Some takeaways

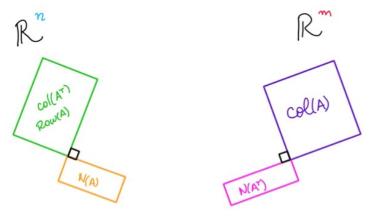
$N(A)$ and $col(A^T) \subset \mathbb{R}^n$

$dim(N(A)) + dim(col(A^T)) = n$

$v \in col(A^T) \perp$ basis of $N(A)$

orthogonal

An overview of the Subspaces



Fund. subsp. and linear systems

$Ax = b$

solution exists only when $b \in col(A)$

$\hookrightarrow$ find basis $\to$ dim $\hookrightarrow$ $\begin{cases} dim < dim(b) \Rightarrow \text{only some } b \text{ have sol.} \\ dim = dim(b) \Rightarrow \text{any } b \text{ has sol.} \end{cases}$

$N(A^T) \ni v \perp w \in col(A) \Rightarrow$ if $\vec{v}$ has any comp. along $N(A^T)$, it's $\notin col(A)$

1. Determined that a system $Ax=b$ will have a solution only if b is in $col(A)$
2. Equivalently, determined that $Ax=b$ will have a solution only if b has no component along $N(A^T)$, which requires checking that $b \cdot \vec{v}_i = 0$ for v_i in the basis of $N(A^T)$

#transformations #theoreticaltools

Rank & transpose

Rank — # lin. $\underbrace{\text{indep cols}}_{\text{pivots}}$ of matrix

$\overbrace{\text{indep. rows}}$

$(A^T)_{ij} = A_{ji}$

Null space

given by v. x

$N(A)$ — subspace satisfy. $Ax = \vec{0}$ (homog. syst.)

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 1 & 0 & 1 \end{bmatrix}$, $N(A)$ - ?

RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

t

$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ - basis for $N(A)$

Null-space:

1. $\vec{0}$
2. closed $\oplus$
3. closed $\odot$

$\underset{m \times n}{A} \underset{n \times 1}{x} = \underset{m \times 1}{\vec{0}}$ $\to$ $N(A)$ - subsp. of $\mathbb{R}^n$

Four fundamental subspaces

$A = M_{m \times n}$:

1. $col(A) \subset \mathbb{R}^m$
2. $N(A) \subset \mathbb{R}^n$
3. $row(A) \subset \mathbb{R}^n$
4. $N(A^T) \subset \mathbb{R}^m$ — left null-space

Four Fundamental Subspaces

Instructions: There are two sections of problems. Complete as many problems as time allows, recording your final answers in this document. You are responsible for all the problems in the first section, and will be asked a related question in the reflection poll at the end of the class session. The second section contains more challenging tasks or enriching material.

Notes: 1) Check in with your instructor occasionally to have your work checked or to ask questions. 2) Be sure to use the different capabilities of the platform to facilitate your work. It is recommended that you manually place your group in a 2-up configuration and open a Sage workbook in the second frame. 3) All problems in this lesson are adapted from G. Strang, Linear Algebra and Its Applications, 4e., an excellent reference for the material in this course.

Section A: Main Material

Record your answers after each prompt as appropriate. It is recommended that you write them in a color other than black for easy reference.

1. Find a matrix A that has V as its row space, and a matrix B that has V as its nullspace, if V is the subspace spanned by

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\5\\0\end{bmatrix}$$

   A = [[1, 1, 0],
   [1, 2, 0]]

   N(A) = { [1, 1, 0], [1, 2, 0] }

   v in N(A), v = <a + b, a + 2b, 0>
   B*v = <0, 0, 0>
   B can be [[0, 0, 1],
   [0, 0, 1],
   [0, 0, 1]
   Since we cannot find x and y such that xa + xb = ya + 2yb for all a and b
2. If the columns of A are linearly independent (A is mxn), then the rank is n, the nullspace is R^0, and the row space is R^m. Find a basis for Row(A).

3. **Dimension counting**
   a. If a 7x9 matrix has rank 5, what are the dimensions of the four subspaces?
   Col(A) = 5, N(A^T) = 9 - 5 = 4
   Row(A) = 5, N(A) = 7 - 5 = 2

   b. If a 3x4 matrix has rank 3, what are its column space and left nullspace?
   Col(A) = R^3
   N(A^T) = R

Section B: Enrichment and Challenge Material

1. Construct a matrix with the required property, or explain why you can't.
    a. Column space contains

    $$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix},$$

    Row space contains

    $$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}.$$

    b. $$\text{Column space has basis } \begin{bmatrix}1\\2\\3\end{bmatrix}, \text{ nullspace has basis } \begin{bmatrix}3\\2\\1\end{bmatrix}.$$

    c. Dimension of nullspace = 1 + dimension of left nullspace.

    d. Row space = column space, nullspace not equal to left nullspace.

2. A is an mxn matrix of rank r. Suppose there are right-hand sides b for which Ax=b has **no solution.**
    a. What inequalities (< or ≤) must be true between m, n, and r?

    b. How do you know that $A^T y=0$ has a nonzero solution?

3. **Block Matrices Revisited** What are the dimensions of the four subspaces for A, B, and C, if I is the 3x3 identity matrix and 0 is the 3x2 zero matrix?

$$A = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} I & I \\ 0^T & 0^T \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \end{bmatrix}$$