Dated: Monday 20th Nov., '23

Lecture#

7.1 Inverse Functions And Their Derivatives–

→ One-To-One Functions–

* A function 'f' is one-to-one if each 'x' in the domain has exactly one image in the range.

* A function that has distinct values at distinct elements in its domain is called one-one

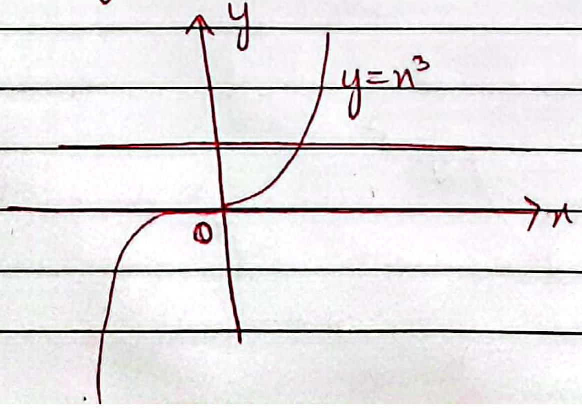
* A function $f(x)$ is one-one on a domain 'D' if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in 'D'.

→ The Horizontal Line Test for One-One Functions–

A function $y=f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

→ ~~Example#1~~

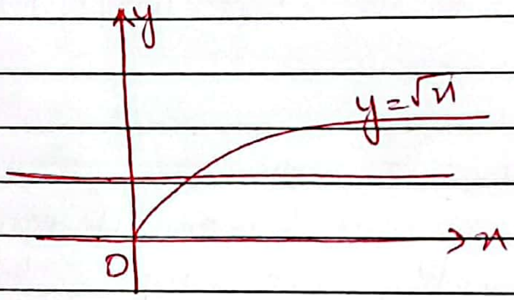
~~e.g~~ * $y=x^3$



One-to-One function

Dated:

* $y=\sqrt{x}$ is one-to-one on any domain of non-negative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$

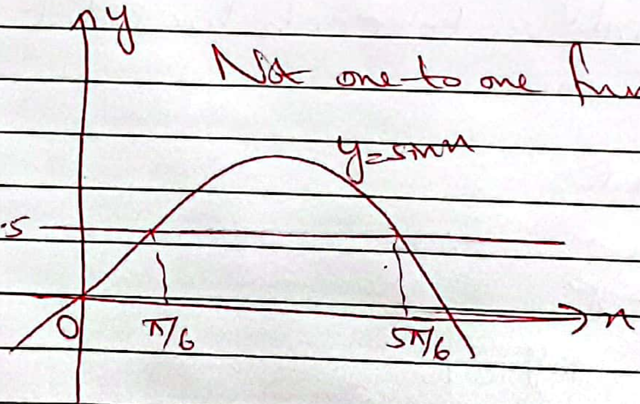


One-to-One function.

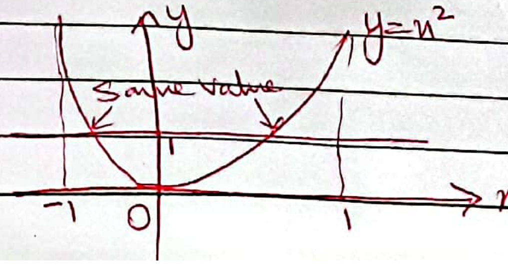
* $g(x) = \sin x$ is not one-to-one on the interval $[0,\pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$

* $g(x) = \sin x$ is one-to-one on the interval $[0,\pi/2)$ because it is an increasing function and, therefore, gives distinct outputs for distinct inputs in the interval.

* Not one-to one function.



* Not one-to-one function



Dated:

→ Inverse Functions

Suppose that 'f' is a one-to-one function on a domain D with range R.
The Inverse function f^{-1} is defined by,

$$f^{-1}(b) = a \quad \text{if } f(a) = b$$

The domain of f^{-1} is R and the range of f^{-1} is D.

→ Example #2

Suppose a one-to-one function $y = f(x)$ is given by a table of values:

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can be obtained by simply interchanging the values in each column of the table for 'f'.

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

Dated:

→ FINDING INVERSES.

1) Solve the equation $y = f(x)$ for 'x'. This gives a formula $x = f^{-1}(y)$, where 'x' is expressed as a function of 'y'.

2) Interchange 'x' and 'y', obtaining a formula $y = f^{-1}(x)$, where f^{-1} is expressed in the conventional format – 'x' as the independent variable and 'y' as the dependent variable.

→ EXAMPLE #3
Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of 'x'.

SOLUTION

1) Solve for 'x' in terms of 'y',

$$y = \frac{1}{2}x + 1$$

$$2y = x + 2$$

$$\Rightarrow x = 2y - 2.$$

2) Interchange 'x' and 'y',

$$\Rightarrow y = 2x - 2$$

⇒ The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$

Dated:

* Checking:

Verify that both compositions give the identity function

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x+1\right)-2 = x+\cancel{2}-\cancel{2} = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x-2)+1 = x-\cancel{1}+\cancel{1} = x.$$

→ EXAMPLE # 4

find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of 'x'.

SOLUTION

→ For $x \geq 0$, the graph satisfies the horizontal line test

⇒ The function is one-to-one and has an inverse.

1) Solve for 'x' in terms of 'y',

$$y = x^2$$

$$\Rightarrow \sqrt{y} = x.$$

2) Interchange 'x' and 'y'.

$$\Rightarrow y = \sqrt{x} \quad , \quad x \geq 0.$$

Dated:

→ The Derivative Rule For Inverses

If 'f' has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f).

The value $(f^{-1})'$ at a point 'b' in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$.

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or,

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

→ Example#5

The function $f(x) = x^2$, $x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and, $(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$.

Solution

Verify: $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Dated:

$$(f^{-1})'(x) = \frac{1}{2(f^{-1}(x))}$$

$$= \frac{1}{2\sqrt{x}} \quad \text{(verified!)}$$

Let, $x = 2 \implies f(2) = 4$ and $f'(2) = 4$

$(f^{-1})'$ at $f(2) = (f^{-1})'(4)$. * derivative of 'f' at 2 is '4'

$$\implies (f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))}$$

* derivative of 'f^{-1}' at $f(2)$ is '4'.

$$= \frac{1}{f'(2)}$$

$$= \frac{1}{4}.$$

→ Example #6

Let, $f(x) = x^3 - 2$, $x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution:

$$f'(x) = f'(2) = 3x^2$$
$$= 3(2)^2$$
$$= 12.$$

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$(f^{-1})'(x)$ at $x = f(2) = \dfrac{1}{f'(f^{-1}(x))}$

$= \dfrac{1}{12}$