Lec-14

Topic 4.3 Monotonic Functions and the First Derivative Test

① Increasing Functions and Decreasing Function

Suppose that f is cont on $[a,b]$ and f is differentiable on (a,b).

- If $\boxed{f'(x) > 0}$ at each point $x \in (a,b)$, then f is $\boxed{\text{increasing}}$ on $[a,b]$
- If $\boxed{f'(x) < 0}$ at each point $x \in (a,b)$, then f is $\boxed{\text{decreasing}}$ on $[a,b]$.

② Monotonic Function

A function that is increasing or decreasing on an interval is said to be $\boxed{\text{monotonic}}$ on the interval.

Example 1 Find critical points of

$$f(x) = x^3 - 12x - 5$$

and identify the open intervals on which f is increasing and on which f is decreasing.

(P.T.O).

Sol: As given $f(x)$ is a polynomial function so f is cont. and diff. everywhere.

(I) Critical points:

$f'(x) = 0$ (Condition to find critical pt)

$f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

$\Rightarrow f'(x) = 0$

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$\boxed{x = \pm 2}$ — Critical Points

(II) Identifying open Intervals:

These critical pts. subdivide the domain of f to create non-overlapping open intervals

$(-\infty, -2), (-2, 2), (2, \infty)$

on which f' is either +ve or −ve.

(III) Determining Sign of f':

We determine the sign of f' by evaluating f' at a convenient point in each subinterval.

Intervals (i) $-\infty < x < -2$

We evaluate f' at $x = -3$

i.e. $f'(-3) = 3(-3)^2 - 12 = 15$

(ii) $-2 < x < 2$

Evaluate f' at $x = 0$ i.e $f'(0) = 3(0)^2 - 12$

$= -12$

(iii) $2 < x < \infty$

Evaluate f' at $x = 3$. i.e $f'(3) = 3(3)^2 - 12$

$= 15$.

- Sign of f' $\longrightarrow$ $+$ $\quad$ $-$ $\quad$ $+$
- Behaviour of f' $\rightarrow$ increasing $\quad$ decreasing $\quad$ increasing

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

☆ **Note:**

- We could use "$\leq$" inequalities i.e. increasing on $-\infty < x \leq -2$, decreasing on $-2 \leq x \leq 2$ and increasing on $2 \leq x < \infty$.
- We do not talk about whether a function is increasing or decreasing at a single point.

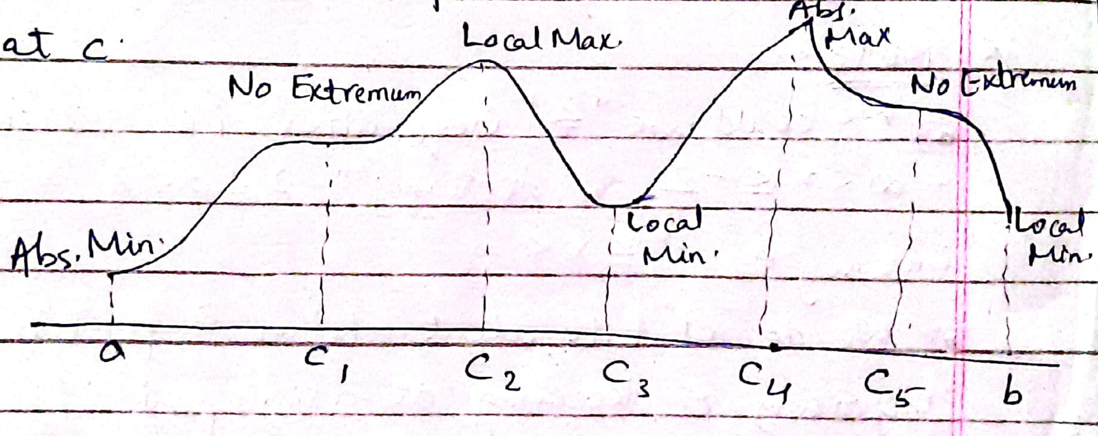
(3) First Derivative Test for Local Extrema

Suppose that c is a critical pt of Cont. function and that is differentiable at every point in some interval containing 'c' except possibly at c itself. Moving across this interval from left to right.

(i) If f' changes from negative to positive at c then f has a local minimum at c

(ii) If f' changes from positive to negative at c, then f has local maximum at c.

(iii) If f' does not change sign at c i.e. f' is positive on both sides of c or negative on both sides then f has no local extremum at c.



*** Note:**

- The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

Example 2 } Class Activity (Home Task).
Example 3 } Pg 199 + 200

Ex 4.3

Analyzing Functions from Derivatives

a) What are critical points?

b) On what open interval f is increasing or decreasing?

c) At what points, if any, does f has local minimum and local maximum?

Q#4 $f'(x) = (x-1)^2(x+2)^2$

Sol:

(a) $f'(x) = 0$

$(x-1)^2(x+2)^2 = 0$

$f'(x) = 0$ iff $x = 1$ or $x = -2$

Critical pts of f is $\boxed{x = 1, -2.}$

(b) Open intervals

$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
$f'(-3) = (-3-1)^2(-3+2)^2$	$f(0) = (0-1)^2(0+2)^2$	$f(2) = (2-1)^2(2+2)^2$
$= (16)(1) = 16$	$= 1+4 = 5$	$= (1)(16) = 16$
(+ve)	(+ve)	(+ve).
increasing	increasing	increasing

(c) There is no changes in sign, so it has not contain any local extreme point in their open domain

7) $f'(x) = \dfrac{x^2(x-1)}{x+2}$ $\quad x \neq -2$

Sol: (a) Critical Points. — $f'(x) = 0$ where

if f' is zero or

$f'(x) = \dfrac{x^2(x-1)}{x+2} = 0$ $\quad$ f' is undefined

$\Rightarrow$ $x = 1$, $x = 0$, $x = -2$ are the

critical points.

(b) Open Intervals where f is increasing or decreasing

(i) $(-\infty, -2)$ $\quad f'(-3) = \dfrac{9(-4)}{-1} = 36$. (Increasing)

(ii) $(-2, 0)$ $\quad f(-1) = \dfrac{1(-2)}{1} = -2$ (Decreasing)

(iii) $(0, 1)$ $\quad f(\tfrac{1}{2}) = \dfrac{\frac{1}{4}(-\frac{1}{2})}{\frac{1}{2}+2} = \dfrac{-\frac{1}{8}}{\frac{5}{2}} = \dfrac{-1}{8} \times \dfrac{2}{5} = \dfrac{-1}{20}$

(Decreasing)

(iv) $(1, \infty)$ $\quad f(2) = \dfrac{4(1)}{4} = 1$ (Increasing)

(c) Local Max. $x = -2$

Local Min. $x = 1$

$\longleftrightarrow$

12) $f'(x) = x^{-1/2}(x-3)$, ~~x≠0~~

Sol: $f'(x) = \frac{x-3}{\sqrt{x}}$

(a) Critical points:

$f'(x) = 0$

$\frac{x-3}{\sqrt{x}} = 0$

$x = 3$ (f' is zero), $x = 0$ (f' is undefined)

Critical Points ~~is~~ are $\boxed{x = 0, 3}$

(b) Open Interval of f ~~which~~ where ~~is~~ increasing or decreasing

(i) $-\infty < x < 0$ $f'(-1) =$ Not define in domain

(ii) $0 < x < 3$ $f'(1) = -2$ (Decreasing)

(iii) $3 < x < \infty$ $f'(4) = \frac{1}{2}$ (Increasing)

(c) No Local Max.

Local Min at $x = 3$.

13) $f'(x) = (\sin x - 1)(2\cos x + 1)$, $0 \le x \le 2\pi$.

Sol: a) Critical points-

$f'(x) = 0$

$(\sin x - 1)(2\cos x + 1) = 0$

$\sin x - 1 = 0$ | $2\cos x + 1 = 0$

$\sin x = 1 \Rightarrow x = \sin^{-1}(1)$ | $2\cos x = -1$

$x = \frac{\pi}{2}$ | $\cos x = -\frac{1}{2} \Rightarrow x = \cos^{-1}\left(-\frac{1}{2}\right)$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Critical Points at $\boxed{x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}}$

b) Open Interval

$0 < x < \pi/2$ $f'(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$ (Decreasing)

$\pi/2 < x < \frac{2\pi}{3}$ $f(\frac{5\pi}{9}) = -9.91 \times 10^{-3}$ (Decreasing)

$\frac{2\pi}{3} < x < \frac{4\pi}{3}$ $f(\pi) = 1$ (Increasing).

$\frac{4\pi}{3} < x < 2\pi$ $f(\frac{3\pi}{2}) = -2$ (Decreasing)

(c) Local Max. at $x = \frac{4\pi}{3}$

Local Min at $x = \frac{2\pi}{3}$.

Q31 (a) Find open intervals on which f is increasing or decreasing.

(b) Identify the function's local and absolute extreme values

$f(x) = x - 6\sqrt{x-1}$

Sol: a) Critical Points

$$f'(x) = 1 - \cancel{6}^{3}\left(\frac{1}{\cancel{2}}(x-1)^{-1/2}\right)$$

$$= 1 - \frac{3}{\sqrt{x-1}}$$

$$f'(x) = 0 \Rightarrow 1 - \frac{3}{\sqrt{x-1}} = 0$$

$$1 = \frac{3}{\sqrt{x-1}}$$

$$\sqrt{x-1} = 3$$

$$x - 1 = 9 \quad \text{(Sq. on both sides)}$$

$$\boxed{x = 10}$$

$f'(x)$ — undefined

$x = 1$ — $f'(x)$ is undefined.

Critical Points at $\boxed{x = 1, 10}$

Open Intervals

$-\infty < x < 1$ $f'(0)$ = Not defined in the domain

$1 < x < 10$ $f'(2) = -2$ (Decreasing)

$10 < x < \infty$ $f'(11) = 0.05$ (Increasing)

(b) At $x = 10$ (Local Min.)

No local Max.

At $x = 10$ (Abs. Min.)

$\longleftrightarrow$

50) a) Identify local extreme values

b) Which of the extreme values are absolute?

Q $f(x) = \sqrt{x^2 - 2x - 3}$; $3 \leq x < \infty$

Sol: a) $f'(x) = \frac{1}{2}(x^2 - 2x - 3)^{-1/2}(2x - 2)$

$$= \frac{2x-2}{2\sqrt{x^2-2x-3}} = \frac{\cancel{2}(x-1)}{\cancel{2}\sqrt{x^2-2x-3}}$$

$f'(x) = 0$

$$\frac{x-1}{\sqrt{x^2-2x-3}} = 0$$

$$\frac{x-1}{\sqrt{(x-3)(x+1)}} = 0$$

Critical Points. At $\boxed{x = 1}$ (f' is zero)

At $\boxed{x = 3, -1}$ (f' is undefined)

As $x = -1$ and $x = 1$ are not in the given domain i.e $3 \leq x < \infty$ so critical point is $\boxed{x = 3}$

Open Interval

(negative, complex value) f'(x) not defined | (Positive value)

$(-\infty, 3)$ 3 $(3, \infty)$

$$f'(2) = \frac{2-1}{\sqrt{2^2 - 2(2) - 3}} = \frac{1}{\sqrt{4-4-3}}$$

$$= \frac{1}{\sqrt{-3}} \text{ (Not defined)}$$

$$f'(4) = \frac{4-1}{\sqrt{4^2 - 2(4) - 3}} = 1.34 \text{ (+ve)} \text{ — increasing}$$

(a)+(b) At $x = 3$ — local min.

$x = 3$ — Abs. Min.

Practice Questions of 4.3

(Try these questions)

Q#1 – 14.

Q# 15 – 40

Q# 41 – 52

Q# 53 – 60