Motion in 2D/3D

- **Position vector:** Vector identifying position of sth from a reference point

e.g: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

- $(-3, 2, 5) \Rightarrow -3\hat{i} + 2\hat{j} + 5\hat{k}$

- **Displacement:** Change in position vector is called displacement →

- $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ ①
- $\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ ②
- $\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$ ③

Average & Instantaneous Velocity

- **Average Velocity:** Displacement divided by time interval.

- $\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$

- $\vec{v}_{avg} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} \Big/ \frac{\Delta x\hat{i}}{\Delta t} + \frac{\Delta y\hat{j}}{\Delta t} + \frac{\Delta z\hat{k}}{\Delta t}$

Unlike displacement

Note: Velocity vector does not extend from one point to another ; only shows magnitude + direction

(no physical length involved)

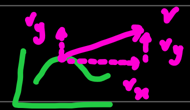
Q- A particle moves through displacement $(12m)i + (3.0m)k$ in 2.0s. Find avg velocity.

$$\vec{v}_{avg} = \frac{12\hat{i} + 3\hat{k}}{2} = (6\,ms^{-1})\,\hat{i} + (1.5\,ms^{-1})\,\hat{k}$$

- **Instantaneous Velocity:** Velocity at a specific point of time.

- $\vec{v} = \lim_{\Delta t \to 0} \frac{dr}{dt}$

- Velocity vector is always tangent to the path at the particle's position



$$\to \vec{v} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k}$$

$$\to \vec{v} = v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k}$$

Average & Instantaneous Acceleration

- **Average Acceleration:** Change in velocity divided by time interval

- $\vec{a}_{avg} = \dfrac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \dfrac{\Delta \vec{v}}{\Delta t}$

- **Instantaneous Acceleration:** $\vec{a} = \dfrac{d\vec{v}}{dt}$; limit $t \to 0$

- $\vec{a} = \dfrac{d}{dt}\left(v_x \hat{i} + v_y \hat{j} + v_z \hat{k}\right)$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

- Acceleration only shows magnitude + direction; does not extend from one point to another (no physical length involved)

Q- 4 descriptions about a particle are given:

1. $x = -3t^2 + 4t - 2$, $y = 6t^2 - 4t$
2. $x = -3t^3 - 4t$, $y = -5t^2 + 6$
3. $\vec{r} = (2t^2)i - (4t+3)j$
4. $\vec{r} = (4t^3 - 2t)i + 3j$

Q- Are x, y acceleration components constant? Is acceleration constant?

1. $v_x = -6t + 4$, $v_y = 12t - 4$

$a_x = \underline{-6}$ $a_y = \underline{12}$

→ All constant

2. $v_x = -9t^2 - 4$, $v_y = -10t$

$a_x = \underline{-18t}$ $a_y = \underline{-10}$

→ a: no, a_y: yes , a_x: no

3. $\vec{v} = 4t\,i - 4j$

$\vec{a} = \underline{\underline{4i}}$

→ All constant

4. $\vec{v} = (12t^2 - 2)\,i$

$\vec{a} = (24t)\,i$

→ a_x: no a_y: yes , a: $\underline{\underline{\text{no}}}$

· Sample Problems: 4.01, 4.02, 4.03

Projectile Motion

- A particle moving with some initial velocity in a vertical plane whose acceleration is always freefall acceleration

- Initial velocity (v_0) → $v_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$

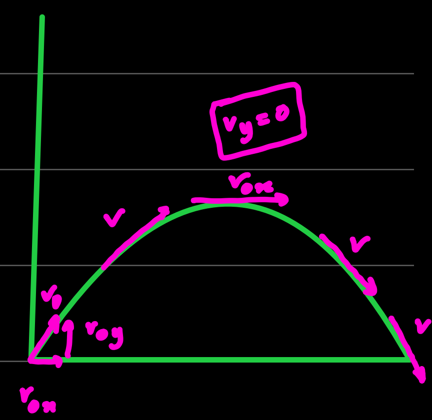
→ $v_{0x} = v_0 \cos\theta$, $v_{0y} = v_0 \sin\theta$

- Horizontal & vertical components are independant of each other.

Q- Velocity at an instant is $25\hat{i} - 4.9\hat{j}$. Has the ball passed its highest point?

- Yes, y component is -ve.

- V_{0x} = Horizontal Component
- V_{0y} = Vertical component



Horizontal Motion

★ Memorize derivations

★ No acceleration so velocity is constant (neglect air resistence)

- $x - x_0 = v_{0x} t$; $s = vt$

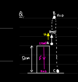
$\Delta x = v_{0x} t$

$\Delta x = (v_0 \cos\theta)\, t$

Only vertical component

+/− SIGNS

ACCELERATION	DISPLACEMENT
STAND AT intialvelocity u	Mark an arrow from START → END and label it s.
IF BODY IS ABOUT TO Speed up ⟶ $a = +ve$ Slow down ⟶ $a = -ve$	Now compare this arrow with initial velocity u If both arrows are in: same direction → $s = +ve$ opp direction → $s = -ve$.



① $v = u + at$

② $s = ut + \frac{1}{2}at^2$

③ $s = \frac{u+v}{2} \times t$

④ $v^2 = u^2 + 2as$

⑤ $s = vt$ $(a = 0)$

Vertical Motion

- Acceleration always acts

- $y - y_0 = v_{0y} t - \frac{1}{2} g t^2$; $s = ut + \frac{1}{2} a t^2$

$\Delta y = v_{0y} t - \frac{1}{2} g t^2$

$\Delta y = (v_0 \sin\theta\, t) - \frac{1}{2} g t^2$

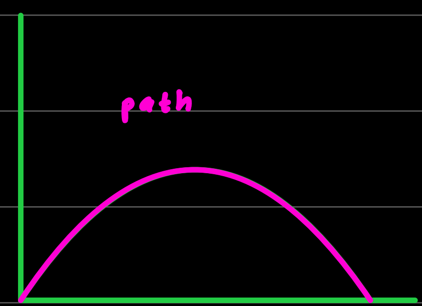
- $v_f = v_0 \sin\theta - gt$; $v = u + at$

- $v_f^2 = (v_0 \sin\theta)^2 - 2gs$; $v^2 = u^2 + 2as$

$$v_f^2 = (v_0 \sin\theta)^2 - 2g(y - y_0)$$

Trajectory

- Distance covered in the parabolic path



① $x - x_0 = v_0 \cos\theta \times t$; $s = vt$

$$t = \frac{x - x_0}{v_0 \cos\theta} \quad - ①$$

→ $y - y_0 = v_0 \sin\theta \times t - \frac{1}{2} g t^2$; $s = ut + \frac{1}{2} a t^2$

$$y - \cancel{y_0}^{\,0} = \cancel{v_0} \sin\theta \times \left(\frac{x - \cancel{x_0}^{\,0}}{\cancel{v_0} \cos\theta} \right) - \frac{1}{2} g \left(\frac{x - \cancel{x_0}^{\,0}}{v_0 \cos\theta} \right)^2$$

$$y = \frac{\sin\theta}{\cos\theta}(x) - \frac{1}{2} g \times \frac{x^2}{(v_0 \cos\theta)^2}$$

$$y = \underbrace{(\tan\theta)}_{a} x + \underbrace{\left(\frac{-g}{2v_0^2\cos^2\theta}\right)}_{b} x^2$$

$$y = ax + bx^2$$

Horizontal Range (Displacement)

- $R = x - x_0$

$R = v_0\cos\theta \times t$; $s = vt$

$t = \frac{R}{v_0\cos\theta}$ — (1)

- $y - y_0 = 0$

- $v_0\sin\theta t - \frac{1}{2}gt^2 = 0$; $s = ut + \frac{1}{2}at^2$

$\frac{1}{2}gt^{\cancel{2}} = v_0\sin\theta \cancel{t}$

$\frac{1}{2}g\left(\frac{R}{v_0\cos\theta}\right) = v_0\sin\theta$ ∴ subbing

$\frac{1}{2}gR = v_0^2\sin\theta.\cos\theta$

$R = \frac{v_0^2 \times 2\sin\theta.\cos\theta}{g}$

$R = \frac{v_0^2\sin 2\theta}{g}$

v_0^2 → const

$\sin 2\theta$ → vary

g → const

∴ $\sin 2\theta = 1$ (max Range when $\theta = 45°$)

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$ / $45°$

Maximum Height

$v_f^2 = v_{oy}^2 - 2g(y - y_0)$; $v^2 = u^2 + 2as$

$v_f^2 = (v_0 \sin\theta)^2 - 2g(y - y_0)$

$\therefore \boxed{y - y_0 = H \;,\; v_{oy} = 0 \quad \text{(at max vertical height)}}$

$0 = (v_0 \sin\theta)^2 - 2gH$

$2gH = (v_0 \sin\theta)^2$

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

Total time of flight

- $v_y = 0$

$\rightarrow$ $0 = v_0 \sin\theta - gt$; $v = u + at$

$gt = v_0 \sin\theta$

$$\boxed{t = \frac{v_0 \sin\theta}{g}}$$ (Half-journey)

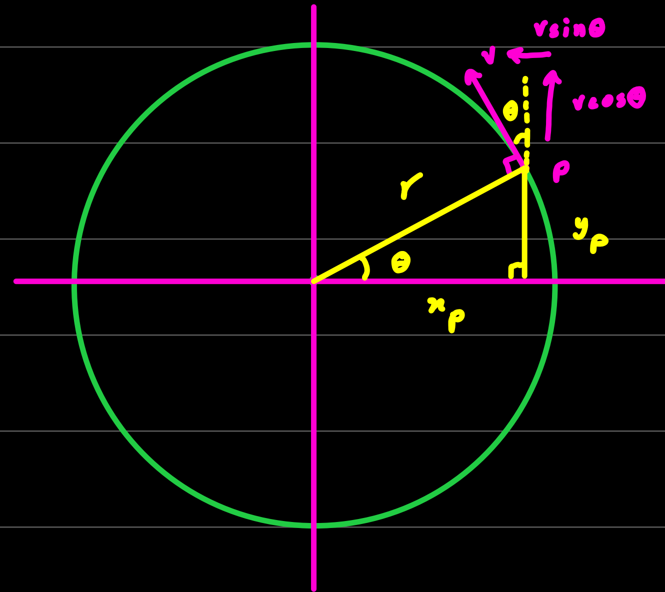
- $$\boxed{2t = \frac{2v_0 \sin\theta}{g}}$$ (Full-journey)

Uniform Circular Motion

- Particle / object travels in a circular path or circular arc
- Particle's speed remains constant, but its velocity changes as its direction changes.
- Velocity is always tangent to the circular path followed by the particle.
- Since velocity changes, particle accelerates
- Acceleration is always radially inward, hence it is called centripetal acceleration

- $a = \frac{v^2}{r}$ (centripetal acceleration)
- $T = \frac{2\pi r}{v}$ (Time-period)

Derivation



$$v_x = -v\sin\theta \quad , \quad v_y = v\cos\theta \quad - \text{①}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$\vec{v} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j} \quad - \boxed{\text{Sub 1}}$$

$$\sin\theta = \frac{y_p}{r} \quad , \quad \cos\theta = \frac{x_p}{r} \quad - \text{②}$$

$$\vec{v} = \left(-\frac{v y_p}{r}\right)\hat{i} + \left(\frac{v x_p}{r}\right)\hat{j} \quad - \boxed{\text{Sub 2}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(-\frac{v y_p}{r}\right)\hat{i} + \frac{d}{dt}\left(\frac{v x_p}{r}\right)\hat{j}$$

$$\vec{a} = -\frac{v}{r}\left(\frac{dy_p}{dt}\right)\hat{i} + \frac{v}{r}\left(\frac{dx_p}{dt}\right)\hat{j}$$

$$\frac{dy_p}{dt} = v\cos\theta \quad , \quad \frac{dx_p}{dt} = -v\sin\theta \quad - \text{③}$$

$$\vec{a} = -\frac{v}{r}\left(v\cos\theta\right)\hat{i} + \frac{v}{r}\left(-v\sin\theta\right)\hat{j}$$

$$\boxed{\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{i} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{j}}$$

$$|\vec{a}| = \sqrt{\left(-\frac{v^2}{r}\cos\theta\right)^2 + \left(-\frac{v^2}{r}\sin\theta\right)^2}$$

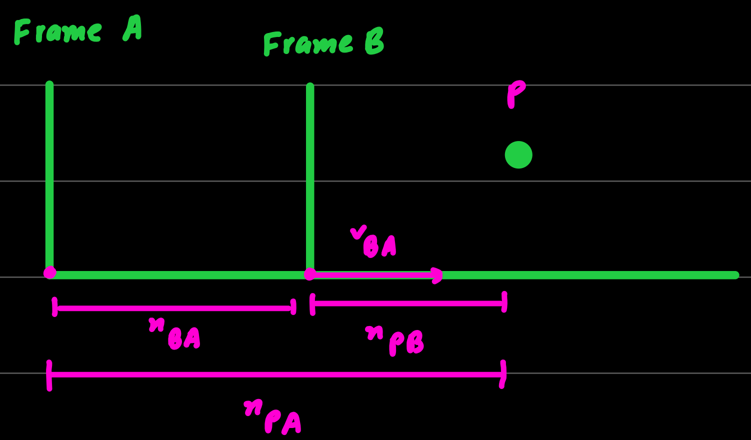
$$a = \sqrt{\frac{v^4}{r^2}\cos^2\theta + \frac{v^4}{r^2}\sin^2\theta}$$

$$a = \sqrt{\frac{v^4}{r^2}\left(\sin^2\theta + \cos^2\theta\right)}$$

$$\boxed{a = \sqrt{\frac{v^4}{r^2}(1)} = \frac{v^2}{r}}$$

$$\cdot\ \tan\phi = \frac{-\cancel{\frac{v^2}{r}}\sin\theta}{-\cancel{\frac{v^2}{r}}\cos\theta} = \underline{\underline{\tan\theta}}$$

Relative Motion in 1-D



; v_{BA} is constant

$$\cdot\ x_{PA} = x_{BA} + x_{PB}$$

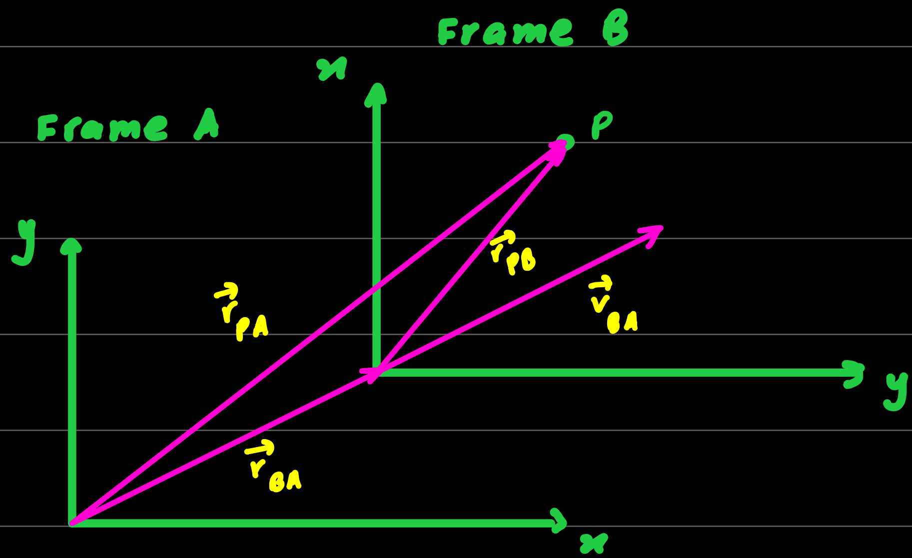
$$\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{BA}) + \frac{d}{dt}(x_{PB})$$

$$v_{PA} = v_{BA} + v_{PB}$$

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{BA}) + \frac{d}{dt}(v_{PB})$$

$\boxed{a_{PA} = a_{PB}}$ v_{BA} is constant, so no acceleration

Relative Motion in 2 Dimensions



- $\vec{r}_{PA} = \vec{r}_{BA} + \vec{r}_{PB}$

$$\frac{d}{dt}\left(\vec{r}_{PA}\right) = \frac{d}{dt}\left(\vec{r}_{BA}\right) + \frac{d}{dt}\left(\vec{r}_{PB}\right)$$

- $\vec{v}_{PA} = \vec{v}_{BA} + \vec{v}_{PB}$

constant

$$\frac{d}{dt}\left(\vec{v}_{PA}\right) = \cancel{\frac{d}{dt}\left(\vec{v}_{BA}\right)} + \frac{d}{dt}\left(\vec{v}_{PB}\right)$$

- $\boxed{\vec{a}_{PA} = \vec{a}_{PB}}$

- Note: For One-Dimensional motion, Observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

- Sample Probs: 4.04 , 4.05

- Exercise Probs: 2, 3, 6, 7, 11, 13, 14, 15, 16, 22, 23, 24, 25, 27, 28, 31, 33