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Improper Integrals

Upto now, we have solved proper definite integrals, which have two main properties

(i) Domain of integration is finite
↳ $[a,b] \rightarrow \int_a^b f(x)\,dx$

(ii) Range (answer) of integral is finite.

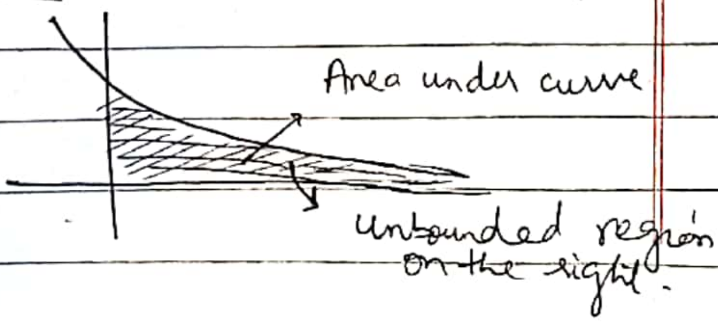
But, sometimes we encounter ~~pro~~ problems that fail to meet one or both of these conditions.

✗ Infinite Limits of Integration

~~There are two types.~~

- Having infinite region (unbounded).
- You might think this region has infinite area, but we will see that the value ~~of~~ is finite.

✗ For example: We have a curve $y = e^{-x/2}$



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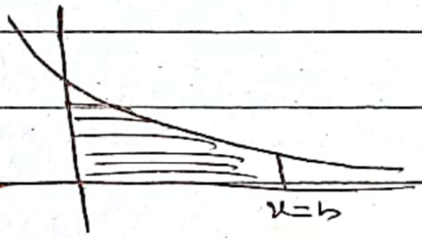
* How to find area of unbounded region

(*) We assign a value to the area: ~~as~~

~~A(b)~~ (I) Find the area $A(b)$ of the portion of the region that is bounded on the right $x=b$



$$A(b) = \int_0^b e^{-x/2}\, dx$$

$$= \left.\frac{e^{-x/2}}{-1/2}\right|_0^b = -2e^{-x/2}\Big|_0^b$$

$$= -2\left[e^{-b/2} - e^0\right]$$

$$= -2e^{-b/2} + 2.$$

(II) Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty}\left(-2e^{-b/2} + 2\right)$$

$$= -2e^{-\infty/2} + 2$$

$$= \frac{-2}{e^{\infty/2}} + 2 = \frac{-2}{\infty} + 2$$

$$= 0 + 2 = \boxed{2} \quad \text{Ans}$$

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(III) The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^{\infty} e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^{b} e^{-x/2}\,dx = 2$$

* Definition: Integrals with infinite limits of integration are improper integrals of Type I

(1) If $f(x)$ is cont: on $[a,\infty)$ then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^{b} f(x)\,dx$$

(2) If $f(x)$ is cont. on $(-\infty, b]$ then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^{b} f(x)\,dx$$

(3) If $f(x)$ is cont. on $(-\infty,\infty)$ then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

where 'c' is any real number.

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* **Note** In each case, [if limit exists and is finite we say that the improper integral converges and that the limit is the value of the improper integral.]*

* [If the limit fails to exist, the improper integral diverges]*

* [If $f \geq 0$ and the improper integral diverges, we say the area under the curve is infinite]*.

① Example 1: From (book)

② Example 2: From (book).

* The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = \frac{1}{x}$ is the boundary between the convergence and divergence improper integrals with integrands of the form $y = \frac{1}{x^p}$.

* **Note** The improper integral converges if $\boxed{p > 1}$ and diverges if $\boxed{p \leq 1}$

3 Example 3 From (book)

* Integrands with Vertical Asymptotes

→ Another type of ~~ind~~ improper integral arises when the integrand has a vertical asymptote — an infinite discontinuity — at a limit of integration, or at some point between the limits of integration.

→ If the integrand f is (+ve) over the interval of integration, we can say that ~~the~~ ~~ind~~ it is the area under the graph f and above x-axis.

* **For example** We have a curve $y = \frac{1}{\sqrt{x}}$ from $x = 0$ to $x = 1$.

At $x = 0$, $y = \frac{1}{\sqrt{x}}$ is discontinuous.

(I) So, first we find the area of the portion from a to 1

$$\int_a^1 \frac{dx}{\sqrt{x}} = \int_a^1 x^{-1/2}\, dx = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_a^1$$

$$= \frac{x^{1/2}}{1/2}\Bigg|_a^1 = 2\sqrt{x}\,\Bigg|_a^1$$

$$= 2 - 2\sqrt{a}.$$

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(II) Then find the limit of this area as $a \to 0^+$ (3)

$$\lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} (2 - 2\sqrt{a})$$

$$= 2 - 2(0) = \boxed{2} \text{ Ans}$$

(III) Therefore the area under curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

★ Definition: Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

(1) If $f(x)$ is cont. on $(a, b]$ and discontinuous at 'a', then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

(2) If $f(x)$ is cont. on $[a, b)$ and discont. at 'b' then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

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(3) If $f(x)$ is discont. at c where $a<c<b$ and cont. on $[a,c)\cup(c,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

'c' — the value at which the func. becomes infinite

Note: In each case, *[if the limit exists and is finite, we say the Improper integral converges]* and *[that the limit is the value of the improper integral.]*.

[If the limit does not exist, the integral diverges].

* Example 4 (From book)
* Example 5 (From book)

* Tests for Convergence and Divergence

→ When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges.

→ The principle tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

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Theorem 2 Direct Comparison Test

Let f and g be cont - on $[a, \infty)$ with $0 \leq f(x) \leq g(x) \quad \forall x \geq a$. Then

(1) If $\int_a^\infty g(x)\,dx$ Converges, then $\int_a^\infty f(x)\,dx$ also converges

(2) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

- How to choose $g(x)$?
  - → If you power in the power then ignore that power e.g. $e^{x^2} \to e^x$.
  - → If you trig function then ignore that and write remaining func e.g. $\frac{\sin^2 x}{x^2} \to \frac{1}{x^2}$

Example 6: Does $\int_1^\infty e^{-x^2}\,dx$ converges?

Sol:

$$f(x) = e^{-x^2}$$

$$g(x) = e^{-x}$$

$$x \leq x^2$$

$$-x \geq -x^2$$

$$e^{-x} \geq e^{-x^2}$$

$$g(x) \geq f(x)$$

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b e^{-x^2}\,dx \quad \text{(Can't solve directly)}$$

$$\leq \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} \left.\frac{e^{-x}}{-1}\right|_1^b$$

$$= \lim_{b\to\infty} \left[-e^{-b} + e^{-1}\right]$$

$$= -e^{-\infty} + \frac{1}{e}$$

$$= -\frac{1}{\infty} + \frac{1}{e} = e^{-1}$$

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* Theorem 3 — Limit Comparison Test

If the positive functions f and g are cont. on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \qquad 0 < L < \infty$$

- positive
- finite
- non negative

Then

$$\int_a^{\infty} f(x)\, dx \text{ and } \int_a^{\infty} g(x)\, dx$$

either both converge or both diverge.

Ex # 8 → from book

Ex # 9 → from book.

Example $\displaystyle\int_0^{\pi} \frac{dx}{\sqrt{x} + \sin x}$

$$f(x) = \frac{1}{\sqrt{x} + \sin x}.$$

$$g(x) = \frac{1}{\sqrt{x}}$$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\frac{1}{\sqrt{x} + \sin x}}{\frac{1}{\sqrt{x}}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{x} + \sin x} =$$

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$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}} + \cos x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{x}}}{\frac{1 + 2\sqrt{x}\cos x}{2\sqrt{x}}}$$

$$= \lim_{x \to 0} \frac{1}{1 + 2\sqrt{x}\cos x}$$

$$= 1 \text{ (finite)}$$

$$\int_0^{\pi} g(x)\,dx = \int_0^{\pi} x^{-1/2}\,dx$$

$$= \left.\frac{x^{-1/2+1}}{-1/2+1}\right|_0^{\pi}$$

$$= \left.\frac{x^{1/2}}{1/2}\right|_0^{\pi} = \left.2\sqrt{x}\right|_0^{\pi}$$

$$= \boxed{2\sqrt{\pi}} \text{ } \mathrm{thus}$$

Both $f(x)$ and $g(x)$ converge