Lec-15

Topic 4.4 Concavity and Curve Sketching

Overview In this section, we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behaviour, we can now draw an accurate graph of a function.

① **Concavity**: The graph of a diff. function $y = f(x)$ is

Concavity means turning or bending behavior of the curve

(a) $\boxed{\text{Concave up}}$ on an open interval if f' is increasing on the interval.

(b) $\boxed{\text{Concave down}}$ on an open interval if f' is decreasing on the interval.

Note: A function whose graph is concave up is also often called convex.

② **The Second Derivative Test for Concavity:**

Let $y = f(x)$ be twice differentiable on an interval

(I) If $\boxed{f'' > 0}$ on interval, the graph of f over interval is $\boxed{\text{concave up}}$

②

II) If $\boxed{f'' < 0}$ on interval, the graph of f over interval is $\boxed{\text{Concave down}}$

Example 1: (a) The curve $y = x^3$.

Sol: $y' = 3x^2$

$y'' = 6x$.

$y'' = 0$

$6x = 0$

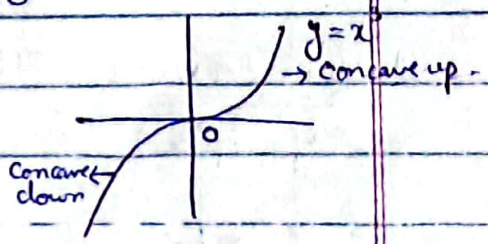
$\boxed{x = 0}$ Point of Inflection.

$(-\infty, 0) \rightarrow y''(-1) = -6 < 0$

↳ Concave down.

$(0, \infty) \rightarrow y''(1) = 6 > 0$

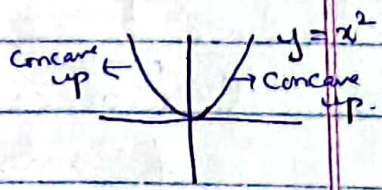
↳ concave up



(b) The curve $y = x^2$

Sol: $y' = 2x$

$y'' = 2 > 0$



Concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always +ve.

* Example 2 — (Do it yourself) *

③ Point of Inflection:

A Point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a point of inflection.

→ In general, if the second derivative exists at a point of inflection $(c, f(c))$ then

$$\boxed{f''(c) = 0}$$

→ At a point of inflection $(c, f(c))$ either
$\boxed{f''(c)=0}$ or $\boxed{f''(c) \text{ fails to exist}}$ (undefined)

$\longleftrightarrow$

- Example 3: Find point of inflection and determine its concavity of → See its Graph by yourself. (Pg 204)

$$f(x) = x^3 - 3x^2 + 2$$

Sol: $f' = 3x^2 - 6x$

$f'' = 6x - 6$

$0 = 6x - 6$ $\quad (\because f''(x) = 0$ Point of Inflection)

$\boxed{x = 1}$

$-\infty \quad\quad 1 \quad\quad \infty$

$(-\infty, 1)$ — $f''(0) = -6 < 0$ (concave down)

$(1, \infty)$ — $f''(0) = 6(2) - 6 = 12 - 6 = 6 > 0$ (concave up)

At $x = 1$ or $(1, 0)$, point of inflection exists

$\longleftrightarrow$

- Example 4: (From Book) — ~~Without~~ A function can have a point of inflection where first derivative exists but the second derivative fails to exist.

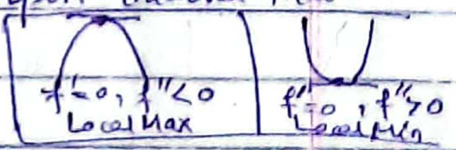
(Pg 204)

- <u>Example 5</u> — (from book) — An inflection pt. need not occur even though both derivatives exist and $f''=0$
- <u>Example 6</u> — (from book) — A pt. of inflection occurs at a vertical tangent to the curve whether neither the first nor the second derivative exists.
- Example 7 — (Home Task Do it yourself)

<u>Theorem 6 (Pg 206)</u>

(4) <u>Second Derivative Test for Local Extrema:</u>

Suppose f'' is cont. on an open interval that contains $x=c$.



(I) If $f'(c)=0$ and $f''(c)<0$ then f has a local maximum at $x=c$

(II) If $f'(c)=0$ and $f''(c)>0$ then f has a local minimum at $x=c$.

(III) If $f'(c)=0$ and $f''(c)=0$ then the Test fails. The function may have a local max., a local min. or neither

- <u>Example 8</u> (from book)

Imp. topic for Exam point of view

5. Procedure for Graphing $y = f(x)$:

1. Identify domain and any symmetries of the curve
2. Find derivatives y' and y'' (first and second derivative)
3. Find critical points of f and identity the functions's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the point of inflection and determine the concavity of the curve.
6. Identity any asymptotes that may exist.
7. Plot key points such as intercepts and points found in Steps 3-5 and sketch the curve together with any asymptotes that exist.

Example 9 (from book).
Example 10 } Task (Present by Students on Board)
Example 11