Part A

A.1

Task (a)

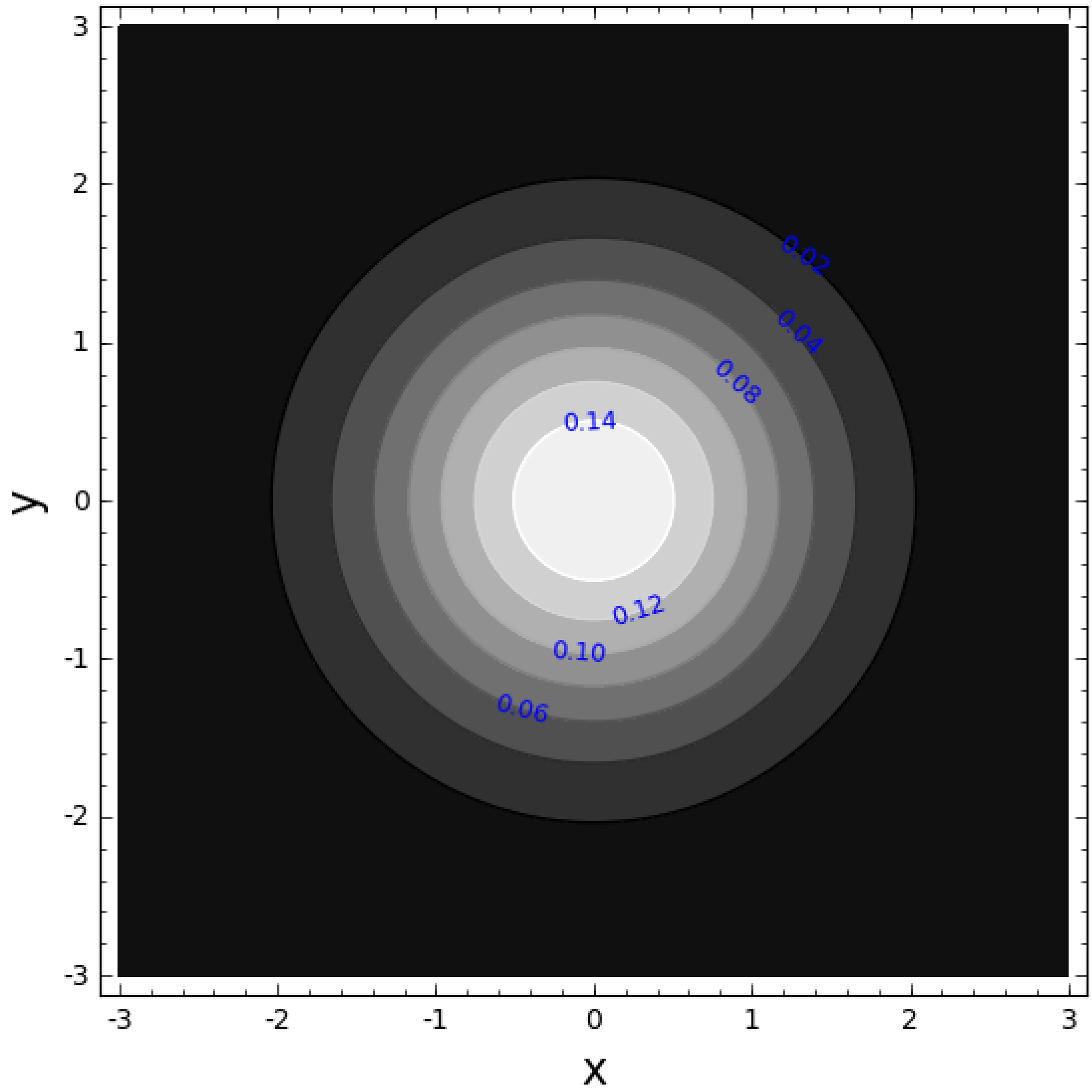
In [119]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = 0

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

contour_plot(P(x,y), (x,-3,3), (y,-3,3), frame=True, axes_labels=['x','y'], axes=Fal
```

Out[119]:



Contour plot of probability density function (PDF) of two variables.

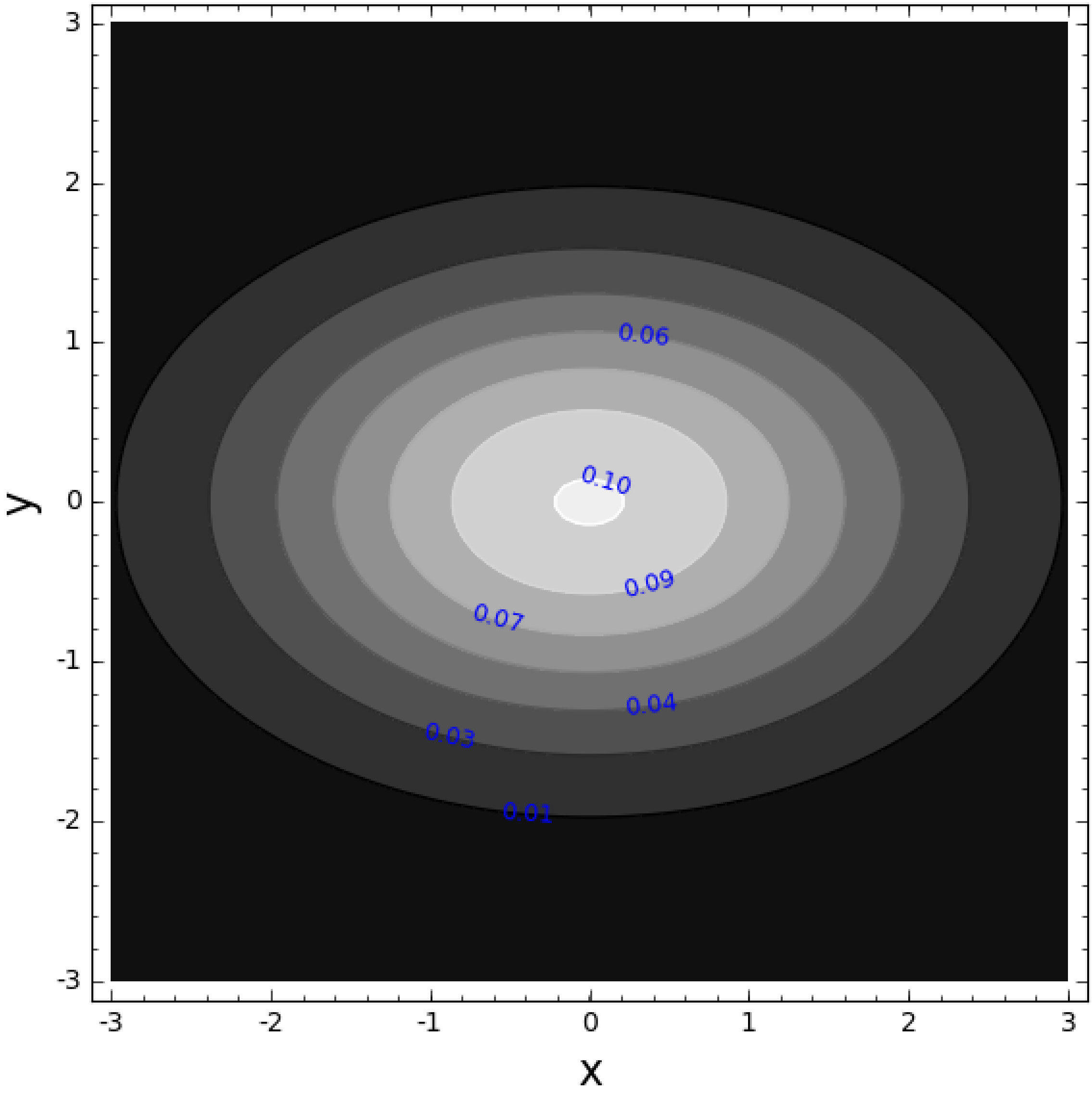
In [118]:

```
var('sd_x,sd_y,ro')
sd_x = 1.5
sd_y = 1
ro = 0

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

contour_plot(P(x,y), (x,-3,3), (y,-3,3), frame=True, axes_labels=['x','y'], axes=Fal
```

Out[118]:



We can observe that increasing standard deviation makes the distribution "more stretched out." As a way to compare, initially, when $SD_x = SD_y = 1$, the distribution's projection was "circular." Setting $SD_x > SD_y$ has made the circles to stretch into an oval along the x axis. Also, we can notice that since the distribution is "squished" out, the overall height decreases.

Task (b)

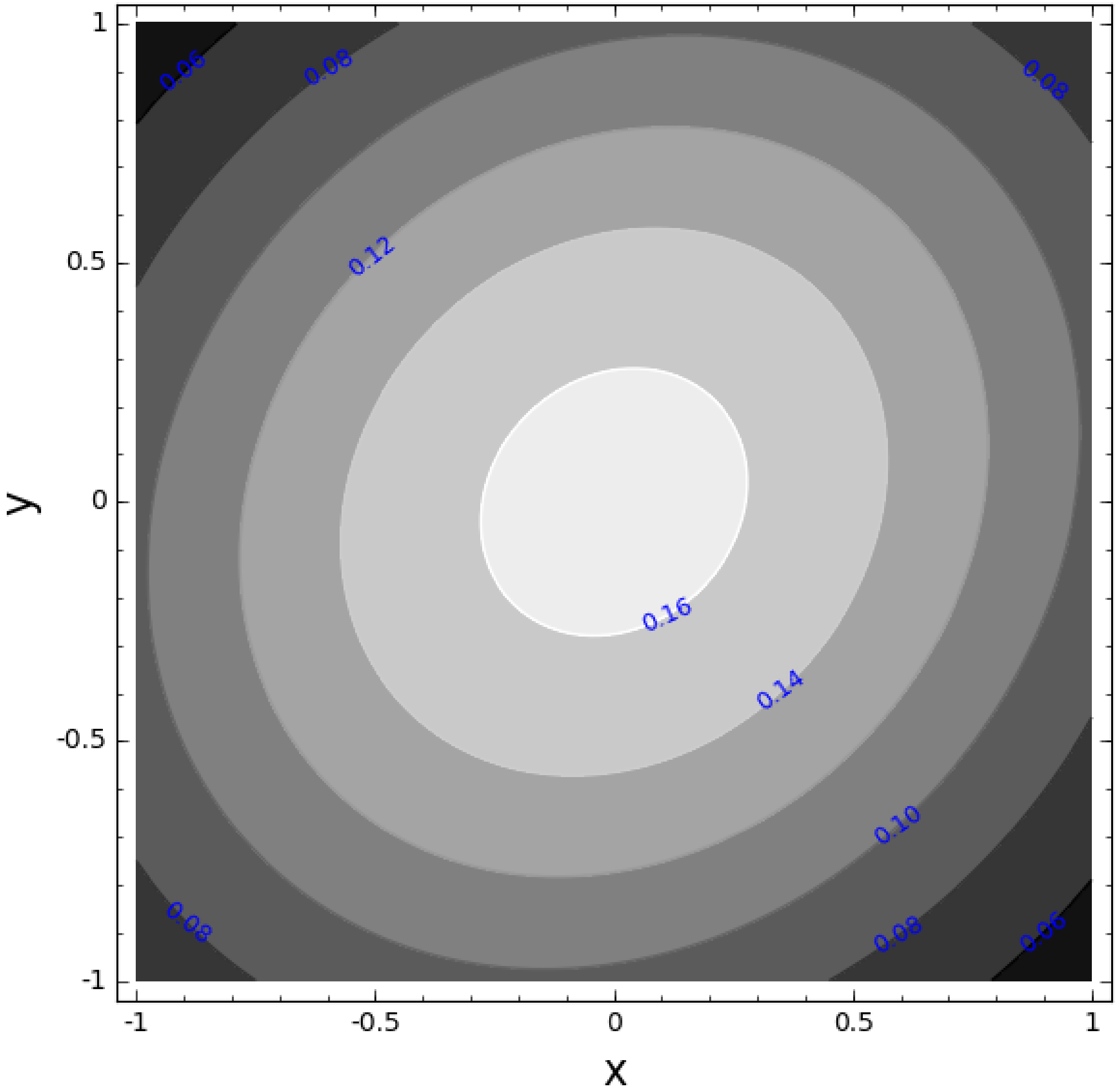
In [71]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = 0.3

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

contour_plot(P(x,y), (x,-1,1), (y,-1,1), frame=True, axes_labels=['x','y'], axes=Fa
```

Out[71]:



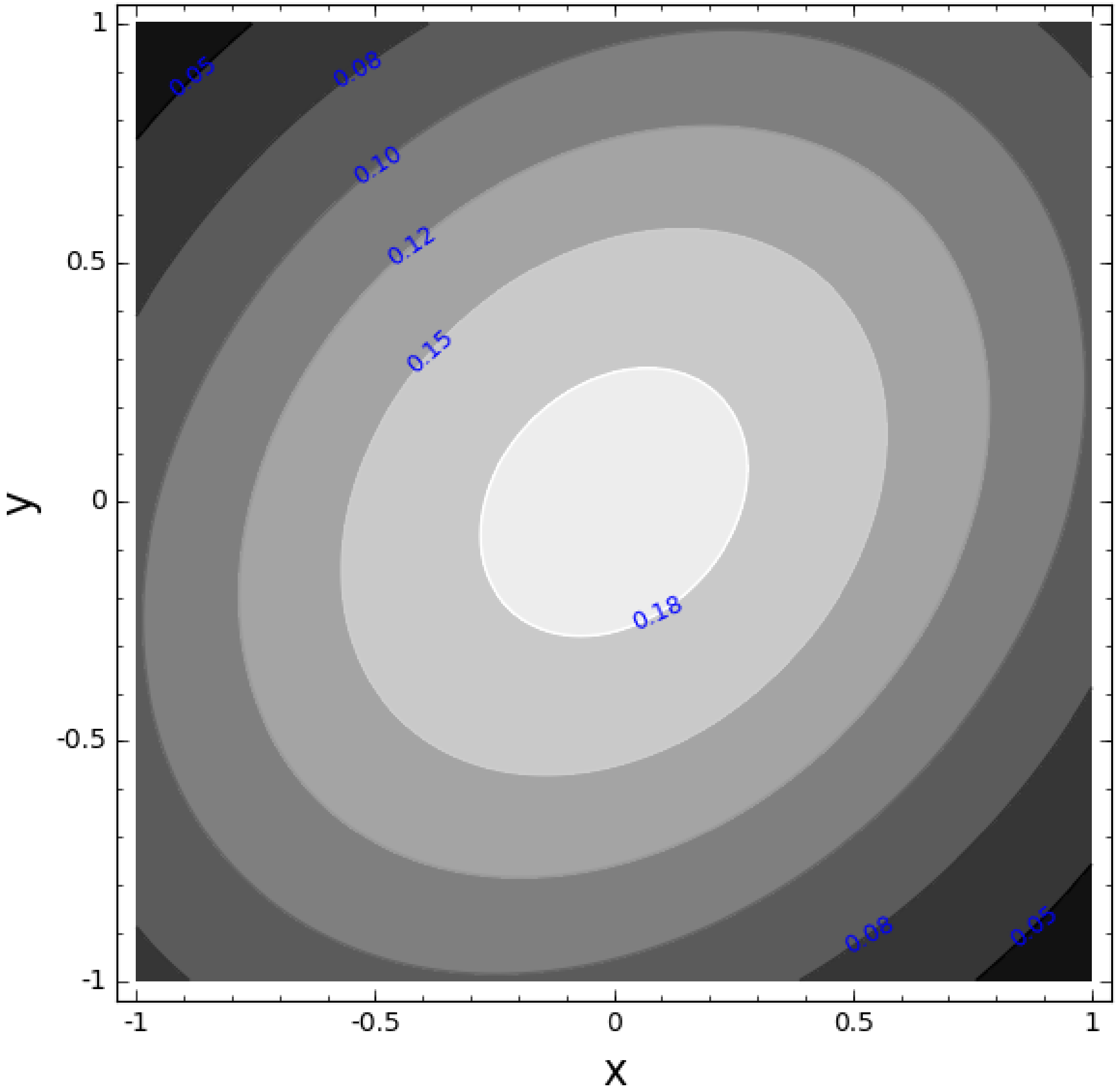
In [70]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = 0.5

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

contour_plot(P(x,y), (x,-1,1), (y,-1,1), frame=True, axes_labels=['x','y'], axes=Fal
```

Out[70]:



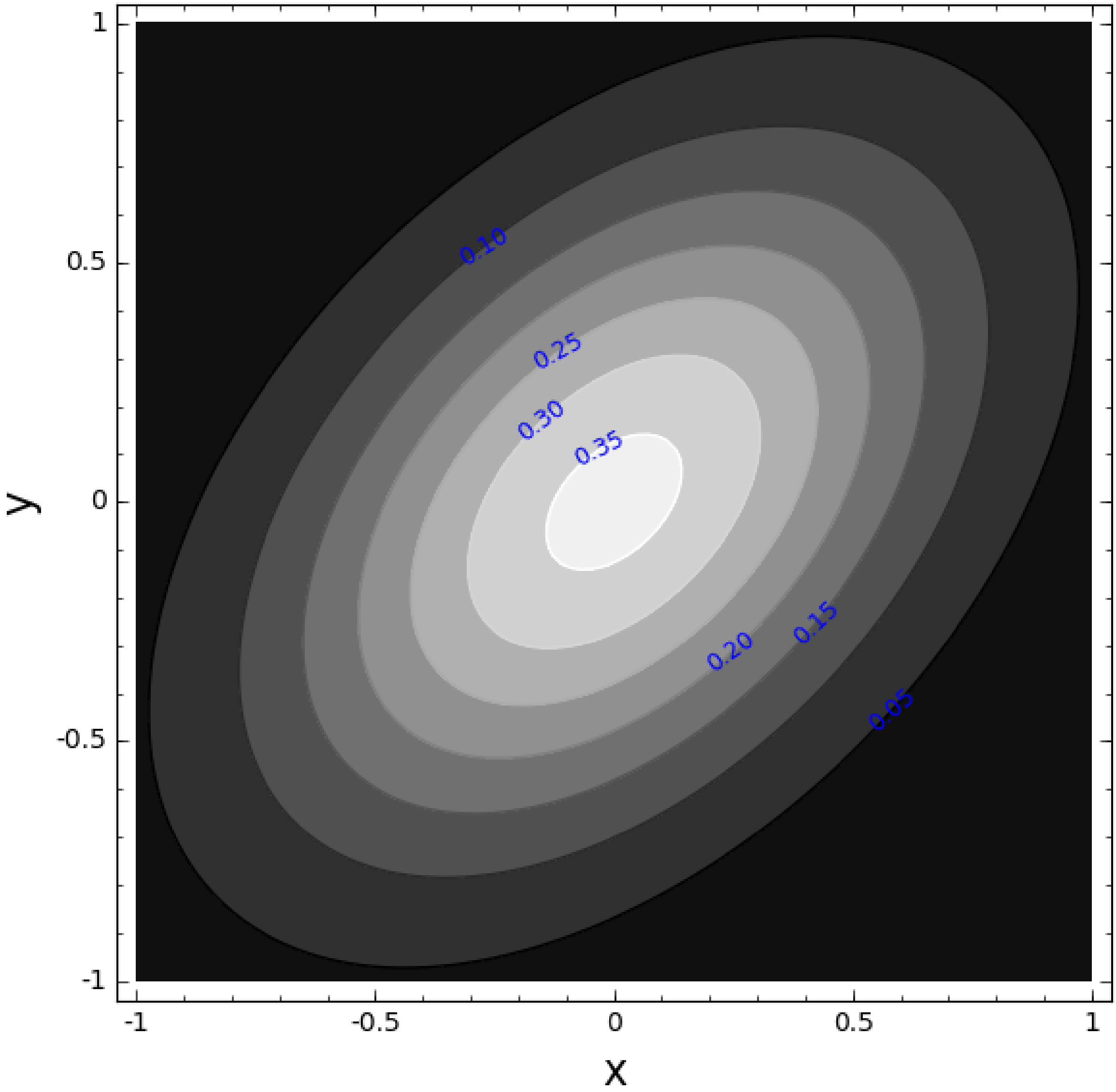
In [5]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = 0.9

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

contour_plot(P(x,y), (x,-1,1), (y,-1,1), frame=True, axes_labels=['x','y'], axes=Fal
```

Out[5]:



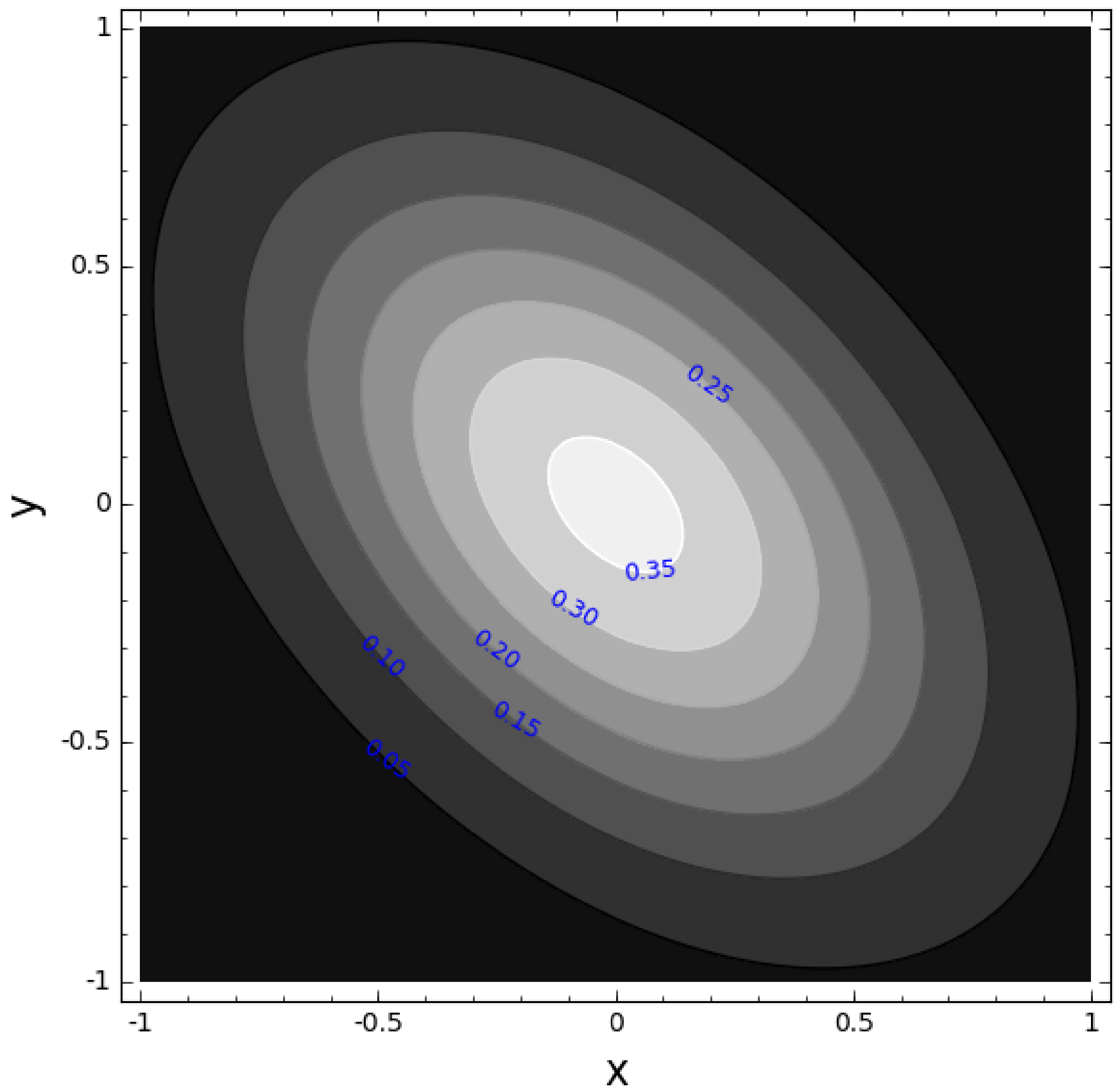
In [70]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = -0.9

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

contour_plot(P(x,y), (x,-1,1), (y,-1,1), frame=True, axes_labels=['x','y'], axes=Fal
```

Out[70]:



The three graphs above represent different values of ρ in an increasing order (meaning, higher correlation coefficient). One observation is that the distribution is stretched along the line $y = x$, the closer is ρ to 1 (which makes sense because y is closely correlated with x, so points where y ≈ x will have a higher probability density). Also, as ρ increases in absolute value, the center becomes taller (also can be implied from the mentioned reason). When ro is less than 0, the distribution bell stretches along the y = -x axis which, again, makes sense because they are invertedly correlated, so then there is a higher probability that y would be equal to -x.

Task (c)

As x and y tend to infinity, based on the graph, we can see that the function approaches 0. To be more precise, we can take a look at the formula and see that $z \to \infty \Rightarrow e^{-\frac{z}{const}} \to 0 \Rightarrow P(x, y) = const * e^{-\frac{z}{const}} \to 0.$

In other words, considering the finite standard deviation and mean at 0, it is infinitely improbable (probability = 0) that we will get an infinite value on either x or y or both.

Task (d)

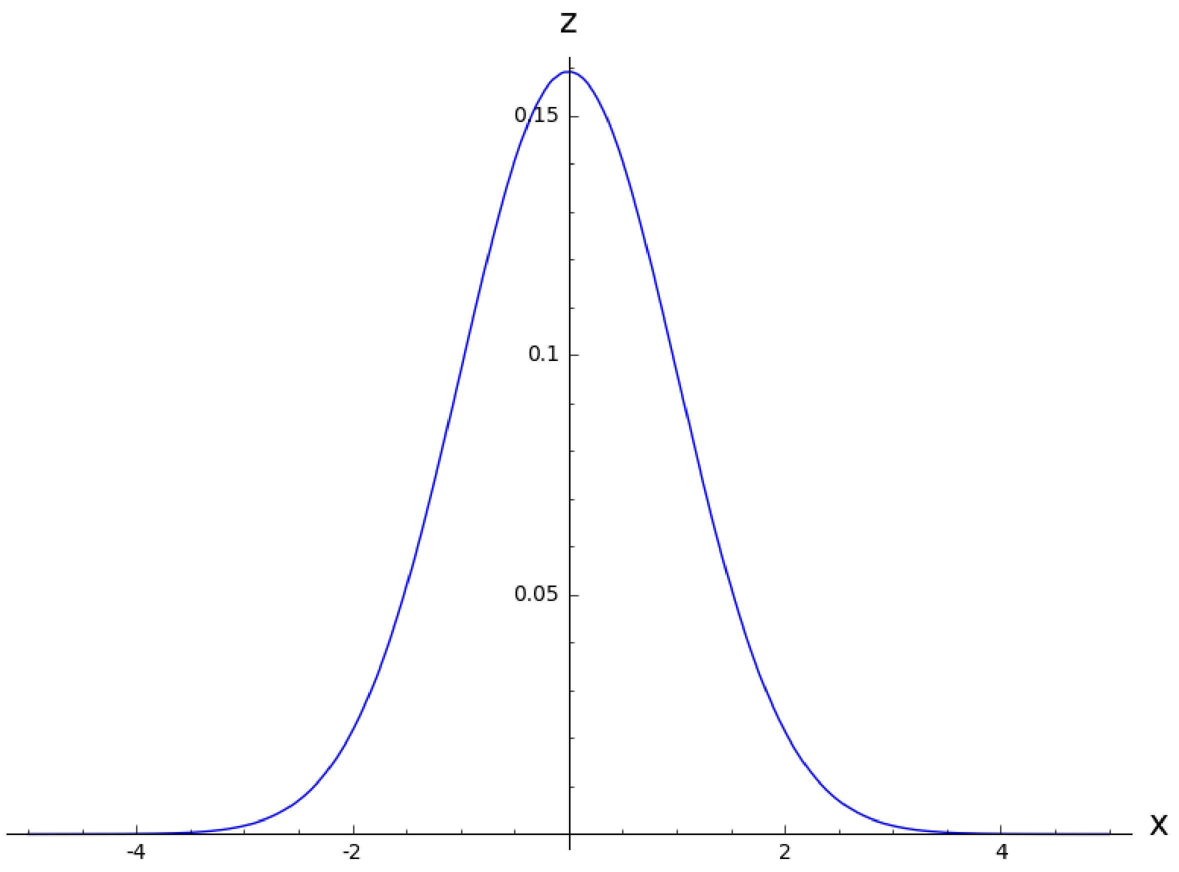
In [154]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = 0

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

plot(P(x,0), (x,-5,5), axes_labels=['x', 'z'], axes=True)
# contour_plot(P(x,0), (x,-1,1), (y,-1,1), frame=True, axes_labels=['x','y'], axes=1
```

Out[154]:



In [176]:

```
print(integrate(P, x, -oo, oo)) # = 1 / sqrt(2*pi)
print(integrate(sqrt(2*pi) * P, x, -oo, oo)) # = 1
```

```
y |--> 1/2*sqrt(2)*e^(-1/2*y^2)/sqrt(pi)
y |--> e^(-1/2*y^2)
```

It is almost a PDF of one variable (when $\sigma_y = 1$ and $\rho = 0$) because when y = 0, all the terms related to it cancel to zero. Then, we are left with $P(x, 0) = \frac{1}{2\pi\sigma_x} * exp[-\frac{x^2}{2\sigma_x^2}]$. The area that we get when we integrate this 2D distribution is $\frac{1}{\sqrt{2\pi}}$ (which is what the above thing is equal to, just replace y with 0). Therefore, we would need to mulptiply the PDF function by $\sqrt{2\pi}$ to get the sum of all probabilities = 1. Then, this would be a PDF function of variable x.

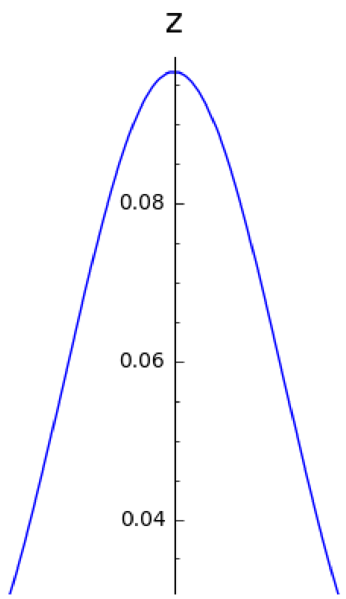
In [179]:

```
var('sd_x,sd_y,ro')
sd_x = 1
sd_y = 1
ro = 0

z(x,y) = x^2/sd_x^2 - ro*x*y/(sd_x*sd_y) + y^2/sd_y^2
P(x,y) = 1/(2*pi*sd_x*sd_y*sqrt(1-ro^2)) * e^(-z(x,y) / (2 * (1-ro^2)))

plot(P(x,1), (x,-5,5), axes_labels=['x', 'z'], axes=True)
```

Out[179]:



In [182]:

```
print(integrate(P, x, -oo, oo)) # = 1 / (e^0.5 * sqrt(2*pi))
print(integrate(e**.5 * sqrt(2*pi) * P, x, -oo, oo)) # = 1
```

```
y |--> 1/2*sqrt(2)*e^(-1/2*y^2)/sqrt(pi)
y |--> 1.6487212707001282*e^(-1/2*y^2)
```

When y = 1, $\sigma_y = 1, \rho = 0 \Rightarrow P(x, 1) = \frac{1}{2\pi\sigma_x} * exp[-\frac{x^2+1}{2\sigma_x^2}] = \frac{1}{2\pi\sigma_x e^{\frac{1}{2\sigma_x}}} * exp[-\frac{x^2}{2\sigma_x^2}]$ - this is proportional to the first one, just squished in the z-axis. Can still work as a PDF for x if we multiply the PDF function by $e^{\frac{1}{2}}\sqrt{2\pi}$ to equalize it with the single-variable PDF function above.

A.2

Task (a)

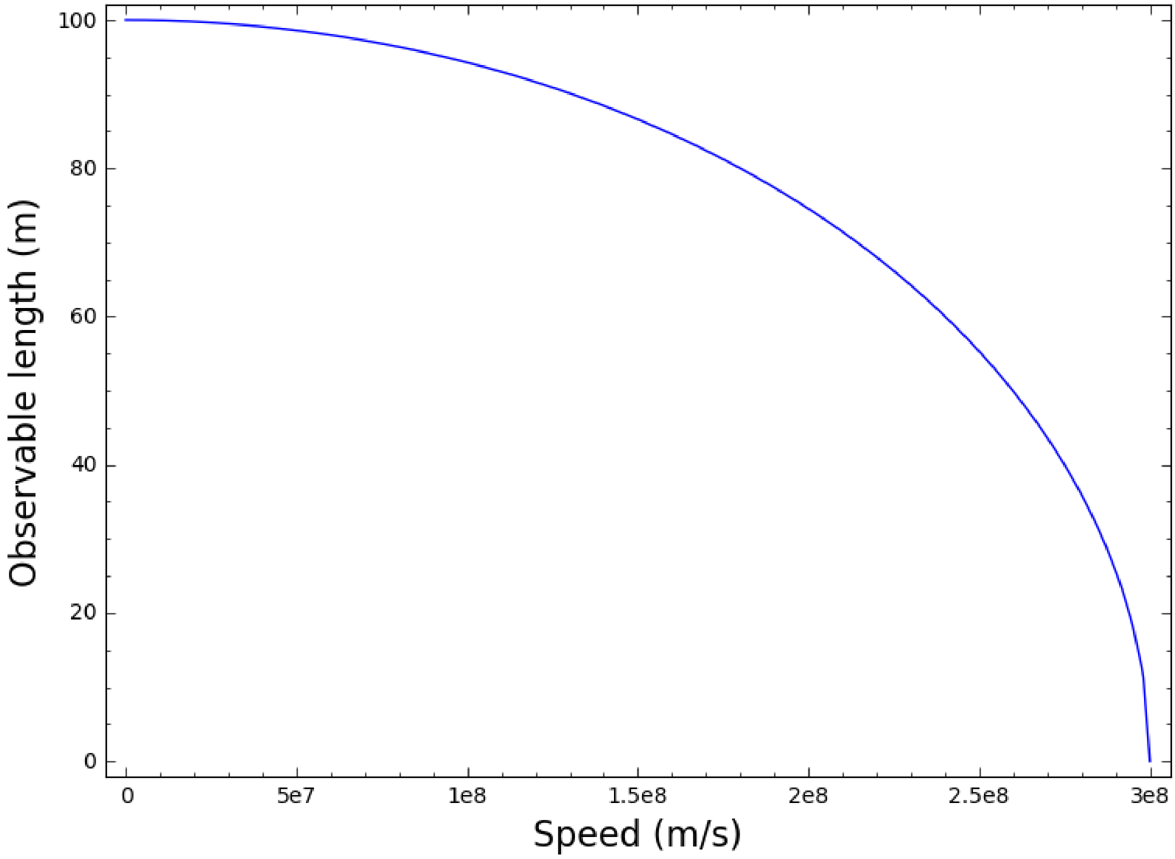
In [183]:

```
var('c, L_0')
c = 3 * 10^8 # m/s
L_0 = 100 # spaceship is 100m long at rest

L(v) = L_0 * sqrt(1 - v^2/c^2)

plot(L(v), (v,0,3*10^8), frame=True, axes_labels=['Speed (m/s)', 'Observable length
```

Out[183]:



Observable length of a moving spaceship based on special relativity principle.

Task (b)

As v increases, L decreases subquadratically $\Rightarrow$ the higher is the speed (closer is to the speed of light), the higher is the rate, at which L drops. This makes sense because $L'(v) = -\frac{L_o v}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}$ $\Rightarrow$ as v increases, the numerator will increase, and the denominator will decrease. We can also notice that observable length $L = 0$ at $v = 3 * 10^8$.

Task (c)

We can observe that in the graph, the speed tends to zero when the speed of the ship is near the speed of light. This can also be observed in the formula: this function's domain $D = [0; 3 * 10^8]$ $\Rightarrow$ it is continuous at the point $3 * 10^8 \Rightarrow \lim_{v \to c-} L_0 * \sqrt{1 - \dfrac{v^2}{c^2}} = L(c) = 0$

In [9]:

```
lim(L(v), v=c, dir='-') # v -> c-
```

Out[9]:

0

Task (d)

We needed left-side limit for the following reasons:

(1) If we surpass the speed of light, we will get a negative number under the square root sign, which will result in complex numbers (not suitable for our length)

(2) According to the laws of Physics, when we get closer to the speed of light, the mass of the rocket gets closer to infinity, so is the energy required to accelerate it goes to infinity. Since we do not have infinite amount of energy in a rocket, we cannot reach *c*, that's why we would want to calculate the limit to see to which value the length tends to as the speed tends to infinity

A.3

Task (a)

$\lim_{(x,y) \to (0,0)} \dfrac{x^2}{x^2 + y^2}$ is in indefinite form, therefore, let's calculate the limit from different paths:

Path 1: y = 0 $\Rightarrow \lim_{x \to 0} \dfrac{x^2}{x^2 + 0} = 1$

Path 2: y = x $\Rightarrow \lim_{y \to 0} \dfrac{y^2}{y^2 + y^2} = \dfrac{1}{2}$

As we can see, these two paths that approach the same $\Rightarrow$ this limit **DNE**

Task (b)

$\lim_{(x,y) \to (1,1)} \dfrac{x^2}{x^2 + y^2}$ is a composition of elementary functions ($f(x) = x^2, g(x) = x^2 + y^2$) which are continuous $\Rightarrow h(x) = \dfrac{f(x)}{g(x)}$ is continuous on its domain $D = \mathbb{R}^2 \backslash \{(0, 0)\}$.

Since $(1, 1) \neq (0, 0)$ $\Rightarrow$ f(x) is continuous at this point $\Rightarrow \lim_{(x,y) \to (1,1)} f(x) = \dfrac{1^2}{1^2 + 1^2} = \dfrac{1}{2}$

Part B

B.1

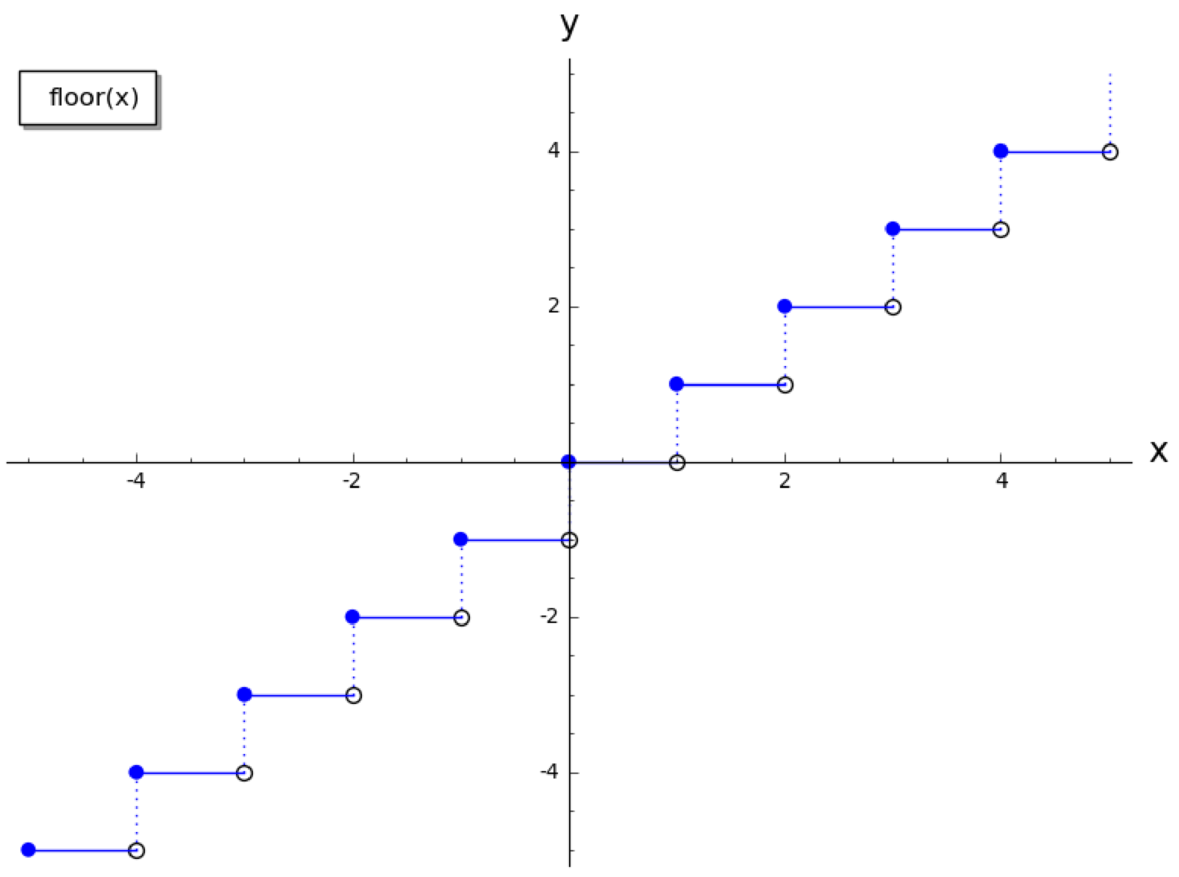
Task (a)

In [27]:

```
# REALISTIC FLOOR GRAPH
graph = point((0,0), pointsize=0, legend_label='floor(x)', axes_labels=['x','y'])
for i in range(-5, 5):
    graph += line([(i, i), (i + 1, i)]) # floor
    graph += point((i, i), rgbcolor='blue', pointsize=50)
    graph += point((i + 1, i), rgbcolor='white', faceted=True, pointsize=50) # cut
    graph += line([(i + 1, i), (i + 1, i + 1)], linestyle=':')

graph
```

Out[27]:

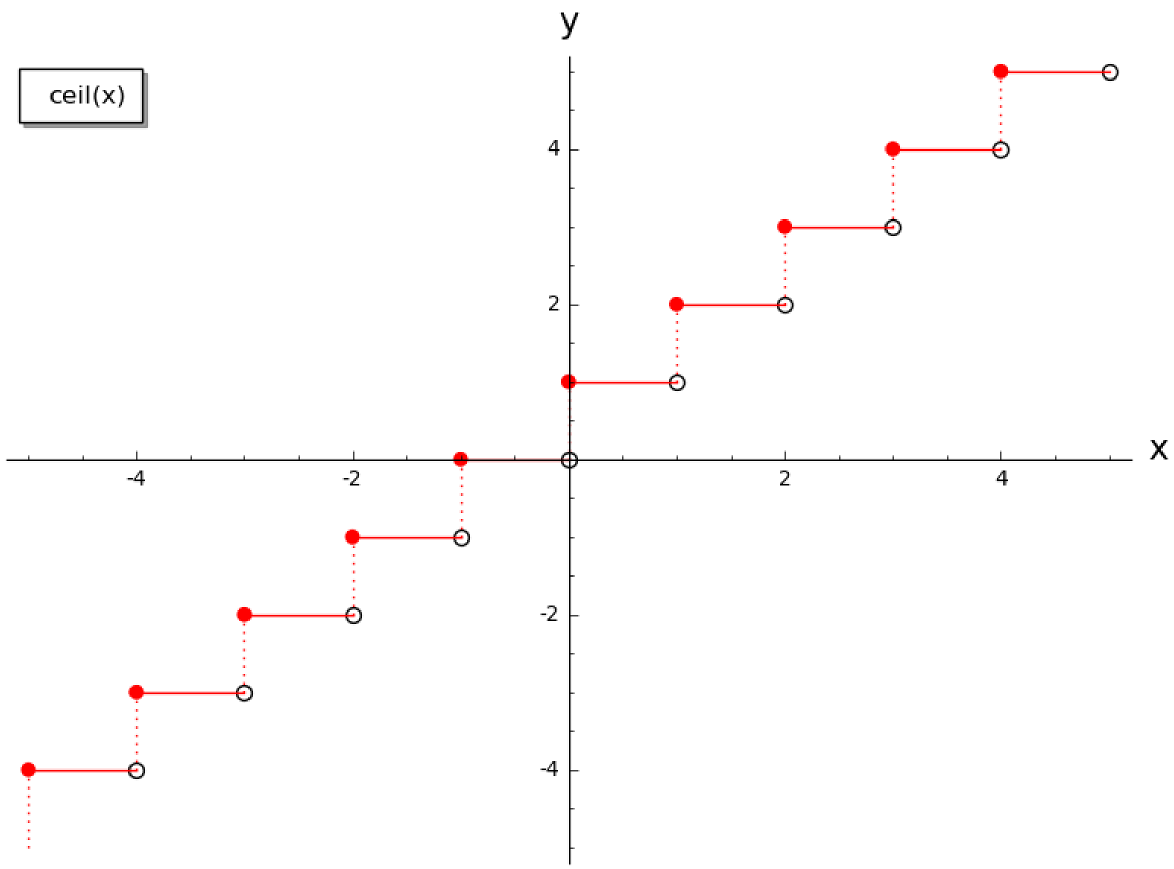


In [37]:

```
# REALISTIC CEIL GRAPH
graph = point((0,0), pointsize=0, legend_label='ceil(x)', axes_labels=['x','y'])
for i in range(-5, 5):
    graph += line([(i, i + 1), (i + 1, i + 1)], rgbcolor='red') # ceil
    graph += point((i, i + 1), rgbcolor='red', pointsize=50)
    graph += point((i + 1, i + 1), rgbcolor='white', faceted=True, pointsize=50) # c
    graph += line([(i, i), (i, i + 1)], linestyle=':', rgbcolor='red')

graph
```

Out[37]:

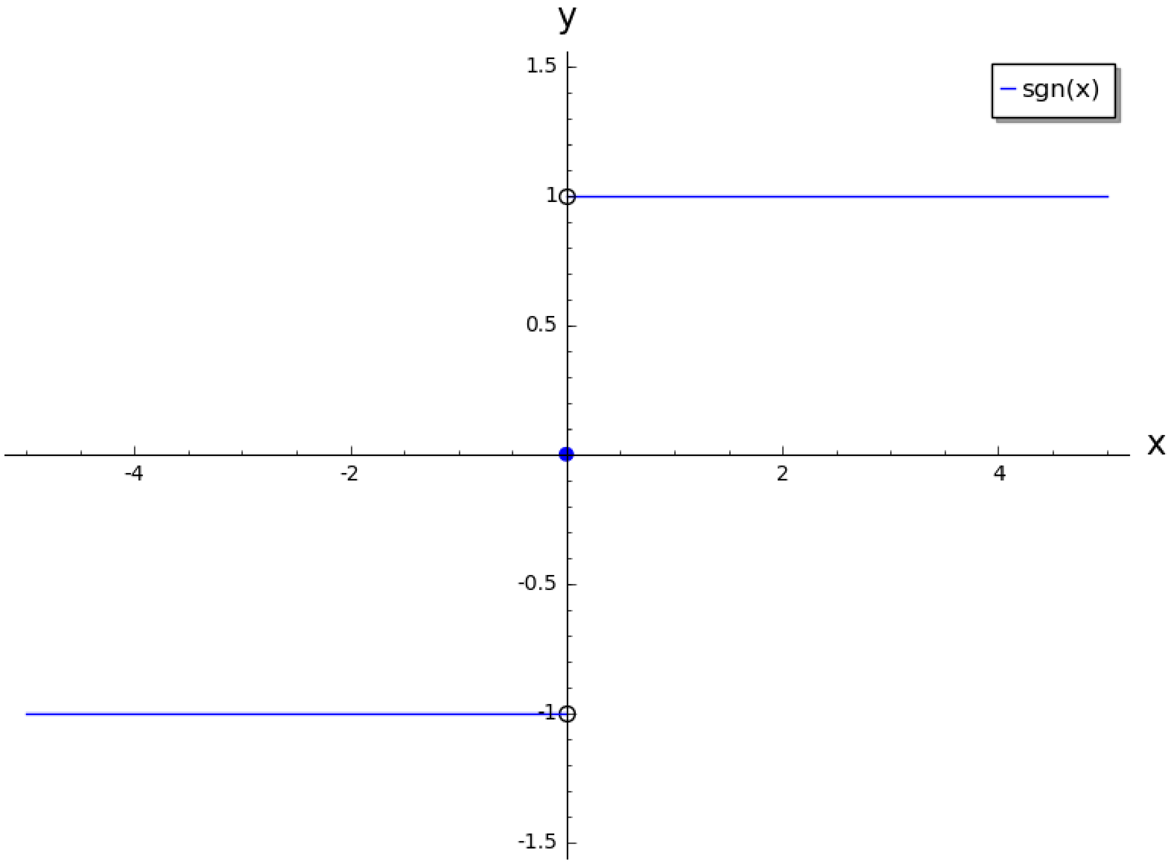


In [14]:

```
# REALISTIC GRAPH
bef_zero = line([(-5,-1), (0,-1)], axes_labels=['x','y'], ymin=-1.5, ymax=1.5, leger
bef_zero_undef = point((0,-1), rgbcolor='white', faceted=True, pointsize=50)
zero = point((0,0), rgbcolor='blue',pointsize=50)
aft_zero_undef = point((0,1), rgbcolor='white', faceted=True, pointsize=50)
aft_zero = line([(0,1), (5,1)])

bef_zero + bef_zero_undef + zero + aft_zero_undef + aft_zero
```

Out[14]:



Functions floor and ceil are **not** continuous on $\mathbb{Z}$ because the values we get when approaching from left and right sides are different.
Function signum is **not** continuous on {0} because the values we get when approaching from left and right sides to zero are different.

Task (b)

Floor function is useful when we want to calculate integer division. Scenario: there are n apples that we would like to split among k people. We would like for everyone to receive the same and maximum amount of apples. To do this, we would do tmp = n / k, num_of_apples = floor(tmp)

Ceiling function is useful when we would like to provide an upper bound estimation. For example, each sofa can fit k people, and we have n people. We would like to have everyone seated, even if one sofa might be half-empty. Therefore, we would do tmp = n / k, num_of_sofas = ceil(tmp)

Lastly, signum function is used in perceptron (or more generally, as an activation function for artificial neurons). For example, we would like our perceptron with inputs x and y (binary digits) to calculate x & y. Having weights w_x, w_y, we would like to train this perceptron so that $w_x * x + w_y * y > 0$ would lead to an answer 1, and $w_x * x + w_y * y \leq 0$ would lead to an answer 0.

Task (c)

In [60]:

```
# DNE, limits exist for x -> 0- and x -> 0+ and are not equal
print(lim(sgn(x), x=0))
print(lim(sgn(x), x=0, dir='-')) # lim from left side
print(lim(sgn(x), x=0, dir='+')) # lim from right side
```

```
und
-1
1
```

In [187]:

```
# limits from both sides are equal
lim(abs(sgn(x)), x=0)
```

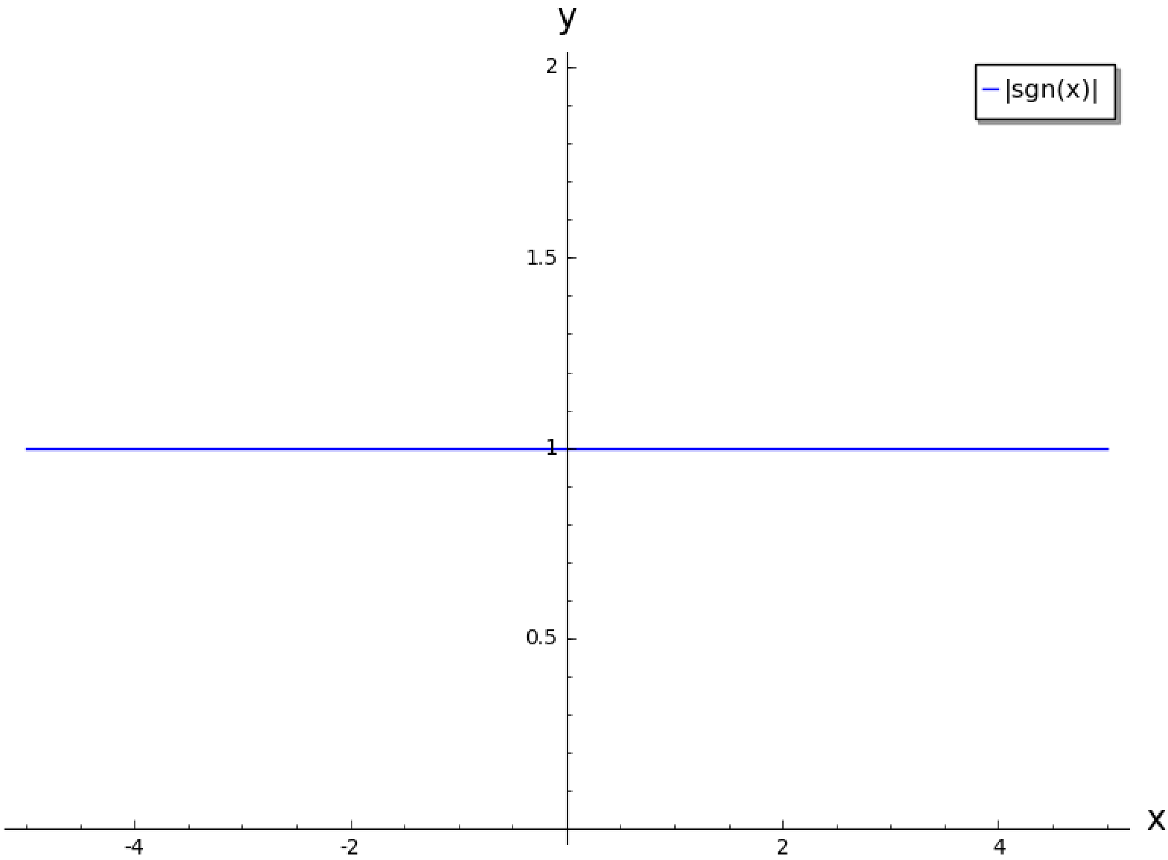
Out[187]:

```
1
```

In [188]:

```
# one line with a hole at (0, 1) and a real point at (0, 0)
plot(abs(sgn(x)), (x,-5,5), legend_label='|sgn(x)|', axes_labels=['x','y'])
```

Out[188]:



In [190]:

```
# At x = -1.5, floor = -2, ceil = -1 => (floor + ceil) / 2 = -1.5
lim((floor(x)+ceil(x)) / 2, x=-1.5)
```

Out[190]:

```
-3/2
```

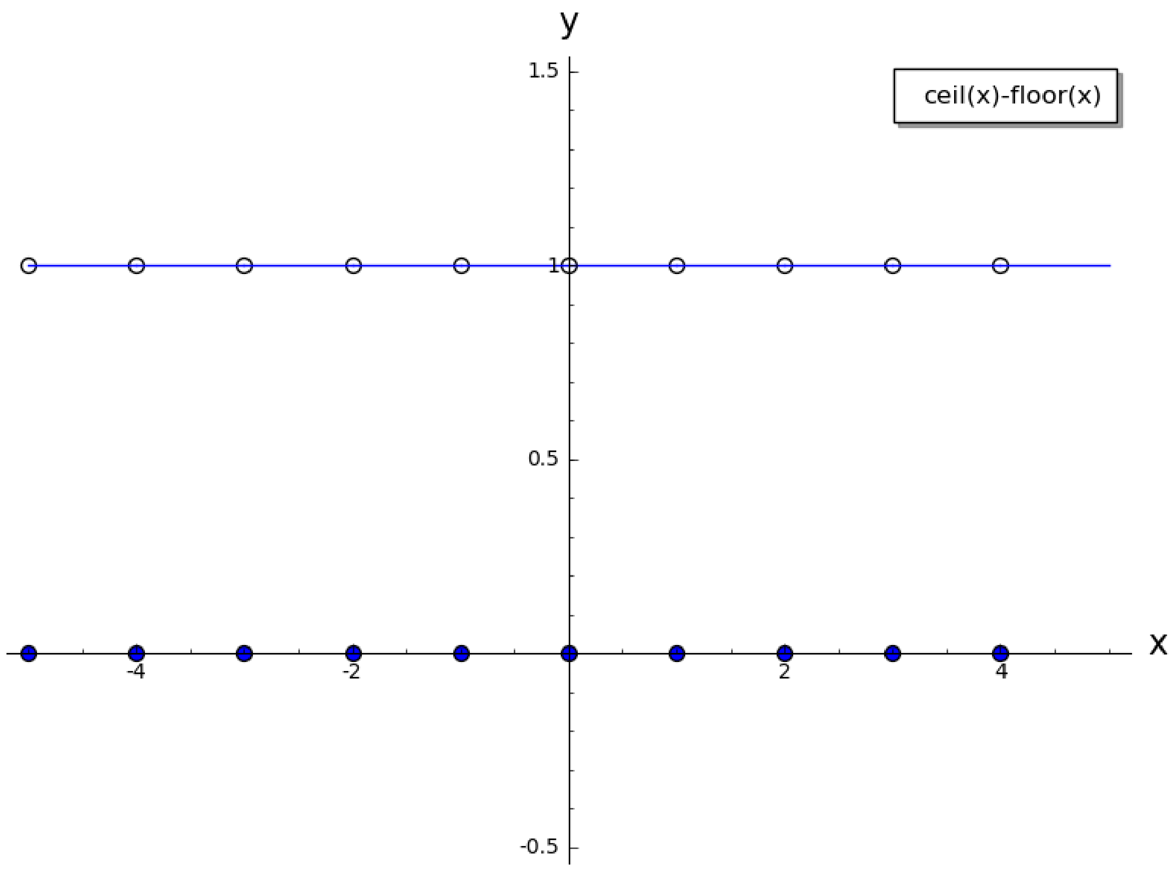
Task (d)

In [69]:

```
graph = point((0,0), pointsize=0, legend_label='ceil(x)-floor(x)', axes_labels=['x',
for i in range(-4, 6):
    graph += line([(i - 1, 1), (i, 1)]) # ceil - floor
    graph += point((i - 1, 1), rgbcolor='white', faceted=True, pointsize=50) # cut
    graph += point((i - 1, 0), rgbcolor='blue', faceted=True, pointsize=50) # real v

graph
```

Out[69]:



One observation is that when $x \in \mathbb{Z}$, ceil(x) = floor(x), then the difference would be 0. Therefore, we have these "discontinuations." $lim_{x \to C} f(x)$ where C is any integer does not exist because at $x \in \mathbb{Z}$, the limits from two sides are unequal. Therefore, the limit at integer points does not exist.

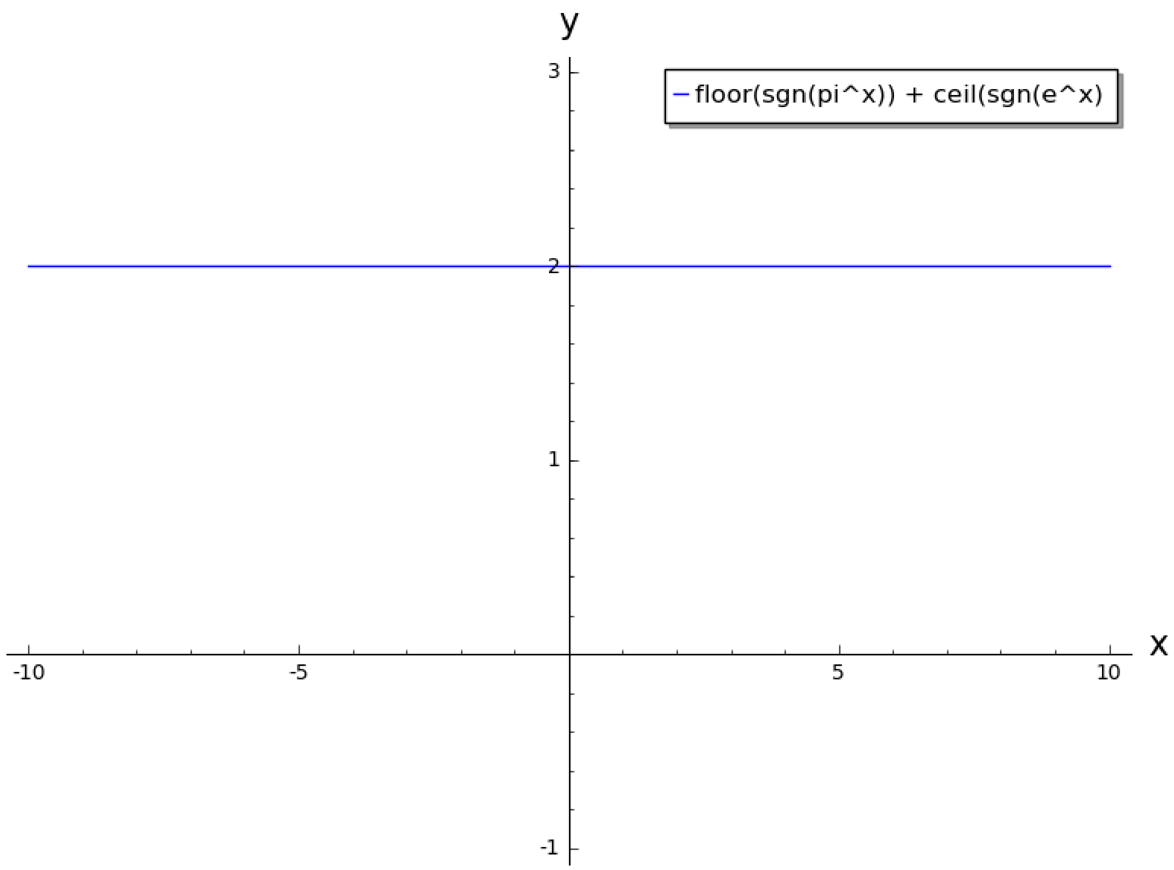
Task (e)

Simple solution

In [202]:

```
# a^x is always a positive number
# sgn(a) = 1, a > 0
# floor(1) + ceil(1) = 1 + 1 = 2
plot(floor(sgn(pi^x)) + ceil(sgn(e^x)), (x,-10,10), ymin=-1, ymax=3, legend_label='f
```

Out[202]:

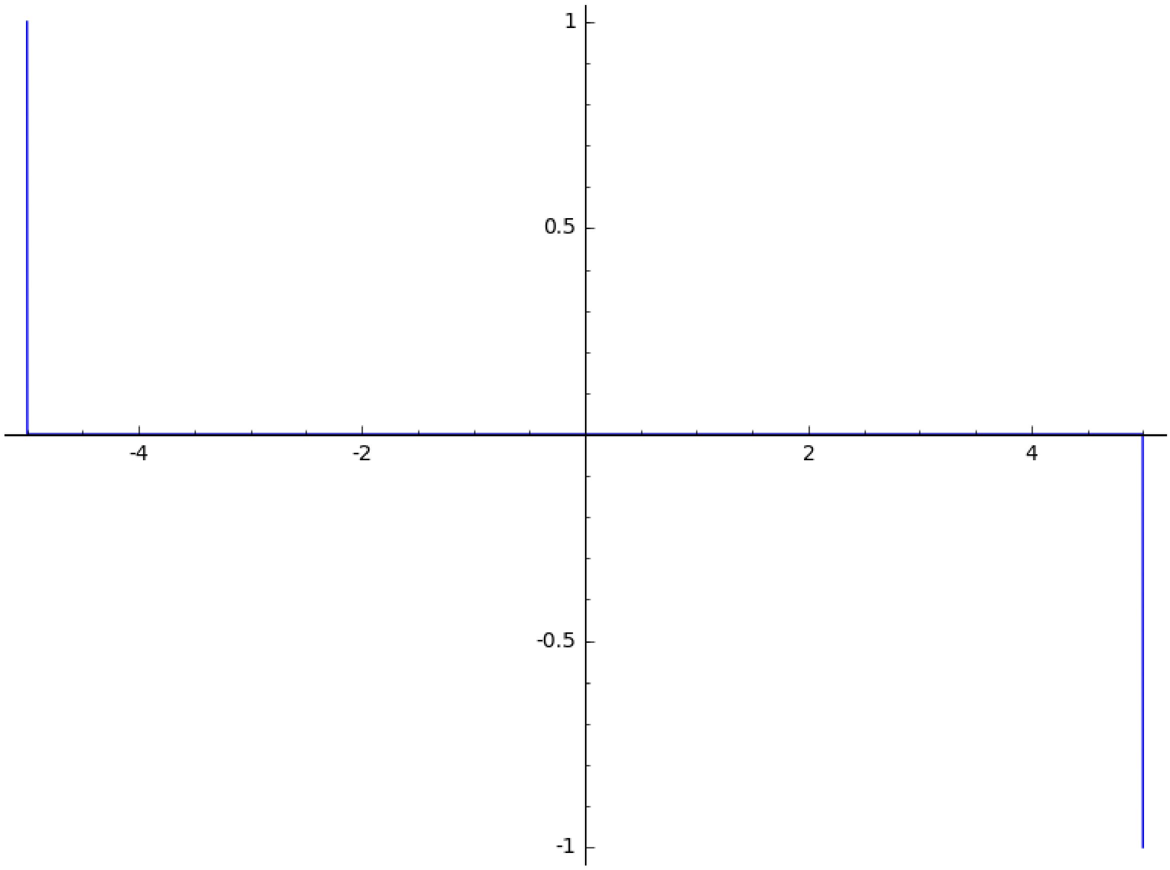


The "why not?" solution

In [207]:

```
# Let's imagine we have such a function:
plot(sgn(x)*(ceil(x) - floor(x) - 1), (x, -5, 5))
```

Out[207]:



In [251]:

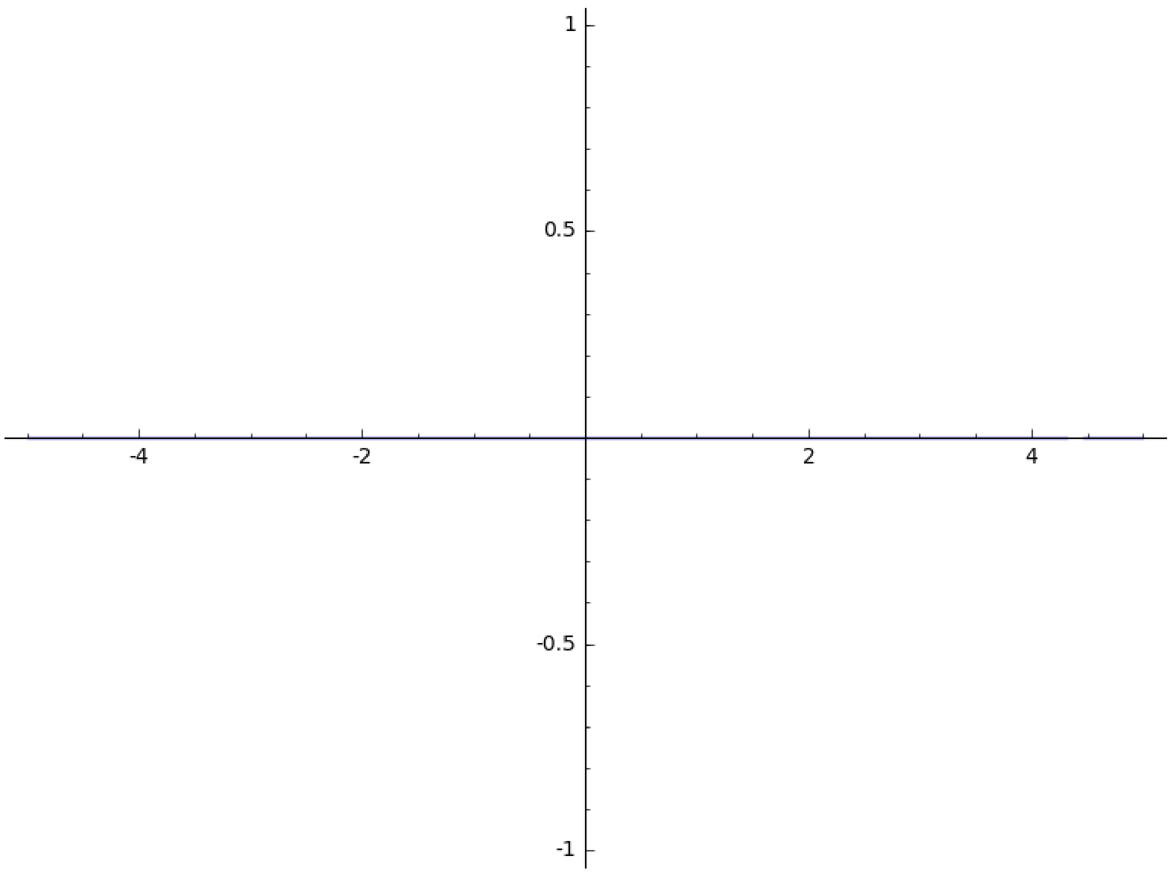
```
# We can observe for all integer x, ceil(x) = floor(x)
# Therefore, function value = -1*sgn(x) != 0 as other points
# Thus, this function is not continuous
# Now, we would like to somehow redefine the function when x in Z
# To do so, let's figure out how to determine if x is an integer

var = 6.5 # some number to work with
is_int = var % 2 # integer would give 0 or 1 while real would give smth in b/w
is_int = abs(is_int - 0.5) # 0, 1 -> 0.5, real -> != 0.5

# Now we would want a function that can transform 0.5 into 1 and != 0.5 into 0 => Pl
f = 1/(sqrt(2*pi)) * e^(-is_int^2) # peak is at 1 / sqrt(2*pi), we want 1
f = e^(-is_int^2) # yay, now let's center it at 0.5
f = e^(-(is_int - 0.5)^2) # now how do we transform as promised??
f = floor(e^(-(is_int - 0.5)^2)) # wow, x < 0.5 -> f = 0, x = 0.5 -> f = 1

# So what do we want to do with this?
# Well, just add it to make the original function 0 instead of -1 when x is int
plot(sgn(x)*(ceil(x) - floor(x) - 1 + 1 * (floor(e^(-(abs(x - floor(x/2)*2 - 0.5) -
```

Out[251]:



In [253]:

```
f = e^(-(x - 0.5)^2)
plot(f, (x, -5, 5))
f(0.5)
```

Out[253]:

1

Now we have a function that is 0 on x that are not int and -1 + 1 = 0 when x is int! 🎉

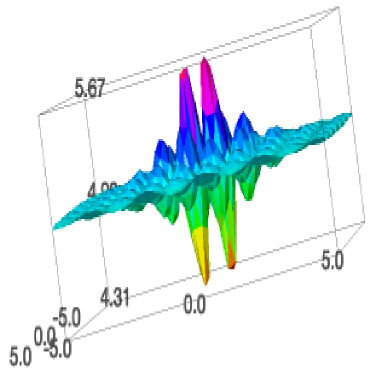
B.2

Task (a)

In [48]:

```
d(x,y) = 5 + sin(4*x)*sin(3*y) / (1+x^2+y^2)
plot3d(d(x,y), (x,-5,5), (y,-5,5), adaptive=True)
```

Out[48]:



Task (b)

(1) $lim_{(x,y)\to(\infty,\infty)} 5 + \dfrac{sin4x * sin3x}{1 + x^2 + y^2} = lim_{(x,y)\to(\infty,\infty)} 5 + \dfrac{1}{1 + x^2 + y^2}$ (since $sin4x * sin3y \leq 1 * 1 = 1$) = $lim_{(x,y)\to(\infty,\infty)} 5 + 0 = 5$

(2) $lim_{(x,y)\to(0,\infty)} 5 + \dfrac{sin4x * sin3x}{1 + x^2 + y^2} = lim_{(x,y)\to(0,\infty)} 5 + \dfrac{1}{1 + x^2 + y^2}$ (since $sin4x * sin3y \leq 1 * 1 = 1$) = $lim_{(x,y)\to(0,\infty)} 5 + 0 = 5$

(3) $lim_{(x,y)\to(\infty,0)} 5 + \frac{sin4x * sin3x}{1 + x^2 + y^2} = lim_{(x,y)\to(\infty,0)} 5 + \frac{1}{1 + x^2 + y^2}$ (since $sin4x * sin3y \leq 1 * 1 = 1$) = $lim_{(x,y)\to(\infty,0)} 5 + 0 = 5$

It's worth noticing that no matter what configuration of infinity we have for (x, y), as long as one of them approaches infinity, the denominator becomes infinity, making the fraction equal to zero.

Based of the 3D plot, we can see that $\exists x_0, \exists y_0 : \forall x > x_0, \forall y > y_0, |d(x, y)| < |d(x - \pi, y - \pi)|$ (since π is half-period of the *sine* function, meaning that $|sin(x)| = |sin(x - \pi)|$)

Or, a bit barbarically, the function tends to 5 as we get further from the center

Task (c)

(1) $lim_{t\to 0-} \eta(x, y, t)$ is the level of water right before the rock hits the surface. Because we assume that the surface of the ocean is still, then the wave surface height would be 0 (counting from the flat ocean surface level).

(2) $lim_{t\to\infty} \eta(x, y, t)$ is the level of water when the time since the rock hit is tending to infinity. For this case, the wave surface height would be 0 (intuitively, after very very long time, the water will return to being still like originally).

(3) $lim_{x\to\infty} \eta(x, y, t)$ is the level of water infinitely far away on the x-axis. This would be equal to zero because it is "so far" away that the wave heights there would tend to 0.

(4) $lim_{y\to\infty} \eta(x, y, t) = 0$ - same reasoning as in (3)

Task (d)

(1) $lim_{t\to\infty} Z(x, y, t)$ is the surface water height when the time since the rock hit is tending to infinity (c)-(2). The water height would be 0 $\Rightarrow lim_{t\to\infty} Z(x, y, t) = d(x, y)$. As we can see, as there is no surface wave height anymore, the ocean depth depends only on the geometry of the bottom.

(2) $lim_{x\to\infty} Z(x, y, t)$ is the level of water infinitely far away on the x-axis. For this case, the water height would be $lim_{x\to\infty} d(x, y) + lim_{x\to\infty} \eta(x, y, t) = 5 + 0 = 5$ (c)-(3).

(3) $lim_{y\to\infty} Z(x, y, t) = 5$ - same explanation as in (2).

Reflection

#professionalism: I use LaTeX formatting to create concise and readable formulas. I use headers of different size to indicate sections, problems, and subproblems.
#dataviz: When plotting, I made sure that the scale is relevant to show the important features of the graphs and axes are signed. For the first task, I chose contour graphs to make the features of graphs easily comprehensible.
#deduction: I used known premises (e.g. implications from continuity of a function) and derived definite conclusions (e.g. in A.3.b, B.2).
#algorithm: In task B.1.e, I describe an algorithm to transform a piecewise function into a mathematical formula that would redefine value of the function for integer arguments.