2.3

- $k^n \det(A) = \det(kA)$ * only for $n \times n$ matrices
- $\det(A+B) \neq \det(A) + \det(B)$
- If A, B matrices differ only in a single row, then $\det(C) = \det(A) + \det(B)$
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$
- $\operatorname{adj}(A)$ = matrix of cofactors of A.

EXAMPLE 8 | Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve

$$\begin{aligned} x_1 + + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

- replace the jth column with the solutions

Q1- $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$

$kA = \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix}$

$\det(kA) = \underline{\underline{-40}}$ ✓

$k^2 \det(A) = 2^2 \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = 4 \cdot (-4-6) = \underline{\underline{-40}}$ ✓

Q5- $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix}$

$\det(AB) = 10 \begin{vmatrix} -1 & 8 \\ 1 & 17 \end{vmatrix} + 0 + 2 \begin{vmatrix} 9 & -1 \\ 31 & 1 \end{vmatrix}$

$= 10(-25) + 2(40)$

$= -250 + 80 = \underline{\underline{-170}}$

$BA = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix}$

$\det(BA) = -1 \begin{vmatrix} 11 & 4 \\ 5 & 2 \end{vmatrix} + 3 \begin{vmatrix} 17 & 4 \\ 10 & 2 \end{vmatrix} + 6 \begin{vmatrix} 17 & 11 \\ 10 & 5 \end{vmatrix}$

$\begin{array}{r} 110 \\ 85 \\ \hline 25 \end{array}$

$85 - 110$

$= -2 + 3(-6) + 6(-25)$

$= -2 - 18 - 150$

$= \underline{-170}$

Q15- $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

$|A| = (k-3)(k-2) - 4$

$k^2 - 2k - 3k + 6 - 4 \neq 0$

$k^2 - 5k + 2 \neq 0$ ✓

Q17- $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 6 \\ k & 2 \end{vmatrix} + 4 \begin{vmatrix} 3 & 1 \\ k & 3 \end{vmatrix}$

$= -16 - 2(6-6k) + 4(9-k)$

$-16 - 12 + 12k + 36 - 4k \neq 0$

$\begin{array}{r} 36 \\ 28 \\ \hline 8 \end{array}$

$8k + 8 \neq 0$

$\underline{\underline{k \neq -1}}$ ✓

Q30- $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A) = \cos^2\theta + \sin^2\theta = \underline{\underline{1}}$

cofactor matrix $= \begin{bmatrix} \cos\theta & +\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\operatorname{adj}(A) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

Q 19- $A = \begin{bmatrix} 2 & 5 & 5^{+} \\ -1 & -1 & 0^{-} \\ 2 & 4 & 3^{+} \end{bmatrix}$ 15-20

$$\det(A) = 5 \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} + 0 + 3 \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix}$$

$$= -10 + 9 = -1$$

cofactor matrix $= \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}$

$\text{adj}(A) = \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}$

$\det(A) = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$ ✓

Q24- $7x_1 - 2x_2 = 3$
$3x_1 + x_2 = 5$

$A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \rightarrow 13$, $A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$

$x_1 = \dfrac{\det(A_1)}{\det(A)}$, $x_2 = \dfrac{\det(A_2)}{\det(A)}$

$= \dfrac{13}{13} = 1$ ✓ $= \dfrac{26}{13} = 2$ ✓

Q25- $4x + 5y = 2$
$11x + y + 2z = 3$
$x + 5y + 2z = 1$

$A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

$A_3 = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$

$\det(A) = 4 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix}$

$= -32 - 5(20)$
$= \underline{\underline{-132}}$

$\det(A_1) = 2 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

$= -16 - 20 = \underline{\underline{-36}}$

$\det(A_2) = 4 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 11 & 2 \\ 1 & 2 \end{vmatrix}$

$= 16 - 40 = -24$

$\det(A_3) = \begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix} - 5 \begin{vmatrix} 4 & 2 \\ 11 & 3 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 11 & 1 \end{vmatrix}$

$= 13 + 50 - 51$
$= \underline{\underline{12}}$

$x = \dfrac{-36}{-132}$, $y = \dfrac{-24}{-132}$, $z = \dfrac{12}{-132}$

$= \dfrac{3}{11}$ ✓ $= \dfrac{2}{11}$ ✓ $= -\dfrac{1}{11}$ ✓

Q 29- $3x_1 - x_2 + x_3 = 4$
$-x_1 + 7x_2 - 2x_3 = 1$
$2x_1 + 6x_2 - x_3 = 5$

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 7 & -2 \\ 2 & 6 & -1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 4 & -1 & 1 \\ 1 & 7 & -2 \\ 5 & 6 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & 4 & 1 \\ -1 & 1 & -2 \\ 2 & 5 & -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 7 & 1 \\ 2 & 6 & 5 \end{bmatrix}$

$\det(A) = 15 + 5 - 20 = 0$

$\det(A) = 0$, so cramers rule does not apply

Q 26- $x - 4y + z = 6$
$4x - y + 2z = -1$
$2x + 2y - 3z = -20$

$A = \begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{bmatrix}$

$\det(A) = -1 - 64 + 10 = -55$

$\det(A_1) = -6 + 172 - 22 = 144$ $\quad -\frac{144}{55} \quad -\frac{61}{55} \quad \frac{46}{11}$

$\det(A_2) = 43 + 96 - 78 = 61$

$\det(A_3) = 22 - 312 + 60 = -230$

$x = -\frac{144}{55}$, $y = -\frac{61}{55}$, $z = \frac{\cancel{230}^{46}}{\cancel{55}_{11}} = \frac{46}{11}$