Topic 5.5

Indefinite Integrals and the Substitution Method

(I) **Indefinite Integral:** The collection of all antiderivatives of f is called indefinite integral of f w.r.t 'x'. (Pg 236).

- It is represented by $\boxed{\int f(x)\,dx = F(x) + c}$ where c is any arbitrary constant.

- The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad (\because 2^{nd} \text{ Fundamental Theorem})$$
$$= [F(b) + C] - [F(a) + C]$$
$$= [F(x) + C]_a^b$$
$$= \left[\int f(x)\,dx\right]_a^b$$

(Example 1 Topic 5.6 (Pg 297))

- When finding the indefinite integral of a function 'f', remember that is always includes an arbitrary constant 'C'.
- A definite integral $\int_a^b f(x)\,dx$ is a number.
- An indefinite integral $\int f(x)\,dx$ is a function plus an arbitrary constant 'C'.

(II) Theorem 6 – The Substitution Rule :

If $u = g(x)$ is a diff. func. whose range is an interval I, and f is cont. on I, then

$$\boxed{\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du}$$

- The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$
  (i) Substitute $u = g(x)$ and $du = \left(\frac{du}{dx}\right) dx = g'(x)$ to obtain $\int f(u)\, du$.
  (ii) Integrate with respect to (w.r.t) 'u'.
  (iii) Replace 'u' by 'g(x)'.

- Also known as U-Substitution Method.

- If u is a diff. func. of x and n is any number different from −1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{d}{dx}u^{n+1}$$

$$= \frac{1}{\cancel{n+1}}\left(\cancel{(n+1)}u^{n+\cancel{1}-\cancel{1}}\right)\left(\frac{du}{dx}\right)$$

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$$

(P.T.O)

From another point of view, this same equation says that:

$\frac{u^{n+1}}{n+1}$ is one of the anti-derivatives of the function $u^n\left(\frac{du}{dx}\right)$. Therefore

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + c \quad \text{— ①}$$

The integral in eq. ① is equal to the simpler integral

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C.$$

which suggests that the simpler expression 'du' can be substituted for $\left(\frac{du}{dx}\right)dx$ when computing an integral i.e

$$du = \left(\frac{du}{dx}\right)dx.$$

* **Example 1** — (DIY).

* **Example 2**: Find $\int \sqrt{2x+1}\,dx$.

Sol: Let $u = 2x+1$

$du = 2\,dx.$

$\frac{du}{2} = dx.$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + c$$

$$= \frac{1}{2} \times \frac{2}{3}\,u^{3/2} + c$$

$$= \frac{1}{3}\,u^{3/2} + c = \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$$ Ans

✱ Example 3 – (DIY)

✱ Example 4: Find $\int \cos(7\theta+3)\,d\theta$.

Sol: Let $u = 7\theta + 3$

$$du = 7\,d\theta$$

$$\frac{du}{7} = d\theta.$$

$$\int \cos(7\theta+3)\,d\theta = \int \cos(u)\,\frac{du}{7}$$

$$= \frac{1}{7}\int \cos(u)\,du.$$

$$= \frac{1}{7}\sin(u) + c.$$

$$= \boxed{\frac{1}{7}\sin(7\theta+3) + c.}$$ Ans.

✱ Example 5 – (DIY)

* **Example 6:** Evaluate $\int x\sqrt{2x+1}\,dx$.

Sol: Let $u = 2x+1$ —— (i)

$du = 2\,dx$.

$\frac{du}{2} = dx$.

$$\int x\sqrt{2x+1}\,dx = \int x\sqrt{u}\,\frac{du}{2}. \quad \text{—— (ii)}$$

To convert x in terms of 'u', use eq (i)

$u = 2x+1$

$u-1 = 2x \Rightarrow x = \frac{u-1}{2}$.

Put the value of x in eq (ii)

$$\int x\sqrt{2x+1}\,dx = \int \left(\frac{u-1}{2}\right)\sqrt{u}\,\frac{du}{2}$$

$$= \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u-1)u^{1/2}\,du$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)du = \frac{1}{4}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + C$$

$$= \frac{1}{\cancel{4}_2}\times\frac{\cancel{2}}{5}u^{5/2} - \frac{1}{\cancel{4}_2}\times\frac{\cancel{2}}{3}u^{3/2} + C.$$

$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C$$

$$= \boxed{\frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.} \quad \text{Ans.}$$

* **Example 7 — (DIY)**

- Important Formulas used in Example 7.

1) $\sin^2 x = \dfrac{1-\cos 2x}{2}$

2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$

3) $\sin 2x = 2\sin x \cos x$
4) $\cos 2x = \cos^2 x - \sin^2 x.$
5) $\cos 2x = 1 - 2\sin^2 x$
6) $1 - 2\sin^2 x = \cos^2 x - \sin^2 x.$

* **Example 8**: Evaluate $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Sol: • **Method 1**: Let $u = z^2+1$

$du = 2z\,dz.$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du$$

$$= \frac{u^{\frac{2}{3}}}{2/3} + C.$$

$$= \frac{3}{2}u^{2/3} + C.$$

$$= \boxed{\frac{3}{2}\left(z^2+1\right)^{2/3} + C.}$$ Ans

- Method 2: Let $u = \sqrt[3]{z^2+1}$

$$u^3 = z^2+1$$

$$3u^2 du = 2z dz$$

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^{\cancel{2}}du}{\cancel{u}}$$

$$= 3\int u\,du = 3\frac{u^2}{2} + c$$

$$= \frac{3}{2}\left(\sqrt[3]{z^2+1}\right)^2 + C$$

$$= \boxed{\frac{3}{2}(z^2+1)^{2/3} + C.}$$ Ans

$\longleftrightarrow$

Practice Questions Q# 1 – 50.
of Ex# 5.5

(Q# 51 – 54) – Try it

Topic 5.6

Definite Integral Substitutions and the Area Between Curves

(I) Theoram 7 – Substitution in Definite Integrals:

If g' is cont. on the interval $[a,b]$ and f is cont. on the range of $g(x) = u$, then

$$\boxed{\int_a^b f(g(x)).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

- This formula shows how the limits of integration change when we apply a substitution to an integral.

Example 1: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Sol: • Method 1:

Let $u = x^3+1$

$du = 3x^2\,dx$

$\frac{du}{3x^2} = dx$.

When $x=-1 \Rightarrow u=(-1)^3+1 = 0$

When $x=1 \Rightarrow u=(1)^3+1 = 2$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \int_0^2 u^{1/2}du$$

$$= \left.\frac{u^{3/2}}{3/2}\right]_0^2 = \frac{2}{3}\left(2^{3/2}-0\right)$$

$$= \frac{2}{3}\left(2\sqrt{2}\right)$$

$$= \boxed{\frac{4\sqrt{2}}{3}} \text{ Ans}$$

- Method 2: Let $u = x^3+1$

$du = 3x^2\,dx.$

Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0\right] = \frac{2}{3}(2\sqrt{2}) = \boxed{\frac{4\sqrt{2}}{3}} \text{ Ans}$$

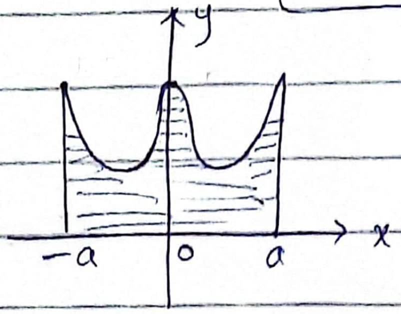
☆ Example 2 — (DIY)

(II) Definite Integrals of Symmetric Functions:

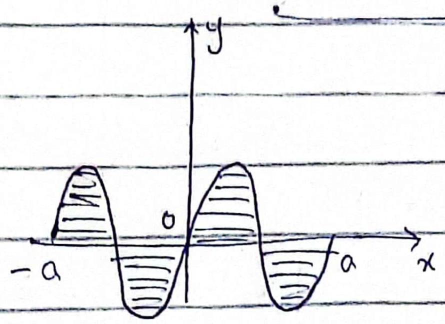
The Substitution Formula in Theorem 7 simplifies the calculation of definite integrals of even and odd functions over a symmetric interval $[-a, a]$.

• Theorem 8: Let f be cont. on the symmetric interval $[-a, a]$

(a) If f is even, then $\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx}$



(b) If f is odd, then $\boxed{\int_{-a}^{a} f(x)\,dx = 0}$



*Example 3: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$

Sol: Check if $f(x)$ is even or odd.

Let $f(x) = x^4 - 4x^2 + 6$.

$f(-x) = (-x)^4 - 4(-x)^2 + 6$

$= x^4 - 4x^2 + 6 = f(x)$.

$f(x)$ is even on symmetric interval $[-2, 2]$

11

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5}\Big|_0^2 - \frac{4x^3}{3}\Big|_0^2 + 6x\Big|_0^2\right]$$

$$= 2\left[\frac{2^5}{5} - 0 - \frac{4(2)^3}{3} + 0 + 6(2-0)\right]$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \boxed{\frac{232}{15}} \text{ Ans.}$$

Example: Evaluate $\int_{-2}^{2} (4x - x^3)\,dx$.

(Out of book)

Sol: Check if $f(x)$ is even or odd

Let $f(x) = 4x - x^3$

$$f(-x) = 4(-x) - (-x)^3 = -4x + x^3$$

$$f(-x) = -(4x - x^3)$$

$$f(-x) = -f(x).$$

$f(x)$ is odd on symmetric interval $[-2, 2]$.

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} (4x - x^3)\,dx = \boxed{0} \quad \text{Ans}$$

or

$$\int_{-2}^{2} (4x - x^3)\,dx = \frac{4x^2}{2}\Big|_{-2}^{2} - \frac{x^4}{4}\Big|_{-2}^{2}$$

$$= 2\left[(2)^2 - (-2)^2\right] - \frac{1}{4}\left[(2)^4 - (-2)^4\right].$$

$$= 2[4-4] - \frac{1}{4}[16-16] = \boxed{0}.$$

III) Areas Between Curves:

- **Definition:** If f and g are cont. with $f(x) \geq g(x)$ throughout $[a, b]$ then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from 'a' to 'b' is the integral of $(f - g)$ from 'a' to 'b':

$$\boxed{A = \int_a^b [f(x) - g(x)]\, dx}$$

- To find areas b/t curves, we have to follow some rules i.e.

(i) When applying this def., it is usually helpful to graph the curves.

(ii) The graph reveals which curve is the upper curve f and which is the lower curve g.

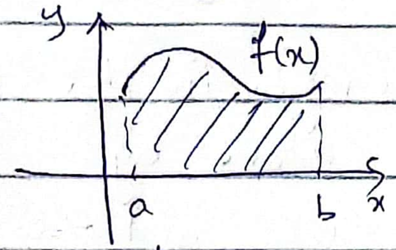
(iii) It also helps you to find the limit of integration if they are not given.

(iv) You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x.

(v) Then you can integrate the function $f - g$ for the area b/t the intersection.

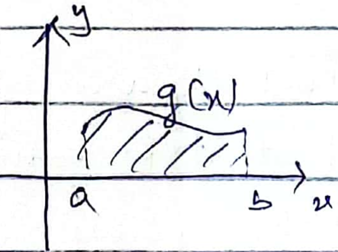
Geometric Representation:

(I)



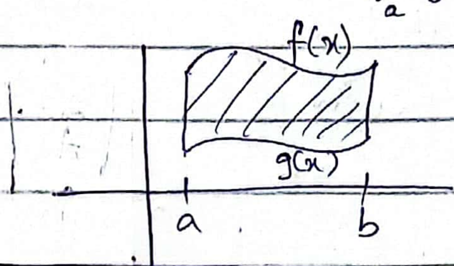
$$A = \int_a^b f(x)\,dx$$

(II)



$$A = \int_a^b g(x)\,dx$$

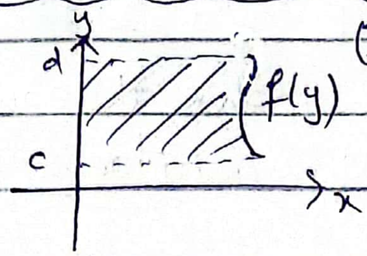
(III)



$$\boxed{A = \int_a^b [f(x) - g(x)]\,dx} \rightarrow$$ Integration with respect to 'x'

Integration with Respect to y:

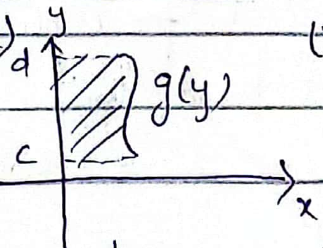
(I)



$$A = \int_c^d f(y)\,dy.$$

$X = f(y)$

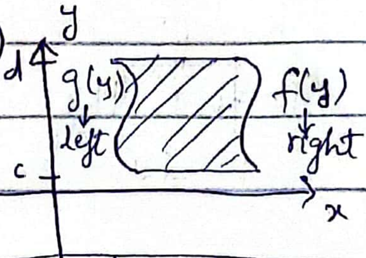
(II)



$$A = \int_c^d g(y)\,dy$$

$X = g(y)$.

(III)



$$\boxed{A = \int_c^d [f(y) - g(y)]\,dy}$$

↓

Integration with respect to 'y'

→ f always denotes the right hand curve and g the left hand curve, so $f(y) = g(y)$ is non-negative.

→ Example # 4

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution

→ The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x,

$$2 - x^2 = -x$$
$$x^2 - x - 2 = 0$$
$$x^2 + x - 2x - 2 = 0$$
$$x(x+1) - 2(x+1) = 0$$
$$(x+1)(x-2) = 0$$
$$\Rightarrow x = -1, \quad x = 2$$

→ The Area between the curves is,

$$A = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\,dx$$

$$= 2x + \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}.$$

→ Example#5

find the area of the region in the first quadrant that is bounded above by $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$

Solution

→ Point of intersection:

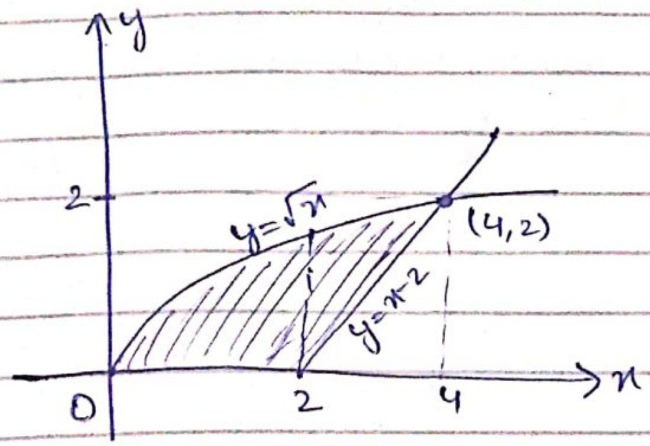
$$\sqrt{x} = x-2$$
$$x = (x-2)^2$$
$$x = x^2+4x+4$$
$$x^2-5x+4=0$$
$$x^2-x-4x+4=0$$
$$x(x-1)-4(x-1)=0$$
$$(x-1)(x-4)=0$$
$$\Rightarrow x=1,\ x=4$$

* $x=4$ → Upper Limit
* $x=1$ → Doesnot Satisfy the equations (Extraneous Root → Not a root of the original equation)

* x-intercept:

$$y=x-2$$
$$0=x-2$$
$$\Rightarrow x=2$$
$(2,0)$

* Sketch:



* For $0 \le x \le 2$:
Shaded Region $= f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

* for $2 \le x \le 4$:
Shaded Region $= f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

→ Total Area $= \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$

$$= \frac{2}{3}x^{3/2}\Big|_0^2 + \frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\Big|_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

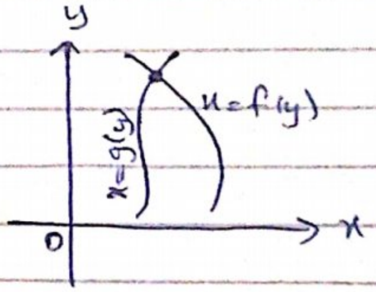
$$= \frac{2}{3}(8) - 2$$

$$= \frac{10}{3}$$

5.6 INTEGRATION WITH RESPECT TO 'y':

If a region's bounding curves are described by functions of 'y', the approximating rectangles are horizontal instead of vertical, and the basic formula has 'y' inplace of 'x'.

$$A = \int_c^d [f(y) - g(y)]\, dy$$



→ **Example #6**

Find the area of the region (in Example #5) with respect to 'y'.

SOLUTION

Given, $y = \sqrt{x}$ and $y = x - 2$

$\Rightarrow x = y^2$ and $x = y + 2$

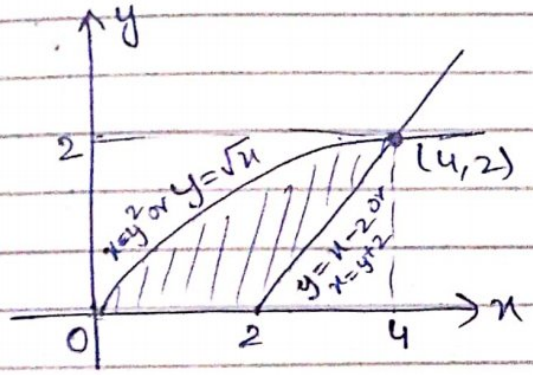
→ **POINT OF INTERSECTION** -

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$y^2 + y - 2y - 2 = 0$$
$$y(y+1) - 2(y+1) = 0$$
$$(y+1)(y-2) = 0$$

$\rightarrow y=-1, \; y=2$

* $y=2 \rightarrow$ Upper limit.
* $y=-1 \rightarrow$ Gives a point of intersection below the x-axis.

* Sketch:



* for $0 \le y \le 2$,
Shaded Region $= f(y) - g(y) = (y+2) - y^2$

$\rightarrow$ The Area of the region is,

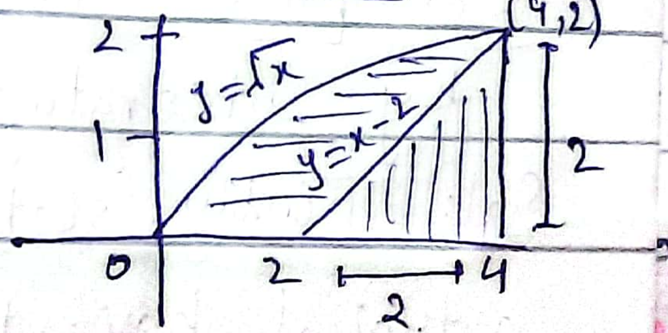
$$A = \int_c^d [f(y) - g(y)] \, dy$$

$$= \int_0^2 (y + 2 - y^2) \, dy$$

$$= 2y + \frac{y^2}{2} - \frac{y^3}{3} \Big|_0^2 = 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}$$

* Another Method

$$\text{Area} = \underbrace{\int_0^4 \sqrt{x}\,dx}_{\text{Area under curve } y=\sqrt{x}} - \underbrace{\frac{1}{2}(2)(2)}_{\text{Area of triangle}}$$



$$= \frac{2}{3}x^{3/2}\Big]_0^4 - 2$$

$$= \frac{2}{3}(8) - 0 - 2 = \boxed{\frac{10}{3}}$$ Ans.

→ Example #7

Find the area of the region bounded below by the line $y = 2-x$ and above by the curve $y = \sqrt{2x-x^2}$.

Solution

* Integration w.r.t 'x',

→ Intersection Point :

$2-x = \sqrt{2x-x^2}$

$(2-x)^2 = 2x-x^2$

$4+x^2-4x = 2x-x^2$

$2x^2-6x+4=0$

$x^2-3x+2=0$

$x^2-x-2x+2=0$

$x(x-1)-2(x-1)=0$

$(x-1)(x-2)=0$

$\Rightarrow x=1, \; x=2$. (Both satisfy the equation)

$$\Rightarrow A = \int_1^2 \left(\sqrt{2x-x^2} + x-2\right) dx$$

* $\int \sqrt{2x-x^2}\, dx$ is complex and no simple substitution is apparent.

* x-intercept:

$y=2-x$	$y=\sqrt{2x-x^2}$
$0=2-x \Rightarrow x=2$, $(2,0)$	$0 = x(2-x)$
	$x=2, (2,0)$

* INTEGRATION w.r.t 'y':

→ INTERSECTION POINT

→ Given, $y = 2-x$ and $y = \sqrt{2x-x^2}$

$\Rightarrow x = 2-y$ and $y^2 = 2x - x^2$

$y^2 = -x^2 + 2x$

$y^2 = -(x^2 - 2x)$

$y^2 = -[x^2 - 2x + 1 - 1]$

$y^2 = -[x^2 - 2x + 1] + 1$

$y^2 = -(x-1)^2 + 1$

$\therefore -(y^2 - 1) = (x-1)^2$

$\Rightarrow x = 1 + \sqrt{1-y^2}$

→ $2-y = 1 + \sqrt{1-y^2}$

$1-y = \sqrt{1-y^2}$

$(1-y)^2 = 1-y^2$

$1 + y^2 - 2y = 1 - y^2$

$2y^2 - 2y = 0$

$2y(y-1) = 0$

$\Rightarrow y = 0, y = 1$ (Both satisfy the equation).

→ The area of the region is given by,

$$A = \int_0^1 (1 + \sqrt{1-y^2}) - (2-y)\,dy$$

$$= \int_0^1 \sqrt{1-y^2} + y - 1\,dy$$

* $\int \sqrt{1-y^2}\,dy$ is complex.

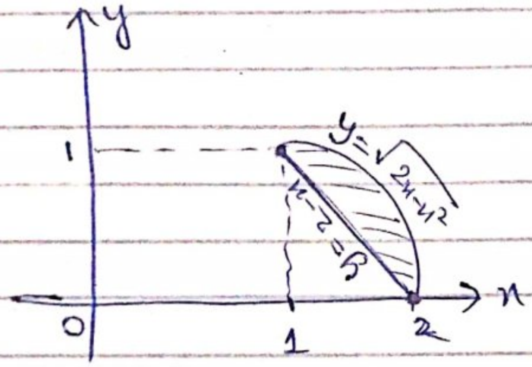
→ Use Equation of Unit Circle.

→ from $y^2 = 2x - x^2$, we obtain $-(y^2-1) = (x-1)^2$

$\Rightarrow 1 - y^2 = (x-1)^2$

$\Rightarrow (x-1)^2 + y^2 = 1$, which is the Equation of the Unit Circle with Centre shifted to the point $(1,0)$.

* Sketch:



* Shaded Area = (Area of upper right quarter of unit circle) − (Area of triangle)

$\Rightarrow A = \pi/4 - \frac{1}{2}$

$= \frac{\pi - 2}{4}$

$\simeq 0.285$.

OR

$$\Rightarrow \text{Area} = \int_0^1 \sqrt{1-y^2}\,dy + \int_0^1 (y-1)\,dy$$

$$= \left[\frac{\pi}{4}\right] + \left[\frac{y^2}{2} - y\Big|_0^1\right]$$

$$= \pi/4 - \frac{1}{2} \simeq 0.285$$