Dated: Monday 27th Nov,'23

Lecture#28

7.5 Indeterminate Forms

The indeterminate forms are forms whose value cannot be determined.

e.g. $0/0$, ∞/∞, $\infty-\infty$, 1^∞, ∞^0

→ $0/0$ form:

$0\times1=0 \Rightarrow 0/0=1$

$0\times2=0 \Rightarrow 0/0=2$

$0\times3=0 \Rightarrow 0/0=3$

and so on.

$\Rightarrow 0/0=1=2=3 \ldots$ (A contradiction!)

$\Rightarrow$ The value of $0/0$ cannot be determined.

→ L'Hospital's Rule

Suppose that $f(a)=g(a)=0$, that 'f' and 'g' are differentiable on an open interval I containing 'a', and that $g'(x)\neq0$ on I if $x\neq a$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Dated:

Example#1

(a) $\lim_{x\to 0} \frac{3x-\sin x}{x}$ $(0/0)$

Sol. L'Hospital's Rule:

$= \lim_{x\to 0} \frac{3-\cos x}{1}$

$= \frac{3-\cos x}{1}\Big|_{x=0}$

$= 3-\cos 0$

$= 3-1$

$= 2$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ $(0/0)$

Sol. $= \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1}$

$= \frac{1}{2}$

Dated:

(c) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $(0/0)$

Sol. $\lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$ $(0/0)$ (L'Hospital's Rule again)

$= \lim_{x\to 0} \frac{(-1/4)(1+x)^{-3/2}}{2}$ (Not 0/0)

$= -\frac{1}{8}$.

(d) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$ $(0/0)$

Sol. $= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$ $(0/0)$ (L'Hospital's Rule again)

$= \lim_{x\to 0} \frac{\sin x}{6x}$ $(0/0)$ (L'Hospital's Rule again)

$= \lim_{x\to 0} \frac{\cos x}{6}$ (Not 0/0)

$= 1/6$.

Dated:

→ Example #2

$$\lim_{x \to 0} \frac{1-\cos x}{x+x^2} \quad (0/0 \text{ form})$$

Sol. $$= \lim_{x \to 0} \frac{\sin x}{1+2x} \quad (\text{Not } 0/0)$$

$$= 0/1 = 0$$

→ Example #3

(a) $$\lim_{x \to 0^+} \frac{\sin x}{x^2} \quad (0/0)$$

Sol. $$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty \quad (+\text{ve for } x > 0)$$

(b) $$\lim_{x \to 0^-} \frac{\sin x}{x^2}$$

Sol. $$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty \quad (-\text{ve for } x < 0)$$

Dated:

→ EXAMPLE #4

(a) $\lim_{x\to \pi/2} \frac{\sec x}{1+\tan x}$ (∞/∞ form)

* $\lim_{x\to\pi/2}$ can be replaced by the one-sided limits

$x \to \pi/2^{+}$ which gives ($-\infty/-\infty$ form) or
$x \to \pi/2^{-}$ which gives (∞/∞ form)

(LHL = RHL).

→ L'Hospital's Rule.

$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x}$

$= \lim_{x\to(\pi/2)^-} \sin x$

$= 1$

(b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$

Sol. $= \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}}$

$= \lim_{x\to\infty} \frac{\sqrt{x}}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

Dated:

(C) $\lim_{x\to\infty} \frac{e^x}{x^2}$

Sol: $= \lim_{x\to\infty} \frac{e^x}{2x}$

$= \lim_{x\to\infty} \frac{e^x}{2}$

$= \infty$

→ Example # 5

(a) $\lim_{x\to\infty} (x \sin 1/x)$ ($\infty \cdot 0$ form)

Sol. ~~Let~~, $h = 1/x$

$= \lim_{h\to 0^+} \left(\frac{1}{h} \sin h\right)$

$= \lim_{h\to 0^+} \frac{\sin h}{h}$

$= 1$

Dated:

(b) $\lim_{x \to 0^+} \sqrt{x} \ln x$ ($\infty . 0$ form)

Sol. $\lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}}$ (∞/∞ form)

(L' Hospital's Rule)

$= \lim_{x \to 0^+} \frac{1/x}{-1/2\, x^{3/2}}$

$= \lim_{x \to 0^+} (-2\sqrt{x})$

$= 0$

→ Example #6

find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

Sol. * If $x \to 0^+$, then $\sin x \to 0^+$ and,

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty$$

* If $x \to 0^-$, then $\sin x \to 0^-$ and,

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty)$$

$$= -\infty + \infty$$

Neither form reveals what happens in the limit

Dated:

To find out, we combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

L'Hospital's Rule

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{x-\sin x}{x \sin x}\right) \quad (0/0 \text{ form})$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \quad (0/0 \text{ form})$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x}$$

$$= \frac{0}{2} = 0.$$

→ Intermediate Powers–

Limits that lead to the indeterminate forms $1^\infty, 0^0, \infty^0$ can sometimes be handled by first taking the logarithm of the function.

* If $\lim_{x\to a} \ln f(x) = L$, then:

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Dated:

→ Example #7

Show that $\lim_{x\to 0^+}(1+x)^{1/x} = e$ (1^∞ form)

Solution

Let, $f(x) = (1+x)^{1/x}$

$\ln f(x) = \ln(1+x)^{1/x}$

$= \frac{1}{x}\ln(1+x)$

L'Hospital's Rule

$= \lim_{x\to 0^+}\ln f(x) = \lim_{x\to 0^+}\frac{\ln(1+x)}{x}$ ($0/0$ form)

$= \lim_{x\to 0^+}\frac{1/1+x}{1}$

$= \frac{1}{1} = 1$

$\Rightarrow \lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e$

→ Example #8

find $\lim_{x\to\infty} x^{1/x}$ (∞^0 form)

Solution

Let, $f(x) = x^{1/x}$

$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x}$

Dated:

L' Hospital's Rule,

$$= \lim_{n\to\infty} \ln f(n) = \lim_{n\to\infty} \frac{\ln n}{n} \quad (\infty/\infty \text{ form})$$

$$= \lim_{n\to\infty} \frac{1/n}{1}$$

$$= \frac{0}{1}$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} f(n) = \lim_{n\to\infty} e^{\ln f(n)} = e^0 = 1$$

Dated:

8.1 Using Basic Integration Formulas

→ Memorize Table 8.1 (Book Page # 448) (Integration Formulas)

→ Example #1

Evaluate the integral $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx$.

Solution

Let, $u = x^2-3x+1$

$\Rightarrow du = (2x-3)dx$

* When $x=3$, $u = (3)^2-3(3)+1 = 1$
* When $x=5$, $u = (5)^2-3(5)+1 = 11$

$$\Rightarrow \int_1^{11} \frac{du}{\sqrt{u}}$$

$$= \int_1^{11} u^{-1/2}\,du$$

$$= 2\sqrt{u}\,\Big]_1^{11}$$

$$= 2(\sqrt{11}-1) \approx 4.63$$

CamScanner

Dated:

→ Example #2
Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x-x^2}}$$

Solution

* $8x-x^2 = -x^2+8x$
$= -(x^2-8x)$
$= -(x^2-8x+4^2-4^2)$
$= -(x-4)^2+16$
$= 16-(x-4)^2.$

$$\Rightarrow \int \frac{dx}{\sqrt{16-(x-4)^2}}$$

Let, $u = x-4$
$\Rightarrow du = dx$

$a = 4.$

$$\Rightarrow \int \frac{du}{\sqrt{a^2-u^2}}$$

$= \sin^{-1}\left(\frac{u}{a}\right)+c$

$= \sin^{-1}\left(\frac{x-4}{4}\right)+c.$

Dated:

→ Example #3

Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x)\,dx$$

Solution

* $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(x+2x) = \sin x \cos 2x + \cos x \sin 2x$

$\Rightarrow \sin(3x) = \cos x \sin 2x + \sin x \cos 2x$

$\longrightarrow \int \sin 3x\,dx$

Let, $u = 3x$

$\Rightarrow du = 3dx$

$\Rightarrow \frac{du}{3} = dx$

$= \int \sin u \times \frac{du}{3}$

$= \frac{1}{3}\int \sin u\,du$

$= \frac{1}{3}(-\cos u) + c$

$= \frac{-\cos 3x}{3} + c$

Dated:

→ EXAMPLE # 4

Find $\int_0^{\pi/4} \frac{dx}{1-\sin x}$

SOLUTION

$$= \int_0^{\pi/4} \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\, dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec^2 x\,(1+\sin x)\, dx$$

$$= \int_0^{\pi/4} \left(\sec^2 x + \sec x \times \frac{\sin x}{\cos x}\right) dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx$$

* $\int \sec^2 x\, dx = \tan x + c$

* $\int \sec x \tan x\, dx = \sec x + c$

$$= \left[\tan x + \sec x\right]_0^{\pi/4}$$

$$= (1+\sqrt{2} - (0+1)) = \sqrt{2}$$

Dated:

→ Example #5

Evaluate $\int \frac{3x^2-7x}{3x+2}\,dx$

Solution

$$
\begin{array}{r}
x-3 \\
3x+2 \,\big)\, \overline{3x^2-7x} \\
\underline{\cancel{3x^2} + 2x} \\
-\cancel{9x} \\
\underline{-\cancel{9x} - 6} \\
6
\end{array}
$$

$$\rightarrow \int \frac{3x^2-7x}{3x+2}\,dx = \int \left((x-3) + \frac{6}{3x-2}\right) dx$$

$$= \frac{x^2}{2} - 3x + 2\int \frac{3\cdot dx}{3x-2} + c$$

$$= \frac{x^2}{2} - 3x + 2\ln|3x-2| + c$$

Dated:

→ Example #6

Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

Solution

* Since; the derivative of the denominator is not in the numerator, we write it separately.

$$= \int \left(\frac{3x}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}} \right) dx$$

$$= 3\int \frac{x}{\sqrt{1-x^2}}\,dx + 2\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$I = I_1 + I_2$$

* $I_1 = -\frac{3}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$$= -\frac{3}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$$

$$= -\frac{3}{2}\left[\frac{(1-x^2)^{1/2}}{1/2}\right] + c$$

* $\int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$

$$= -3\sqrt{1-x^2} + c$$

Dated:

$$\star \quad I_2 = 2\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\star \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$a=1$$

$$\Rightarrow I_2 = 2\sin^{-1}x + c.$$

$$I = I_1 + I_2$$
$$= -3\sqrt{1-x^2} + 2\sin^{-1}x + c.$$

→ Example #7

Evaluate $\int \frac{dx}{(1+\sqrt{x})^3}$

Solution

$\star$ Let $u = 1+\sqrt{x} \Rightarrow u-1 = \sqrt{x}$

$$\Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow dx = 2\sqrt{x}\,du$$

$$\Rightarrow dx = 2(u-1)\,du$$
$$= (2u-2)\,du$$

$$= \int \frac{(2u-2)\,du}{u^3}$$

$$= \int \left(\frac{2u}{u^3} - \frac{2}{u^3}\right)du$$

$$= \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right)du$$

CamScanner

Dated:

$$= \int \left[2u^{-2} - 2u^{-3}\right] du$$

$$= 2\int u^{-2} du - 2\int u^{-3} du$$

$$= \frac{2u^{-1}}{-1} - \cancel{2}\frac{u^{-2}}{-\cancel{2}} + c$$

$$= \frac{-2}{u} + \frac{1}{u^2} + c$$

$$= \frac{1-2u}{u^2} + c$$

$$= \frac{1-2(1+\sqrt{x})}{(1+\sqrt{x})^2} + c$$

$$= C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}$$

→ EXAMPLE #8

Evaluate $\int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx$

SOLUTION

* $x^3 \to$ odd function
* $\cos x \to$ even function because $\cos(-x) = \cos(+x)$

$\Rightarrow x^3 \cos x$ is odd function.

* If 'f' is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

$\rightarrow \int_{-\pi/2}^{\pi/2} x^3 \cos x \, dx = 0$

CamScanner

Dated:

→ 8.2 INTEGRATION BY PARTS

→ FORMULA –

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE #1

find $\int \underbrace{x}_{1^{st}} \underbrace{\cos x}_{2^{nd}}\,dx$

SOLUTION

* To Simplify the selection of the first function, we use the ILATE RULE

I → Inverse Trignometric functions ($\sin^{-1}x$, $\cos^{-1}x$)

L → Logarithmic functions ($\log x$, $\ln x$)

A → Algebraic functions (x^2, $\sqrt{x}$)

T → Trignometric functions ($\sin x$, $\cos x$)

E → Exponential functions (e^x, 2^x)

Dated:

Let, $u = x$, $dv = \cos x\,dx$
$\Rightarrow du = dx$, $\Rightarrow v = \sin x$

$\rightarrow \int x \cos x\,dx = uv - \int v\,du$

$= x \sin x - \int \sin x\,dx$

$= x \sin x + \cos x + c$

→ Example#2
Find $\int \ln x\,dx$

Solution

$= \int \ln x . 1\,dx$

Let, $u = \ln x$, $dv = dx$
$\Rightarrow du = \frac{1}{x}\,dx$, $\Rightarrow v = x$

$\rightarrow \int \ln x . 1\,dx = uv - \int v\,du$

$= x \ln x - \int \cancel{x} \times \frac{1}{\cancel{x}}\,dx$

$= x \ln x - x + c$

Dated:

→ Example #3

Evaluate $\int x^2 e^x dx$

Solution

Let $u = x^2$, $dv = e^x dx$

$\Rightarrow du = 2x dx$, $\Rightarrow v = e^x$.

$\rightarrow \int x^2 e^x dx = uv - \int v du$

$= x^2 e^x - \int e^x \times 2x dx$

$= x^2 e^x - 2 \int x e^x dx$

Integration By Parts Again

Let $u = x$, $dv = e^x dx$

$\Rightarrow du = dx$, $\Rightarrow v = e^x$.

$* \int x e^x dx = uv - \int v du$

$= xe^x - \int e^x dx$

$= xe^x - e^x + c$.

$\Rightarrow \int x^2 e^x dx = x^2 e^x - 2(xe^x - e^x) + c$

$= x^2 e^x - 2xe^x + 2e^x + c$

Dated:

→ EXAMPLE # 4

Evaluate $\int e^x \cos x\, dx$

SOLUTION

Let, $u = e^x$, $dv = \cos x\, dx$

$\Rightarrow du = e^x dx$, $\Rightarrow v = \sin x$

$\rightarrow \int e^x \cos x\, dx = uv - \int v\, du$

$= e^x \sin x - \int \sin x \times e^x dx$

$= e^x \sin x - \int e^x \sin x\, dx$

INTEGRATION BY PARTS AGAIN,

Let, $u = e^x$, $dv = \sin x\, dx$

$\Rightarrow du = e^x dx$, $\Rightarrow v = -\cos x$.

$* \int e^x \sin x\, dx = uv - \int v\, du$

$= e^x(-\cos x) - \int -\cos x \times e^x dx$

$= -e^x \cos x + \underbrace{\int e^x \cos x\, dx}_{I} + c$

$\Rightarrow \quad I = e^x \sin x - (-e^x \cos x + I) + c$

$I = e^x \sin x + e^x \cos x - I + c$

Dated:

$$2I = e^x \sin x + e^x \cos x + c$$

$$I = \frac{e^x \sin x + e^x \cos x}{2} + c$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + c$$

→ Example #5

* Example #5 is solved to make a general formula for $\cos^n x$

→ Solve $\int \cos^3 x \, dx$

Solution

$$= \int \underset{1^{st}}{\cos^2 x} \times \cos x \, dx$$

Let, $u = \cos^2 x$, $dv = \cos x \, dx$

$\Rightarrow du = 2\cos x(-\sin x)\,dx$, $\Rightarrow v = \sin x$

$du = -2\sin x \cos x \, dx$,

$$\rightarrow \int \cos^2 x \cdot \cos x \, dx = uv - \int v \, du$$

$$= (\cos^2 x)(\sin x) - \int \sin x \times -2\sin x \cos x \, dx$$

$$I = (\cos^2 x)(\sin x) + 2\int \sin^2 x \cos x \, dx$$

Dated:

$$I = (\cos^2 x)(\sin x) + 2\int (1-\cos^2 x)\cos x \, dx + c$$

$$I = \cos^2 x \sin x + 2\int (\cos x - \cos^3 x) dx + c$$

$$I = \cos^2 x \sin x + 2\left[\int \cos x \, dx - \int \cos^3 x \, dx\right] + c$$

$$I = \cos^2 x \sin x + 2[\sin x - I] + c$$

$$I = \cos^2 x \sin x + 2\sin x - 2I + c$$

$$3I = \cos^2 x \sin x + 2\sin x + c$$

$$\Rightarrow I = \frac{1}{3}(\cos^2 x \sin x) + \frac{2}{3}\sin x + c.$$

→ EVALUATING DEFINITE INTEGRALS BY PARTS

→ EXAMPLE #6

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x=0$ to $x=4$.

SOLUTION

$$\Rightarrow \int_0^4 xe^{-x} \, dx.$$

Let, $u = x$, $dv = e^{-x} dx$

$\Rightarrow du = dx$; $\Rightarrow v = -e^{-x}$

Dated:

$$\rightarrow \int_0^4 xe^{-x}dx = uv\Big]_0^4 - \int_0^4 v\,du$$

$$= -xe^{-x}\Big]_0^4 - \int_0^4 -e^{-x}\,dx$$

$$= \left[-4e^{-4} - (-0e^{-0})\right] + \int_0^4 e^{-x}\,dx$$

$$= -4e^{-4} + (-e^{-x})\Big]_0^4$$

$$= -4e^{-4} + \left[-e^{-4} - (-e^{-0})\right]$$

$$= -4e^{-4} - e^{-4} + 1$$

$$= 1 - 5e^{-4} \approx 0.91$$