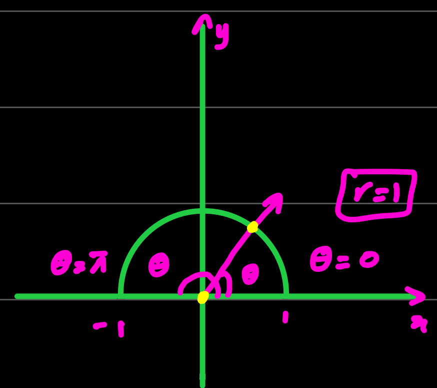
Double Integrals In Polar form
(15.4)

- $$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Q. Evaluate $\iint_R e^{x^2+y^2}\, dy\, dx$ where R is the semi circular reigon bounded by the x-axis and the curve $\sqrt{1-x^2}$

* Difficult to solve in cartesian form, so convert into polar form.

$\underline{\underline{r^2 = x^2+y^2}} \rightarrow \iint_R e^{x^2+y^2}\, dy\, dx$



$$= \iint_R e^{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{1} r.e^{r^2}\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\left[e^{r^2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{\pi} e-1\, d\theta = \frac{1}{2}\left[\theta.e-\theta\right]_0^{\pi} = \frac{1}{2}(\pi.e-\pi) = \underline{\underline{\frac{\pi}{2}(e-1)}}$$

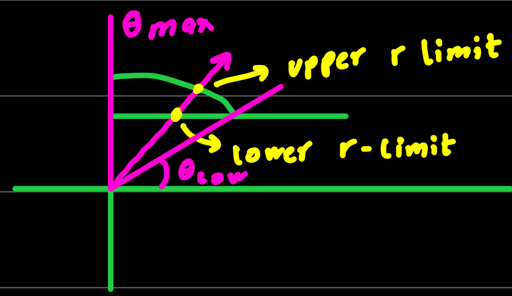
→ Finding limits

① Sketch the graphs

② Find r-limits

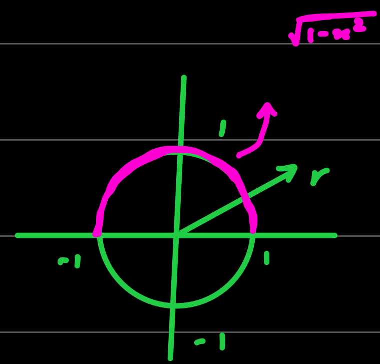
③ Find θ-Limits ; draw rays for min and max θ

④ Also convert the function in terms of r.



Q- Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cdot r \, dr\, d\theta$$



r=0
r=1

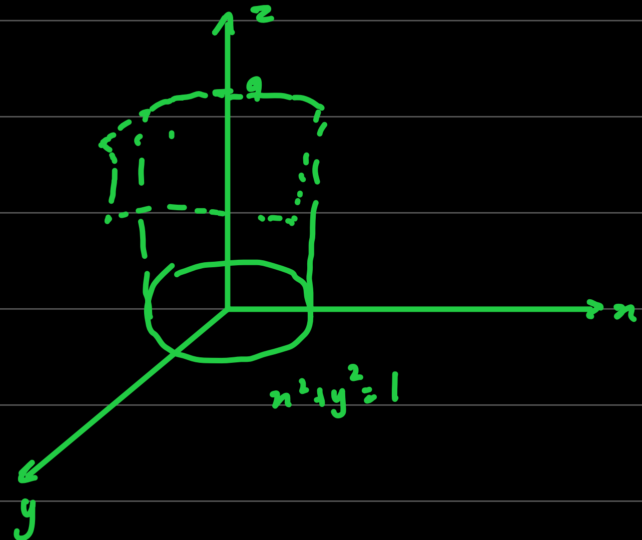
θ=0
θ=2π

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^1$$

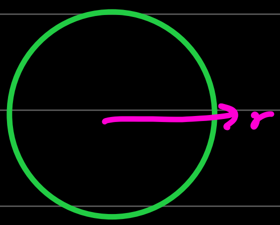
$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\theta = \frac{\pi}{8}$$

Q- Find volume bounded above by paraboloid $z = 9 - x^2 - y^2$, and below by unit circle in xy plane.

$$\iint_R 9 - x^2 - y^2 \, dy \, dx$$



· See shadow in xy plane and act on it.



$$= \int_0^{2\pi} \int_0^1 (9 - r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 9r - r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{1}{4} \right) d\theta$$

$$= \left[\frac{17}{4} \theta \right]_0^{2\pi} = \frac{17}{\cancel{4}_2} \times \cancel{2}^1 \pi = \frac{17}{2} \pi$$

Q- Find Area of reigon R in xy plane enclosed by circle $x^2+y^2=4$, above line $y=1$, and below line $y=\sqrt{3}\,x$

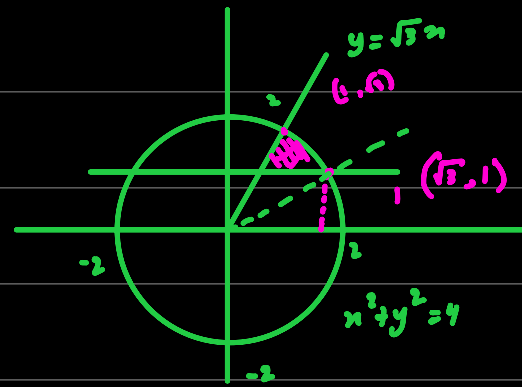
$y = r\sin\theta$

$r = \csc\theta$ — ①

$r = 2$ — ②

$\theta = \frac{\pi}{6}$ — ①

$\theta = \frac{\pi}{3}$ — ②



$x = r\cos\theta$

$y = r\sin\theta$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\frac{1}{2}r^2\right]_{\csc\theta}^{2} d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2 - \frac{\csc^2\theta}{2}\,d\theta$$

$$= \left[2\theta + \frac{\cot\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \cancel{\frac{2\pi}{3}} + \frac{1}{2\sqrt{3}} - \cancel{\frac{\pi}{3}} - \frac{\sqrt{3}}{2}$$

$2 \,|\, 2\sqrt{3} - 2$

$\sqrt{3} \,|\, \sqrt{3} - 1$

$1 - 1$

$$= \frac{\pi}{3} + \frac{1-3}{2\sqrt{3}}$$

$$= \frac{\pi}{3} - \frac{1}{\sqrt{3}} = \frac{\pi\sqrt{3} - 3}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{3\pi - 3\sqrt{3}}{9} = \underline{\underline{\frac{\pi - \sqrt{3}}{3}}} \checkmark$$

Triple Integrals (15.5)

- $\iint_R \overbrace{f(x,y)}^{\text{height}} \overbrace{dx\,dy}^{\text{Area}}$ = Volume under the surface ; when $f(x,y) = 1$, and integrate then Area is given

- $\iiint_D \overbrace{f(x,y,z)}^{\text{height}} \overbrace{dz\,dy\,dx}^{\text{volume}}$ = hyper volume

$\rightarrow$ $f(x,y,z) = 1$

↳ $\iiint 1 \, dz\,dy\,dx$ = $\underline{\underline{\text{volume}}}$

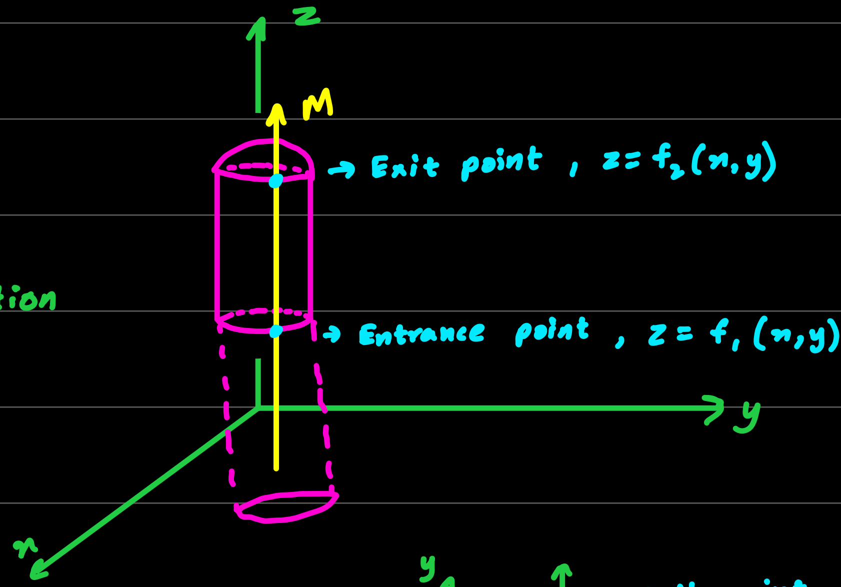
· **Steps:**

① sketch domain + shadow in xy plane

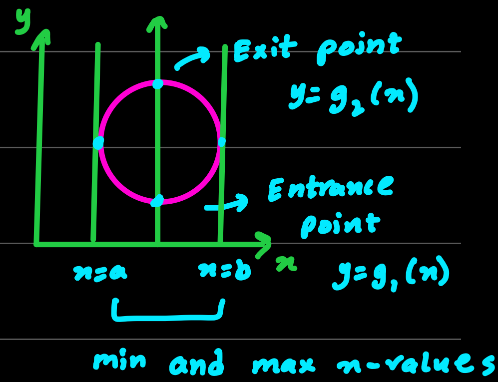
② Verticle line M, parallel and towards to the axis of 1st integration

③ Verticle line, parallel and towards to the axis of 2nd integration

④ x-limits / limits for final integration



$$\text{Volume} = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} 1 \, . \, dz\,dy\,dx$$



Q- Evaluate $\int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,x$

$$= \int_0^1 \int_x^1 y - x \; dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2} - yx \right]_x^1$$

$$= \int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} + x^2 \right) dx$$

$$= \left[\frac{1}{2}x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^3}{3} \right]_0^1$$

$$= \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} - \frac{1}{6} + \frac{1}{3} = \frac{-1+2}{6} = \frac{1}{6}$$

Q- Let S be the sphere of radius 5 centered at the origin, and let D be the reigon under the sphere that lies above the plane $z=3$. Set up limits for integration

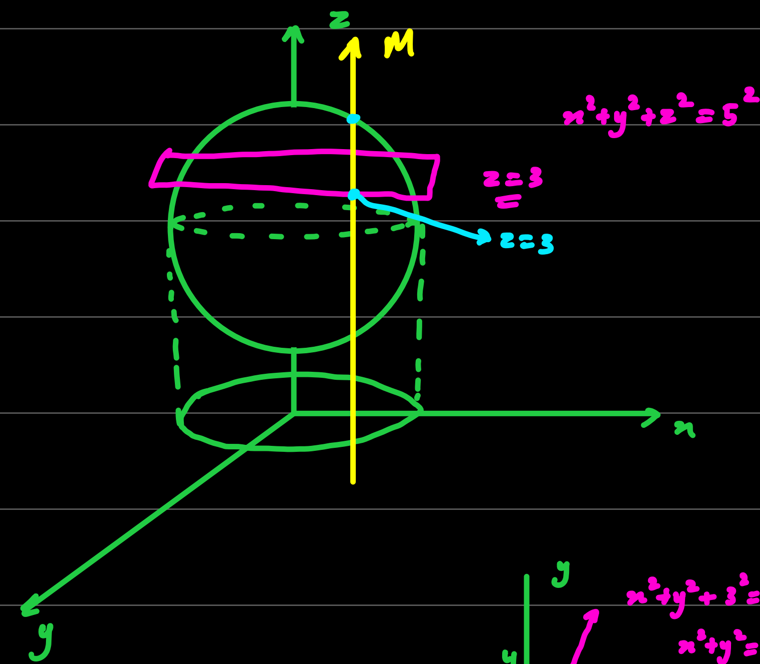
z - limits :

Enter : $z=3$

Exit : $x^2+y^2+z^2=25$

$$z = \pm\sqrt{25-x^2-y^2}$$

$$z = \sqrt{25-x^2-y^2}$$ going towards +ve z-axis



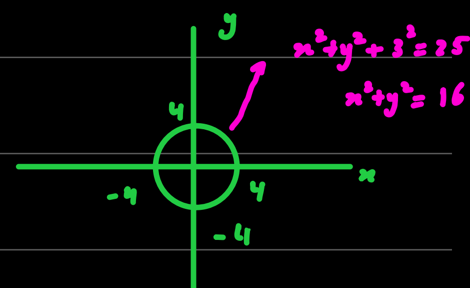
y- limits

→ $x^2+y^2=16$

$y = \pm\sqrt{16-x^2}$

$y=\sqrt{16-x^2}$ - Exit

$y=-\sqrt{16-x^2}$ - Entrance

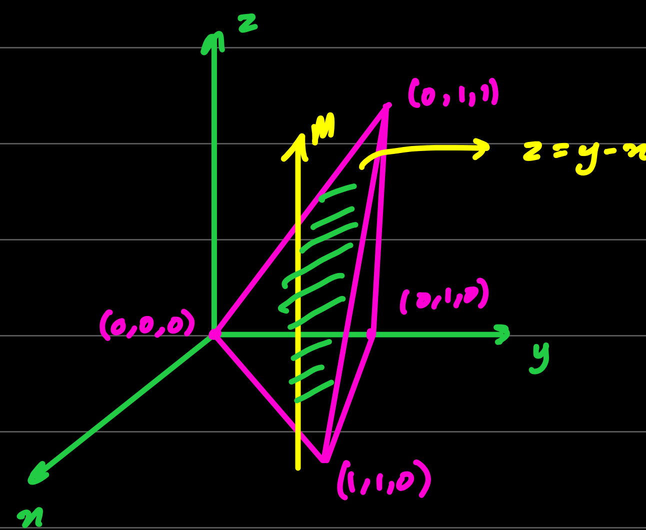


x- limits

$x=-4$, $x=4$

$$\boxed{\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{3}^{\sqrt{25-x^2-y^2}} f(x,y,z)\; dz\; dy\; dx}$$

Q- Set up limits of integration for evaluating the triple integral of a function over the tetrahedron D whose vertices are $(0,0,0)$, $(1,1,0)$ $(0,1,0)$, $(0,1,1)$. Use order of integration $dz\,dy\,dx$, and then the order $dy\,dz\,dx$



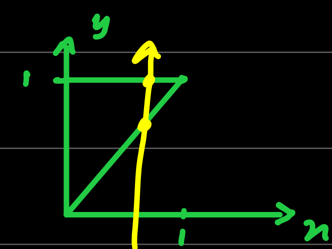
z - limits , y - limits , x limits

$z_1=0$, $z_2=y-x$ $\qquad$ $y_1=x$ $\qquad$ $x=0$, $x=1$

$y_2=1$

$$\int_0^1\int_x^1\int_0^{y-x} 1\; dz\; dy\; dx$$

$$=\int_0^1\int_x^1 (y-x)\; dy\; dx$$



$$= \int_0^1 \left[\frac{y^2}{2} - yx \right]_x^1 dx$$

$$= \int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} + x^2 \right) dx$$

$$= \left[\frac{1}{2}x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^3}{3} \right]_0^1$$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{6} + \frac{1}{3} = \frac{1}{6} \checkmark$$

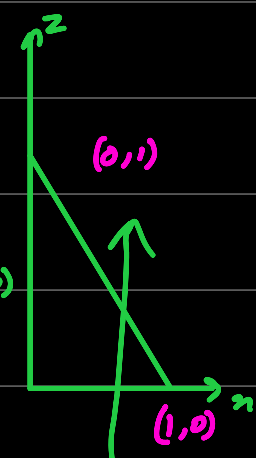
1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$

The derivatives of the other trigonometric functions:

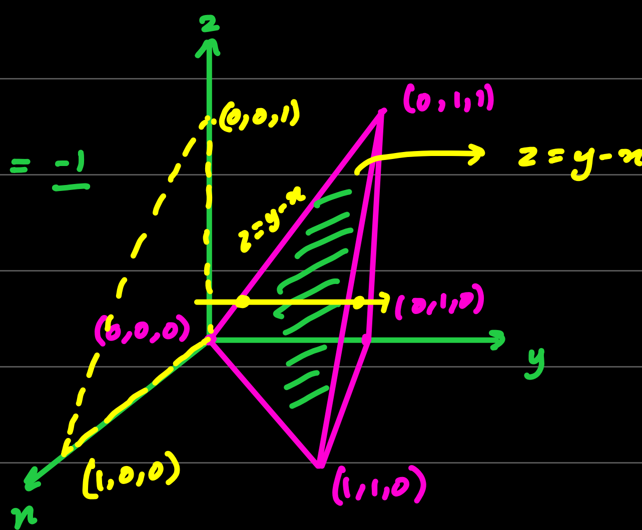
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$dy\ dz\ dx \rightarrow$



$\frac{0-1}{1-0} = -1$



y-limits , z-limits

$z = y - x$ $\quad z_1 = 0$

$y_1 = z + x$ $\quad z_2 = 1 - x$ $\quad z - 1 = -1(x - 0)$

$y_2 = 1$ $\quad z = -x + 1$

x - limits

$x_1 = 0 \quad , x_2 = 1$

$$\iiint 1.\, dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x} \int_{z+x}^1 dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x} (1-z-x)\, dz\, dx$$

$$= \int_0^1 \left[z - \frac{z^2}{2} - xz \right]_0^{1-x}$$

$$= \int_0^1 1-x - \frac{(1-x)^2}{2} - x(1-x) \quad dx$$

$$= \int_0^1 1-x - \frac{(1-2x+x^2)}{2} - x+x^2\, dx$$

$$= \int_0^1 1 - \cancel{x} - \frac{1}{2} + \cancel{x} - \frac{x^2}{2} - x + x^2\, dx$$

$$= \int_0^1 \frac{1}{2} - x + \frac{x^2}{2} \quad dx$$

$$= \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right]_0^1$$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6} - 0$$

$$= \underline{\underline{\frac{1}{6}}}$$