Assignment 3

Description

This assignment is a problem set consisting of two parts. In **Part A**, you will revisit some of the problems from the classroom activities. In **Part B**, you will apply the concepts and techniques discussed in class to solve novel problems.

Grading

This assignment has a significant weight (3x), so make sure to allot your time accordingly. Each exercise will be graded on the tagged LO. Additionally, you will receive a single holistic score on the LO **#mathtools**, for appropriate usage of SageMath and other ancillary mathematical techniques, and a single holistic score on the HC **#professionalism**.
Include an appendix in which you highlight 2 to 5 specific HC applications. For each highlighted use, give the HC hashtag and a 1 to 2 sentence explanation that demonstrates the strength of your application. The total length of this appendix should be no more than 200 words.

Collaboration

Math is a lot more fun with friends. We encourage you to collaborate and to discuss the problem sets with each other. However, it is key that you are able to apply the skills and concepts you learn in class and explain your approach on your own. For this reason, you should not be sharing **any** written work with each other outside of class and are required to write up all solutions independently.

Submission

Consider these options for presenting your work neatly. You may choose to type your solutions using LaTeX, Google Documents or other software. You may also write on a graphic tablet and submit a PDF of your work. Hand-written solutions on paper are also acceptable, so long as they are scanned/photographed clearly and are fully legible. There exist scanning apps that will allow you to take photos of pages and create a PDF out of them (Office Lens, for example).

- Low scores will be assigned for illegible work and/or images of poor quality.
- All graphs should be included in your PDF file, with a relevant caption.
- The common conventions for writing an essay will apply. For example, if you use external resources, make sure to cite them using proper formatting.

Part A

A.1 *Cobb-Douglas: Round 2* (**#Diffapplication**)

Suppose you are given the production function

$$Y = 2L^{0.75}K^{0.25},$$

where Y is the real \$-value of goods produced in a year, L is the total number of person-hours worked in a year, and K is the real \$-value of all capital input. See here if you would like more context.

(a) Extend the notion of linear approximation to multivariable functions by doing the following:

  (i) Find the tangent plane to the graph of $Y(L, K)$ at the point (1000 hours, \$1 million).

  (ii) Evaluate how much you expect Y to change when you raise L by 1 hour, and K by \$1 when $(L, K) =$ (1000 hours, \$1 million). Describe how this relates to the tangent plane.

  (iii) How much does Y actually change when you raise L by 1 hour, and K by \$1 when $(L, K) =$ (1000 hours, \$1 million). Was your answer for (ii) a decent approximation? (Feel free to define decent how you see fit!)

(b) Interpret what directional derivatives and the gradient mean in the context of the Cobb-Douglas function for total production that we have seen previously.

(c) Select another multivariable function of your choice from a context different from any we have explored in class. Interpret what directional derivatives and the gradient mean in this new context.

A.2 *Heavy Metal* (**#Diffapplication**, **#Difftheory**)

[Adapted from J. Stewart, *Multivariable Calculus, 6E*]

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120°.

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

Part B

B.1 *Euro Trip* (**#Diffapplication**)

You want to continue your travels through Europe and for that you decide you need to improve the conversion function you discussed in class. You now want a function that given the price of any item in Euros, y, and a Euro/Dollar conversion factor, x, will return the value of the item in dollars.

(a) Write the function $f(x, y)$ that converts y Euros to Dollars, when the Euro/Dollar conversion factor is given by x. Plot your function using Sage for suitable ranges of x and y.

(b) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Explain the meaning of each partial derivative in the context of currency conversion.

(c) You are worried the exchange rate will fluctuate during your travels. Use partial derivatives to estimate how your conversion function would be affected by a 5% decrease in the conversion factor.

(d) You enter a store holding a spectacular sale. Your amazing gadget that once cost 200 Euro is now discounted, going for 150 Euro. However, your fears were confirmed: the conversion factor changed from 0.81 Euro per Dollar to 0.77 Euro per Dollar. Find the tangent plane to your function, and use it to estimate the combined effects of the discount and the new conversion factor.

B.2 *Mixing Partials* (**#Difftheory**, **#limitscontinuity**)

[Adapted from Marsden et.al. *Basic Multivariable Calculus*]

In many cases you can take the mixed second partial derivative in either order and get the same function. This is a result of Clairaut's Theorem (aka. the Equality of Mixed Partials): If $f(x, y)$ has continuous second partial derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(a) Show that Clairaut's Theorem holds for the following examples:

(i) $C(x, y) = \cos(x^2 y^2)$

(ii) $r(\theta, t) = \dfrac{1}{\sin^2(\theta) + e^{-t}}$

(b) Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(i) Plot $f(x,y)$. Where is f continuous? Where is f differentiable? Explain your reasoning.

(ii) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x,y) \neq (0,0)$.

(iii) Show that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. *(Hint: Use the limit definition of a partial derivative.)*

(iv) Using the last two parts, write expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as piecewise functions.

(v) Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$.

(vi) What went wrong? Why aren't the mixed partials equal?