DIFFERENTIAL EQUATIONS (10 Marks)

1% DIFFERENTIAL EQUATIONS

99% INTEGRATION

$\boxed{P1}$ Reverse working:

$$\frac{dy}{dx} = 2x+1$$

$$y = \int (2x+1)\,dx$$

$\boxed{P3}$

$$\frac{dy}{dx} = \frac{2x+1}{y^2}$$

$$y = \cancel{\int \frac{2x+1}{y^2}\,dx}$$

TO SOLVE SUCH PROBLEMS THE METHOD USED IS CALLED DIFFERENTIAL EQUATIONS.

Q: $\dfrac{dy}{dx} = \dfrac{2x+1}{y^2}$ Solve this differential equation.

$$dy = \frac{2x+1}{y^2}\,dx$$

$$\int y^2\,dy = \int (2x+1)\,dx$$

SEPARATION OF VARIABLES

$$\boxed{\frac{y^3}{3} = \frac{2x^2}{2} + x + c}$$

Place $+c$ on any one side of your

choice.

TYPE1: WITHOUT PROOF

7 Given that $y = 0$ when $x = 1$, solve the differential equation

$$xy\frac{dy}{dx} = y^2 + 4,$$

obtaining an expression for y^2 in terms of x. [6]

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DIFF. EQUATION.

$$xy\,dy = (y^2+4)\,dx$$

$$\int \frac{y}{y^2+4}\,dy = \int \frac{1}{x}\,dx$$

INTEGRATION

$$\frac{1}{2}\int \frac{2y}{\boxed{y^2+4}}\,dy = \int \frac{1}{x}\,dx$$

$\square = y^2+4$

$\square' = 2y$

$$\boxed{\frac{1}{2}\ln(y^2+4) = \ln x + c}$$

$y=0$

$x=1$

$$\frac{1}{2}\ln(0^2+4) = \ln 1 + c$$

$$\boxed{\frac{1}{2}\ln 4 = c}$$

This must always stay EXACT. (NO calculator usage)

$$\frac{1}{2}\ln(y^2+4) = \ln x + \frac{1}{2}\ln 4$$

ION.

$$\frac{1}{2}\ln(y^2+4) - \frac{1}{2}\ln 4 = \ln x$$

LOGS SIMPLIFICAT

$$\frac{\ln(y^2+4)-\ln(4)}{2} = \ln x$$

$$\ln(y^2+4)-\ln(4) = 2\ln x$$

$$\cancel{\ln}\left(\frac{y^2+4}{4}\right) = \cancel{\ln} x^2$$

$$\frac{y^2+4}{4} = x^2$$

$$y^2+4 = 4x^2$$

$$\boxed{y^2 = 4x^2-4}$$

8 The variables x and t are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

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$$e^{2t}\frac{dx}{dt} = \cos^2 x$$

$$e^{2t}\,dx = \cos^2 x\,dt$$

$$\int \frac{1}{\cos^2 x}\,dx = \int \frac{1}{e^{2t}}\,dt$$

$$\int \sec^2 \boxed{x}\, dx = \frac{1}{-2}\int -2\, e^{\boxed{-2t}}\, dt$$

$\square = -2t$

$\square' = -2$

$\square = x$

$\square' = 1$

$$\boxed{\tan x = -\frac{1}{2}e^{-2t} + c}$$

$t = 0$

$x = 0$

$$\tan 0 = -\frac{1}{2}e^{-2(0)} + c$$

$$0 = -\frac{1}{2}(1) + c$$

$$\boxed{c = \frac{1}{2}}$$

$$\boxed{\tan x = -\frac{1}{2}e^{-2t} + \frac{1}{2}}$$

PROOF BASED: SIMPLE

3 A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9-h)^{\frac{1}{3}}$. It is given that, when $t = 0$, $h = 1$ and $\frac{dh}{dt} = 0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{dh}{dt} = 0.1(9-h)^{\frac{1}{3}}.$$ [2]

(ii) Solve this differential equation, and obtain an expression for h in terms of t. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

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Rate of increase of height $\propto (9-h)^{\frac{1}{3}}$

$$\frac{dh}{dt} = K(9-h)^{\frac{1}{3}} \qquad h=1,\ \frac{dh}{dt}=0.2$$

$$0.2 = K(9-1)^{\frac{1}{3}}$$

$$K = 0.1$$

$$\boxed{\frac{dh}{dt} = 0.1(9-h)^{\frac{1}{3}}}$$

(ii)

$$dh = 0.1(9-h)^{\frac{1}{3}}dt$$

$$\int \frac{1}{(9-h)^{\frac{1}{3}}}\,dh = \int 0.1\,dt$$

$$-1\int (-1)(\boxed{9-h})^{-\frac{1}{3}}dh = 0.1t + c$$

$\square = 9-h$

$\square' = -1$

$$-1\,\frac{(9-h)^{\frac{2}{3}}}{2/3} = 0.1t + c$$

$$\boxed{-\frac{3}{2}(9-h)^{\frac{2}{3}} = 0.1t + c} \qquad \begin{array}{l} t=0 \\ h=1 \end{array}$$

$$-\frac{3}{2}(9-1)^{\frac{2}{3}} = 0.1(0) + c$$

$$\boxed{-6 = c}$$

$$-\frac{3}{2}(9-h)^{\frac{2}{3}} = 0.1t - 6$$

$$(9-h)^{\frac{2}{3}} = -\frac{2}{3}(0.1t-6)$$

$$(9-h)^{\frac{2}{3}} = \frac{12-0.2t}{3}$$

$$9-h = \left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}}$$

$$h = 9-\left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}}$$

(iii) Max height:

$$\frac{dh}{dt} = 0$$

$$0.1(9-h)^{\frac{1}{3}} = 0$$

$$9-h = 0$$

$$\boxed{h=9}$$

$$9 = 9-\left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}}$$

$$\left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}} = 0$$

$$12-0.2t = 0$$

$$t = 60$$

(iv) Max height $= 9$

Half Max height $= 4.5$

$$h = 9-\left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}}$$

$$4.5 = 9-\left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}}$$

$$\left(\frac{12-0.2t}{3}\right)^{\frac{3}{2}} = 4.5$$

$$\frac{12-0.2t}{3} = (4.5)^{\frac{2}{3}}$$

$$12-0.2t = 8.177$$

$$\frac{12-8.177}{0.2} = t$$

$$t = 19.115$$

5 A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance (x) is proportional to $(20-x)$. When $t=0$, $x=0$ and $\frac{dx}{dt}=1$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20-x).$$ [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when $t=10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

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Rate of formation of substance $(x) \propto 20-x$

$$\boxed{\frac{dx}{dt} = k(20-x)} \qquad \frac{dx}{dt}=1, \ x=0$$

$$1 = k(20-0)$$

$$k = 0.05$$

$$\boxed{\frac{dx}{dt} = 0.05(20-x)}$$

$$dx = 0.05(20-x)\,dt$$

$$-1\int \frac{-1}{\boxed{20-x}}\,dx = \int 0.05\,dt$$

$\square = 20-x$

$\square' = -1$

$$\boxed{-\ln(20-x) = 0.05t + c} \qquad t=0 \quad x=0$$

$$-\ln(20-0) = 0.05(0) + c$$

$$c = -\ln 20$$

$$-\ln(20-x) = 0.05t - \ln 20$$

$$\ln 20 - \ln(20-x) = 0.05t$$

$$\ln\left(\frac{20}{20-x}\right) = 0.05t$$

e

$$\frac{20}{20-x} = e^{0.05t}$$

$$\frac{20}{e^{0.05t}} = 20-x$$

$$\boxed{x = 20 - \frac{20}{e^{0.05t}}}$$

(iii) $t=10$ $\qquad x = 20 - \dfrac{20}{e^{0.05(10)}} = 7.9$

(iv)

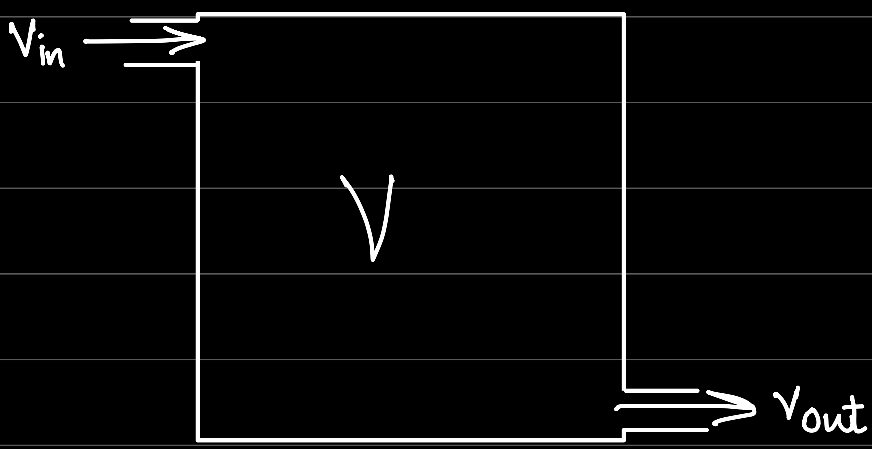
$$x = 20 - \frac{20}{\ldots}$$

$e^{0.05t}$

t becomes large (inifinite)

$x = 20 - 0 = 20$

$$\boxed{\text{PROOF TYPE 2}}$$



$V = V_{in} - V_{out}$

$$\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt}$$

Rate of change of volume = Rate of inflow − Rate of outflow.

2 In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

Rate of formation of substance (x) $\propto x$, $\frac{dx_{out}}{dt} = 25$

$$\frac{dx_{in}}{dt} = kx$$

$$\frac{dx}{dt} = \frac{dx_{in}}{dt} - \frac{dx_{out}}{dt}$$

$$\boxed{\frac{dx}{dt} = Kx - 25}$$

$\frac{dx}{dt} = 75, \quad x = 1000$

$$75 = K(1000) - 25$$

$$100 = 1000K$$

$$K = 0.1$$

$$\frac{dx}{dt} = 0.1x - 25$$

$$\boxed{\frac{dx}{dt} = 0.1(x - 250)}$$

PROOF TYPE 3: CHAIN RULE

LHS OF OUR PROOF WILL NOT BE SAME AS QUESTION.

11 In a model of the expansion of a sphere of radius r cm, it is assumed that, at time t seconds after the start, the rate of increase of the surface area of the sphere is proportional to its volume. When $t = 0$, $r = 5$ and $\frac{dr}{dt} = 2$.

(i) Show that r satisfies the differential equation

$$\frac{dr}{dt} = 0.08r^2. \qquad [4]$$

[The surface area A and volume V of a sphere of radius r are given by the formulae $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$.]

Rate of increase of surface area $\propto$ Volume.

$$\frac{dA}{dt} = K\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dr}{dt} \times \frac{dA}{dr} = K\left[\frac{4\pi r^3}{3}\right]$$

$$\frac{dr}{dt} \times 8\pi r = K\left(\frac{4}{3}\pi r^3\right)$$

LHS of our proof: A

LHS AQ: r

Connecting EQ.

$A = 4\pi r^2 \qquad \frac{dA}{dr} = 8\pi r$

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$

$$\frac{dr}{dt} = k\left(\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}\right) \times \frac{1}{2\,\cancel{8}\cancel{\pi}\cancel{r}}$$

$$\boxed{\frac{dr}{dt} = \frac{kr^2}{6}}$$

$\frac{dr}{dt} = 2, \; r = 5$

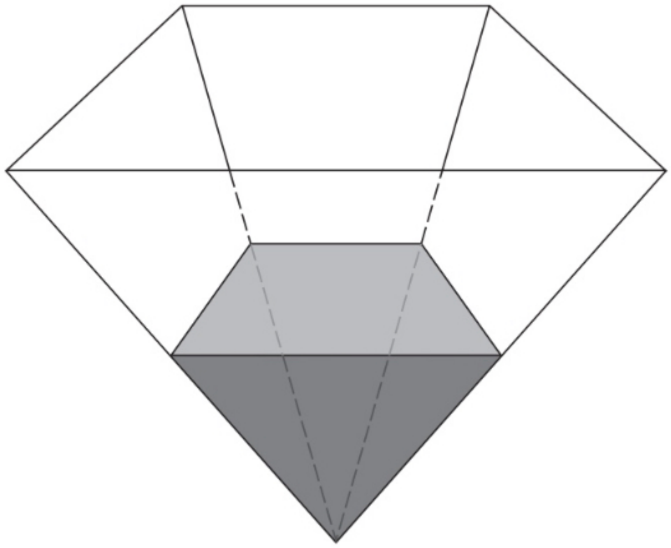
$$2 = \frac{k(5)^2}{6}$$

$$k = \frac{12}{25} = 0.48$$

$$\frac{dr}{dt} = \frac{0.48 r^2}{6}$$

$$\boxed{\frac{dr}{dt} = 0.08 r^2}$$

9



$t=0$

$V=0$
$h=0$

An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is $V\,\text{m}^3$ and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of $20\,\text{m}^3$ per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{5}{h^2} - \frac{1}{20}.$$

[4]

$$\boxed{\frac{dV_{in}}{dt} = 20}$$

$$\frac{dV_{out}}{dt} \propto h^2 \Rightarrow \boxed{\frac{dV_{out}}{dt} = kh^2}$$

$$\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt}$$

$$\frac{dV}{dt} = 20 - kh^2$$

$$\frac{dh}{dt} \times \frac{dV}{dh} = 20 - kh^2$$

$$\frac{dh}{dt} \times 4h^2 = 20 - kh^2$$

Chain Rule

$$V = \frac{4}{3}h^3 \rightarrow \frac{dV}{dh} = 4h^2$$

$$\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$$

$$\frac{dh}{dt} = \frac{20}{4h^2} - \frac{kh^2}{4h^2}$$

$$\boxed{\frac{dh}{dt} = \frac{5}{h^2} - \frac{K}{4}}$$

$h = 1$, $\frac{dh}{dt} = 4.95$

$$4.95 = \frac{5}{1^2} - \frac{K}{4}$$

$$\frac{K}{4} = 0.05$$

$$K = 0.2$$

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{0.2}{4}$$

$$\boxed{\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}}$$