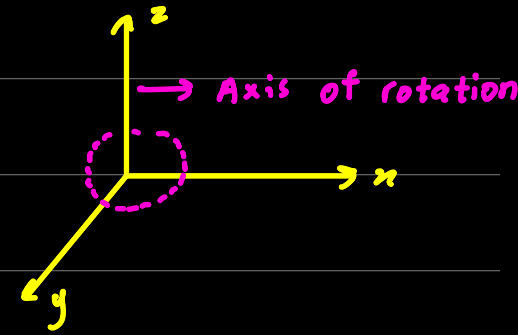
Greens Theorem

(16.4)

· (Curl of $\vec{F}$) · $\hat{k}$

$\underbrace{\vec{\nabla} \times \vec{F}}_{\text{curl}}$ = circulation, rotation

$= (\vec{\nabla} \cdot \vec{F}) \cdot \hat{k}$ → circulation density



· $\vec{F} = M\hat{i} + N\hat{j}$; $\vec{F}$ is in xy plane

· $\vec{\nabla} = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y}$

→ $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

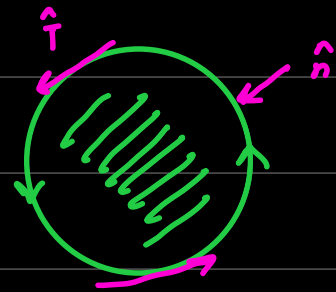
$$= 0\hat{i} - 0\hat{j} + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$(\vec{\nabla} \times \vec{F}) \cdot \hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k} \cdot \hat{k}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

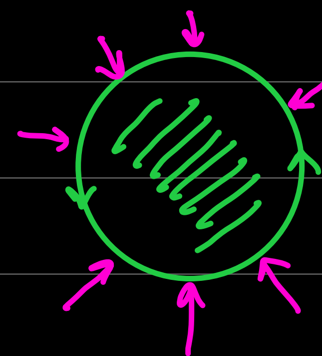
Tangential Form



· $\underbrace{\oint_C \vec{F}\cdot\hat{T}\,ds}_{\text{line Integral}} = \int M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\hat{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$

Normal Form



· If normal, then flux.

$\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx$

Expansion/compression/static

$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot\left(M\hat{i} + N\hat{j}\right)$

$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

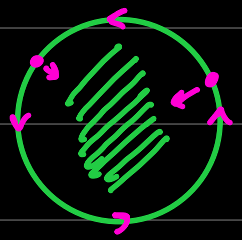
$= \iint_R \vec{\nabla}\cdot\vec{F}\,dx\,dy$

$= \iint_R \left(\overbrace{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}^{\text{divergence}}\right) dx\,dy$

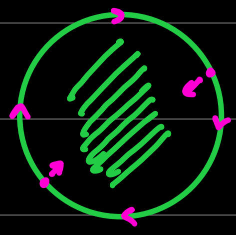
$\vec{\nabla}\cdot\vec{F} < 0$ compression

$\vec{\nabla}\cdot\vec{F} > 0$ expansion

$\vec{\nabla}\cdot\vec{F} = 0$ static



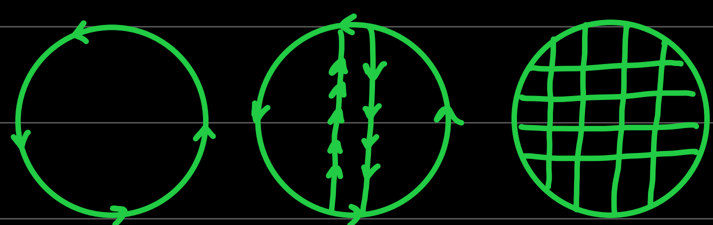
positive orientation, reigon on left side of object



negative orientation, reigon on right side of the object

$\oint$ Proof

- Suppose an area is enclosed by the closed path C in a vector field $\vec{A}$. This can be considered to be made up of infinite number of closed paths



· Consider one element of area $\Delta s \hat{i}$

curl $= \vec{\nabla} \times \vec{A}$ = line integral of A per unit Area

$$\text{curl } \vec{A} = \oint_c \frac{\vec{A} \cdot d\vec{r}}{\text{Area}}$$

$$(\vec{\nabla} \times \vec{A}) \cdot \vec{s}_i = \oint_{c_i} \vec{A} \cdot d\vec{r} \quad \text{— (1)}$$

Adding all such pieces

$$\sum_{i=1}^{n} (\vec{\nabla} \times \vec{A})\, \Delta \vec{s}_i = \sum_{i=1}^{n} \oint_{c_i} \vec{A} \cdot d\vec{r} \quad \text{— (2)}$$

If $n \to \infty$, then $\Delta s \to 0$

$$\iint_s (\vec{\nabla} \times \vec{A})\, d\vec{s} = \int_c \vec{A} \cdot d\vec{r}$$