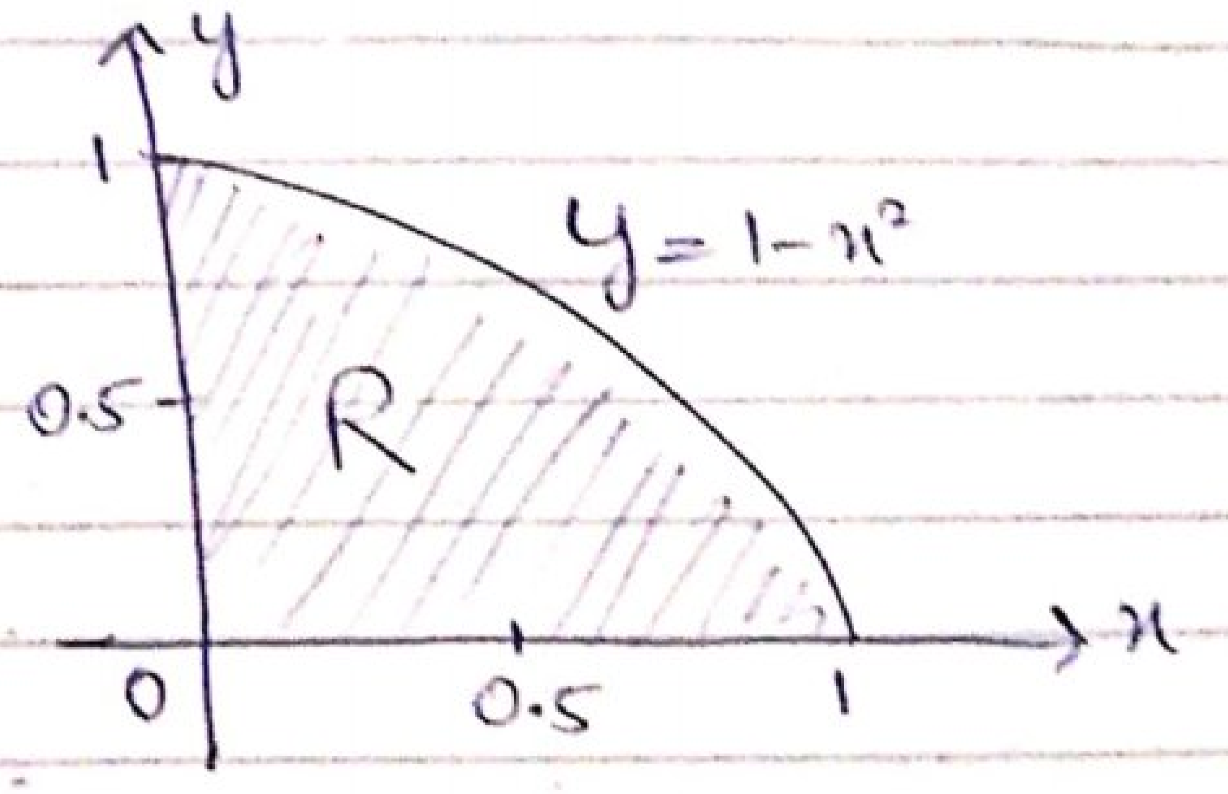
Chapter #5 → INTEGRALS

→ The definite integral is defined as a limit or the area under a curve between two fixed limits.

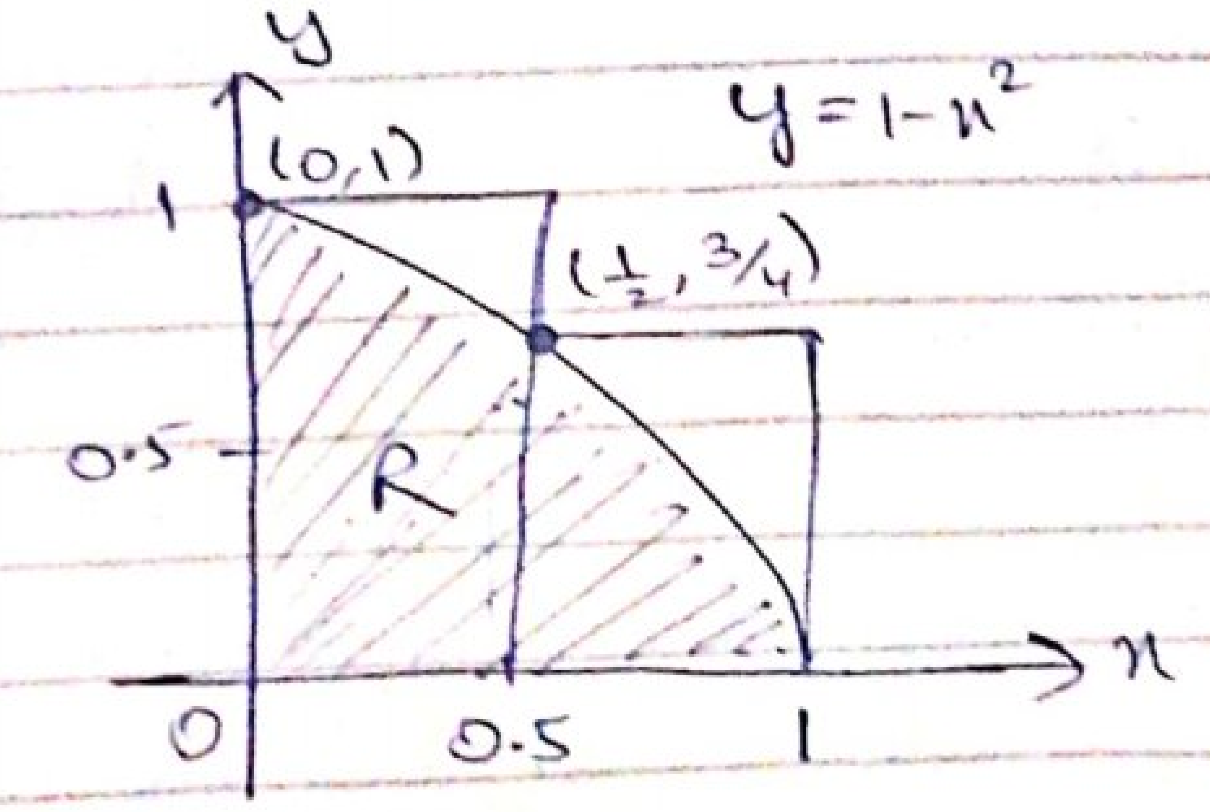
→ Area:
Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$



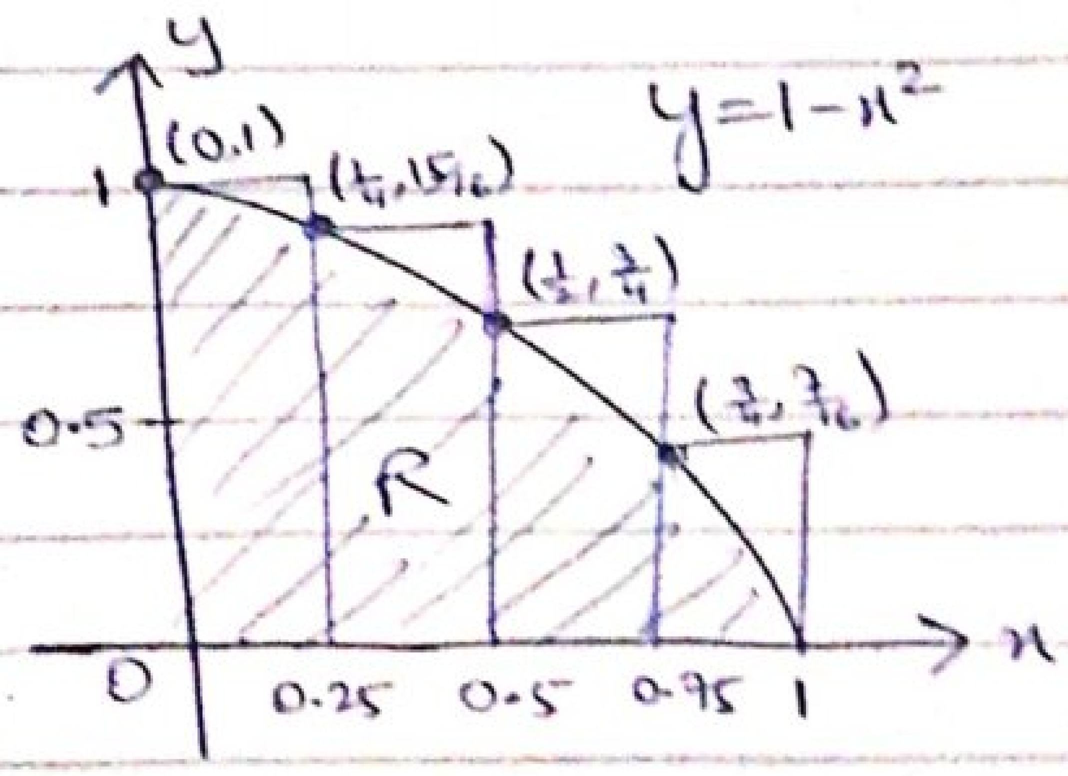
1) * We get an upper estimate of the area of R by using two rectangles containing R.

2) * Four rectangles give a better upper estimate.

1)



2)



→ UPPER SUM – is an overestimate (estimation is larger than the actual value) of the total area.

1) ⇒ The total Area of 2 rectangles:

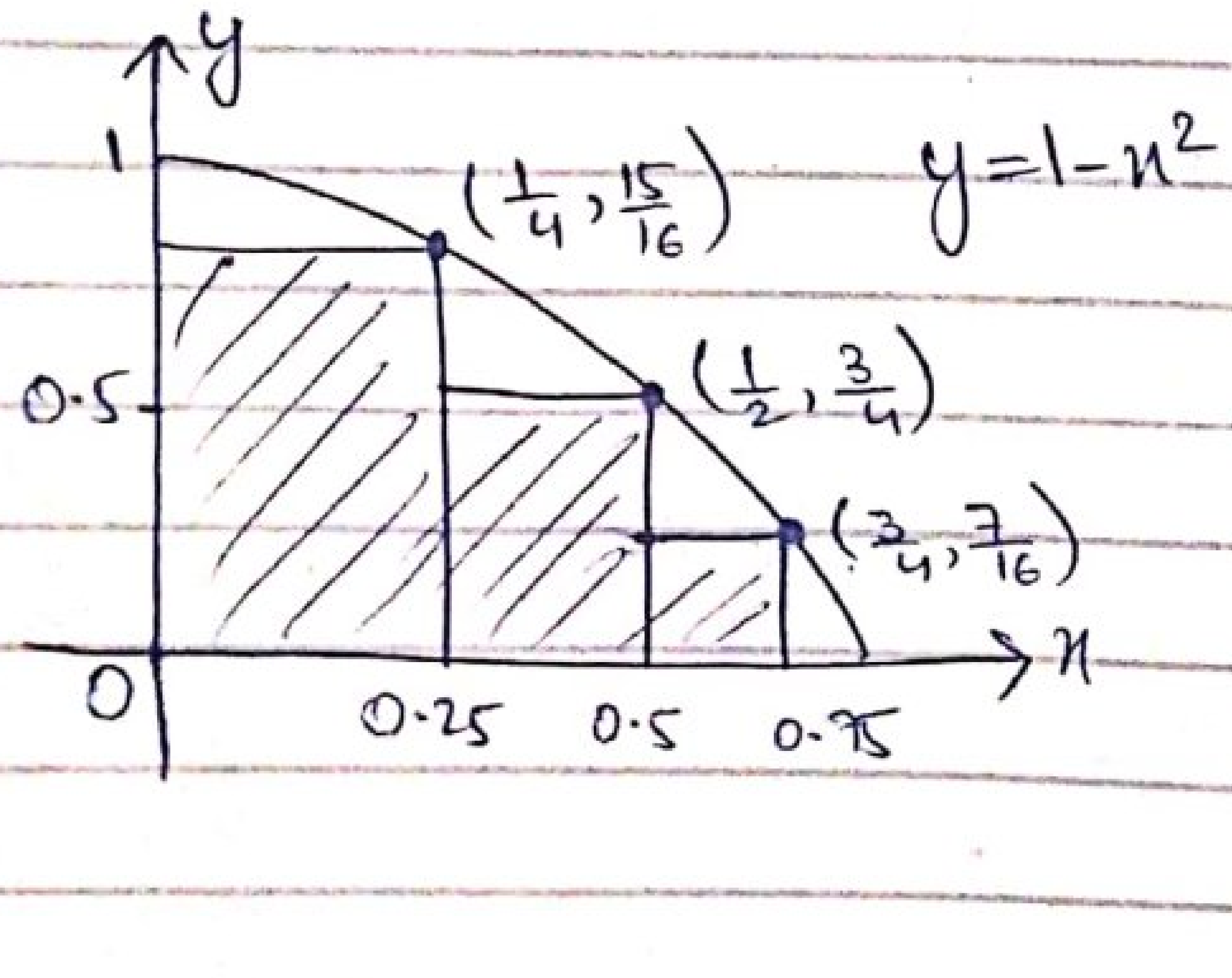
$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875$$

2) ⇒ The total Area of 4 rectangles:

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = 0.78125$$

→ LOWER SUM – is an underestimate (estimation is smaller than the actual value) of the total area.

3)



$$\star \text{ Area} \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125$$

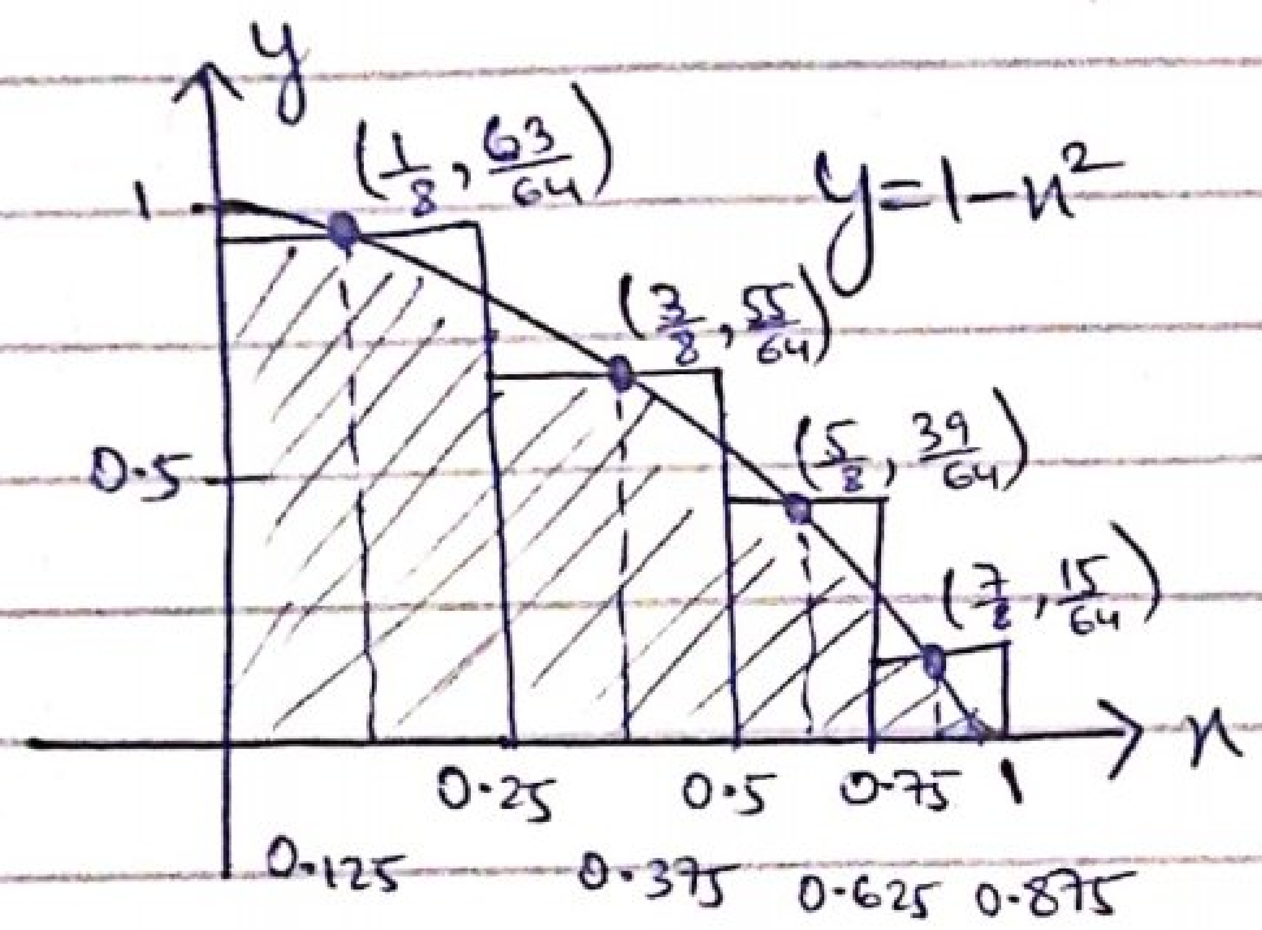
→ The true value of A lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125$$

⇒ Error cannot be greater than $0.78125 - 0.53125 = 0.25$

→ MIDPOINT RULE – The mid-point rule gives an estimate that is between a lower sum and an upper sum, but it is not quite so clear whether it overestimates or under-estimates the true area.

4)

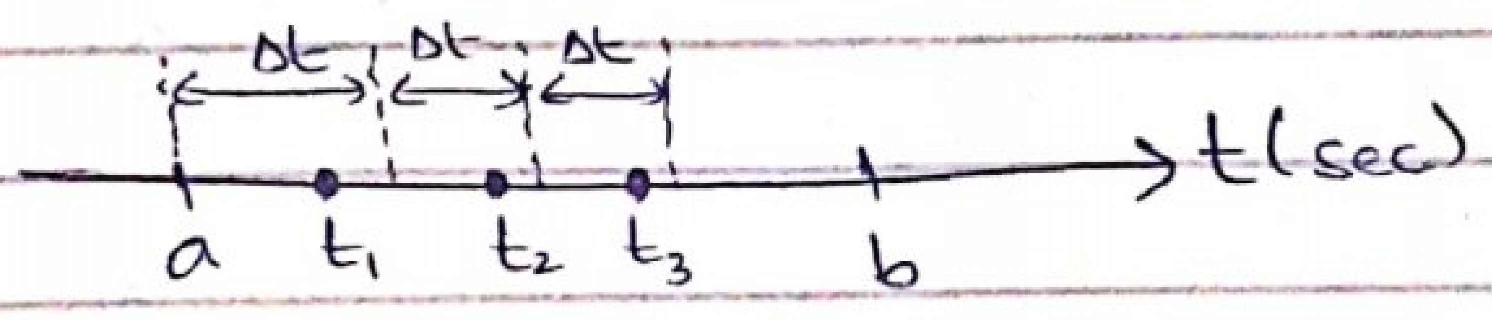


$$\star A = \frac{63}{64}\cdot\frac{1}{4} + \frac{55}{64}\cdot\frac{1}{4} + \frac{39}{64}\cdot\frac{1}{4} + \frac{15}{64}\cdot\frac{1}{4} = \frac{172}{64}\cdot\frac{1}{4} = 0.671875$$

→ DISTANCE TRAVELED –

Distance = Velocity × Time , $t \in [a, b]$
$= v(t) \times \Delta t$

→ Suppose the sub-intervals are all of equal length, Δt.



$$D \simeq v_1(t_1)\Delta t + v_2(t_2)\Delta t + \cdots + v(t_n)\Delta t$$

$\star$ Δt is so small that the velocity barely changes.

where, 'n' is the total number of sub-intervals. This sum is only an approximation to the true distance D, but the approximation increases in accuracy as we take more and more sub-intervals.

→ Example#1

The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Use the summation technique to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m?

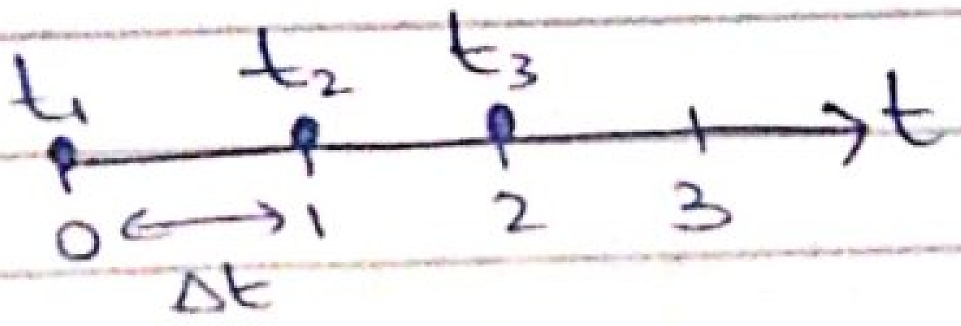
Solution

→ $f(t)$ is decreasing

⟹ Choosing left endpoints gives an upper sum estimate, and

choosing right endpoints gives a lower sum estimate.

(a) '3' sub-intervals of length '1', with 'f' evaluated at left endpoints giving an upper sum



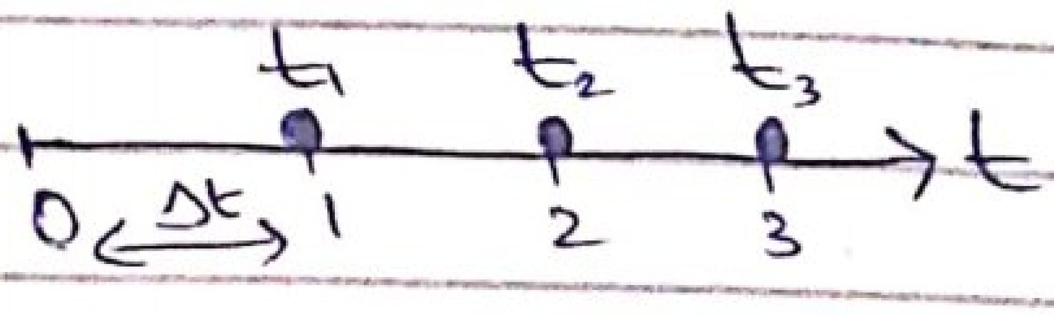
→ With 'f' evaluated at $t = 0, 1,$ and 2, we have

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$

$$= [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1)$$

$$= 450.6$$

(b) '3' Subintervals of length '1', with 'f' evaluated at right endpoints giving a lower sum,



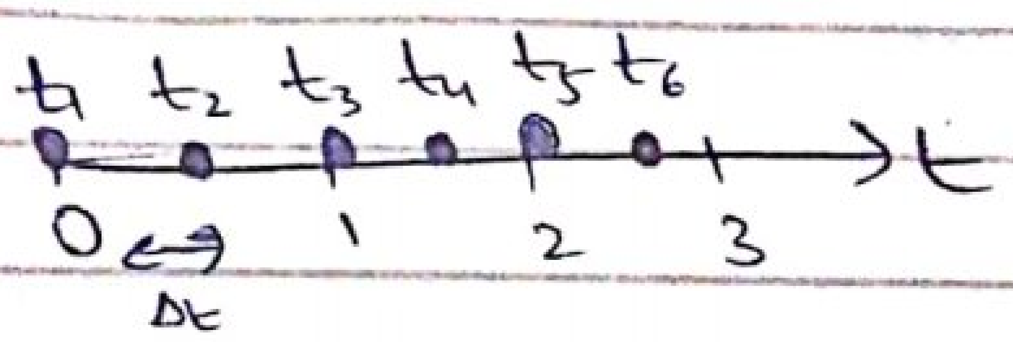
→ With 'f' evaluated at $t = 1, 2$, and 3, we have

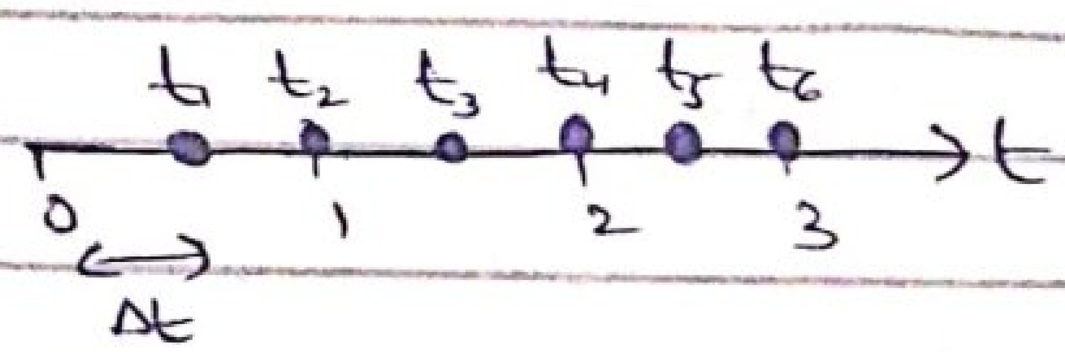
$D \simeq f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$

$= [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1)$

$= 421.2$

(c) '6' Subintervals of length $1/2$, we get





Left Endpoints (Upper Sum)

$\Rightarrow D \simeq 443.25$

Right Endpoints (Lower Sum)

$\Rightarrow D \simeq 428.55.$

→ The results IMPROVE as the subintervals get shorter.

Number of Subintervals	Length of each subinterval	Upper Sum	Lower Sum
3	1	450.6	421.2
6	$1/2$	443.25	428.55
12	$1/4$	439.58	432.23

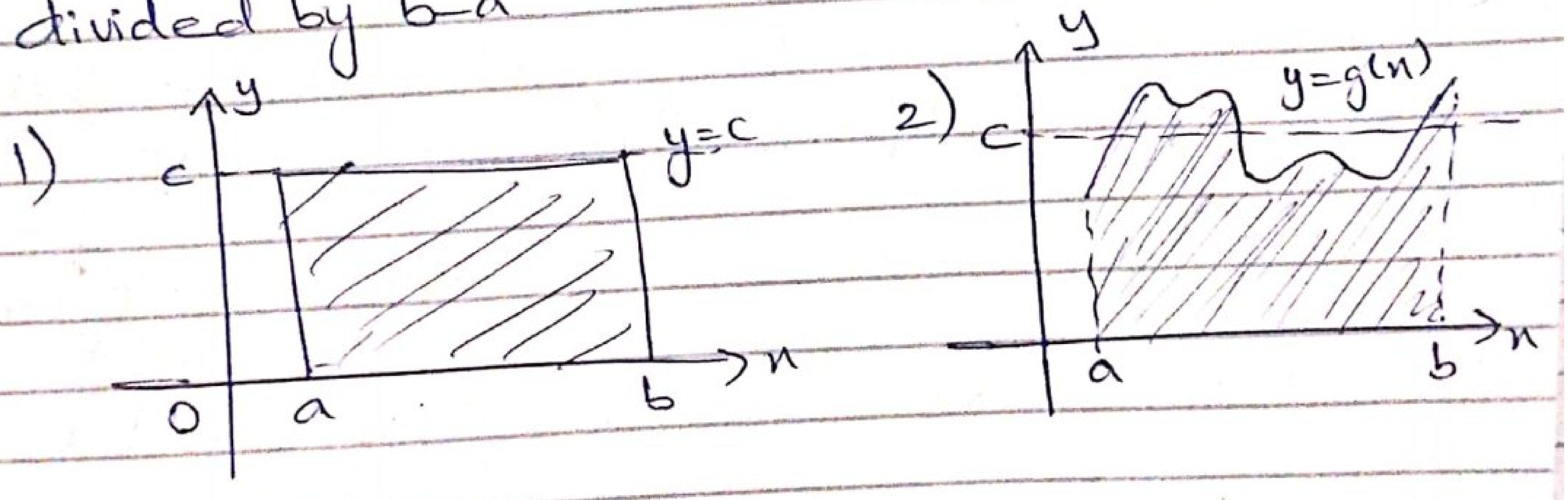
→ Error Magnitude = |true value − calculated value|
$= |435.9 - 435.67| = 0.23$

→ Error Percentage $= \frac{0.23}{435.9} = 0.05\%$

Example # 2 X

→ **AVERAGE VALUE OF A NON-NEGATIVE CONTINUOUS FUNCTION-**

1) → The average value of a constant function $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b-a$.



2) → The average value of a non-constant function $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b-a$.

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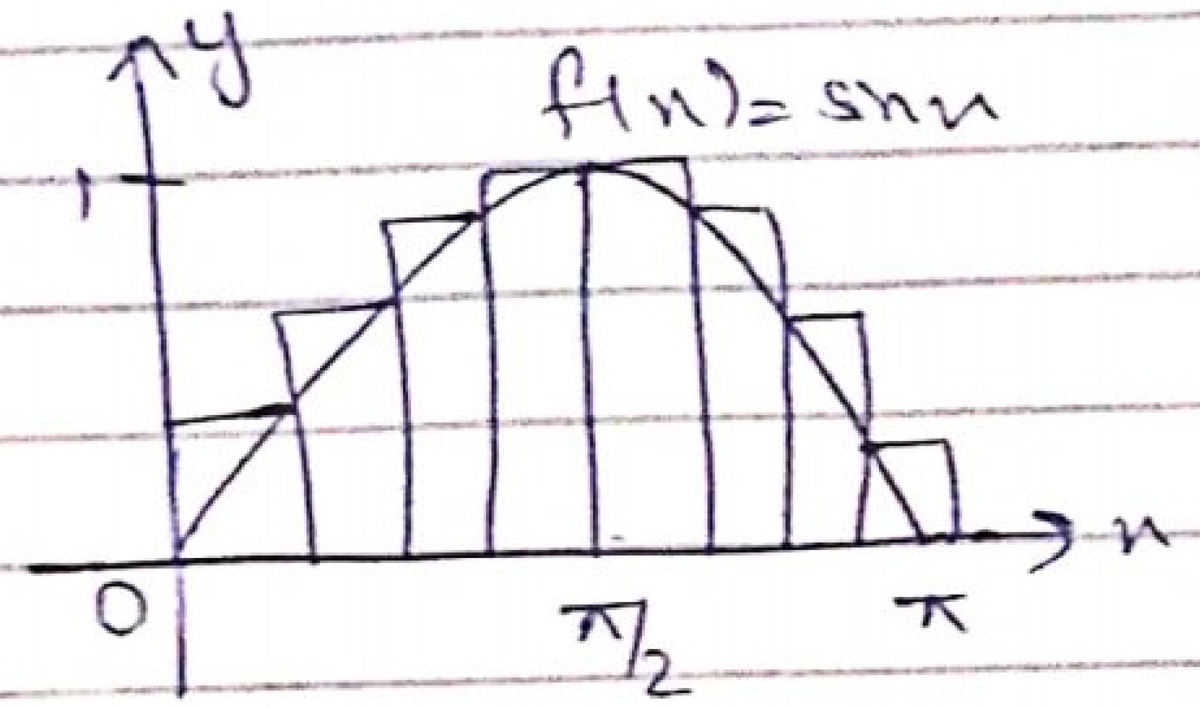
Example#3

Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution



→ To find the average, find area under graph and divide by $\pi - 0 = \pi$.

→ Upper sum approximation → Add area of 8 rectangles of equal width $\pi/8$.

$$A \simeq \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8} \right) \cdot \frac{\pi}{8}$$

$$\simeq \frac{2.364}{\pi} \simeq 0.753 \text{ (Average)}.$$

Topic 5.3 (Pg 265 – 275)

(1) **Definite Integral:** An integral that contains the upper and the lower limits.

→ It is represented by "$\int_a^b f(x)\,dx$" as "the integral from a to b of f of x with respect to x"

→ It helps to find the area of a curve in a graph

Example $\int_2^3 x\,dx = \left.\frac{x^2}{2}\right|_2^3 = \frac{3^2}{2} - \frac{2^2}{2}$

$= \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$ Ans

(2) **Theorem 1 – Integrability of Cont. Functions:**

If f – cont. over interval [a, b] then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over [a, b]

(3) **Theorem 2:**

When f and g are integrable over the interval [a, b], the definite integral satisfies the rules listed in table.

(4) **Table:**

1) Order of Integration: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

2) Zero Width interval: $\int_a^a f(x)\,dx = 0$.

3) Constant Multiple: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4) Sum and difference: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5) Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

6) Max-Min Inequality: If f has maximum value ($\max f$) and minimum value ($\min f$) on $[a,b]$ then

$$(\min f)\cdot(b-a) \leq \int_a^b f(x)\,dx \leq (\max f)\cdot(b-a)$$

7) Domination: If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq 0$.

Example 2 Given that

$\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)dx = -2$, $\int_{-1}^{1} h(x)dx = 7$.

Find (i) $\int_4^1 f(x)\,dx$, (ii) $\int_{-1}^{1}[2f(x)+3h(x)]$ and (iii) $\int_{-1}^{4} f(x)dx$.

Sol:

(i) $\int_4^1 f(x)\,dx = -\int_{1}^{4} f(x)dx = -(-2) = 2$. Ans

(ii) $\int_{-1}^{1}[2f(x)+3h(x)]dx = 2\int_{-1}^{1} f(x)dx + 3\int_{-1}^{1} h(x)dx$

$= 2(5)+3(7) = 10+21$

$= 31$. Ans

(iii) $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$

$= 5-2 = 3$. Ans.

⑤ Area under the graph of a Non-Negative Function:

Definition: If $y=f(x)$ is nonnegative and integrable over the closed interval $[a,b]$, then the area under the curve $y=f(x)$ over $[a,b]$ is the integral of 'f' from 'a' to 'b'

$$A = \int_a^b f(x)\,dx.$$

The area under the graph of a non-negative integrable function to be the value of that definite integral.

Average Value of Continuous Function

Definition If f is integrable on $[a,b]$, then its average value on $[a,b]$ which is also called mean is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Mean

$$m = \frac{\text{Sum of terms}}{\text{no. of terms}}$$

Example 5 Find average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$.

Sol:

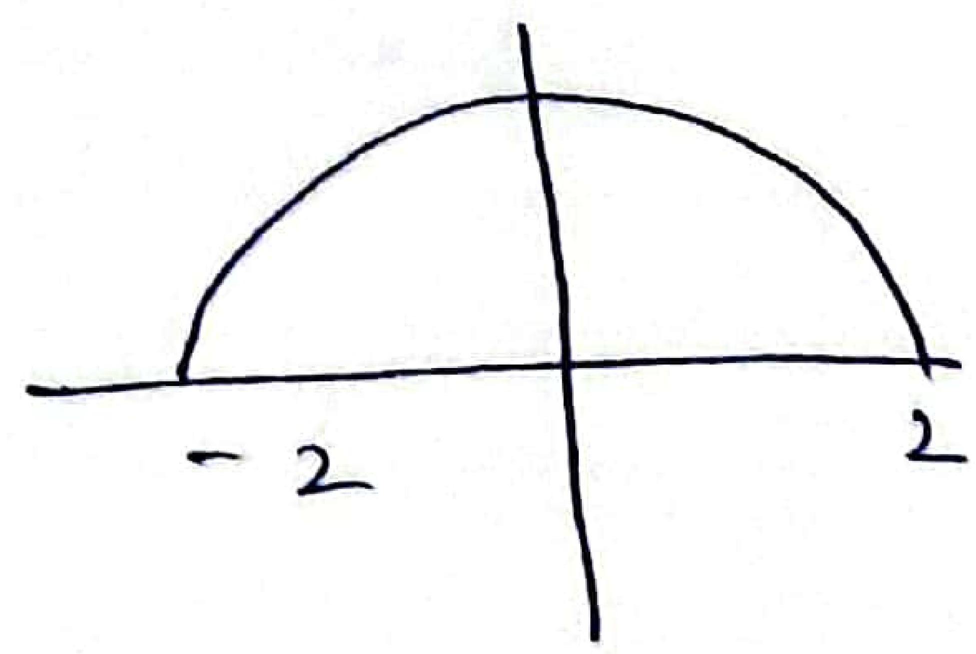
$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} \sqrt{4-x^2}\,dx.$$

$$= -\frac{1}{2}\int_{-2}^{2} \sqrt{4-x^2}\; 2\,dx$$

$$= -\frac{1}{2}\int_{-2}^{2}$$

Sol:

$$A = \int_{-2}^{2} f(x)\,dx = \frac{\pi r^2}{2}$$



$$= \frac{\pi(2)^2}{2}$$

$$= 2\pi.$$

$$av(f) = \frac{1}{2+2}\int_{2}^{-2} f(x)\,dx.$$

$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$ Ans

$$= 1.57.$$

Practice Questions of Ex 5.3

Q#9 – 62

Practice Questions of Ex 5.1

Q#1 – 9.