INDICES | P3

LOGS } Base Ch.
TRIG
DIFF
INT
DIFF EQ

45 Marks

IN $\boxed{\text{ADD / SUBTRACT}}$ NO PROPERTIES RELATED TO POWERS APPLY. EVER.

CAUTION

$\boxed{1}$ $a^m \times a^n = a^{m+n}$ $\qquad a^m + a^n = \times$

$a^m \div a^n = a^{m-n}$ $\qquad a^m - a^n = \times$

$\boxed{2}$ $a^m \times b^m = (a \times b)^m$ $\qquad a^m + b^m = \cancel{(a+b)^m}$

$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$ $\qquad a^m - b^m = \cancel{(a-b)^m}$

$a^2 + b^2 = \cancel{(a+b)^2}$

$a^2 - b^2 = \cancel{(a-b)^2}$

You can take power common in MULTIPLY / DIVIDE BUT do not take power common in ADD / SUB.

$\boxed{3}$ $(a \times b)^m = a^m \times b^m$ $\qquad (a+b)^m = \cancel{a^m + b^m}$

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $\qquad (a-b)^m = \cancel{a^m - b^m}$

$(x+4)^2 = \cancel{x^2 + 4^2}$

$(x-6)^2 = \cancel{x^2 - 6^2}$

$\boxed{4}$ $(a^m)^n = a^{m \times n}$

POWER KI POWER MULTIPLY HOGI.

$(a^3)^2 \rightarrow$ weak $= a^9$ $\times$

$\searrow$ good $= a^{3 \times 2} = a^6$

5 . Negative Powers (Reciprocal of base changes +/- sign of power)

$$a^{-m} = \frac{1}{a^m} \quad , \quad \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^{m}$$

6 $\boxed{a^0 = 1}$, $\boxed{1^a = 1}$

7 $\sqrt{x} = x^{\frac{1}{2}}$

$\sqrt[3]{x} = x^{\frac{1}{3}}$

$\sqrt[n]{x} = x^{\frac{1}{n}}$

$x^{\frac{3}{2}} \longrightarrow \sqrt{x^3}$

$(x^3)^{\frac{1}{2}}$

$\sqrt{x^3}$

$x^{\frac{m}{n}} = \sqrt[n]{x^m}$

SURDS (SQUARE ROOTS)

1 SIMPLIFY

$\sqrt{8} \rightarrow$ SQUARE Factors

1 → MISS
2
4
8

$\sqrt{4 \times 2}$

$\sqrt{4}\ \sqrt{2}$

$2\sqrt{2}$

$\sqrt{27} \rightarrow$ Factors

1
3
9
27

$\sqrt{9 \times 3}$

$\sqrt{9} \times \sqrt{3}$

$\boxed{3\sqrt{3}}$

$\sqrt{180} \rightarrow$ factors

1
2
3
4
5
6
9
10
12
15
18
20 ✓
36

$\sqrt{36 \times 5}$

$\sqrt{36} \times \sqrt{5}$

$6\sqrt{5}$

45
60
90

ADD / SUB

1) $3\sqrt{7} + 5\sqrt{7} = 8\sqrt{7}$

$\sqrt{7}(3+5)$

2) $2\sqrt{8} + 5\sqrt{2}$

$\sqrt{4\times 2}$

$2(\sqrt{4}\sqrt{2})$

$2(2\sqrt{2})$

$4\sqrt{2} + 5\sqrt{2}$

$9\sqrt{2}$

MULTIPLY

$(3+\sqrt{2})(\sqrt{3}-5)$

$3\sqrt{3} - 15 + \sqrt{6} - 5\sqrt{2}$

4 **DIVIDE**: MATHS PPL DO NOT LIKE SQUARE ROOTS IN DENOMINATOR.

RATIONALIZING PROCESS OF SHIFTING ROOTS FROM DENOMINATOR → NUMERATOR.

(i) $\dfrac{3}{\sqrt{2}}$

$\dfrac{3}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$\dfrac{3\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$

(ii) $\cos 45 = \dfrac{1}{\sqrt{2}}$

$\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$\dfrac{\sqrt{2}}{\sqrt{4}}$

$$\frac{3\sqrt{2}}{\sqrt{4}} \qquad \frac{\sqrt{2}}{2}$$

$$\frac{3\sqrt{2}}{2}$$

(iii)
$$\frac{3\sqrt{2}+1}{\sqrt{7}-2}$$

Conjugate:
Change middle sign

$$\frac{3\sqrt{2}+1}{\sqrt{7}-2} \times \frac{\sqrt{7}+2}{\sqrt{7}+2}$$

$$\left.\begin{matrix}\sqrt{7}-2\\ \sqrt{7}+2\end{matrix}\right\}$$ BOTH ARE CONJUGATES OF EACH OTHER

$$\frac{(3\sqrt{2}+1)(\sqrt{7}+2)}{(\sqrt{7}-2)(\sqrt{7}+2)}$$

$$\frac{3\sqrt{14}+6\sqrt{2}+\sqrt{7}+2}{(\sqrt{7}+2)(\sqrt{7}-2)}$$

$$(\sqrt{7}-2)(\sqrt{7}-2)$$
$$(\sqrt{7}-2)^2$$
$$a^2-2ab+b^2$$
$$(\sqrt{7})^2-\underbrace{2\sqrt{7}(2)}_{??}+(2)^2$$

$$\frac{3\sqrt{14}+6\sqrt{2}+\sqrt{7}+2}{(\sqrt{7})^2-(2)^2}$$

$$\frac{3\sqrt{14}+6\sqrt{2}+\sqrt{7}+2}{7-4}$$

$$\frac{3\sqrt{14}+6\sqrt{2}+\sqrt{7}+2}{3}$$

LOGS

$2^x = 8$

$2^x = 2^3$

$\boxed{x = 3}$

$2^x = 11$

Now you cannot use indices.

Now we need logs to solve this.

$a^x = b$ Exponential form / Index power.

$x = \log_a b$ LOG FORM

$2^x = 11 \longrightarrow x = \log_2 11$

$5^{2x} = 8 \longrightarrow 2x = \log_5 8$

$\log_{\square} \boxed{}$ → ARGUMENT

1- Cannot be zero

2- Cannot be negative

BASE:

1- Cannot be zero

2- Cannot be negative

3. Cannot be 1

IMPORTANT REVERSE STEP

Diff + Integ

LOG FORM $\longrightarrow$ INDEX FORM.

disappear

$$\log_a b = x \longrightarrow b = a^x$$

Disappear

(1) $\log_2 (x-3) = 4$ (Reverse step)

$x - 3 = 2^4$

$x - 3 = 16$

$x = 19$

$$\boxed{\log_{any} 1 = 0}$$

$$\boxed{\log_a a = 1}$$

1 POWER RULE

$$\log_a \square^n = n \log_a \square$$

eg: $\log_2 x^7 = 7 \log_2 x$

IMPORTANT CAUTION.

$\times$ $\log 3x^2$

$3\log_2 x = \log_2 x^3$

$\times \; (\log_a 3x)^2$

$\checkmark \; \log_a (3x)^2 = 2\log_a (3x)$

$\boxed{2}$ $\log_a x + \log_a y = \log_a (xy)$

$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

For these to apply there must be no cofficient behind either logs.

eg: $2\log_a 3 + 3\log_a 4$

$\log_a 3^2 + \log_a 4^3$

$\log_a 9 + \log_a 64$

$\log_a (9 \times 64)$

$= \log_a 576$

$4\log_a 2 - 3\log_a 3$

$\log_a 2^4 - \log_a 3^3$

$\log_a 16 - \log_a 27$

$\log_a \left(\frac{16}{27}\right)$

Reverse

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

CAUTION !!!

$$\log_a(x+y) = X$$

$$\log_a(x-y) = X$$

$$\log_a x \times \log_a y = X$$

$$\frac{\log_a x}{\log_a y} = X$$

Super imp for solving in DiFF and INTEG.

BASES

COMMON (base = 10)	NATURAL (base = e = 2.718..)
$\log_{10}\square \longrightarrow$ Book/Paper = $\lg\square$ Calculator = $\boxed{\text{Log}}$	$\log_e\square \longrightarrow$ Book, Calculator = $\ln\square$
$\lg 10 = 1$	$\ln e = 1$
$2^x = 11$	$2^x = 11$
Introduce log on both sides	Introduce log on both sides
$\lg 2^x = \lg 11$	$\ln 2^x = \ln 11$

$x \lg 2 = \lg 11$	$x \ln 2 = \ln 11$
$x = \frac{\lg 11}{\lg 2} = 3.459$	$x = \frac{\ln 11}{\ln 2} = 3.459.$

$\boxed{\text{TYPE1}}$

1) one term on each side
2) No logs on either side.

Method : Introduce lg/ln on both sides.

(i) $2^x = 11$

Introduce log on both sides

$\lg 2^x = \lg 11$

$x \lg 2 = \lg 11$

$x = \frac{\lg 11}{\lg 2} = 3.459$

(ii) $e^{2x} = 5$

$\ln e^{2x} = \ln 5$

$2x \ln e = \ln 5$

$2x(1) = \ln 5$

$x = \frac{\ln 5}{2}$

$x = 0.805$

(iii) $3^{2x+1} = 7$

$\ln 3^{2x+1} = \ln 7$

$(2x+1)\ln 3 = \ln 7$

$2x+1 = \frac{\ln 7}{\ln 3}$

$2x = \frac{\ln 7}{\ln 3} - 1$

$x = \frac{1}{2}\left(\frac{\ln 7}{\ln 3} - 1\right)$

$x = 0.386$

$\boxed{\text{TYPE2}}$

1- One term on each side
2- Log on one side.

Method : Reverse step.

$\boxed{\text{most important for DIFF \& INTEG}}$

(i) $\log_2(x+3) = 4$

$x+3 = 2^4$

(ii) $\lg(2x-4) = 2$

$\lg(2x-4) = 2$

(iii) $\ln(3x+1) = 2$

$\ln(3x+1) = 2$

$x+3=16$	$2x-4=10^2$	$3x+1=e^2$
$x=13$	$2x=104$	$3x=e^2-1$
	$x=52$	$x=\dfrac{e^2-1}{3}$

$x=$

$\boxed{\text{TYPE 3}}$

1. More terms on either side
2. Logs on either side.

CASE 1: All terms are with log.	CASE 2: one term without log.
1) Combine logs till you have a single log on both sides 2) cancel the log and solve.	1. Isolate term without log. 2. Combine logs on other side 3. Reverse step & solve.
(i) $\log_3 x+\log_3(x+1)=2\log_3 x-\log_3 5$	(ii) $2\log_{10} x=\log_{10} x+1+2$
$\log_3(x)(x+1)=\log_3 x^2-\log_3 5$	isolate term without log.
$\cancel{\log_3}(x^2+x)=\cancel{\log_3}\left(\frac{x^2}{5}\right)$	$2\log_{10} x-\log_{10}(x+1)=2$
$x^2+x=\dfrac{x^2}{5}$	$\log_{10} x^2-\log_{10}(x+1)=2$
Solve Quadratic Equation.	$\log_{10}\left(\frac{x^2}{x+1}\right)=2$

now use reverse step (Type 2)

$$\frac{x^2}{x+1}=10^2$$

$x^2 = 100(x+1)$

Solve Quadratic.

TYPE 4: MORE TERMS ON EITHER SIDE

NO LOGS ON ON EITHER SIDE

TERMS BEING ADDED / SUBTRACTED.

METHOD : SUBSTITUTION.

Q: $2^x - 2^{-x} = 5$

$2^x = a$

$$2^x - \frac{1}{2^x} = 5$$

$$a - \frac{1}{a} = 5$$

$$\frac{a^2 - 1}{a} = 5$$

$$a^2 - 1 = 5a$$

$$a^2 - 5a - 1 = 0$$

$$a = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-1)}}{2(1)}$$

$$a = \frac{5 \pm \sqrt{29}}{2}$$

$a = 5.193$ $\qquad$ $a = -0.193$

$\lg 2^x = \lg 5.193$

$x \lg 2 = \lg 5.193$

$x = \dfrac{\lg 5.193}{\lg 2}$

$x = 2.377$

$\lg 2^x = \lg -0.193$

NO REAL SOLUTIONS

IN SUBSTITUTION THERE ARE SOME VARIATIONS:

$\boxed{3^x = a}$

(a) $3^{-x} = \dfrac{1}{3^x} = \dfrac{1}{a}$

(b) $3^{2x} = (3^x)^2 = a^2$

$3^{3x} = (3^x)^3 = a^3$

(c) $3^{x+2} = 3^x \cdot 3^2 = a \cdot 9 = 9a$

(d) $3^{x-3} = \dfrac{3^x}{3^3} = \dfrac{a}{27}$

(e) $3^{2x-1} = \dfrac{3^{2x}}{3^1} = \dfrac{(3^x)^2}{3^1} = \dfrac{a^2}{3}$

(f) $3^{3x+2} = 3^{3x} \cdot 3^2 = (3^x)^3 \cdot 3^2 = a^3 \cdot 9 = 9a^3$

PRACTICE:

20 It is given that $\ln(y+1) - \ln y = 1 + 3\ln x$. Express y in terms of x, in a form not involving logarithms. [4]

9709/33/M/J/13

$$\ln(y+1) - \ln y - \ln x^3 = 1$$

$$\ln a + \ln b - \ln c - \ln d$$

$$\ln\left(\frac{ab}{cd}\right)$$

$$\ln\left(\frac{y+1}{yx^3}\right) = 1$$

e

$$\frac{y+1}{yx^3} = e^1$$

$$y+1 = eyx^3$$

$$1 = eyx^3 - y$$

$$1 = y(ex^3 - 1)$$

$$\boxed{y = \frac{1}{ex^3 - 1}}$$

10 Solve the equation

$$\ln(1 + x^2) = 1 + 2\ln x,$$

-0.763 (reject)

giving your answer correct to 3 significant figures. [4]

9709/31/O/N/10

$$\ln(1+x^2) - \ln x^2 = 1$$

$$\ln\left(\frac{1+x^2}{x^2}\right) = 1$$

e

$$\frac{1+x^2}{x^2} = e$$

$$1 + x^2 = ex^2$$

$$1 = ex^2 - x^2$$

$$1 = (e-1)x^2$$

$$x^2 = \frac{1}{e-1}$$

$$x = \pm\sqrt{\frac{1}{e-1}}$$

(1 mark)

$x = 0.763$ or $x = -0.763$ (Reject)

Always check these values in main equation of Question.

17 Solve the equation

$$5^{x-1} = 5^x - 5,$$

giving your answer correct to 3 significant figures. [4]

9709/31/O/N/12

SUBSTITUTION $5^x = a$

$$5^{x-1} = 5^x - 5$$

$$\frac{5^x}{5^1} = 5^x - 5$$

$$\frac{a}{5} = a - 5$$

$$a = 5a - 25$$

$$25 = 4a$$

$$a = 6.25$$

$$\lg 5^x = \lg 6.25$$

$$x \lg 5 = \lg 6.25$$

$$x = \frac{\lg 6.25}{\lg 5} = 1.139. = 1.14 \text{ (3sf)}.$$

MODULUS

$|3| = 3$ $\qquad$ $|-3| = 3$

$|2x-3| = |x+5|$

SQUARE BOTH SIDES (WHOLE EQUATION) AND MODULUS SIGN DISAPPEARS. YOU WILL STILL HAVE TO SQUARE TERMS.

(Do same step even if you have modulus on one side or both).

$(|2x-3|)^2 = (|x+5|)^2$

$(2x-3)^2 = (x+5)^2$

$4x^2 - 12x + 9 = x^2 + 10x + 25$

$3x^2 - 22x - 16 = 0$

Solve the quadratic.

30 (i) Solve the equation $2|x-1| = 3|x|$. [3]

(ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

9709/31/M/J/16

(i) $(2|x-1|)^2 = (3|x|)^2$

$2^2(x-1)^2 = 3^2(x)^2$

$4(x^2-2x+1) = 9x^2$

$4x^2 - 8x + 4 = 9x^2$

$0 = 5x^2 + 8x - 4$

$0 = 5x^2 + 10x - 2x - 4$

(ii) We can see a substitution of x with 5^x, So from last part

$x = 0.4$ $\qquad$ $x = -2$

$\lg 5^x = \lg 0.4$ $\qquad$ $\lg 5^x = \lg -2$

$x \lg 5 = \lg 0.4$ $\qquad$ NO SOLUTIONS

$0 = 5x(x+2) - 2(x+2)$

$x = 0.4 \quad , x = -2$

$x = \frac{\lg 0.4}{\lg 5} = -0.569.$