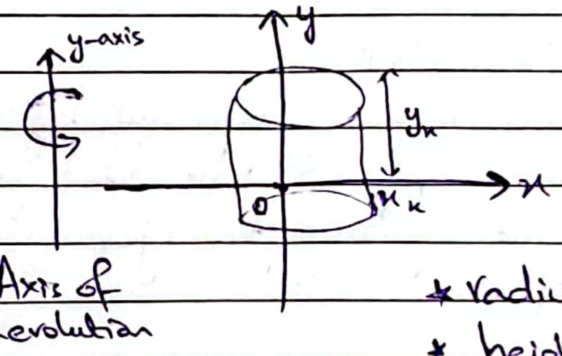
Dated: Wednesday 15th Nov,'23

Lecture

6.2 Volumes Using Cylindrical Shells

→ Slicing With Cylinders

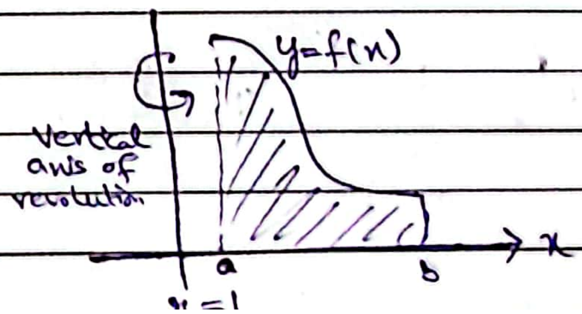
* Suppose, we slice through the solid using circular cylinders of increasing radii.
* Slicing straight down through the solid so that the axis of each cylinder is parallel to y-axis.
* The radii of the cylinders increase with each slice.



* radius = x_k
* height = y_k

→ The Shell Method

Suppose that the region bounded by the graph of a non-negative continuous function $y=f(x)$ and the x-axis over the finite closed interval $[a,b]$ lies to the right of the vertical line $x=L$. We generate a solid S by rotating this region about the vertical line 'L'.



Dated:

Shell Formula For Revolution About A Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y=f(x)\geq 0$, $L\leq a\leq x\leq b$, about a vertical line $x=L$ is,

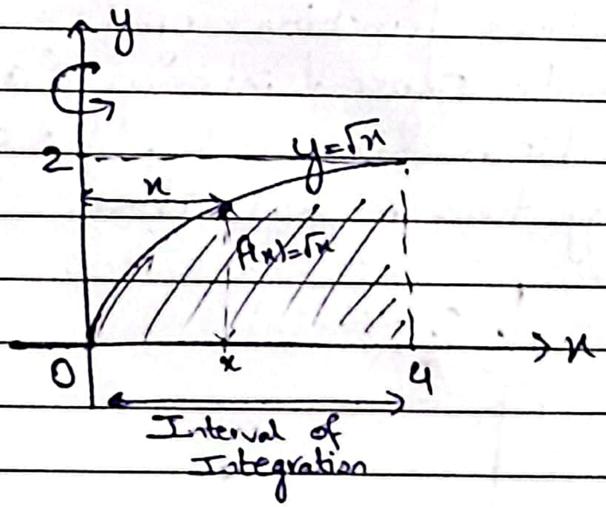
$$V=\int_a^b 2\pi\begin{pmatrix}\text{Shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx$$

Example#2

The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

Sketch:



(Distance from axis of Revolution)

* ↳ Shell Radius $=x$
* Shell Height $=y=\sqrt{x}$

Dated:

$$V = \int_a^b 2\pi \left(\begin{matrix}\text{Shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$

$$= \int_0^4 2\pi (x)(\sqrt{x})\, dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx$$

$$= 2\pi \left[\frac{2}{5} x^{5/2}\right]_0^4$$

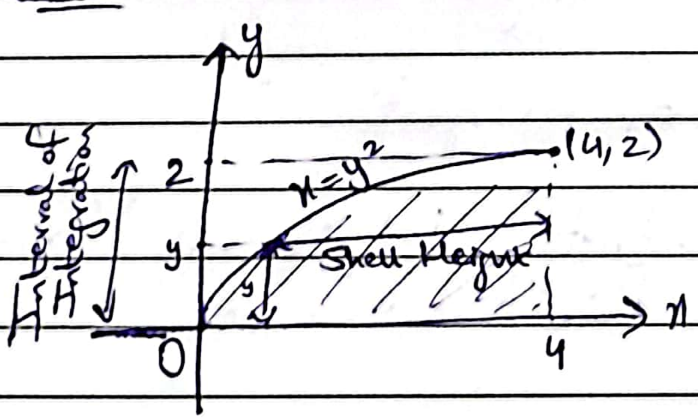
$$= \frac{128\pi}{5}$$

→ Example # 3

The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method

Solution

Sketch:



* Shell Height $= 4 - y^2$
* Shell Radius $= y$

Dated:

$$V = \int_a^b 2\pi \begin{pmatrix}\text{Shell}\\ \text{Radius}\end{pmatrix}\begin{pmatrix}\text{Shell}\\ \text{Height}\end{pmatrix} dy$$

$$= \int_0^2 2\pi (y)(4-y^2)\,dy$$

$$= 2\pi \int_0^2 (4y-y^3)\,dy$$

$$= 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2$$

$$= 8\pi.$$

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* SUMMARY OF THE SHELL METHOD *

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for the shell method are:

1) Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the shell height and shell radius.

2) Find the limits of integration.

3) Integrate the product 2π(shell radius)(shell height) w.r.t 'x' or 'y' to find the volume.

Dated:

6.3 ARC LENGTH

LENGTH OF A CURVE $y=f(x)$:

→ Suppose the curve whose length we want to find is the graph of the function $y=f(x)$ from $x=a$ to $x=b$.
Inorder to derive an integral formula for the length of the curve, we assume that 'f' has a continuous derivative at every point of $[a,b]$. Such a function is called SMOOTH and its graph is a smooth curve because it does not have any breaks.

DEFINITION

→ If f' is continuous on $[a,b]$, then the arc length of the curve $y=f(x)$ from the point $A=(a,f(a))$ to the point $(b,f(b))$ is the value of the integral

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Dated:

→ Example#1

find the length of the curve, which is the graph of the function

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \quad , \quad 0 \le x \le 1$$

Solution:

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \times \frac{3}{2} x^{1/2} = 2\sqrt{2}\, x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}\, x^{1/2}\right)^2 = 8x$$

→ The length of the curve over $x=0$ to $x=1$ is,

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1+8x}\, dx$$

$$= \frac{2}{3} \cdot \frac{1}{8} (1+8x)^{3/2} \Big|_0^1$$

$$= 13/6 \approx 2.17$$

Dated:

→ Example #2
Find the length of the graph of:

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \quad , \quad 1 \le x \le 4$$

Solution

$$\frac{dy}{dx} = \frac{1}{12} \times 3x^2 - \frac{1}{x^2}$$

$$= \frac{x^2}{4} - \frac{1}{x^2}$$

$$* \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2$$

$$= 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}$$

$$= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

→ The length of the graph over $[1,4]$ is:

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

Dated:

$$\Rightarrow L = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4$$

$$= \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{72}{12}$$

$$= 6$$

→ **Dealing With Discontinuities In dy/dx:**
Even if the derivative dy/dx does not exist at some point on a curve, it is possible that dx/dy could exist. This can happen, for example, when a curve has a vertical tangent.

→ **Formula For The Length Of $x = g(y)$, $c \le y \le d$**

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B(g(d), d)$ is,

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Dated:

→ Example #3
Find the length of the curve $y = (x/2)^{2/3}$ from $x=0$ to $x=2$.

Solution

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x=0$. So, we cannot find the curve's length.

→ We rewrite the equation to express 'x' in terms of 'y',

$$y^{3/2} = \frac{x}{2}$$

$$\Rightarrow x = 2y^{3/2}$$

→ When $x=0 \Rightarrow y=0$
→ When $x=2 \Rightarrow y=1$

→ $\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$ is continuous on $[0,1]$.

$$\Rightarrow L = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1+9y}\, dy$$

$$= \frac{1}{9} \times \frac{2}{3}(1+9y)^{3/2}\Big|_0^1 = \frac{2}{27}(10\sqrt{10}-1) \approx 2.27$$

Dated:

→ **Example # 4**

Find the arc length for the curve

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \quad , \quad 1 \le x \le 4$$

taking $A = (1, 13/12)$ as the starting point.

Solution

→ (Example # 2)

$$1 + [f'(x)]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\Rightarrow \text{Arc length} = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt = \int_1^x \left(\frac{t^4}{4} + \frac{1}{t^2}\right) dt$$

$$= \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}$$

→ To compute the arc length along the curve from $A = (1, 13/12)$ to $B(4, 67/12)$, for instance,

$$\text{Arc length} = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12} = 6.$$