Cumulative Final Assessment

CS111A Fall 2018

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Part 1: High dimensional chains

(a)

The given equations describe the motion of the particle. The variables are x, y and t, where x and y are parametrized by t So, we can say that x and y are describing the position of the particle as coordinates in the x-y plane. These positions vary with time. So, variable t must represent time. Time, t, is the independent variable here as it doesn't depend on any other factors in the given context. The dependent variables are x and y, that depend on time, t. Hence, the given set of composite functions describe the path of the particle moving in the x-y plane with respect to time.

Since t is time, $t = 0$ should be when the observation is started. So, the graph that shows the path of the particle in the x-y plane for $t \geq 0$ is:

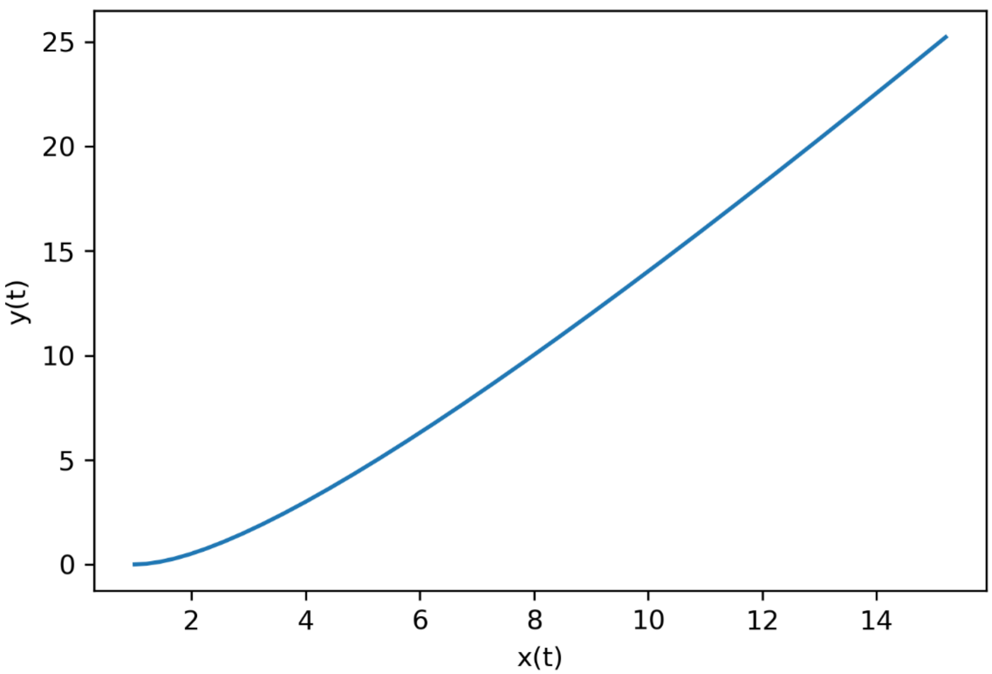

Figure 1-1a: Parametric plot of $y(t)$ vs $x(t)$ for $0 \leq t \leq 3$.

In the given context, no boundary condition is given for t. So, the equations should actually be able to predict the position of the particle for any $t < 0$ as well. Hence, the plot for negative t to positive t should look as below.

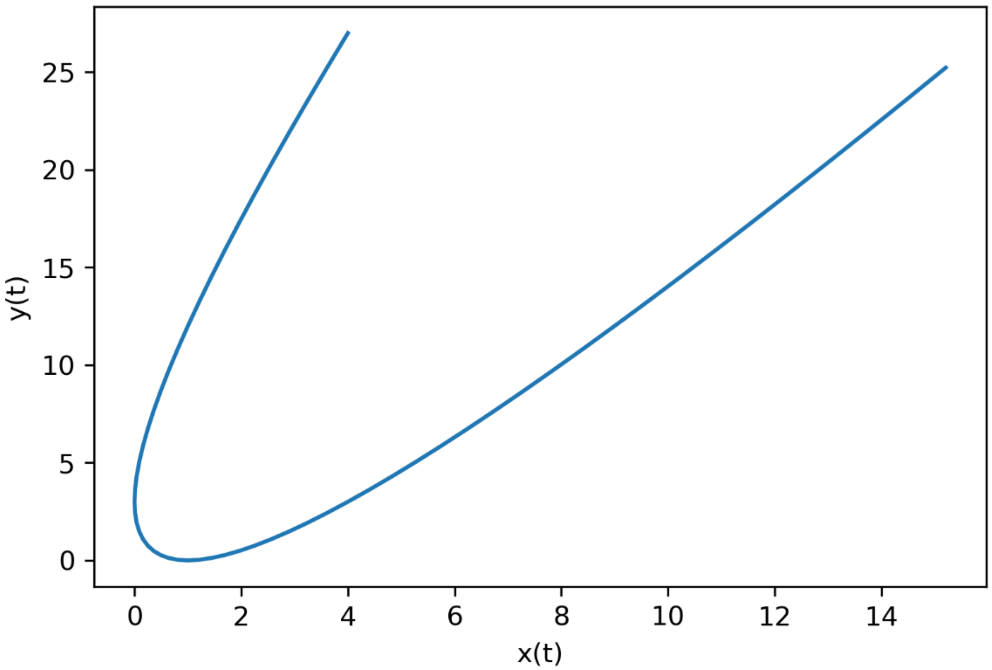

Figure 1-1b: Parametric plot of $y(t)$ vs $x(t)$ for $-3 \leq t \leq 3$.

(See Appendix A for the Python code of both these graphs.)

(b)

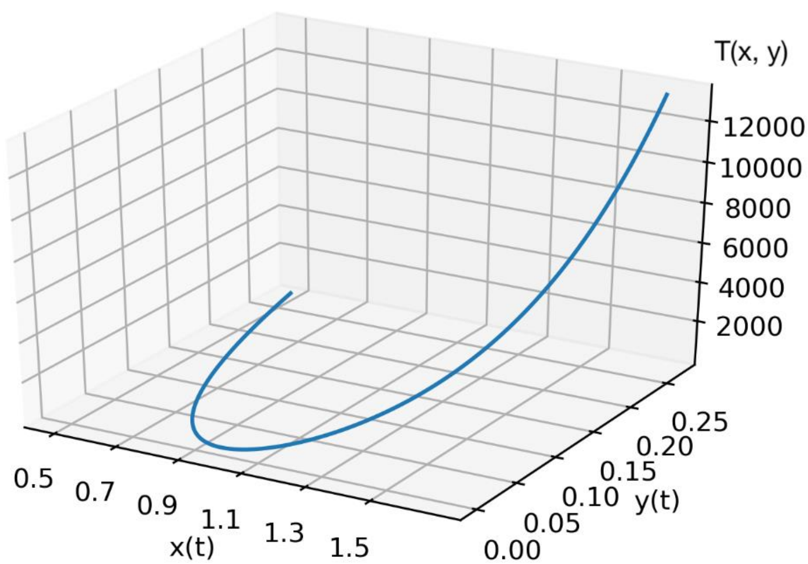

Figure 1-2: Parametric plot of $T\big(x(t), y(t)\big)$ for $-0.3 \leq t \leq 0.3$.

(See Appendix B for the Python code of this graph.)

$T(x, y)$ is a third variable (temperature) that is dependent on both x and y. So, the plot is a 3-d plot. It is also parametric as x and y are parametrized by t.

(c)

T is the changing temperature that varies based on the function $T(x, y)$. So, $\frac{dT}{dt}$ is first derivative of $T(x, y)$ that gives the rate of change of the temperature with time. As $T(x, y)$ depends on the location of the particle, and each of x and y depend on t, the temperature change is dependent on the location of the particle and, in turn, the change in time.

(d)

$$T(x, y) = e^{5x}(x^2 + y^2)$$

Let's first make $T(x, y)$ a single variable function by decomposing the composite functions in terms of the independent variable, $\boldsymbol{t}$ so that we can use single variable differentiation and chain rule for getting the derivative.

Replacing $x = (t + 1)^2$ and $y = 3t^2$,

$$T(\mathrm{x(t)}, \mathrm{y(t)}) = e^{5(t+1)^2}((t + 1)^4 + 9t^4)$$
$$\Rightarrow \frac{dT}{dt} = 10(t + 1) \times e^{5(t+1)^2} \times ((t + 1)^4 + 9t^4) + e^{5(t+1)^2}(4(t + 1)^3 + 36t^3)$$
$$\Rightarrow \frac{dT}{dt} = e^{5(t+1)^2}(10(t + 1)((t + 1)^4 + 9t^4) + 4(t + 1)^3 + 36t^3) \qquad (1)$$

Now, let's use the given formula for multivariable chain rule on $T(x, y)$ to see if they output the same result.

$$T(x(t), y(t)) = e^{5x}(x^2 + y^2)$$
$$\Rightarrow \frac{d}{dt}T(x(t), y(t)) = (5e^{5x}(x^2 + y^2) + 2xe^{5x})2(t + 1) + 12ye^{5x}t$$

Replacing x and y with their respective equations,

$$\frac{d}{dt}T(x(t), y(t)) = (5e^{5(t+1)^2}((\mathrm{t} + 1)^4 + 9\mathrm{t}^4) + 2(t + 1)^2 e^{5(t+1)^2})2(t + 1) + 12 \times 3t^2 \times e^{5(t+1)^2}t$$
$$\Rightarrow \frac{d}{dt}T(x(t), y(t)) = e^{5(t+1)^2}(10(t + 1)((t + 1)^4 + 9t^4) + 4(t + 1)^3 + 36t^3) \qquad (2)$$

It is apparent that (1) and (2) are the same. So, the chain rule holds for $T(t)$.

■

(e)

As proven before, it is true that,

$$\frac{d}{dt}\Big(f\big(x(t),y(t)\big)\Big)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

$\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ can be computed using this formula.

For computing $\frac{\partial z}{\partial u}$:

Let's hold v constant as the partial derivative is being taken with respect to u. So, taking derivatives of x and y with respect to u will also be partial derivatives (unlike in the given chain rule where x and y were single variable functions). Therefore, applying these two principles in the given chain rule:

$$\boxed{\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$

Similarly, for computing $\frac{\partial z}{\partial v}$:

This time let's have u constant and the same principle for derivates of x and y. Hence,

$$\boxed{\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

(f)

$$x^2\frac{\partial z}{\partial x}-y^2\frac{\partial z}{\partial y}=G(x,y)z \qquad (1)$$

$$z=xyf\left(\frac{x+y}{xy}\right) \qquad (2)$$

Let's call $\frac{x+y}{xy}$ as b(x, y):

$$\begin{aligned}\frac{\partial z}{\partial x}&=yf(b)+xy\frac{\partial f}{\partial b}\frac{\partial b}{\partial x}\\&=yf(b)+xy\frac{\partial f}{\partial b}\frac{xy-y(x+y)}{x^2y^2}\\&=yf(b)-\frac{\partial f}{\partial b}\frac{y}{x}\end{aligned}$$

Since $z=xyf\left(\frac{x+y}{xy}\right)$ is symmetrical with respect to x and y, we can trivially infer that,

$$\frac{\partial \mathrm{z}}{\partial \mathrm{y}}=\mathrm{xf(b)}-\frac{\partial \mathrm{f}}{\partial \mathrm{b}}\frac{\mathrm{x}}{\mathrm{y}}$$

Plugging back into the equation (1):

$$
\begin{aligned}
x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} &= x^2 \times \left(yf(b) - \frac{\partial f}{\partial b}\frac{y}{x}\right) - y^2 \times \left(xf(b) - \frac{\partial f}{\partial b}\frac{x}{y}\right) \\
&= x^2yf(b) - xy\frac{\partial f}{\partial b} - y^2xf(b) + xy\frac{\partial f}{\partial b} \\
&= xyf(b)(x-y) \\
&= xyf\left(\frac{x+y}{xy}\right) \times (x-y)
\end{aligned}
$$

Replacing $z = xyf\left(\frac{x+y}{xy}\right)$ from (2):

$$
\boxed{x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = (x-y)z}
$$

Therefore, the given equation is in fact in the form of $G(x,y)z$ and $\boxed{G(x,y) = (x-y)}$.

Part 2: Taking it up a notch

(a)

(i)

This situation can be explained as an analogy of double integral. In double integral, having $f(x,y) = 1$ would mean the height of a solid is set to be one. So, numerically speaking (that is, ignoring the dimensions), the volume will be equal to the area under $f(x,y)$ (because volume = area of base × height, and if height = 1, then volume = area of base). This is the same for triple integral. Having $f(x,y,z) = 1$ means the 4th dimension is set to be 1 unit. So, numerically, the integral should be equal to the volume of the solid formed by the boundaries of $f(x,y,z)$.

One critique of this method is the dimensional inaccuracy. Because the triple integral does compute the hyper-volume. But setting the 4th dimension to 1 unit just makes it numerically accurate but wrong according to dimensions.

On the other hand, it might be more computationally convenient to just take triple integrals of just 1 instead of taking the double integral of a function. But that varies based on context.

(ii)

Let's recall first that density $= \frac{\text{mass}}{\text{volume}}$. So, mass = density × volume.

The triple integral is integrating density function over unit volume. So, we have a solid with variable density. Let's start by piecing the solid into really small cubes of volume ΔV. As the side of ΔV reduces, it becomes a better representative of the density of mass inside it. So, multiplying the value of density function at a point (x_i, y_j, z_k) that is the center of the infinitesimally small cube gives the mass of the solid at that point (looking back at the formula, mass = density × volume). In other words,

$$Mass = \sum_{k=1}^{r} \sum_{j=1}^{q} \sum_{i=1}^{p} f(x_i, y_j, z_k) \times \Delta V$$

The value will be more accurate as more values of (x, y, z) are taken, which will make ΔV tend to zero and p, q, and r tend to infinity. That is, we have,

$$Mass = \lim_{p,q,r \to \infty} \sum_{k=1}^{r} \sum_{j=1}^{q} \sum_{i=1}^{p} f(x_i, y_j, z_k) \times \Delta V$$

From the definition of integrals (Riemann's sum to integrals),

$$Mass = \iiint f(x,y,z)\ dV$$

This means, the triple integral of density function does give the total mass of the solid. So, in our case of region E,

$$Mass = \iiint_E f(x, y, z)\, dV$$

(b)

For the region, E, the upper and lower limit of z and y are given. In terms of x, the limits of z is $\left[0, 2 - \frac{x^2}{2}\right]$ and the limits of y is $[0, x]$. The definite relationships of x itself that we know are $2 - \frac{x^2}{2} \geq 0$ and $x \geq 0$. From the first inequality:

$$\begin{aligned} &2 - \frac{x^2}{2} \geq 0 \\ &\Rightarrow -x^2 \geq -4 \\ &\Rightarrow x^2 \leq 4 \\ &\Rightarrow -2 \leq x \leq 2 \end{aligned}$$

But given, $x \geq 0$. So, combining the two inequalities of x, we have, $0 \leq x \leq 2$.

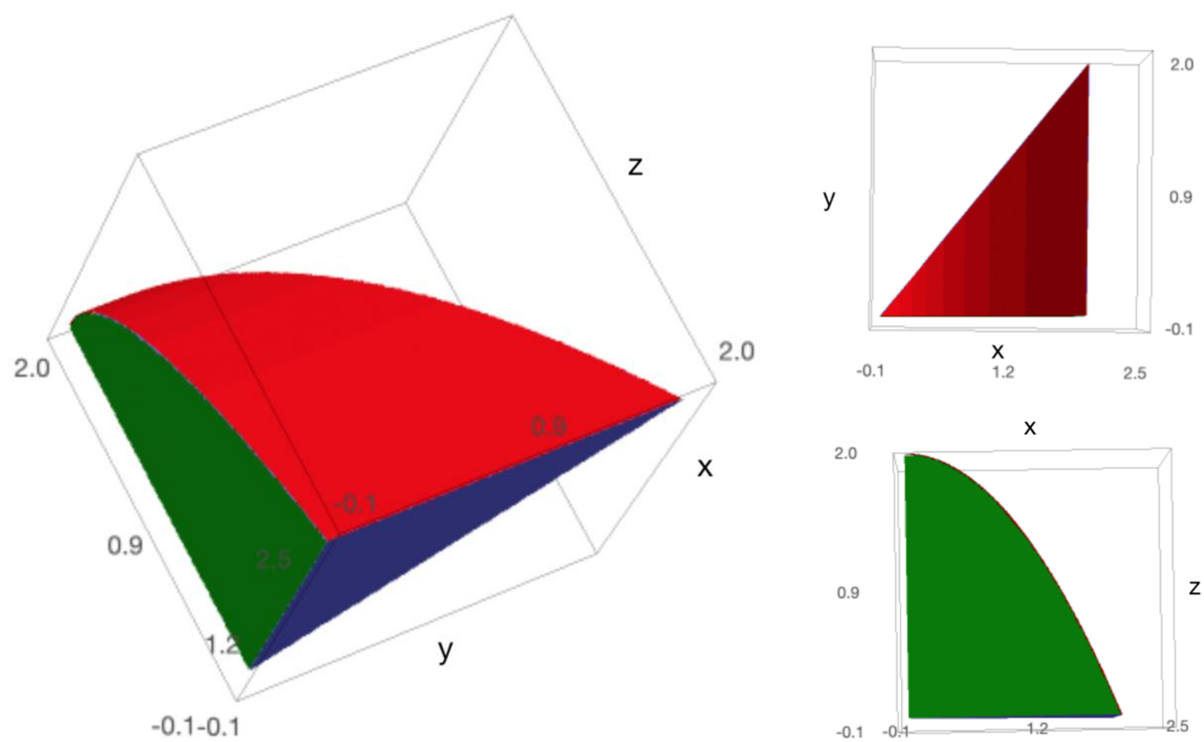

Figure 2-1: Views of region E from three different perspectives. The image in the left shows the 3-D topology of the region. The top-right image shows the top-view along z-axis which is the projection on the x-y plane (showing the relationship between x and y). The bottom-right image shows the view along y-axis which is the projection on the x-z plane (showing the relationship between x and z).

So, now we have the boundary conditions of the region, E. As discussed in a(i), triple integration of $\delta(x, y, z) = kz$ within the above derived boundary conditions should give out the total mass of the solid, and triple integral of 1 using the boundaries will give out the volume of the solid. To arrive at a numerical value, we need the integration of independent variable x to be the last step in the integration. For integration along z, y axes, the order does not matter as proven by Fubini's theorem. (Fubini's theorem, n.d.)

$$Mass, m = \int_0^2 \int_0^x \int_0^{2-\frac{x^2}{2}} kz \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^x \left(\left[\frac{kz^2}{2} \right]_0^{2-\frac{x^2}{2}} \right) dy \, dx$$

$$= \int_0^2 \int_0^x \left(\frac{k\left(2 - \frac{x^2}{2}\right)^2}{2} - \frac{k \times 0^2}{2} \right) dy \, dx$$

$$= \int_0^2 \int_0^x \left(\frac{k\left(2 - \frac{x^2}{2}\right)^2}{2} \right) dy \, dx$$

$$= \int_0^2 \left(\left[\frac{k\left(2 - \frac{x^2}{2}\right)^2}{2} \times y \right]_0^x \right) dx$$

$$= \int_0^2 \left(\frac{k\left(2 - \frac{x^2}{2}\right)^2}{2} \times x - 0 \right) dx$$

$$= \int_0^2 \left(\frac{kx\left(2 - \frac{x^2}{2}\right)^2}{2} \right) dx$$

$$= \frac{k}{2} \int_0^2 x \left(4 - 2x^2 + \frac{x^4}{2} \right) dx$$

$$= \frac{k}{2} \int_0^2 \left(4x - 2x^3 + \frac{x^5}{2} \right) dx$$

$$= \frac{k}{2} \left[2x^2 - x^4 + \frac{x^6}{12} \right]_0^2$$

$$= \frac{k}{2} \times \left(-\frac{8}{3}\right)$$
$$= -\frac{4k}{3}$$

We can observe that the density function is kz and $z \geq 0$. Material density has to be non-negative, meaning $k \geq 0$; therefore, $-\frac{4k}{3} \leq 0$. However, mass has to be non-negative. So, we only take the magnitude of the integral: $m = \left|-\frac{4k}{3}\right| = \frac{4k}{3}$.

$$Volume = \int_0^2 \int_0^x \int_0^{2-\frac{x^2}{2}} 1\, dz\, dy\, dx$$
$$= \int_0^2 \int_0^x [z]_0^{2-\frac{x^2}{2}} dy\, dx$$
$$= \int_0^2 \int_0^x \left(2 - \frac{x^2}{2} - 0\right) dy\, dx$$
$$= \int_0^2 \int_0^x \left(2 - \frac{x^2}{2}\right) dy\, dx$$
$$= \int_0^2 \left[2y - \frac{x^2 y}{2}\right]_0^x dx$$
$$= \int_0^2 \left(2x - \frac{x^2 \times x}{2} - 0\right) dx$$
$$= \int_0^2 \left(2x - \frac{x^3}{2}\right) dx$$
$$= \left[x^2 - \frac{x^4}{8}\right]_0^2$$
$$= 2$$

So, $\boxed{\text{volume of the solid} = 2 \text{ (cubic unit)}}$ and $\boxed{\text{mass of the solid} = \frac{4k}{3} \text{ (mass unit)}}$.

(c)

$$
\begin{aligned}
M_{yz} &= \int_0^2 \int_0^x \int_0^{2-\frac{x^2}{2}} xkz\,dz\,dy\,dx \\
&= \int_0^2 \int_0^x \left(\left[\frac{xkz^2}{2} \right]_0^{2-\frac{x^2}{2}} \right) dy\,dx \\
&= \int_0^2 \int_0^x \left(\frac{xk\left(2-\frac{x^2}{2}\right)^2}{2} - \frac{xk \times 0^2}{2} \right) dy\,dx \\
&= \int_0^2 \int_0^x \left(\frac{xk\left(2-\frac{x^2}{2}\right)^2}{2} \right) dy\,dx \\
&= \int_0^2 \left(\left[\frac{xk\left(2-\frac{x^2}{2}\right)^2}{2} \times y \right]_0^x \right) dx \\
&= \int_0^2 \left(\frac{xk\left(2-\frac{x^2}{2}\right)^2}{2} \times x - 0 \right) dx \\
&= \int_0^2 \left(\frac{kx^2\left(2-\frac{x^2}{2}\right)^2}{2} \right) dx \\
&= \frac{k}{2} \int_0^2 x^2 \left(4 - 2x^2 + \frac{x^4}{4}\right) dx \\
&= \frac{k}{2} \int_0^2 \left(4x^2 - 2x^4 + \frac{x^6}{4}\right) dx \\
&= \frac{k}{2} \left[\frac{4}{3}x^3 - \frac{2}{5}x^5 + \frac{x^7}{28} \right]_0^2 \\
&= \frac{k}{2} \times \frac{256}{105} \\
&= \frac{128k}{105}
\end{aligned}
$$

$$
\begin{aligned}
M_{xz} &= \int_0^2 \int_0^x \int_0^{2-\frac{x^2}{2}} ykz \, dz \, dy \, dx \\
&= \int_0^2 \int_0^x \left(\left[\frac{ykz^2}{2} \right]_0^{2-\frac{x^2}{2}} \right) dy \, dx \\
&= \int_0^2 \int_0^x \left(\frac{yk\left(2-\frac{x^2}{2}\right)^2}{2} - \frac{yk \times 0^2}{2} \right) dy \, dx \\
&= \int_0^2 \int_0^x \left(\frac{yk\left(2-\frac{x^2}{2}\right)^2}{2} \right) dy \, dx \\
&= \int_0^2 \left[\frac{y^2k\left(2-\frac{x^2}{2}\right)^2}{4} \right]_0^x dx \\
&= \int_0^2 \left(\frac{x^2k\left(2-\frac{x^2}{2}\right)^2}{4} - 0 \right) dx \\
&= \frac{k}{4} \int_0^2 x^2 \left(2-\frac{x^2}{2}\right)^2 dx \\
&= \frac{k}{4} \int_0^2 x^2 \left(4 - 2x^2 + \frac{x^4}{4}\right) dx \\
&= \frac{k}{4} \int_0^2 \left(4x^2 - 2x^4 + \frac{x^6}{4}\right) dx \\
&= \frac{k}{4} \left[\frac{4}{3}x^3 - \frac{2}{5}x^5 + \frac{x^7}{28} \right]_0^2 \\
&= \frac{k}{4} \times \frac{256}{105} \\
&= \frac{64k}{105}
\end{aligned}
$$

$$M_{xy} = \int_0^2 \int_0^x \int_0^{2-\frac{x^2}{2}} zkz\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^x \left[\frac{kz^3}{3}\right]_0^{2-\frac{x^2}{2}} dy\, dx$$

$$= \int_0^2 \int_0^x \left(\frac{k\left(2-\frac{x^2}{2}\right)^3}{3} - 0\right) dy\, dx$$

$$= \int_0^2 \int_0^x \left(\frac{k\left(2-\frac{x^2}{2}\right)^3}{3}\right) dy\, dx$$

$$= \int_0^2 \left[\frac{k\left(2-\frac{x^2}{2}\right)^3}{3} y\right]_0^x dx$$

$$= \int_0^2 \left(\frac{kx\left(2-\frac{x^2}{2}\right)^3}{3} - 0\right) dx$$

$$= \frac{k}{3}\int_0^2 \left(8x - 6x^3 + \frac{3x^5}{2} - \frac{x^7}{8}\right) dx$$

$$= \frac{k}{3}\left[4x^2 - \frac{3x^4}{2} + \frac{x^6}{4} - \frac{x^8}{64}\right]_0^2$$

$$= \frac{k}{3} \times 4$$

$$= \frac{4k}{3}$$

So, the center of mass of the solid object is:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\frac{\frac{128k}{105}}{\frac{4k}{3}}, \frac{\frac{64k}{105}}{\frac{4k}{3}}, \frac{\frac{4k}{3}}{\frac{4k}{3}}\right) = \boxed{\left(\frac{32}{35}, \frac{16}{35}, 1\right)}$$

(d)

Expected value is the weighted average of a set of data. The weightage is based on the probabilities. The higher probability of occurrence any data point has, the higher is its weightage. The weightage here is the probability of that event occurring (that is, the outcome being that data-point) divided by the total probability (which in trivial cases is 1). So, the expected value is a point about which the weightage (the probability) will be equal.

For independent data, these probabilities will be independent and hence, the expected values for each independent sets of data should not be affecting each other.

This situation is analogous to the calculation of center of mass. The weightage can be defined as the part of the total mass carried by that point (that is, mass of that point divided by the total mass). Hence, the center of mass is giving the point about which the weightage (here, the mass) is equal (the object can be balance about that point).

Centers of mass of independent directions ($\bar{x}, \bar{y}, \bar{z}$) don't affect each other. So, like expected value, center of mass is giving the average position of all the points of the solid (weighted based on the mass instead of probability).

Part 3: Euler's solver

(a)

$$\frac{dy}{dt} = f(t, y) \quad (1)$$

Equation for linear approximation from the initial point (t_0, y_0):

$$y(t) = y_0 + \frac{d}{dt}y(t_0)(t - t_0)$$
$$\Rightarrow \boxed{y(t) = y_0 + f(t_0, y_0)(t - t_0)} \qquad (2)$$

(b)

Let $t_1 = t_0 + h$. Then, the equation (2) would approximate the function y at t_1 as

$$y(t_1) = y_0 + f(t_0, y_0)(t_1 - t_0)$$
$$\Rightarrow \boxed{y(t_1) = y_0 + f(t_0, y_0)h}$$

(c)

$$\frac{y - y_1}{t - t_1} = f\big(t_1, y(t_1)\big)$$
$$\Rightarrow \boxed{y = y_1 + f\big(t_1, y(t_1)\big)(t - t_1)}$$

This line passes through $\big(t_1, y(t_1)\big)$ and has the slope that the differential equation (1) describes for point $\big(t_1, y(t_1)\big)$.

(d)

Estimating y_2 through (t_1, y_1):

$$y_2 - y_1 = f(t_1, y_1)(t_2 - t_1)$$
$$\Rightarrow \boxed{y_2 = y_1 + f(t_1, y_1)h}$$

(e)

The pattern we can see from (c) and (d) is that for each estimation of y_n, we have to compute the y_{n-1} first. The differential equation, value of a t_0 and corresponding y_0 has to be known. If h is kept constant, then the following algorithm can be used:

```
import math

def euler(f, t0, tn, y0, h): # f = differential equation, t0 = initial t, tn
= final t, y0 = initial y, n = number of steps
    t = t0
    y = y0
    steps = int(math.ceil((tn - t0) / h))

    for i in range(steps):
        y += h * f(t, y)
        t += h
    return y

var('y, t')
f(t, y) = 2*y + t
euler(f, 0, 5, 10, 1) # usage
```

In this code, an initial value of t and y are given, along with f(t,y) and number of steps, n (number of steps will be used to determine a fixed value of step-size). The code will run for n times and output a value of y using the Euler's method.

(f)

$$\frac{dy}{dt} = ry + q(t)$$

Given, $r = 0.01$, $q(t) = 250$, $t_0 = 0$ and $y_0 = 100$. Using the previous algorithm, for up to $t_n = 12$:

Applying the algorithm from (e) for step size=0.5:

```
import math

def euler_savings(h, t0=0, tn=12, y0=100):
    t = t0
    y = y0
    r = 0.01
    q = 250
    steps = int(math.ceil((tn - t0) / h))

    for i in range(steps):
        y += h * (r*y + q)
        t += h
    return y

euler_savings(0.5)
```

Using the algorithm, we get the estimation for the money in the checking account after 12 months as $\boxed{\$3291.71}$.

Now we will range h from 4 to 0.1 with interval 0.1 to see how the approximation changes.

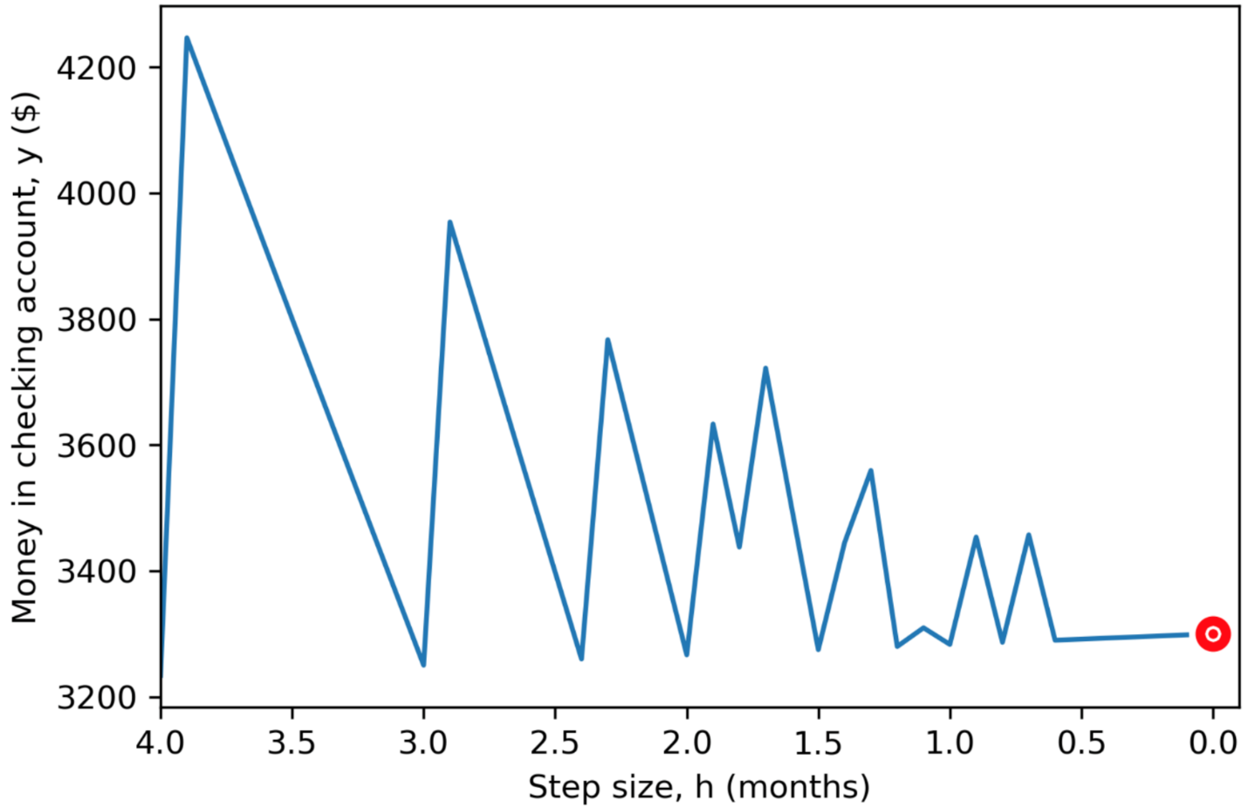

Figure 3-1: Graph of step size versus the linearly approximated amount of money in the checking account. The red dot represents the actual money value in the account. As we can see, the approximation is "spiky." This happens because for certain step sizes, 12 months is not divisible by that step size (e.g. $12 \ mod \ 3.9 \ \neq 0$), then, we would over-estimate using the derived formula (see Appendix C for the Python code).

On a large scale, we can see that as the step size decreases, the estimation will reach the actual function value. This makes sense, because as we make the argument interval smaller, the change in the function will be similar to the slope of a tangent line from the point of estimation. We can notice that when the step size is a divisor of $(t_n - t_0) = 12$, the estimated value increases slowly and gets towards the value ~$3300.17 (exact value will be determined in (g)). The reason we get underestimation is that this is a compounding interest, so the order of growth is exponential as it will be seen in Fig. 3-2 (even though with a small exponent term in the beginning months), while we use linear approximation to estimate the values.

Since in this task, differential equation is defined, we can further group certain calculations and simplify the algorithm.

(Since $q(t)$ in this case is a constant, let's rewrite it as q for simplicity.)

Let's write the estimation of money in the checking account as a linear approximation for n months from h months ago:

$$y_n = y_{n-h} + (ry_{n-h} + q)(t_n - t_{n-h})$$

$$\Rightarrow y_n = y_{n-h} + (ry_{n-h} + q)h$$

$$\Rightarrow y_n = y_{n-h}(hr + 1) + qh$$

Then, we can plug in the estimation of y_{n-h} as a linear approximation from y_{n-2h} to see the pattern:

$$y_n = (y_{n-2h}(hr + 1) + q)(hr + 1) + qh$$

$$y_n = y_{n-2h}(hr + 1)^2 + qh(hr + 1) + qh$$

Seeing this pattern, we can infer that to approximate y_n from y_0, we can use the following formula:

$$y_n = y_0(hr + 1)^{\frac{n}{h}} + \sum_{i=0}^{\frac{n}{h}-1} qh(hr + 1)^i$$

Let's prove this using mathematical induction. To do so, we need to first check that for base case y_0 if this formula is true. Then, assuming that for y_k this is true, we should be able to determine y_{k+h} through this formula.

1. At base case, $t_0 = 0$.

$$\begin{aligned} RHS &= y_0(hr + 1)^{\frac{0}{h}} + \sum_{i=0}^{\frac{0}{h}-1} qh(hr + 1)^i \\ &= y_0 \\ &= LHS \end{aligned}$$

2. Now assuming y_k is true:

$$\begin{aligned} y_k &= y_0(hr + 1)^{\frac{k}{h}} + \sum_{i=0}^{\frac{k}{h}-1} qh(hr + 1)^i \\ y_{k+h} &= y_k * (hr + 1) + qh \\ &= \left(y_0(hr + 1)^{\frac{k}{h}} + \sum_{i=0}^{\frac{k}{h}-1} qh(hr + 1)^i \right)(hr + 1) + q \\ &= y_0(hr + 1)^{\frac{k}{h}+1} + \sum_{i=1}^{\frac{k}{h}} qh(hr + 1)^i + qh \\ &= y_0(hr + 1)^{\frac{k+h}{h}} + \sum_{i=0}^{\frac{k}{h}} qh(hr + 1)^i \end{aligned}$$

We have proved the correctness of the base case and the induction step, therefore, verifying the correctness of the formula for all k, when $h \neq 0$.

■

(g)

Let us start with finding the equation of the money in the checking account using the given differential equation:

$$\begin{aligned}
&\frac{dy}{dt} = ry + q \\
&\Rightarrow \frac{dy}{ry+q} = dt \\
&\Rightarrow \int \frac{1}{ry+q}\,dy = \int 1\,dt \\
&\Rightarrow \frac{ln(ry+q)}{r} = t + C \\
&\Rightarrow ln(ry+q) = tr + Cr \\
&\Rightarrow e^{tr+Cr} = ry + q \\
&\Rightarrow y = \frac{e^{tr+Cr} - q}{r} \quad (3.1)
\end{aligned}$$

Plugging in the initial condition $y(0) = 100, r = 0.01, q = 250$, we can find C in the equation above:

$$\begin{aligned}
&100 = \frac{e^{0+0.01C} - 250}{0.01} \\
&\Rightarrow C = 100 \ln 251
\end{aligned}$$

Putting this C and the other parameters back into equation 3.1, we get the equation for the amount of money in checking account as such:

$$y = 100\left(e^{\frac{t+100\ln 251}{100}} - 250\right) \quad (3.2)$$

Using this formula, we can get y_{12}:

$$\begin{aligned}
y_{12} &= 100\left(e^{\frac{12+100\ln 251}{100}} - 250\right) \\
&= 100\left(e^{0.12+\ln 251} - 250\right) \\
&\approx 3300.17
\end{aligned}$$

Therefore, amount saved in the account after 12 months is about $\boxed{\$3300.17}$.

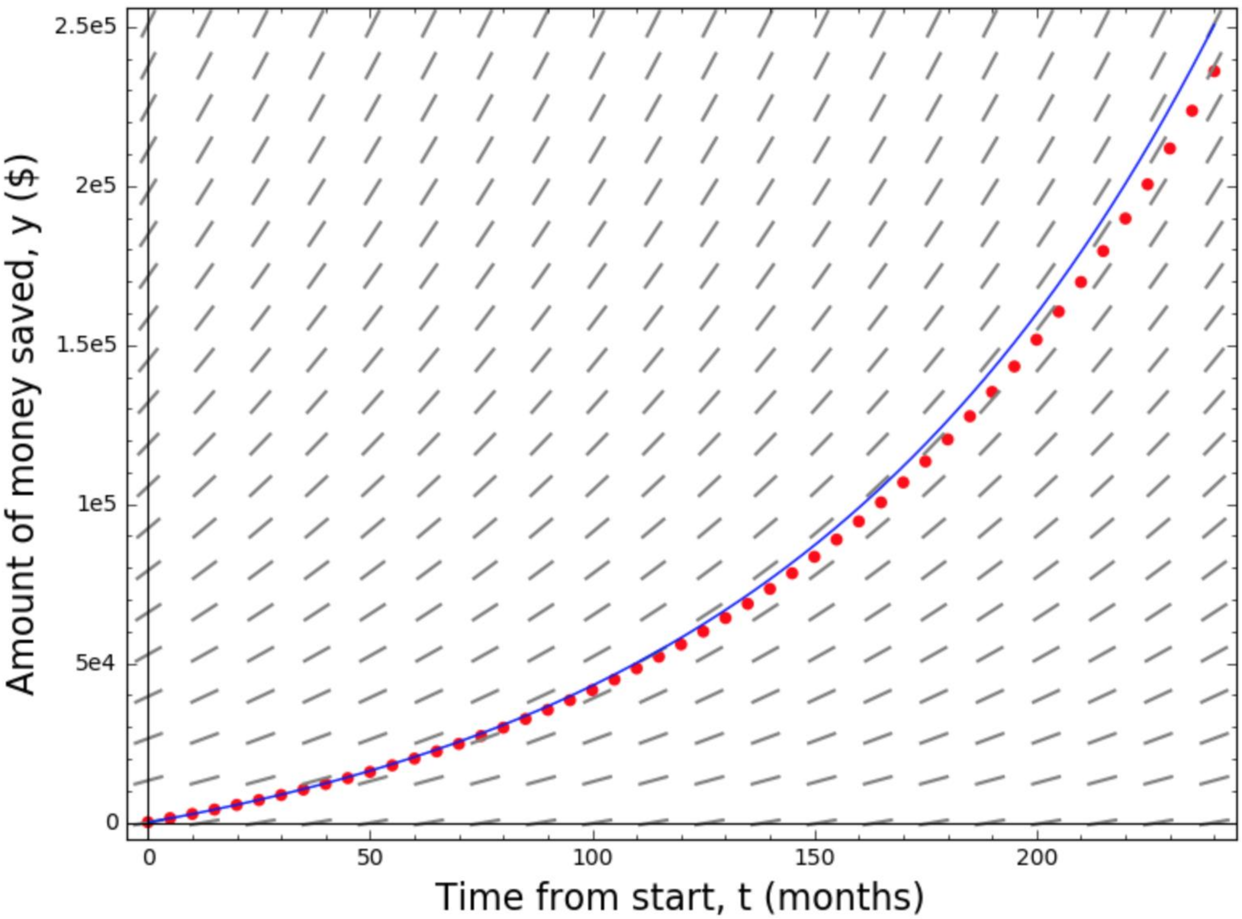

Figure 3-2: Slope field of the amount of money saved over time. Red dots represent the estimate of the amount of money using linear approximation with step-size h = 5 months. As we can observe, each pair of consecutive dots form almost the same angle as the slope line for the same interval. The blue line represents the actual amount of money over time (plotted using the function 3.2). (See Appendix D for Python code).

As noted in (f), the actual function is exponential while the approximation is linear. Consequently, as we took constant steps, the approximations deviated from the actual function over time towards underestimation. That is why, using small step size and checking the trend for 240 months in Fig. 3-2, we see the dotted line grows slower than the blue line, more noticeably when t is large.

Part 4: Something fishy here

(a)

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{N}\right) - H$$
$$= rP - \frac{rP^2}{N} - H$$

Let's assume the differential equation is dimensionally homogenous. Time, t, is the independent variable and population, P, is the dependent variable. So, the parameters r, N and H are constants.

Let's say the dimension of population is $[P]$. So, dimension of the left-hand side of the equation is $\frac{[P]}{[T]}$. Therefore, each term in the right-hand side should have the same dimension. Therefore, we have:

$$Dimensions\ of\ rP = \frac{[P]}{[T]}$$
$$\Rightarrow Dimension\ of\ r \times [P] = \frac{[P]}{[T]}$$
$$\Rightarrow Dimension\ of\ r = \frac{1}{[T]}$$

Unit of time is given as week. So, unit of r in the equation is 'per week'. So, it is the frequency of change of the population. In other words, r should represent the per capita growth rate of the population. But in the given equation, r is a constant. If r is set to be the maximum per capita growth rate, then for small values of r, the logistic growth equation should be accurate (as then, this parameter should affect the other parameters the least). Therefore, r is the maximum per capita growth rate of the fish population.

$$Dimensions\ of\ \frac{rP^2}{N} = \frac{[P]}{[T]}$$
$$\Rightarrow Dimension\ of\ \frac{1}{N} \times \frac{1}{[T]} \times [P]^2 = \frac{[P]}{[T]}$$
$$\Rightarrow Dimension\ of\ \frac{1}{N} = \frac{1}{[P]}$$
$$\Rightarrow Dimension\ of\ N = [P]$$

As calculated, the dimension of N is the same as that of population, P. Hence, the unit of N is number of fish and the range has to be greater or equal to zero. In the differential equation, P is the population, and $\left(1 - \frac{P}{N}\right) = \left(\frac{N-P}{N}\right)$ is scaling the population, P based on a parameter, N. N seems to be a special value of the population. As P approaches N, $\frac{dP}{dt}$ increases till N=P. When P>N, $\frac{dP}{dt}$ starts decreasing. So, we can infer that N is the maximum population capacity of the fishing reserve.

$$Dimensions\ of\ H = \frac{[P]}{[T]}$$

It is apparent that the unit of H is population per unit time. This term contributes by scaling down the rate of the population change, $\frac{dP}{dt}$. In other words, it is taking into account the decrease in fish population that is not due to natural reasons that are accounted for in r. One prominent reason might be fishing. Then, H will be the number of fish fished every week.

(b)

When $H = 0$,

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{N}\right)$$

$$\Rightarrow \begin{cases} \int \frac{1}{P\left(1 - \frac{P}{N}\right)} dP = \int r\, dt; \text{ when } P \neq N & (4.1) \\ \frac{dP}{dt} = 0; \text{ when } P = N \text{ } or \text{ } P = 0 & (4.2) \end{cases}$$

Solving for equation 4.1:

For the left side of the equation:

$$\int \frac{1}{P\left(1 - \frac{P}{N}\right)} dP = N \int \frac{1}{P(N - P)} dP$$

$$= N \int \frac{1}{P^2\left(\frac{N}{P} - 1\right)} dP$$

Let, $u = \frac{N}{P} - 1 \Rightarrow dP = \left(-\frac{P^2}{N}\right) du$. Substituting,

$$N \int \frac{1}{P^2\left(\frac{N}{P} - 1\right)} dP = N \int \frac{1}{P^2 u}\left(-\frac{P^2}{N}\right) du$$

$$= -\int \frac{1}{u} du$$

$$= -\ln|u| + C_1$$

$$= -\ln\left|\frac{N}{P} - 1\right| + C_1$$

$$= -\ln\left|\frac{N - P}{P}\right| + C_1$$

$$= \ln\left|\frac{P}{N - P}\right| + C_1$$

Going back to the differential equation,

$$\int \frac{1}{P\left(1 - \frac{P}{N}\right)} dP = \int r\, dt$$

$$\Rightarrow \ln\left|\frac{P}{N - P}\right| + C_1 = rt + C_2$$

$$\Rightarrow rt = \ln\left|\frac{P}{N - P}\right| + C$$

$$\Rightarrow rt = \ln\left|\frac{P}{N - P}\right| + \ln e^C$$

$$\Rightarrow rt = \ln\left|\frac{Pe^C}{N-P}\right|$$

Let $e^C = K > 0$. Then,

$$e^{rt} = \left|\frac{PK}{N-P}\right|$$

$$\Rightarrow \begin{cases} Ke^{-rt} = \frac{N}{P} - 1 \ ; \ 0 < P < N: Ke^{-rt} > 0 \ \forall (r,t) \in \mathbb{R} \\ Ke^{-rt} = 1 - \frac{N}{P}; \ P > N: Ke^{-rt} > 0 \ \forall (r,t) \in \mathbb{R} \end{cases}$$

Taking into account equation 4.2, the system of equations above becomes the following set of equations, which are also the general solutions of the logistic differential equation.

$$P = \begin{cases} \dfrac{N}{Ke^{-rt}+1} & ; 0 < \ P < N \\ \dfrac{N}{1-Ke^{-rt}} & ; \ P > N \\ N & ; \ P = N \\ 0 & ; \ P = 0 \end{cases}$$

As described in (a), r should have a small value. N depends on the natural holding capacity of the reserve. As other parameters like volume of the reserve and requirements are not known, let's assume N can be any moderately large number.

Let's use Fermi estimation to determine the values of r and N. Assuming we have an average size lake with carrying capacity of 10000 fish. (So, $N = 10000$). Knowing that an average fish can give birth 5 times in a lifetime (~5 years) and each time about 100 new fish come to life, we can infer that on average one fish gives birth to $\frac{5\times 100 \ fish}{5 \ years} = \frac{5\times 100 \ fish}{5\times 52 \ weeks} \approx 1.92$ fish per week. So, we are taking $r = 2$ for ease of calculations. #####Estimation

So, let's have $r = 2$ when $N = 10000$. So, the general solution of the differential equation becomes,

$$\boxed{P = \begin{cases} \dfrac{10000}{Ke^{-2t}+1} & ; 0 < P < 10000 \\ \dfrac{10000}{1-Ke^{-2t}} & ; P > 10000 \\ 10000 & ; P = 10000 \\ 0 & ; P = 0 \end{cases}} \qquad (4.3)$$

At $P(0)$, $t = 0$. So, the equation becomes,

$$P(0) = \begin{cases} \dfrac{10000}{K+1} & ; 0 < P < 10000 \\ \dfrac{10000}{1-K} & ; P > 10000 \\ 10000 & ; P = 10000 \\ 0 & ; P = 0 \end{cases}$$

As for the differential equation:

$$\frac{dP(0)}{dt} = rP(0)\left(1 - \frac{P(0)}{10000}\right)$$

(i)

$P(0) < 10000$. So, $\left(1 - \frac{P(0)}{10000}\right)$ is positive. Hence, $\frac{dP(0)}{dt}$ becomes positive.

This means, when the initial population of the fish is less than the carrying capacity of the reserve, the population will start growing.

Then, as t increases, P will approach N from below, but will never be equal to N, as seen in first equation of the system of general solutions (4.3). Henceforth, $\frac{dP(0)}{dt}$ decreases in magnitude over time. That is, the rate of change of population approaches zero from the positive side over time when initial population was less than the carrying capacity of the lake.

(ii)

$P(0) > N$. So, $\left(1 - \frac{P(0)}{N}\right)$ is negative. So, $\frac{dP(0)}{dt}$ is negative.

This means, when the initial population of the fish is larger than the carrying capacity of the reserve, the population will start to decline over time.

Then, as t increases, P will approach N from above, but will never be equal to N, as seen in second equation of the system of general solutions (4.3). Henceforth, $\frac{dP(0)}{dt}$ decreases in magnitude over time. That is, the rate of change of population approaches zero from the negative side over time when initial population was greater than the carrying capacity of the lake.

For both of these situations, the rates of change of population tends to zero as the population tends to the carrying capacity as time approaches infinity.

(iii)

$P(0) = N$. So, $\left(1 - \frac{P(0)}{10000}\right)$ is zero. So, $\frac{dP(0)}{dt}$ is zero.

Looking at the third equation of 4.3, we can see that the population count is independent of time and is a constant which is equal to the carrying capacity of the lake. The growth rate is zero.

Meaning, if the population starts at the carrying capacity of the lake, then it will remain constant over time.

(iv)

$P(0) = 0$. So, rP is zero. So, So, $\frac{dP(0)}{dt}$ is zero.

Looking at the fourth equation of 4.3, we can see that the population count is independent of time and is a constant which is equal to zero. The growth rate is zero. Meaning, if the population starts at zero, and no other fish is introduced in the lake at any time, then it will remain constant at zero over time, which is same as what obvious common sense suggests.

For real life, these situations are idealistic. But in an ideal world, it would be the best to start with $P = N$ (scenario (iii)) for getting maximum yield of fish in the long term (without having to feed too much when we need more fish to grow as in (i), and without having so many fish deaths incurring loss as in (ii), or no fish at all like in (iv)).

(c)

At steady state solution, $\frac{dP}{dt} = 0$. So,

$$\begin{aligned}
&\frac{dP}{dt} = rP\left(1 - \frac{P}{N}\right)\\
&\Rightarrow rP\left(1 - \frac{P}{N}\right) = 0\\
&\Rightarrow rP = 0 \ or \ \left(1 - \frac{P}{N}\right) = 0\\
&\Rightarrow P = 0 \ \ or \ \ P = N
\end{aligned}$$

Therefore, $P = 0$ and $P = N$ are the two steady state solutions for the logistic differential equation.

Now, let’s do the second derivative test for finding the nature of the steady state solutions.

$$\begin{aligned}
&\frac{dP}{dt} = rP\left(1 - \frac{P}{N}\right)\\
&\Rightarrow \frac{d^2P}{dt^2} = r\left(1 - \frac{P}{N}\right) + rP\left(-\frac{1}{N}\right)
\end{aligned}$$

At $P = 0$,

$$\frac{d^2P}{dt^2} = r(1 - 0) + r \times 0 \times \left(-\frac{1}{N}\right) = r$$

r is a positive constant. So, $\frac{d^2P}{dt^2}$ is positive. This means the slopes are increasing about $P = 0$. Meaning, the points are diverging. So, $P = 0$ is an unstable steady state solution. In case of the fish reserve situation, this would mean that any small changes in population from $P = 0$ would cause the population to increase and never come back to this steady state.

At $P = N$,

$$\frac{d^2P}{dt^2} = r\left(1 - \frac{N}{N}\right) + rN\left(-\frac{1}{N}\right) = -r$$

Both of r is a positive number. So, $\frac{d^2P}{dt^2}$ is negative at $P = N$. This means the steady state solution at $P = N$ is a stable solution. In the context of the fish population, this means that any value of P about N will tend to reach N. And if the population starts at $P = N$, then it will ideally remain at this number unless a change is made externally, after which it will tend to reach $P = N$ again.

The following graph shows this situation when $N = 10000$ and $r = 2$.

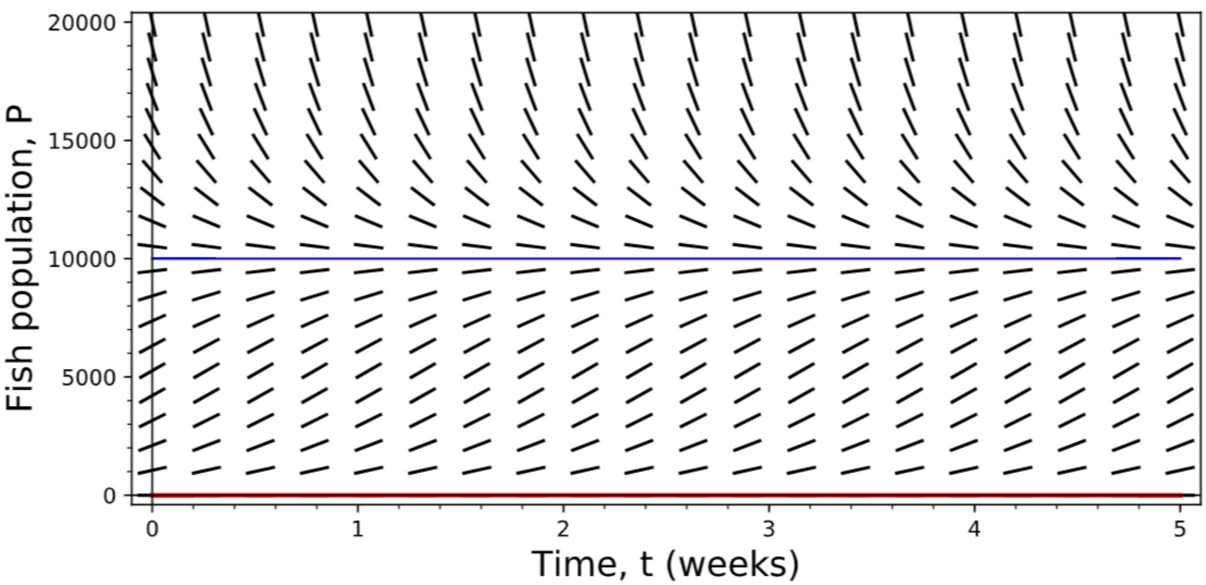

Figure 4-1: Slope field of the logistic growth equation. The horizontal lines show the steady states (blue is stable, and red is unstable).

It is apparent from Fig. 4-1 that $P = N$ is in fact a stable solution as the slope fields converge towards this point and $P = 0$ is indeed an unstable solution as the slope fields around it diverge.

(d)

$$\begin{aligned}
&\frac{dP}{dt} = 0 \\
&\Rightarrow rP\left(1 - \frac{P}{N}\right) - H = 0 \\
&\Rightarrow rP - \frac{rP^2}{N} - H = 0 \\
&\Rightarrow \frac{r}{N}P^2 - rP + H = 0
\end{aligned}$$

$$\Rightarrow P = \frac{-(-r) \pm \sqrt{(-r)^2 - 4 \times \frac{r}{N} \times H}}{2 \times \frac{r}{N}}$$

$$\Rightarrow P = \frac{N}{2} \pm \frac{1}{2r}\sqrt{N^2r^2 - 4rNH}$$

So, the steady state solutions of the logistic differential equation are:

$$\boxed{P = \begin{cases} \frac{N}{2} + \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} \\ \frac{N}{2} - \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} \end{cases}}$$

(e)

For population to never go extinct, we must have $P > 0$. That is, $\frac{N}{2} + \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} > 0$ and $\frac{N}{2} - \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} > 0$.

For $\frac{N}{2} + \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} > 0$:

$$\frac{N}{2} + \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} > 0$$
$$\Rightarrow -Nr < \sqrt{N^2r^2 - 4rNH}$$

Breaking down the inequality for values of Nr:

$$\begin{cases} N^2r^2 - 4rNH > Nr^2 & when\ Nr < 0 \\ N^2r^2 - 4rNH > 0 & when\ Nr \geq 0 \end{cases}$$

But $Nr > 0$. So,

$$N^2r^2 - 4rNH > 0$$
$$\Rightarrow Nr(Nr - 4H) > 0$$
$$\Rightarrow Nr > 0\ or\ H < \frac{Nr}{4}$$

Therefore, one relationship of H is $H < \frac{Nr}{4}$.

For $\frac{N}{2} - \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} > 0$:

$$\frac{N}{2} - \frac{1}{2r}\sqrt{N^2r^2 - 4rNH} > 0$$
$$\Rightarrow Nr > \sqrt{N^2r^2 - 4rNH}$$
$$\Rightarrow (Nr)^2 > \left(\sqrt{N^2r^2 - 4rNH}\right)^2$$
$$\Rightarrow N^2r^2 > N^2r^2 - 4rNH$$
$$\Rightarrow 4rNH > 0$$
$$\Rightarrow H > 0$$

Hence, the other relationship for H is $H > 0$.

Combining the calculated solutions of H, we get, $\boxed{0 < H < \frac{Nr}{4}}$. That is, to prevent the fish from going extinct from the reserve, no more than $\frac{Nr}{4}$ fishes should be harvested per week.

Part 5: Taylor approximations

(a)

To determine such quantities, calculators use Taylor polynomials up to a certain degree to give an approximated answer, basing of off prior known values; in this case, calculators know e^1 and $\sin 0$. If we were to do such approximation by hand, we can use linear approximation (which is Taylor polynomial of degree 1) since the values we are calculating are relatively close to the ones we know.

(i)

$$f(x) = e^x$$

$$\begin{aligned} f_{approx}(1.3) &= f(1) + f'(1)(1.3 - 1) \\ &= e + e(1.3 - 1) \\ &= 1.3e \end{aligned}$$

(ii)

$$f(x) = sinx$$

$$\begin{aligned} f_{approx}(0.14) &= f(0) + f'(0)(0.14 - 0) \\ &= 0 + 1 * 0.14 \\ &= 0.14 \end{aligned}$$

(b)

Let us denote $x_0 \in Dom(f)$ as a point, for which we know $f(x_0)$. Then, to linearly approximate $f(x)$ from point x_0, we would use the following equation:

$$\begin{aligned} f_{approx}(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &= f(x_0) + f'(x_0)x - f'(x_0)x_0 \\ &= f'(x_0)x + (f(x_0) - f'(x_0)x_0) \end{aligned}$$

$$\begin{aligned} &p(x) = ax + b, \quad (a, b) = ((f'(x_0), f(x_0) - f'(x_0)x_0)) \\ &\Rightarrow p(x) = f'(x_0)x + f(x_0) - f'(x_0)x_0 \quad (5.1) \end{aligned}$$

That is,

$$\boxed{a = f'(x_0)}$$

$$\boxed{b = f(x_0) - f'(x_0)x_0}$$

(c)

To find the linear approximation for $f(x) = sinx$, we will utilize formula (5.1) with $x_0 = 0$:

$$p_{sin}(x) = xcos0 + (sin0 - 0cos0) = x$$

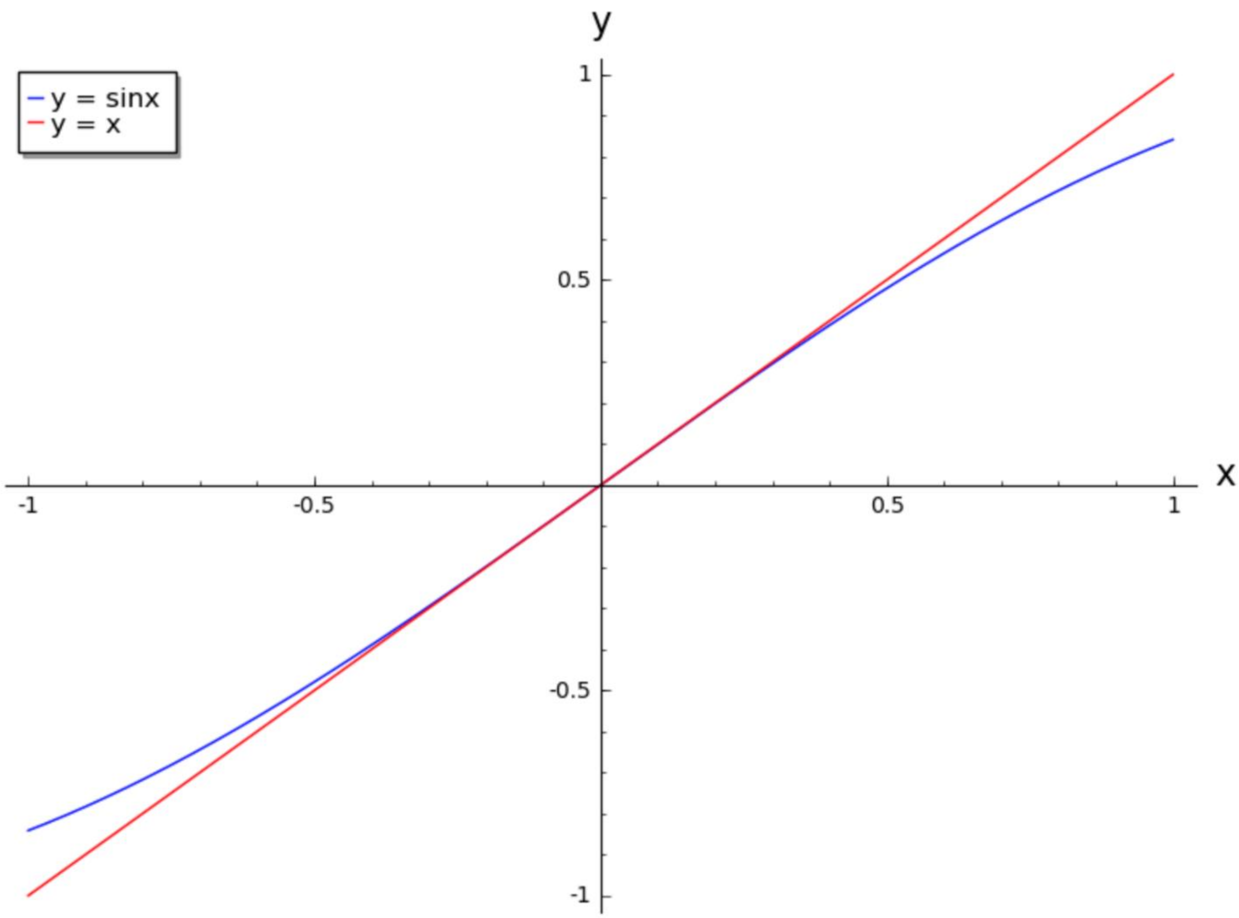

Figure 5-1: Graph of sine on the interval of $[-1,1]$ (blue curve) and its linear approximation (red line).

At $x = \frac{\pi}{6}$, the actual value is $sin\frac{\pi}{6} = \frac{1}{2} = 0.5$. The approximated value is $p_{sin}(x) = x = \frac{\pi}{6} \approx 0.5236$. We can determine how good is this estimation by finding how much it differs from the actual solution (relative error). In this case, relative error, $\epsilon = \left|\frac{p_{sin}(x) - sinx}{sinx}\right| = \left|\frac{\pi/6 - 1/2}{1/2}\right| = \frac{\pi}{3} - 1 \approx 0.047 =$ 4.7%.

This is a relative error. Because at small values of $\sin x$, the output is really close to the output of $f(x) = x$ (since, $\lim_{x \to 0} \frac{\sin x}{x} = 1$). That's why approximating values of $\sin x$ as $\sin x \approx x$ for small values of x gives fairly accurate results.

(d)

$$p(x) = ax^2 + bx + c$$

We have here,

$$p(0) = c$$

$$p'(0) = b$$

$$p''(0) = 2a$$

Now let's compute the values for $f(x) = sinx$:

$$f(0) = \sin 0 = 0$$

$$f'(0) = \cos x\,|_0 = \cos 0 = 1$$

$$f''(0) = -\sin x\,|_0 = -\sin 0 = 0$$

So, $(a, b, c) = (0,1,0)$. Using that, $\boxed{p(x) = x}$ is the quadratic approximation for the function $sinx$.

(e)

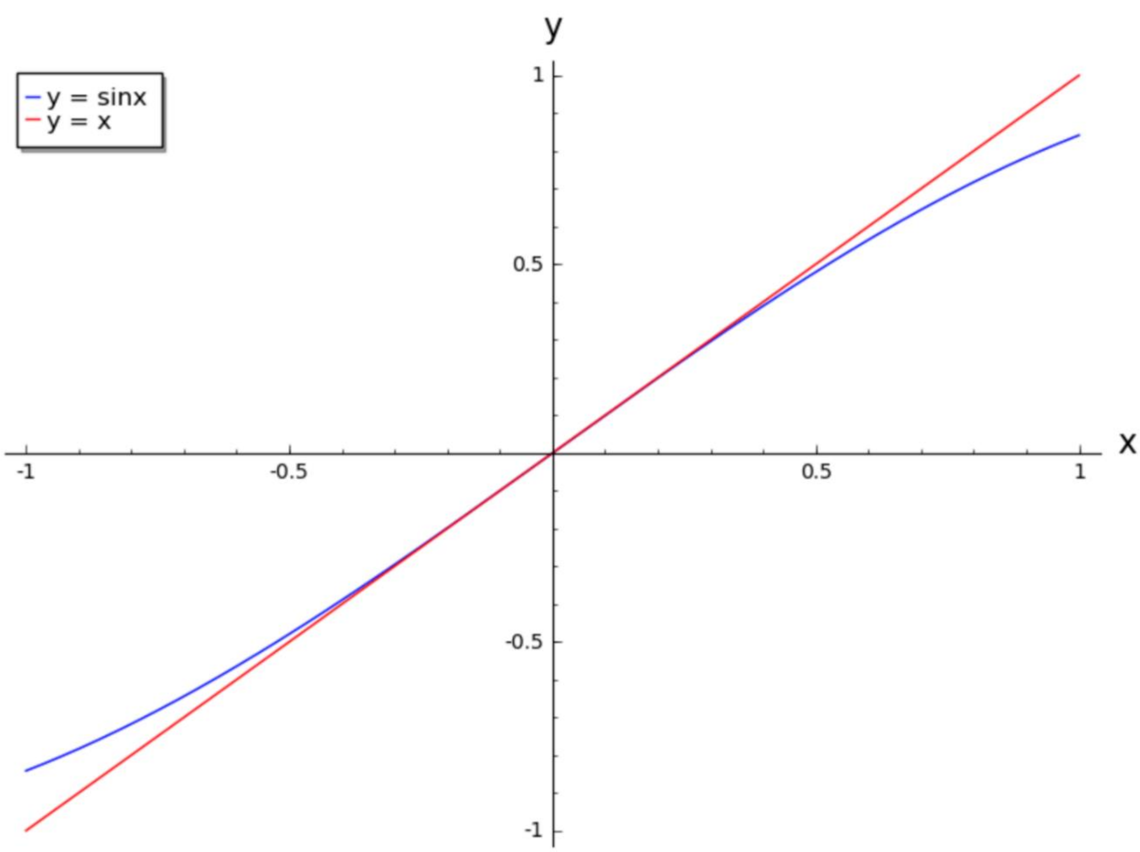

Figure 5-2: Graph of sine on the interval of $[-1,1]$ (blue curve) and its quadratic approximation (red line).

Since the quadratic approximation formula is exactly the same as the one for linear approximation in case of $sinx$, estimated from $x_0 = 0$, the graph will be just like Fig. 5-1, and the error will still be $\epsilon = 4.7\%$. The reason the quadratic and linear approximation matched is that sine is an odd function, while a quadratic function with non-zero parameters can never be odd. Then, using only the linear approximation will suffice.

(f)

$$p(x) = ax^3 + bx^2 + cx + d$$

We have here,

$$p(0) = d$$

$$p'(0) = c$$

$$p''(0) = 2b$$

$$p'''(0) = 6a$$

Now for $f(x) = sinx$:

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -1$$

So, $(a, b, c, d) = (-1,0,1,0)$. Using that, $\boxed{p(x) = -\frac{x^3}{6} + x}$ is the cubic approximation of the function $sin\,x$.

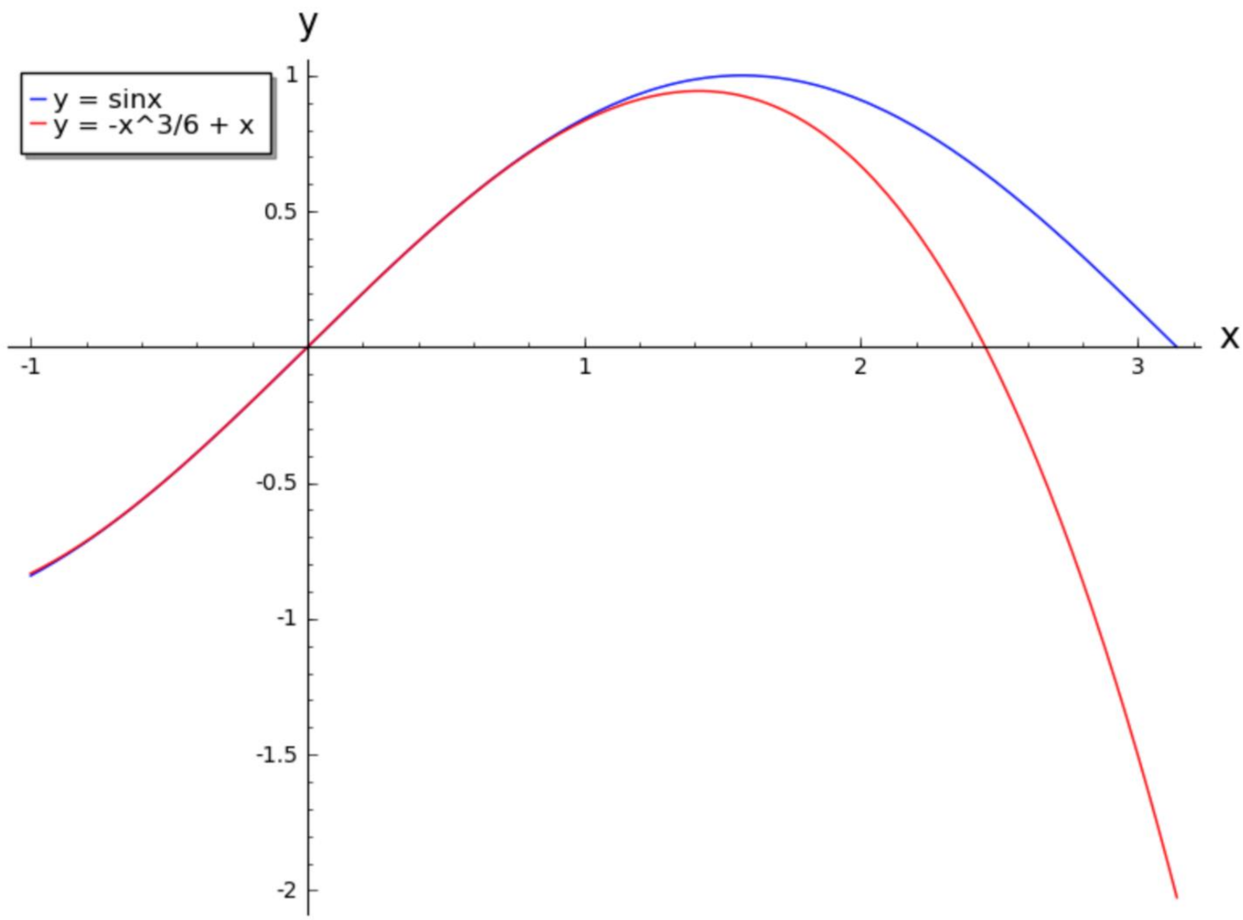

Figure 5-3: Graph of sine on the interval of $[-1, \pi]$ (blue curve) and its cubic approximation (red curve). As we can see, for values nearby 0 (in range [-1, 1]), the approximated value closely matches the actual value, but deviating further from that range the cubic approximation does a poor job.

At $x = \frac{\pi}{6}$, the actual value is $sin\frac{\pi}{6} = \frac{1}{2}$. The approximated value is $p_{sin}(x) = -\frac{x^3}{6} + x = -\frac{\pi^3}{6^4} + \frac{\pi}{6} = \frac{\pi}{6} - \frac{\pi^3}{1296} \approx 0.4997$. We can determine how good is this estimation by finding how much it differs from the actual solution (relative error). In this case, relative error, $\epsilon = \left|\frac{p_{sin}(x) - \sin x}{\sin x}\right| = \left|\frac{\frac{\pi}{6} - \frac{\pi^3}{1296} - \frac{1}{2}}{\frac{1}{2}}\right| \approx 0.00065 \approx 0\%$. Since we have a very small relative error for the approximation, we can give ourselves a tap on the shoulder for doing a good approximation.

(g)

When we center the finite Taylor polynomials at a point other than zero, we will generally get more precise estimations near that new point. An example is centering Taylor polynomial of 2nd degree at pi and getting an estimate of $sin(9pi/8)$ compared to estimating from $sin(0)$ which would give a big error as seen in Fig. 5-3. Sometimes, it is not possible to center at zero if we do not know the exact value there or the function does not exist there, such as for function $f(x) = log_x 2$.

Now let's use the following lines of code for graphing the functions $f(x) = sinx$, $g(x) = e^x$ and $h(x) = 1/(1-x)$.

```
def taylor_series(func, ymin, ymax):
    graph_colors = ["blue", "aqua", "green", "lime", "red", "yellow"]
    taylor = plot(func, (x, -3*pi/2, 1), ymax=ymax, ymin=ymin, thickness=2,
color="black", legend_label="Actual graph of "+str(func), axes_labels=["x",
"y"]) + plot(func, (x, 1, 3*pi/2), thickness=2, color="black")

    for i, col in zip(range(1, 7), sorted(graph_colors)):
        tlr = func.taylor(x, 0, i)
        taylor += plot(tlr, (x, -3*pi/2, 3*pi/2), color=col,
legend_label="Taylor polynomial of order "+str(i)+": "+str(tlr)[:],
linestyle="--")
    return taylor

f(x) = sin(x)
taylor_series(sin(x), -3, 2)
```

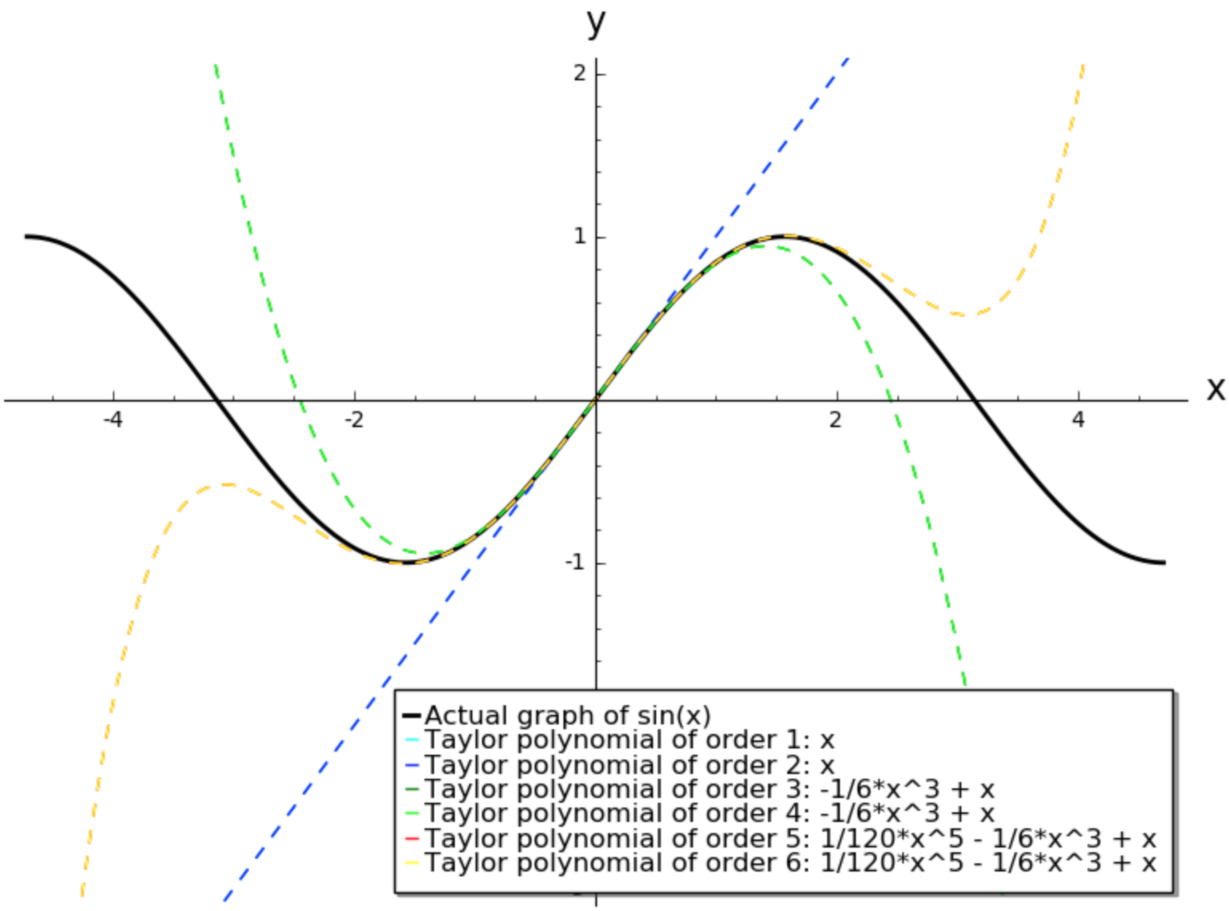

Figure 5-4-1: Graph of $f(x) = sinx$ with Taylor polynomials centered at 0.

```
g(x) = e^x
taylor_series(g(x), -4, 6)
```

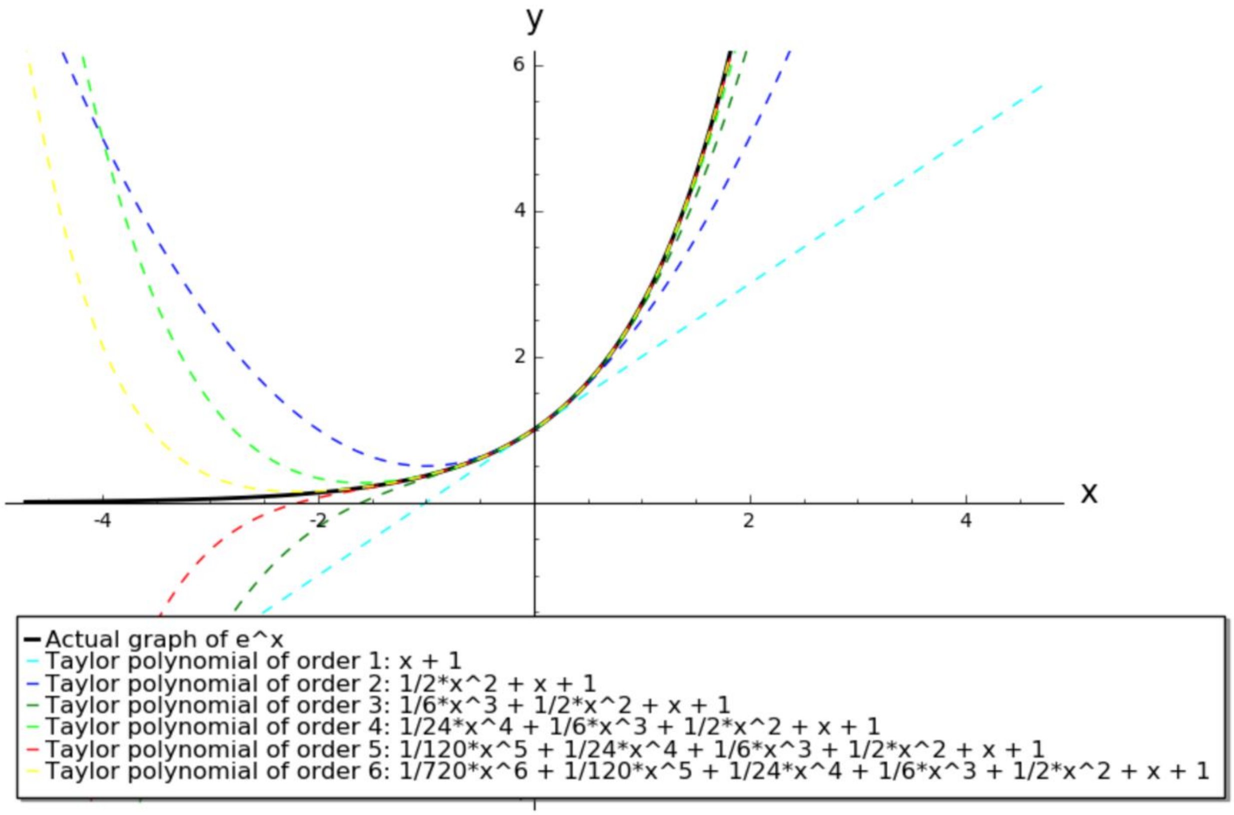

Figure 5-4-2: Graph of $g(x) = e^x$ with Taylor polynomials centered at 0.

```
h(x)=1/(1-x)
taylor_series(h(x), -5, 5)
```

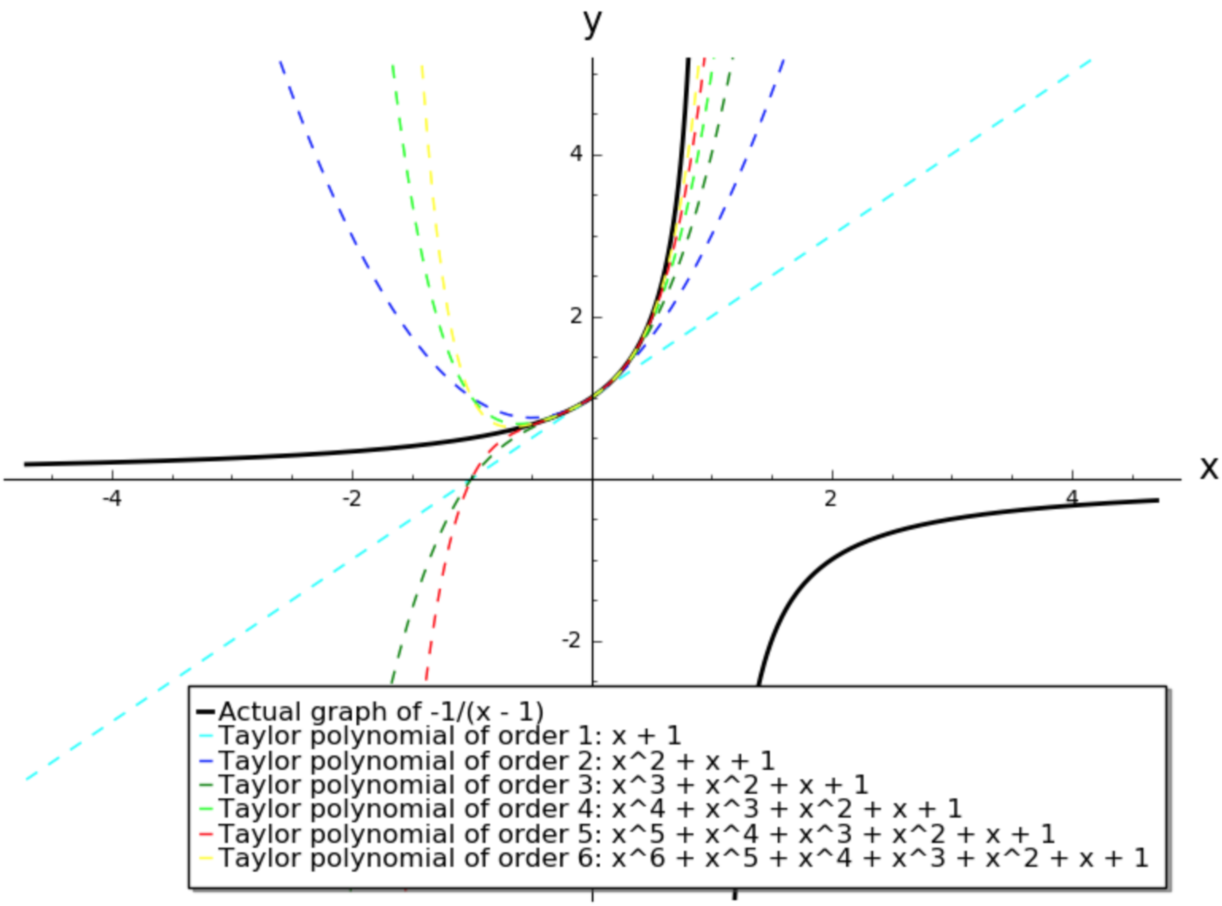

Figure 5-4-3: Graph of $h(x) = \frac{1}{x-1}$ with Taylor polynomials centered at 0.

In these three graphs, we can observe the general trend is that when the order of Taylor polynomial is higher, it is more likely to have a smaller relative approximation error (since the function can "bend" more). For all of these Taylor polynomials, values near zero are approximated with minimal error, since the polynomials are calculated from (centered at) 0.

(h)

Since sine is a periodic function with the period of 2π, for any value x: $|x| \geq 2\pi, \sin(x) = \sin(x \bmod 2\pi)$. Then, we only have to approximate the function of sine in the interval $[0, 2\pi)$ which includes all the values that this function can reach. So, the next step is to pick a Taylor polynomial of high enough order to "simulate" the curvature of sine. The order of 200 gives very high precision when making estimates from 0. Besides, to make the calculations execute in $O(k)$ time complexity, where k is the order of the Taylor polynomial, we cache the values for $f(0), f'(0), \ldots, f^{(k)}(0)$.

```
# caching of f(0), f'(0), ... f^(200)(0)
params = f.taylor(x, 0, 100)
```

Then the following code will prompt the user to input a value for x for which it will compute the value of sin(x) using its Taylor polynomial of up to 200th order.

```
import math

def calc_sin(inp):
    tmp = inp % (2*math.pi)
    f(x) = sin(x)
    return float(params(tmp))

calc_sin(int(input("sin")))
```

(i)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times (-x) = -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow f''(x) = \left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) + \left(-\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(-x)\right) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow f'''(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(-x) + \sqrt{\frac{2}{\pi}} x e^{-\frac{x^2}{2}} + \frac{x^2}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(-x) = \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sqrt{\frac{2}{\pi}} x e^{-\frac{x^2}{2}} - \frac{x^3}{2\pi} e^{-\frac{x^2}{2}}$$

$$\Rightarrow f^{(4)}(x) = \sqrt{\frac{2}{\pi}} x e^{-\frac{x^2}{2}} - \frac{x^2}{2\pi} e^{-\frac{x^2}{2}} x + \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} + \sqrt{\frac{2}{\pi}} x e^{-\frac{x^2}{2}}(-x) - \frac{3x^2}{2\pi} e^{-\frac{x^2}{2}} + \frac{x^3}{2\pi} e^{-\frac{x^2}{2}}(-x)$$

$$= \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} + \sqrt{\frac{2}{\pi}} x e^{-\frac{x^2}{2}} - \left(\sqrt{\frac{2}{\pi}} - \frac{3}{2\pi}\right) x^2 e^{-\frac{x^2}{2}} - \frac{1}{2\pi} x^3 e^{-\frac{x^2}{2}} - \frac{1}{2\pi} x^4 e^{-\frac{x^2}{2}}$$

Centering at zero. So,

$$f(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{0}{2}} = \frac{1}{\sqrt{2\pi}}$$

$$f'(0) = -\frac{0}{\sqrt{2\pi}} e^{-\frac{0}{2}} = 0$$

$$f''(0) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{0}{2}} + \frac{0}{\sqrt{2\pi}} e^{-\frac{0}{2}} = -\frac{1}{\sqrt{2\pi}}$$

$$f'''(0) = \frac{0}{\sqrt{2\pi}} e^{-\frac{0}{2}} + \sqrt{\frac{2}{\pi}} 0 e^{-\frac{0}{2}} - \frac{0}{2\pi} e^{-\frac{0}{2}} 0 = 0$$

$$f^4(0) = \sqrt{\frac{2}{\pi}}e^{-\frac{0}{2}} + \sqrt{\frac{2}{\pi}}0e^{-\frac{0}{2}} - \left(\sqrt{\frac{2}{\pi}} - \frac{3}{2\pi}\right)0e^{-\frac{0}{2}} - \frac{1}{2\pi}0e^{-\frac{0}{2}} - \frac{1}{2\pi}0e^{-\frac{0}{2}} = \sqrt{\frac{2}{\pi}}$$

Let's denote the subscript of $p(x)$ as the order of the approximation. Since at least for $n = 4$ every other term is zero (seen by the trend above), we will get $p_{n+1}(x) = p_n(x)\ \forall\, n \equiv 0\ (mod\ 2)$. (It has been shown below for the first two terms, i.e., $p_0(x)$ and $p_1(x)$.) For this, we can ignore calculating every other polynomial. So, we have,

$$p_0(x) = f(0) = \frac{1}{\sqrt{2\pi}}$$

$$p_1(x) = f(0) + f'(0)x = \frac{1}{\sqrt{2\pi}} + 0x = \frac{1}{\sqrt{2\pi}}$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = \frac{1}{\sqrt{2\pi}} + 0x - \frac{1}{2\sqrt{2\pi}}x^2 = \frac{1}{\sqrt{2\pi}} - \frac{1}{2\sqrt{2\pi}}x^2$$

$$p_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^4(0)}{24}x^4 = \frac{1}{\sqrt{2\pi}} - \frac{1}{2\sqrt{2\pi}}x^2 + \frac{1}{24}\sqrt{\frac{2}{\pi}}x^4$$

Now, let's integrate each of $p_0(x), p_2(x)$ and $p_4(x)$ within $[-1, +1]$.

$$\begin{aligned}
&\int_{-1}^{1} p_0(x)\,dx = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}}\,dx \\
&= \left[\frac{1}{\sqrt{2\pi}}x\right]_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}} - \left(-\frac{1}{\sqrt{2\pi}}\right) \\
&= \sqrt{\frac{2}{\pi}} \\
&\approx 0.7979
\end{aligned}$$

$$\begin{aligned}
&\int_{-1}^{1} p_2(x)\,dx = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} - \frac{1}{2\sqrt{2\pi}}x^2\,dx \\
&= \left[\frac{1}{\sqrt{2\pi}}x - \frac{1}{6\sqrt{2\pi}}x^3\right]_{-1}^{1} \\
&= \frac{5}{6\sqrt{2\pi}} - \left(-\frac{5}{6\sqrt{2\pi}}\right) \\
&\approx 0.6649
\end{aligned}$$

$$
\int_{-1}^{1} p_4(x)\,dx = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} - \frac{1}{2\sqrt{2\pi}}x^2 + \frac{1}{24}\sqrt{\frac{2}{\pi}}x^4\,dx
$$

$$
= \left[\frac{1}{\sqrt{2\pi}}x - \frac{1}{6\sqrt{2\pi}}x^3 + \frac{1}{120}\sqrt{\frac{2}{\pi}}x^5\right]_{-1}^{1}
$$

$$
= \frac{8\sqrt{\frac{2}{\pi}}}{15} - \left(-\frac{8\sqrt{\frac{2}{\pi}}}{15}\right)
$$

$$
= \frac{17}{20\sqrt{2\pi}} - \left(-\frac{17}{20\sqrt{2\pi}}\right)
$$

$$
\approx 0.6782
$$

Seems like the sums are converging towards a value near 0.7979, 0.6649 and 0.6742.

In fact, it is converging towards ~ 0.6827. Because the range we used is $[-1, 1]$, which is looking at the area under the probability density function $f(x)$ for one standard deviation away from the mean. This gives the probability within one standard deviation away from the mean, which is, according to the '68-95-99.7 rule', about 0.6827 (68–95–99.7 rule, 2018).

References

68–95–99.7 rule. (2018, December 12). Retrieved from https://en.wikipedia.org/wiki/68–95–99.7_rule

Fubini Theorem. (n.d.). Retrieved from http://mathworld.wolfram.com/FubiniTheorem.html

HC Appendix

#variables: In 1(a) and (b), we identified t as the independent variable, and x and y as the dependent variables that are both parametrized by t. We also described what each of these variables mean in the given context.

#critique: In 2(a)(i), we described a method of finding volumes and then pointed an inconsistency of the method which arises due to the dimensional inaccuracy. We further described why the method is justified, and sometimes even preferable, despite this flaw.

#deduction: In 2(a)(ii), we reached the conclusion that the triple integral of density function gives the mass of the 3-D region based on mathematical logic in every step. For instance, we used Riemann's sum as a valid premise for reaching the integral definition from the summation formula that we devised from another premise of the basic relation of density with volume and mass.

Python Appendix

Appendix A

For Fig. 1-1a:

```
from matplotlib import pyplot as plt
from matplotlib.pyplot import figure
import numpy as np

X_arr = []
Y_arr = []

for t in np.arange(-0.3, 0.3, 0.001):
    X_arr.append((t + 1)**2)
    Y_arr.append(3*t**2)

figure(num=None, figsize=(6, 4), dpi=300, facecolor='w', edgecolor='k')
plt.plot(X_arr[len(X_arr) // 2:], Y_arr[len(Y_arr) // 2:])
plt.ylabel("y(t)")
plt.xlabel("x(t)")
plt.show()
```

For Fig. 1-1b:

```
figure(num=None, figsize=(6, 4), dpi=300, facecolor='w', edgecolor='k')
plt.plot(X_arr, Y_arr)
plt.ylabel("y(t)")
plt.xlabel("x(t)")
plt.show()
```

Appendix B

For Fig. 1-2:

```
from mpl_toolkits.mplot3d import Axes3D
import math

Z_arr = []
for i in range(len(X_arr)):
    x = X_arr[i]
    y = Y_arr[i]
    Z_arr.append(math.e**(5*x) * (x**2 + y**2))

fig = figure(num=None, figsize=(6, 4), dpi=300, facecolor='w', edgecolor='k')
ax = fig.add_subplot(111, projection='3d')
ax.set_xlabel('x(t)')
ax.set_ylabel('y(t)')
plt.xticks(np.arange(min(X_arr) + 0.01, max(X_arr), 0.2))
plt.yticks(np.arange(min(Y_arr), max(Y_arr), 0.05))

ax.plot(X_arr, Y_arr, Z_arr)
plt.show()
```

Appendix C

For Fig. 3-1:

```
import numpy as np
from matplotlib import pyplot as plt
from matplotlib.pyplot import figure

def balance_est(y_0, k, h, r, q):
    term_2 = 0
    for i in np.arange(0, k / h, 1):
        term_2 += q*h * (h*r + 1)**i

    y = y_0 * (h*r + 1)**(k/h) + term_2
    return y

r = 0.01 # interest rate
q = 250 # [dollars/week] constant savings rate
k = 12 # number of months
y_0 = 100 # [dollars] initial amount of money

h_arr = []
y_arr = []
for h in np.arange(0.1, 4.1, 0.1):
    h_arr.append(h)
    y_arr.append(balance_est(y_0, k, h, r, q))

h = 0.00001 # precise estimation
y = balance_est(y_0, k, h, r, q)

figure(figsize=(6, 4), dpi=300)
plt.xlim(4, -0.1) # decreasing h
plt.plot(h_arr, y_arr)
plt.xlabel('Step size, h (months)')
```

```
plt.ylabel("Money in checking account, y ($)")
plt.scatter(h, y, marker='.', color='r', linewidth=8)
plt.show()
```

Appendix D

For Fig. 3-2:

```
import numpy as np

var('y, t')
r = 0.01
q = 250

y_arr = []
k_arr = []
for k in np.arange(0, 12.5, 0.5):
    k_arr.append(k)
    y_arr.append(balance_est(100, k, 5, 0.01, 250))

A = point(zip(k_arr, y_arr), color="red", size=30)
B = plot_slope_field(lambda t, y: r*y + q, (t, 0.1, 12), (y, 0, 3600),
color='gray', axes_labels=["Time from start, t (months)", "Amount of money
saved, y ($)"])
C = plot((100*(e^((t+100*log(251))/100) - 250)) / 0.01, 0.1, 12)
C + B + A
```