Chapter 4

Imp for Exams

Topic 4.5 : Applied Optimization

Solving Applied Optimization Problems : Pg 214.

1) Read the problem
   - What is given?
   - What is the unknown quantity to be optimized?

2) Draw a picture — Label it

3) Introduce variables — Make relationship between known and unknown variables.

4) Write an equation for unknown quantity — express the unknown as a function of single variable.

5) Test the critical points and endpoints in the domain of the unknown / Use First Derivative Test / Use Second Derivative Test to find local extreme values

Optimization
- Objective — Minimization or Maximization
- Constraint — Constant parameter (cannot change it)

$\boxed{\text{Example 1}}$ Pg 214

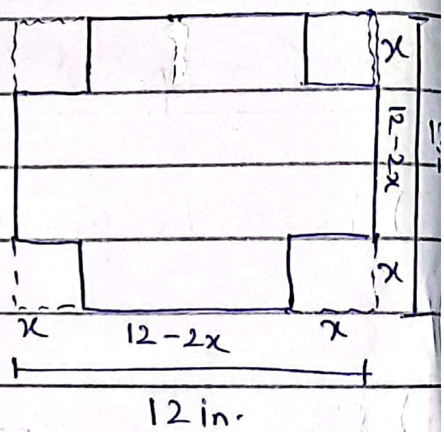
(Statement from book)

Sol: Here

Finite Volume — Constraint

To choose value of x of max. size of square — objective.

$x = ?$



$V = L \times W \times H$.

$$V(x) = (12-2x)(12-2x)x \quad \text{— (1)}$$

$$= (144 - 24x - 24x + 4x^2)\,x$$

$$V(x) = 4x^3 - 48x^2 + 144x$$

Interval: From eq (1)

Method 1 $\quad V(x) = x(12-2x)^2$.

$V(x) = 0$.

$x(12-2x)^2 = 0$

$\boxed{x = 0}$, $\quad (12-2x)^2 = 0$

$12 = 2x \Rightarrow \boxed{x = 6}$

At $x = 0$, $V(0) = 0$

At $x = 6$, $V(6) = 0$

Domain of x $\quad 0 \le x \le 6$.

Critical point: (Interior point on $0 < x < 6$).

$$V'(x) = 12x^2 - 96x + 144.$$

$$V'(x) = 0.$$

$$12x^2 - 96x + 144 = 0.$$

$$12(x^2 - 8x + 12) = 0 \Rightarrow x^2 - 8x + 12 = 0.$$

$x^2 - 6x - 2x + 12 = 0$.

$x(x-6) - 2(x-6) = 0 \Rightarrow (x-6)(x-2) = 0$

$\boxed{x=2}$, $\boxed{x=6}$

Only $x=2$ lies in the interior of the function's domain and makes the critical point list.

The values of $V(x)$ at this critical point $(x=2)$ and two endpoints $(x=0, x=6)$ are.

$V(2) = 128$. , $V(0) = 0$, $V(6) = 0$.

Maximum Volume is 128 (inches)^3 at $x = 2$.

The cutout squares should be 2 in. on a side.

$\longleftrightarrow$

Method 2. $V'(x) = 12x^2 - 96x + 144$.

$V'(x) = 0 \Rightarrow 12x^2 - 96x + 144 = 0$.

$12(x^2 - 8x + 12) = 0$

$\boxed{x=2}$, $\boxed{x=6}$

In these questions, we consider ~~positive values~~ (finite)

Using First Derivative Test:

✓ ✓ x.

$(0, 2)$, $(2, 6)$, $\underbrace{(6, \infty)}_{\text{Reason}}$ → ~~This is the~~ Domain of x. If we put it in $12-2x$, it will become $-$ve. So ignore this interval.

$V'(1) = 60$ (+ve)

$V'(3) = -36$ ($-$ve).

Max. at $x = 2$.

$V(2) = 4(2)^3 - 48(2)^2 + 144(2)$

$= 128$.

Max. Volume $128 in^3$ at $x = 2$.

$\longleftrightarrow$

Method 3 $V'(x) = 12x^2 - 96x + 144$,

$V'(x) = 0 \Rightarrow 12x^2 - 96x + 144 = 0$.

$12(x^2 - 8x + 12) = 0$

$\boxed{x = 2}$, $\boxed{x = 6}$

$V''(x) = 24x - 96$.

Using Second Derivative Test:

At $x = 2 \Rightarrow V'(2) = 0$. So $V''(2) = -48 < 0$ Local Max.

At $x = 6 \Rightarrow V'(6) = 0$. So $V''(6) = 48 > 0$ Local Min. (not required).

$V(2) = 4(2)^3 - 48(2)^2 + 144(2) = 128$.

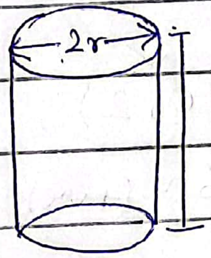
Max Volume 128 in^3 if $x = 2$.

$\boxed{\text{Example 2}}$ Pg 215

(Statement from book)

Sol: → Use minimum material (objective)

→ $V = 1000 \text{cm}^3 = 1$ Litre (constraint)



$V = \pi r^2 h$. — ①

Surface Area → Minimum (Required).

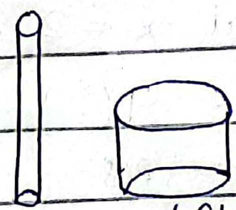
$A = \underbrace{2\pi r^2}_{(2 \text{ circles})} + \underbrace{2\pi r h}_{(\text{Wall})} \Rightarrow$ Total Surface Area.

↳ ②

From eq ①

$1000 = \pi r^2 h$

$h = \dfrac{1000}{\pi r^2}$



$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$A(r) = 2\pi r^2 + \frac{2000}{r}.$$

$$A'(r) = 4\pi r - \frac{2000}{r^2}$$

Critical point : $A'(r) = 0.$

$$4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$4\pi r^3 = 2000$$

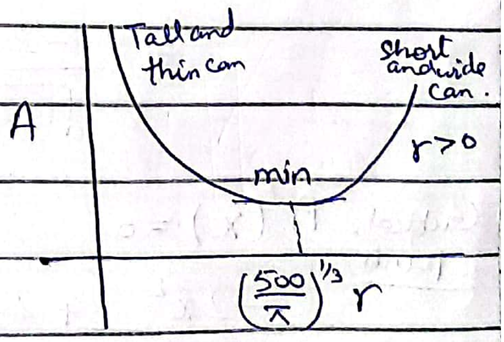
$$r^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \simeq 5.42 \text{ cm}$$

$$A''(r) = 4\pi + \frac{4000}{r^3}$$

Using Second Derivative Test,

$$A''(5.42) = 4\pi + \frac{4000}{(5.42)^3}$$



$$= 37.69 \; (+ve) > 0 \; (\text{Local Min})$$

$\downarrow$

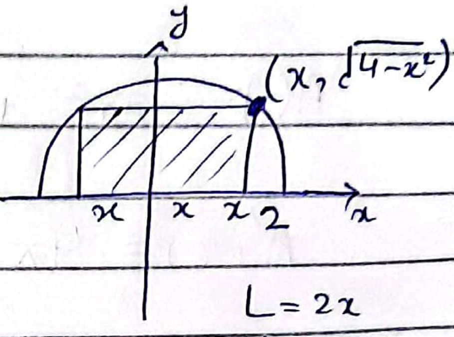
concave up.

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(5.42)} = 10.84 \text{ cm.}$$

The one litre can that uses the least material has height equal to twice the radius.

$\boxed{\text{Example 3}}$ Pg 216

(Statement from book).



Sol: Objective → Max. area of rectangle

Constraint → rectangle should be Inscribed in semi-circle

$L = 2x$

$H = \sqrt{4-x^2}$

$x^2+y^2 = r^2$ (eq. of circle)

$x^2+y^2 = 4.$

$y^2 = 4-x^2$ (eq. of semi-circle)

$y = \sqrt{4-x^2}$ (consider positive value).

Domain of x $\quad 0 \leq x \leq 2$ (considering positive side only).

$A = L \times W = 2x\sqrt{4-x^2}.$

(Length x Height)

$A(x) = 2x\sqrt{4-x^2}$

$A'(x) = \cancel{2}x\left(\frac{1}{\cancel{2}}(4-x^2)^{-1/2}(-2x)\right) + 2\sqrt{4-x^2}$

$= \dfrac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}.$

Critical points: $A'(x) = 0$

$\dfrac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0.$

$\dfrac{-2x^2 + 2(4-x^2)}{\sqrt{4-x^2}} = 0.$

$-2x^2 + 2(4-x^2) = 0.$

$-2x^2 + 8 - 2x^2 = 0$

$$8-4x^2=0$$
$$x^2=2$$
$$x=\pm\sqrt{2}.$$

At $x=2$, $x=\sqrt{2}$, $x=-\sqrt{2}$ $\Rightarrow$ $A'(x)=0$.

Only $x=\sqrt{2}$ lies in the interior of function's domain and makes the critical point list.

The values of $A(x)$ at the endpoints and at this one critical point are

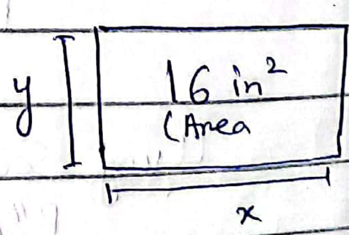
$$A(\sqrt{2})=2\sqrt{2}\sqrt{4-2}=4.$$
$$A(0)=0\quad,\quad A(2)=0.$$

The area has a max. value of 4 when the rectangle is $\sqrt{4-x^2}=\sqrt{2}$ units high and $2x=2\sqrt{2}$ unit long.

$\longleftrightarrow$ EX 4.5

$\boxed{\text{EX \# 4.5}}$ Pg 220 (Practice Q# 1-20)

$\boxed{\text{Q\#1}}$ (Statement from book).



Sol: $A=16\,in^2 \longrightarrow (1)$

$x=?\ ,\ y=?\ ,\ \text{Min. } P=?$

$$A=xy \longrightarrow (2)$$
$$P=2(x+y)=2x+2y \longrightarrow (3)$$

From (1) and (2)

$$16=xy$$
$$y=\frac{16}{x}$$

Put in (3)

$$P = 2x + 2y = 2x + 2\left(\frac{16}{x}\right)$$

$$P = 2x + \frac{32}{x} = 2x + 32x^{-1}$$

$$P(x) = 2x + 32x^{-1}$$

$$P'(x) = 2 - \frac{32}{x^2}$$

Critical Point : $P'(x) = 0$

$$2 - \frac{32}{x^2} = 0$$

$$2 = \frac{32}{x^2}$$

$$x^2 = 16$$

$$x = \pm 4.$$

Here we consider $\boxed{x = 4}$ because side of rectangle is never negative.

$$P''(x) = \frac{64}{x^3}.$$

Using Second Derivative Test

$$P''(4) = \frac{64}{(4)^3} = 1 > 0.$$ Local Min.

Thus at $x = 4$, perimeter is min.

Value of sides at min. perimeter is $x = 4$ and $y = 4$.

Q#2 Sol:

$P = 8m.$

Largest Area is a Square = ?

Perimeter of rectangle $P = 2x + 2y.$

$8 = 2(x+y)$

$4 = x+y$

$y = 4-x.$

$A = xy = x(4-x)$

$A = 4x - x^2.$

$A'(x) = 4-2x.$

$A'(x) = 0$

$4-2x = 0 \Rightarrow \boxed{x=2}$

$A''(x) = -2 < 0$ Local Max.

Value of y when area of rectangle is max.

$y = 4-2 = 2$

$\boxed{y=2}$

We get equal values of sides at which area of rectangle is max. it means that square has largest area in among all rectangle of perimeter of 8m.

Q#3 Sol: Since θ is an angle of a triangle $0 < \theta < 180^\circ$. Since the max. value of $\sin\theta$ is 1 and the only angle between 0 and 180° for $\sin\theta$ to be 1 is 90°. $\theta = 90^\circ$ will maximize the triangle's area. In that case, the max. area of the triangle will be $A = \frac{1}{2}ab.$