Ex 1 $\sum_{0}^{\infty} \frac{2^n}{(2n)!}$

Sol:-

$a_n = \frac{2^n}{(2n)!}$; $a_{n+1} = \frac{2^{n+1}}{(2n+2)!}$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}}{(2n+2)!} \times \frac{(2n)!}{2^n}$

$= \lim_{n \to \infty} \frac{2}{(2n+2)(2n+1)} = 0 < 1$

So the given series is convergent by ratio test.

Ex 2 $\sum_{1}^{\infty} \frac{n!}{n^n}$

Sol:- $a_n = \frac{n!}{n^n}$

$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$

$= \lim_{n \to \infty} \frac{\cancel{(n+1)}}{\cancel{(n+1)}} \times \left(\frac{n}{n+1}\right)^n$

$= \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$

$$\sum_{1}^{\infty} \frac{7^n}{n \cdot 5^{n+1}}$$

$$\sum_{1}^{\infty} \frac{2^n}{n(n+2)}$$

$$\sum_{1}^{\infty} \frac{n}{n^2+1}$$

$$\sum_{1}^{\infty} \frac{1}{n^n}$$

$$\sum_{1}^{\infty} \frac{(2n)!}{4^n}$$

$$\sum_{1}^{\infty} \frac{\ln n}{e^n}$$

Ex. $\displaystyle\sum_{1}^{\infty} \left(\frac{3n+2}{2n-1}\right)^n$

$$a_n = \left(\frac{3n+2}{2n-1}\right)^n$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{3n+2}{2n-1}\right)$$

$$= \lim_{n\to\infty} \frac{3 + 2/n}{2 - 1/n}$$

$$= 3/2 > 1$$

Hence the given series is divergent by root test.

Ex. $\displaystyle\sum_{1}^{\infty} \frac{1}{n^n}$

$$a_n = \frac{1}{n^n}$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

$\circledast$ $\sum_{1}^{\infty} \frac{3^n}{n^3}$

$\circledast$ $\sum_{1}^{\infty} \frac{e^n}{(\ln n)^n}$

$\circledast$ $\sum_{1}^{\infty} n \cdot \left(\frac{\pi}{n}\right)^n$

$\circledast$ $\sum_{1}^{\infty} \frac{(n!)^2 2^n}{(2n+2)!}$

Ex. $\sum_{1}^{\infty} \frac{e^{\tan^{-1} n}}{1+n^2}$

$$f(x) = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)\, e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} - e^{\tan^{-1} x} \cdot 2x}{(1+x^2)^2}$$

$$= \frac{e^{\tan^{-1} x}(1-2x)}{(1+x^2)^2} < 0 \; ; \forall\; x \geq 1$$

Now we calculate

$$\int_1^\infty \frac{e^{\tan^{-1} x}}{1+x^2}\, dx = \lim_{t\to\infty} \left. e^{\tan^{-1} x}\right|_1^t = e^{\pi/2} - e^{\pi/4}$$

By integral test given series is convergent.

Ex. $\sum_{2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

$$f(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$f'(x) = \frac{1-2x^2}{x^2(x^2-1)} < 0 \quad \forall\; x \in [2,\infty)$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \left. \sec^{-1} x \right|_2^t = \sec^{-1}\infty - \sec^{-1} 2$$

$$\sum_{1}^{\infty} \frac{n^2}{e^n}$$

$$\sum_{1}^{\infty} \frac{1}{(n+1)(\ln(n+1))^2}$$

Ex. $\sum_{1}^{\infty} (-1)^{n-1} \left(\frac{n+2}{3n-1}\right)^n$

$$|a_n| = \left|\frac{n+2}{3n-1}\right|^n$$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left|\frac{n+2}{3n-1}\right|$$

$$\lim_{n\to\infty}\left(\frac{1+2/n}{3-1/n}\right) = 1/3 < 1$$ cgt. absolutely

Ex. $\sum_{1}^{\infty} (-1)^{n-1} \frac{n^2}{(n+2)!}$

$$|a_n| = \left|\frac{n^2}{(n+2)!}\right|$$

$$|a_{n+1}| = \left|\frac{(n+1)^2}{(n+3)!}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(n+3)!} \times \frac{(n+2)!}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{n+3}$$

$$= 0 < 1$$

Hence. A.C

Ex. $\sum_{1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$.

$|a_n| = \left|\frac{\sqrt{n}}{n+1}\right|$

$|a_{n+1}| = \frac{\sqrt{n+1}}{n+2}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}}{n+2} \times \frac{n+1}{\sqrt{n}}$

$= \lim_{n\to\infty}\frac{(n+1)^{3/2}}{n^{3/2}+2n^{1/2}}$

$1+\frac{1}{2}$

$n^{1/2} \cdot n^{-3/2}$

$\frac{1-3}{2} = -1$

$= \lim_{n\to\infty}\frac{(1+1/n)^{3/2}}{1+\frac{2}{n}}$

$= 1$ test fail.

$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Find also radius of convergence.

$\downarrow$

$|x|<1$

$|a_n| = \left|\frac{x^{2n}}{2^{2n}(n!)^2}\right|$

use ratio test.