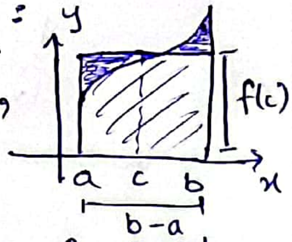
Lec - 18

Topic 5.4: The Fundamental Theorem of Calculus

① **Theorem 3 – The Mean Value Theorem for Definite Integrals:**

If f is cont. on $[a,b]$, then at some point c in $[a,b]$,

$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



$A = L \times W$

$\int_a^b f(x)dx = f(c) \times b-a$

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

② **Theorem 4 – The Fundamental Theorem of Calculus (Part 1)**

If f is cont. on $[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is cont. on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$\boxed{F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Here $F(x) = \int_a^x f(t)dt$

Example 2: Use Theorem 4 to find dy/dx if (Pg 280)

(a) $y = \int_a^x (t^3+1)\,dt$ (b) $y = \int_x^5 3t\sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$ (d) $y = \int_{1+3x^2}^4 \frac{1}{2+t}\,dt$.

Sol: (a)

$$\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1. \quad \text{Ans}$$

(b)

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\,dt = \frac{d}{dx}\left[-\int_5^x 3t\sin t\,dt\right]$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x. \quad \text{Ans}$$

(P.T.O)

(c) $y = \int_1^{x^2} \cos t \, dt$

Sol:

Let $u = x^2$.

$du = 2x\,dx \Rightarrow \frac{du}{dx} = 2x$

$y = \int_1^{u} \cos t \, dt$

We must apply Chain Rule to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left[\frac{d}{du}\left(\int_1^u \cos t \, dt\right)\right] \cdot \frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx} = \left[\cos(x^2)\right] \cdot 2x$$

$$= 2x \cos x^2 \quad \text{Ans.}$$

d) $y = \int_{1+3x^2}^{4} \frac{1}{2+t} \, dt$

Sol:

Let $u = 1+3x^2$

$du = 6x\,dx \Rightarrow \frac{du}{dx} = 6x$

$y = \int_u^4 \frac{1}{2+t} \, dt$

We must apply Chain Rule to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[\frac{d}{du}\left(\int_u^4 \frac{1}{2+t} dt\right)\right] \cdot \frac{du}{dx}$$

$$= \left[\frac{d}{du}\left(-\int_4^u \frac{1}{2+t} dt\right)\right] \cdot \frac{du}{dx}$$

$$= \left[-\frac{d}{du}\int_4^u \frac{1}{2+t} dt\right] \cdot \frac{du}{dx}$$

$$= -\frac{1}{2+u} \cdot \frac{du}{dx}$$

$$= -\frac{1}{2+(1+3x^2)} \cdot (6x)$$

$$= -\frac{6x}{2+1+3x^2} = \frac{-6x}{3+3x^2} = \frac{-6x}{3(1+x^2)}$$

$$= \frac{-2x}{1+x^2} \quad \text{Ans}$$

Theorem 4 – The Fundamental Theorem of Calculus (Part 2)

If f is cont. over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

→ This theorem is also known as The Evaluation Theorem.

→ To use this theorem, we have to calculate two things:

(i) Find an antiderivative F of f

(ii) Calculate the number $F(b) - F(a)$

Example 3: Use Evaluation Theorem to find

(a) $\int_0^{\pi} \cos x\,dx$

Sol:
$$\int_0^{\pi} \cos x\,dx = \sin x\Big]_0^{\pi} = \sin\pi - \sin 0$$
$$= 0 - 0 = 0 \quad \text{Ans}$$

$\frac{d}{dx}\sin x = \cos x$

(b) $\int_{-\pi/4}^{0} \sec x \tan x\,dx$

Sol:
$$\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^{0}$$
$$= \sec(0) - \sec\left(-\frac{\pi}{4}\right)$$
$$= 1 - \sqrt{2} \quad \text{Ans}$$

$\frac{d}{dx}\sec x = \sec x \cdot \tan x.$

(c) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx$

Sol:
$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx = \frac{3}{2}\int_1^4 \sqrt{x}\,dx - 4\int_1^4 \frac{1}{x^2}\,dx.$$
$$= \frac{\cancel{3}}{\cancel{2}}\,\frac{x^{3/2}}{\cancel{3}/\cancel{2}}\Big]_1^4 + \frac{4}{x}\Big]_1^4$$
$$= \left[(4)^{3/2} - (1)^{3/2}\right] + 4\left[\frac{1}{4} - 1\right]$$
$$= (8 - 1) + 4\left(-\frac{3}{4}\right)$$
$$= 7 - 3 = 4 \quad \text{Ans}$$

4 Theorem 5 – The Net Change Theorem:

The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$\boxed{F(b) - F(a) = \int_a^b F'(x)\,dx}$$

5 The Relationship Between Integration and Differentiation:

(i) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

If you first integrate the function f and then differentiate the result, you get the function f back again.

(ii) $\int_a^x F'(t)\,dt = F(x) - F(a)$

If you first differentiate the function F and then integrate the result, you get the function F back.

6 Total Area:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a,b]$:

(i) Subdivide $[a,b]$ at the zeros of f.
(ii) Integrate f over each subinterval.
(iii) Add the absolute value of the integrals.

Example 6 – (DIY)

Example 7: The graph of $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) The definite integral of $f(x)$ over $[0, 2\pi]$
(b) The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$

Solution:

(a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big]_0^{2\pi}$$

$$= -\left[\cos 2\pi - \cos 0\right]$$

$$= -[1-1] = 0 \text{. Ans}$$

(b) The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: $[0, \pi]$ and $[\pi, 2\pi]$.

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0]$$

$$= -[-1-1] = 2.$$

$$\int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi]$$

$$= -[1-(-1)] = -[1+1] = -2.$$

The area between the graph and the axis is obtained by adding the absolute values of the definite integrals,

$$\text{Area} = |2| + |-2| = 2 + 2 = 4. \quad \text{Ans}$$

Example 8: Find area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

Solution: First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x^2 - 2x + x - 2)$$

$$= x[x(x-2) + 1(x-2)]$$

$$= x(x-2)(x+1).$$

The zeros are $x = 0, -1, 2$. Subdivide the interval $[-1, 2]$ as $[-1, 0]$ and $[0, 2]$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}.$$

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}. \text{ Ans.}$$

Ex 5.4

(13) $\int_{\pi/2}^{0} \frac{1+\cos 2t}{2}\, dt$.

Sol: $\int_{\pi/2}^{0} \frac{1+\cos 2t}{2} dt = \int_{\pi/2}^{0}\left(\frac{1}{2} + \frac{\cos 2t}{2}\right)dt$

$= \int_{\pi/2}^{0} \frac{1}{2} dt + \int_{\pi/2}^{0} \frac{\cos 2t}{2} dt$

Let $2t = u$ $\Rightarrow 2t = u$ (For finding limits) When $t=0 \Rightarrow u=0$; When $t=\pi/2 \Rightarrow u=\pi$.

$2dt = du$

$dt = \frac{du}{2}$

$= \int_{\pi/2}^{0} \frac{1}{2} dt + \int_{\pi}^{0} \frac{\cos u}{2} \frac{du}{2} = \frac{1}{2}\int_{\pi/2}^{0} dt + \frac{1}{4}\int_{\pi}^{0} \cos u\, du$

$= \frac{1}{2} t\Big]_{\pi/2}^{0} + \frac{1}{4} \sin u\Big]_{\pi}^{0}$

$= \frac{1}{2} t\Big]_{\pi/2}^{0} + \frac{1}{4} \sin(2t)\Big]_{\pi}^{0}$

$\because$ As $u = 2t$.

$= \frac{1}{2}\left(0 - \frac{\pi}{2}\right) + \frac{1}{4}(\sin 0 - \sin \pi)$

$= -\frac{\pi}{4} + 0 = \boxed{-\frac{\pi}{4}}$ Ans

(24) $\int_{1}^{8} \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}} dx$

Sol: $= \int_{1}^{8} \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}} dx$

$= \int_{1}^{8} \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3}\right) dx$.

$= 2\int_{1}^{8} dx - \int_{1}^{8} x^{2/3} dx + 2\int_{1}^{8} x^{-1/3} dx - \int_{1}^{8} x^{1/3} dx$.

$= 2x\Big]_{1}^{8} - \frac{x^{5/3}}{5/3}\Big]_{1}^{8} + 2\frac{x^{2/3}}{2/3}\Big]_{1}^{8} - \frac{x^{4/3}}{4/3}\Big]_{1}^{8}$

$= 2(8-1) - \frac{3}{5}(8^{5/3} - 1^{5/3}) + 3(8^{2/3} - 1^{2/3}) - \frac{3}{4}(8^{4/3} - 1^{4/3})$

$= \boxed{-\frac{137}{20}}$ Ans

37) $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)dx$.

Sol:

$= \frac{1}{2}\int_0^{\pi}(\cos x + |\cos x|)dx$.

$= \frac{1}{2}\int_0^{\pi} f(x)\,dx$. Let $f(x) = \cos x + |\cos x|$.

$f(x) = \cos x + \cos x = 2\cos x$ for $0 \le x \le \frac{\pi}{2}$.

$f(x) = \cos x - \cos x = 0$ for $\frac{\pi}{2} \le x \le \pi$.

$= \frac{1}{2}\left[\int_0^{\pi/2} 2\cos x\,dx + \int_{\pi/2}^{\pi} 0\,dx\right]$

$= \frac{2}{2}\int_0^{\pi/2}\cos x\,dx + 0 = \int_0^{\pi/2}\cos x\,dx = \sin x\Big]_0^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin 0$

$= \boxed{1}$ Ans

38) Find derivative

a) by evaluating the integral and differentiating the result.

b) by differentiating the integral directly.

$\frac{d}{dt}\int_0^{\sqrt{t}}\left(x^4 + \frac{3}{\sqrt{1-x^2}}\right)dx$.

Sol:

(a) $\int_0^{\sqrt{t}}\left(x^4 + \frac{3}{\sqrt{1-x^2}}\right)dx = \left[\int_0^{\sqrt{t}} x^4\,dx + 3\int_0^{\sqrt{t}}\frac{1}{\sqrt{1-x^2}}dx\right]$

$= \left[\left[\frac{x^5}{5}\right]_0^{\sqrt{t}} + 3\left[\sin^{-1}x\right]_0^{\sqrt{t}}\right]$. $\left(\because \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}\right)$

$= \left[\frac{(\sqrt{t})^5}{5} - 0 + 3\left(\sin^{-1}\sqrt{t} - \sin^{-1}0\right)\right]$

$= \left[\frac{t^{5/2}}{5} + 3\sin^{-1}\sqrt{t}\right]$

Now $\frac{d}{dt}\int_0^{\sqrt{t}}\left(x^4 + \frac{3}{\sqrt{1-x^2}}\right)dx = \frac{d}{dt}\left[\frac{t^{5/2}}{5} + 3\sin^{-1}\sqrt{t}\right]$

(P.T.O)

$$= \frac{d}{dt}\frac{t^{5/2}}{5} + \frac{d}{dt} 3\sin^{-1}\sqrt{t}.$$

$$= \frac{1}{5}\frac{d}{dt} t^{5/2} + 3\frac{d}{dt}\sin^{-1}\sqrt{t}$$

$$= \frac{1}{\cancel{5}}\left(\frac{\cancel{5}}{2} t^{3/2}\right) + 3\left(\frac{1}{\sqrt{1-(\sqrt{t})^2}}\right)\frac{1}{2\sqrt{t}}$$

$$= \boxed{\frac{t^{3/2}}{2} + \frac{3}{2}\frac{1}{\sqrt{t-t^2}}} \quad \text{Ans}$$

(b) Using the First Fundamental Theorem of Calculus i.e.

$$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

$$\frac{d}{dt}\int_0^{\sqrt{t}}\left(x^4 + \frac{3}{\sqrt{1-x^2}}\right)dx = \left[(\sqrt{t})^4 + \frac{3}{\sqrt{1-(\sqrt{t})^2}}\right]\frac{1}{2\sqrt{t}}$$

$$= \boxed{\frac{t^{3/2}}{2} + \frac{3}{2\sqrt{t-t^2}}} \quad \text{Ans} \longrightarrow \text{Chain Rule is used.}$$

$\longleftrightarrow$

(43) Find dy/dx

$$y = \int_{-1}^{x}\frac{t^2}{t^2+4}\,dt - \int_3^x \frac{t^2}{t^2+4}\,dt$$

Sol: Using formula.

$$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{-1}^{x}\frac{t^2}{t^2+4} - \int_3^x \frac{t^2}{t^2+4}\right]dt$$

$$= \frac{x^2}{x^2+4} - \frac{x^2}{x^2+4} = \boxed{0}.\ \text{Ans}$$

$\longleftrightarrow$

(45) $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$ $|x| < \pi/2$.

Sol: Using formula

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Let $u = \sin x$

$du = \cos x\, dx$.

$\frac{du}{dx} = \cos x$.

We must apply Chain Rule to find dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left[\frac{d}{du}\int_0^u \frac{dt}{\sqrt{1-t^2}}\right]\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} = \frac{1}{\sqrt{1-\sin^2 x}}\cos x$$

$$= \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|} = \boxed{1}$$ Ans $\because$ for $|x| < \frac{\pi}{2}$

$|\cos x| = \cos x$.

(50) Find total area between the region and the x-axis.

$y = x^{1/3} - x$ $-1 \le x \le 8$.

Sol: We need to find zeros of y.

$$y = x^{1/3}(1 - x^{2/3})$$

So, zeros are $x = 0$ and $x = \pm 1$. The zeros subdivide $[-1, 8]$ into three subintervals: $[-1, 0]$, $[0, 1]$, $[1, 8]$. (P.T.O)

$$\cancel{A = \int_{-1}^{0}(x^{1/3} - x)\,dx + \int_0^1 (x^{1/3} - x)\,dx + \int_1^8 (x^{1/3} - x)\,dx}$$

$$\cancel{= \left.\frac{x^{4/3}}{4/3}\right]_{-1}^{0} - \left.\frac{x^2}{2}\right]_{-1}^{0} + \left.\frac{x^{4/3}}{4/3}\right]_0^1 - \left.\frac{x^2}{2}\right]_0^1 + \left.\frac{x^{4/3}}{4/3}\right]_1^8 - \left.\frac{x^2}{2}\right]_1^8}$$

$$\cancel{= \left[0 - \frac{3}{4}\left((-1)^{4/3}\right)\right] - \left[0 - \frac{1}{2}(-1)^2\right] + \frac{3}{4}\left[(1)^{4/3} - 0\right] - \frac{1}{2}\left[(1)^2 - 0\right]}$$

$$\cancel{+ \frac{3}{4}\left[(8)^{4/3} - (1)^{4/3}\right] - \frac{1}{2}\left[(8)^2 - (1)^2\right].}$$

$$\cancel{= -\frac{3}{4} + \frac{1}{2} + \frac{3}{4} - \frac{1}{2} + 12 - \frac{3}{4} - 32 + \frac{1}{2} =}$$

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$$\int_{-1}^{0}(x^{1/3}-x)\,dx = \frac{x^{4/3}}{4/3}\Big]_{-1}^{0} - \frac{x^2}{2}\Big]_{-1}^{0}$$

$$= \frac{3}{4}\left[0-(-1)^{4/3}\right] - \frac{1}{2}\left[0-(-1)^2\right]$$

$$= -\frac{3}{4} + \frac{1}{2} = \boxed{-\frac{1}{4}}$$

$$\int_{0}^{1}(x^{1/3}-x)\,dx = \frac{x^{4/3}}{4/3}\Big]_{0}^{1} - \frac{x^2}{2}\Big]_{0}^{1}$$

$$= \frac{3}{4}\left[(1)^{4/3}-0\right] - \frac{1}{2}\left[(1)^2-0\right]$$

$$= \frac{3}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}$$

$$\int_{1}^{8}(x^{1/3}-x)\,dx = \frac{x^{4/3}}{4/3}\Big]_{1}^{8} - \frac{x^2}{2}\Big]_{1}^{8}$$

$$= \frac{3}{4}\left[(8)^{4/3}-(1)^{4/3}\right] - \frac{1}{2}\left[(8)^2-(1)^2\right]$$

$$= \frac{3}{4}[16-1] - \frac{1}{2}[64-1]$$

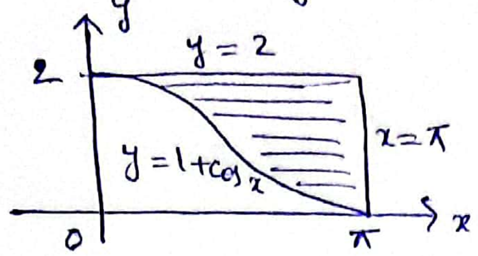
$$= \frac{3}{4}(15) - \frac{1}{2}(63)$$

$$= \frac{45}{4} - \frac{63}{2} = \boxed{-\frac{81}{4}}$$

Total Area $= \left|-\frac{1}{4}\right| + \left|\frac{1}{4}\right| + \left|-\frac{81}{4}\right| = \frac{1}{4} + \frac{1}{4} + \frac{81}{4}$

$$= \boxed{\frac{83}{4}}$$ Ans

(51) Find the areas of the shaded regions



To find
Shaded Area = Area of rectangle − Area under the curve y.

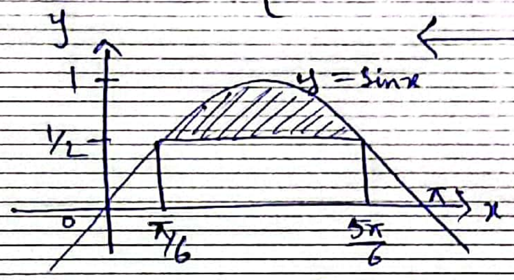
$$A = (\text{width} \times \text{length}) - \int_0^{\pi} (1+\cos x)\,dx$$

$$= (2\pi) - (x + \sin x)\Big]_0^{\pi}$$

$$= 2\pi - [(\pi + \sin\pi - 0 - \sin 0)]$$

$$= 2\pi - [\pi + 0 - 0 - 0] = \boxed{\pi} \text{ Ans}$$

(52) Find the areas of the shaded regions



To find Shaded region Area = Area under curve y − Area of rectangle − Area under the curve y from $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, \pi)$

$$A = \int_0^{\pi} \sin x\,dx - (\text{width} \times \text{length}) - \int_0^{\pi/6} \sin x\,dx - \int_{5\pi/6}^{\pi} \sin x\,dx$$

$$= -\cos x\Big]_0^{\pi} - \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \left(-\cos x\Big]_0^{\pi/6}\right) - \left(-\cos x\Big]_{5\pi/6}^{\pi}\right)$$

$$= -(\cos\pi - \cos 0) - \frac{1}{2}\left(\frac{4\pi}{6}\right) + \left(\cos\frac{\pi}{6} - \cos 0\right) + \left(\cos\pi - \cos\frac{5\pi}{6}\right)$$

$$= -(-1-1) - \frac{\pi}{3} + \left(\frac{\sqrt{3}}{2} - 1\right) + \left(-1 + \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\sqrt{3} - \frac{\pi}{3}} \text{ Ans}$$

Practice Questions (Q1 − 52) of Ex# 5.4

1) if $f(x)$ gives relationship between f and x

then $f'(x)$ gives rate of change between f and x

then $\int_a^b f'(x)\,dx$ give the net change in f over $[a,b]$

$\hookrightarrow f(x)\Big|_a^b = \boxed{f(b) - f(a)}$

2) Area of a regular triangle is $A(a) = \underbrace{\frac{\sqrt{3}}{4} a^2}$ (a is side)

write an integral to represent the net change of the area as the side increases from 2" to 3"

$$\int_2^3 A'(a)\,da = \boxed{\int_2^3 \frac{\sqrt{3}}{2} a\,da} = \underbrace{A(3) - A(2)} = \frac{\sqrt{3}}{4}(3)^2 - \frac{\sqrt{3}}{4}(2)^2$$

$$= \boxed{\frac{5\sqrt{3}}{4}}$$