8.5 Integration Of Rational Functions By Partial Fractions

The method of re-writing rational functions as a sum of simpler fractions is called the Method of Partial fractions.

→ **General Description Of The Method**

Writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

1) The degree of $f(x)$ must be less than the degree of $g(x)$ ⟹ The fraction must be proper.
2) We must know the factors of $g(x)$.

→ **Integration By Partial Fractions Formula**

The list of formulas used to decompose the given improper rational functions is given:

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Form Of The Rational Function	Form Of The Partial Fraction
1) $\dfrac{px+q}{(x-a)(x-b)}, a \neq b$	$\dfrac{A}{x-a} + \dfrac{B}{x-b}$
2) $\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2}$
3) $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$
4) $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$
5) $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a} + \dfrac{Bx+C}{x^2+bx+c}$ where, x^2+bx+c cannot be factorised further

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→ How To Do Integration By Partial Fractions –

Step-1: Check whether the given integrand is a proper or improper rational function.

Step-2: If the given function is an improper rational function, identify the type of denominator.

Step-3: Decompose the integrand using a suitable expression by comparing it with the five different forms (in the table)

Step-4: Divide the integration into parts and integrate the individual functions.

→ Example #1

Use partial fractions to evaluate

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

→ finding the values of the undetermined coefficients A, B, and C.

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$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$
$$= A(x^2+4x+3) + B(x^2+2x-3) + C(x^2-1)$$
$$= (A+B+C)x^2 + (4A+2B)x + (3A-3B-C)$$

→ Equating coefficients of like powers of x:

Coefficients of x^2 : $A+B+C = 1$
Coefficients of x^1 : $4A+2B = 4$
Coefficients of x^0 : $3A-3B-C = 1$

→ Solving these simultaneously gives,

$A = 3/4$, $B = 1/2$, $C = -1/4$.

$$\Rightarrow \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + c$$

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→ EXAMPLE #2

Use partial fractions to evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx$$

SOLUTION

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x+7 = A(x+2)+B$$
$$= Ax+(2A+B)$$

Coefficients of x: $6 = A$

Coefficients of x^0: $7 = 2A+B$

$$7 = 2(6)+B$$
$$\Rightarrow B = -5$$

$$\Rightarrow \int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right)dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + c$$

Dated:

→ Example #3

Use partial fractions to evaluate

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$$

Solution

First, we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r|l} & 2x \\ \hline x^2-2x-3 & \cancel{2x^3}-\cancel{4x^2}-x-3 \\ & -\cancel{2x^3}\mp\cancel{4x^2}\mp 6x \\ \hline & 5x-3 \end{array}$$

$$\Rightarrow \frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx$$

$$= x^2 + \int \frac{5x-3}{x^2-2x-3}\,dx$$

$$* \int \frac{5x-3}{x^2-2x-3} = \int \frac{A}{x+1}\,dx + \int \frac{B}{x-3}\,dx$$

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Or, $\dfrac{5x-3}{x^2-2x-3} = \dfrac{A}{x+1} + \dfrac{B}{x-3}$

$\Rightarrow 5x-3 = A(x-3) + B(x+1)$
$= Ax - 3A + Bx + B$
$= (A+B)x + (-3A+B)$

$\rightarrow$ Comparing coefficients of x: $5 = A+B$
Comparing coefficients of x^0: $-3 = -3A+B$

$\Rightarrow A = 2, \ B = 3.$

$\Rightarrow \displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = x^2 + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$

$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C$

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EXAMPLE #4

Use partial fractions to evaluate:

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

SOLUTION

→ The denominator has an irreducible quadratic factor x^2+1 as well as a repeated linear factor $(x-1)^2$

$$\Rightarrow \frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\Rightarrow -2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$
$$= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D)$$

→ Equating coefficients of like terms,

Coefficients of x^3: $0 = A+C$ ——— (1)
Coefficients of x^2: $0 = -2A+B-C+D$ ——— (2)
Coefficients of x^1: $-2 = A-2B+C$ ——— (3)
Coefficients of x^0: $4 = B-C+D$ ——— (4)

From (1.), $C = -A$

put in Eq (3.),

$$\Rightarrow -2 = \cancel{A} - 2B - \cancel{A}$$
$$\Rightarrow \boxed{B=1}$$

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Put $C=-A$ and $B=1$ in Eq (4),

$$\Rightarrow 4 = 1-(-A)+D$$
$$3 = A+D \quad \text{--- (5)}$$

Put $C=-A$ and $B=1$ in Eq (2),

$$\Rightarrow 0 = -2A+1-(-A)+D$$
$$0 = -2A+1+A+D$$
$$0 = 1-A+D$$
$$-1 = -A+D \quad \text{--- (6)}$$

$$\Rightarrow A=2, \quad C=-2, \quad D=1$$

$$\Rightarrow \frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

$$\Rightarrow \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int\left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int\left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \ln|x^2+1| + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + c$$

Dated:

→ EXAMPLE #5

Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}$$

SOLUTION

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow 1 = A(x^2+1)^2 + (Bx+C)(x^2+1) + (Dx+E)x$$
$$= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$
$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

→ Equating coefficients gives,

$$\Rightarrow A+B=0 \;,\; C=0 \;,\; 2A+B+D=0 \;,\; C+E=0 \;,$$
$$A=1$$

→ Solving simultaneously,

$$\Rightarrow A=1,\; B=-1,\; C=0,\; D=-1,\; E=0$$

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx$$

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

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$$\Rightarrow \int \frac{dx}{x(x^2+1)^2} = \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + c$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + c$$

$*$ let, $u = x^2+1$
$\Rightarrow du = 2x\,dx$

$$= \ln|x| - \ln(x^2+1)^{1/2} + \frac{1}{2(x^2+1)} + c$$

$$= \frac{\ln|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + c$$

→ Example #6 (Method #2 to determine constants)
find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Solution

* Multiply both sides of the Equation by $x-1$ to get,

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x=1$. The resulting equation gives the

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Value of A,

$$\frac{(1)^2+1}{(1-2)(1-3)} = A+0+0$$

$$\Rightarrow A=1$$

→ Similarly, multiply both sides by $(x-2)$ and then, Substitute in $x=2$. This gives,

$$\frac{(2)^2+1}{(2-1)(2-3)} = B$$

$$\Rightarrow B=-5.$$

* Multiply both sides by $(x-3)$ and then, Substitute in $x=3$,

$$\Rightarrow \frac{(3)^2+1}{(3-1)(3-2)} = C$$

$$\Rightarrow C=5.$$

Dated:

→ Example #7 (Method #3 to determine constants)
Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

by clearing fractions, differentiating the result, and substituting $x=-1$.

Solution

$$x-1 = A(x+1)^2 + B(x+1) + C$$

→ Substituting $x=-1$ ⇒ $C=-2$

→ We then differentiate both sides with respect to 'x';

$$\Rightarrow 1 = 2A(x+1) + B$$

→ Substituting $x=-1$ ⇒ $B=1$

→ We differentiate again

$$\Rightarrow 0 = 2A$$
$$\Rightarrow A = 0$$

$$\Rightarrow \frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}$$

Dated:

→ Example # 8 (Method # 4)
Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

by assigning numerical values to x.

Solution

$$\Rightarrow \quad x^2+1 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

Then, let $x = 1, 2, 3$ Successively to find A, B, and C:

* $x = 1$: $(1)^2+1 = A(-1)(-2)+B(0)+C(0)$
$2 = 2A$
$A = 1$

* $x = 2$: $(2)^2+1 = A(0)+B(1)(-1)+C(0)$
$5 = -B$
$B = -5$

* $x = 3$: $(3)^2+1 = A(0)+B(0)+C(2)(1)$
$10 = 2C$
$C = 5$

$$\rightarrow \quad \frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$$