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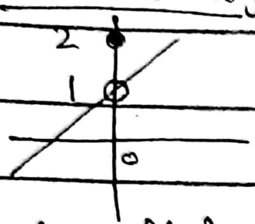
Types of Discontinuity

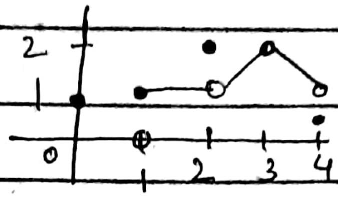
① Removable Discontinuity



$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x).$$

$$f(0) \neq \lim_{x\to 0} f(x).$$

② Jump Discontinuity



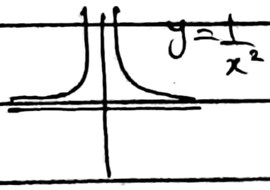
At $x = 1$.

$$\lim_{x\to 1^-} f(x) = 0$$

$$\lim_{x\to 1^+} f(x) = 1$$

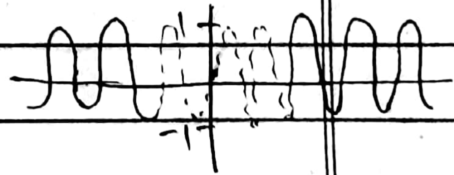
$$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$$

③ Infinite Discontinuity



$$\lim_{x\to c} f(x) = \pm\infty.$$

④ Oscillating Discontinuity



- Rapid changes in the function's values.
- Oscillates so much as $x \to c$.
- Does not approach a single number, so limit as $x \to c$ does not exist.

Continuity of Composition of Functions :

Theorem 9 : $f \to$ cont. at $x = c$

$g \to$ cont at $f(c)$

Then $g \circ f \to$ cont at $x = c$.

$$\boxed{\lim_{x \to c} g \circ f = g(f(c))}$$

→ Holds for any finite number of compositions of function.

→ Only requirement is that each function be cont. where it is applied.

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Example 8 Show that $y = f(x)$ is cont. on its natural domain.

(d) $$y = \left| \frac{x \sin x}{x^2+2} \right|$$

→ $\sin x$ — cont.

→ $x \cdot \sin x$ — cont. (Product Rule)

→ x^2+2 — cont. (polynomial func.)

→ Composition of two functions

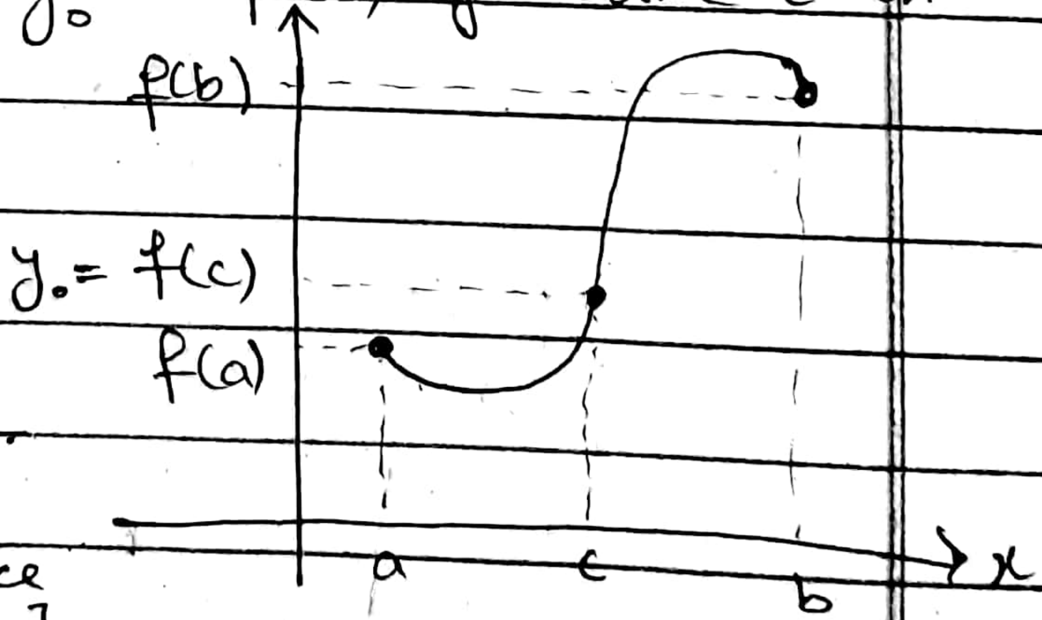
$f(x) = \frac{x \sin x}{x^2+2}$, $g(x) = |x|$.

$$g \circ f = \left| \frac{x \sin x}{x^2+2} \right|$$ → quotient of cont. func. with absolute value func.

Intermediate Value Theorem for Cont. Functions

If f is cont on closed interval $[a, b]$ and if y_0 is any value b/t $f(a)$ and $f(b)$ then $y_0 = f(c)$ for some 'c' in $[a, b]$

Geometrically, theorem says that any horizontal line $y = y_0$ crossing the y-axis b/t the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$



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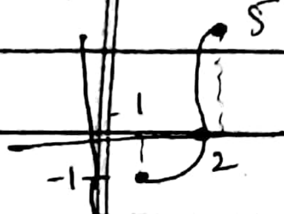
(1) → Graph will be connected

(2)(a) → IVT tells us if f is cont.
then any interval on which f changes
sign contains a zero of the function → (the value of x that makes $f(x)=0$)
(i.e $f(x)=0$).

(2)(b) One point where a cont. func.
is +ve and second point where it is -ve
the function must be equal to zero

Example 10 Show that there is a root of the
equation $x^3-x-1=0$ b/t 1 and 2



Sol:

$$f(1) = 1^3-1-1=-1 < 0$$

$$f(2) = 2^3-2-1=5 > 0$$

we see that $y_0=f(c)=0$ is
the value b/t $f(1)$ and $f(2)$.

Since f → polynomial → cont.
IVT → there is a zero of f b/t 1 & 2.

Continuous Extension to a Point

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f, to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$, since $x = 0$ is not in its domain. Since $y = (\sin x)/x$ has a finite limit as $x \to 0$ (Theorem 7), we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \quad \text{Same as original function for } x \neq 0 \\ 1, & x = 0. \quad \text{Value at domain point } x = 0 \end{cases}$$

The new function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \to 0} \frac{\sin x}{x} = F(0),$$

so it meets the requirements for continuity (Figure 2.47).

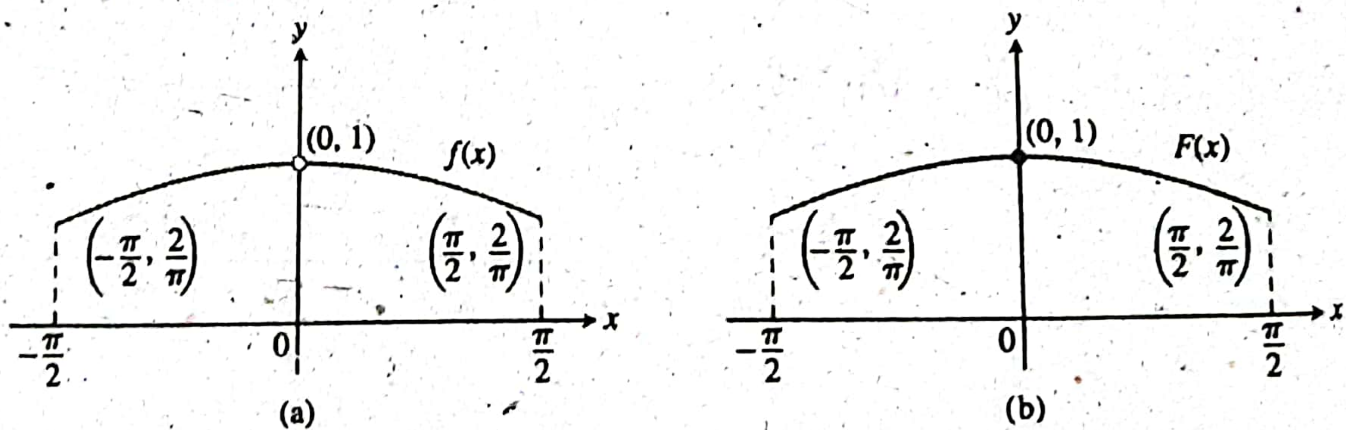


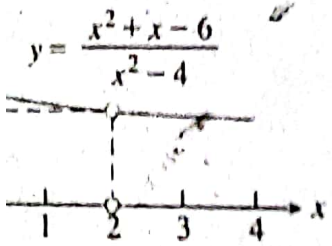
FIGURE 2.47 (a) The graph of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can extend the domain to include $x = 0$ by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$ and $F(x)$ is a continuous function at $x = 0$.

ontinuous at $x = c$. It is called the **continuous extension of f** to $x = c$. For rational functions f, continuous extensions are often found by canceling common factors in the numerator and denominator.

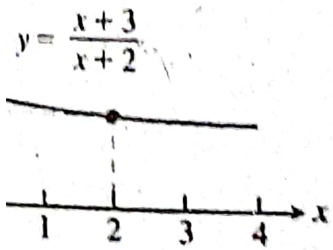
EXAMPLE 12 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.



(a)



(b)

2.48 (a) The graph
d (b) the graph of
ious extension $F(x)$
12).

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}.$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.48. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f extended across the missing domain point at $x = 2$ so as to give a continuous function over the larger domain. ■

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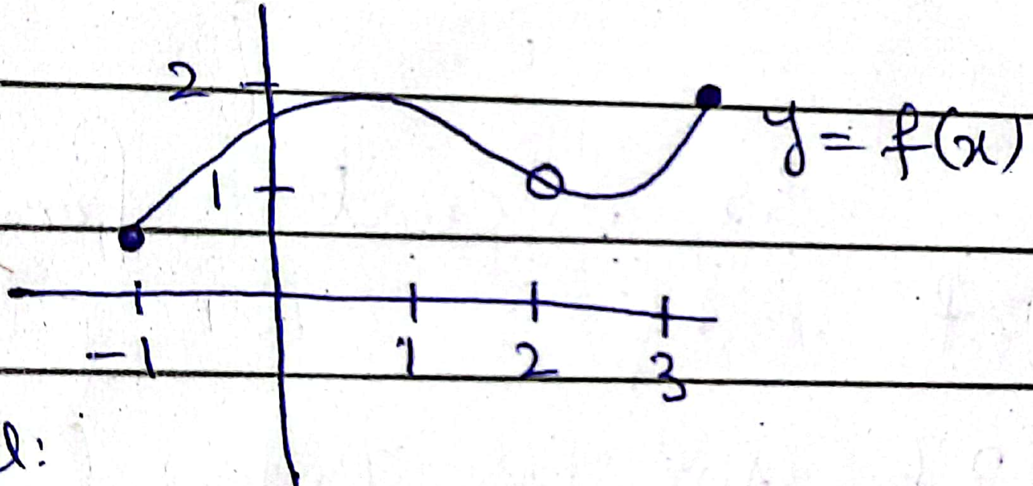
Exercise 2.5

Q1 Graph of function is cont. or not ~~at~~ on $[-1, 3]$?



Sol:

The graph of function is not cont. at $x = 2$

It fails the 1^{st} condition of continuity test

i.e. the value of function at $x = 2$ is not defined.

function at $x = 2$ is the ...

Q13 At what points are the function continuous?

$$y = \frac{1}{x-2} - 3x.$$

Sol: $$y = \frac{1}{x-2} + (-3x).$$

$$y = f(x) + g(x).$$

We know if $f(x)$ & $g(x)$ are cont. at $x = c$ then their sum i.e $y = f(x) + g(x)$ is also cont.

$\Rightarrow f(x) = \frac{1}{x-2}$ is a rational function and is cont. at every point except $x = 2$.

i.e $\mathbb{R} - \{2\}$.

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$\Rightarrow$ $g(x) = -3x$ is a polynomial function and is continuous on entire real number line i.e. $\mathbb{R}$.

Thus $y = \frac{1}{x-2} - 3x$ is cont. for $x \in \mathbb{R} - \{2\}$ Ans

$x \in \mathbb{R} - \{2\}$ Ans

Same question statement as Q13.

Q17 → $y = |x-1| + \sin x$

Sol: $y = |x-1| + \sin x = f(x) + g(x)$

$\Rightarrow f(x) = |x-1|$ is absolute value and is cont. $\forall x \in \mathbb{R}$.

$\Rightarrow g(x) = \sin x$ and is cont. $\forall x \in \mathbb{R}$.

Thus $y = |x-1| + \sin x$ is cont. for $x \in \mathbb{R}$. Ans

Q23 $y = \dfrac{x \tan x}{x^2+1}$ (Same question statement as Q13)

Sol: $y = \dfrac{x \sin x}{\cos x(x^2+1)} = \dfrac{P(x)}{Q(x)}$ (rational function),

$P(x)$ is defined $\forall x \in \mathbb{R}$.

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$Q(x) \neq 0$

$\cos x(x^2+1) \neq 0$

$\cos x \neq 0 \quad [x^2+1 \neq 0 \because x^2 \geq 0]$

$x \neq (2n-1)\frac{\pi}{2} \quad [n \in \mathbb{Z}]$

Thus $y = \frac{x \tan x}{x^2+1}$ is cont. for

Ans

$x \in \mathbb{R} - \{(2n-1)\frac{\pi}{2}\}$ where n is an integer.

Q29 $g(x) = \begin{cases} \dfrac{x^2-x-6}{x-3}, & x \neq 3 \\ 5, & x = 3. \end{cases}$

Sol: When $x \neq 3$.

$$g(x) = \frac{x^2-x-6}{x-3} = \frac{x^2-3x+2x-6}{x-3}$$

$$= \frac{x(x-3)+2(x-3)}{x-3}$$

$$= \frac{(x+2)(x-3)}{x-3} = x+2$$

$$\therefore \; g(x) = \begin{cases} x+2, & x \neq 3 \\ 5, & x = 3 \end{cases}$$

~~For x≠~~ $\lim\limits_{x \to 3^-} g(x) = \lim\limits_{x \to 3^+} g(x) = 5.$

$\lim\limits_{x \to 3} g(x) = 5. \quad , \quad g(3) = 5$

$$\Rightarrow \lim_{x \to 3} g(x) = g(3)$$

Hence $g(x)$ is cont. at $x = 3$. and is cont. for $x \in (-\infty, \infty)$. Ans

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Q41 For what value of a is

$$f(x) = \begin{cases} x^2-1 & x<3 \\ 2ax & x \geq 3 \end{cases}$$

Cont. at every x?

Sol: The function is cont. for $x>3$ and $x<3$ since both subfunctions are polynomials which are cont. over ~~entire~~ entire real number line. However we need to ensure continuity at $x=3$. For $f(x)$ to be cont. at $x=3$ we need to ensure

$$f(3) = \lim_{x\to 3} f(x) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2-1 = 3^2-1 = 8$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2ax = 6a$$

$$f(3) = 2a(3) = 6a$$

Now $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$$8 = 6a$$

$$a = \frac{8}{6} = \frac{4}{3} \quad \text{Ans}$$

Lec- 7

Topic 2.6 Limits involving Infinity + Asymptotes of Graphs.

(1) Finite limits as $x \to \pm\infty$

describible the behaviour of a function when the values in its domain or range outgrow all finite bounds.

(i) $f(x)$ has the limit L as $x \to \infty$ as

If x moves increasingly far from the origin in the +ve direction, $f(x)$ gets arbitrarily close to L.

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

(ii) $f(x)$ has the limit L as $x \to -\infty$ as

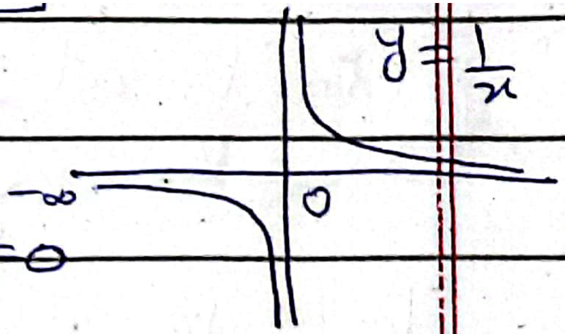
If x moves increasingly far from the origin in the -ve direction, $f(x)$ gets arbitrarily close to L.

$$\boxed{\lim_{x \to -\infty} f(x) = L}$$

|| $y = L$

★ Example : $f(x) = \frac{1}{x}$

is defined $\forall x \neq 0$



(i) When 'x' is +ve and becomes increasingly large, '$\frac{1}{x}$' becomes increasingly small.

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

(ii) When 'x' is −ve and becomes increasingly large, '$\frac{1}{x}$' again becomes small.

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Theorem 12 → All the limit laws in Theorem 1 (Sec 2.2) are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\pm\infty}$ i.e. the variable x may approach a finite number c or $\pm\infty$.

Example 2 a) $\lim_{x\to\infty}\left(5+\frac{1}{x}\right)$.

$$= \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{\infty} = 5+0 = 5.$$

(b) $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3} \times \lim_{x\to-\infty}\frac{1}{x^2}$

$$= \pi\sqrt{3} \times (0) = 0.$$

2 Limits at Infinity of Rational Functions

To determine the limit of rational functions as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator.

Example 3 a) $\lim_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$

When degree of the numerator is less or than or equal to degree of denominator.

$$= \lim_{x\to\infty} \frac{\frac{5x^2}{x^2} + \frac{8x}{x^2} - \frac{3}{x^2}}{\frac{3x^2}{x^2} + \frac{2}{x^2}} = \lim_{x\to\infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}}$$

$$= \frac{5+(0)-(0)}{3+(0)} = \frac{5}{3}$$

Ans.

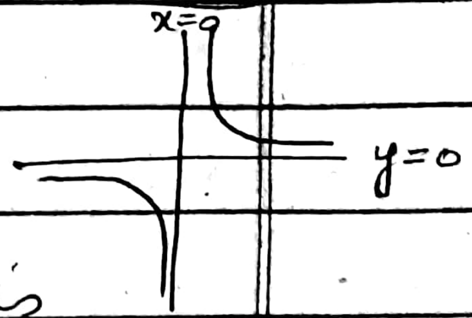
(3) Horizontal Asymptotes

A line $y = b$ is a horizontal asymptote of the graph of a a function $y = f(x)$ if either

$$\boxed{\lim_{x \to \infty} f(x) = b} \quad \text{or} \quad \boxed{\lim_{x \to -\infty} f(x) = b.}$$

$x \to -\infty$

★ <u>Example</u> $f(x) = \frac{1}{x}$.



We observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

We can say that x-axis (y=0) is a horizontal asymptote of $f(x) = \frac{1}{x}$.

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★ Example 4 Find horizontal asymptotes of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$.

Sol:

For $x \geq 0$ $\lim\limits_{x \to \infty} \dfrac{x^3 - 2}{|x|^3 + 1} = \lim\limits_{x \to \infty} \dfrac{x^3 - 2}{x^3 + 1}$

$= \lim\limits_{x \to \infty} \dfrac{1 - 2/x^3}{1 + 1/x^3} = 1$. Ans

For $x < 0$ $\lim\limits_{x \to -\infty} \dfrac{x^3 - 2}{|x|^3 + 1} = \lim\limits_{x \to -\infty} \dfrac{x^3 - 2}{(-x)^3 + 1}$

$= \lim\limits_{x \to -\infty} \dfrac{1 - 2/x^3}{-1 + 1/x^3} = -1$ Ans

The horizontal asymptotes are $y = 1$ and $y = -1$.

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④ Oblique Asymptote → also known as Slant line asymptote.

- If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote.

- We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

as $x \to \pm\infty$

Example 9 Find oblique asymptote of

$$f(x) = \frac{x^2-3}{2x-4}$$

Sol:

We divide x^2-3 by $2x-4$.

$$\begin{array}{r|l} & \frac{x}{2}+1 \\ \hline 2x-4 & \cancel{x^2}+0x-3 \\ & \pm \cancel{x^2} \mp 2x \\ \hline & \cancel{2x}-3 \\ & \pm \cancel{2x} \mp 4 \\ \hline & 1 \end{array}$$

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$$f(x) = \frac{x^2-3}{2x-4} = \underbrace{\frac{x}{2}+1}_{\substack{\text{linear function} \\ g(x)}} + \underbrace{\frac{1}{2x-4}}_{\text{remainder}}$$

As $x \to \pm\infty$ the remainder goes to zero making the slanted line

$$g(x) = \frac{x}{2}+1.$$

an asymptote of $f(x) = \frac{x^2-3}{2x-4}$.

The line $y = g(x)$ is an asymptote both to the right and to the left. (See fig. 2.57 on Page 88)

5 Infinite Limits

(i) $f(x)$ approaches infinity as $x \to c$ as

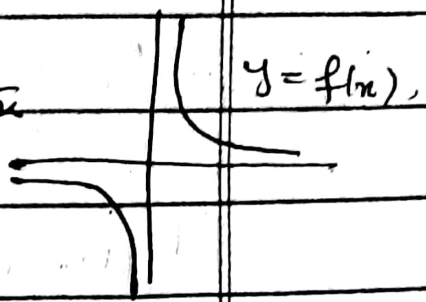
$$\boxed{\lim_{x \to c} f(x) = \infty}$$

(ii) $f(x)$ approaches negative infinity as $x \to c$ as

$$\boxed{\lim_{x \to c} f(x) = -\infty}$$

Example $y = f(x) = \frac{1}{x}$

As $x \to 0^+$ the value of $f(x) = \frac{1}{x}$ becomes arbitrarily large (f grows without bound)



$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

As $x \to 0^-$, the value of $f(x) = \frac{1}{x}$ becomes arbitrarily large and negative.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

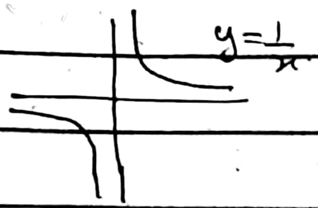
- We are not saying that the limit exists and equals to $\pm\infty$.
- We are describing the behaviour of a function whose limits as $x \to 0^+$ and $x \to 0^-$ do not exist because ~~its~~ $\frac{1}{x}$ value become arbitrarily large and positive (as $x \to 0^+$) and negative (as $x \to 0^-$)

Example 10 Find $\lim\limits_{x \to 1^+} \frac{1}{x-1}$ and

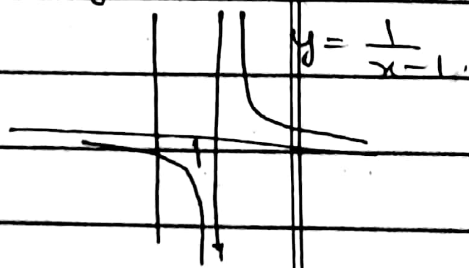
$\lim\limits_{x \to 1^-} \frac{1}{x-1}$?

Sol: (i) Geometric Solution

$y = \frac{1}{x}$



$y = \frac{1}{(x-1)}$ (shift right 1 unit)



Therefore, $y = \frac{1}{x-1}$

behaves near 1 exactly

the way $y = \frac{1}{x}$ behaves near 0.

$\lim\limits_{x \to 1^+} \frac{1}{x-1} = +\infty$ and $\lim\limits_{x \to 1^-} \frac{1}{x-1} = -\infty$.

(ii) Analytic Solution

As $x \to 1^+$, we have $(x-1) \to 0^+$

and $1/x-1 \to +\infty$.

As $x \to 1^-$, we have $(x-1) \to 0^-$

and $1/x-1 \to -\infty$.

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6 Vertical Asymptotes

A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\boxed{\lim_{x \to a^+} f(x) = \pm\infty}$$

and

$$\boxed{\lim_{x \to a^-} f(x) = \pm\infty}$$

★ Example 15 Find horizontal and vertical asymptotes of $y = \frac{x+3}{x+2}$.

Sol: We are interested in the behaviour as $x \to \pm\infty$ and the behaviour as $x \to -2$ where the denominator is zero.

Using long division

$$
\begin{array}{r|l}
 & 1 \\
\hline
x+2 & x+3 \\
 & x+2 \\
\hline
 & 1
\end{array}
$$

$$y = 1 + \frac{1}{x+2}$$

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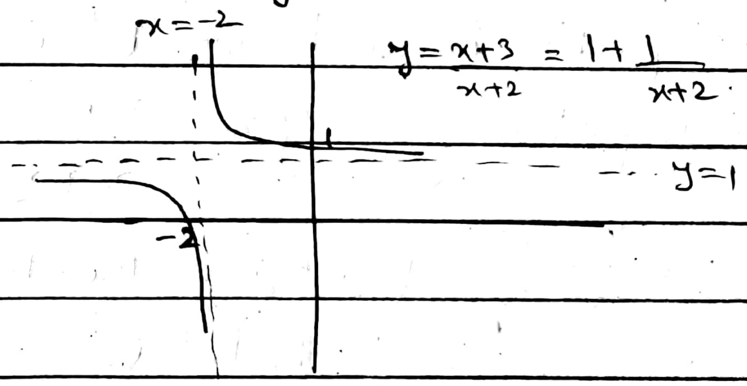
⇒ As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$

⇒ As $x \to -2$, the curve approaches the vertical asymptote $x = -2$.

⇒ The graph of $f(x) = 1 + \frac{1}{x+2}$ is

the graph of $f(x) = \frac{1}{x}$ shifted 1 unit up

and 2 units left.



⇒ The asymptotes instead of being the coordinate axes are now the lines $y = 1$ and $x = -2$.

Exercise 2.6

Q7 Find limit of function as $x \to \infty$ and $x \to -\infty$.

$$h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$$

Sol:

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{-5 + (7/x)}{3 - (1/x^2)}$$

$$= -\frac{5}{3} \quad \text{Ans}$$

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{-5 + (7/x)}{3 - (1/x^2)}$$

$$= -\frac{5}{3} \quad \text{Ans}$$

3

Q11 Find limit

$$\lim_{t \to -\infty} \frac{2-t+\sin t}{t+\cos t}$$

Sol:

$$= \lim_{t \to -\infty} \frac{2/t - t/t + \sin t/t}{t/t + \cos t/t}$$

$$= \lim_{t \to -\infty} \frac{2/t - 1 + \sin t/t}{1 + \cos t/t} = \frac{0 - 1 + 0}{1 + 0}$$

$$= -1 \quad \text{Ans}$$

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Q21 Find limit of rational function as $x \to \infty$ and $x \to -\infty$

$$f(x) = \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$$

Sol: As $x \to \infty$

$$= \lim_{x \to \infty} \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$$

$$= \lim_{x \to \infty} \frac{3x^7/x^3 + 5x^2/x^3 - 1/x^3}{6x^3/x^3 - 7x/x^3 + 3/x^3}$$

$$= \lim_{x \to \infty} \frac{3x^4 + 5/x - 1/x^3}{6 - 7/x^2 + 3/x^3} = \frac{3(\infty) + 0 - 0}{6 - 0 + 0}$$

$$= \infty \quad \text{Ans}$$

As $x \to -\infty$

$$= \lim_{x \to -\infty} \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$$

$$= \lim_{x \to -\infty} \frac{3x^4 + 5/x - 1/x^3}{6 - 7/x^2 + 3/x^3} \quad \left(\text{Dividing by } x^3\right)$$

$$= \frac{3(-\infty)^4 + 0 - 0}{6 - 0 + 0} = \infty \quad \text{Ans}$$

Limit of this function is '∞' at both points.

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Q23 Find limits

$$\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$$

Sol:

$$= \lim_{x\to\infty} \sqrt{\frac{8x^2/x^2 - 3/x^2}{2x^2/x^2 + x/x^2}}$$

$$= \lim_{x\to\infty} \sqrt{\frac{8-3/x^2}{2+1/x}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2. \ \underline{\text{Ans}}$$

Q35 Find limit

Q35 Find limit

$$\lim_{x \to \infty} \frac{x-3}{\sqrt{4x^2+25}}$$

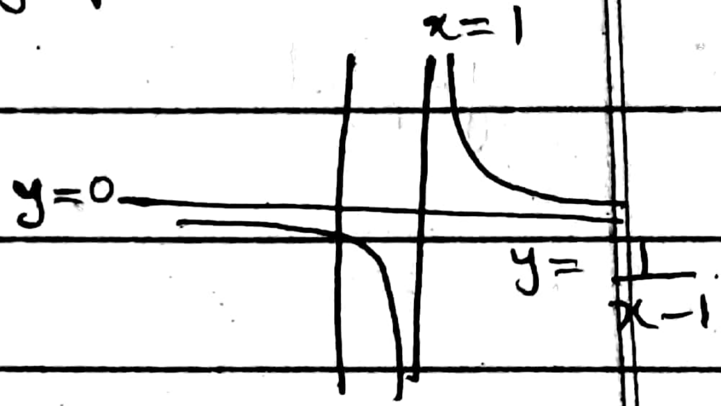
Sol:

$$= \lim_{x \to \infty} \frac{x/x - 3/x}{\sqrt{4x^2/x^2 + 25/x^2}} = \lim_{x \to \infty} \frac{1 - 3/x}{\sqrt{4 + 25/x^2}}$$

$$= \frac{1-0}{\sqrt{4+0}} = \frac{1}{2} \quad \text{Ans}$$

Q63 Graph the rational function. Include graph and equation of the asymptote and dominant terms.

$$y = \frac{1}{x-1}$$



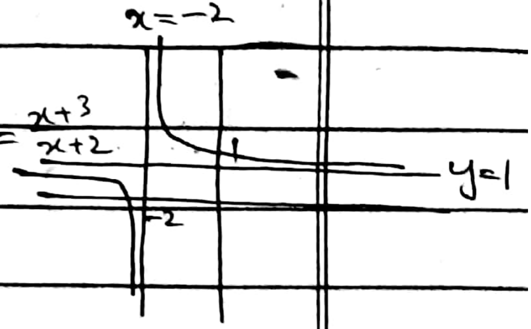
Sol: As $x \to \pm\infty$,

$$\lim_{x\to\pm\infty} \frac{1}{x-1} = 0 \quad (y=0 \text{ is horizontal asymptote}).$$

As $x \to 1$,

$$\lim_{x\to 1} \frac{1}{x-1} = \infty \quad (x=1 \text{ is its vertical asymptote}).$$

Q67 Same statement as Q63



Sol:

$$y = \frac{x+3}{x+2}$$

Using long division

$$x+2 \,\overline{)\, \cancel{x}+3} \quad 1$$
$$-\cancel{x} \pm 2$$
$$1$$

$$y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$$

As $x \to \pm\infty$, dominant term is $\boxed{y=1}$. This ~~is~~ also becomes its horizontal asymptote

When $x \to -2$, the term $1/x+2$ becomes ~~is~~ very large (goes to infinity) and hence the dominant term. Also $\boxed{x=-2}$ becomes its vertical asymptote.

$\longleftrightarrow$