Lec-8

~~Topic 3.1 Tangent Lines and the Derivative at a Point~~

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Topic 2.1 Rates of change and Tangent lines to Curves

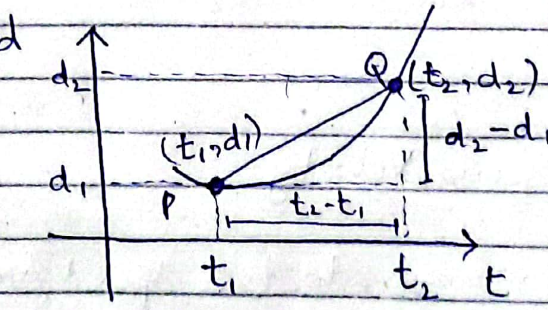
(I) Average Speed
or
Average Velocity: → this is measured / calculated over the interval of time $[t_1, t_2]$

$$V_{av} = \frac{\text{distance travelled}}{\text{time elapsed}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

$$\boxed{V_{av} = \frac{d_2 - d_1}{t_2 - t_1}}$$

- Average velocity → Slope of secant line.

- Average velocity is defined as the dist. travelled over a given time period.



In general, distance 'd' is the function of time 't'

$$d = f(t)$$

$$\boxed{V_{av} = \frac{d_2 - d_1}{t_2 - t_1} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}}$$

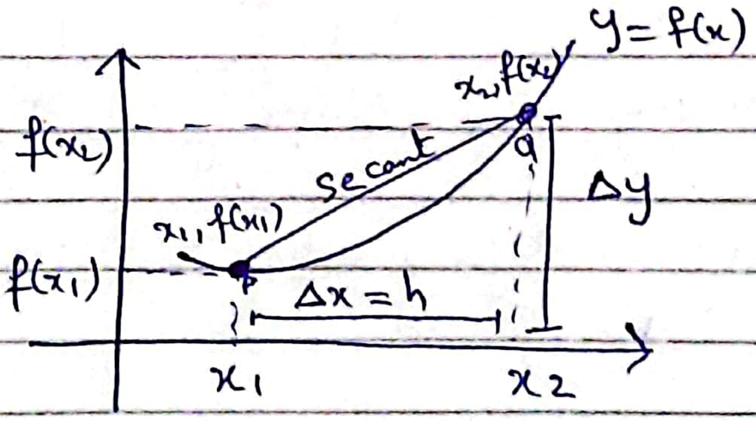
- Average velocity is just the slope of the Secant line joining the points (t_1, d_1) and (t_2, d_2)

(II) Average Rate of Change :

If $y = f(x)$ then avergerate of change of y w.r.t x over the interval $[x_1, x_2]$ is the slope of Secant line (m_{sec}) joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f

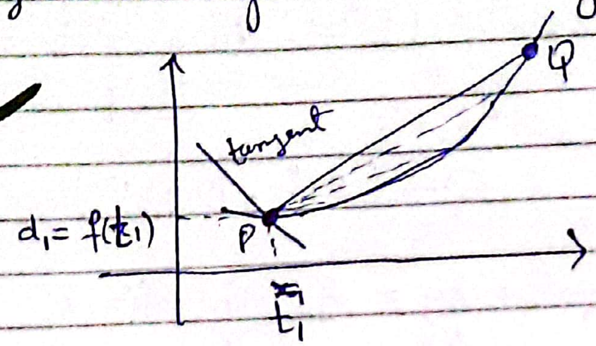
Slope of secant line $\leftarrow m_{sec} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h}, \quad h \neq 0}$$



III Instantaneous Velocity :

→ From average velocity, we can find instantaneous velocity. As $Q \to P$, the secant line becomes a tangent line and we get the slope of tangent line from the slope of secant line.



$$\boxed{\cancel{m_{tan} = \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}}$$

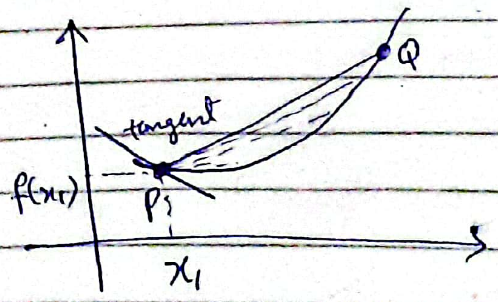
As t_2 gets very close to t_1, our approximation of inst. velocity will get better

$$\boxed{V_{ins} = \lim_{t_2 \to t_1} v_{av} = \lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1}}$$

IV Instantaneous Rate of Change :

If $y = f(x)$ then the instantaneous rate of change of y w.r.t x at the point x_1 is the slope of the tangent line m_{tan} to the graph of f at the point x_1 i.e

$$\boxed{m_{tan} = \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$



Example 3 → from book.

Find slope of tangent line of $y = x^2$ at $(2,4)$ by analyzing slope of secant line through $(2,4)$. Also write eq. for the tangent line to the parabola.

Sol: $P \to (2,4)$, $Q \to (2+h, (2+h)^2)$

$$\text{Secant line slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h}$$

$$= \frac{(2+h)^2 - 2^2}{h}$$

$$= \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$

$$= h + 4.$$

$$\text{Tangent line slope} = \lim_{h \to 0} \frac{\Delta y}{\Delta x}$$

$$m = \lim_{h \to 0} h + 4 = 4.$$

Eq. for tangent line to the parabola at this point is

$y - y_1 = m(x - x_1)$ → Point slope eq.

$$y - 4 = 4(x - 2)$$

$$y = 4 + 4x - 8$$

$$\boxed{y = 4x - 4}$$

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Topic 3.1 Tangent lines and the Derivative at a Point

① **Slope of curve** (Slope of tangent line at P).

If $y = f(x)$ at the point $P(x_0, f(x_0))$

then
$$\boxed{\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Example 1 (a) Find slope of the curve $y = \frac{1}{x}$ at $x = a \neq 0$
What is the slope at the point $x = -1$.

Sol:
$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \frac{\cancel{a} - \cancel{a} - \cancel{h}}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = \frac{-1}{a^2} \quad \text{Ans}$$

As $x = a \Rightarrow a = -1$.

slope at $-1 = \frac{-1}{(-1)^2} = -1$. Ans

$\longleftrightarrow$

(2) Rate of Change : Derivative of f at a point

$$\boxed{f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Summary

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

(1) Slope of graph / $y = f(x)$ at $x = x_0$

(2) Slope of tangent line / to the curve $y = f(x)$ at $x = x_0$

(3) The rate of change of $f(x)$ w.r.t x at $x = x_0$.

(4) The derivative $f'(x_0)$ / at $x = x_0$.

$\longleftrightarrow$

Topic 3.2 The Derivative as a Function

① Derivative of function f at each point x in the domain

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

In other words if the derivative of a function exists at all points of its domain then $f(x)$ is said to be differentiable.

- The process of calculating a derivative is called differentiation.

② Alternative Formula for Derivative

$$\boxed{f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}}$$

where $z = x+h$ and
$h = z - x$.

Example 1: $f(x) = \dfrac{x}{x-1}$ $f'(x) = ?$

Sol: $f(x+h) = \dfrac{(x+h)}{(x+h)-1}$ $\Rightarrow f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \frac{-\cancel{h}}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

Example 2 a) Find $f'(x)$ if $f(x) = \sqrt{x}$.

b) Find tangent line of $y = \sqrt{x}$ at $x = 4$. (from book). $(4, 2)$.

③ Differentiable on an Interval

(i) → Derivative at endpoints of closed interval are one sided limits

(ii) → $y = f(x)$ is differentiable on an open interval if it has a derivative at each point of the interval

(i) It is differentiable on a closed interval $[a, b]$ if it is diff. on the interior (a, b) and if limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \rightarrow \text{Right hand derivative at 'a'}$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \rightarrow \text{Left hand derivative at 'b'}$$

Topic 3.2 Continuation

Example 4 Show $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Sol: $x > 0$ $\frac{d}{dx}(x) = 1$

$x < 0$ $\frac{d}{dx}(-x) = -1$

At $x = 0$

R.H.D at $x=0$ $= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$

$= \lim_{h \to 0^+} \frac{|a+h| - |a|}{h} = \lim_{h \to 0^+} \frac{a+h-a}{h}$

$= 1$

L.H.D $= \lim_{h \to 0^-} \frac{|a+h| - |a|}{h}$

$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$ $\lim_{h \to 0^-} \frac{-h}{h} = -1$

$\longleftrightarrow$

Pg 110 – When does a function not have a derivative at a point? (DIY)

(4) Differentiable functions are Continuous:

If f has a derivative at $x = c$ then f is cont. at $x = c$

Note: A differentiable func. is always cont. but a cont. function may not always be differentiable..

$\longleftrightarrow$

Topic 3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, rational functions and certain combinations of them, simply and directly without having to take limits each time.

① Derivative of a Constant Function

If $y = f(x) = c$ then

$$\boxed{\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(c) = 0}$$

② Derivative of a Positive Integer Power
or
Power Rule

If n is any real number then

$$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$$

$\forall x$ where the powers of x^n and x^{n-1} are defined

Example 1:

c) $x^{\sqrt{2}}$.

$= \frac{d}{dx} x^{\sqrt{2}} = \sqrt{2}\, x^{\sqrt{2}-1}$ Ans

Example 1: f) $\sqrt{x^{2+\pi}}$

$$= \frac{d}{dx}\sqrt{x^{2+\pi}} = \frac{d}{dx}\left(x^{2+\pi}\right)^{1/2}$$

$$= \frac{d}{dx}\left(x^{1+\pi/2}\right)$$

$$= \left(\frac{1+\pi}{2}\right) x^{1+\frac{\pi}{2}-1}$$

$$= \frac{2+\pi}{2} x^{\pi/2} = \frac{1}{2}(2+\pi)\sqrt{x^{\pi}}$$

Ans

↔

(3) Derivative Constant Multiple Rule

If u is a differiable function of x and c is a constant then

$$\boxed{\frac{d}{dx}(cu) = c\frac{du}{dx}}$$

Example 2:

a) $y = 3x^2$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2)$$

$$= 3(2x) = 6x$$

Ans

↔ .

(4) Derivative Sum Rule

If u & v are diff func. of x then their sum u+v is diff at every point

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}}$$

(5) Derivative Product Rule

If u and v are diff at x then their product uv

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v}$$

or

$$\boxed{(uv)' = uv' + u'v}$$

Example 5: Find $\frac{dy}{dx} = (x^2+1)(x^3+3)$.

$= (x^2+1)\ (3x^2) + (2x)(x^3+3)$

$= 3x^4 + 3x^2 + 2x^4 + 6x$

$= 5x^4 + 3x^2 + 6x.$ Ans.

or

$= \frac{d}{dx} x^5 + 3x^2 + x^3 + 3. \quad = 5x^4 + 6x + 3x^2.$

$= 5x^4 + 3x^2 + 6x$

$\longleftrightarrow$

(6) Derivative Quotient Rule

If u and v are diff at x and if $v(x) \neq 0$ then quotient u/v

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Example 7: $y = \frac{(x-1)(x^2-2x)}{x^4}$ $\quad \frac{dy}{dx} = ?$

Sol. $y = \frac{(x^3 - 3x^2 + 2x)}{x^4} \Rightarrow \frac{dy}{dx} = \frac{x^4(3x^2-6x+2) - (x^3-3x^2+2x)(4x^3)}{x^8}$

$= \frac{3x^6 - 6x^5 + 2x^4 - 4x^6 + 12x^5 - 8x^4}{x^8}$

$= \frac{-x^6 + 6x^5 - 6x^4}{x^8} = \frac{-1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$

Ans

7 Higher Order Derivative

$$\boxed{y^{(n)} = \frac{d}{dx}\left(y^{(n-1)}\right)} = \frac{d^n y}{dx^n} = D^n y$$

denoting the nth derivative of y with respect to x for any positive integer n.

Example 8: Find first four derivatives of $y = x^3 - 3x^2 + 2$

Sol:

$$y' = 3x^2 - 6x$$
$$y'' = 6x$$
$$y''' = 6$$
$$y^{(iv)} = 0$$

Topic 3.4 : The Derivative as a Rate of Change

① Instantaneous Rate of Change of f (w.r.t x at x_0)

$$\boxed{f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

Thus instantaneous rates are limits of average rates.

Example 1: Area of a circle is related to its diameter D

$$A = \frac{\pi}{4} D^2$$

How fast does the area change w.r.t diameter D when D is 10m?

Sol:

$$\frac{dA}{dD} = \frac{\pi}{4}(2D)$$

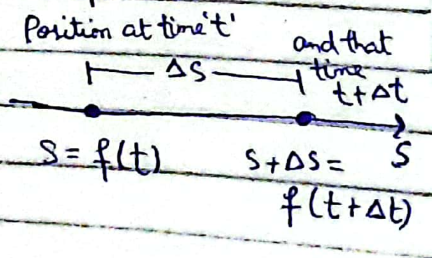
$$= \frac{\pi D}{2}$$

When $D = 10m \Rightarrow \frac{\pi(10)}{2} = 5\pi \ m^2/m$. Ans

② Motion Along a Line: Displacement, Velocity, Speed, Acceleration and Jerk

Suppose that an object is moving along a coordinate line usually horizontal or vertical so that we know its position 's' on that line as a function of time 't'

$$\boxed{s = f(t)}$$



(i) Displacement → of the object over time interval from t to $t+\Delta t$

$$\boxed{\Delta s = f(t+\Delta t) - f(t)}$$

(ii) Average Velocity

$$\boxed{V_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}}$$

(iii) Velocity (instantaneous velocity)

$$\boxed{v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}}$$

(iv) Speed – absolute value of velocity.

$$\boxed{\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|}$$

(v) Acceleration – derivative of velocity w.r.t time.

$$\boxed{a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}}$$

(vi) Jerk – derivative of acceleration w.r.t time.

$$\boxed{j(t) = \frac{da}{dt} = \frac{d^2 v}{dt^2} = \frac{d^3 s}{dt^3}}$$

Topic 3.5 Derivatives of Trigonometric Functions

① Derivative of the Sine Function:

$$\boxed{\frac{d}{dx}(\sin x) = \cos x.}$$

Example 1:

(c) $y = \frac{\sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(\cos x) - \sin x(1)}{x^2}$

$= \frac{x\cos x - \sin x}{x^2}$ Ans

② Derivative of the Cosine Function:

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example 2:

(c) $y = \frac{\cos x}{1-\sin x}$ $\frac{dy}{dx} = ?$

$\Rightarrow \frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$= \frac{\cancel{1} - \sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}$ Ans

③ Derivatives of Other Trigonometric Functions:

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}, \quad \boxed{\frac{d}{dx}(\cot x) = -\csc^2 x}$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}, \quad \boxed{\frac{d}{dx}(\csc x) = -\csc x \cot x}$$

Example 6: Find y'' if $y = \sec x$

$$y' = \sec x \tan x.$$

$$y'' = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x. \quad \text{Ans}$$

Topic 3.6: The Chain Rule

1) Theorem 2 – Chain Rule

If $y = f(u)$ and $u = g(x)$ then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Example 2: $x(t) = \cos(t^2+1)$. Find velocity?

Sol: Let

$x = \cos u$ and $u = t^2+1$.

$\frac{dx}{dt} = v = ?$

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt} = -\sin u \cdot (2t)$$

$$= -2t\sin u = -2t\sin(t^2+1)$$

Ans

(I) Direct Way to Write Chain Rule

$$\boxed{\frac{d}{dx} f(u) = f'(u)\, \frac{du}{dx}}$$

or $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

Example 3: Diff $\sin(x^2+x)$ w.r.t 'x'

Sol: $\frac{d}{dx}\left(\sin(x^2+x)\right) = \cos(x^2+x)\cdot(2x+1)$

$= 2x+1 \cos(x^2+x)$

Ans

Example 4 Find $g'(t)$ if $g(t) = \tan(5 - \sin 2t)$.

Sol:

$$\frac{d}{dt} g(t) = \sec^2(5 - \sin 2t)(-\cos 2t \cdot 2)$$

$$= \sec^2(5 - \sin 2t)(-2\cos 2t)$$

$$= -2\cos 2t \sec^2(5 - \sin 2t) \quad \text{Ans}$$

(2) The Chain Rule with Powers of a Function

$$\boxed{\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}}$$

Example 5: (b)

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -1(3x-2)^{-2}(3)$$

$$= \frac{-3}{(3x-2)^2} \quad \text{Ans}$$

Example 7: Find slope of $y = \frac{1}{(1-2x)^3}$.

Sol:

$$\frac{dy}{dx} = \frac{d}{dx}\frac{1}{(1-2x)^3} = -3(1-2x)^{-4}(-2)$$

$$= \frac{6}{(1-2x)^4} \quad \text{Ans}$$