ALEVELS P3

COMPLEX NUMBERS WITH DIAGRAM
(HARD)

1 The complex number u is given by

$$u = \frac{3+\mathrm{i}}{2-\mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

(ii) Find the modulus and argument of u. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

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2 The variable complex number z is given by

$$z = 2\cos\theta + \mathrm{i}(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leqslant \pi$.

(i) Show that $|z - \mathrm{i}| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. [3]

(ii) Prove that the real part of $\dfrac{1}{z + 2 - \mathrm{i}}$ is constant for $-\pi < \theta < \pi$. [4]

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3 The complex number w is given by $w = -\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w. [2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\dfrac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\dfrac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2\mathrm{i}$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

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4 The complex numbers $-2 + \mathrm{i}$ and $3 + \mathrm{i}$ are denoted by u and v respectively.

(i) Find, in the form $x + \mathrm{i}y$, the complex numbers

(a) $u + v$, [1]

(b) $\dfrac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\dfrac{u}{v}$. [1]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, v and $u + v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments OA and BC. [2]

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5 The complex number $2 + 2\mathrm{i}$ is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leqslant |z - \mathrm{i}|$ and $|z - u| \leqslant 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which $\arg z$ is least. [3]

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6 The complex number u is defined by $u = \dfrac{6 - 3\mathrm{i}}{1 + 2\mathrm{i}}$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-\frac{1}{2}\pi$. [4]

(ii) For complex numbers z satisfying $\arg(z - u) = \frac{1}{4}\pi$, find the least possible value of $|z|$. [3]

(iii) For complex numbers z satisfying $|z - (1 + \mathrm{i})u| = 1$, find the greatest possible value of $|z|$. [3]

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7 **(a)** Showing your working, find the two square roots of the complex number $1 - (2\sqrt{6})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are exact. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z - 3\mathrm{i}| \leqslant 2$. Find the greatest value of $\arg z$ for points in this region. [5]

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8 The complex number w is defined by $w = -1 + \mathrm{i}$.

(i) Find the modulus and argument of w^2 and w^3, showing your working. [4]

(ii) The points in an Argand diagram representing w and w^2 are the ends of a diameter of a circle. Find the equation of the circle, giving your answer in the form $|z - (a + b\mathrm{i})| = k$. [4]

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9 **(a)** The complex numbers u and w satisfy the equations

$$u - w = 4\mathrm{i} \quad \text{and} \quad uw = 5.$$

Solve the equations for u and w, giving all answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 + 2\mathrm{i}| \leqslant 2$, $\arg z \leqslant -\frac{1}{4}\pi$ and $\operatorname{Re} z \geqslant 1$, where $\operatorname{Re} z$ denotes the real part of z. [5]

(ii) Calculate the greatest possible value of $\operatorname{Re} z$ for points lying in the shaded region. [1]

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10 **(a)** Without using a calculator, solve the equation $\mathrm{i}w^2 = (2 - 2\mathrm{i})^2$. [3]

(b) **(i)** Sketch an Argand diagram showing the region R consisting of points representing the complex numbers z where

$$|z - 4 - 4\mathrm{i}| \leqslant 2.$$ [2]

(ii) For the complex numbers represented by points in the region R, it is given that

$$p \leqslant |z| \leqslant q \quad \text{and} \quad \alpha \leqslant \arg z \leqslant \beta.$$

Find the values of p, q, α and β, giving your answers correct to 3 significant figures. [6]

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11 **(a)** Without using a calculator, solve the equation

$$3w + 2\mathrm{i}w^* = 17 + 8\mathrm{i},$$

where w^* denotes the complex conjugate of w. Give your answer in the form $a + b\mathrm{i}$. [4]

(b) In an Argand diagram, the loci

$$\arg(z - 2\mathrm{i}) = \tfrac{1}{6}\pi \quad \text{and} \quad |z - 3| = |z - 3\mathrm{i}|$$

intersect at the point P. Express the complex number represented by P in the form $r\mathrm{e}^{\mathrm{i}\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. [5]

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12 **(a)** The complex number $\boldsymbol{w}$ is such that $\operatorname{Re}\boldsymbol{w} > 0$ and $\boldsymbol{w} + 3\boldsymbol{w}^* = \mathrm{i}\boldsymbol{w}^2$, where $\boldsymbol{w}^*$ denotes the complex conjugate of $\boldsymbol{w}$. Find $\boldsymbol{w}$, giving your answer in the form $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2\mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

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13 **Throughout this question the use of a calculator is not permitted.**

(a) The complex numbers u and v satisfy the equations

$$u + 2v = 2\mathrm{i} \quad \text{and} \quad \mathrm{i}u + v = 3.$$

Solve the equations for u and v, giving both answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + \mathrm{i}| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$. Find the least value of $|z - w|$ for points on these loci. [5]

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14 **(a)** Without using a calculator, use the formula for the solution of a quadratic equation to solve

$$(2 - \mathrm{i})z^2 + 2z + 2 + \mathrm{i} = 0.$$

Give your answers in the form $a + b\mathrm{i}$. [5]

(b) The complex number w is defined by $w = 2\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$. In an Argand diagram, the points A, B and C represent the complex numbers w, w^3 and w^* respectively (where w^* denotes the complex conjugate of w). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. [5]

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15 **(a)** The complex number $\dfrac{3 - 5\mathrm{i}}{1 + 4\mathrm{i}}$ is denoted by $\boldsymbol{u}$. Showing your working, express $\boldsymbol{u}$ in the form $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are real. [3]

(b) **(i)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 2 - \mathrm{i}| \leqslant 1$ and $|z - \mathrm{i}| \leqslant |z - 2|$. [4]

(ii) Calculate the maximum value of $\arg z$ for points lying in the shaded region. [2]

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16 The complex number $\boldsymbol{w}$ is defined by $\boldsymbol{w} = \dfrac{22 + 4\mathrm{i}}{(2 - \mathrm{i})^2}$.

(i) Without using a calculator, show that $\boldsymbol{w} = 2 + 4\mathrm{i}$. [3]

(ii) It is given that $\boldsymbol{p}$ is a real number such that $\frac{1}{4}\pi \leqslant \arg(\boldsymbol{w} + \boldsymbol{p}) \leqslant \frac{3}{4}\pi$. Find the set of possible values of $\boldsymbol{p}$. [3]

(iii) The complex conjugate of $\boldsymbol{w}$ is denoted by $\boldsymbol{w}^*$. The complex numbers $\boldsymbol{w}$ and $\boldsymbol{w}^*$ are represented in an Argand diagram by the points $\boldsymbol{S}$ and $\boldsymbol{T}$ respectively. Find, in the form $|z - \boldsymbol{a}| = \boldsymbol{k}$, the equation of the circle passing through $\boldsymbol{S}$, $\boldsymbol{T}$ and the origin. [3]

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17 The complex number u is given by $u = -1 + (4\sqrt{3})\mathrm{i}$.

(i) Without using a calculator and showing all your working, find the two square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the real numbers a and b are exact. [5]

(ii) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying the relation $|z - u| = 1$. Determine the greatest value of $\arg z$ for points on this locus. [4]

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18 The complex number $1 - \mathrm{i}$ is denoted by $\boldsymbol{u}$.

(i) Showing your working and without using a calculator, express

$$\frac{\mathrm{i}}{\boldsymbol{u}}$$

in the form $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are real. [2]

(ii) On an Argand diagram, sketch the loci representing complex numbers z satisfying the equations $|z - \boldsymbol{u}| = |z|$ and $|z - \mathrm{i}| = 2$. [4]

(iii) Find the argument of each of the complex numbers represented by the points of intersection of the two loci in part **(ii)**. [3]

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19 The complex number $3 - \mathrm{i}$ is denoted by $\boldsymbol{u}$. Its complex conjugate is denoted by $\boldsymbol{u}^*$.

(i) On an Argand diagram with origin $\boldsymbol{O}$, show the points $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ representing the complex numbers $\boldsymbol{u}$, $\boldsymbol{u}^*$ and $\boldsymbol{u}^* - \boldsymbol{u}$ respectively. What type of quadrilateral is $\boldsymbol{OABC}$? [4]

(ii) Showing your working and without using a calculator, express $\dfrac{\boldsymbol{u}^*}{\boldsymbol{u}}$ in the form $\boldsymbol{x} + \mathrm{i}\boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are real. [3]

(iii) By considering the argument of $\dfrac{\boldsymbol{u}^*}{\boldsymbol{u}}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right)$$

3]

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20 **(a)** It is given that $(1 + 3\mathrm{i})w = 2 + 4\mathrm{i}$. Showing all necessary working, prove that the exact value of $|w^2|$ is 2 and find $\arg(w^2)$ correct to 3 significant figures. [6]

(b) On a single Argand diagram sketch the loci $|z| = 5$ and $|z - 5| = |z|$. Hence determine the complex numbers represented by points common to both loci, giving each answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$. [4]

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21 **(a)** Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) **(i)** On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2\mathrm{i}| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

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22 **(a)** Showing all necessary working, solve the equation $\mathrm{i}z^2 + 2z - 3\mathrm{i} = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) **(i)** On a sketch of an Argand diagram, show the locus representing complex numbers satisfying the equation $|z| = |z - 4 - 3\mathrm{i}|$. [2]

(ii) Find the complex number represented by the point on the locus where $|z|$ is least. Find the modulus and argument of this complex number, giving the argument correct to 2 decimal places. [3]

9709/32/M/J/16/Q10

23 **Throughout this question the use of a calculator is not permitted.**

The complex numbers $-1 + 3\mathrm{i}$ and $2 - \mathrm{i}$ are denoted by u and v respectively. In an Argand diagram with origin O, the points A, B and C represent the numbers u, v and $u + v$ respectively.

(i) Sketch this diagram and state fully the geometrical relationship between OB and AC. [4]

(ii) Find, in the form $x + \mathrm{i}y$, where x and y are real, the complex number $\dfrac{u}{v}$. [3]

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

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24 **Throughout this question the use of a calculator is not permitted.**

The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})\mathrm{i}$. The complex conjugate of z is denoted by z^*.

(i) Find the modulus and argument of z. [2]

(ii) Express each of the following in the form $x + \mathrm{i}y$, where x and y are real and exact:

(a) $z + 2z^*$;

(b) $\dfrac{z^*}{\mathrm{i}z}$.

[4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and $\mathrm{i}z$ respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

9709/33/O/N/16/Q7

25 **Throughout this question the use of a calculator is not permitted.**

The complex numbers u and w are defined by $u = -1 + 7\mathrm{i}$ and $w = 3 + 4\mathrm{i}$.

(i) Showing all your working, find in the form $x + \mathrm{i}y$, where x and y are real, the complex numbers $u - 2w$ and $\dfrac{u}{w}$. [4]

In an Argand diagram with origin O, the points A, B and C represent the complex numbers u, w and $u - 2w$ respectively.

(ii) Prove that angle $AOB = \frac{1}{4}\pi$. [2]

(iii) State fully the geometrical relation between the line segments OB and CA. [2]

9709/31/M/J/17/Q7

26 **Throughout this question the use of a calculator is not permitted.**

The complex number $2 - \mathrm{i}$ is denoted by u.

(i) It is given that u is a root of the equation $x^3 + ax^2 - 3x + b = 0$, where the constants a and b are real. Find the values of a and b. [4]

(ii) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - u| < 1$ and $|z| < |z + \mathrm{i}|$. [4]

9709/32/M/J/17/Q6

27 **Throughout this question the use of a calculator is not permitted.**

(a) The complex numbers z and w satisfy the equations

$$z + (1 + \mathrm{i})w = \mathrm{i} \qquad \text{and} \qquad (1 - \mathrm{i})z + \mathrm{i}w = 1.$$

Solve the equations for z and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. [6]

(b) The complex numbers u and v are given by $u = 1 + (2\sqrt{3})\mathrm{i}$ and $v = 3 + 2\mathrm{i}$. In an Argand diagram, u and v are represented by the points A and B. A third point C lies in the first quadrant and is such that $BC = 2AB$ and angle $ABC = 90°$. Find the complex number z represented by C, giving your answer in the form $x + \mathrm{i}y$, where x and y are real and exact. [4]

9709/33/M/J/17/Q11

28 **(a)** Find the complex number z satisfying the equation

$$3z - \mathrm{i}z^* = 1 + 5\mathrm{i},$$

where z^* denotes the complex conjugate of z. [4]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z| \leqslant 3$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of z. Calculate the greatest value of $\arg z$ for points in this region. Give your answer in radians correct to 2 decimal places. [5]

9709/33/M/J/18//Q9

29 **(a)** Showing all necessary working, express the complex number $\dfrac{2+3\mathrm{i}}{1-2\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give the values of r and θ correct to 3 significant figures. [5]

(b) On an Argand diagram sketch the locus of points representing complex numbers z satisfying the equation $|z - 3 + 2\mathrm{i}| = 1$. Find the least value of $|z|$ for points on this locus, giving your answer in an exact form. [4]

9709/31/O/N/18/Q8

30 **(a)** **(i)** Without using a calculator, express the complex number $\dfrac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in the form $x + \mathrm{i}y$, where x and y are real. [2]

(ii) Hence, without using a calculator, express $\dfrac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ. [3]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying both the inequalities $|z - 3\mathrm{i}| \leqslant 1$ and $\operatorname{Re} z \leqslant 0$, where $\operatorname{Re} z$ denotes the real part of z. Find the greatest value of $\arg z$ for points in this region, giving your answer in radians correct to 2 decimal places. [5]

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31 **(a)** The complex number u is given by $u = -3 - (2\sqrt{10})\mathrm{i}$. Showing all necessary working and without using a calculator, find the square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5]

(b) On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$, $\arg z \geqslant \frac{1}{4}\pi$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of the complex number z. [5]

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32 **(a)** Find the complex number z satisfying the equation

$$z + \frac{\mathrm{i}z}{z^*} - 2 = 0,$$

where z^* denotes the complex conjugate of z. Give your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) **(i)** On a single Argand diagram sketch the loci given by the equations $|z - 2\mathrm{i}| = 2$ and $\operatorname{Im} z = 3$, where $\operatorname{Im} z$ denotes the imaginary part of z. [2]

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33 **Throughout this question the use of a calculator is not permitted.**

The complex number with modulus 1 and argument $\frac{1}{3}\pi$ is denoted by w.

(i) Express w in the form $x + \mathrm{i}y$, where x and y are real and exact. [1]

The complex number $1 + 2\mathrm{i}$ is denoted by u. The complex number v is such that $|v| = 2|u|$ and $\arg v = \arg u + \frac{1}{3}\pi$.

(ii) Sketch an Argand diagram showing the points representing u and v. [2]

(iii) Explain why v can be expressed as $2uw$. Hence find v, giving your answer in the form $a + \mathrm{i}b$, where a and b are real and exact. [4]

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