8.4 Trignometric Substitutions

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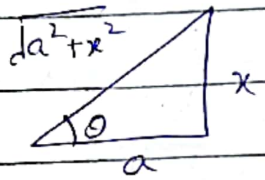
→ Trignometric Substitutions occur when we replace the variable of integration by a trignometric function.

→ The most common substitutions are:

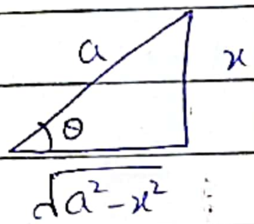
1. $x = a\tan\theta$

$\sqrt{a^2+x^2} = a|\sec\theta|$.



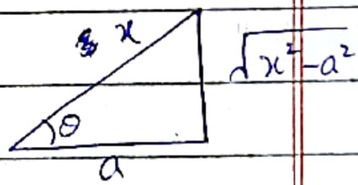
2. $x = a\sin\theta$

$\sqrt{a^2-x^2} = a|\cos\theta|$



3. $x = a\sec\theta$

$\sqrt{x^2-a^2} = a|\tan\theta|$



→ These substitutions are effective in transforming integrals involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ into integrals we can evaluate directly

Trignometric function

$$1+\tan^2\theta = \sec^2\theta$$

$$1-\sin^2\theta = \cos^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

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* Read details from book (Pg 467)

$$x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right) \text{ with } \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1 \\ \frac{\pi}{2} < \theta \leq \pi & \text{if } \frac{x}{a} \leq -1 \end{cases}$$

Procedure for a Trignometric Substitution

(1) Write down the substitution for x, calculate dx and specify the selected values of θ for substitution.

(2) Substitute the trignometric-expression and the calculated differential into the integrand and then simplify the results algebraically.

(3) Integrate the trignometric integral, keeping in mind the restrictions on the angle θ for reversibility.

(4) Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

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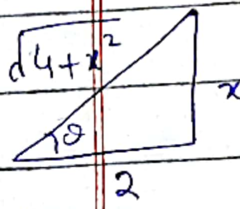
Example 1: $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{2^2+x^2}}$

Sol: Let $x = a\tan\theta$

$x = 2\tan\theta$



$dx = 2\sec^2\theta \, d\theta \quad , -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

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- $x = 2\tan\theta$
  $\tan\theta = \frac{x}{2}$
- $4+x^2 = 4\sec^2\theta$
  $\sec^2\theta = \frac{4+x^2}{4}$
  $\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$4+x^2 = 4 + 4\tan^2\theta$

$= 4(1+\tan^2\theta) = 4\sec^2\theta.$

$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{2^2+x^2}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4\sec^2\theta}}$

$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{2}\cancel{\sec}\theta}$

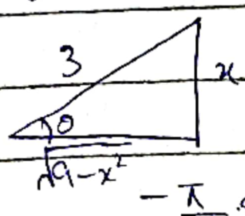
$= \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C.$

$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$

Example 3: Evaluate $\int \frac{x^2 \, dx}{\sqrt{9-x^2}} = \int \frac{x^2 \, dx}{\sqrt{3^2-x^2}}$

Sol: Let $x = a\sin\theta$



$\theta = \sin^{-1}\left(\frac{x}{3}\right) \Leftarrow x = 3\sin\theta.$

$\frac{x}{3} = \sin\theta \Leftarrow dx = 3\cos\theta \, d\theta. \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

3 ~~9+x~~ $\quad 9-x^2 = 9-(3\sin\theta)^2$

$= 9 - 9\sin^2\theta$

$9-x^2 = 9\cos^2\theta$ $\quad = 9(1-\sin^2\theta) = 9\cos^2\theta$

$\cos^2\theta = \frac{9-x^2}{9} \Leftarrow$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

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$$\int \frac{x^2}{\sqrt{9-x^2}}dx = \int \frac{9\sin^2\theta \cdot 3\cos\theta d\theta}{\sqrt{9\cos^2\theta}}$$

$$= 9\int \frac{\cancel{3}\sin^2\theta \cancel{\cos\theta} d\theta}{\cancel{3}\cancel{\cos\theta}}$$

$$= 9\int \sin^2\theta \, d\theta \qquad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= 9\int \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{9}{2}\left(\theta - \frac{\cancel{2}\cos\theta \sin\theta}{\cancel{2}}\right) + C \qquad \left(\because \sin 2\theta = 2\cos\theta\sin\theta\right)$$

$$= \frac{9}{2}\left(\theta - \sin\theta\cos\theta\right) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

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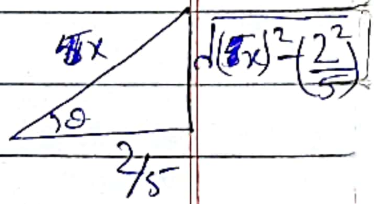
* Example 4: $\int \frac{dx}{\sqrt{25x^2-4}}$ $\quad x > \frac{2}{5}$

Sol. $\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$



$$= \sqrt{25\left(x^2 - \frac{2^2}{5^2}\right)}$$

$$= \sqrt{25}\sqrt{x^2-\left(\frac{2}{5}\right)^2} = 5\sqrt{x^2-\left(\frac{2}{5}\right)^2}$$

$x = a\sec\theta \Rightarrow x = \frac{2}{5}\sec\theta$ $\quad 0<\theta<\frac{\pi}{2}$

$$dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$$

$$x^2-\left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\sec\theta\right)^2-\left(\frac{2}{5}\right)^2$$

$$= \frac{4}{25}\sec^2\theta - \cancel{3}\frac{4}{25} = \frac{4}{25}\left(\sec^2\theta - 1\right)$$

$$= \frac{4}{25}\tan^2\theta$$

$$\int\frac{dx}{\sqrt{25x^2-4}} = \int\frac{dx}{5\sqrt{x^2-\left(\frac{2}{5}\right)^2}} = \int\frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{5\cdot\sqrt{\frac{4\tan^2\theta}{25}}}$$

$$= \int\frac{\cancel{2}/5\ \sec\theta\ \cancel{\tan\theta}\,d\theta}{\cancel{5}\cdot\frac{\cancel{2}}{\cancel{5}}\cancel{\tan\theta}}$$

$$= \frac{1}{5}\int\sec\theta\,d\theta = \frac{1}{5}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right| + C$$