Ex.1 Find a parameterization of the cone

$z = \sqrt{x^2+y^2}$; $0 \le z \le 1$

Let $x = r\cos\theta$ $y = r\sin\theta$ and $z = \sqrt{x^2+y^2} = \sqrt{r^2} =$

where $0 \le r \le 1$ & $0 \le \theta \le 2\pi$

we have the parameterization

$\vec{r}(r,\theta) = (r\cos\theta)\underline{i} + (r\sin\theta)\underline{j} + r\,\underline{k}$

with $0 \le r \le 1$; $0 \le \theta \le 2\pi$

$$\vec{r}_r = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} + \underline{k}$$

$$\vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\underline{i} + r\cos\theta\,\underline{j} + \underline{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$\vec{r}_r \times \vec{r}_\theta = -r\cos\theta\,\underline{i} - r\sin\theta\,\underline{j} + r\,\underline{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\,r$$

$$\text{Surface Area} = \int_0^{2\pi}\int_0^1 \sqrt{2}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\,d\theta = \frac{\sqrt{2}}{2}\Big|\theta\Big|_0^{2\pi}$$

$$= \sqrt{2}\,\pi\ (\text{unit})^2$$

Ex.2 Find a Parameterization and surface area of the sphere $x^2+y^2+z^2=a^2$

We consider

$x = a\sin\phi\cos\theta \quad y = a\sin\phi\sin\theta \quad z = a\cos\phi$

$0 \le \phi \le \pi \; ; \; 0 \le \theta \le 2\pi$

$\vec{r}(\phi,\theta) = (a\sin\phi\cos\theta)\underline{i} + (a\sin\phi\sin\theta)\underline{j} + a\cos\phi\,\underline{k}$

with $0 \le \phi \le \pi \; ; \; 0 \le \pi \le 2\pi$

$$\vec{r}_\phi \times \vec{r}_\theta' = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$\vec{r}_\phi \times \vec{r}_\theta = (a^2\sin^2\phi\cos\theta)\underline{i} + (a^2\sin^2\phi\sin\theta)\underline{j} + a^2\sin\phi\cos\phi\,\underline{k}$

$|\vec{r}_\phi \times \vec{r}_\theta| = a^2\sin\phi$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left|-a^2\cos\phi\right|_0^{\pi} d\theta = \int_0^{2\pi} 2a^2 d\theta = 4\pi a^2 \; (\text{unit})^2$$

Ex-3 Find S.A from the bottom of Paraboloid

$x^2+y^2-z=0$ by the Plane $z=4$

Here we can choose

$x^2+y^2 \leq 4$ & $\vec{p} = \underline{k}$

& $F(x,y,z) = x^2+y^2-z$

$\nabla F = 2x\underline{i} + 2y\underline{j} - \underline{k}$

$|\nabla F| = \sqrt{4x^2+4y^2+1}$

$|\nabla F \cdot P| = |-1| = \sqrt{1} = 1$

$$S.A = \int_0^{2\pi}\int_0^{2} (4r^2+1)^{1/2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{12}\right] (4r^2+1)^{3/2} \Big|_0^2 \, d\theta$$

$$= \frac{\pi}{6}\left(17\sqrt{17}-1\right) (\text{unit})^2$$

Ex.4 $F(x,y,z) = x^2$ over the cone $z = \sqrt{x^2+y^2}$;

$0 \le z \le 1$

from Ex1.

$$\iint_S x^2 \, d\sigma = \int_0^{2\pi}\int_0^1 (r^2 \cos^2\theta)\sqrt{2}\, r \, dr \, d\theta$$

Because

(1) $\because \iint_S F(x,y,z)\, d\sigma = \iint_R F(f(u,v), g(u,v), h(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, du \, dv$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3 \cos^2\theta \, dr \, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}$$