Assignment 4

Part A

A.1

Heartwarming (***#diffapplication***)

$$y = \frac{Q}{D}$$

1. First, let's determine variables we know and need to find:
$$Q = 233\frac{ml}{min};\ D = 41\frac{ml}{L};\ \frac{dD}{dt} = 2\frac{ml}{L * min};\ \frac{dQ}{dt} = 0\frac{ml}{min^2};\ \frac{dy}{dt} - ?$$
2. Let's differentiate the two parts of the original equation since they are equal to each other, so are their derivatives:
$$\frac{dy}{dt} = \frac{\frac{dQ}{dt} * D - \frac{dD}{dt} * Q}{D^2} = \frac{0\frac{ml^2}{L*min^2} * -(-2) * 233\frac{ml^2}{L*min^2}}{41^2\frac{ml^2}{L^2}} \approx 0.277\frac{L}{min^2}$$
3. With D decreasing at a rate of $2\frac{ml}{L*min}$ when $Q = 233\frac{ml}{min}$ and $D = 41\frac{ml}{L}$, the cardiac output y would increase at a rate of $0.277\frac{L}{min^2}$ (or $0.277\frac{L}{min}$ per minute).

A.2

Least squares (***#diffapplication***)

Let $f(m, b)$ be the cost function of the regression line with slope m and bias unit b:

$$f(m, b) = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n}(y_i - (mx_i + b))^2$$
$$f_m = \sum_{i=1}^{n} -2x_i(y_i - (mx_i + b))$$
$$f_b = \sum_{i=1}^{n} 2(y_i - (mx_i + b))$$

To minimize cost function, we need both the partial derivatives to be zero:

$$\begin{cases} \sum_{i=1}^{n} -2x_i(y_i - (mx_i + b)) = 0 \\ \sum_{i=1}^{n} 2(y_i - (mx_i + b)) = 0 \end{cases}$$

$$\begin{cases} \sum_{i=1}^{n} x_i(y_i - (mx_i + b)) = 0 \\ \sum_{i=1}^{n} y_i - (mx_i + b) = 0 \end{cases}$$

$$\begin{cases} \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} mx_i^2 + bx_i = 0 \\ \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} mx_i + b = 0 \end{cases}$$

$$\begin{cases} \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} m x_i^2 + \sum_{i=1}^{n} b x_i \\ \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} m x_i + bn \end{cases}$$

$$\begin{cases} \sum_{i=1}^{n} y_i = m \sum_{i=1}^{n} x_i + bn \\ \sum_{i=1}^{n} x_i y_i = m \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i \end{cases}$$

Let's run the second derivative test to make sure that these m and b describe the minimum of the function.

$$f_{m^2} = \sum_{i=1}^{n} 2x_i^2 > 0$$
$$f_{b^2} = \sum_{i=1}^{n} 2 > 0$$
$$f_{b,m} = \sum_{i=1}^{n} -2x_i$$
$$D = f_{m^2} f_{b^2} - f_{b,m} = 4n \sum_{i=1}^{n} x_i^2 - 4 \sum_{i=1}^{n} x_i^2$$

We know that n > 1 because that is the number of data points we are optimizing for, therefore, D > 0, meaning that it is not a saddle point. Then, we can see that one of the partial second derivatives is positive, meaning that the function is concave up on both axes, making this a point of local minimum.

Since the partial second derivatives are always positive, this point is a global minimum because the function is concave up, meaning that any point excluding the optimal (m, b) would yield a larger value.

These are the equations of method of least squares that the task asked to prove. Proven.

B.1

Airplane! (***#diffapplication***)

Task (a)

Let's name variables: $r = p * q$ - revenue function that depends on p - price per checked bag and q - quantity of checked bags.

Now's we know that that for $2 increase in price per bag, there will be 2 less bags checked, so we can define the relation of p and q:

$$2q = -2p + const$$
$$q = -p + const$$

We also know that at $20, there will be 50 bags checked, so we can find the $const$ coefficient:

$$50 = -20 + 70$$
$$q = -p + 70$$

From this relationship, we can see that the price **cannot** be more than $70, otherwise, the quantity of checked bags would be negative. Theoretically, the company can give money to people to check in, but then in reality, no one would do so, so we can set the minimum value of p at 0. Therefore, $p \in [0; 70]$

Now, let's plug this dependency in the original price function and model the revenue of price per bag:

$$r = p(-p + 70) = -p^2 + 70p$$

In [10]:

```
var('p')
r = -p^2 + 70*p
plot(r, (p, 0, 70), axes_labels=['p ($/bag)', 'r ($)'])
```

Out[10]:

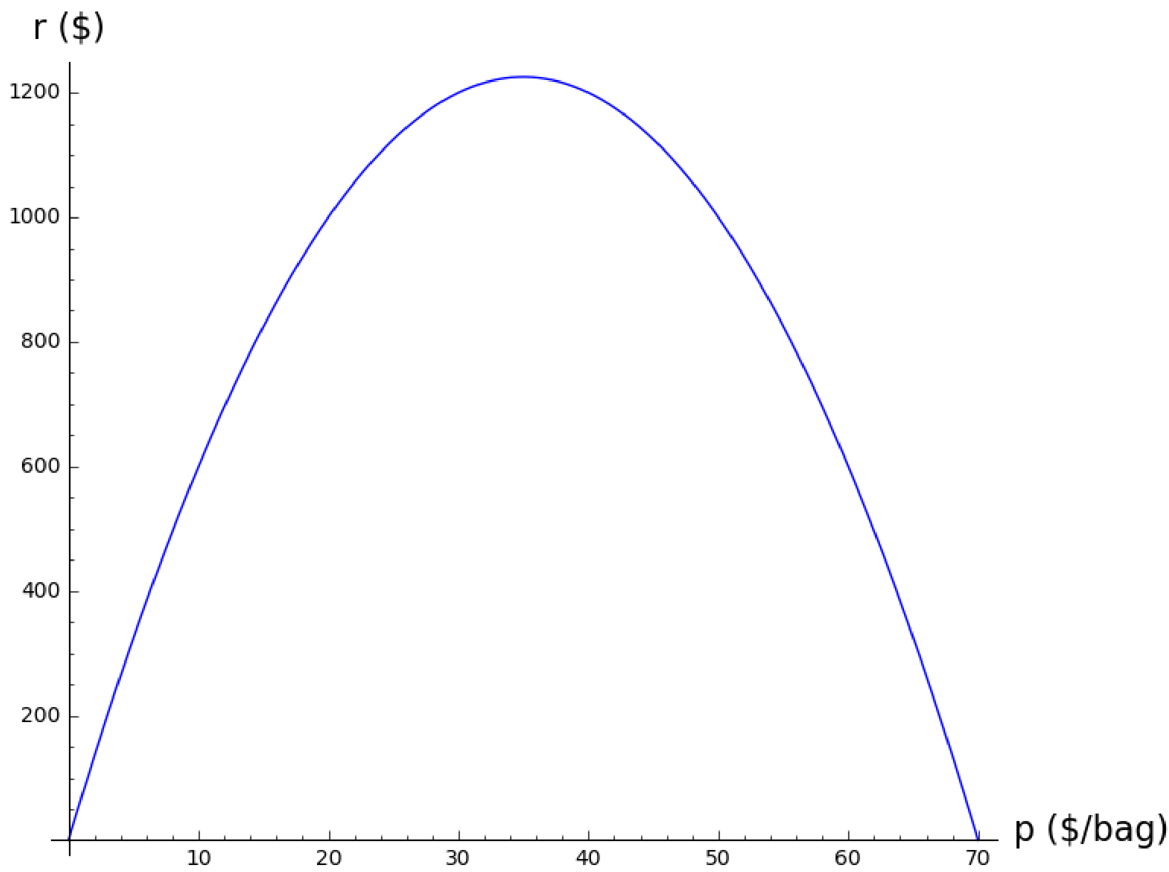

Figure 1. Graph of revenue from charging for checked bags based on the price the company charges per bag.

Task (b)

$$C(q) = \frac{q^2}{20} + 7q$$

To find the profit, we can plug in the definition of q in terms of p:

$$C(p) = \frac{(-p+70)^2}{20} + 7(-p+70) = \frac{p^2}{20} - 7p + 245 - 7p + 490 = \frac{p^2}{20} - 14p + 735$$

Then, profit would be equal to:

$$A(p) = r(p) - C(p) = -p^2 + 70p - \frac{p^2}{20} + 14p - 735 = -\frac{21}{20}p^2 + 84p - 735$$

In [14]:

```
A = -21*p^2/20 + 84*p - 735
plot(A, (p, 0, 70), axes_labels=['p ($/bag)', 'A ($)'])
```

Out[14]:

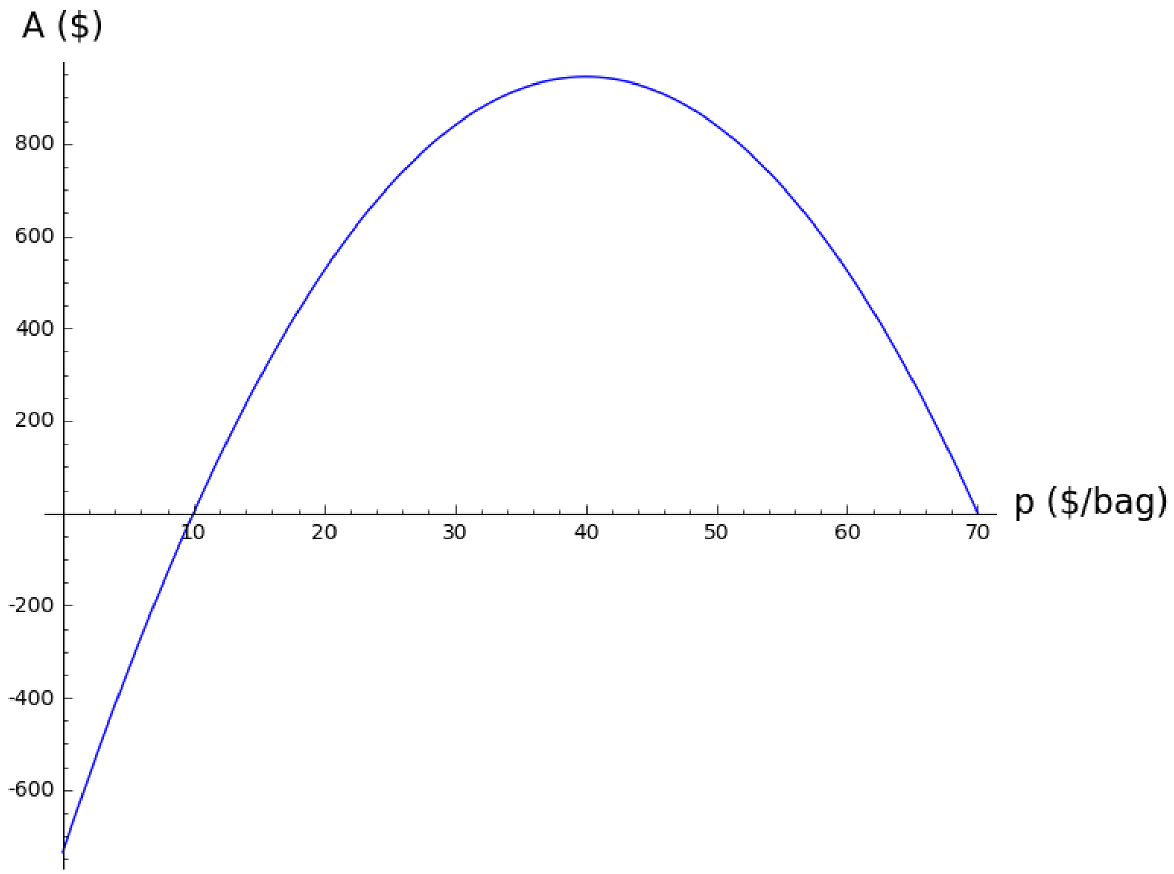

Figure 2. Graph of calculated profit from charging for checked bags based on the price the company charges per bag.

Task (c)

To find the maximum profit, we would need to find the maxima for the profit function:

$$A'(t) = -\frac{21}{10}p + 84 = 0$$

We equalize this function to zero because at points of extrema, the slope of the tangent line is 0 (the tangent is horizontal).

$$p_0 = 40$$

We know that p_0 is the extremum point, but we need to make sure that it is the maximum. Therefore, we check the second derivative to check if the function is concave down (the tip is facing up and is a maximum).

$$A''(t) = -\frac{21}{10} < 0$$

⇒ the function is concave down on the closed interval of 0 to 70, so the point p_0 is the point of local and global maximum. The graph can confirm that it is in fact true.

Therefore, the optimal baggage fee is $40 and the number of bags that will be taken to the flight is $q = -40 + 70 = 30$ bags.

B.2

Allele alliteration (***#diffapplication***)

$$P = 2pq + 2pr + 2rq$$
$$p + q + r = 1$$

To solve this problem, let's redefine P through two variables since only two of them are lineraly independent.

$$p = 1 - q - r$$
$$P(r, q) = 2q(1 - q - r) + 2r(1 - q - r) + 2rq = -2q^2 - 2r^2 - 2rq + 2q + 2r$$

Next, let's find the points of local extrema by finding points where both partial derivatives are equal to zero and checking if that's a saddle point (by using the second derivative test).

$$\begin{cases} \frac{\partial P}{\partial r} = -4r - 2q + 2 = 0 \\ \frac{\partial P}{\partial q} = -4q - 2r + 2 = 0 \end{cases} \Rightarrow \begin{cases} -2r = q - 1 \\ -2q = r - 1 \end{cases} \Rightarrow \begin{cases} r = \frac{1}{3} \\ q = \frac{1}{3} \end{cases}$$

Second derivative test:

$$\frac{\partial^2 P}{\partial r^2} = -4$$
$$\frac{\partial^2 P}{\partial q^2} = -4$$
$$\frac{\partial^2 P}{\partial q \partial r} = -2$$

$$D = \frac{\partial^2 P}{\partial r^2} * \frac{\partial^2 P}{\partial q^2} - (\frac{\partial^2 P}{\partial q \partial r})^2 = 16 - 4 = 12 > 0$$

We got $D > 0$ which means that this is not a saddle point, and because one of the partials ($\frac{\partial^2 P}{\partial r^2}$) is negative, this means that the function is concave down throughout its domain, meaning that the point of extremum can only be the local maximum.

Now, we can see that both partial derivatives are negative, and mixed partials derivative is also negative. This implies that if we slice the graph with any plane, parallel to P and passing through point ($\frac{1}{3}, \frac{1}{3}$), we will always get negative concavity. Therefore, this point is also a global maximum.

In [29]:

```
var('r, q')
P = -2*q^2 - 2*r^2 - 2*r*q + 2*q + 2*r
global_max = point3d((1/3, 1/3, 2/3), color="red", size=40)
func_graph = plot3d(P, (r, 0, 1), (q, 0, 1))
func_graph + global_max
```

Out[29]:

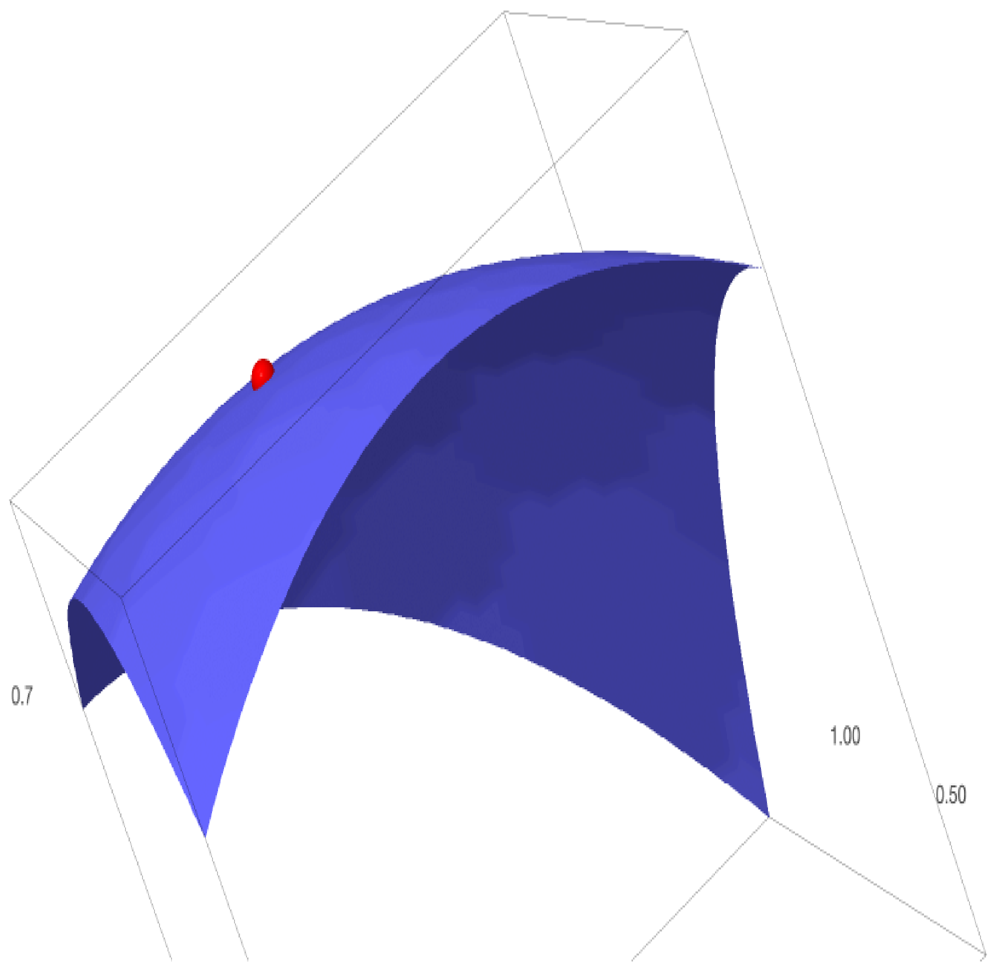

Figure 3. Graph of proportions of individuals with two different alleles depending on proportion of people with B and O blood types in the population. Global maximum is at (1/3, 1/3) and reaches P = 2/3.

Now, let's find the maximal P at $(r, q) = (\frac{1}{3}, \frac{1}{3})$.

$$P = -2 * \frac{1}{9} - 2 * \frac{1}{9} - 2 * \frac{1}{9} + 2 * \frac{1}{3} + 2 * \frac{1}{3} = \frac{2}{3}$$
$$max(P) = \frac{2}{3}$$

Appendix

#variables: In optimization problems, I identified which are the variables we need to optimize for, and which ones we have. Knowing the behavior of them, I used the zero-derivative and concavity principles to find extrema points of the dependent variables.
#decisionheuristics: In each problem, I divide the problem into several steps and describe what we do at each

step to reach the next goal (e.g. finding partial derivatives to then do the second derivative test) to ensure that I am getting closer to the goal of finding points of extrema.

#analogies: I generalize these problems to the problem of finding mix/max of a function by defining what that function is and what are the independent variables and how they correspond to each other. Since the scenarios are quite analogous to finding mix/max of a funtion, I can apply the same methodologies (e.g. second derivative test) to solve for the answer.

#deduction: In task A.2 I prove the provided equations by coming to these equations through a round of established premises, from differentiating to find points of extrema of the sum of squares to making a point why we have identified a point of minimum rather than maximum using constraint conditions.