Lec-12

## Topic 4.1 Extreme Values of Functions on Closed Intervals

① **Absolute Maximum**

$$f(x) \leq f(c) \quad \forall x \text{ in } D$$

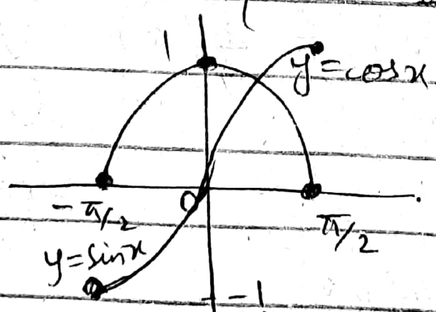
② **Absolute Minimum**

$$f(x) \geq f(c) \quad \forall x \text{ in } D$$

* Max and min. values are extreme values.
* Absolute maxima or minima are global maxima or minima.

Example: Closed Interval $[-\pi/2, \pi/2]$

$f(x) = \cos x$ | $f(x) = \sin x$




Abs. Max. $= 1$ $(x=0)$

Abs. Min. $= 0$ $(x = -\frac{\pi}{2}, x = \pi/2)$

Max. Value $= 1$ $(x = \pi/2)$

Min. Value $= -1$ $(x = -\pi/2)$

* Functions defined by same eq. or formula can have diff. extrema depending on the domain.

* A Function might not have a max. or min. if domain is unbounded or fails to contain an endpoint.

Example 1 : $y = x^2$

( From book).

(3) Theorem 1 — Extreme Value Theorem

( from book )

(4) Local Maximum :

$f(x) \leq f(c) \quad \forall x \in D$ lying in some open interval containing 'c'

(5) Local Minimum :

$f(x) \geq f(c) \quad \forall x \in D$ lying in some open interval containing 'c'

* Local extrema are relative extrema.

Fig 4.5 (from book).

⑥ Finding Extrema :

Theorem 2 — First Derivative Theorem for Local Extreme Values.

If $f$ — local max. or min.
at an interior pt. 'c' of its domain.

$f'$ is defined at c

Then

$$f'(c) = 0.$$

Only places where $f$ can have extreme values

i) interior pts $f'$ is zero

ii) interior pts where $f'$ is undefined

iii) endpoints of domain of $f$.

⑦ Critical pts : An interior pt of the domain of a function $f$ where $f'$ is zero or undefined is a critical pt. of $f$.

* Only domain pts where a function can assume extreme values are critical pts. and endpoints.

* A function may have a critical pt at $x = c$ without having a local extreme value.

e.g $y = x^3$ and $y = x^{1/3}$ have critical pts at the origin but neither function has a local extrema value at the origin (See Fig. 4.7

⑧ Finding absolute extrema of a Cont. function on a Finite closed interval.

i) Find all critical pts on interval.
ii) Evaluate f at all critical pts and endpoints
iii) Take the largest and smallest of these values

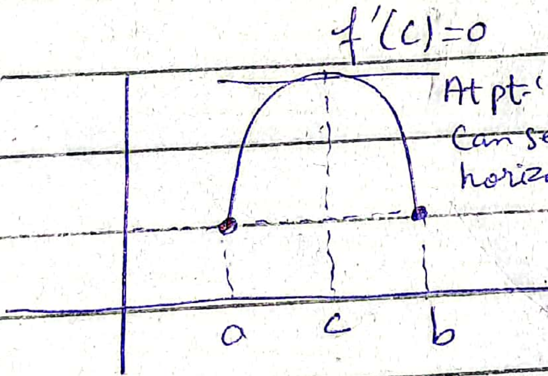
Example 2 – (from book).
Example 3 – Class Activity.
Example 4 – from book.

# Lec - 13

## Topic 4.2 The Mean Value Theorem

### I) Rolle's Theorem: (Statement from book) Pg 191.

$f(x)$ — ① $[a, b]$ continuous.
② $(a, b)$ differentiable.
③ If $f(a) = f(b)$ then $\boxed{f'(c) = 0}$.
(horizontal tangent at some pt 'c' b/t 'a' and 'b')




At pt 'c', we can see that slope of horizontal tangent is zero.

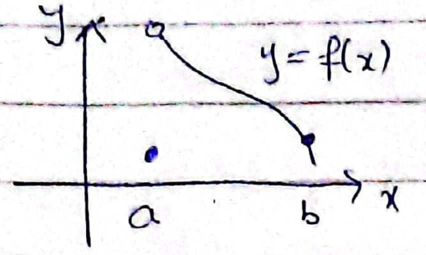
→ It is cont. on $[a, b]$ — there is no holes, no vertical asymptotes, ~~no jumps in~~ that curve

→ It is diff. on $(a, b)$ — smooth curve, no sharp turns, no jumps.

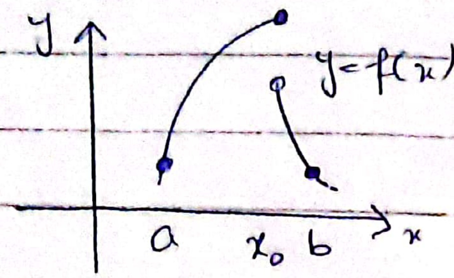
→ $f(a) = f(b)$

If these 3 conditions are met then there ~~is~~ exists a horizontal tangent somewhere b/t 'a' and 'b'

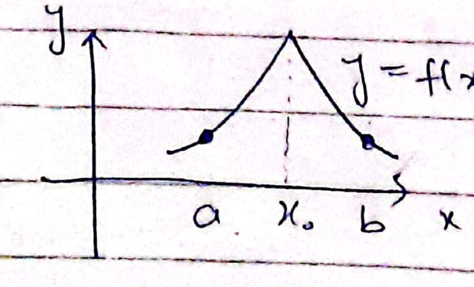
* The hypotheses of theorem 3 (Rolle's Theorem) are essential. If they fail at even one point, the graph may not have a horizontal tangent.




Discont. at an endpoint of [a, b]




Discont. at an interior point of [a, b].



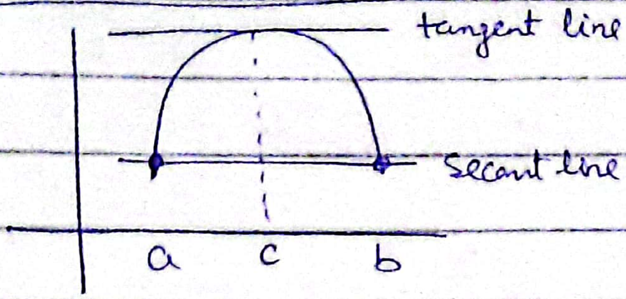

Cont. on [a, b] but not diff. at an interior pt.

## II Mean Value Theorem: (Statement from book Pg 192)

f(x)
1. cont. on [a, b]
2. Diff on (a, b)
3. Then there is at least one pt. 'c' in (a, b) s.t

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

## Geometrical Interpretation :




Geometrically Mean Value Theorem says that Somewhere b/t 'a' and 'b' the curve has at least one tangent line parallel to Secant line that joins 'a' and 'b'

## Physical Interpretation :

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$\frac{f(b)-f(a)}{b-a}$ ↓ Slope of Secant line (average rate of Change)

$f'(c)$ ↓ Slope of tangent line (instantaneous rate of rate).

Mean value Theorem says that the instantaneous change at some interior pt is equal to the average change over the entire interval.

**Ex 4.2** Find value of c using Mean value Theorem

(1) $f(x) = x^2+2x-1$, $[0,1]$.

Sol: MVT $\Rightarrow$ $\frac{f(b)-f(a)}{b-a} = f'(c)$.

$f(x) = x^2+2x-1$

$f'(x) = 2x+2$.

$$f'(c) = 2c+2 = \frac{f(1)-f(0)}{1-0}$$

$$2c+2 = \frac{2-(-1)}{1}$$

$$2c+2 = 2+1$$

$$2c+2 = 3$$

$$2c = 1$$

$$\boxed{c = \frac{1}{2}}$$

(3) $f(x) = \sqrt{x-1}$ , $[1,3]$

Sol:

$$f'(x) = \frac{1}{2}(x-1)^{-1/2} = \frac{1}{2\sqrt{x-1}}$$

$$f'(c) = \frac{1}{2\sqrt{c-1}} = \frac{f(3)-f(1)}{3-1}$$

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}-0}{2}$$

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sqrt{2} = 2\sqrt{c-1}$$

$$\sqrt{c-1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$c-1 = \frac{1}{2} \quad \text{(Sq. on b/s)}$$

$$c = \frac{1}{2}+1 = \frac{3}{2}$$

$$\boxed{c = \frac{3}{2}}$$

(6) $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Sol:

$\cdot\ g'(x) = \begin{cases} 3x^2 & -2 \le x \le 0 \\ 2x & 0 < x \le 2 \end{cases}$

$g'(c) = \frac{g(0) - g(-2)}{0-(-2)}$ and $\frac{g(2)-g(0)}{2-0}$

$3c^2 = \frac{0-(-8)}{2}$ and $2c = \frac{4-0}{2}$

$3c^2 = \frac{8}{2}$ and $2c = \frac{4}{2}$

$3c^2 = 4$ and $2c = 2$

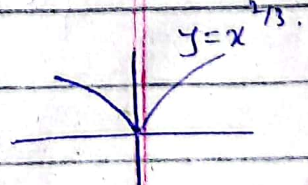
$c^2 = \frac{4}{3}$ and $\boxed{c = 1}$

$\boxed{c = \frac{2}{\sqrt{3}}}$

$\longleftrightarrow$

Satisfy the hypotheses of Mean Value Theorem on given interval.

(7) $f(x) = x^{2/3}$ $[-1, 8]$.




Sol:

$f(x)$ – continuous on $[-1, 8]$.

$f(x)$ – not diff on $(-1, 8)$. because at $x=0$ it has ~~sharp~~ corner

$f'(c) = \frac{2}{3\sqrt[3]{c}} = \frac{f(8)-f(-1)}{8+1}$

$\frac{2}{3\sqrt[3]{c}} = \frac{4-1}{9} = 8.$

$\boxed{c = 8}$ → which is not in $(-1, 8)$.

Func. is not satisfied MVT hypothesis $f(x)$ because not diff

(11) $f(x) = \begin{cases} x^2 - x & -2 \le x \le -1 \\ 2x^2 - 3x - 3 & -1 < x \le 0 \end{cases}$

Sol:

Check cont. on $[-2,0]$. at $x=-1$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 - x$$

$$= 2$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x^2 - 3x - 3$$

$$= 2.$$

$$f(-1) = \cancel{x^2}\ (-1)^2 - (-1) = 2.$$

$$\Rightarrow \lim_{x \to -1} f(x) = f(-1)$$

Above $f(x)$ is cont in interval $[-2,0]$

Using MVT condition

$$f'(x) = \begin{cases} 2x - 1 & -2 \le x \le -1 \\ 4x - 3 & -1 < x \le 0 \end{cases}$$

$$f'(c) = 2c - 1 = \frac{f(-1) - f(-2)}{-1 + 2}$$

$$2c - 1 = \frac{2 - 6}{1} = -4$$

$$2c - 1 = -4$$

$$2c = -3$$

$$\boxed{c = -\frac{3}{2}}$$

$$f'(c) = 4c - 3 = \frac{f(0) - f(-1)}{0+1}.$$

$$4c - 3 = \frac{-3 - 2}{1}$$

$$4c - 3 = -5$$

$$4c = -2$$

$$\boxed{c = \frac{-1}{2}}$$

Function is satisfied MVT hypothesis $f(x)$ and values of $c' = \frac{-1}{2}, \frac{-3}{2}$

$\longleftrightarrow$.