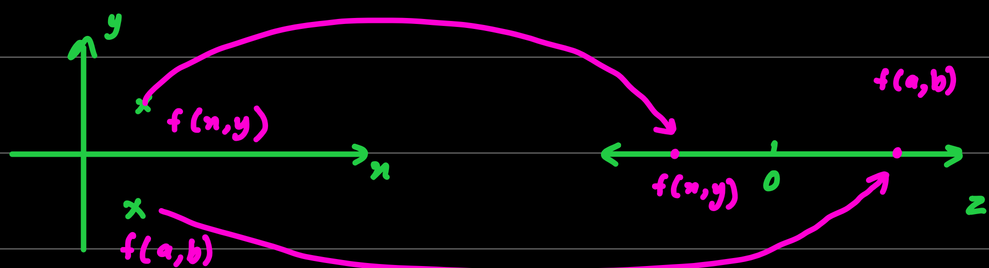
Functions of several variables

(14.1)

- functions having several input variables till n-tuples $\rightarrow f(x_1, x_2 \cdots x_n)$
- If $D = \{x_1, x_2 \cdots x_n\}$ and $w = f(x_1, x_2 \cdots x_n)$, then D is Domain and $f(x_1, x_2 \cdots x_n)$ will give range

- Output variable = w
- Input variables = $x_1, x_2 \cdots x_n$
- Dependant variable = w
- Independant variables = $x_1, x_2 \cdots x_n$

Q- $V(r,h) = \pi r^2 h$



Q- Domain & Range $z = \sqrt{y - x^2}$

Domain : $y - x^2 \geq 0$

Range : $[0, \infty)$

b) $z = \frac{1}{xy}$ $\boxed{\frac{1}{xy} \overset{?}{\neq} \infty}$

★ Use invalid conditions to set domain

Domain : $x \cdot y \neq 0$

Range : $(-\infty, 0) \cup (0, \infty)$

c) $z = \sin(xy)$

Domain: x-y plane / Entire plane

Range : $[-1, 1]$

d) $w = \sqrt{x^2 + y^2 + z^2}$

Domain : Entire space / xyz plane

Range: $[0, \infty)$

e) $w = \frac{1}{x^2+y^2+z^2}$

Domain: $(x,y,z) \neq (0,0,0)$

Range: $(0,\infty)$

f) $w = xy \ln z$

Domain: * Half space $z>0$

Range: $(-\infty,\infty)$

Interior & Exterior Points

- A point in a reigon is an interior point if all disks having the point as their centre of positive radius lies entirely within the reigon

- A point in a reigon is a boundary point if at least one disk having the point as its centre contains points inside and outside the reigon

Open set: Contains only and all interior points

Closed set: Contains all boundary and interior points

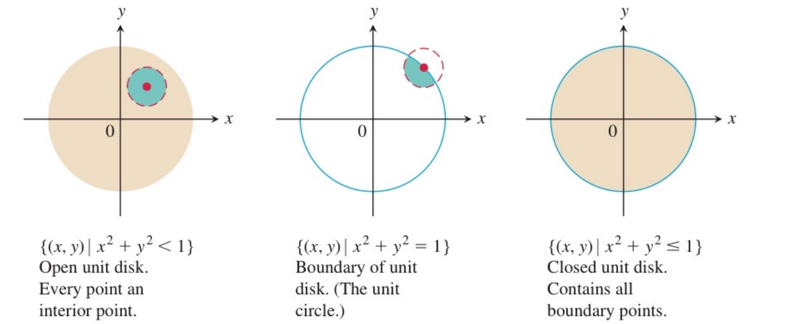


FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

Q- [a,b)

→ neither open, nor closed

· Empty set and xy plane are both open and closed.

· 3 var → sphere, reigon

· 2 var → Disc, reigon

Interior pts

Bounded / Unbounded reigon: A reigon is bounded if it lies inside another reigon, else it is unbounded.

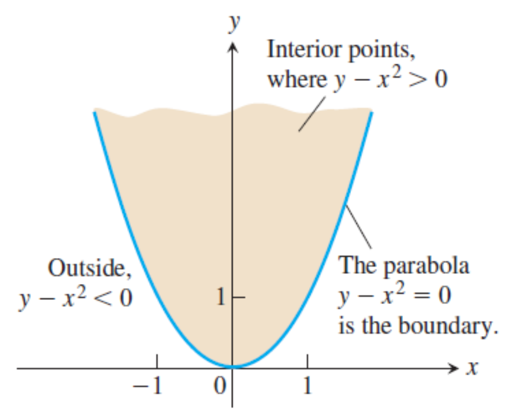


FIGURE 14.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

Level Curve

- $f(x,y) = c$ (Level Curve)
  $\downarrow$
  constant

- $f(x,y,z) = c$ (Level Surface)
  $\downarrow$
  constant

- Set of all points $(x, y, f(x,y))$ in space of (x,y) in domain of f is called the graph of f.

→ Also called "surface $z = f(x,y)$"

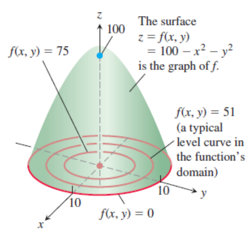


FIGURE 14.5 The graph and selected level curves of the function $f(x, y)$ in Example 3. The level curves lie in the xy-plane, which is the domain of the function $f(x, y)$.

EXAMPLE 3 Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure 14.5.

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \quad \text{or} \quad x^2 + y^2 = 100,$$

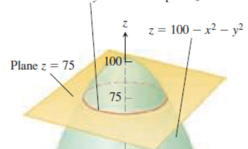
which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure 14.5) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \quad \text{or} \quad x^2 + y^2 = 49$$
$$f(x, y) = 100 - x^2 - y^2 = 75, \quad \text{or} \quad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of f. ■

The contour curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the plane $z = 75$.



The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f. Figure 14.6 shows the contour curve $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour curve lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.

The distinction between level curves and contour curves is often overlooked, and it is common to call both types of curves by the same name, relying on context to make it clear which type of curve is meant. On most maps, for example, the curves that represent constant elevation (height above sea level) are called contours, not level curves (Figure 14.7).

Limits (14.2)

Continuity

Q-
$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Show that the function is continuous at every point except origin

$y=mx \rightarrow \dfrac{2x(mx)}{x^2+m^2x^2}$

$= \dfrac{2m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \dfrac{2m}{1+m^2}$ $\therefore$ There can be any value of m, hence limit does not exist and the function is not continuous

Q- Show that the function has no limit as (x,y) approaches $(0,0)$

$f(x,y) = \dfrac{2x^2y}{x^4+y^2}$

$\cdot \quad \dfrac{2x^2(mx)}{x^4 + m^2x^2} \quad ; \; y = mx$

$= \dfrac{2\cancel{x^3}^{1} m}{\cancel{x^2}(x^2+m^2)} = \dfrac{2xm}{x^2+m^2} \rightarrow \dfrac{2(0)m}{0^2+m^2} = \underline{\underline{0}}$

$\therefore$ If an answer comes verify with another path, both answers must be same

$\cdot \quad \dfrac{2x^2(kx^2)}{x^4 + k^2x^4} \quad ; \; y = kx^2$

$= \dfrac{2k\cancel{x^4}}{\cancel{x^4}(1+k^2)} = \dfrac{2k}{1+k^2} \rightarrow$ limit doesn't exist due to any value of k, and the function is not continuous.

* only applicable on (0,0) paths

Proving & Finding Limits

Q- $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

$\boxed{x=0} \rightarrow \; = \frac{0^2}{0^2+y^2} = \underline{\underline{0}}$

$\boxed{y=0} \rightarrow \; = \frac{x^2}{x^2+0^2} = \underline{\underline{1}}$

$0 \neq 1$, mismatch , limit doesn't exist

Q- $\lim_{x\to(0,0)} \frac{x^2 y^2}{x^4+y^4}$

$\boxed{x=0} \rightarrow \; = \frac{0^2 y^2}{0^4+y^4} = \underline{\underline{0}}$

$\boxed{y=0} \rightarrow \; = \frac{x^2(0)^2}{x^4+0^4} = \underline{\underline{0}}$

$\boxed{0=0}$

$\boxed{y=x} \rightarrow \frac{x^2 \cdot x^2}{x^4+x^4}$

$= \frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2}$

$\frac{1}{2} \neq 0$, hence limit doesn't exist

→ Different equations are taken for different limits.

→ If first two limits from eq's $x=0$, $y=0$ come equal, then confirm using equation $y=x$, then any other eq which satisfies the limit. 4-5 times

- $\lim_{(x,y,z)\to(0,0,0)}$ $x=t^2$, $y=t^2$, $z=t^2$

14.1 : 5-12, 13-16, 17-30, 65-68

14.2 : 1-48, 71-74

- $\lim_{(x,y)\to(0,0)}$ → $y^2=x$