orthognal Functions

* Inner Product

If u and v are two vectors in R^3 or 3-space then the inner product (u, v) (in calculas it is called $u \cdot v$) possesses the following properties

(i) $(u, v) = (v, u)$

(ii) $(ku, v) = k(u, v)$ k is a scalar

(iii) $(u, u) = 0$ if $u = 0$ and $(u, u) > 0$ if $u \neq 0$

(iv) $(u + v, w) = (u, w) + (v, w)$

* Inner Product of two functions

The inner product of two functions f_1, f_2 on the interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x) \cdot f_2(x)\, dx$$

* orthognal functions

Two function f_1 and f_2 are orthognal on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) \cdot f_2(x)\, dx = 0$$

Ex. 1. (a) Show that the functions $f_1(x) = x^2$ & $f_2(x) = x^3$ are orthognal on the interval $[-1, 1]$

$$(f_1, f_2) = \int_{-1}^{1} f_1(x) \cdot f_2(x)\, dx$$

$$= \int_{-1}^{1} x^2 \cdot x^3\, dx = \int_{-1}^{1} x^5\, dx$$

$$= \left| \frac{x^6}{6} \right|_{-1}^{1} = \frac{1}{6}(1-1) = 0$$

(b) $f_1(x) = x^2$ & $f_2(x) = x^4$ over $[-1, 1]$

$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^4\, dx = \int_{-1}^{1} x^6\, dx = \left| \frac{x^7}{7} \right|_{-1}^{1} = \frac{2}{7} \neq 0$$

* Orthognal sets

A real valued functions $\{\phi_0(x), \phi_1(x), \ldots\}$ is said to orthognal on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0 \quad m \neq n$$

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Transform
Fourier Series

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Ex. Show that $\{1, \cos x, \cos 2x, \dots\}$ is orthogonal on the interval $[-\pi, \pi]$

(ii) Find norm of each function in the orthogonal set.

(iii) Find orthonormal set on the interval $[-\pi, \pi]$.

Sol:- Let $\phi_0(x) = 1$ and $\phi_n(x) = \cos nx$ we have to show $\int_{-\pi}^{\pi} \phi_0(x).\phi_n(x)\,dx = 0$ & $\int_{-\pi}^{\pi} \phi_n(x).\phi_m(x)\,dx = 0$; $m \neq n$

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\,dx = \int_{-\pi}^{\pi} 1.\cos nx\,dx = \left|\frac{\sin nx}{n}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{n}(\sin n(\pi) + \sin(n\pi)) = \frac{1}{n}(0+0) = 0$$

$$(\phi_n, \phi_m) = \int_{-\pi}^{\pi} \phi_n(x).\phi_m(x)\,dx = \int_{-\pi}^{\pi} \cos nx.\cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} 2\cos nx \cos mx\,dx$$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} (\cos(m+n)x + \cos(n-m)x)\,dx\right.$$

$$= \frac{1}{2}\left(\frac{\sin(m+n)x}{m+n} + \frac{\sin(n-m)x}{n-m}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{2}(0) = 0 \; ; \; m \neq n$$

Sol(ii) Let $\phi_0(x) = 1$

$$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} 1 \cdot 1 \cdot dx = |x|_{-\pi}^{\pi} = \pi - (-\pi) = 2\pi$$

& $\|\phi_0(x)\| = \sqrt{2\pi}$

For $\phi_n(x) = \cos nx$

$$\|\phi_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx)\, dx$$

$$= \frac{1}{2}|x|_{-\pi}^{\pi} + \frac{1}{2}\left|\frac{\sin 2nx}{2n}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{2}[\pi - (-\pi)] + \frac{1}{2}[0]$$

$$= \frac{1}{2}(2\pi) = \pi$$

& $\|\phi_n(x)\| = \sqrt{\pi}$

Sol(iii):- For the orthonormal set we divide each function to its norm

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots \right\}$$

which is orthonormal on $[-\pi, \pi]$

(*) A set of real valued function $\{\phi_0(x), \phi_1(x), \dots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on $[a,b]$ if

$$\int_a^b w(x)\, \phi_m(x) \cdot \phi_n(x)\, dx = 0 \; ; \; m \neq n$$

1 $x_1(t) =$
2 Sinwo
3 $f_1(x) =$
4 $f_1(x) =$
5 $f_1(x) =$