r_1, \ldots, r_p)\}$:

$$F(P) = pF + \sum_{j=1}^{p} \phi(r_j).$$

Consider a facility covering subset I of $k \leqslant n$ demand points. Since the facility cost is non-decreasing function of the radius, it is advantageous to cover the set I with the circle of the smallest possible radius. Therefore, the optimal set of covering circles must be based on subsets, each covered by the smallest possible radius. These subsets are termed "minimal subsets". In the following theorems we establish the number of minimal subsets and show that the optimal solution is based on minimal subsets.

Theorem 1. *There are at most $n(n^2 + 5)/6$ minimal subsets.*

Proof. There are $2^n - 1$ non-empty subsets of n points. Each subset defines a 1-center problem that has a unique solution point. This solution point is the center of the smallest circle enclosing all the points of the subset. Each subset is covered by a circle with the smallest possible radius passing through either one point (radius of zero) or two points (the segment connecting the two points is the diameter of the circle) or three points (the circle passing through these three points contains all the points of the subset). Therefore, there are at most $n(n-1)(n-2)/6 + n(n-1)/2 + n = n(n^2 + 5)/6$ minimal subsets. □

Theorem 2. *There exists an optimal solution to the variable radius problem which is a union of minimal subsets.*

Proof. Suppose that a subset I, which is not a minimal subset, is part of an optimal solution. We prove that there exists a minimal subset with an equal radius that covers all the points in I and possibly some additional points. The one center solution to the problem based on subset I is the center of the circle of the smallest radius that encloses all points in I. The points enclosed in such a circle form a minimal subset M by definition. M must include all the points in I and may include additional points. A solution where I is replaced by M can only be a better solution. Therefore, subsets which are not minimal subsets can be removed from consideration. □

The problem can be solved by "brute force" as follows:

1. Find all minimal subsets.
2. Calculate the radius and consequently the cost for each minimal subset.
3. Find the subset of points covered by each minimal subset.
4. Construct a set covering problem: the columns of the table consist of minimal subsets and for each minimal subset we have a list of covered points and a cost determined by the radius which is the distance to the farthest point of the subset.
5. Solving this set covering problem can be done by integer programming or available special programs such as Sitation (Daskin, 2003).

Note that the classical reduction rules for set covering problems are implicitly embedded in the construction of the minimal subsets.

2.1. The number of minimal subsets

The "brute force" procedure described above depends on the number of minimal subsets in a particular problem instance. While Theorem 1 provides an upper bound on this number, the actual number of minimal subsets can be substantially smaller. In this section we calculate the number of minimal subsets for some randomly generated problems. Demand points were generated in a unit square from the uniform distribution and the number of minimal subsets counted. Ten random problems were generated for each value of n. For $n \geqslant 100$ the number of minimal subsets is about 28% of the upper bound (see Table 1). For $n = 1000$, while the upper bound is 167 million minimal subsets, in our experiments we found an average of about 46 million minimal subsets with a standard deviation of about 197 hundred thousand (the average number is about 27.7% of the maximal possible number). There are elimination schemes which can remove some minimal subsets. For example, if a union of several subsets contains subset S and the sum of the costs for the several subsets is less than or equal to the cost of subset S, subset S can be eliminated from the set covering problem. However, we believe that the number of minimal subsets will still be too large for a reasonable number of demand points and thus the "brute force" approach is impractical.

2.2. Heuristic approaches

The brute force approach described above can only be used for relatively small problems. In this section we propose several heuristic algorithms for the variable radius covering problem. These algorithms are tested on problems of up to $n = 10,000$ demand points in Section 3.

All three heuristic algorithms share components for generating an initial solution and improving a given solution. We describe these common components below before presenting details of individual heuristics.

2.2.1. Generating a starting solution

To generate a random starting solution, randomly select the initial number of facilities p. For example, generate p in the range $[0.1n, 0.3n]$. Randomly generate in the area p centers forming a set P. Improve the solution by Algorithm 1 which is the alternate approach suggested by Cooper (1963, 1964).

Algorithm 1. Improving a given solution

1. Assign each demand point to the closest center getting a partition of the set of demand points.
2. All empty centers are removed from P. Therefore, the number of facilities may decrease but cannot increase.
3. Calculate the centers of the subsets by solving the one center problem for each subset that has changed from the previous iteration (otherwise the center remains the "old" one).
4. If the centers did not change (or equivalently all subsets remain unchanged), stop. Otherwise, go to Step 1.

Table 1
Number of minimal subsets for random problem instances

n	Max $n(n^2+5)/6$	Average	Standard deviation	Percent of Max
5	25	18	(2)	70.0
10	175	88	(6)	50.4
20	1350	521	(27)	38.6
50	20,875	6662	(94)	31.9
100	166,750	49,411	(965)	29.6
200	1,333,500	379,690	(3744)	28.5
500	20,833,750	5,816,895	(37,099)	27.9
1000	166,667,500	46,144,500	(196,751)	27.7

Solving the one center problem can be efficiently done by the Elzinga and Hearn (1972) algorithm. Note that in Step 2 of the algorithm some of the subsets may be removed from consideration and the number of subsets reduced. This phenomenon may present a difficulty in solving the p-center problem by a similar approach (Brimberg and Mladenovic, 1999). However, it is not an issue for the variable radius problem because the number of facilities is not fixed. Metaheuristic algorithms which apply an improvement phase similar to Algorithm 1 for the p-center problem are proposed in Brimberg et al. (2000).

2.2.2. A descent algorithm

Our first heuristic, which we call the "descent algorithm", consists of first generating an initial solution, improving it by applying Algorithm 1, and then performing a neighborhood search. The neighborhood for the search is defined by the following possible moves:

Type 1: Remove one center (unless the cardinality of P is one) and apply Algorithm 1 possibly improving the solution.
Type 2: Randomly add one center and apply Algorithm 1 possibly improving the solution.

The Type 1 moves are attempted for each center. The Type 2 moves are repeated p times where p is the cardinality of P to allow an equal chance of adding or removing a facility. In total, $2p$ moves are evaluated.

If an improving move has been found, perform the *best* improving move. If all moves of types 1 and 2 fail to improve the solution, the algorithm stops with the last P as the solution.

2.2.3. Simulated annealing

Our second heuristic is based on the simulated annealing metaheuristic. The simulated annealing (Kirkpatrick et al., 1983) simulates the cooling process of hot melted metals. The variant used in this paper depends on three parameters: the starting temperature T_0, the number of iterations N, and the factor $\alpha < 1$ by which the temperature is lowered every iteration.

There are two types of moves described in the descent approach. We suggest to select each type with a probability of $\frac{1}{2}$. If the cardinality of P is 1, a Type 2 move is performed. For Type 1 moves, the center to be removed is randomly selected with equal probability for each center. For Type 2 moves the location of the additional center is randomly selected. Note that each iteration of the simulated annealing requires one application of Algorithm 1.

2.2.3.1. The simulated annealing algorithm.

1. Set the temperature $T = T_0$. Select a starting solution P and evaluate the objective function $F(P)$. Set the best found solution to $F(P)$.
2. Repeat the following iterations N times:
   (a) Randomly select a move as described above and apply Algorithm 1 to get a set P'. Evaluate $F(P')$. Let $\Delta F = F(P') - F(P)$.
   (b) If $\Delta F \leqslant 0$, perform the move to P' and go to Step 2d.
   (c) If $\Delta F > 0$, perform the move to P' with probability $e^{-\frac{\Delta F}{T}}$ and go to Step 2e. Otherwise, retain P and go to Step 2e.
   (d) If $F(P')$ is better than the best found solution update the best found solution.
   (e) Multiply $T = \alpha T$.
3. The best found solution is the result of the algorithm.

2.2.4. A genetic algorithm

Our final heuristic is a genetic algorithm. Genetic algorithms (Holland, 1975; Goldberg, 1989; Drezner and Drezner, 2005) maintain a population of solutions, which is traced over several generations. A generation is defined as merging a pair of parents and generating one offspring. In the proposed genetic algorithm each solution is defined by a list of locations for the facilities. The details of the merging operation are described below. The parameters of the genetic algorithm are the population size S and the number of generations G.

2.2.4.1. The genetic algorithm.

1. Randomly generate a starting population of S solutions and apply Algorithm 1 on each selected solution.
2. Repeat the following for G generations:
   (a) Randomly select two population members as parents.
   (b) Merge the two parents creating an offspring (as described below).
   (c) If the offspring is better than the worst population member, then
      - if the offspring is not identical to an existing population member, replace the worst population member with the offspring,
      - otherwise go to step 2a.
3. The best population member is the solution of the algorithm.

The merging of two solutions in order to create an offspring is not straight-forward. We tried many merging rules. The following merging rule (which converts the algorithm to a hybrid genetic algorithm) performed best:

1. Create the union of the two parents by creating a combined list of all locations.
2. Check all pairs of locations in the list and if the two locations are the same, drop one location from the list.
3. Apply the descent algorithm with only Type 1 moves.

3. Computational experiments with planar problems

We coded programs in Fortran compiled by Intel Fortran 9.0 and run on a 2.8 GHz Pentium IV desktop computer.

3.1. Randomly generated problems

Twenty problems were randomly generated with $n = 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, 10,000$. One set of 10 problems were generated from the uniform distribution on a unit square and the second set of problems were generated from the same distribution on a rectangle of dimensions 1 by 2. The cost function used consisted of a fixed cost of $F = 0.01$ and a variable cost $\phi(r) = r^2$. First, for the unit square problems, we experimented with generating a grid of 1, 4, 9, 16, and 25 points by dividing the unit square into equal small squares, locating a facility at the center of each small square and applying Algorithm 1 on this starting configuration. Note that Algorithm 1 may reduce the number of facilities and thus the actual value of p for the reported solution may not be the value of p at the top of the column. It should be noted (see Suzuki and Drezner, 1996) that the p-center solution for a uniformly distributed demand in a square is not symmetric for $n = 9, 16, 25, \ldots$ (for example, the p-center objective function for a symmetric grid of 9 centers in a 3 by 3 squares grid is $\frac{\sqrt{2}}{6} = 0.23570$ while a non-symmetric solution of 0.23064 does exist). Thus one should not expect the symmetric grid-based solutions to necessarily be optimal for the current problem, even when demand points are generated from the uniform distribution.

Table 2
Solutions starting from a grid

n	*p*					Best known	
	1	4	9	16	25	*p*	Obj.
10	0.26827	0.10809	0.11265	**0.07486**	0.08417	6	0.07086
20	0.34267	0.20429	0.17166	**0.14200**	0.15529	9	0.12360
50	0.34274	0.29288	0.27408	**0.24083**	0.28501	10	0.21557
100	0.39402	0.35581	**0.33211**	0.34527	0.36038	13	0.28621
200	0.44483	**0.41419**	0.44163	0.41780	0.48064	16	0.38378
500	**0.46462**	0.46515	0.48711	0.52028	0.56100	9	0.44128
1000	0.48343	**0.47657**	0.49926	0.54761	0.60055	4	0.46537
2000	**0.49019**	0.49765	0.52511	0.57391	0.64506	4	0.48788
5000	**0.49573**	0.50993	0.54599	0.59856	0.67491	1	0.49573
10000	**0.50369**	0.52735	0.55603	0.61487	0.69166	1	0.50369

The results for the problems starting from a square grid in a square are summarized in Table 2. The best of these solutions for each value of n are depicted in boldface. The best known solution (reported later in Table 3) is given in the last two columns of the table for comparison purposes.

The best found solution to the $n = 10$ problem is depicted in Fig. 1. By inspection of the figure it is quite clear that no better solution exists.

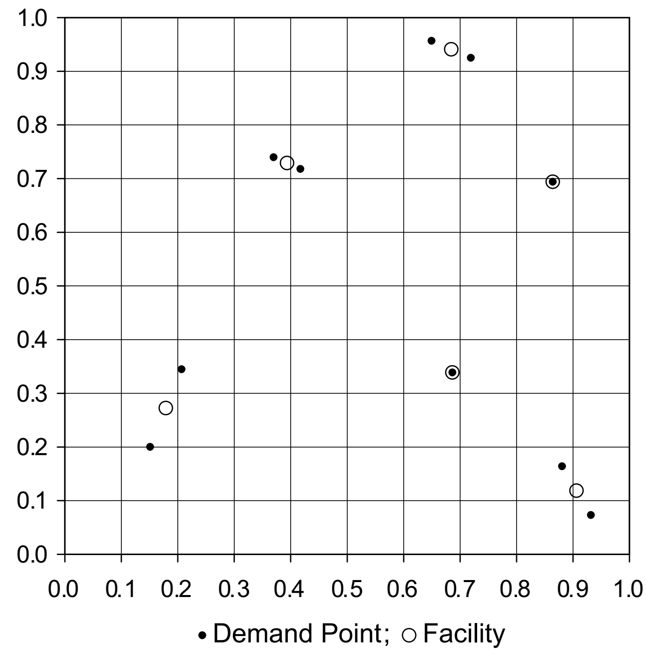

Fig. 1. Solution to the $n = 10$ problem.

3.2. One hundred and fifty largest cities in the united states

We also solved problems based on the 150 largest cities in the United States (Daskin, 1995). For each city the latitude ϕ and longitude θ are given. Distances should be measured by spherical distances. Since our formulation is based on Euclidean distances, we approximated the location of the cities as follows. The spherical distance d on a sphere of radius R between two points (ϕ_1, θ_1) and (ϕ_2, θ_2) is

$$\sin^2 \frac{d}{2R} = \sin^2 \frac{\phi_1 - \phi_2}{2} + \sin^2 \frac{\theta_1 - \theta_2}{2} \cos\phi_1 \cos\phi_2. \quad (2)$$

We selected a point around the center of the United States at $(\phi_0, \theta_0) = (36.7, 95.2)$ and assuming that $\sin x \approx x$ we converted the spherical locations (ϕ, θ) to planar locations (x, y) by the transformation,

$$x = R(\theta - \theta_0)\sqrt{\cos\phi_0 \cos\phi}; \quad y = R(\phi - \phi_0). \quad (3)$$

Table 3
Heuristic algorithms results for randomly generated problems

n	Best known		Descent			Sim. annealing			Genetic algorithm		
	p	Obj.	[a]	[b]	Time[c]	[a]	[b]	Time[c]	[a]	[b]	Time[c]
Points in a Square of 1 by 1											
10	6	0.07086	366	59.94	0.04	10	0.00	0.07	10	0.00	0.01
20	9	0.12360	262	11.14	0.06	10	0.00	0.20	10	0.00	0.11
50	10	0.21557	141	2.95	0.05	10	0.00	0.87	10	0.00	1.52
100	13	0.28621	5	2.84	0.40	3	0.39	2.55	3	0.29	8.69
200	16	0.38378	37	3.49	4.20	1	1.31	6.92	10	0.00	38.64
500	9	0.44128	0	9.89	23.66	1	0.90	22.93	7	0.11	62.14
1000	4	0.46537	0	11.69	59.01	10	0.00	55.81	3	0.36	139.05
2000	4	0.48788	0	12.85	148.04	9	0.02	144.87	8	0.09	318.67
5000	1	0.49573	0	17.52	483.80	10	0.00	438.38	10	0.00	987.56
10000	1	0.50369	0	19.66	1182.63	10	0.00	1148.24	10	0.00	2354.63
Average			81.1	15.20	190.19	7.4	0.26	182.09	8.1	0. 09	391.10
Points in a rectangle of 1 by 2											
10	7	0.08214	279	89.29	0.00	10	0.00	0.07	10	0.00	0.02
20	11	0.15204	387	21.86	0.01	10	0.00	0.20	10	0.00	0.11
50	18	0.30021	9	9.87	0.09	5	0.43	0.86	9	0.08	1.43
100	20	0.41795	1	3.96	0.43	0	0.76	2.48	10	0.00	9.42
200	20	0.60761	0	4.33	4.37	0	2.65	7.60	2	0.87	61.07
500	18	0.77084	0	8.31	29.58	0	1.41	27.72	1	0.10	185.74
1000	8	0.84272	0	8.95	78.72	1	0.61	67.68	0	0.75	420.80
2000	8	0.89549	0	11.28	199.12	9	0.03	161.10	10	0.00	450.24
5000	8	0.95073	0	12.01	658.80	1	0.23	487.48	0	0.29	1274.33
10000	8	0.97461	0	13.08	1573.60	5	0.04	1204.72	0	0.12	2247.58
Average			67.6	18.29	254.47	4.1	0.62	195.99	5.2	0. 22	465.07

[a] No. of times (out of 1000 for descent, 10 for others) that best known solution found.
[b] Percentage of average solution over the best known one (the relative error).
[c] Total time in minutes.

Since the angles ϕ and θ are given in degrees, R in (3) should be replaced by the distance in miles of one degree latitude which is 69.09 miles.

Following the transformation (3) distances between the cities are measured in miles. The cost function used consisted of a fixed cost of $F = 100,000$ and a variable cost $\phi(r) = r^2$. We solved 11 problems with the largest 50,60,...,150 cities.

3.3. Parameters for the heuristic algorithms

Following extensive experiments the following parameters were selected for the heuristic algorithms.

3.3.1. The descent algorithm

The value of p for the starting solution was randomly generated in $[0.1n, 0.3n]$ and if the generated value was 40 or larger, it was replaced by a random value in [20,60].

3.3.2. Simulated annealing

The same rule as in the descent algorithm for generating the starting p was adopted. The number of iterations was set to $N = 200,000$. The initial temperature was set to $T_0 = 1$. The temperature reduction factor was set to $\alpha = 1 - \frac{5}{N} = 0.999975$ leading to a final temperature of about $e^{-5} \approx 0.0067$.

3.3.3. The genetic algorithm

The starting population was generated by randomly generating one solution with $p = 1$, four solutions with $p = 2$, nine solutions with $p = 3$ and so on until 49 solutions with $p = 7$ for a total of 140 population members. Algorithm 1 was applied on each of these solutions. The number of generations was set to 5000.

3.4. Results of the heuristic approaches

3.4.1. Randomly generated problems

The results for the heuristic approaches are reported in Table 3. The algorithms performed faster for the problem generated in a square rather than in a rectangle. This is likely due to the fact that the solutions found for the problems in a rectangle consist of more facilities than the solutions found for the square problem and thus Algorithm 1 requires more iterations. The descent approach failed to find the best known solution even once for all large problems with $n > 200$ demand points in a square and $n > 100$ in a rectangle. Moreover, the relative errors for the best found solutions were large. Both simulated annealing and the genetic algorithm found the best known solution at least once (genetic at least three times) for each problem in a square but each failed to find the best known solution for three problems each in a rectangle. However, the relative errors were generally less than 1%, indicating that both algorithms consistently found solutions that were close to the best known ones.

Overall, the genetic algorithm appears to produce the best results (lowest relative errors), but required longer run times than the other two algorithms.

It is interesting to note that the search starting from a grid performed quite well for larger values of n on a square (but not well for small n).

3.4.2. One hundred and fifty largest cities in the united states

The results of the heuristic algorithms for the 50,60,...,150 largest cities in the United States are depicted in Table 4. The algorithms appear to perform somewhat differently on this real-life data set: the solution quality is generally higher, but the running times were somewhat longer than for the randomly generated problems of similar dimensionality used in the previous experiment. Once again, the genetic algorithm yielded the best solutions but took the longest time to produce them.

It is interesting to note that the solution for the 140 largest cities is the same as the solution to the 130 largest cities. The ten additional cities (131–140) are located inside the six circles of the solution to the largest 130 cities.

4. The discrete variable radius covering problem

In this section we assume that there is a finite sets of potential facility locations available—that is, we assume a discrete location setting. Without lost of generality we assume that the demand points and the potential locations are set of nodes $N(|N| = n)$ of a network G with shortest distances $d_{ij}\ \forall i,j \in N$. We redefine the decision variables X and Y as follows (note that the new definitions refer to the potential facility locations, whereas the ones used earlier referred to the actual locations):

$$X_j = \begin{cases} 1, & \text{if a facility is located at node } j \in N, \\ 0, & \text{otherwise}, \end{cases}$$

$$Y_{ij} = \begin{cases} 1, & \text{if node } i \text{ is assigned to node } j, \\ 0, & \text{otherwise}. \end{cases}$$

As in Section 2, we assume that there exists a non-negative, non-decreasing coverage function $\phi_j(r)$ which represents the cost of locating a facility with coverage radius r at node j. Note that since we have a finite set of possible locations, we can formulate the problem with a different fixed and variable components of the cost function for different nodes. Specifically, we assume that the cost of building a facility at node j with radius r is

$$F_j + \phi_j(r),$$

where F_j is the fixed cost of locating a facility at node j.

Table 4
Heuristic algorithms results for largest cities in the United States

n	Best known		Descent			Sim. annealing			Genetic algorithm		
	p	Obj.	[a]	[b]	Time[c]	[a]	[b]	Time[c]	[a]	[b]	Time[c]
50	6	1,289,473	101	2.48	0.95	10	0.00	31.43	10	0.00	43.51
60	6	1,293,431	146	1.79	1.40	10	0.00	37.28	10	0.00	52.67
70	8	1,313,939	108	2.35	2.20	10	0.00	44.17	10	0.00	73.35
80	4	1,316,576	17	3.16	3.40	10	0.00	50.74	10	0.00	84.52
90	7	1,356,957	43	1.43	4.72	10	0.00	59.93	10	0.00	90.48
100	7	1,373,426	2	1.62	6.58	4	0.23	65.33	10	0.00	106.73
110	7	1,399,790	2	1.08	9.26	9	0.00	73.69	10	0.00	132.91
120	7	1,400,167	0	1.21	12.25	10	0.00	80.28	10	0.00	149.30
130	6	1,410,268	2	0.33	16.39	7	0.01	87.22	9	0.00	148.58
140	6	1,410,268	0	0.47	20.05	7	0.01	94.00	10	0.00	157.39
150	6	1,422,214	115	2.08	24.14	10	0.00	112.63	10	0.00	201.71

[a] No. of times (out of 1000 for descent, 10 for others) that best known solution found.
[b] Percentage of average solution over the best known one (the relative error).
[c] Total time in seconds.

The problem of minimizing the total cost of the system can now be formulated as follows:
(P_1)

$$\min \quad \sum_{j=1}^{n} F_j X_j + \sum_{j=1}^{n} \phi_j(\max_{i=1,\dots,n}(d_{ij}Y_{ij})) \tag{4}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} Y_{ij} = 1, \quad i = 1,\dots,n, \tag{5}$$

$$Y_{ij} \leqslant X_j, \quad i,j = 1,\dots,n, \tag{6}$$

$$X_j, Y_{ij} \in \{0,1\}, \quad i,j = 1,\dots,n. \tag{7}$$

The connection of this formulation to formulation (1) presented earlier is clear. As before, the coverage radii r_j are not explicit decision variables in the model but are determined through the decision variables $\{Y_{ij}\}$ via $r_j = \max_i(d_{ij}Y_{ij})$. This observation is similar to the minimal subset property in the planar case—i.e., for each facility j, there exists one (or more) nodes for which the travel distance to j is maximized among all nodes assigned to this facility; these "critical" nodes determine the coverage radius r_j.

Constraints (5) ensure that each node is assigned to a facility while constraints (6) forbid an assignment of node i to a node which does not host a facility. The constraints are linear whereas the objective function is non-linear.

Example 1. Consider the 5-node network in Fig. 2 with the shortest distance matrix D:

$$D = \begin{pmatrix} 0 & 3 & 2 & 4 & 5 \\ 3 & 0 & 4 & 7 & 8 \\ 2 & 4 & 0 & 6 & 5 \\ 4 & 7 & 6 & 0 & 3 \\ 5 & 8 & 5 & 3 & 0 \end{pmatrix}.$$

We assume that $\phi_j(r) = C_j r^2$. Suppose $F_1 = 150$, $F_2 = 100$, $F_3 = 150$, $F_4 = 100$, $F_5 = 150$, $C_1 = 15$, $C_2 = 10$, $C_3 = 30$, $C_4 = 10$, and $C_5 = 25$. The optimal solution (using the enhanced non-linear solver for Excel) is to place two facilities, one at node 2 and one at node 4. Each demand node is assigned to the closest open facility: nodes $\{1,2,3\}$ are assigned to the facility at node 2, and nodes $\{4,5\}$ are assigned to the facility at node 4. Thus, $r_2 = \max\{3,0,4\} = 4$, and $r_4 = \max\{0,3\} = 3$, and therefore the objective function value is: $F_2 + F_4 + C_2 r_2^2 + C_4 r_4^2 = 100 + 100 + 160 + 90 = 450$.

While problem (P_1) has a non-linear objective function, it can be re-formulated as a linear mixed integer program. Let $W_j,\ j = 1,\dots,n$ be continuous decision variables where W_j is actually $\phi(\max_i\{Y_{ij}d_{ij}\}) = \phi(r_j)$. The problem can now be formulated as follows:

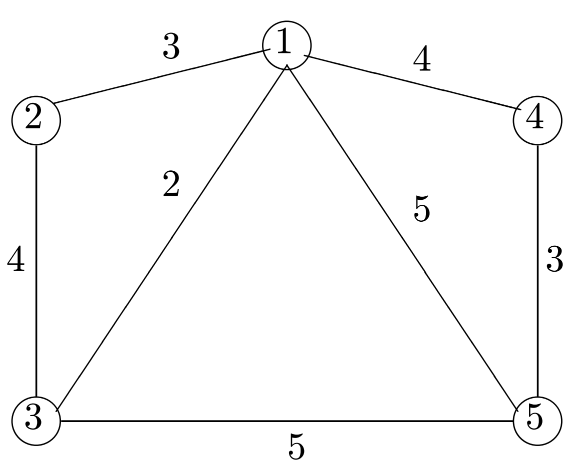

Fig. 2. The example network.

(P_2)

$$\min \quad \sum_{j=1}^{n} F_j X_j + \sum_{j=1}^{n} W_j \tag{8}$$

$$\text{s.t.} \quad W_j \geqslant Y_{ij}\phi_j(d_{ij}), \quad i,j = 1,\dots,n. \tag{9}$$

Constraints (5) and (6)

$$W_j \geqslant 0, \quad X_j, Y_{ij} = 0,1. \tag{10}$$

Obviously since at optimality $W_j = \max_i\{Y_{ij}\phi_j(d_{ij})\}$, (P_1) and (P_2) are equivalent. Next we show that (P_2) can be re-formulated so that the binary variables X_j and constraint (6) are not even required. We can re-define W_j as the quantity $X_j F_j + \phi(\max_{i=1,\dots,n} d_{ij}Y_{ij})$. Therefore,

$$\sum_{j=1}^{n} W_j = \sum_{j=1}^{n} X_j F_j + \sum_{j=1}^{n} \phi_j(\max_{i=1,\dots,n} d_{ij}Y_{ij})$$
$$= \sum_{j=1}^{n} \max_{i=1,\dots,n}\{Y_{ij}\}F_j + \sum_{j=1}^{n} \max_{i=1,\dots,n}\{\phi_j(d_{ij})Y_{ij}\}.$$

Thus, the problem is (P_3)

$$\min \quad \sum_{j=1}^{n} W_j \tag{11}$$

$$\text{s.t.} \quad W_j \geqslant Y_{ij}[\phi_j(d_{ij}) + F_j], \quad i,j = 1,\dots,n, \tag{12}$$

$$\sum_{j=1}^{n} Y_{ij} = 1, \quad i = 1,\dots,n, \tag{13}$$

$$W_j \geqslant 0, Y_{ij} = 0,1, \quad i,j = 1,\dots,n. \tag{14}$$

Obviously, in this formulation a facility is located at node j whenever $\sum_i Y_{ij} > 0$.

4.1. Solving the discrete variable radius covering problem

Here we discuss a procedure to solve the problem as a set covering problem. Consider a potential facility location j. Since the coverage radius r_j is determined by the farthest node assigned to this facility, there are at most n potential values for r_j ranging from 0 to $\max_i d_{ij}$ (note that only distinct values need to be considered). We can order these values in ascending order and think of them as potential "levels of coverage" for facility j. Let $l = 1,\dots,L(j)$ be the indices of the possible coverage levels for location j and $d_{l(j)} = d_{ij}$ be the corresponding travel distance, where i is the lth furthest node from j (in case of ties we only include in the arrangement one of the tied nodes so $1 \leqslant L(j) \leqslant n$). Thus, for each facility we must decide which coverage level should be assigned to it. Note that if level l is used for facility at node j, the contribution to the total cost is

$$f_l(j) = F_j + \phi_j(d_{l(j)}).$$

Corresponding to each potential facility location j and coverage level $l(j)$ we define a binary decision variable $Z_l(j)$, which is equal to 1 if a facility is located at node j and coverage level l is assigned to this facility. We also define n binary parameters,

$$b_{i,l(j)} = \begin{cases} 1, & \text{if } d_{ij} \leqslant d_{l(j)}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, $b_{il(j)} = 1$ if node i is covered from facility j when coverage level $l(j)$ is used at j.

The set covering problem we consider is(P_4)

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \sum_{l=1}^{L(j)} Z_l(j) f_l(j) \\ s.t. \quad & \sum_{j} \sum_{l} b_{i,l(j)} Z_l(j) \geqslant 1, \quad i = 1,\dots,n, \\ & Z_l(j) = 0,1 \quad \forall l,j. \end{aligned} \tag{15}$$

In the objective function of (P_4), we sum up the total cost of selecting a certain coverage level for each facility. In constraint (15) we ensure that for any node i there is at least one facility j with assigned coverage level $l(j)$ such that i is covered from j at that coverage level. To show that (P_4) provides the solution to our problem, it is sufficient to prove the following lemma.

Lemma 1. *For any given j, at most one decision variable $Z_l(j)$ can be equal to 1 in the optimal solution.*

Proof. Suppose $Z_{l'}(j) = Z_{l''}(j) = 1$ and $l'' > l'$. Since $b_{i,l''(j)} \geqslant b_{i,l'(j)}\ \forall i$, a saving of $f_{l'}(j)$ can be realized if setting $Z_{l'}(j) = 0, X_{l''}(j) = 1$. □

Let B be a matrix of $b_{i,l(j)}$ values for $i,j = 1,\ldots,n$, and $l = 1,\ldots,L(j)$. Notice that each column of B corresponds to one of the decision variables $Z_l(j)$. Constraints (15) are actually $BZ \geqslant \bar{1}$ where $\bar{1}$ is a column vector of n one's. Since the number of decision variables is $O(n^2)$, it is quite essential to reduce this number as much as possible, particularly for large problem instances. The following observations can reduce the number of decision variables significantly. The next observation is obvious.

Observation 1. Let OF^H be the value of the best known solution to our problem (OF^H can be the result of a greedy heuristic applied to (P_2) or (P_3) as shown later). If $f_l(j) > OF^H$ we can set $Z_l(j) = 0$.

In the next observation we use the fact that if each element in a column of B corresponding to decision variables $Z_{l'}(j')$ is greater than or equal to each corresponding element of another column of B corresponding to decision variable $Z_{l''}(j'')$ and $f_{l'}(j) < f_{l''}(j)$ then $Z_{l''}(j'')$ is dominated by $Z_{l'}(j')$ and can be set to 0.

Observation 2. Let $Z_{l'}(j')$ and $Z_{l''}(j'')$ be two decision variables where $j' \neq j''$. If $f_{l'}(j') < f_{l''}(j'')$ and $b_{i,l''(j)} \leqslant b_{i,l'(j)}\ \ i = 1,\ldots,n$ we can set $Z_{l''}(j'') = 0$. ($Z_{l''}(j'')$ is dominated by $Z_{l'}(j')$). This implies that all but one variables in the set $\{Z_n(j),\ j = 1,\ldots,n\}$ should be deleted since for all of these variables $b_{i,n(j)} = 1,\ i = 1,\ldots,n$.

The next observation relates to the case where each element in a column of B corresponding to decision variables $Z_{l'}(j)$ are identical to the corresponding elements of column B of another decision variable $Z_{l''}(j'')$ except for several demand nodes $I \subset N$ for which the elements of the column corresponding to $Z_{l'}(j')$ are larger than the corresponding element of $Z_{l''}(j'')$ and $f_{l'}(j') > f_{l''}(j'')$. Then if constructing new facilities in I yield a total fixed cost larger than the difference between $f_{l'}(j')$ and $f_{j''}(j'')$, $Z_{l'}(j')$ can be set equal to zero.

Observation 3. Let $Z_{l'}(j')$ and $Z_{l''}(j'')$ be two decision variables. Suppose $b_{i,l'(j')} = b_{i,l'(j'')}$ for all i except for one $\hat{i}$, where $b_{\hat{i},l'(j')} > b_{\hat{i},l''(j'')}$ and $f_{l'}(j') > f_{l''}(j'')$. Then if $f_{l'}(j') - f_{l''}(j'') \geqslant F_{\hat{i}}$, we can set $Z_{l'}(j') = 0$. ($Z_{l'}(j')$ is dominated by $Z_{l''}(j'')$). More generally, if there exists a subset $I \subset N$ such that $b_{i,l'(j')} = b_{i,l''(j'')}$ for all $i \notin I$, $b_{i,l'(j')} > b_{i,l''(j'')}$ for $i \in I$ and $f_{l'}(j') - f_{l''}(j'') \geqslant \sum_{i \in I} F_i$, then we can set $Z_{l'}(j') = 0$.

We now present a greedy heuristic to obtain OF^H. First, to calculate the objective function value for a given set of facilities S^0 we apply the following simple procedure:

(i) For each $i \in N$ set $Y_{ij^0} = 1$ where $j^0 = \arg\min_j\{\phi(d_{ij}) + F_j\}$ and $Y_{il} = 0$ for $l \neq j^0$ where $l, j^0 \in S^0$ (in case of a tie, break the tie arbitrarily)
(ii) The objective function value of S^0 is

$$\sum_{j \in S^0} W_j = \sum_{j \in S^0} (F_j + \phi(\max_i d_{ij} Y_{ij})) = \sum_{j \in S^0} (F_j + \max_i \{Y_{ij} \phi(d_{ij})\}).$$

The following greedy heuristic can be used to obtain OF^H.

Step 0: $p = 1, S^0 = S^* = \{j^0\}$ where $j^0 = \arg\min_j \max_l f_l(j)$ and $OF^H = W_{j^0} = \max_l C_l(j^0)$.
Step 1: $p = p + 1$.
Step 2: Find the best facility k to add to S^0 (k is best if $\sum_{j \in S^0 \cup k} W_j \leqslant \sum_{j \in S^0 \cup t} W_j \forall t \in N$). Set $S^0 = S^0 \cup \{k\}$. If $\sum_{j \in S^0} W_j < OF^H$, set $S^* = S^0, OF^H = \sum_{j \in S^0} W_j$. If $p = n$ stop, S^* is "optimal" with objective function value OF^H. Otherwise go to Step 1.

Example 2. (cont.) As an illustration of our overall solution procedure we use the data from the 5-node example discussed earlier. For $j = 1,\ldots,4, l = 1,\ldots,5$ and for $j = 5, l = 1,\ldots,4$ (since $d_{15} = d_{35}$). Therefore, we have a total of 24 decision variables.

For $j = 1$, the distances corresponding to the different levels of coverage are $0, 2, 3, 4, 5$, which leads to the following objective function coefficients: $f_1(1) = 150,\ f_2(1) = 150 + 2^2 * 15 = 210,\ f_3(1) = 150 + 3^2 * 15 = 285,\ f_4(1) = 150 + 4^2 * 15 = 390,\ f_5(1) = 150 + 5^2 * 15 = 525$. Coefficients for the other decision variables are computed similarly, leading to the following objective function:

$$\begin{aligned}\min\ & 150Z_1(1) + 210Z_2(1) + 285Z_3(1) + 390Z_4(1) + 525Z_5(1)\\ & + 100Z_1(2) + 190Z_2(2) + 260Z_3(2) + 590Z_4(2) + 740Z_5(2)\\ & + 150Z_1(3) + 270Z_2(3) + 630Z_3(3) + 900Z_4(3) + 1230Z_5(3)\\ & + 100Z_1(4) + 190Z_2(4) + 260Z_3(4) + 460Z_4(4) + 590Z_5(4)\\ & + 150Z_1(5) + 375Z_2(5) + 775Z_3(5) + 1750Z_4(5).\end{aligned}$$

Next we compute the first row of matrix B. Note that node 1 is the closest node to itself (i.e., it would be covered at any coverage level by a facility at 1), is second-closest for nodes 2 and 3 (i.e., it would be covered at levels 2 and higher by facilities at 2 or 3), and is third-closest for nodes 4 and 5 (it would be covered at level 3 or higher from these nodes). Thus, $b_{1,l(1)} = 1$ for $l = 1, 2, 3, 4, 5$; $b_{1,l(2)} = b_{1,l(3)} = 1$ for $l = 2, 3, 4, 5$; $b_{1,l(4)} = 1$ for $l = 3, 4, 5$; $b_{1,l(5)} = 1$ for $l = 3, 4$, with all other entries $b_{i,l(j)}$ equal to 0. This allows us to write the coverage constraint for node 1; the constraints for other nodes are developed similarly.

We now show how we can reduce the number of decision variables. The greedy heuristic gives $OF^H = 475$. By Observation 1 and the examination of the objective function coefficients we can set $Z_5(1) = Z_4(2) = Z_5(2) = Z_3(3) = Z_4(3) = Z_5(3) = Z_5(4) = Z_3(5) = Z_4(5)$ equal to 0. By Observation 2 we can set $Z_3(1) = 0$ (dominated by $Z_3(2)$), $Z_2(5) = 0$ (dominated by $Z_3(4)$) and $Z_2(3) = 0$ (dominated by $Z_2(1)$). By Observation 3, $Z_4(4) = 0$ (dominated by $Z_3(4)$) and $Z_4(1) = 0$ (dominated by $Z_3(1)$).

The resulting formulation has 10 decision variables:

$$\begin{aligned}\min\quad & 150Z_1(1) + 210Z_2(1) + 100Z_1(2) + 190Z_2(2) + 260Z_3(2)\\ & \quad + 150Z_1(3) + 100Z_1(4) + 190Z_2(4) + 260Z_3(4) + 150Z_1(5)\\ \text{s.t.}\quad & Z_1(1) + Z_2(1) + Z_2(2) + Z_3(2) \geqslant 1\\ & Z_1(2) + Z_2(2) + Z_3(2) \geqslant 1\\ & Z_2(1) + Z_3(2) + Z_1(3) + Z_3(4) \geqslant 1\\ & Z_1(4) + Z_2(4) + Z_3(4) \geqslant 1\\ & Z_2(4) + Z_3(4) + Z_1(5) \geqslant 1\end{aligned}$$

5. Computational experiments with the discrete problem

We solved the 40 networks that were designed for testing p-median problems in Beasley (1990). The value of p for each problem was not used. The problems using formulation (P_4), after the applying simplifications based on Observations 1–3, were solved on Pentium V PC using CPLEX LP/IP version 8.1. The cost at node j is $F_j + C_j r^2$. Three cost structures were applied where $F_j \sim U[.8\delta, .1.2\delta]$, $C_j \sim U[.8\gamma, 1.2\gamma]$ (where $U[a, b]$ stands for the

Table 5
Computational results using CPLEX

Prob. No.	n	Cost 1			Cost 2			Cost 3		
		Objective function	# of facilities	Time (seconds)	Objective function	# of facilities	Time (seconds)	Objective function	# of facilities	Time (seconds)
1	100	88,737	83	0.355	261,436	6	0.386	468,988	2	0.326
2	100	82,673	73	0.077	252,065	5	0.294	431,226	2	0.341
3	100	87,316	83	0.078	338,616	7	0.448	558,693	2	0.465
4	100	92,585	84	0.031	333,918	6	0.371	526,831	2	0.418
5	100	76,629	69	0.093	230,983	2	0.294	395,856	2	0.326
6	200	77,686	19	1.712	130,569	3	2.134	309,308	2	2.433
7	200	71,808	27	1.527	103,807	2	1.546	256,533	2	1.178
8	200	79,328	17	1.712	135,942	2	1.605	303,101	2	1.069
9	200	78,993	23	2.330	123,449	2	1.991	287,230	2	1.798
10	200	50,715	13	0.972	85,538	3	1.081	243,474	2	0.573
11	300	39,975	10	2.067	63,140	2	2.238	246,879	2	0.853
12	300	49,302	11	4.196	77,341	2	3.750	240,618	2	1.983
13	300	44,817	13	3.224	67,396	2	2.608	218,559	2	1.070
14	300	52,152	8	4.181	82,385	2	3.997	251,677	2	2.774
15	300	41,594	8	2.776	73,221	3	2.731	231,632	2	1.301
16	400	27,099	2	3.151	45,830	2	2.793	196,492	2	0.946
17	400	26,477	4	4.125	46,634	2	4.891	201,996	2	1.674
18	400	35,761	6	6.180	54,543	2	5.725	210,252	2	1.952
19	400	31,631	4	4.805	48,861	2	4.645	213,451	2	1.999
20	400	33,429	4	5.145	55,562	2	4.861	212,233	2	1.395
21	500	22,962	8	5.222	37,576	2	4.614	189,529	2	0.884
22	500	27,251	4	7.663	48,998	2	7.793	224,099	2	3.161
23	500	26,393	5	7.508	46,302	2	9.699	199,662	2	1.519
24	500	25,328	3	7.122	46,305	2	7.001	194,258	2	1.164
25	500	26,842	2	5.067	45,747	2	4.961	206,784	2	1.086
26	600	21,039	4	8.944	38,283	2	7.032	201,038	2	1.304
27	600	19,194	4	8.635	37,780	2	7.759	187,964	2	1.071
28	600	18,642	4	8.850	38,487	2	6.832	194,202	2	1.769
29	600	19,732	3	9.067	37,008	2	7.697	191,845	2	1.521
30	600	21,059	4	11.724	38,390	2	6.940	206,288	2	1.924
31	700	14,297	2	8.434	31,168	2	7.795	258719	3	0.729
32	700	16,984	3	10.755	34,496	2	9.412	204,797	2	2.032
33	700	16,056	2	11.599	32,906	2	7.863	189,238	2	1.924
34	700	16,414	4	9.187	33,157	2	7.925	199,889	2	1.568
35	800	12,405	2	11.027	31,118	2	8.251	181,994	2	1.458
36	800	14,161	2	13.996	31,910	2	12.369	191,785	2	1.078
37	800	14,181	2	13.471	29,406	2	9.897	180,929	2	1.641
38	900	12,163	2	14.739	28,497	2	8.523	186,975	2	1.241
39	900	11,674	3	14.353	27,764	2	9.081	177,324	2	3.414
40	900	13,458	2	14.662	29,451	2	11.607	180,506	2	1.706

uniform distribution over $[a, b]$). In cost structure 1, $\delta = 1,000$, $\gamma = 10$; in cost structure 2, $\delta = 10,000$, $\gamma = 10$ and in cost structure 3, $\delta = 100,000$, $\gamma = 10$. The results are depicted in Table 5. All the problems were solved in a remarkably short computer time. As can be seen from the table, when the fixed costs are increasing, less facilities are open and the objective function values increase. When the number of nodes n increases, less facilities are open and the objective function values decrease. The reason for this is that, in the Beasley problems, when n increases the shortest distances get smaller and therefore the variable costs get smaller as well.

6. Conclusions and future research

The problem of covering a set of demand points with facilities with controllable coverage radius was introduced in the plane and when finite sets of demand points and potential locations are given. The cost of a facility is a monotonically increasing function of the covering radius. Three heuristics for its solution in the plane were introduced and the proposed genetic algorithm is the recommended approach.

For the discrete problem, several formulations were presented. In the first formulation the problem is formulated as a non-linear integer program. In formulations (P_2) and (P_3) it is shown that the problem can be formulated as a mixed integer linear program. Formulations (P_2) and (P_3) lead to the most efficient formulation (P_4) that is a set covering type problem. The 'price paid' is a substantial increase in the number of decision variable ($O(n^2)$ compared to $O(n)$ for (P_3)). However, it is also shown that the number of decision variables for (P_4) can be reduced since some of the decision variables may be dominated by others. Computational experiments with the 40 networks (Beasley (1990)) show that the problems with up to 900 nodes were solved by CPLEX in very short computer times.

As future research we propose the following problem. Suppose that each demand point has a weight. We wish to cover the most weight with a given budget. This is a natural extension of the MCLP model in the variable-radius framework—instead of a fixed number of facilities, we have a given budget and the radii of cover are not given.

Acknowledgement

This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

References

Aboolian, R., Berman, O., Krsss, D., 2008. Competitive facility location and design problem. European Journal of Operational Research 182, 40–62.
Batta, R., Dolan, J., Krishnamurthy, N., 1989. The maximal expected covering location problem: Revisited. Transportation Science 23, 277–287.

Beasley, J.E., 1990. OR-Library – Distributing test problems by electronic mail. Journal of the Operational Research Society 41, 1069–1072 <http://people.brunel.ac.uk/~mastjjb/jeb/orlib/pmedinfo.html> .
Berman, O., 1994. The p-maximal cover – p-partial center problem on networks. European Journal of Operational Research 72, 432–442.
Berman, O., Krass, D., Drezner, Z., 2003. The gradual covering decay location problem on a network. European Journal of Operational Research 151, 474–480.
Berman, O., Krass, D., 2002. The generalized maximal covering location problem. Computers and Operations Research 29, 563–591.
Brimberg, J., Mladenovic, N., 1999. Degeneracy in the multi-source Weber problem. Mathematical Programming 85, 213–220.
Brimberg, J., Hansen, P., Mladenovic, N., Taillard, E., 2000. Improvements and comparison of heuristics for solving the multi-source Weber problem. Operations Research 48, 444–460.
Canovas, L., Pelegrin, B., 1992. Improving some heuristic algorithms for the rectangular *p*-cover problem. In: Moreno-Perez, J.A. (Ed.), Proceedings of the VI Meeting of the EURO Working Group on Locational Analysis. Universidad De La Laguna, Tenerife, Spain, pp. 23–31.
Church, R.L., Eaton, D.J., 1987. Hierarchical location analysis using covering objectives. In: Ghosh, A., Rushton, G. (Eds.), Spatial Analysis and Location–Allocation Models. Van Nostrand Reinhold Company, New York, pp. 163–185.
Carrizosa, E., Plastria, F., 1998. Polynomial algorithms for parametric min-quantile and maxcovering planar location problems with locational constraints. TOP 6, 179–194.
Church, R.L., ReVelle, C., 1974. The maximal covering location problem. Papers of the Regional Science Association 32, 101–118.
Cooper, L., 1963. Location – Allocation problems. Operations Research 11, 331–343.
Cooper, L., 1964. Heuristic methods for location – Allocation problems. SIAM Review 6, 37–53.
Current, J., Daskin, M., Schilling, D., 2002. Discrete network location models. In: Drezner, Z., Hamacher, H. (Eds.), Facility Location: Applications and Theory. Springer-Verlag, Berlin.
Current, J.R., O'Kelly, M.E., 1992. Locating emergency warning sirens. Decision Sciences 23, 221–234.
Daskin, M.S., 1983. A maximum expected covering location model: Formulation, properties and heuristic solution. Transportation Science 17, 48–70.
Daskin, M.S., 1995. Network and Discrete Location: Models, Algorithms, and Applications. John Wiley & Sons, New York.
Daskin, M., 2003. Sitation Software. <http://users.iems.nwu.edu/~msdaskin>.
Daskin, M.S., Stern, E.H., 1981. A hierarchical objective set covering model for emergency medical service vehicle deployment. Transportation Science 15, 137–152.
Drezner, Z., 1981. On a modified one-center problem. Management Science 27, 848–851.
Drezner, Z., 1986. The p-cover problem. European Journal of Operational Research 26, 312–313.
Drezner, T., 1998. Location of multiple retail facilities with a limited budget. Journal of Retailing and Consumer Services 5, 173–184.
Drezner, T., Drezner, Z., 2005. Genetic algorithms: Mimicking evolution and natural selection in optimization models. In: Bar-Cohen, Y. (Ed.), Biomimetics – Biologically Inspired Technologies. CRC Press, Boca Raton, pp. 157–175.
Drezner, Z., Wesolowsky, G.O., 1997. On the best location of signal detectors. IIE Transactions 29, 1007–1015.
Drezner, Z., Wesolowsky, G.O., Drezner, T., 2004. The gradual covering problem. Naval Research Logistics 51, 841–855.
Elzinga, D.J., Hearn, D.W., 1972. Geometrical solutions for some minimax location problems. Transportation Science 6, 379–394.
Fernandez, J., Pelegrin, B., Plastria, F., Toth, B., 2007. Solving a huff-like competitive location and design model for profit maximization in the plane. European Journal of Operational Research 179, 1274–1287.
Goldberg, D.E., 1989. Genetic Algorithms in Search, Optimization and Machine Learning. Addison-Wesley, Wokingham, England.
Holland, J.H., 1975. Adaptation in Natural and Artificial Systems. University of Michigan Press, Ann Arbor.
Kirkpatrick, S., Gelat, C.D., Vecchi, M.P., 1983. Optimization by simulated annealing. Science 220, 671–680.
Megiddo, N., Zemel, E., Hakimi, S.L., 1983. The maximum coverage location problems. SIAM Journal of Algebraic and Discrete Methods 4, 253–261.
Plastria, F., 2002. Continuous covering location problems. In: Drezner, Z., Hamacher, H. (Eds.), Facility Location: Applications and Theory. Springer-Verlag, Berlin.
Plastria, F., Carrizosa, E., 1999. Undesirable facility location in the plane with minimal covering objectives. European Journal of Operational Research 119, 158–180.
Plastria, F., Carrizosa, E., 2004. Optimal location and design of a competitive facility. Mathematical Programming 100, 247–265.
ReVelle, C., 1986. The maximum capture or 'sphere of influence' location problem: Hotelling revisited on a network. Journal of Regional Science 26, 343–357.
ReVelle, C., Toregas, C., Falkson, L., 1976. Applications of the location set covering problem. Geographical Analysis 8, 67–76.
Schilling, D.A., Vaidyanathan, J., Barkhi, R., 1993. A review of covering problems in facility location. Location Science 1, 25–55.
Suzuki, A., Drezner, Z., 1996. The p-center location problem in an area. Location Science 4, 69–82.
Watson-Gandy, C., 1982. Heuristic procedures for the *m*-partial cover problem on a plane. European Journal of Operational Research 11, 149–157.