6.1 Volumes Using Cross-Sections –

→ Volume –

The volume of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ is,

$$V = \int_a^b A(x)\,dx$$

→ Calculating The Volume Of A Solid –

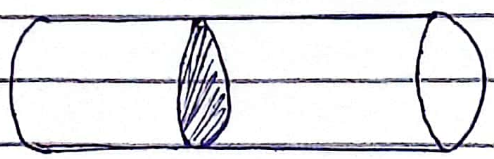
1) Sketch the solid and a typical cross-section.
2) Find a formula for $A(x)$ – the area of a typical cross-section
3) Find the limits of integration.
4) Integrate $A(x)$ to find the volume.

→ Cross-Sectional Area –

The cross-sectional area is the area of a two-dimensional shape that is obtained when a three-dimensional object, such as a cylinder, is sliced perpendicular to some specified axis at a point.

e.g. The cross-section of a cylinder when sliced parallel to its base, is a circle.

Cross-Sectional Area (Circle)

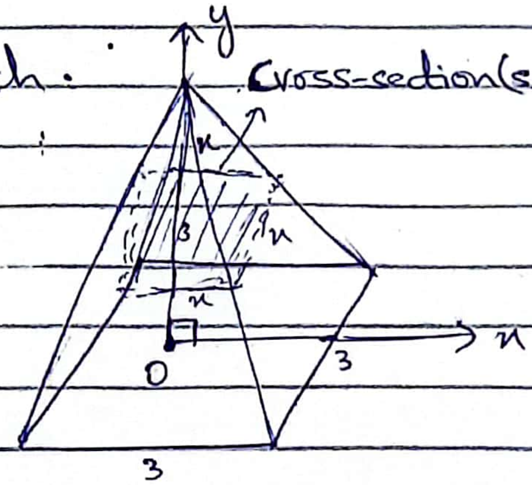


Example#1

A pyramid 3m high has a square base that is 3m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1) Sketch.



2) Cross-sectional Area = Area of Square

$$A(x) = x^2$$

3) The limits of Integration:

The squares lie on the planes from $x=0$ to $x=3$.

4) $$V = \int_0^3 A(x)\,dx$$

$$= \int_0^3 x^2\,dx$$

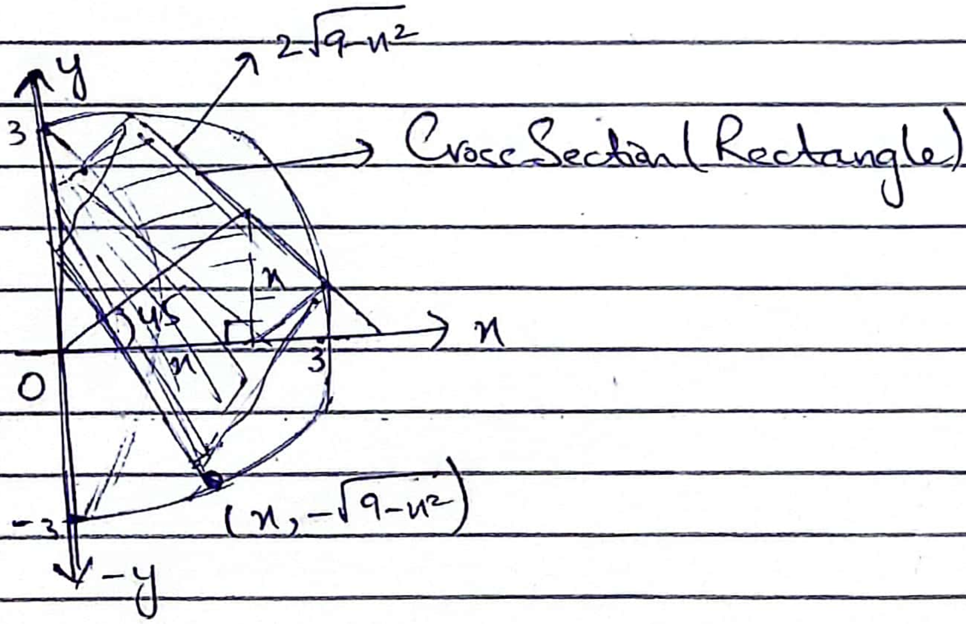
$$= \frac{x^3}{3}\Big|_0^3 = 9m^3.$$

→ Example #2

A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the centre of the cylinder. Find the volume of the wedge.

SOLUTION

1) Sketch:



→ The base of wedge is a semi-circle cut from the circle $x^2+y^2=3^2$ by the $45°$ plane.

$$\Rightarrow y^2 = 9-x^2$$
$$y = \pm\sqrt{9-x^2}$$

→ Height of rectangle $= x$

Width of rectangle $= 2\sqrt{9-x^2}$

→ Area of cross-section is:

$A(x) = \text{Height} \times \text{Width}$

$= x \times 2\sqrt{9-x^2}$

$= 2x\sqrt{9-x^2}$

→ Limit : $x = 0$ to $x = 3$

→ $V = \int_a^b A(x)\,dx$

$= \int_0^3 2x\sqrt{9-x^2}\,dx$

let $u = 9 - x^2$

$\Rightarrow du = -2x\,dx$

→ $V = -\frac{2}{3}(9-x^2)^{3/2}\Big|_0^3$

$= 0 + \frac{2}{3}(9)^{3/2}$

$= 18.$

* Example #3 * ~~X~~

Solids Of Revolution: The Disk Method

The solid generated by rotating (or revolving) a planar region about an axis in its plane is called a Solid Of Revolution.

→ Cross-Sectional Area of a disk of radius $R(x)$, is

$$A(x) = \pi(\text{radius})^2$$
$$= \pi[R(x)]^2$$

→ Volume By Disks For Rotation About The x-Axis.

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx$$

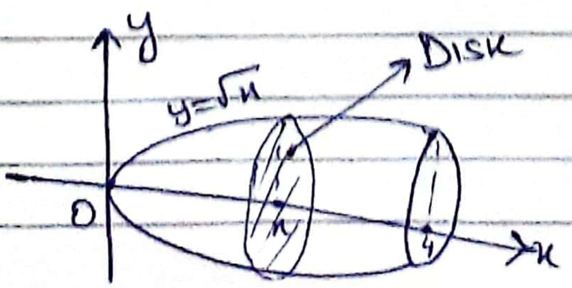
→ This method for calculating the volume of a solid of revolution is called the Disk Method because a cross-section is a circular disk of radius $R(x)$.

→ Example #4-

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution

1) Sketch:



$$\text{Volume} = \int_a^b \pi [R(x)]^2 dx$$

$$= \int_0^4 \pi [\sqrt{x}]^{\cancel{2}} dx$$

$$= \pi \int_0^4 x\, dx$$

$$= \pi \frac{x^2}{2} \Big|_0^4 = \pi \frac{(4)^2}{2} = 8\pi.$$

→ Example #5

The circle $x^2+y^2=a^2$ is rotated about the x-axis to generate a sphere. Find its volume.

Solution

→ Imagine the sphere cut into thin slices by planes perpendicular to the x-axis.

→ The radius is $R(x) = y = \sqrt{a^2-x^2}$

→ The cross-sectional area at a typical point 'x' between $-a$ and a is:

$$A(x) = \pi y^2$$
$$= \pi(a^2 - x^2)$$

$$V = \int_{-a}^{a} A(x)\,dx$$

$$= \int_{-a}^{a} \pi(a^2 - x^2)\,dx$$

$$= \pi \left[a^2 x - \frac{x^3}{3} \right] \Bigg|_{-a}^{a}$$

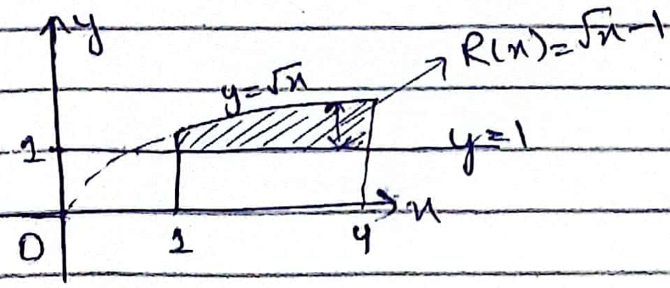
$$= \frac{4}{3}\pi a^3$$

→ Example # 6
Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$

Solution

→ When $y = 1 \Rightarrow y^2 = x \Rightarrow 1^2 = x \Rightarrow x = 1$

→ Sketch:



* Limit of integration:
$x = 1$ to $x = 4$

$$V = \int_1^4 \pi [R(x)]^2 \, dx$$

$$= \int_1^4 \pi [\sqrt{x} - 1]^2 \, dx$$

$$= \pi \int_1^4 (x - 2\sqrt{x} + 1) \, dx$$

$$= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4$$

$$= 7\pi/6$$

→ Volume By Disks For Rotation About The y-Axis

$$V = \int_c^d A(y) \, dy = \int_c^d \pi [R(y)]^2 \, dy$$

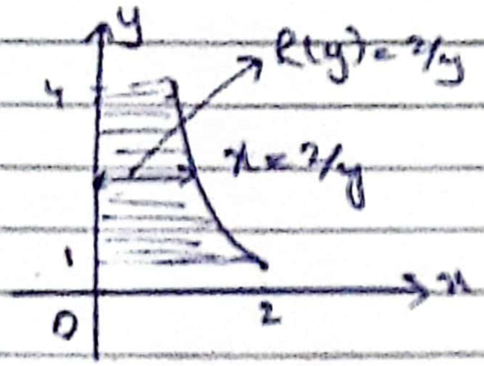
→ Example #7

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution

→ Limit of integration: $y = 1$ to $y = 4$.

Sketch:



$$V = \int_1^4 \pi [R(y)]^2 \, dy$$

$$= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy$$

$$= \pi \int_1^4 \frac{4}{y^2} \, dy$$

$$= 4\pi \left[\frac{-1}{y}\right]_1^4$$

$$= \cancel{4}\pi \left[\frac{3}{\cancel{4}}\right]$$

$$= 3\pi$$