8.3 Trignometric Integrals

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→ Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions.

* Products of Power of Sines & Cosines

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We begin with integral of the form

$$\int \sin^m x \cos^n x \, dx$$

where 'm' and 'n' are non negative integers (positive or zero).

→ We can divide the appropriate substitution into three cases according to 'm' and 'n' being 'odd' or 'even'.

* Case 1: If $\boxed{m \text{ is odd}}$ we write $\boxed{m \text{ as } 2k+1}$ and use the identity $\boxed{\sin^2 x = 1 - \cos^2 x}$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \cdot \sin x$$

$$\boxed{\sin^m x = (1 - \cos^2 x)^k \sin x}$$

We then combine the single $\sin x$ with dx and set $\boxed{\sin x dx}$ equal to $\boxed{-d(\cos x)}$

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* Case 2: If $\boxed{n \text{ is odd}}$ in $\int \sin^m x \cos^n x \, dx$, we write $\boxed{n \text{ as } 2k+1}$ and use the identity $\boxed{\cos^2 x = 1 - \sin^2 x}$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x$$

$$\boxed{\cos^n x = (1 - \sin^2 x)^k \cos x}$$

We then combine the single $\cos x$ with dx and set $\boxed{\cos x \, dx}$ equal to $\boxed{d(\sin x)}$

* Case 3: If both $\boxed{m \text{ and } n \text{ are even}}$ in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\boxed{\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

- Example 1 (Case 1).

Evaluate $\int \sin^3 x \cos^2 x \, dx$.

$m = 2k+1$
$3 = 2k+1$
$3 - 1 = 2k$
$2 = 2k$
$k = 1$.

Sol: Here $m = 3$ (odd).

$\sin x \, dx = -d\cos x$

$$\int \sin^3 x \cos^2 x \, dx = \cancel{\int (1 - \cos^2 x)} \int (\sin^2 x)^k \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^1 \cos^2 x \, (-d(\cos x))$$

$$= \int (1 - u^2)(u^2)(-du) \quad \because u = \cos x$$

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$$\int \sin^3 x \cos^2 x \, dx = \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

- **Example 2** (Case 2).

Evaluate $\int \cos^5 x \, dx$

Sol: Here $n = 5$ (odd)

$n = 5$
$n = 2k+1$
$5 = 2k+1$
$5-1 = 2k$
$4 = 2k$
$k = 2$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx$$

$$= \int (1-\sin^2 x)^2 \cos x \, dx$$

$\cos x \, dx = d(\sin x)$.

$$= \int (1-\sin^2 x)^2 \, d(\sin x)$$

$$= \int (1-u^2)^2 \, du \qquad \because u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

- **Example 3** (Case 3)

Evaluate $\int \sin^2 x \cos^4 x \, dx$

Sol: Here $m = 2$ (even), $n = 4$ (even)

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2 \, dx.$$

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$$\int \sin^2 x \cos^4 x\,dx = \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\,dx$$

$$= \frac{1}{8}\int (1+2\cos 2x+\cos^2 2x-\cos 2x-2\cos^2 2x-\cos^3 2x)\,dx$$

$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\,dx$$

$$= \frac{1}{8}\left[x+\frac{\sin 2x}{2}-\int (\cos^2 2x+\cos^3 2x)\,dx\right]$$

$$\int \cos^2 2x\,dx = \int \left(\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{2}\int (1+\cos 4x)\,dx = \frac{1}{2}\left[x+\frac{\sin 4x}{4}\right]+c$$

$$\int \cos^3 2x\,dx = \int \cos^2 2x\,\cos 2x\,dx$$

$$= \int (1-\sin^2 2x)\cos 2x\,dx$$

Let $u = \sin 2x$

$du = 2\cos 2x\,dx$

$$= \int (1-u^2)\frac{du}{2}$$

$$= \frac{1}{2}\left[u-\frac{u^3}{3}\right]+c$$

$$= \frac{1}{2}\left[\sin 2x-\frac{1}{3}\sin^3 2x\right]+c$$

$$\int \sin^2 x\cos^4 x\,dx = \frac{1}{8}\left[x+\cancel{\frac{\sin 2x}{2}}-\frac{1}{2}x-\frac{\sin 4x}{8}-\cancel{\frac{1}{2}\sin 2x}+\frac{1}{6}\sin^3 2x\right]+c$$

$$= \frac{1}{8}\left[\frac{x}{2}-\frac{1}{8}\sin 4x+\frac{1}{6}\sin^3 2x\right]+c$$

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2. Eliminating Square Roots

→ We use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

to eliminate a square root.

• **Example 4:** Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$.

Sol: To eliminate the sq. root, we use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$1+\cos 2\theta = 2\cos^2\theta.$$

with $\theta = 2x$, this becomes

$$1+\cos 4x = 2\cos^2 2x.$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{2}\cos 2x\, dx.$$

$$= \sqrt{2}\, \frac{\sin 2x}{2} \Bigg|_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 0\right] = \frac{\sqrt{2}}{2}.$$

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3 Integrals of Powers of $\tan x$ and $\sec x$.

→ We know how to integrate $\tan x$ and $\sec x$ functions & their squares.

→ To integrate higher powers, we use the identities

$$\tan^2 x = \sec^2 x - 1$$
$$\sec^2 x = \tan^2 x + 1.$$

and integrate by parts when necessary to reduce the higher powers to lower powers.

- Example 5: Evaluate $\int \tan^4 x\,dx$.

Sol:
$$\int \tan^4 x\,dx = \int \tan^2 x \tan^2 x\,dx$$
$$= \int \tan^2 x(\sec^2 x - 1)dx$$
$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx.$$
$$= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx$$

$$\int \tan^2 x \cdot \sec^2 x\,dx = \int u^2\,du \qquad \because u = \tan x, \quad du = \sec^2 x\,dx$$

$$= \frac{u^3}{3} + c = \frac{\tan^3 x}{3} + c$$

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \tan x + x + C$$

- **Example 6** Evaluate $\int \sec^3 x\,dx = I$.

Sol: Integration by parts.

$u = \sec x$

$v = \sec^2 x$

$I = \int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx = \sec x \tan x - \int (\sec x \tan x)\tan x\,dx$

$= \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$

$= \sec x \tan x - I + \ln|\sec x + \tan x| + c$

$2I = \sec x \tan x + \ln|\sec x + \tan x| + c.$

$I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$

- **Example 7:** Evaluate $\int \tan^4 x \sec^4 x\,dx$.

Sol: $\int \tan^4 x \sec^4 x\,dx = \int \tan^4 x \sec^2 x \sec^2 x\,dx$

$= \int \tan^4 x(\underbrace{1 + \tan^2 x}_{\sec^2 x})\,dx$

$= \int (\tan^4 x + \tan^6 x)\sec^2 x\,dx.$

$= \int \tan^4 x \sec^2 x\,dx + \int \tan^6 x \sec^2 x\,dx$

$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$

(4) Products of Sines & Cosines

The integrals

- $\int \sin mx \sin nx\,dx$, • $\int \sin mx \cos nx\,dx$, • $\int \cos mx \cos nx\,dx$

(a) $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$

(b) $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$.

(c) $\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$

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- **Example 8** : Evaluate $\int \sin 3x \cos 5x \, dx$

Sol:

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int \left[\sin(3x-5x) + \sin(3x+5x)\right] dx$$

$$= \frac{1}{2}\int \left[\sin(-2x) + \sin(8x)\right] dx$$

$$= \frac{1}{2}\int \left[\sin 8x - \sin 2x\right] dx$$

$$= \frac{1}{2}\left(-\frac{\cos 8x}{8}\right) + \frac{1}{2}\frac{\cos 2x}{2} + c$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$