Dated: Wednesday 6th Dec, '23

# LECTURE #31

## 8.6 INTEGRAL TABLES

A table of Integrals is provided at the end of the book (after the index) → MEMORIZE !!!

→ Example #1

find $\int x(2x+5)^{-1}\,dx$

**SOLUTION**

* FORMULA *

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + c$$

Here, $a=2$, $b=5$

$$\Rightarrow \int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + c$$

Dated:

→ **Example #2**

Find $\int \frac{dx}{x\sqrt{2x-4}}$

**Solution**

*Formula*

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1}\sqrt{\frac{ax-b}{b}} + C$$

Here, $a=2$, $b=4$

$$\Rightarrow \int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1}\sqrt{\frac{2x-4}{4}} + C$$

$$= \tan^{-1}\sqrt{\frac{x-2}{2}} + C$$

Dated:

→ EXAMPLE #3

Find $\int x \sin^{-1} x \, dx$

SOLUTION

* FORMULA *

$$\int x^n \sin^{-1} ax \, dx = \frac{x^{n+1}}{n+1} \sin^{-1} ax - \frac{a}{n+1} \int \frac{x^{n+1} \, dx}{\sqrt{1-a^2x^2}} , \quad n \neq -1$$

Here, $n=1$, $a=1$,

$$\Rightarrow \int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2 \, dx}{\sqrt{1-x^2}}$$

* FORMULA *

$$\int \frac{x^2}{\sqrt{a^2-x^2}} \, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2} x \sqrt{a^2-x^2} + c$$

Here, $a=1$,

$$\Rightarrow \int \frac{x^2 \, dx}{\sqrt{1-x^2}} = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C$$

$$\Rightarrow \int x \sin^{-1} x = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2}\left(\frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C\right)$$

$$= \left(\frac{x^2}{2} - \frac{1}{4}\right) \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + c'$$

Dated:

## → Reduction Formulas

1) $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

2) $$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

3) $$\int \sin^n x \cos^m x\,dx = \frac{\sin^{n-1}x \cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2}x \cos^m x\,dx,$$
$(n \neq -m)$

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## → Example #4

Find $\int \tan^5 x\,dx$

**Solution**

Using formula '1', $n=5$,

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$$

Using formula '1' again, $n=3$,

$$\Rightarrow \int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + c$$

$$\Rightarrow \int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + c'.$$

Dated:

## 8.8 Improper Integrals

→ Integrals with infinite limits of integration are Improper Integrals of Type-1.

1) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where, 'c' is any real number.

→ In each case, if the limit exists and is finite, the improper integral converges and the limit is the value of the improper integral.

→ If the limit fails to exist, the improper integral diverges.

ated:

→ **Example #1**

Evaluate $\int_1^\infty \frac{\ln x}{x^2}\,dx$

**Solution**

Integration By Parts:

Let, $u = \ln x$, $dv = dx/x^2$

$\Rightarrow du = \frac{1}{x}\,dx$, $v = -1/x$.

$\Rightarrow \int_1^\infty \frac{\ln x}{x^2}\,dx = uv\Big]_1^b - \int_1^b v\,du$, then take limit as $b \to \infty$

$$= \left[(\ln x)(-1/x)\right]_1^b - \int_1^b -1/x \times \frac{1}{x}\,dx$$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty} \ln b/b\right] - 0 + 1 \quad \text{(L'Hospital's Rule)}$$

$$= -\left[\lim_{b\to\infty} \frac{1/b}{1}\right] + 1$$

$$= 0 + 1 = 1$$

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Dated:

→ EXAMPLE #2

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

SOLUTION

→ DEFINITION (PART-3):

Let, $c = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

Now,

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \tan^{-1}x \Big]_{a}^{0}$$

$$= \lim_{a \to -\infty} (\tan^{-1}0 - \tan^{-1}a)$$

$$= 0 - (-\pi/2) = \pi/2$$

Similarly,

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{b \to \infty} \tan^{-1}x \Big]_{0}^{b}$$

$$= \lim_{b \to \infty} (\tan^{-1}b - \tan^{-1}0) = \pi/2 - 0 = \pi/2$$

Dated:

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi/2 + \pi/2 = \pi.$$

* The integral $\int_1^{\infty} \frac{dx}{x^p}$ converges if $p>1$ and diverges if $p \le 1$, because the function $y = 1/x$ is the boundary between the convergent and divergent improper integrals.

→ Example #3

For what values of 'p' does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution

* If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \left. \frac{x^{-p+1}}{-p+1} \right]_1^b$$

$$= \frac{1}{1-p}\left(b^{-p+1} - 1\right)$$

$$= \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)$$

Dated:

$$\Rightarrow \int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right]$$

$$= \begin{cases} 1/p-1 &, p>1 \\ \infty &, p<1 \end{cases}$$

because $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0 &, p>1 \\ \infty &, p<1 \end{cases}$

$\Rightarrow$ The integral converges to the value $1/(p-1)$ if $p>1$ and it diverges if $p<1$.

* If $p=1$, the integral also diverges.

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x}$$

$$= \lim_{b\to\infty}\int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x \Big]_1^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 1)$$

$$= \infty.$$

Dated:

→ Integrals of functions that become infinite at a point within the interval of integration are Improper Integrals of Type-2.

1) If $f(x)$ is continuous on $(a,b]$ and discontinuous at 'a', then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

2) If $f(x)$ is continuous on $[a,b)$ and dis-continuous at 'b', then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

3) If $f(x)$ is discontinuous at 'c', where $a<c<b$, and continuous on $[a,c) \cup (c,b]$, then:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

→ In each case, if the limit exists and is finite, the improper integral converges and the limit is the value of the improper integral.

→ If the limit does not exist, the integral diverges.

Dated:

→ **EXAMPLE #4**

Investigate the convergence of $\int_0^1 \frac{1}{1-x}\,dx$

**SOLUTION**

→ The integrand $f(x) = 1/(1-x)$ is continuous on $[0,1)$, but is discontinuous on $x=1$, and becomes infinite as $x \to 1^-$

$$\Rightarrow \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx$$

$$= \lim_{b\to 1^-} \Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-} \Big[-\ln|1-b| + 0\Big] = \infty$$

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→ **EXAMPLE #5**

Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

**SOLUTION**

The integrand has a vertical asymptote at $x=1$, and is continuous on $[0,1)$ and $(1,3]$.

→ Using definition (Part-3)

$$\Rightarrow \int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

Dated:

$$\star \int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b\to 1^-} 3(x-1)^{1/3} \Big]_0^b$$

$$= \lim_{b\to 1^-} \left[ 3(b-1)^{1/3} + 3 \right] = 3$$

$$\star \int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{c\to 1^+} 3(x-1)^{1/3} \Big]_c^3$$

$$= \lim_{c\to 1^+} \left\{ 3(3-1)^{1/3} - 3(c-1)^{1/3} \right\}$$

$$= 3\sqrt[3]{2}$$

$$\Rightarrow \int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}$$