DYNAMICS—THE GEOMETRY OF BEHAVIOR

PART 1: PERIODIC BEHAVIOR

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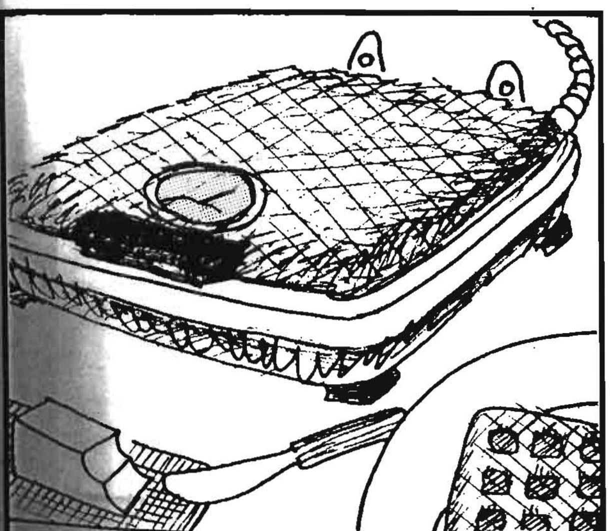
VOLUME 1

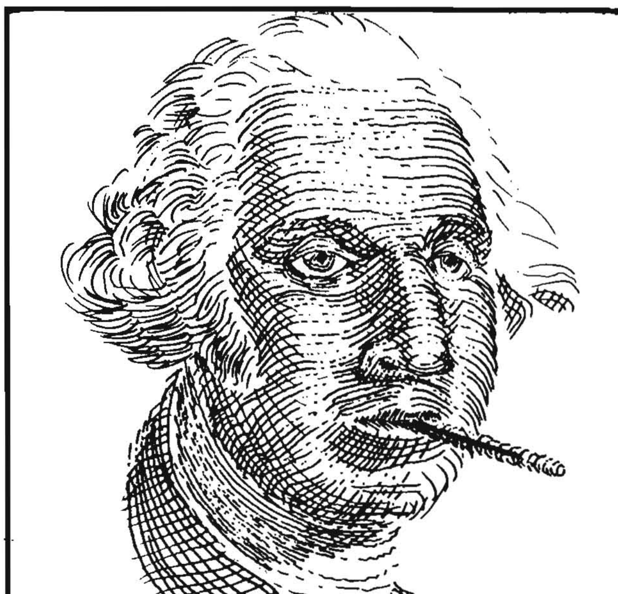
1.1 STATE SPACES

The strategies for making mathematical models for observed phenomena have been evolving since ancient times. An organism--physical, biological, or social--is observed in different states. This *observed system* is the target of the modeling activity. Its states cannot really be described by only a few observable parameters, but we pretend that they can. This is the first step in the process of "mathematical idealization" and leads to a geometric model for the set of all idealized states: the *state space* of the model. Different models may begin with different state spaces. The relationship between the actual states of the real organism and the points of the geometric model is a fiction maintained for the sake of discussion, theory, thought, and so on: this is known as the *conventional interpretation.* This section describes some examples of this modeling process.

The simplest scheme is the one-parameter model. The early history of science used this scheme extensively.

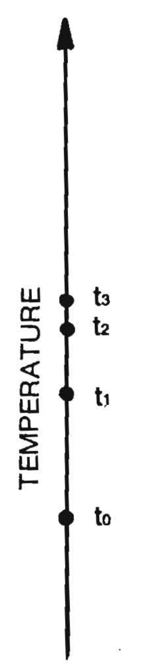


1.1.1. The actual state of this waffle iron cannot be described completely by a single observable parameter, such as the temperature. But usually we find it convenient to pretend that it can. This pretense is an agreement, the *conventional interpretation,* within the modeling process. It is justified by its usefulness in describing the behavior of the device.

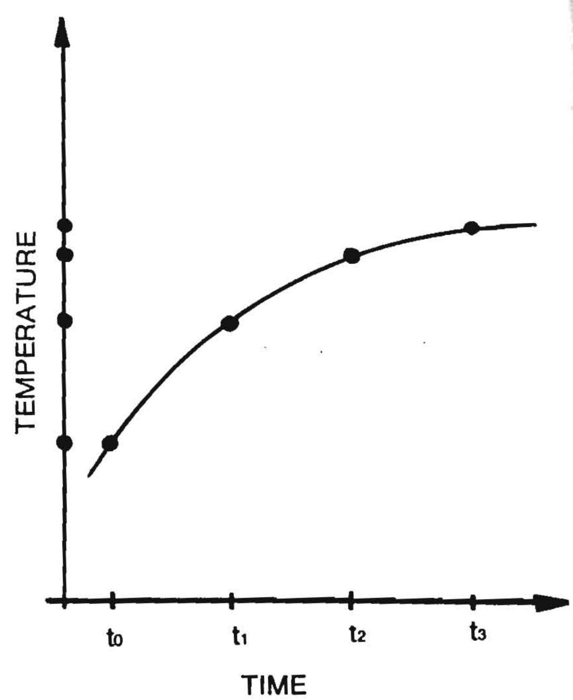


1.1.2. The correlation between the internal state of a complex system, such as a mammal, and a single observed parameter may be very good or very bad, depending on the context. In the case of George Washington, the oral temperature correlates better with his health than with his honesty.

1.1.3. In these examples, the geometric model for the set of all (mathematically idealized) states is the real number line. This is one of the simplest state spaces.

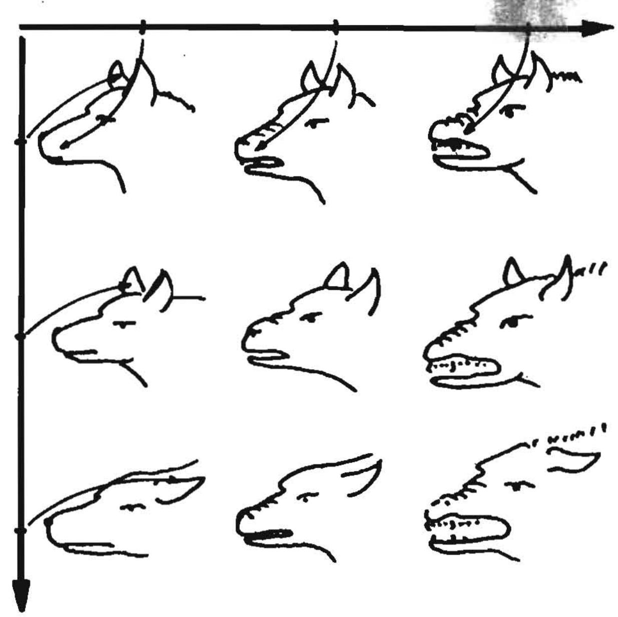


1.1.4. Observing the parameter for a while, it will probably change. The different values observed may be labeled by the time of their observation: the states observed at four different times are shown here.

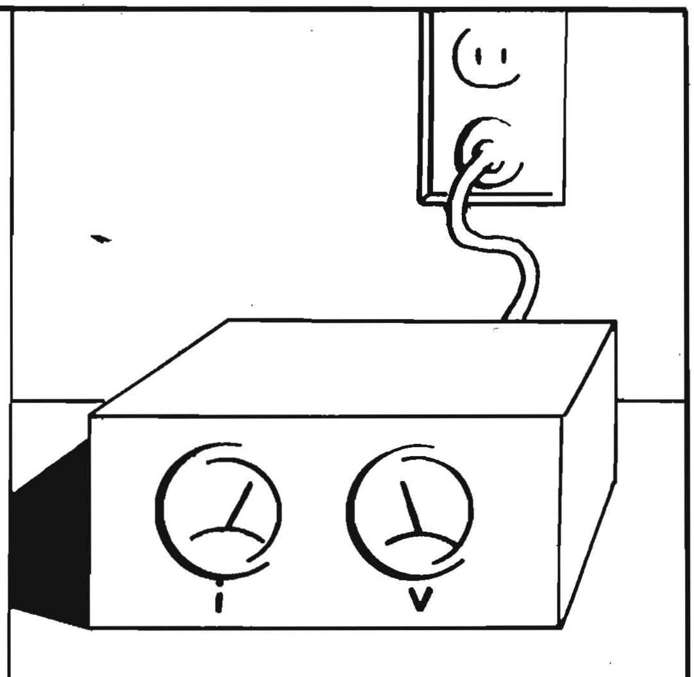


1.1.5. These data comprise a *time series* of observations, and are shown here as a graph. The vertical line represents the state space, the horizontal axis indicates time.

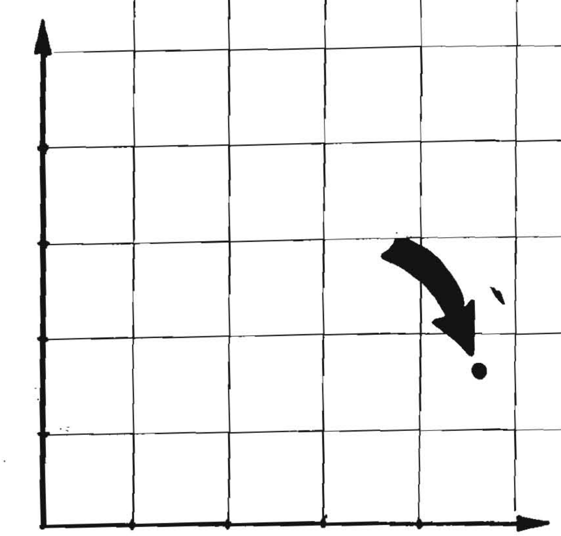
Closer observation may suggest two parameters for the description of a given state of the actual organism.



1.1.6. In this modeling scheme of Konrad Lorenz and Christopher Zeeman (1), two parameters are used for the emotional state of a dog. The two observed parameters are *ear attitude,* which correlates with the emotional state of fear, and *fang exposure*, corresponding to the degree of rage.

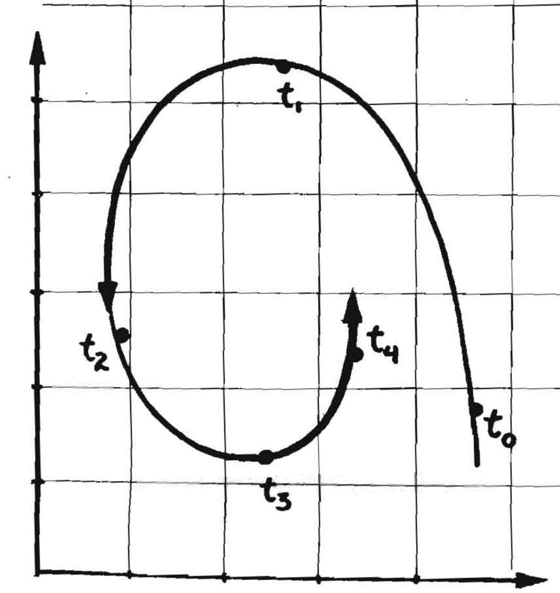


1.1.7. Electronic devices are simple to model, as the observations are explicitly numerical. This electronic 'black box' is provided with panel meters, which indicate the instantaneous values of voltage and current at specific points of the electronic network within the box.



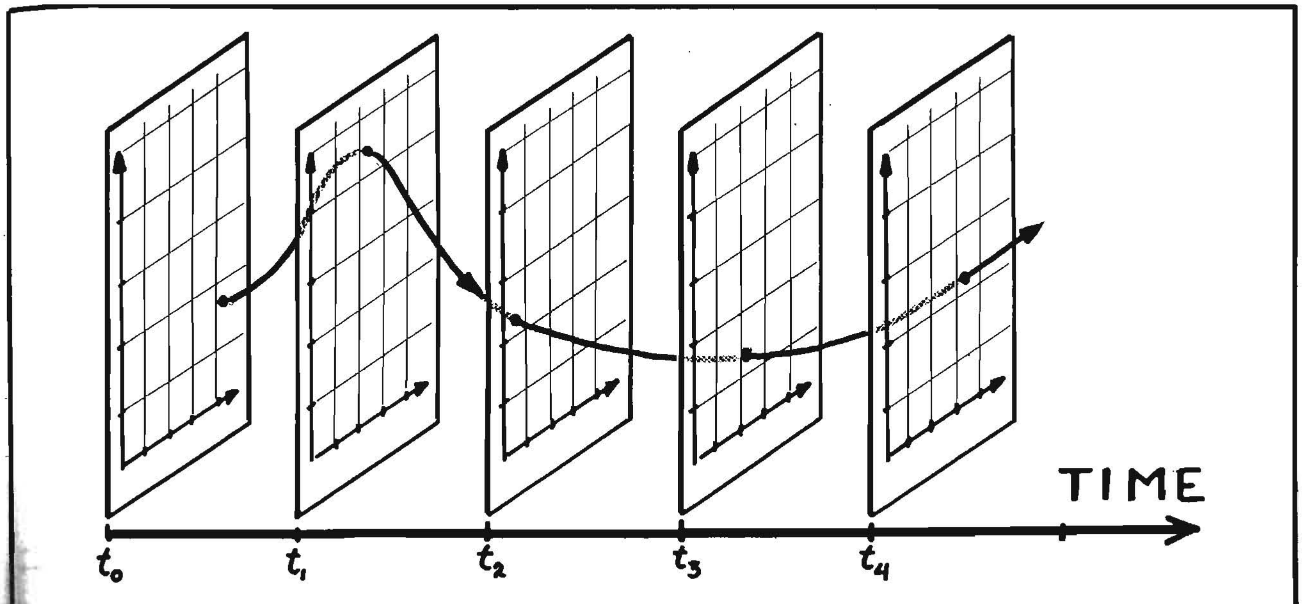
1.1.8. The values of two numerical parameters (of a model such as those of the preceding two examples) may be represented by a single point in this two-dimensional state space, the *plane* of Euclidean geometry.

Changes in the actual state of the system are observed, and are represented as a curve in the state space. Each point on this curve carries (implicitly at least) a label recording the time of observation. This is called a *trajectory* of the model.

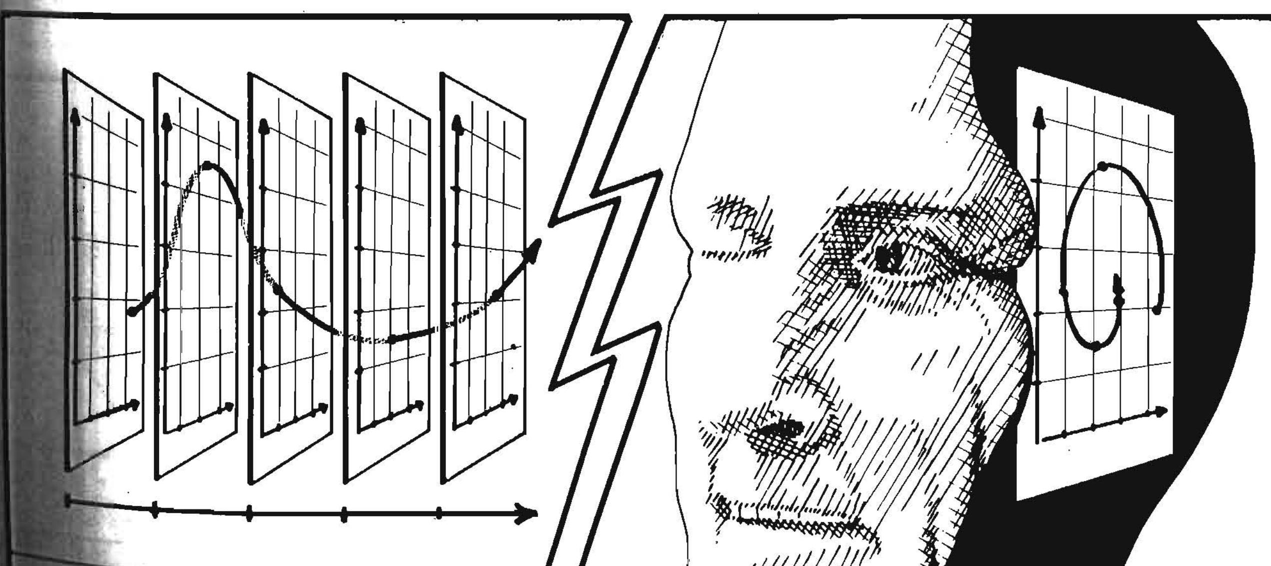


1.1.9. For example, if the two parameters representing the emotional state of a dog, or the internal state of an electronic black box, are observed at successive times and recorded in the plane with labels, a trajectory of the model is obtained.

Another style of representing the changing data is by its *time series,* which means the *graph* of a trajectory. We have already seen a time series, in a one-dimensional context. But this style of data representation may also be used in higher dimensions.

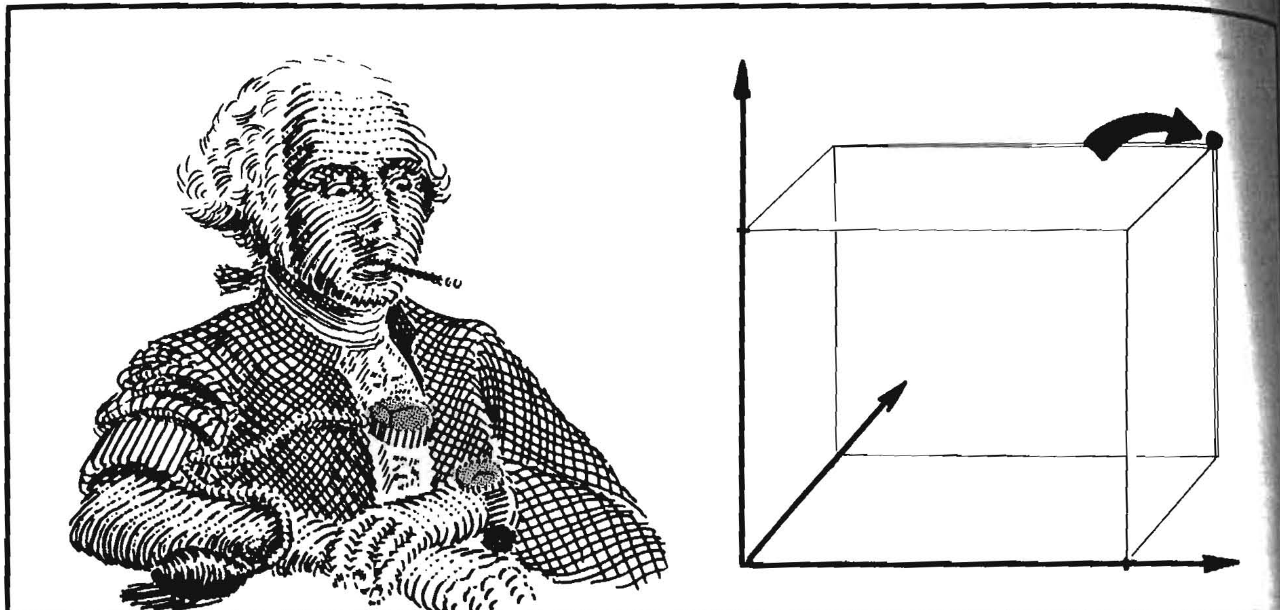


1.1.10. Here the vertical plane represents the state space, and the horizontal axis represents the time of observation. The parameters observed at a given time are plotted in the vertical plane passing through the appropriate point on the time axis.



1.1.11. The trajectory may by obtained from the time series, by simply viewing it from the right angle— straight down the time axis from the end, infinitely far away.

Observing more parameters leads to models of higher dimensions.



1.1.12. Suppose that a 7 a.m., this athlete observes three of his body parameters (say: temperature, blood pressure, and pulse rate), records these three data as a point in three-dimensional space, and labels this point with the time of observation. This is a simple example of a three-dimensional state space.

Many phenomena require geometric models which are not simply coordinate spaces. In dynamical systems theory, the geometric models which are used are *manifolds*.

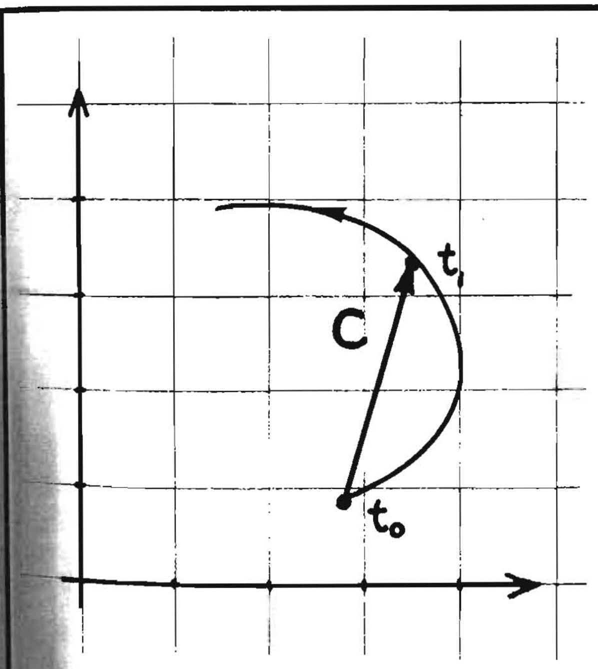
1.1.13. Here are some examples of manifolds. Other examples will arise in the sequel. They are made of pieces of flat spaces, bent, and glued together.

More about manifolds may be found in Volume 0 of this series.

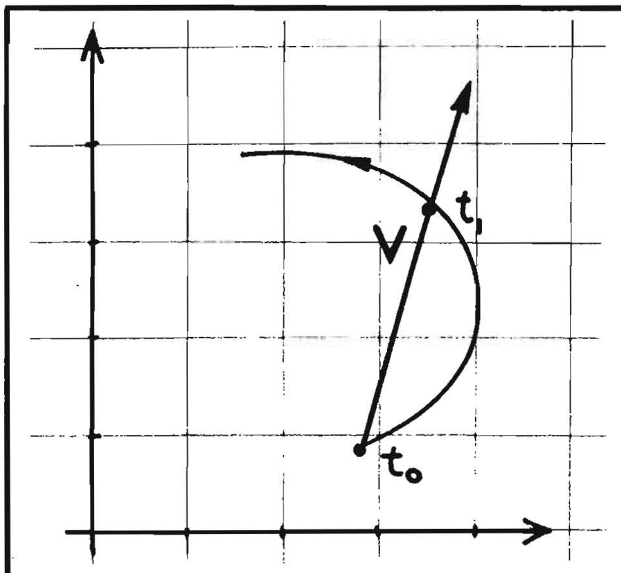
1.2 DYNAMICAL SYSTEMS

At this point, the history of a real system has been represented graphically, as a *trajectory* in a geometric *state space.* An alternative representation is the *time series,* or *graph,* of the trajectory. The dynamical concepts of the middle ages included these kinds of representation. But in the 1660's, something new was added — *the instantaneous velocity, or derivative, of vector calculus* — by Newton. As dynamical systems theory evolved, the *velocity vectorfield* emerged as one of the basic concepts. Trajectories determine velocity vectors, by the *differentiation* process of Calculus. Conversely, velocity vectors determine trajectories, by the *integration* process of Calculus.

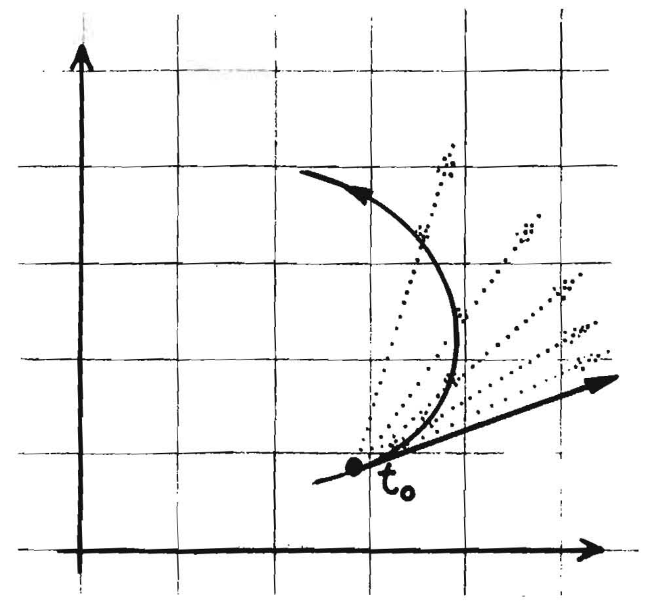
This is the differentiation process, which determines the velocity vectorfield from the trajectories.



1.2.1. On this trajectory, the states observed at two different times, t_0 and t_1, are connected by a *bound vector,* represented here by a line segment pointed on one end. (See Volume 0 for details.) Let C denote this bound vector.



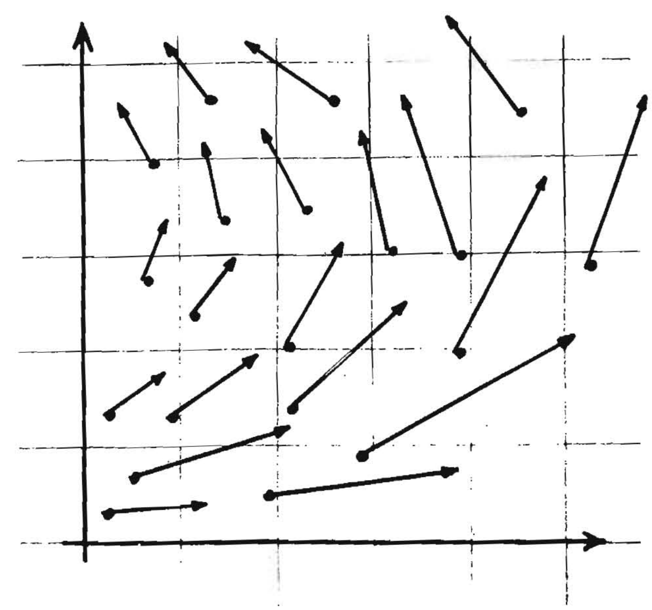
1.2.2. The *average velocity* of the change of state, C, is the vector starting at the point labeled t^0 on the curve, and directed along the vector of change of state, C, but divided by T, the time elapsed between t^0 and t^1. Let V denote this vector, $V=C/T$. It represents the average speed and direction of the change of state.



1.2.3. The *instantaneous velocity* of the trajectory at the time t_0 is the bound vector that V tends to, as the elapsed time, T, shrinks smaller and smaller. This limiting vector, denoted here by C,† is also known as the *tangent vector.* The construction of this velocity, or tangent vector, from the curve is called *differentiation* in Vector Calculus.

The modeling process begins with the choice of a particular state space in which to represent observations of the system. Prolonged observations lead to many trajectories within the state space. At any point on any of these curves, a velocity vector may be derived. This is the new dynamical concept of Newton and Leibniz. It is useful in describing an inherent tendency of the system to move with a habitual velocity, at particular points in the state space.

The prescription of a velocity vector at each point in the state space is called a *velocity vectorfield.*



1.2.4. A *vectorfield* is a field of bound vectors, one defined at (and bound to) each and every point of the state space. Here, only a few of the vectors are drawn, to suggest the full field.

The state space, filled with trajectories, is called the *phase portrait* of the dynamical system. The velocity vectorfield has been derived from the phase portrait by *differentiation.*

We regard this vectorfield as the model for the system under study.
In fact, the phrase *dynamical system* will specifically denote this vectorfield.

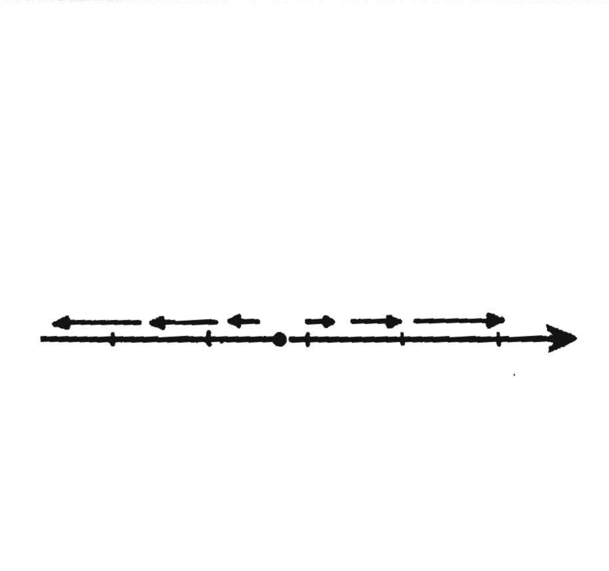
In the practice of this modeling art, the choice of a vectorfield is a difficult and critical step. Extensive observations of the organism being modeled, over a long period of time, will usually reveal tendencies (to a dynamicist, at least) which can be represented as a dynamical system. The history of applied dynamics provides excellent examples of this process. Several of these are described in the next four chapters. The usefulness of this kind of model depends on the following fundamental hypotheses.

Hypothesis 1. The observation of the organism over time, represented as a trajectory in the state space, will have this property, at each of its points: its velocity vector is exactly the same as the vector specified by the dynamical system.

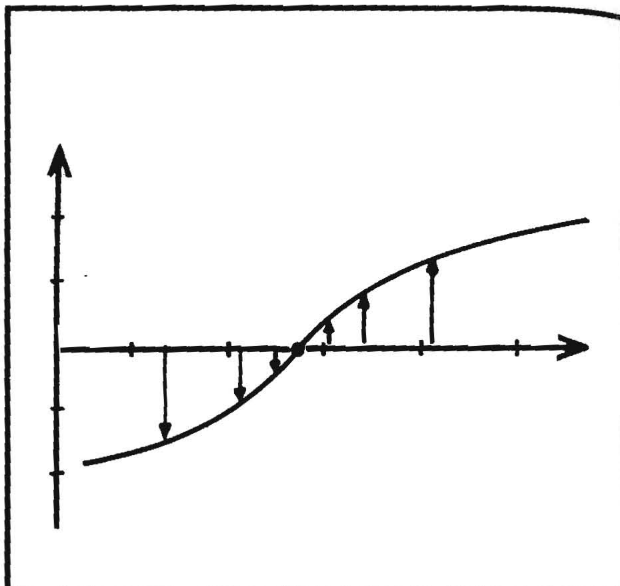
Henceforward, the word *trajectory* will always carry this assumption. That is, the trajectories of the phase portrait have the specified velocity vectors, and further, they will be assumed to represent the behavior of the system being modeled. Further, for technical reasons we also assume:

Hypothesis 2. The vectorfield of the model is smooth.

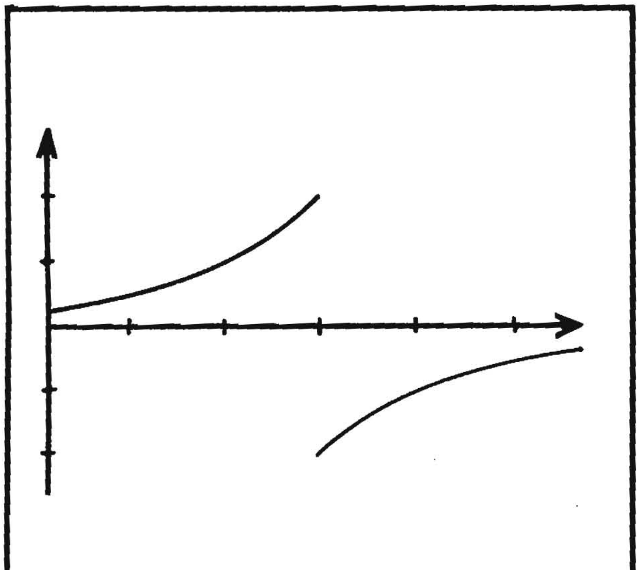
The word smooth, in this context, is most easily seen in the one-dimensional case. On a one-dimensional state space, a vectorfield is specified by a graph in the plane. In this context, the graph is *smooth* if it is continuous, and its derivative is continuous as well: no jumps, no sharp corners. More details are given in Volume 0 of this series.



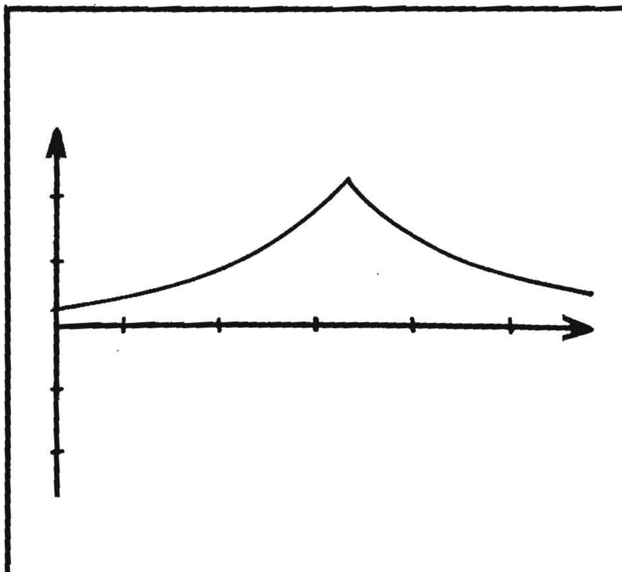
1.2.5. For example, here is a vectorfield (green) on a one-dimensitonal state space (black). The vector at the rest point (red) is the "zero vector", its length is zero.



1.2.6. Stand up each green vector by means of a counterclockwise rotation by a right angle. The arrowheads (green) trace out a curve (red) which is the graph of a function. The vectorfield is completely described by this function.

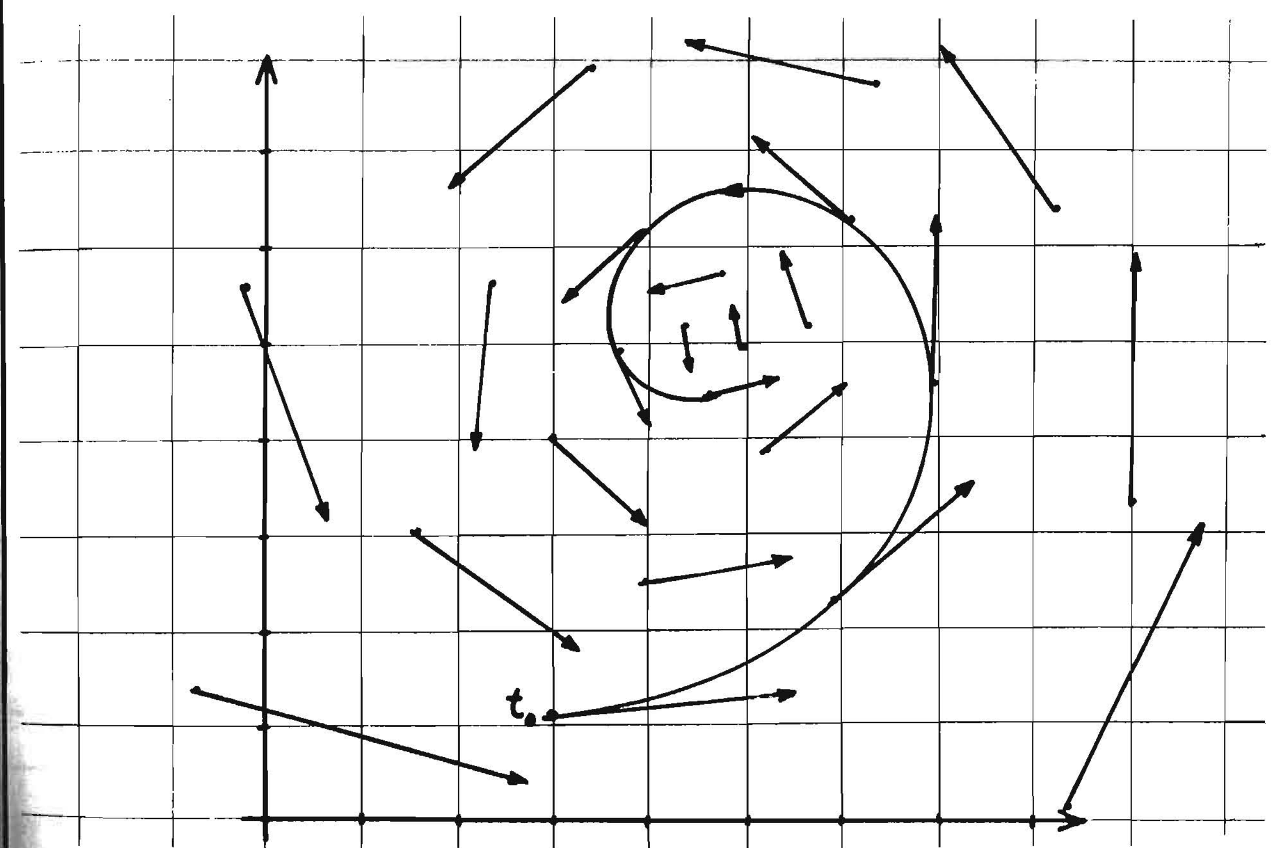


1.2.7. Another vectorfield is described by this function. This function is not continuous, so the vectorfield is not smooth.



1.2.8. Yet another vectorfield is described by this function. This function is continuous, but has a sharp corner. This vectorfield is not smooth either.

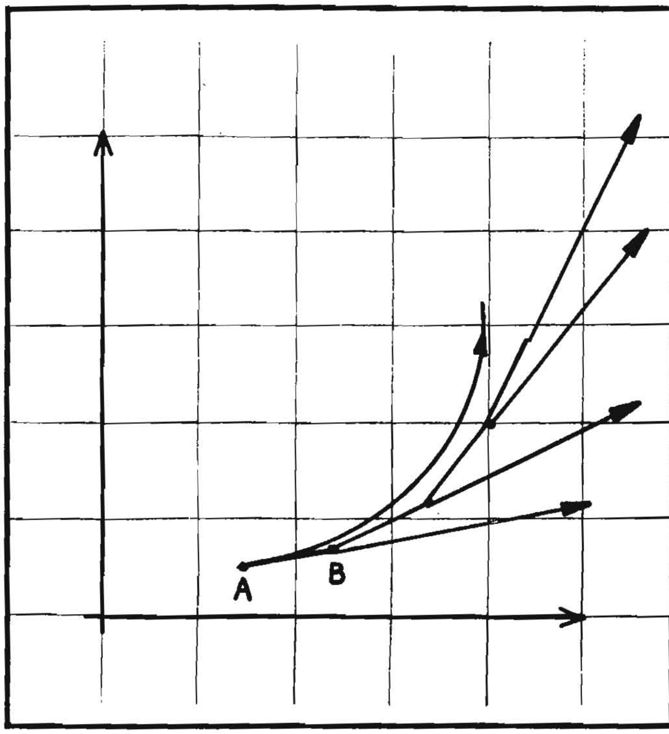
We suppose now that a dynamical system has been chosen as a model for a system. Given this vectorfield, how can we deduce the trajectories, thus the phase portrait, and the behavior of the system?



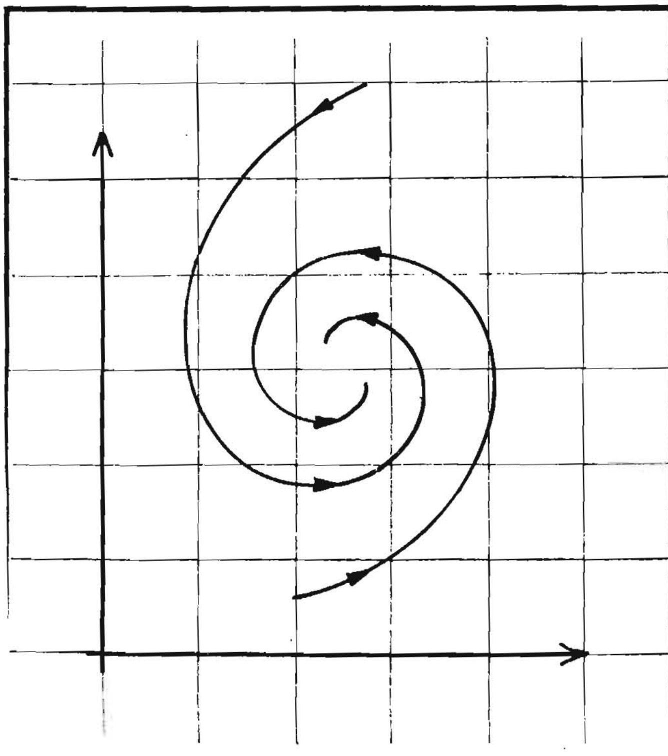
1.2.9. Given a state space and a dynamical system (smooth vectorfield), a curve in the state space is a *trajectory,* or *integral curve,* of the dynamical system if its velocity vector agrees with the vectorfield, at each point along the curve. This means the curve must evolve so as to be tangent to the vectorfield at each point, as shown here. The point on the trajectory corresponding to elapsed time zero, t=0, is the *initial state* of the trajectory.

Given a dynamical system (a smooth vectorfield on a manifold), how can we find its trajectories? Analysis, the mathematical theory which has evolved since Newton and Leibniz, has established that from each initial point, there is a single trajectory of the system. Finding it requires the construction called *integration* in Vector Calculus. Thus, trajectories are sometimes called *integral curves.*

A graphical construction which approximates the integration of a trajectory, or integral curve, was discovered by Euler.



1.2.10. Euler's method approximates an integral curve by a polygon. Starting from the initial point, *A*, a straight line is drawn along the vector of the dynamical system attached to that point, *V(A)*. The length of this straight line is a small proportion of the length of this vector, say 10%. At point *B*, at the far end of this line segment, the construction is repeated, using the vector, *V(B)*, attached to this point by the dynamical system. This construction is repeated as many times as necessary, to draw the polygonal, approximate trajectory.

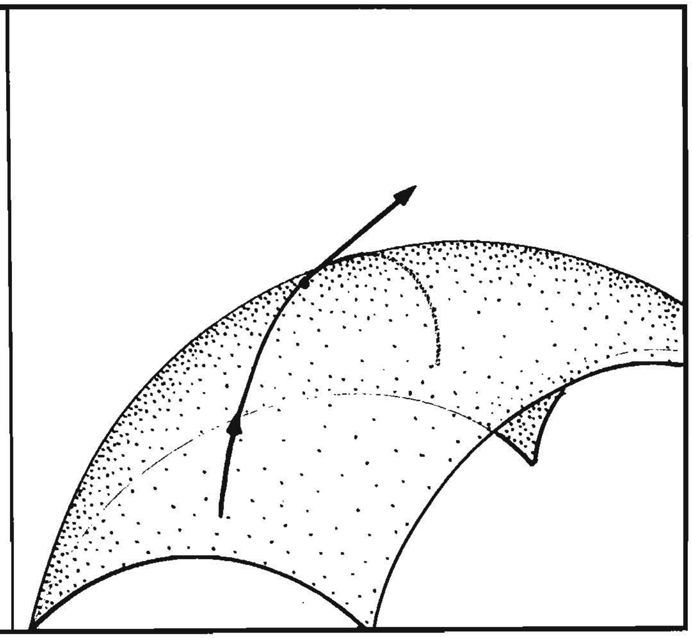


1.2.11. The state space is filled with trajectories, completely determined by the dynamical system. The display of the state space, decomposed into these curves, is the *phase portrait* of the system. Furthermore, the space of states may be imagined to flow, as a fluid, around in itself, with each point following one of these curves. This motion of the space within itself is called the *flow* of the dynamical system.

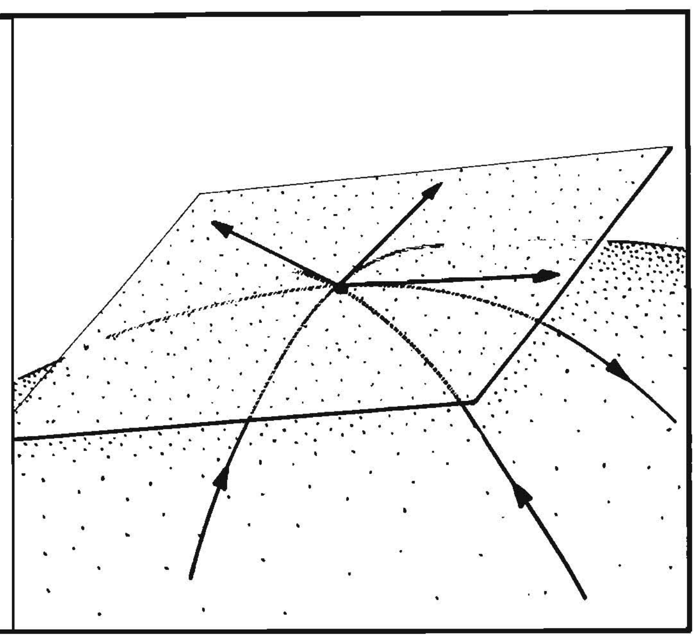
In the next chapter, simple examples will show that flat, Euclidean, spaces will not suffice for all of our geometric models. In some cases, curved spaces (that is, *manifolds)* will be necessary. In Global Analysis, the Calculus of Newton and Leibniz is generalized to the context of manifolds. This generalization provides the basic tools of mathematical dynamics. A description may be found, in pictorial representations, in Volume 0 of this series.

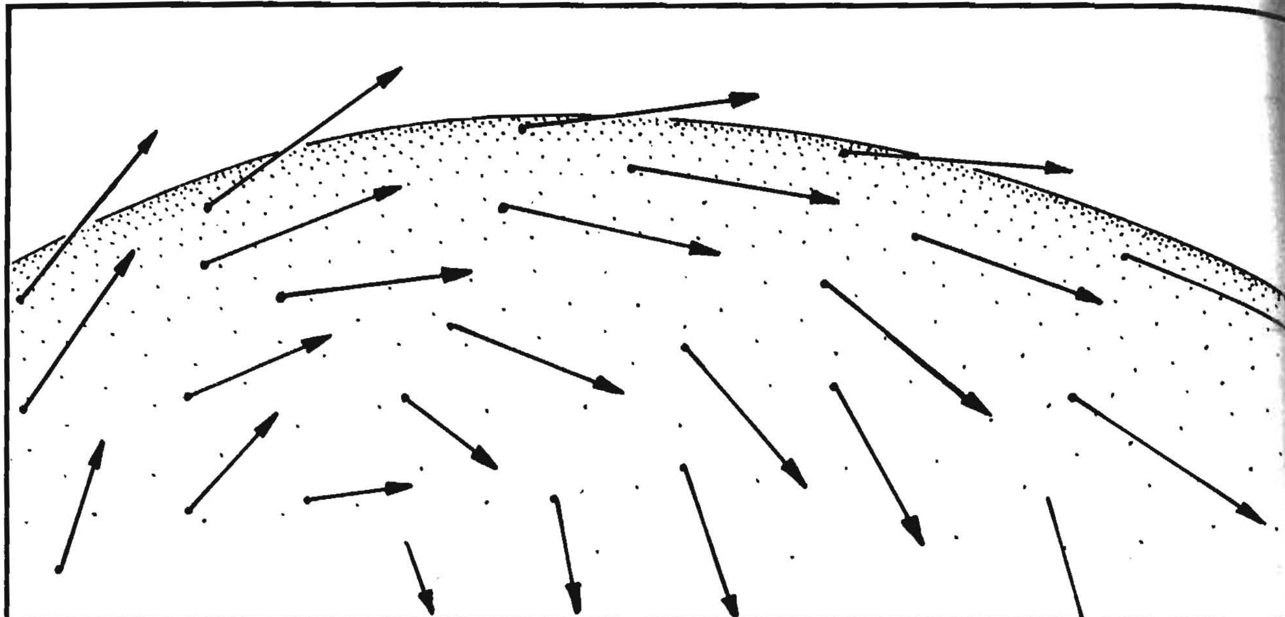
The trajectory and velocity vector concepts fit nicely into the context of manifolds. Here we illustrate these ideas on a two-dimensional manifold, which is simply a curved surface.

1.2.12. Here the instantaneous velocity vector is obtained as a limit of average velocity vectors, as in an earlier illustration. But in this case, the state space is *curved.* The velocity vector does not live in the curved surface. It sticks out into the ambient three-dimensional space. It is *tangent* to the surface.

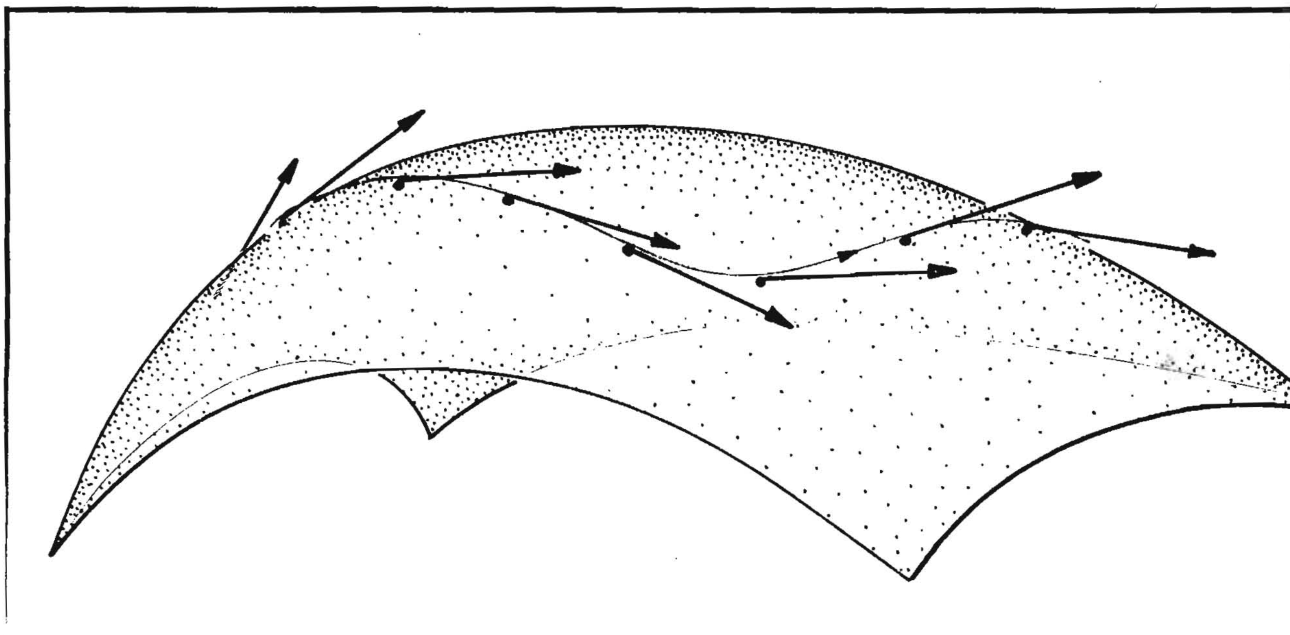


1.2.13. Now repeat this construction many times, with different curves lying in the surface, all passing through the same point. All the vectors lie in the same plane, tangent to the curved surface at a point. This plane is the *tangent space* of the space of states, at that point.





1.2.14. A vectorfield, in this context, means the assignment of a tangent vector to every point of the curved surface.



1.2.15. This is a trajectory, or integral curve, of a vectorfield (dynamical system) on a curved space of states. The tangent vectors at each point project off the surface, yet the integral curve stays within it.

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This completes the introduction of the basic concept of a dynamical system, and the modeling process, which we may summarize as follows.

Suppose a dynamical model has been proposed for some experimental situation. This situation may be a laboratory device, an organism, a social group, or whatever. The model consists of a manifold and a vectorfield. The manifold is a geometrical model for the observed states of the experimental situation, and is called the *state space* of the model. The vectorfield is a model for the habitual tendencies of the situation to evolve from one state to another, and is called the *dynamic* of the model. Now mathematics can be brought out of the tool-box, and used to draw many trajectories of the dynamical system, creating its *phase portrait.* These basic concepts of dynamical systems theory have been illustrated in the preceding two sections.

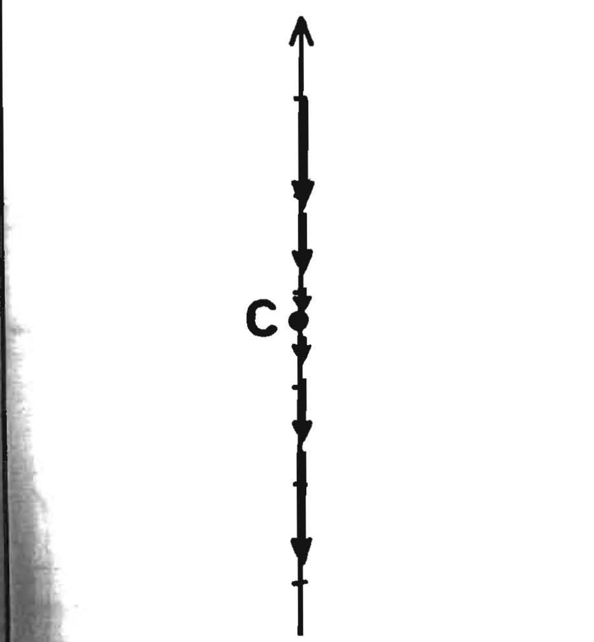
Now you may ask: SO WHAT? Well, according to our *conventional interpretation,* the agreed rules of the game, these trajectories are supposed to describe the behavior of the system, as observed over an interval of time. Either they do this, with an accuracy sufficient to impress you, and be useful for predicting the behavior of the experimental situation, or they do not. In many examples of this art, called *applied dynamics,* they do. These models succeed remarkably well, and have been used by many satisfied customers over the years. Some of these examples are presented in the next four chapters.

But some obstinate reader's may still exclaim: SO WHAT? Well, dynamical systems theory has yet more to offer: PREDICTION FOREVER. Sophisticated techniques from the research frontier of pure mathematics have been employed to yield *qualitative predictions of the asymptotic behavior of the system in the long run, or even forever.* Although qualitative predictions are not as precise as quantitative ones, they are a whole lot better than no predictions at all. *And for most problems of applied dynamics, quantitative predictions are impossible.* So, the remaining sections of this chapter are devoted to the illustration of the concepts of *asymptotic behavior.*

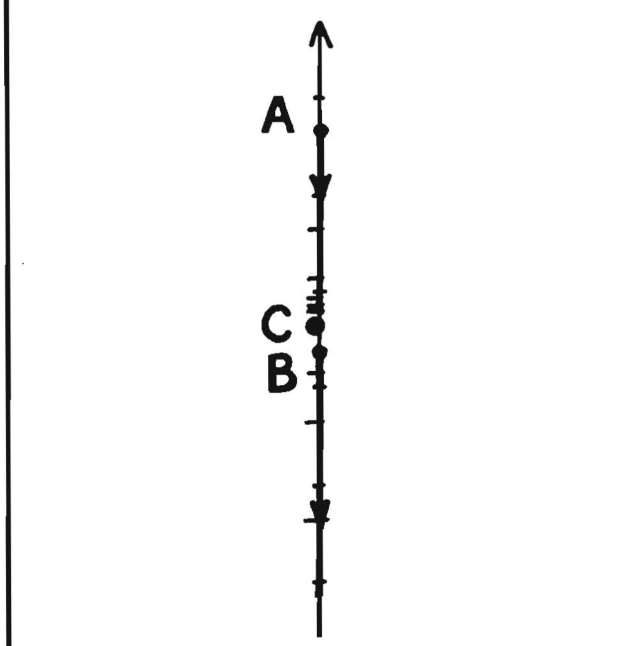
1.3 SPECIAL TRAJECTORIES

The first step in the quest for qualitative predictions of asymptotic behavior is the examination of the phase portrait for special types of trajectories. Here we illustrate some of these special trajectories.

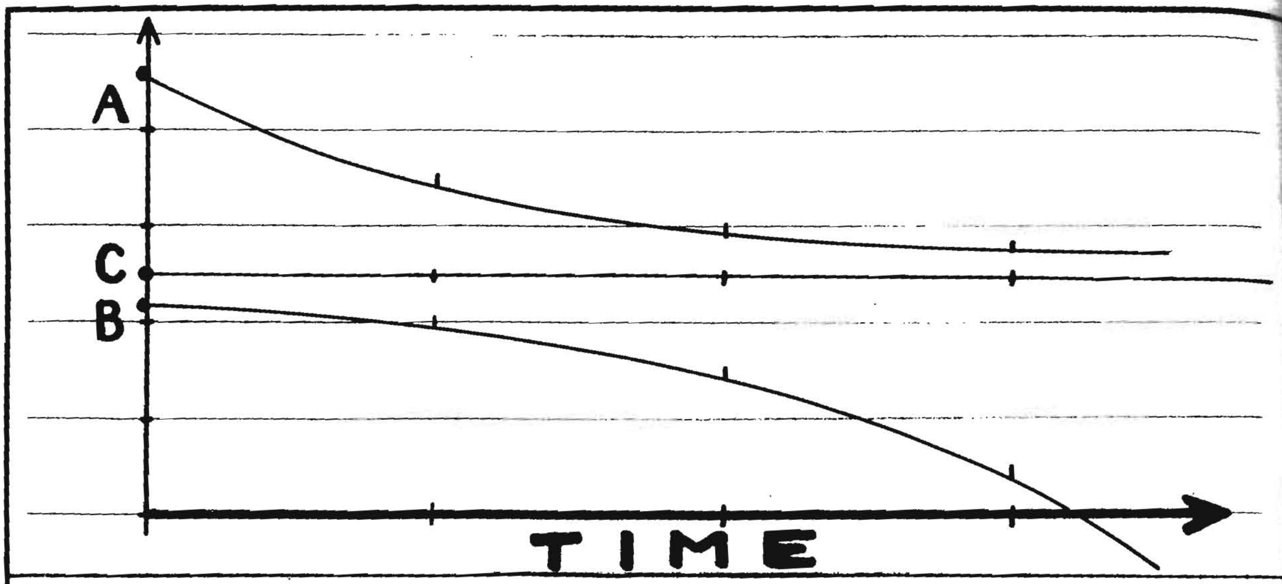
The simplest special trajectory is a point. Let's consider this in a one-dimensional context first.



1.3.1. Here is a vectorfield on a one-dimensional state space. At one point in the state space, marked *C*, the associated velocity vector is the *zero vector.* This vector has length zero. The point, marked *C*, is called a *critical point,* or an *equilibrium point,* of the vectorfield. Because we assumed at the start that the vectorfield is *continuous,* the velocity vectors attached to points near the critical point are very short. (This is explained in Volume 0.)

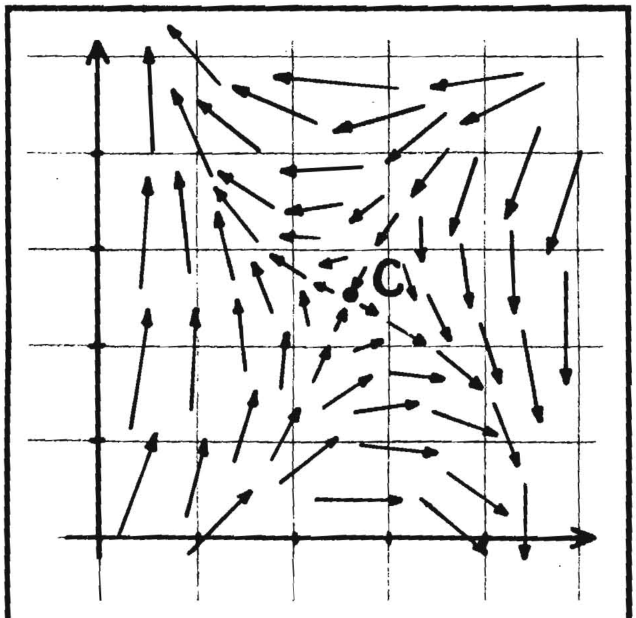


1.3.2. This is the phase portrait of the dynamical system to the left. Three trajectories are shown, starting at the points, *A*, *B*, and *C*. Tick marks along the trajectories indicate the positions at successive seconds. Note that they are closer together near the critical point. One trajectory is piled up on the critical point. The velocity of this trajectory is zero, at all times. It does not move. It is called a *constant trajectory*.

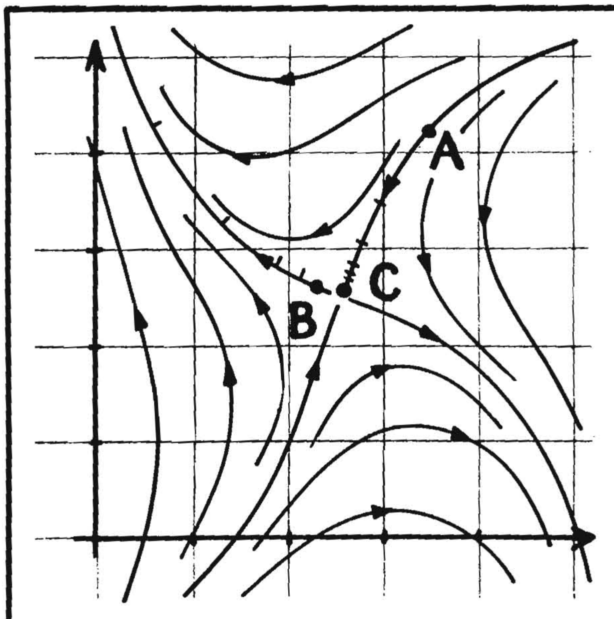


1.3.3. These are the *time series* corresponding to the three trajectories of the preceding phase portrait. The graph (time series) of the trajectory of C is a horizontal line, a constant function of time. This represents the constant trajectory of the critical point.

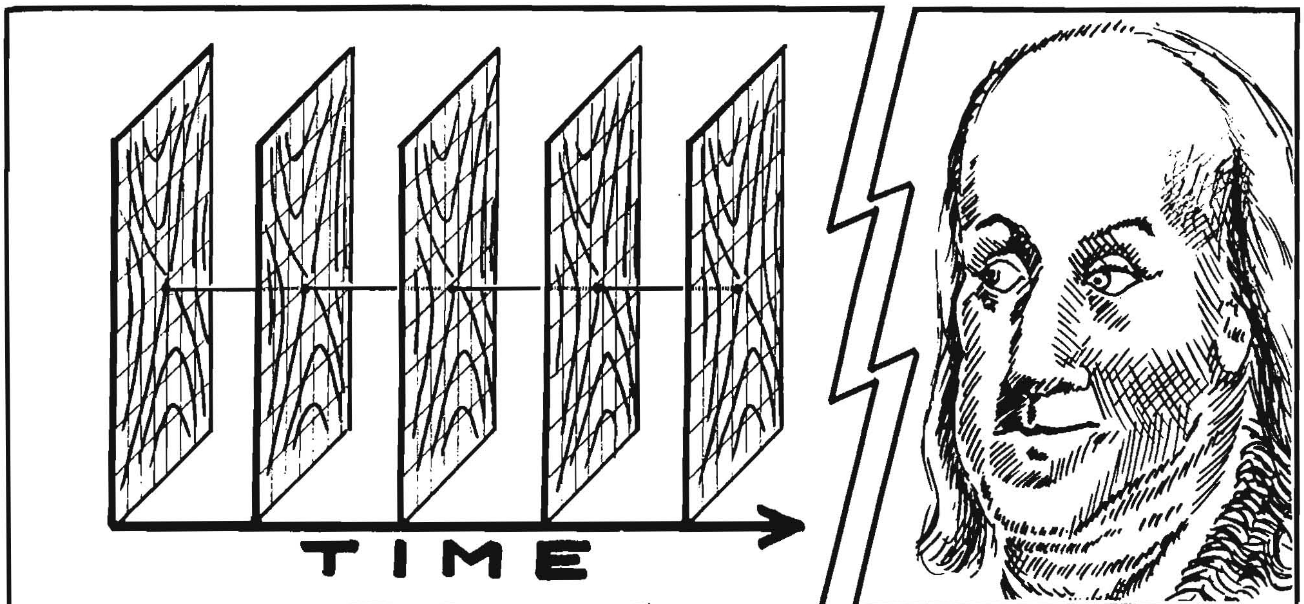
Now let's look at the same idea in a two-dimensional context.



1.3.4. This is a garden-variety vectorfield in the plane. The zero vector appears once, as the velocity vector of the critical point, C. Nearby, the vectors are short.

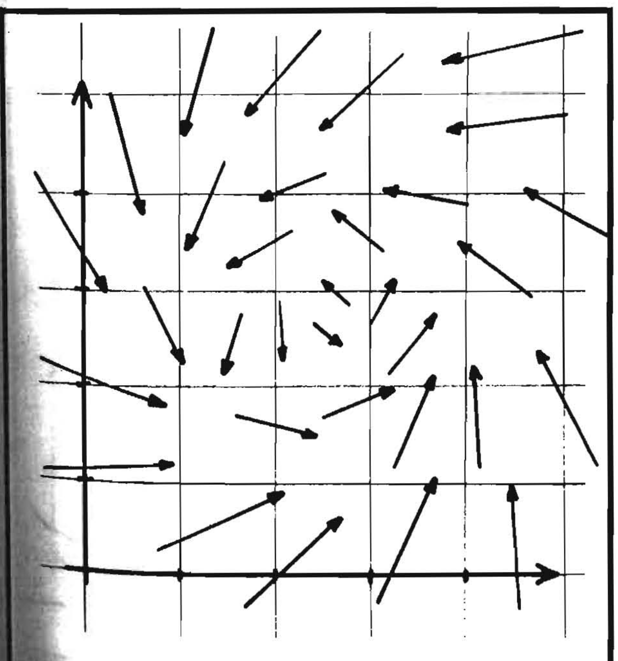


1.3.5. This is the phase portrait of the vectorfield to the left. The trajectory of the critical point is again piled up on the critical point. It is a constant trajectory.

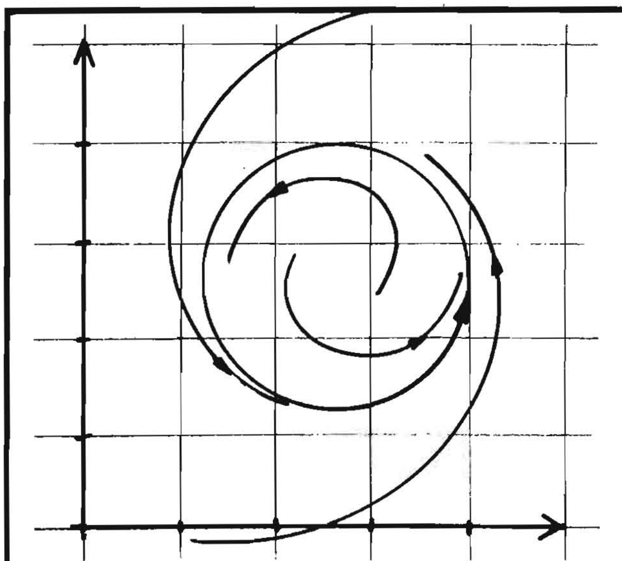


1.3.6. This is the time series of the constant trajectory in the two-dimensional context. Once again, it is a constant function of the time parameter.

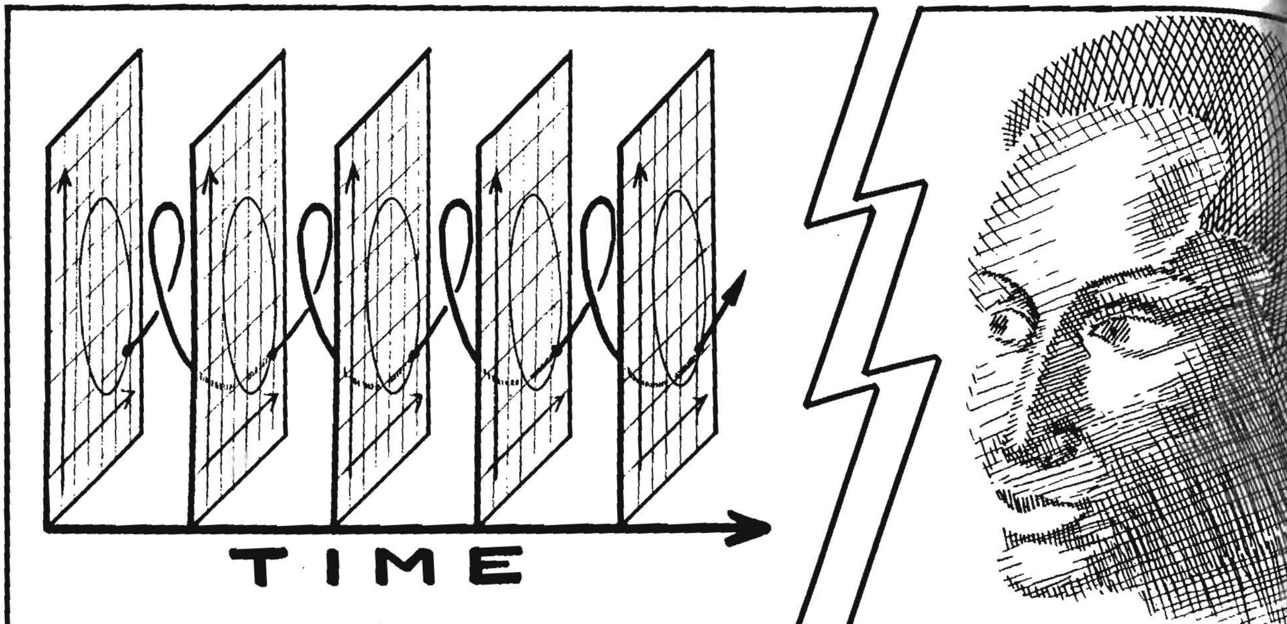
In two dimensions or more, other types of special trajectories frequently occur. Here is a very important one, the *cycle*.



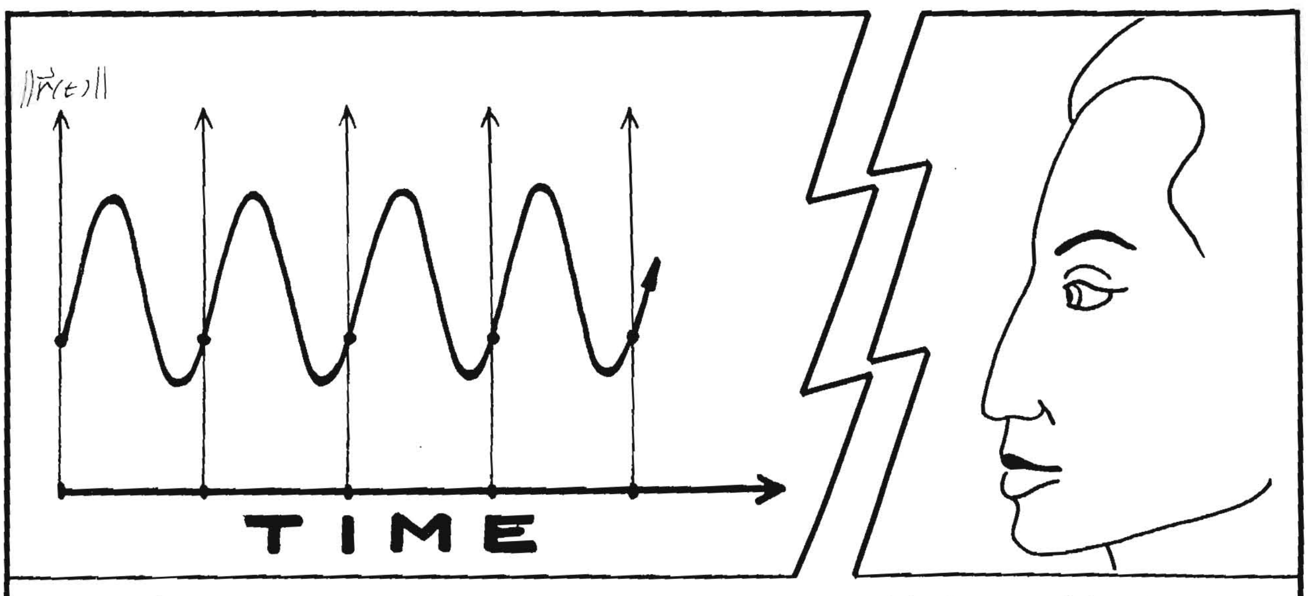
1.3.7. This planar vectorfield has an *eddy*. It seems as if the flow must somehow circle about a point.



1.3.8. This is the phase portrait of the preceding dynamical system. Indeed, here we find a trajectory which is wrapped around and around the same curve. This is called a *closed trajectory*. It is also known as a *closed orbit, periodic trajectory, cycle,* or *oscillation.*

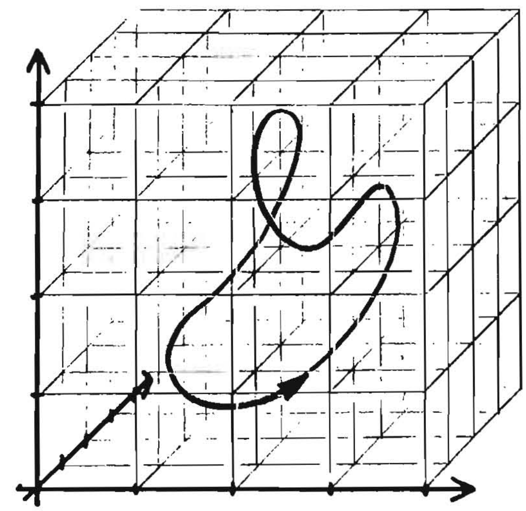


1.3.9. This is the time series representation of a closed trajectory. The graph wraps around a horizontal cylinder. The same interval of time is required to complete each wrap. This time interval is called the *period* of the closed trajectory.

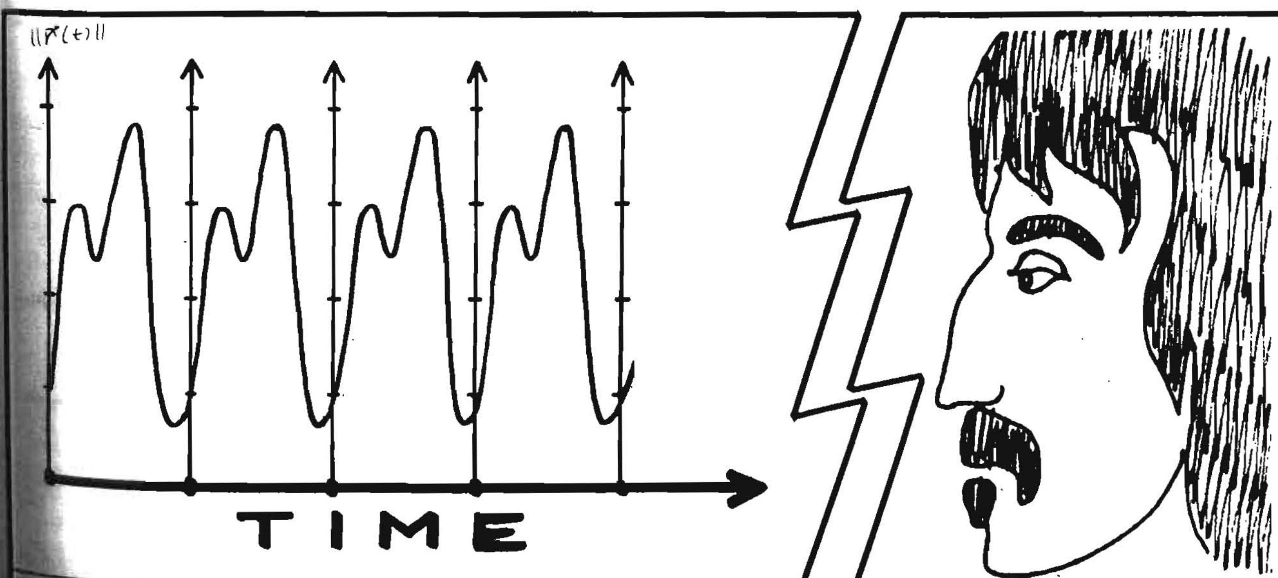


1.3.10. If a single parameter (of the two coordinates in the plane) is chosen, and the other data are forgotten, the time series of the chosen data may be plotted in the plane. The result, the *time series of the preferred parameter,* is a *periodic function.* This means that in every vertical strip corresponding to one period (or wrap, or cycle) of the closed trajectory, the graph exactly repeats itself.

These special types of trajectories, constant and periodic trajectories, also occur in phase portraits of dimension three or more.

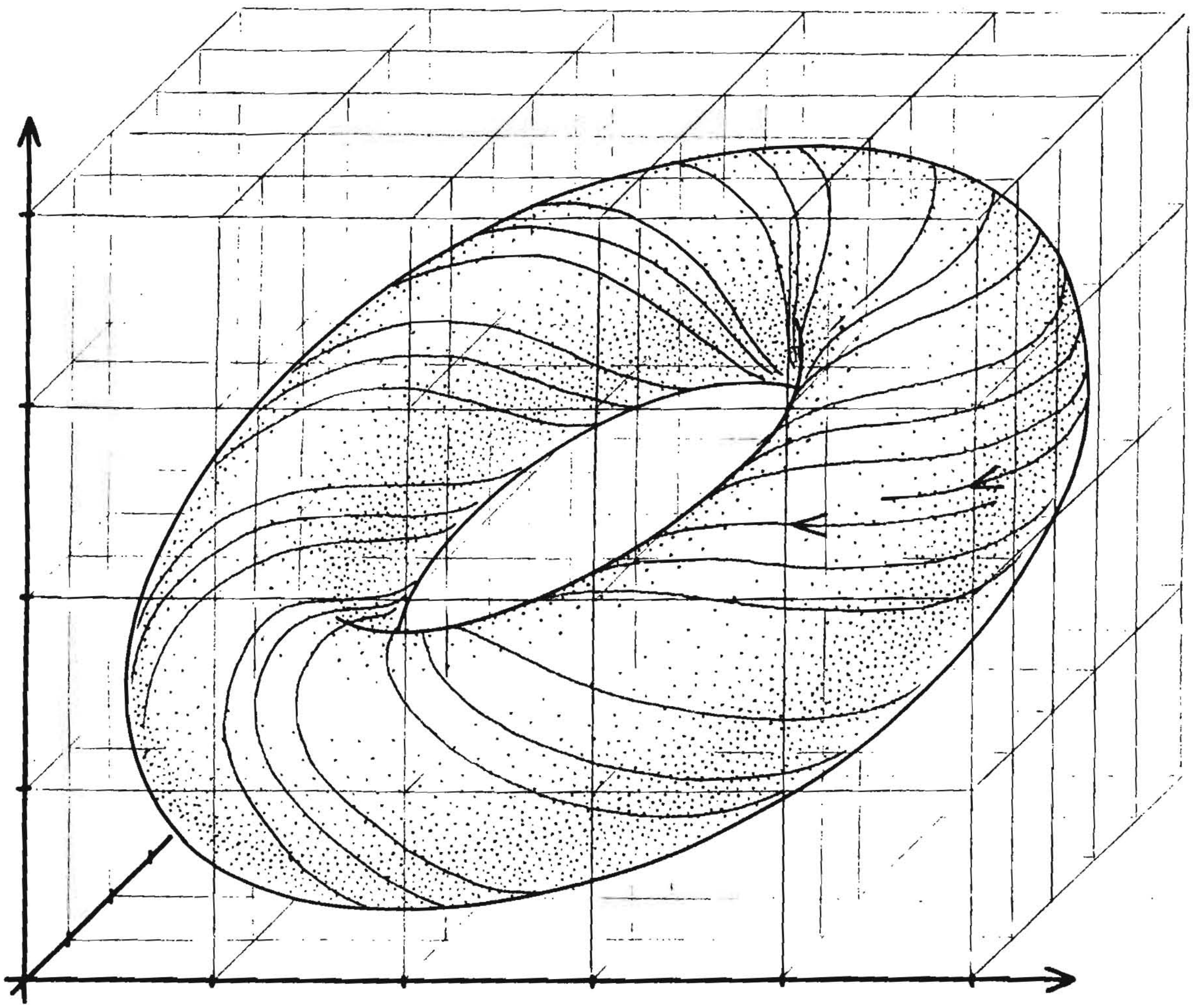


1.3.11. This is what a periodic trajectory looks like in 3D.



1.3.12. Choosing one parameter from the three coordinates, this preferred parameter may be recorded along the periodic trajectory. The time series of these data is again a periodic function.

But in higher dimensional phase portraits, other special trajectories may be found.



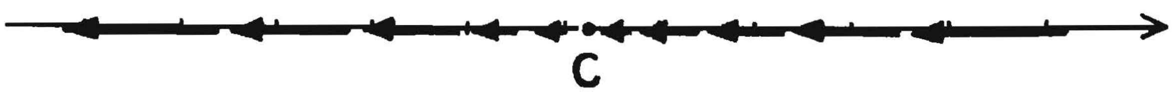
1.3.13. In the three-dimensional state space, imagine a torus (doughnut-like surface, see Volume 0 for details) with an infinitely long coil of wire wrapped endlessly around it, but never crossing itself or piling up. This can occur as a trajectory of a dynamical system, as we shall see in Section 4.4. It is called a *solenoidal* or *almost periodic trajectory*. An application is discussed in Chapter 5.

This finishes our list of special types of trajectory. Their significance in applications depends on the fact they they occur as limit sets, as we describe in the next section.

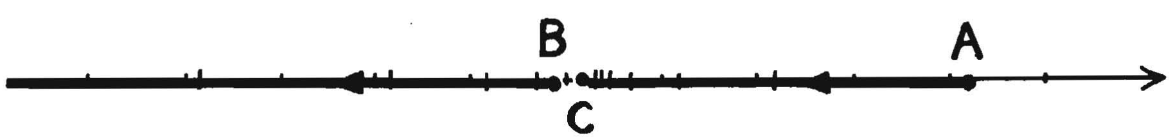
1.4 ASYMPTOTIC APPROACH TO LIMIT SETS

The second step in the dynamical systems quest for qualitative predictions of asymptotic behavior is the examination of the phase portrait for *asymptotic limit sets*. Let's see what this means.

We reconsider critical points in one dimension first, to start with the simplest case.

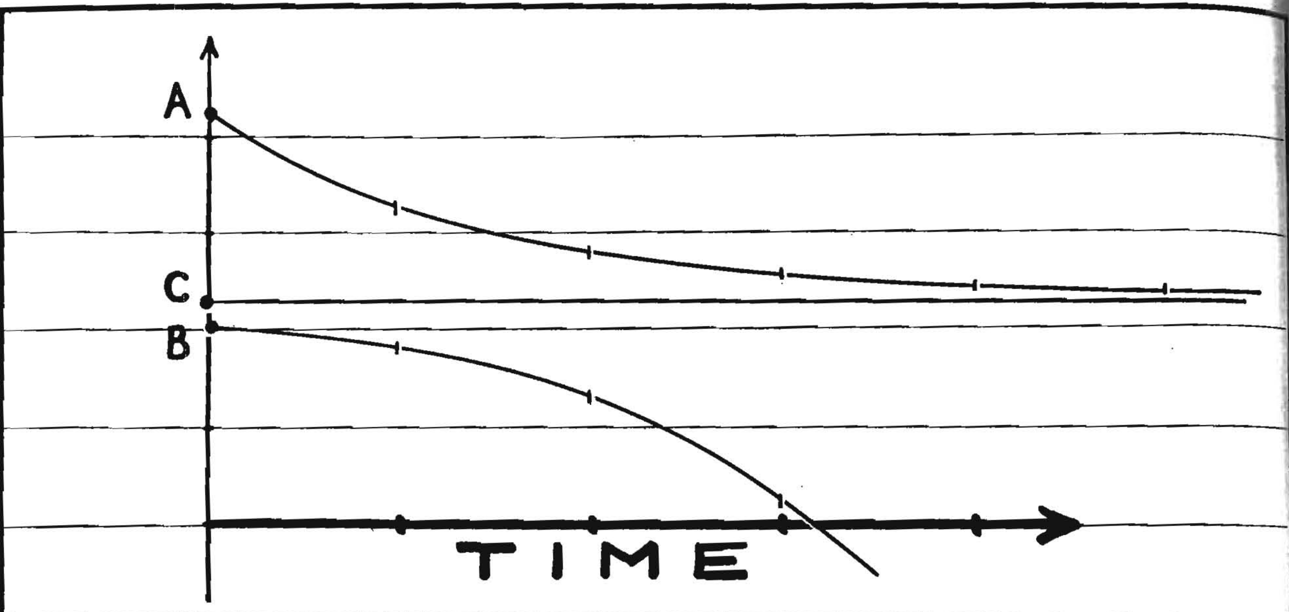


1.4.1. Here is the vectorfield on a one-dimensional state space. Recall that the point marked C is a critical point of the vectorfield. Because the vectorfield is smooth, the velocity vectors attached to points near the critical point are very short.

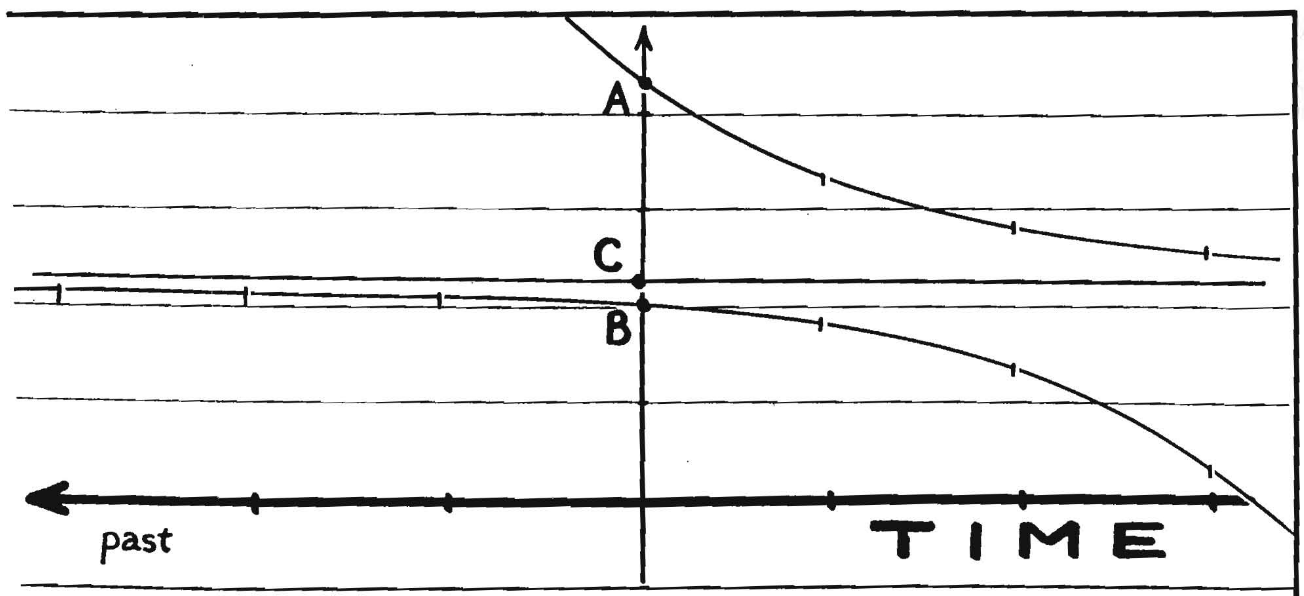


1.4.2. This is the phase portrait of the dynamical system above. Two trajectories are shown, starting at the points, A, and C. Tick marks along the trajectories indicate the positions at successive seconds. Note, again, that they are closer together near the critical point. The trajectory of C is a constant trajectory, piled up on the critical point. As time marches on, the trajectory of A gets ever closer to the point C. As it gets closer, it slows down. It gets closer and slower indefinitely, and approaches the critical point *asymptotically*. That is, it takes forever to reach C. We say that C is the *limit point* of the trajectory through A.

This trajectory approaches its limit point asymptotically.

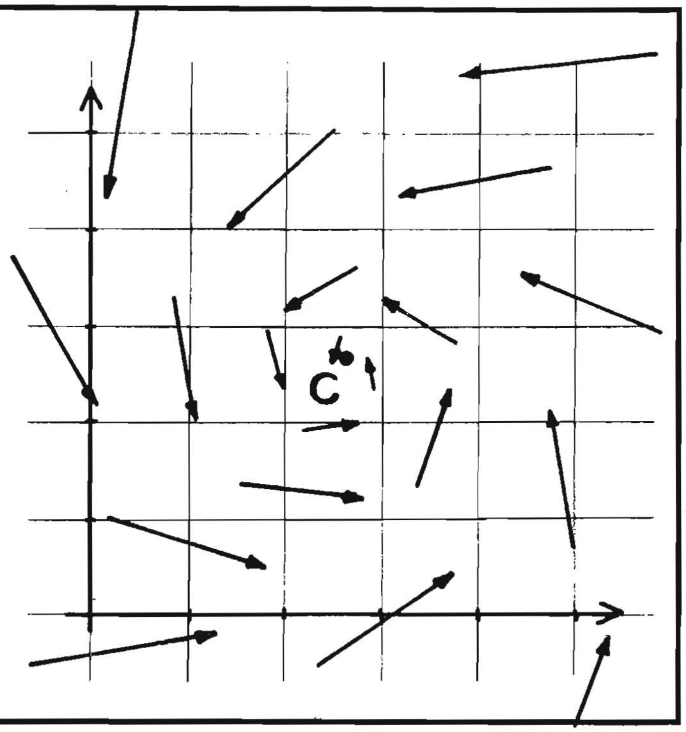


1.4.3. These are the time series corresponding to the three trajectories of the preceding phase portrait. The graph (time series) of the trajectory of C is a horizontal line, a constant function of time. This represents the constant trajectory of the critical point. The graph of the trajectory of A is descending to the right toward the horizontal line. It approaches this line *asymptotically,* as time increases to the right. The horizontal line is the *asymptote* of the graph of the trajectory of point A.

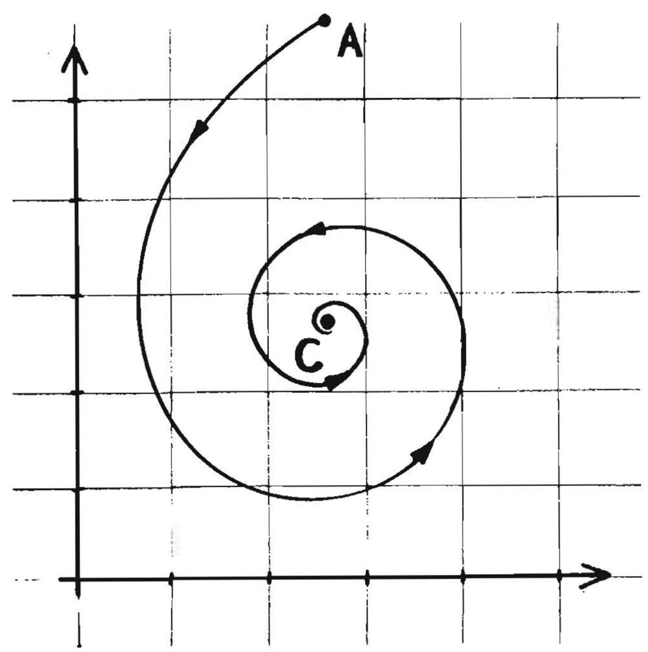


1.4.4. The graph of the trajectory of B similarly approaches asymptotically to the horizontal line, but going *backwards in time.*

Now let's look at these ideas in a two-dimensional context.

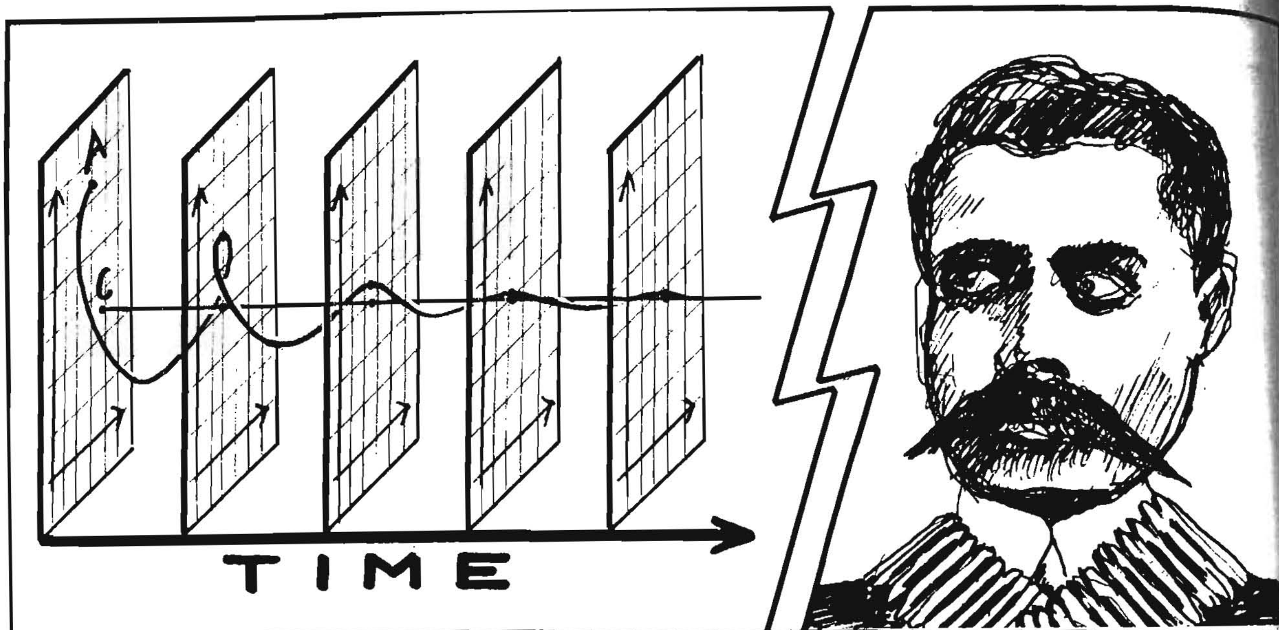


1.4.5. This is a garden-variety vectorfield in the plane. As in the planar example in the preceding section, the zero vector appears once, at the point C. Nearby, the vectors are short. But this vectorfield is different. The vectors spiral around the critical point. It is a critical point of *spiral type.*

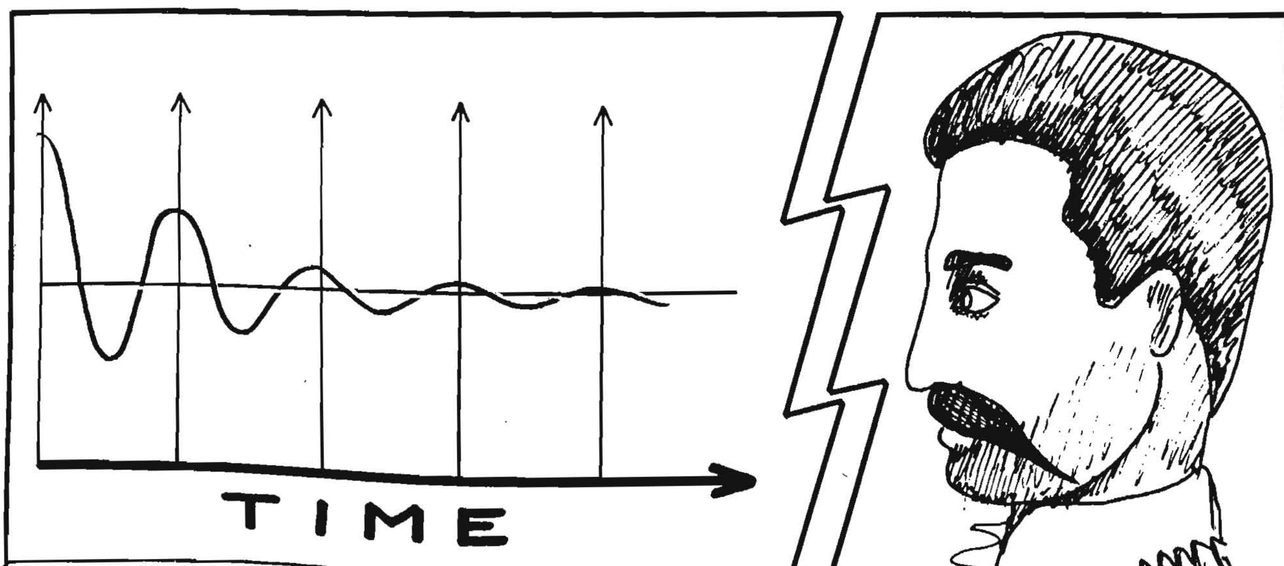


1.4.6. This is the phase portrait of the vectorfield to the left. The trajectory of the critical point is again piled up on the critical point. It is a constant trajectory. The other nearby trajectories, such as the one through the point marked A, spiral around the critical point, getting closer and closer. They approach this point *asymptotically,* slowing down as they close in. We say that the critical point, C, is a *limit point* of the trajectory through the point A.

This trajectory approaches its limit point asymptotically.



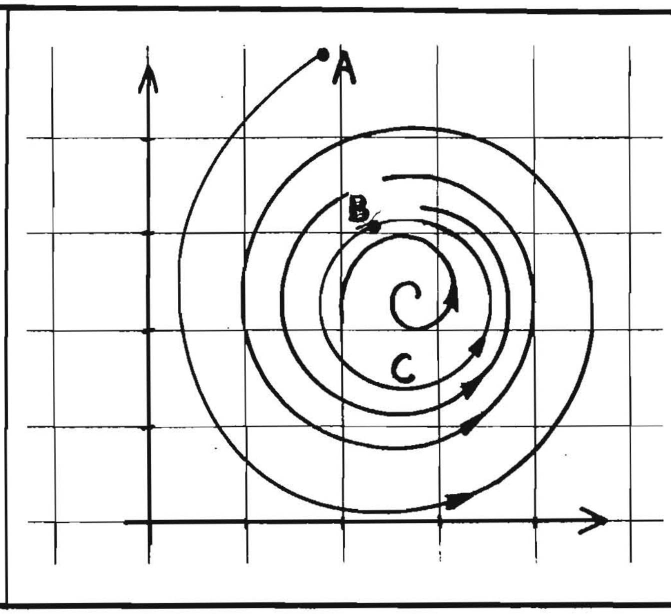
1.4.7. Here are the time series of two trajectories of this phase portrait. The time series of the constant trajectory, piled up on the critical point, *C*, is the graph of a constant (vector-valued) function. This graph is a horizontal, straight line. The time series of the nearby point, *A*, spirals around this straight line, approaching closer and closer as time moves to the right.

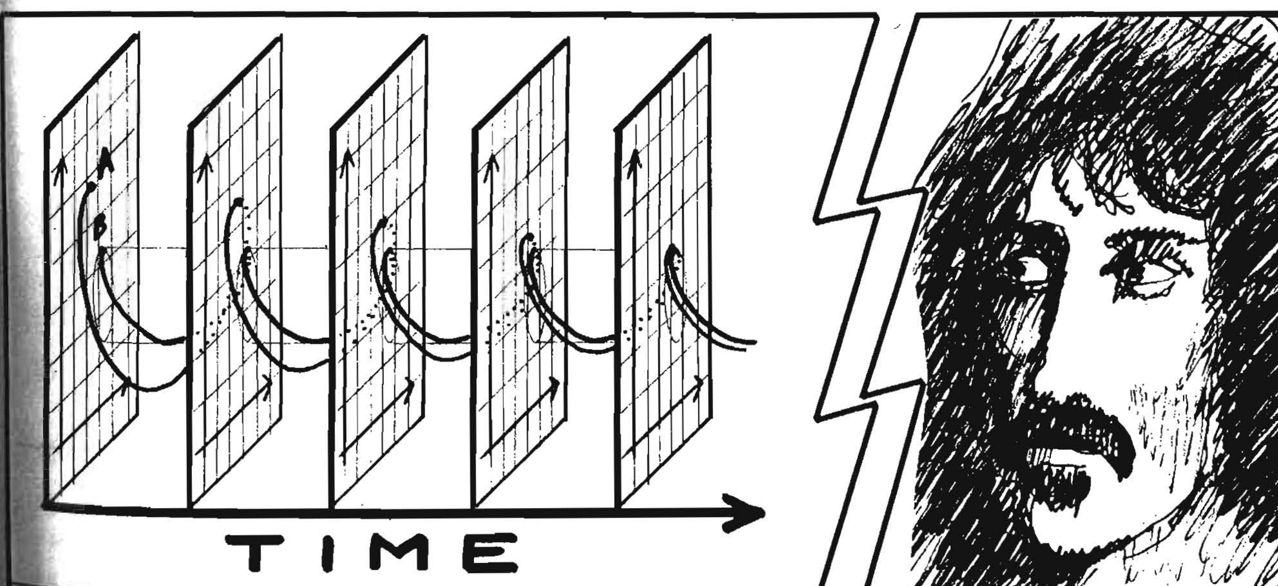


1.4.8. Choosing one of the two coordinates of the plane as a preferred parameter, the time series of this parameter along the two trajectories looks like this. The time series of the trajectory through *A* (wavy curve) approaches asymptotically toward the time series of the constant trajectory (horizontal, straight line) as time increases to the right.

In two dimensions or more, other types of special trajectories frequently occur. One of these, as we have seen in the previous section, is the *cycle*. A cycle may be the asymptotic limit set for a trajectory.

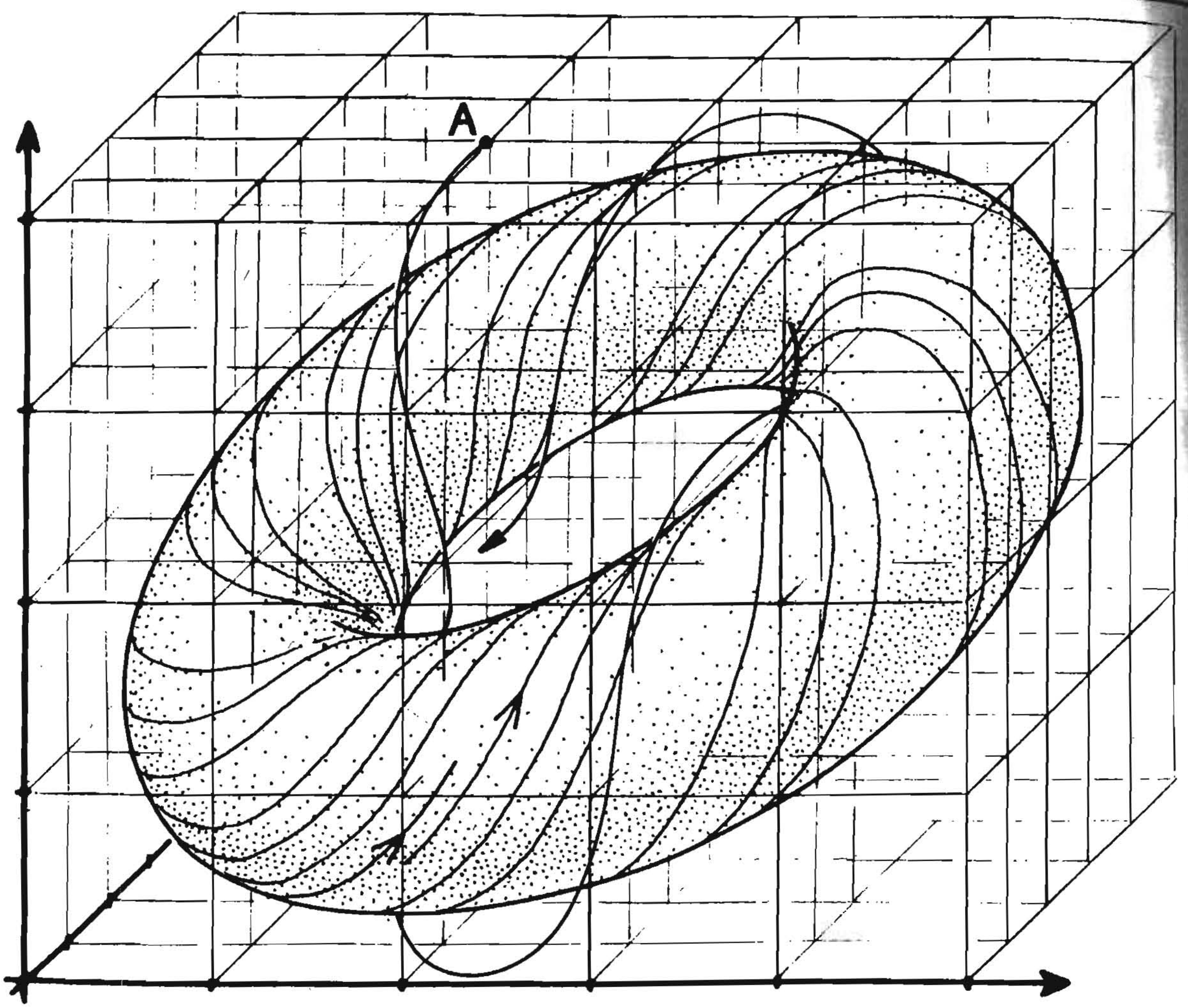
1.4.9. This is the phase portrait of a planar vectorfield with a cycle, marked *C*. A point on the cycle is marked *B*. The trajectory through *B* is a closed trajectory, winding around and around this cycle. Another trajectory is shown, through the point marked *A*. This trajectory spirals around the cycle, getting closer and closer as time goes on. We say that *C* is a *limit cycle*. It is the *limit set* for the trajectory through the point *A*.





1.4.10. This is the time series representation of the two trajectories. A horizontal cylinder is shown, extending to the right from the cycle, *C*. The time series for the closed trajectory, *B*, wraps around this horizontal cylinder. The time series for the spiraling trajectory, *A*, is wound loosely around the cylinder, and gets tighter and tighter as time increases, to the right.

Limit points and limit cycles also occur in phase portraits of higher dimensions. Further, in dimensions greater than two, other limit sets may turn up. For example, a torus can occur as a limit set, in a three-dimensional system. The solenoid, described in the preceding section, is a case in point.



1.4.11. Here, the trajectory through the point marked *A* is wound, like a loose solenoid, around the torus. As time goes on, it winds around tighter and tighter. It approaches its limit set, the torus, asymptotically.

There are many more limit sets. Some of the more exotic ones will be shown in Volume 2. But, we have not yet made our case for the importance of the geometric theory of dynamical systems. To explain what we mean by *prediction forever,* a further concept is needed: that of an *attractor.* This is an outstanding type of limit set. It represents the behavior of a system in *dynamical equilibrium, after transients die away.* So let's go on.

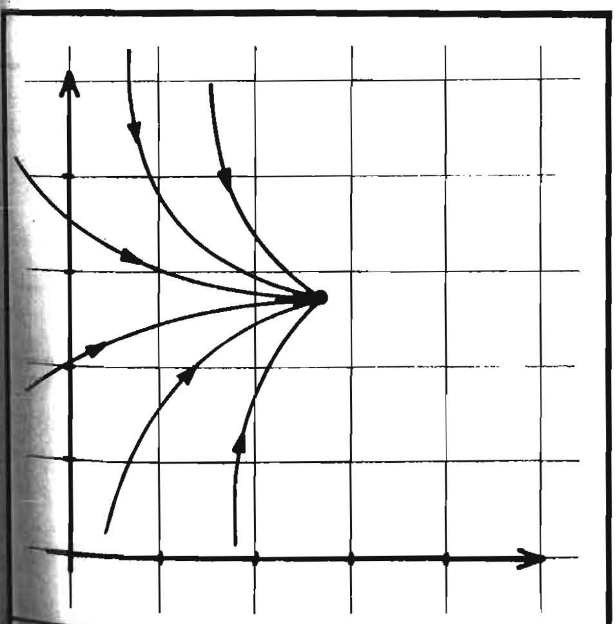
1.5 ATTRACTORS, BASINS, AND SEPARATRICES

If an organism is dropped into a prepared environment, or an experimental device is prepared in an initial state and then turned on, we expect to see a brief settling-in period, before it settles down to an observable behavior. The erratic behavior during the initial settling-in period is called the *start-up transient*. The settled-in, eventual observable behavior is the *equilibrium state* of the experiment. *Warning: Equilibrium, as used here, does not imply a static equilibrium, nor a steady state.*

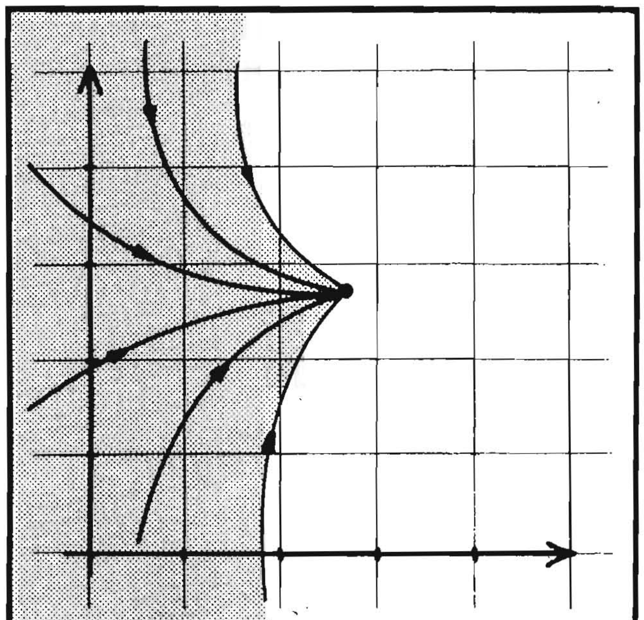
In a dynamical system modeling this experimental situation, a trajectory will model the start-up transient, while its limit set models the equilibrium state which follows. The asymptotic approach of the trajectory to its limit set models the dying away of the transient, as the system settles to its dynamic equilibrium.

For probability reasons to be explained shortly, the only equilibrium states which may be observed experimentally are those modeled by the limit sets which receive most of the trajectories. These are called *attractors*.

Here is the attractor concept, illustrated in two dimensions. The same ideas apply in all dimensions. First, we consider limit points, the simplest limit sets.

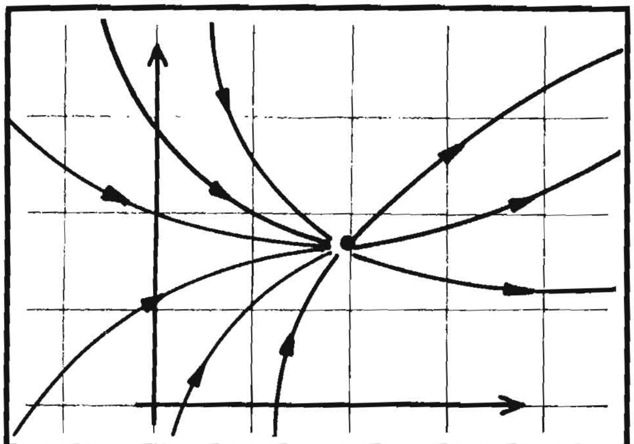


1.5.1. Suppose a dynamical system in the plane has a critical point. And let's suppose further that this critical point is the limit set of some trajectories in the phase portrait.

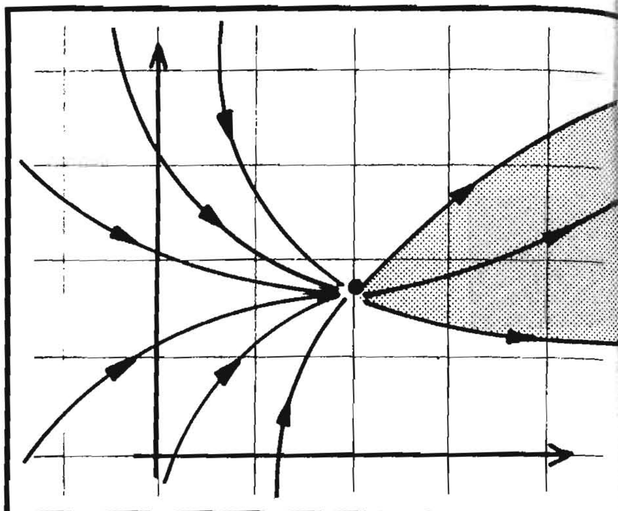


1.5.2. Now, find every single trajectory which approaches this limit point asymptotically, and color it green. The green portion of the plane is the *inset* of the limit set (that is, the critical point).

The inset of a limit set represents, in a dynamical model, all the initial states which end up in the same equilibrium state, after the start-up transient dies away.



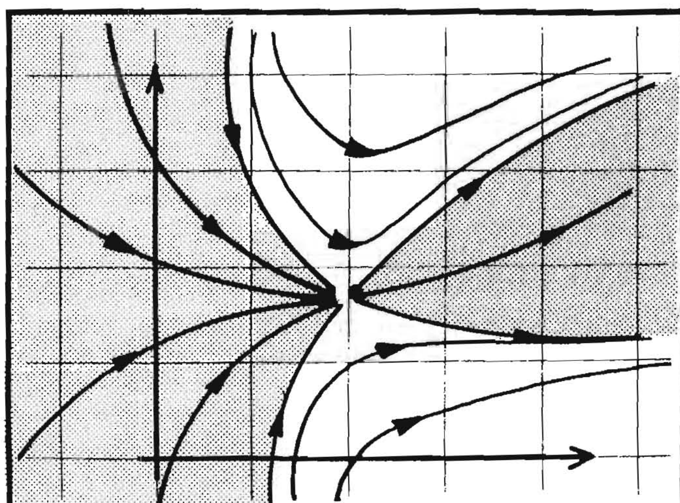
1.5.3. Other trajectories depart from the limit point. That is, *if the direction of time were reversed, these trajectories would approach asymptotically to this limit point.* Restoring the direction of time to normal again, we say these departing trajectories have the critical point as their *alpha-limit set.*



1.5.4. Now find every trajectory which has this critical point as its alpha-limit set, and color it blue. The blue portion of the plane is the *outset* of the limit set (that is, the critical point.)

Sometimes we say *omega-limit,* in place of just plain limit. Then omega-limit set refers to the future asymptotic behavior, while alpha-limit set refers to the past. The trajectory goes from alpha-limit to omega-limit.

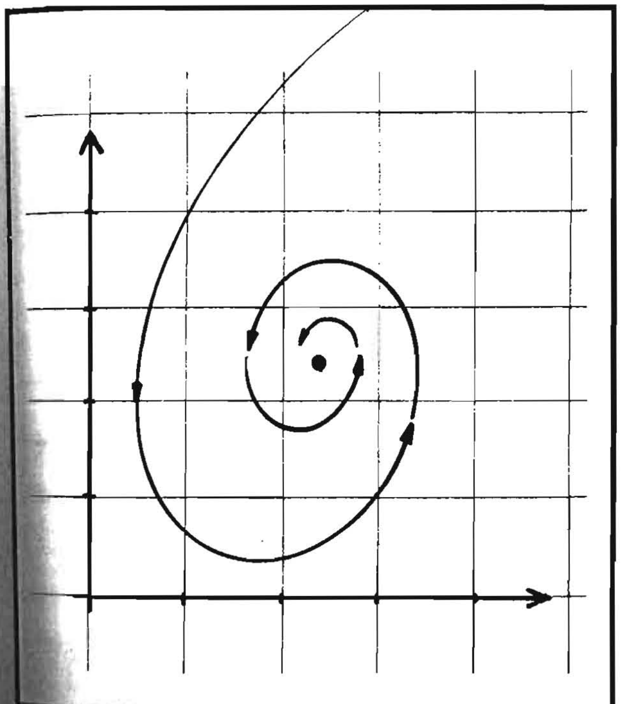
Warning: Some trajectories neither arrive nor depart at a critical point, although they may pass closely by!



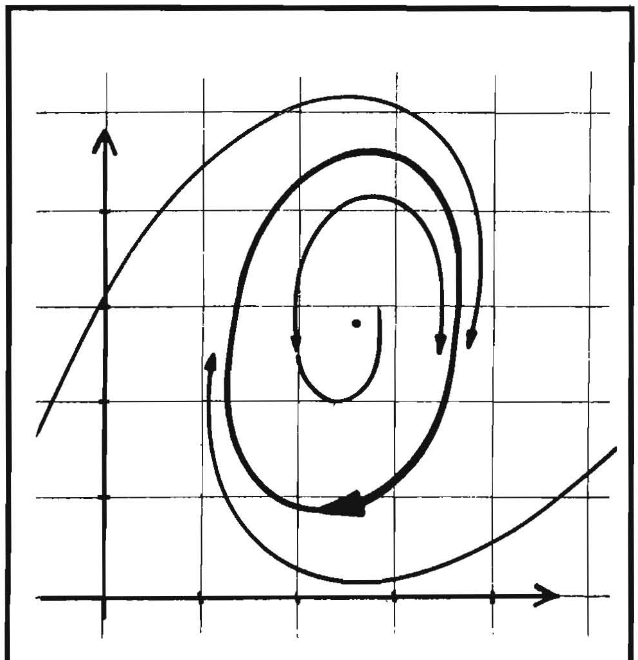
1.5.5. Here we see the same dynamical system in the plane. Both the inset and the outset are colored-in. The black trajectories start near the inset, and go towards the limit set for a while. When they get near the limit point, they feel the influence of the outset, and turn aside. Following the outset, they disappear into the distance.

When a trajectory flies by, it is on its way somewhere. Either it has an omega-limit set elsewhere in the phase portrait, or it departs from the state space, never to be seen again. Similarly, it came from somewhere else. Thus, every trajectory may belong simultaneously to the outset of one limit set (its alpha-limit set) and to the inset of another (its omega-limit set.) In two dimensions, cycles may be limit sets. Limit cycles have insets and outsets, too. (So do the other limit sets, in higher dimensions.)

What if all nearby trajectories are arriving?



1.5.6. Here is a limit point in two dimensions. Every nearby trajectory is arriving. The Inset contains an open disk around the limit point. Every initial point in this disk is captured by the limit point, when its transient dies away. If an initial point is chosen at random from all the points in the state space, the probability of it asymptotically approaching this limit point, instead of some other, is positive. (In fact, it is 100% in this particular example.) This limit point is an *attractor*. As it is a point, and represents static equilibrium, it is also called a *static attractor*.



1.5.7. This is the phase portrait of a different dynamical system in the plane. It has a limit cycle. Again, the inset of this limit set is as large as possible. Except for the solitary red point in the center, which is a critical point, every single initial state evolves to the same limit set. The inset includes an open annulus (ring) around the limit cycle. The probability that an initial state, chosen at random from among all the initial states in the state space, will end up at this limit cycle is positive. (In fact, it is 100% in this particular case.) This limit cycle is an attractor. As it is a cycle, and represents a periodic equilibrium, it is also called a *periodic attractor*.

An *attractor* is a limit set with an *open inset.* That is, there is an open neighborhood of the limit set within its inset. (Refer to Volume 0 if *open set* is unfamiliar.)

Of all limit sets, which represent possible dynamical equilibria of the system, the attractors are the most prominent, experimentally. This is because the probability of an initial state of the experiment to evolve asymptotically to a limit set is proportional to the volume of its inset. We will say that a limit set is *probable* if the volume of its inset is a positive number, instead of zero. Open sets have positive volume, although not every set of positive volume is an open set. Attractors have open insets, so they are probable. They are experimentally discoverable. Other limit sets may be probable, without being attractors in the strict sense of the preceding paragraph. They are called *vague attractors.* This is synonymous with *probable limit set.* These are also experimentally discoverable. Limit sets which have *thin insets* (that is, that have probability zero) are *non-attractors.* They are experimentally insignificant. These are called *exceptional limit sets,* or synonymously, *improbable limit sets*[2].

The inset of an attractor is called its *basin.* In a typical phase portrait, there will be more than one attractor. The phase portrait will be divided into their different basins. The dividing boundaries (or regions) are called *separatrices.* In fact, any point which is not in the basin of an attractor belongs to a separatrix, by definition.

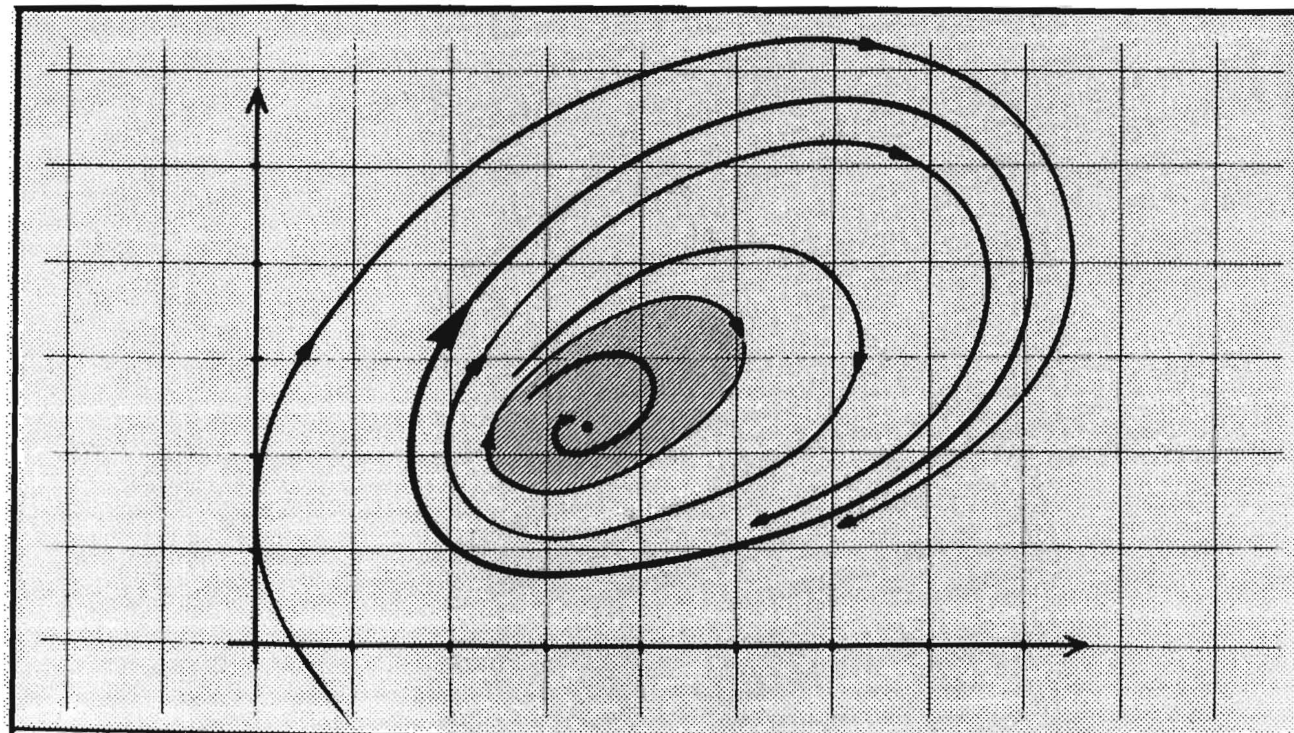
Here are some examples of attractors, basins, and separatrices, in two dimensions. The same concepts apply in three or more dimensions, but are harder to visualize.

COLOR CONVENTION:

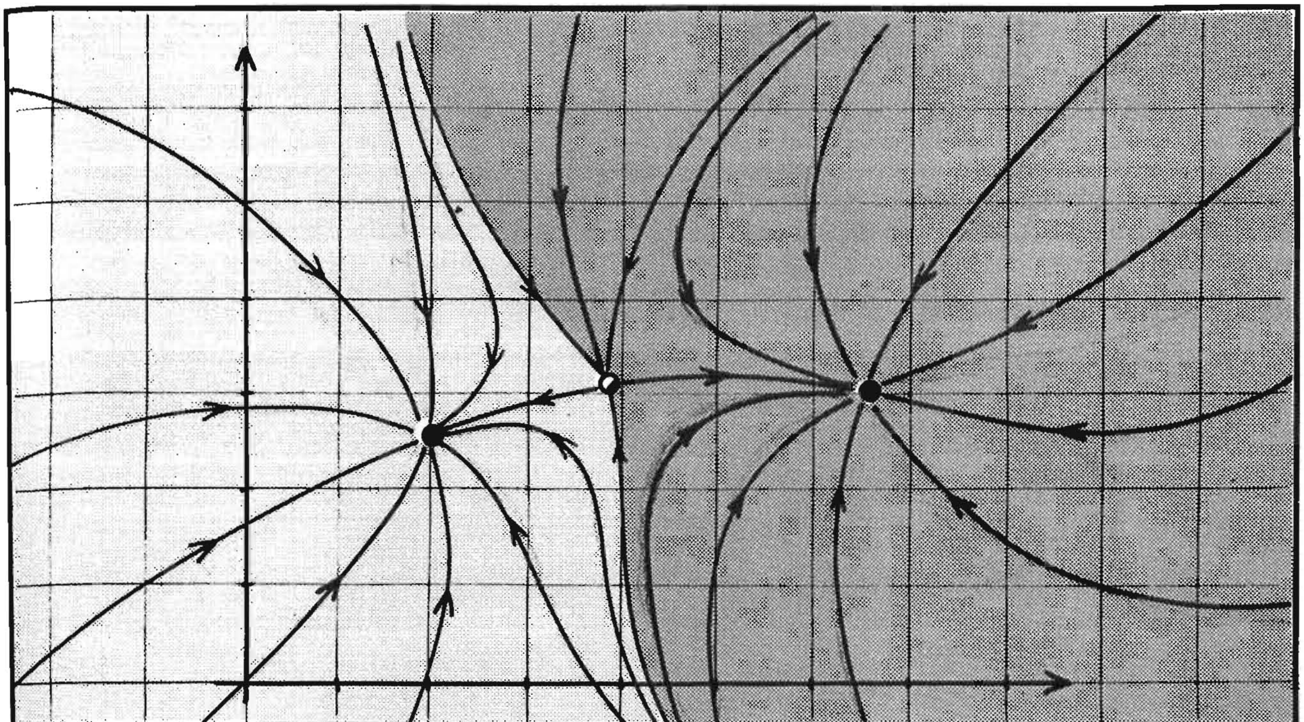
Attractors = red

Basins = blue

Separatrices = green



1.5.8. In this example, there are two attractors: a point and a cycle. A third limit set, a cycle, comprises the separatrix.



1.5.9. Here, also, there are two attractors: both points. A third limit set, also a point, is not an attractor. It is an exceptional limit set. The inset of this exceptional limit set comprises the separatrix.

This reveals the pattern of the general case: the separatrix consists of all points not in a basin. Every point tends to a limit set. If its limit set is an attractor, it belongs to a basin. So, if it belongs to the separatrix (and therefore not to a basin), it must tend to a non-attractor. Thus: ***the separatrix consists of insets of exceptional limit sets.***

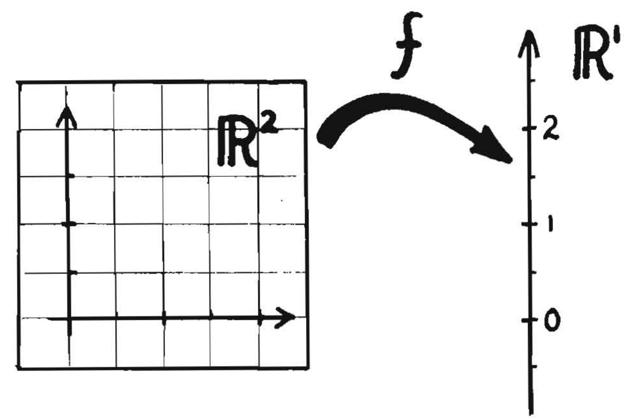
The preceding examples are artificial, made up to illustrate the concepts. But, we are overdue for some more meaningful examples. So, at this point, let's turn to gradient systems—a rich source of simple examples, based on a geometrical construction.

1.6 GRADIENT SYSTEMS

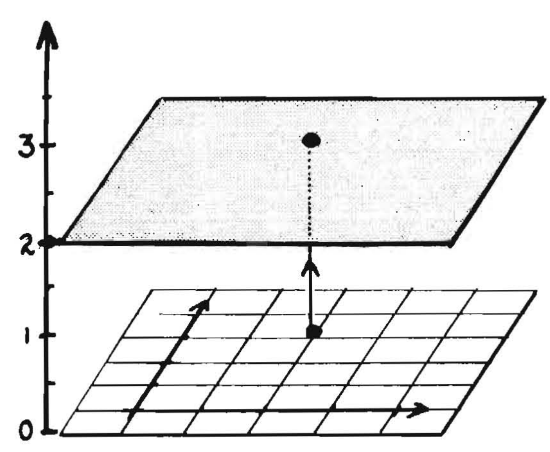
The *gradient* operation of vector calculus provides dynamical systems (vectorfields) of an especially simple type called a *gradient system.* In these, there is an auxiliary function, called the *potential function.* The velocity vectorfield is simply the *gradient vectorfield* of this potential function.

This section develops a typical example of gradient dynamics in the plane.

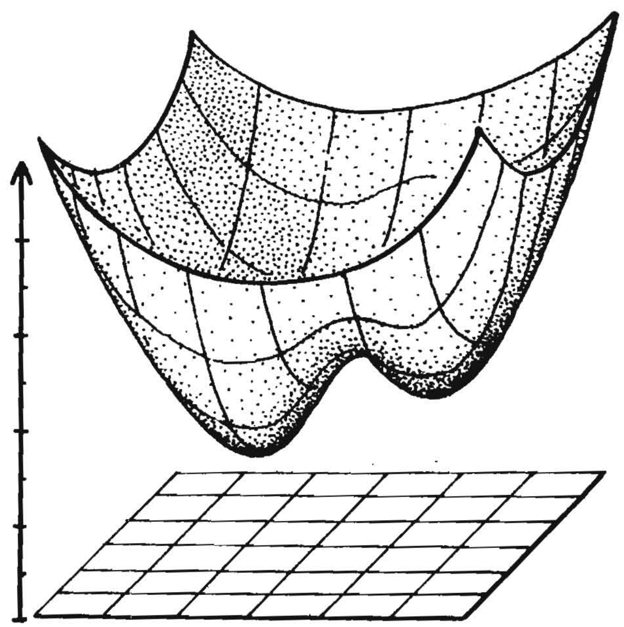
1.6.1. The state space, in this example, is the plane. The potential function is a function from the state space to the real number line. To each point in the state space, it assigns a real number. This number is the *potential* of the corresponding state. In an application of this scheme, this potential would presumably be observable, or deducible from observations.



1.6.2. Represent this function as a graph in three-dimensional space. The state space is the horizontal coordinate plane. From each point in this plane, move vertically a distance equal to the potential of that point—up if the potential is a positive real number, down if it is negative.

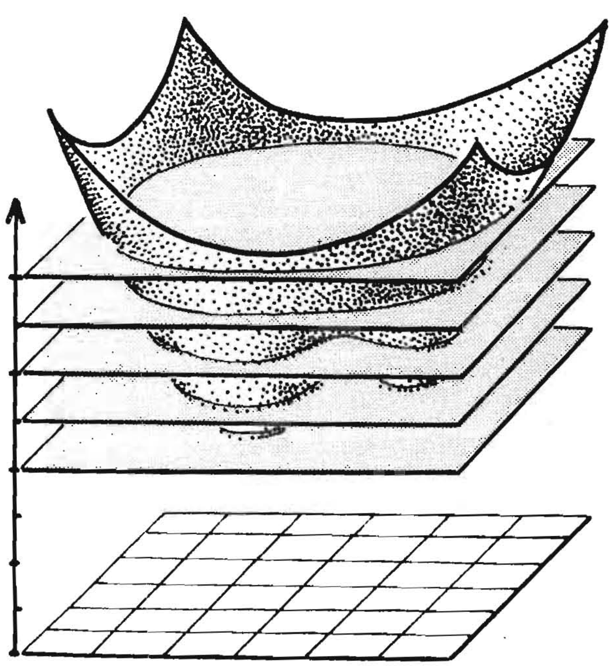


The graph of a potential function on a planar state space is a surface in three-dimensional space, called the *potential surface.* We may think of this as a landscape.



1.6.3. For the sake of definiteness, let's choose a particular potential function, and visualize it as a potential surface. This one, for example, has two valleys, with a saddle ridge in between.

An alternate representation of a function is its *contour map.* Let us represent our exemplary potential this way.



1.6.4. At regular intervals along the vertical axis, draw horizontal, cutting planes. Mark the potential surface with a red curve, where each cutting plane cuts through the surface. These are called the *level curves* of the surface.

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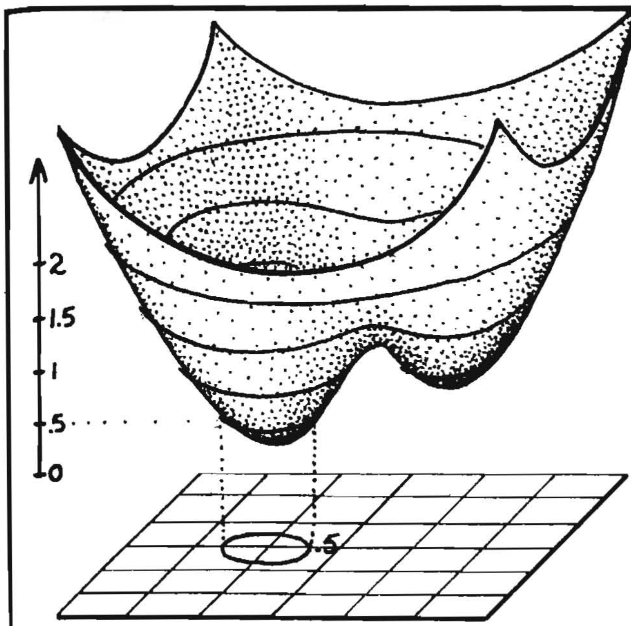
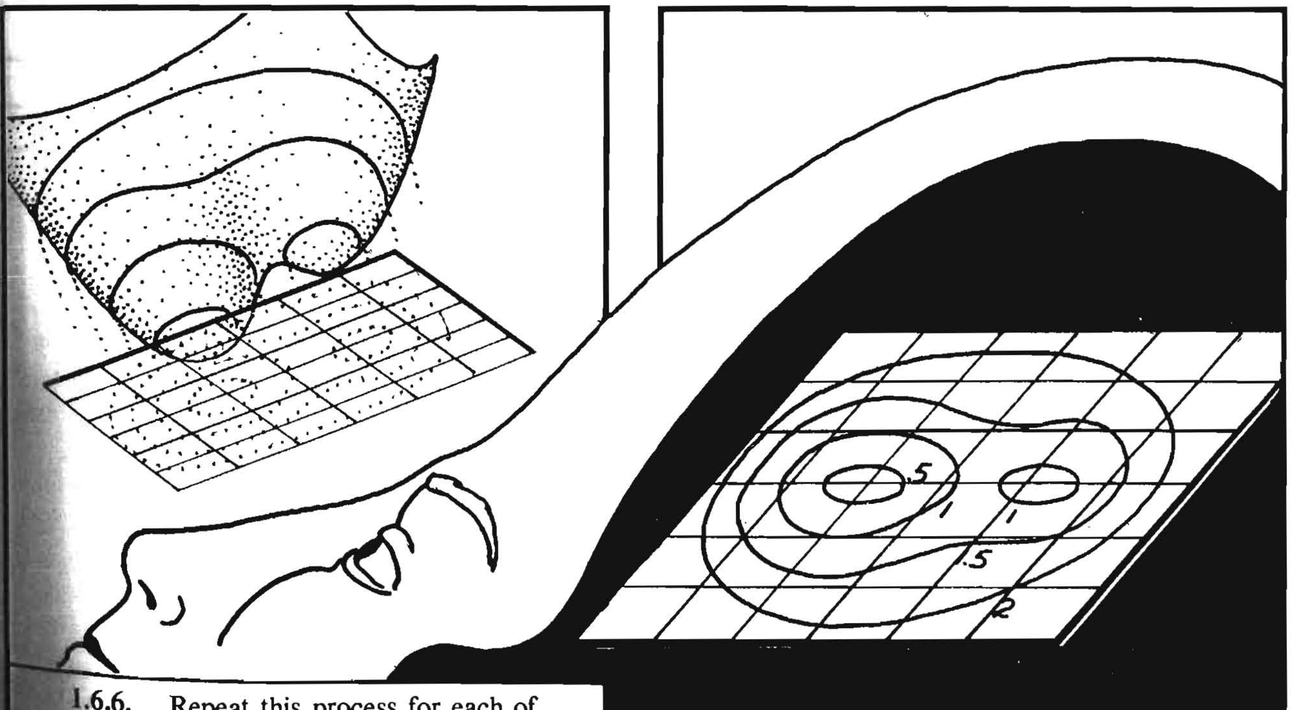


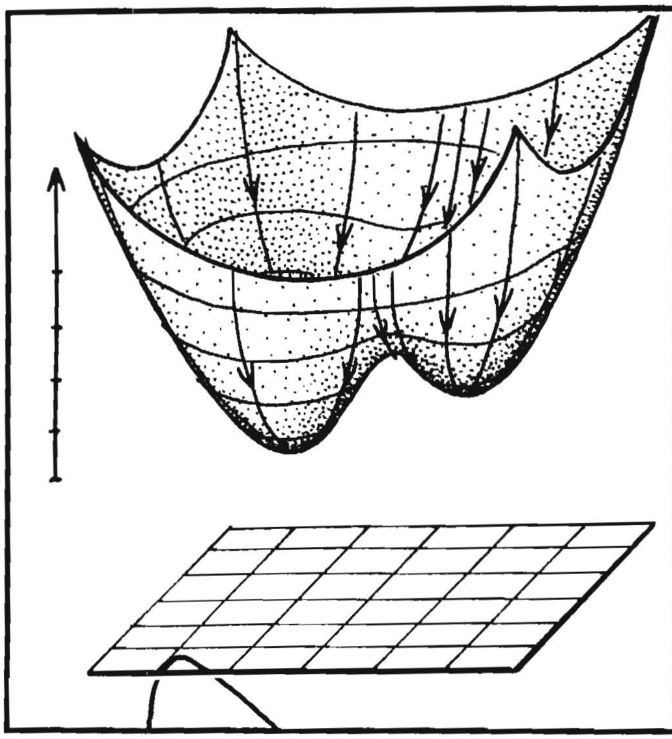
1.6.5. Next, each level curve is projected onto the horizontal coordinate plane. This curve in the state space, also called a *contour curve,* contains every state with the same value of the potential function. Over this curve, the potential surface has a constant height. Label this contour curve with its common value of the potential.



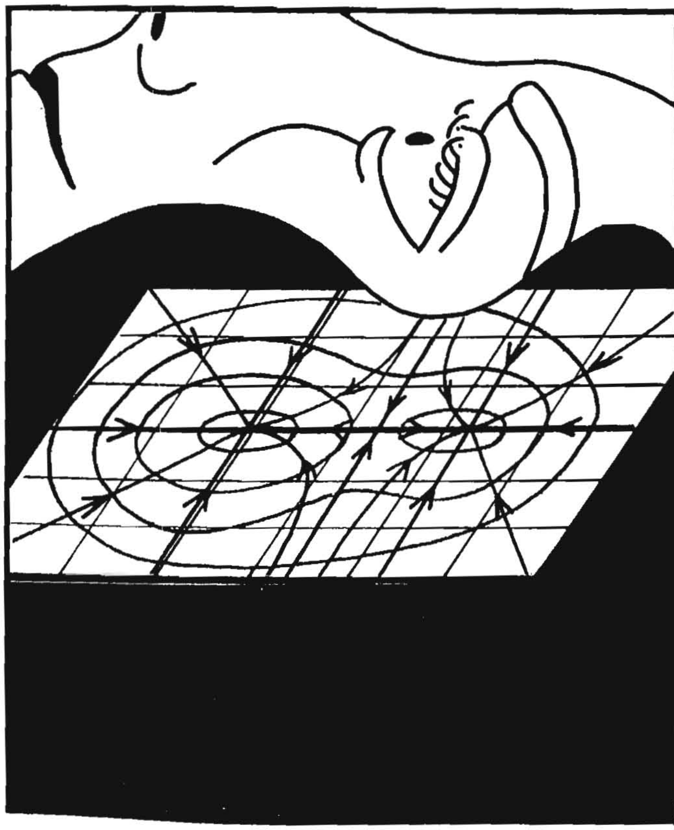
1.6.6. Repeat this process for each of the level curves. The result is the contour map of the potential surface, drawn in the horizontal coordinate plane. This is essentially what you would see, looking up from far below at the potential surface, with level curves drawn upon it.

1.6.7. Finally, extract the state space, with the contour map drawn within it, from the three-dimensional context of the graph. This is the alternate representation.

The gradient dynamical system for this particular potential function is derived as follows.



1.6.8. Sprinkle the potential surface with a fine mist of blue ink. Droplets will run down the fall-lines, that is, the routes of steepest descent. Suppose that the speed of a droplet is exactly the steepness of the slope.



1.6.9. Viewed from far below, the blue droplets appear to move over the contour map, at right angles to the contours.

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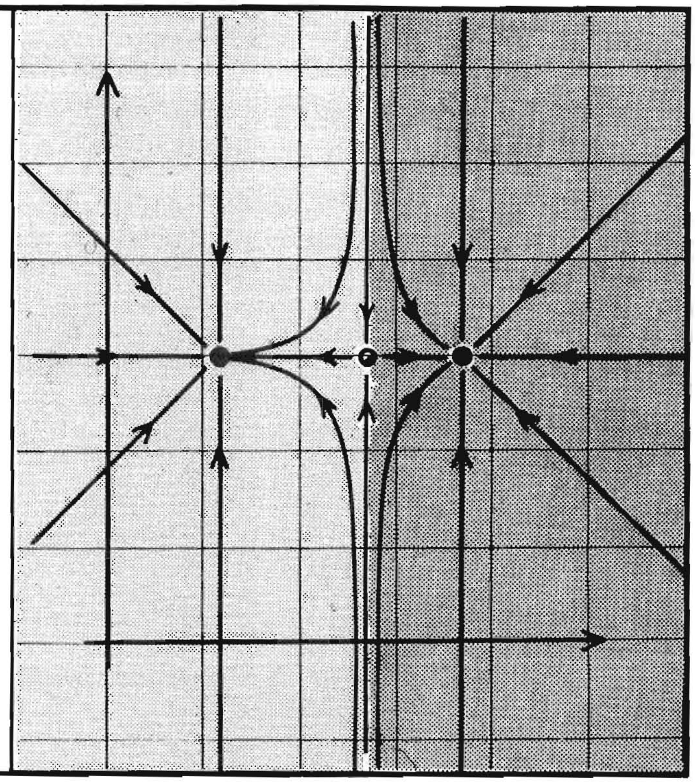
The blue curves, perpendicular to the contours, together with the parameter of time along each, comprise the phase portrait of the gradient dynamical system.



1.6.10. In this example, the gradient phase portrait has two basins, with a point attractor in each. Between them is a limit point of saddle type, or *saddle point,* corresponding to the saddle on the ridge between the two valleys in the landscape. The inset of the saddle point consists of the two green trajectories. This inset is also the separatrix, dividing the state space into the two basins.

Gradient systems, generally, are much like this example. Their limit sets are generally equilibrium points. A limit cycle is impossible in a gradient system, as you cannot go steadily downhill and still return to your starting point (except in an Escher print). Although gradient systems are useful in some elementary physics problems, their usefulness in general applications is severely limited by the lack of limit cycles. The next chapter will show why limit cycles are so important.