5.1

- $A x = \underset{\text{eigenvalue}}{\underset{\downarrow}{\lambda}} x \rightarrow$ eigen vector

$(\lambda I - A)x = 0$

- If and only if "$\underbrace{\det(\lambda I - A) = 0}_{\text{characteristic eq}}$", then λ is an eigenvalue

- Eigenvalues are main diagonals in UT, LT, diagonal matrices
- A is invertible if $\lambda = 0$ is not an eigenvalue

Q1- $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ -\lambda \end{bmatrix}$

$\lambda = -1$ only

Q5- (a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$\rightarrow \begin{vmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{vmatrix} = 0$

$(\lambda-1)(\lambda-3) - 8 = 0$

$\lambda^2 - 3\lambda - \lambda + 3 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$\lambda^2 - 5\lambda + \lambda - 5 = 0$

$\lambda(\lambda-5) + 1(\lambda-5) = 0$

$(\lambda+1)(\lambda-5) = 0$ ✓

$\lambda = -1$, $\lambda = 5$

$\lambda = -1$

$(-I - A) = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x_1 - 4x_2 = 0$

$-2x_1 - 4x_2 = 0$

$\begin{bmatrix} -2 & -4 & 0 \\ -2 & -4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 0 \\ -2 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 = s, \quad x_1 = -2s$

$\begin{bmatrix} -2s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ✓

EXAMPLE 7 | Eigenvectors and Bases for Eigenspaces

Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, or in factored form, $(\lambda - 1)(\lambda - 2)^2 = 0$ (verify). Thus, the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so there are two eigenspaces of A.

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$, or in matrix form,

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

In the case where $\lambda = 2$, Formula (6) becomes

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$$x_1 = -s, \quad x_2 = t, \quad x_3 = s$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent (why?), these vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

If $\lambda = 1$, then (6) becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields (verify)

$$x_1 = -2s, \quad x_2 = s, \quad x_3 = s$$

Thus, the eigenvectors corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{so that} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$.