
KTU
NOTES

MODULE - 4

Connectivity And Planer Graph

Cut - Sets And Cut - Vertices

Cut ~~edge~~ sets

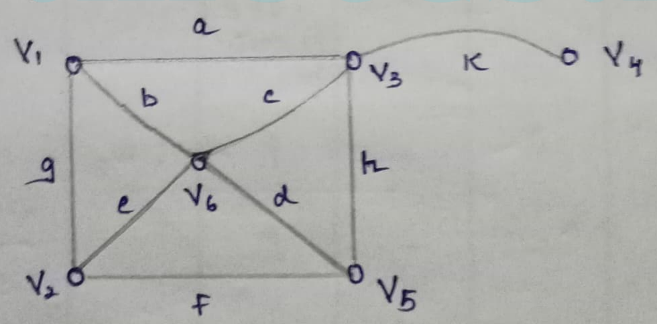
An edge 'e' of graph G is a cut-edge if G - e is disconnected.

~~Cut - set~~

~~An edge 'e' of a graph G is~~

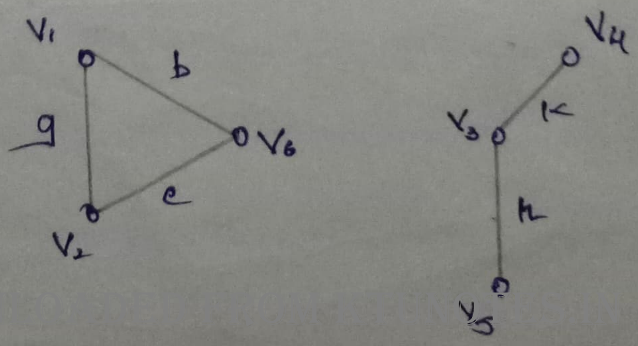
In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these

eg:- 1.



$\{K\} \longrightarrow$ cut set & cut - edge

Consider the set $\{a, c, d, f\}$. Removal of $\{a, c, d, f\}$ from G. cuts it into 2.

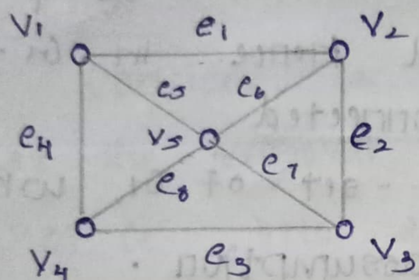


$\therefore \{a, c, d, f\}$ is a cut set of G.

Similarly $\{a, b, g\}$, $\{a, b, e, f\}$, $\{d, h, f\}$ are cut-sets of G

~~But~~ Edge $\{k\}$ alone is also a cut-set

But $\{a, c, d, h\}$ is not a cut-set, since its proper subset $\{a, c, h\}$ makes the Graph Disconnected.

2: 

Consider the Graph and write 5 cut-sets

Ans :- 1) $\{e_1, e_5, e_4\}$

2) $\{e_2, e_3, e_7\}$

3) $\{e_4, e_8, e_3\}$

4) $\{e_5, e_6, e_7, e_8\}$

5) $\{e_1, e_6, e_7, e_3\}$

Some Properties of A Cut-set

1. Every edge of a tree is a cut-set

Proof

By property of a tree, every edge of a tree T is a cut edge.

$\therefore T - \{e\}$ is disconnected also

$\therefore$ Every edge of a tree is a cut-set

2. Every cut-set in a graph G must contain atleast one branch of every spanning tree of G.

Proof

Let F be a cut-set in G and T be any spanning tree of G. If F does not contain any edge of T, then there will be a unique path between any pair of vertices in T and hence in G - F

ie G - F is connected

ie F is not a cut-set of G, which is contradicting our assumption.

∴ every cut-set in a graph G must contain atleast one branch of every spanning tree.

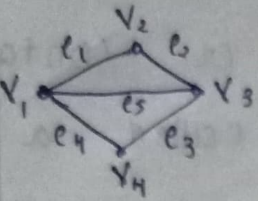
3. In any connected graph G, any minimal set of edges consisting of atleast one branch of every spanning tree of G is a cut-set.

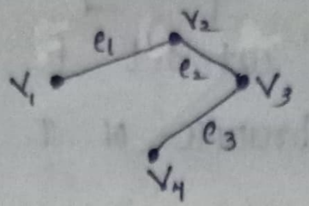
Proof

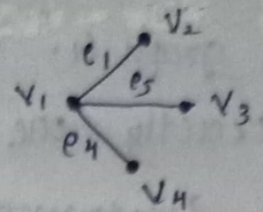
Let F be a minimal set of edges consisting of atleast one branch of every spanning tree T of G. Then G - F will remove atleast one edge from every spanning tree of G

$\therefore$ G - F is disconnected

$\therefore$ F is a cut-set of G







$\{e_1, e_3, e_4\}$ - cut set

4 Every Circuit has even number of edges in common with any cut-set.

Proof

Consider a cut-set F in a graph G. Let the removal of F partition the vertices of G into 2 subsets V_1 & V_2.

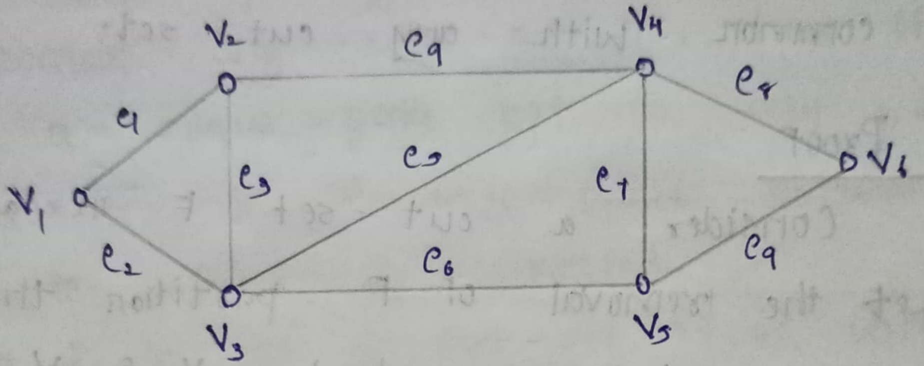
Consider a circuit C in G. If all the vertices in C are entirely within vertex V_1 or V_2, the number of edges common to F and C is 0 (An even no.)

If not, some vertices in C are in V_1 and some in V_2, we traverse back and forth b/w the sets V_1 & V_2 as we traverse the circuit. Because of closed nature of circuit, the number of edges we traverse b/w V_1 and V_2 must be even. Since every edge in F has one end in V_1 & other end in V_2, the number of edges common.

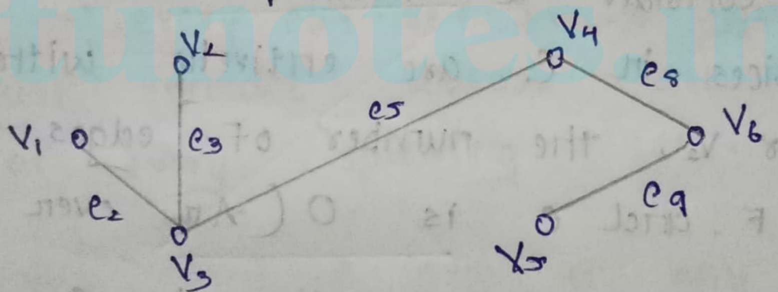
Fundamental Cut - Set

Let T be a spanning tree of a connected graph G. A cut set F of G containing exactly one branch of T is called a Fundamental cut-set of G w.r.t T.

eg 1:-



Consider a spanning tree T of G by



Fundamental cut-sets of G w.r.t T are

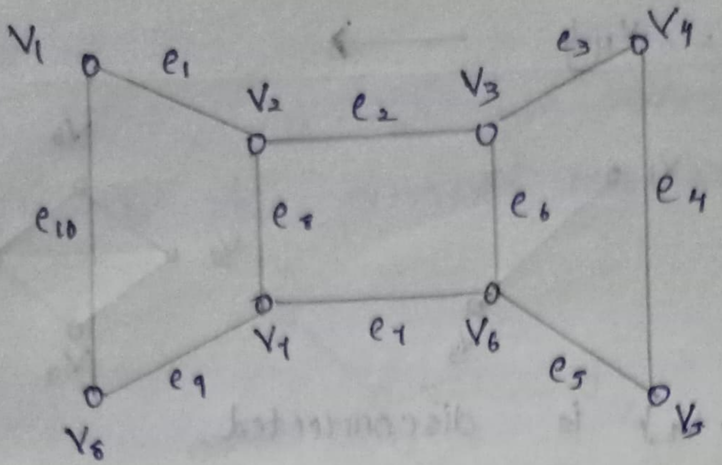
1) $\{e_1, e_3, e_4\}$ 2) $\{e_1, e_2\}$ 3) $\{e_4, e_5, e_6\}$

4) $\{e_8, e_7, e_6\}$ 5) $\{e_6, e_7, e_9\}$

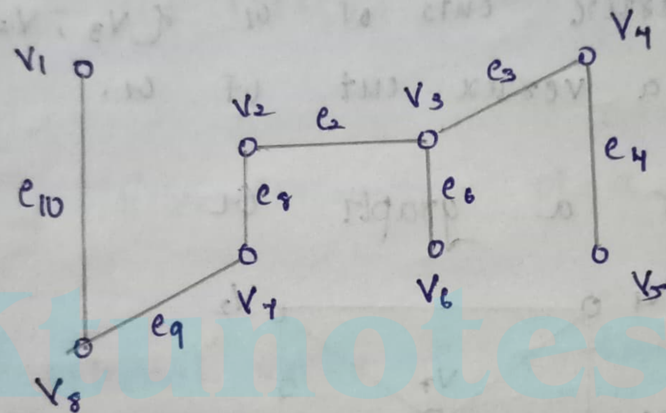
Result

Every connected graph of order n has (n-1) distinct fundamental cut-sets corresponding to any spanning tree of G.

eg 2 :-



Draw a spanning tree T of G and write all the fundamental cut-set of G w.r.t T



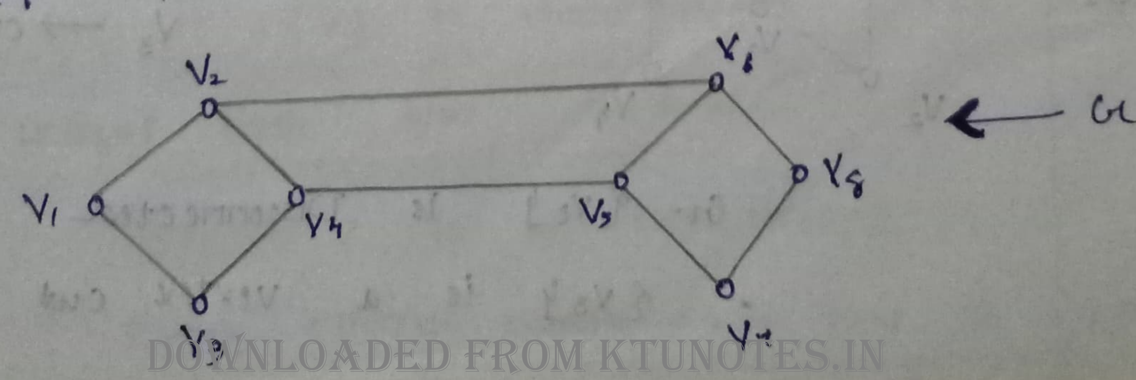
1) $\{e_1, e_{10}\}$ 2) $\{e_2, e_7\}$ 3) $\{e_4, e_5\}$ 4) $\{e_5, e_6, e_7\}$

5) $\{e_3, e_5\}$ 6) $\{e_1, e_8, e_7\}$ 7) $\{e_1, e_9\}$

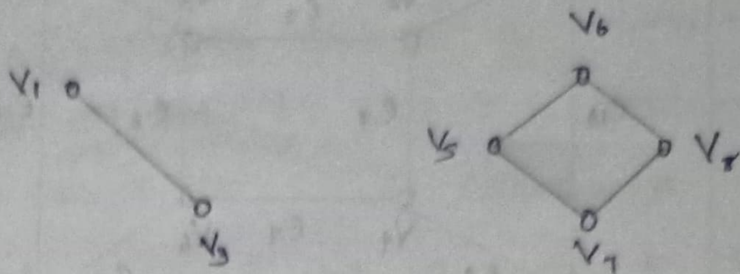
Vertex - Cut (cut Vertices)

A subset W of the vertex-set (V) of a graph G is said to be a vertex set of G if G - W is disconnected.

eg:-



$G - \{v_2, v_4\} \longrightarrow$

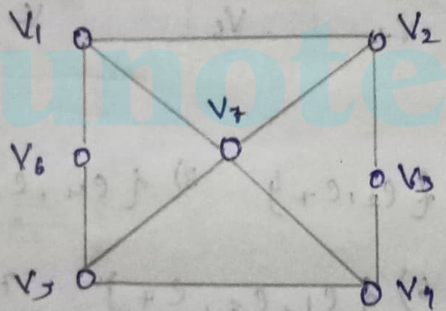


$G - \{v_2, v_4\}$ is disconnected

$\therefore \{v_2, v_4\}$ is a vertex cut of G

Similarly $\{v_6, v_7\}$, $\{v_6, v_5\}$, $\{v_3, v_4, v_6\}$, $\{v_2, v_3\}$ are vertex cuts of G. $\{v_3, v_4, v_5, v_7\}$ is not a vertex cut of G.

eg 2:- Consider a graph G

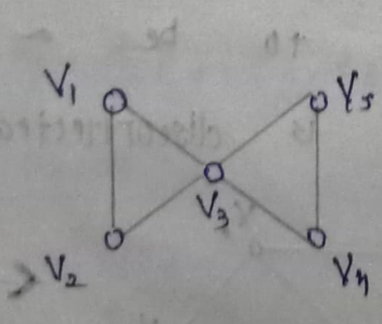


Write any 5 vertex - cut of G

1) $\{v_1, v_5\}$ 2) $\{v_2, v_4\}$ 3) $\{v_2, v_7, v_5\}$

4) $\{v_1, v_7, v_4\}$ 5) $\{v_3, v_6, v_7\}$

eg 3:-



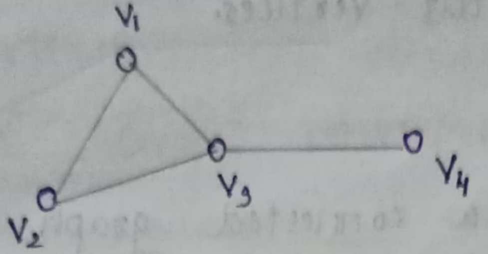
$V_3 \longrightarrow$ cut vertex

$G - \{v_3\}$ is Disconnected

$\therefore \{v_3\}$ is a vertex cut

Result

If v is a cut-vertex of G, then $\{v\}$ is a vertex cut of G.



$G - \{v_3\}$ is Disconnected

$\therefore$ v_3 - cut vertex $\{v_3\}$ vertex cut

Theorm

Every internal vertex of a tree is a cut-vertex

Proof

An internal vertex of a tree is a vertex with degree ≥ 2.

Let 'v' be an internal vertex of a tree.

Since $d(v) \geq 2$. there are atleast 2 neighbours for v in tree T. Let u and w are neighbours of v. Then uvw is a u-w path in G.

We know that, in a tree $\exists$ a unique Path b/w any 2 vertices.

$\therefore$ $T - \{v\}$ is disconnected since theare is a unique Path b/w u & w.

$\therefore$ v is a cut vertex

$\therefore$ Every internal vertex of a tree is a cut vertex.

Theorm

Every connected graph on 3 or more vertices has atleast 2 vertices which are not cut-vertices.

Proof

Let G be a connected graph with order $n \geq 3$. Let T be a spanning tree of G. Then by a theorm, $\exists$ atleast 2 pendant vertices in T.

Note that no pendant vertex can be a cut-vertex of a graph. Let 'v' be one of the pendant vertices in T.

Then $T - \{v\}$ is also a connected graph. Hence T - v is a spanning tree of the graph G - v.

$\therefore$ G - v is connected and ~~have v~~ hence v is not a cut-vertex of G.

ie Pendant vertices in a spanning tree of a graph G will not be the cut vertices of G as well.

$\therefore$ $\exists$ atleast 2 pendant vertices which are not cut vertices.

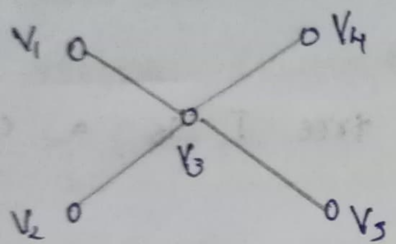
Connectivity in Graphs

Separable Graph

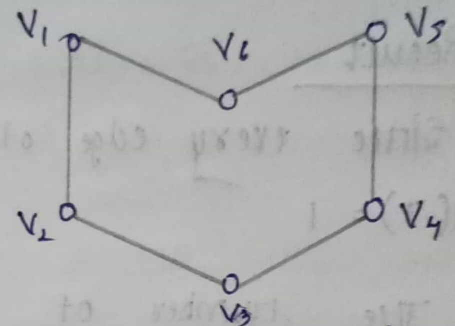
A ~~that~~ connected graph which has a cut vertex.

A Graph which is not separable is called non-separable ~~is called~~ A graph.

eg



Separable Graph

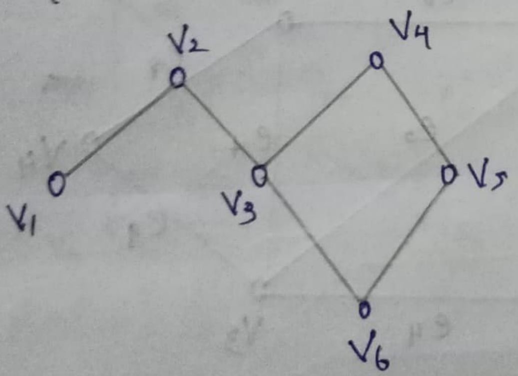


Non-Separable Graph

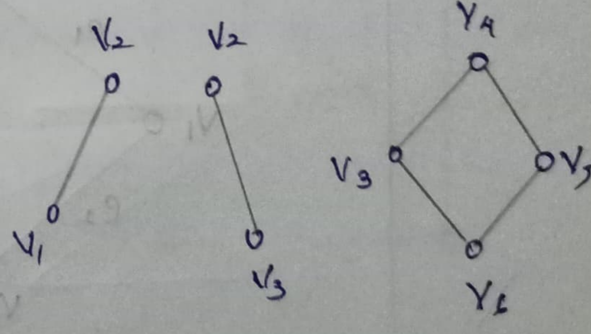
Block

A non-separable sub graph of a separable graph is called Block of Graph G.

eg:-



Separable Graph



Block of G

Edge Connectivity of a Graph

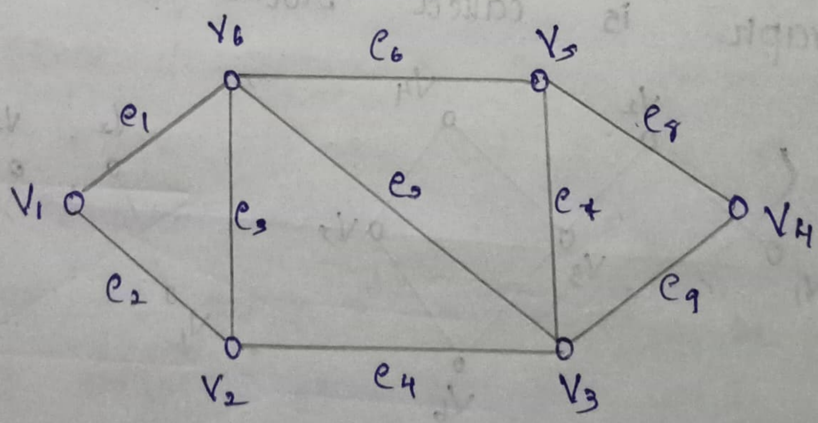
Let G be a graph having k components. The min number of edges whose deletion from G increases the number of components of G is called edge connectivity of G. The edge connectivity of G is denoted by $\lambda(G)$.

Result

1. Since every edge of a tree T is a cut-set $\lambda(T) = 1$

2. The number of edges in the smallest cut-set of a graph is its edge connectivity

List any 5 different cut sets and find the edge connectivity



Cut set of G are

1) $\{e_1, e_2\}$ $\{e_8, e_9\}$ $\{e_6\ e_5\ e_4\}$

$\{e_2, e_3, e_4\}$ $\{e_8\ e_1\ e_5\ e_6\}$ $\{e_1 e_3\ e_5\ e_6\}$

$\lambda(G) = 2$

Vertex Connectivity Of A Graph

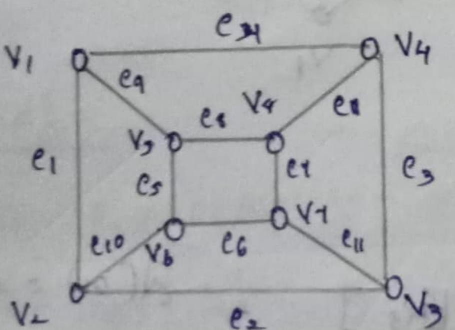
Let G be a graph. The minimum no. of vertices whose deletion from G is increases the no. of components of G is called the vertex connectivity of G. The vertex connectivity of G is denoted by $K(G)$.

Result

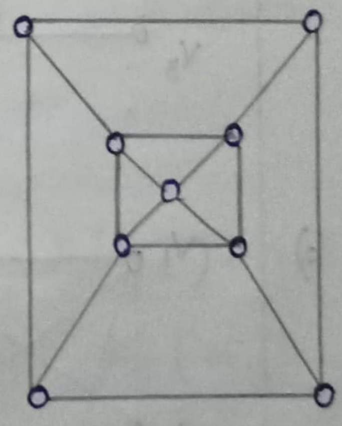
Every Separable Graph is Having $K(G)$

Find edge connectivity and vertex connectivity of the following graph

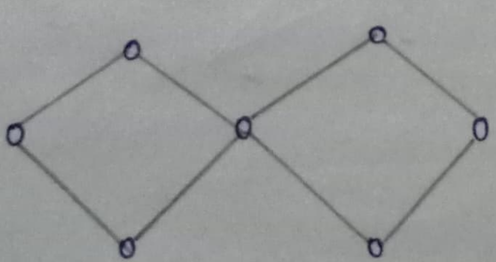
1)



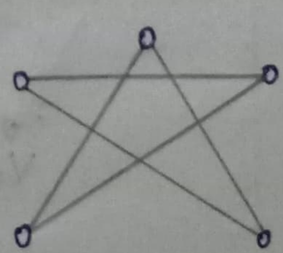
2)



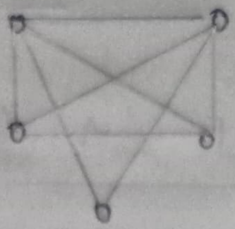
3)



4)



5)



1) $\lambda(G) = 3$ $\quad K(G) = 3$

2)

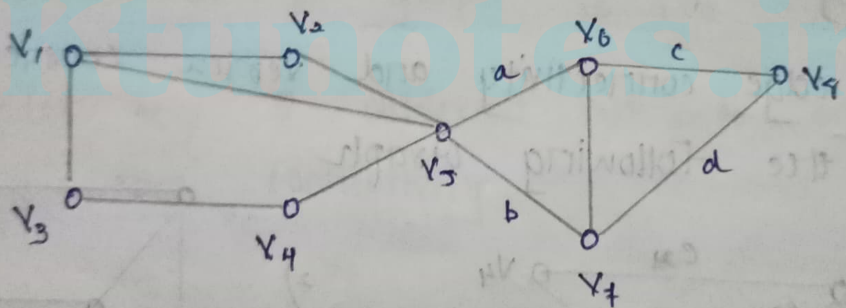
3)

4)

5)

Find the vertex connectivity and edge connectivity of the following graphs.

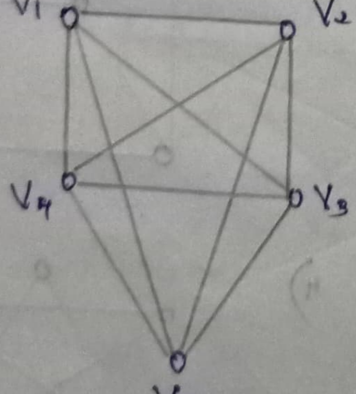
1)



$\lambda(G) = 2$

$K(G) = 1$

2)

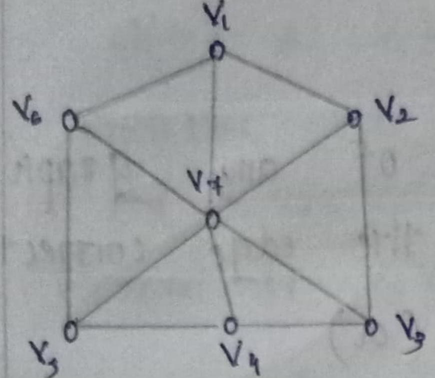


$[K_5]$

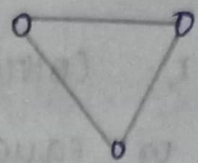
$\lambda(G) = 4$

$K(G) = 4$

3)

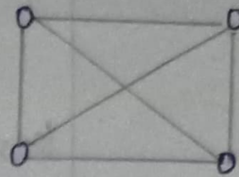


4)



$\lambda(K_3) = 2$

5)



$\lambda(K_4) = 3$

$\lambda(G) = 3$

$K(G) = 3$

For every complete graph K_n on n-vertices has $\lambda(K_n) = n-1$

$K_3 \longrightarrow$ 2-connected $K_4 \longrightarrow$ 3-connected, etc

In a graph G, the graph is K-vertex connected means deletion of that vertices makes a graph with one vertex

K_n has no vertex cut also

Result

1. For a complete graph K_n, $\lambda(K_n) = n-1$
2. For a complete graph K_n, $K_1[K_n] = n-1$
3. K_n has no vertex cut (cut vertices)

Theorm

Vertex connectivity of any graph G is less than or equal to the edge connectivity

ie $K(G) \leq \lambda(G)$

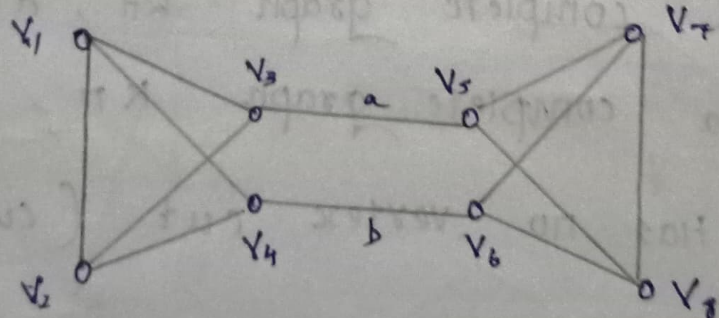
Proof

Let $\lambda = \lambda(G)$ and $K = K(G)$.

Since λ is edge connectivity of G, there exists a cut sets in G with λ edges. Let S partition the vertices of G into subsets V_1 & V_2. By removing atmost λ vertices from V in which the edges in S are incident, we can effect the removal of S in G.

ie Max. number of vertices to the removed from G to delete λ edges from G in λ

$\therefore K \leq \lambda$ $\therefore K(G) \leq \lambda(G)$



$\lambda(G) = 2$ Partition into 2

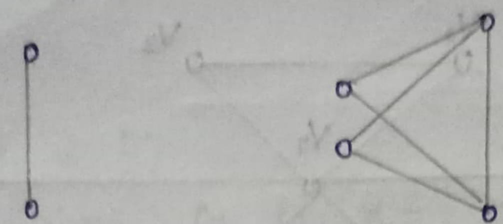
$\{V_1, V_2, V_3, V_4\}$ $\{V_5, V_6, V_7, V_8\}$

$\lambda(G) = 2$ $\{4, 6\}$

Partition into 2

$\{v_1, v_2, v_3, v_4\}$ $\{v_5, v_6, v_7, v_8\}$

Removal of $v_3 v_4$



Theorm

The maximum vertex connectivity of a connected graph G with n vertices and e-edges is $\left[\frac{2e}{n}\right]$ [integerpart of $\frac{2e}{n}$]

Proof

Let G be graph with n vertices and e-edges

By 1st theorm of graph theory,

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$\therefore$ Average degree of vertices in G is $\frac{2e}{n}$

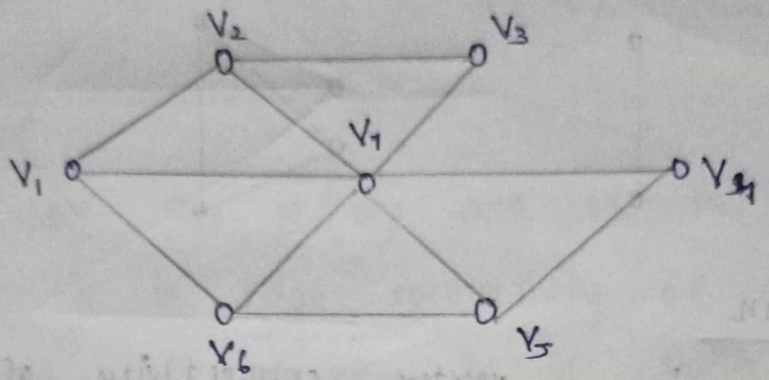
ie, Atleast one vertex in G whose degree is less than or equal to $\lfloor 2e/n \rfloor$

Also we have edge connectivity of a graph be less than or equal to its minimum degree

$$\therefore \lambda(G) \leq \left\lfloor \frac{2e}{n} \right\rfloor$$

We have $K(G) \leq \lambda(G)$ [prev. theorm]

$\therefore K(G) \leq \left\lfloor \frac{2e}{n} \right\rfloor$



$d(v_1) = 3$ $\quad d(v_2) = 3$ $\quad d(v_3) = 2$ $\quad d(v_4) = 2$

$d(v_5) = 3$ $\quad d(v_6) = 3$ $\quad d(v_7) = 6$

Avg. degree $= \left\lfloor \frac{22}{7} \right\rfloor = \lfloor 3.14 \rfloor = 3$

$\lambda(G) \leq$ min. degree of vertex in G

$\lambda(G) \leq 3$ $\qquad K(G) \leq \lambda(G) \leq 3$

1) $K(G) \leq \lambda(G)$

2) $\lambda(G) \leq$ min. degree

3) max. value of $K(G) \leq \left\lfloor \frac{2e}{n} \right\rfloor$

~~Proof~~

A Graph G is said to be K connected if its vertex connectivity is K

Theorm

A graph G is K - connected if and only if there exist atleast K disjoint paths b/w any pair of vertices in G.

Proof

Part 1 :-

Assume that G is K - connected. Then K vertices are to be removed to make G disconnected. If possible assume, there are less than K disjoint paths b/w any pair of vertices from G. This is a contradiction to the fact that G is K - connected

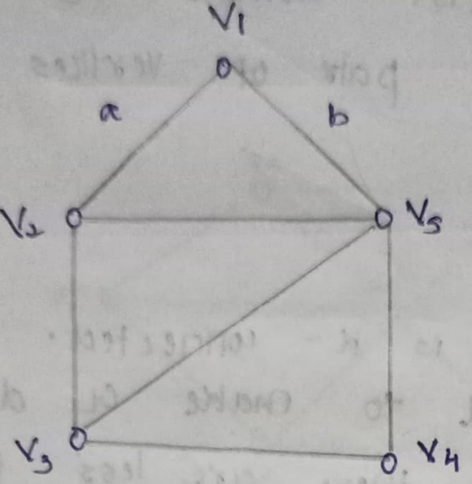
∴ There must be atleast K disjoint paths between any pair of vertices in G.

Part 2 :-

Assume that there exist atleast K disjoint paths b/w any pair of vertices u & v in G. Then note that the removal of K - vertices; one from each one of the K - disjoint paths leaves u & v in 2 different components. ~~Here o~~ More over removing less than K vertices will not leave u & v in 2 different components.

∴ G is K - connected

eg:-



$\lambda(G) \leq 2$ (min)

$K(G) \leq 2$

Here

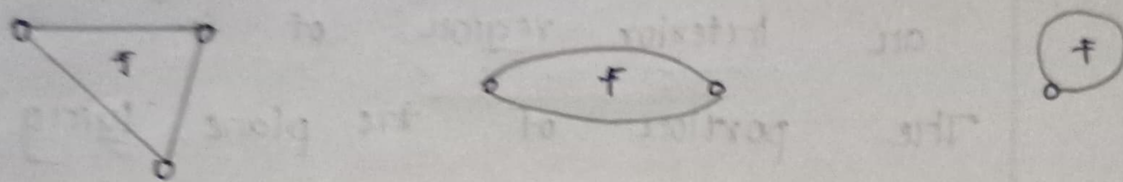
$\lambda(G) = 2$ $\{a, b\}$

$K(G) = 2$ $\{v_5, v_3\}$

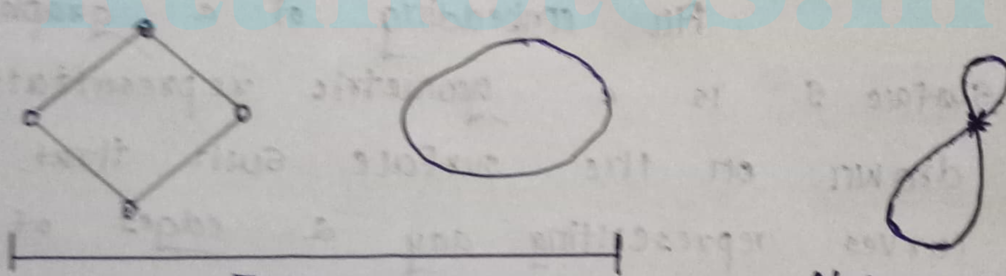
Planar Graph

A face of a graph G is the region formed by a cycle or parallel edges or loops in G. A face of a graph G is also called a region or mesh. It is denoted by 'f'.

eg:-



A Jordan curve is a non-self-intersecting continuous closed curve in the plane. cycles in a graph can be considered to be Jordan Curve.

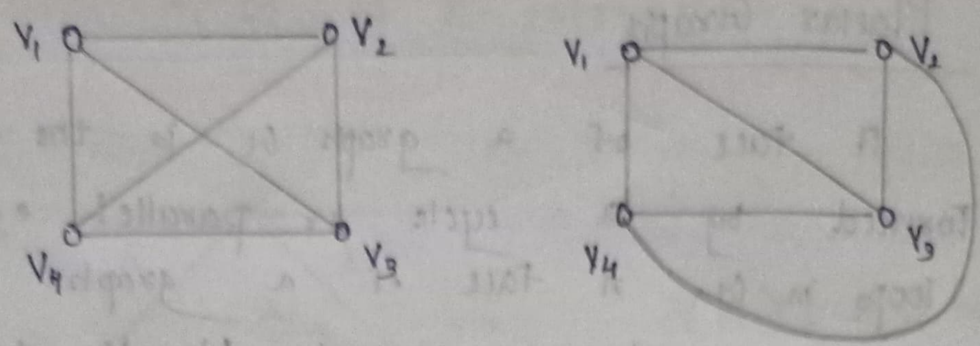


Jordan Curves

Not a Jordan Curve

Planar Graph

A Graph G is called a planar graph if it can be re-drawn on a plane without any crossovers (ie, in such a way that 2 edges intersect only at their end vertices). Such a drawing of G, if exists, is called a plane graph or planar embedding or an embedding of G.



The portion of the plane lying inside the graph G ~~&~~ embedded in a plane is called an interior region of G.

The portion of the plane lying outside the graph G embedded in a plane is called an exterior region of G.

Embedding of a graph.

An embedding of a graph G on a surface S is a geometric representation of G drawn on the surface such that the curves representing any 2 edges of G do not intersect except at a point representing a vertex of G

ie A Graph G is planar if and only if G is isomorphic to a graph which is embedded on a plane.

Non-Planar Graph

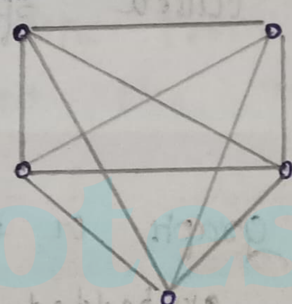
A graph which is not planar is called non-planar graph.

Intersection number of a Graph

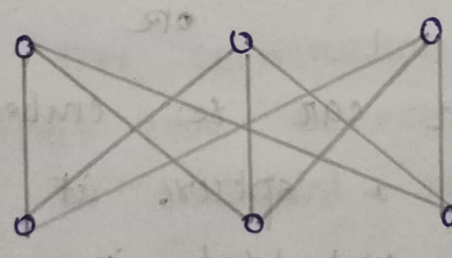
The intersection number of a Graph G is the min. no. of edges crossing when we draw G on a plane.

Examples of Non-Planar Graph

K_5



$K_{3,3}$



→ Non Planar

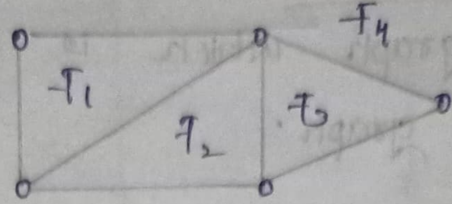
Note

Every Graph is having 2 types of Faces.

1) Interior Face [Region bounded by cycles]
2) Exterior Face

Number of exterior Face in any graph is one.

eg



Spherical Embedding

A graph G is said to be embeddable on a sphere if it can be drawn on the surface of sphere without crossing edges. Such drawing of G is called spherical embedding.

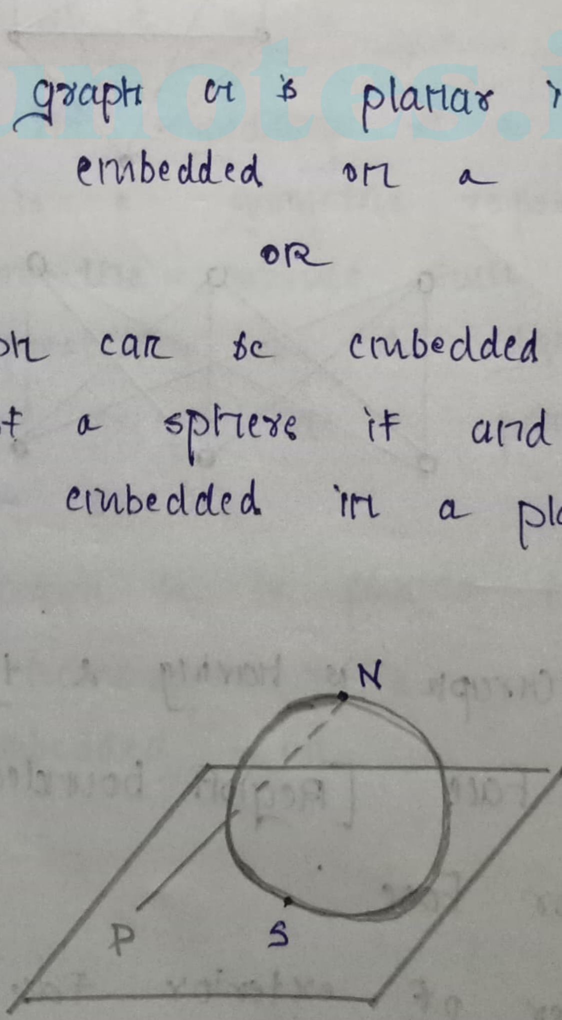
Theorm

A graph G is planar if and only if it can be embedded on a sphere

OR

A graph can be embedded on the surface of a sphere if and only if it can be embedded in a plane.

Proof



Consider the stereographic projection of a sphere on the Plane. Put the sphere on the plane and call the ~~pla~~ point of contact as s (south pole). At s, draw a straight line ⊥ to the plane and let the point where this line intersect the surface of the sphere be called N (North Pole).

Now consider to any point P on the plane there exist a unique pt 'P_0' on the sphere and vice-versa, P_0 is the pt where the straight line from a Point P to the point N intersect the surface of the sphere. Thus there is a one-one correspondance b/w the points on the sphere and points on the plane.

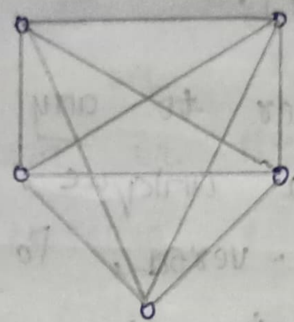
∴ From this construction, it is clear that any graph that can be embedded in a plane can also be embedded on the sphere and vice-versa.

∴ A graph G is planar if and only if it can be embedded on a surface of the sphere.

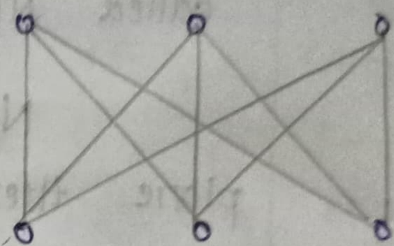
Kuratowski Graphs

The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called Kuratowski's graphs. These 2 graphs are Non-Planar.

K_5



$K_{3,3}$



Kuratowski's Theorm

A graph G is planar if and only if it has no sub-divisions of K_5 and $K_{3,3}$

ie No component of the graph is isomorphic to K_5 and $K_{3,3}$

Note

Exactly one edge each in K_5 and $K_{3,3}$ intersects with other edges while attempting to embed the graph on plane.

The intersection number of these 2 graphs K_5 & $K_{3,3}$ is 1

Properties of K_5 and $K_{3,3}$

1. K_5 & $K_{3,3}$ are non planar
2. K_5 & $K_{3,3}$ are regular. $K_5 \rightarrow$ 4 Regular
   $K_{3,3} \rightarrow$ 3 Regular
3. Removal of one edge or a vertex makes each of K_5 and $K_{3,3}$ a planar graph
4. K_5 is a non-planar graph with the smallest number of vertices and $K_{3,3}$ is the non-planar graph with smallest number of edges.

   ie $K_{3,3}$ & K_5 are the simplest non-planar graphs.

Euler's Theorm on Plane Graph

Theorm

If G is a connected plane graph with n vertices, m edges and F faces then

$$n - m + f = 2$$

or.

A connected planar graph with n vertices and edges has $e - n + 2$ faces.

Proof

We prove the result by mathematical induction on no. of edges 'm'

If $m = 0$, then $n = 1$ & $f = 1$

$n - m + f = 2$

If $m = 1$, then $n = 2$ & $f = 1$

$n - m + f = 2$ $\therefore$ the result is true for $m = 0, 1$

Now assume that theorm is true for all connected graph upto $(m-1)$ edges.

ie $n - (m-1) + f = 2$

Let G be a connected plane graph with m edges. If G is a

If G is a tree, then $n = m+1$ and $f = 1$

Hence $n - m + f = m+1 - m + 1 = 2$

If G is not a tree, then graph contain cycles.

Take any edge 'e' on a cycle and consider graph $H = G - e$.

$$|E(H)| = |E(G)| - 1$$

$$= m - 1 < m$$

$\therefore$ By Induction Hypothesis (from ①)

$$|V(H)| + |F(H)| - |E(H)| = 2$$

We know that

$|E(H)| = m - 1$, $|V(H)| = |V(G)|$

$= n$

And $|F(H)| = |F(G)| - 1$

$= f - 1$

$\therefore |V(H)| + |F(H)| - |E(H)| = 2$

$\Longrightarrow n + (f-1) - (m-1) = 2$

$\Longrightarrow \cancel{n+(f} \; n + f - 1 - m + 1 = 2$

$\Longrightarrow n - m + f = 2$

ie If G be a graph with 'n' vertices, m edges and f faces, then $n - m + f = 2$

$\therefore$ Result ~~is~~ true for 'm' edges also

$\therefore$ By Mathematical induction, the result is true for all connected plane graphs.

Theorm

Let G be a planar graph without parallel edges on n vertices and e edges, where $e \geq 3$. Then $e \leq 3n - 6$

Proof

Let 'f' be the no. of faces of G and let 'm_i' be the no. of edges in the boundary of i^{th} face, $i = 1, 2, \ldots, f$.

Since every face contain atleast 3 edges we have $m_i \geq 3 \;\; \forall i$

Then $3f \le \sum_{i=1}^{f} m_i$ — ①

On the other hand, since every edge can be on the boundary of atmost 2 faces, we have

$$\sum_{i=1}^{f} m_i \le 2e \quad — ②$$

From ① and ② $\longrightarrow$ $3f \le 2e$

$$f \le \frac{2e}{3}$$

By Euler's theorm on planar graph

$$n - m + f = 2$$

$$n - e + \frac{2e}{3} \ge 2$$

$$n - \frac{e}{3} \ge 2$$

$$\frac{-e}{3} \ge 2 - n$$

$$\frac{e}{3} \le n - 2$$

$$e \le 3n - 6$$

Theorm

The Complete Graph K_5 is non-planar.

Proof

If Possible, let K_5 be a planar Graph then by above theorm, $e \le 3n - 6$

In K_5, we have $n = 5$ & $e = 10$

Hence $3n - 6 = 9$ $e = 10$

$10 \nleq 9$ ie $e \nleq 3n-6$

Theorm

If G is a bipartite which is planar, then $e \leq 2n-4$ where e denote the number of edges & n denote the number of vertices.

Proof

If G is bipartite, the shortest cycle is atleast 4. Let F' be the no. of faces of G and m_i be the number of edges in the boundary of i^{th} face. Then we have

$$4F \leq \sum_{i=1}^{F} m_i \quad - \text{①}$$

Also we Have

$$\sum_{i=1}^{F} m_i \leq 2e \quad - \text{②}$$

From ① & ② $4F \leq 2e$ $F \leq e/2$.

By Euler's Theorm

$$n - m + F = 2$$

$$n - e + e/2 \geq 2$$

$$\frac{-e}{2} \geq 2-n$$

$$\frac{e}{2} \leq n-2$$

$$e \leq 2n-4$$

Theorm

A complete Bipartite graph $K_{3,3}$ is non-planar

Proof

By A Graph G is planar bipartite, then $e \leq 2n-4$ where e - no. of edges & n - no. of vertices. If possible assume that $K_{3,3}$ is a planar graph

In $K_{3,3}$ $n=6$, & $e=9$

$\therefore 2n-4 = 2 \times 6 - 4 = 8$

$\therefore e \leq 2n-4 \implies 9 \leq 8$

This is a contradiction, $\because 9 > 8$

$\therefore$ Our Assumption is wrong

$\therefore K_{3,3}$ is non-planar

1. If G is a 5 regular graph with no. of vertices = 10. Prove that G is non-planar

$n = 10$, $e = 25$

By

$$e \leq 3n - 6$$

If Possible Assume, the given graph planar.

$$3n - 6 = 3 \times 10 - 6 = 24$$

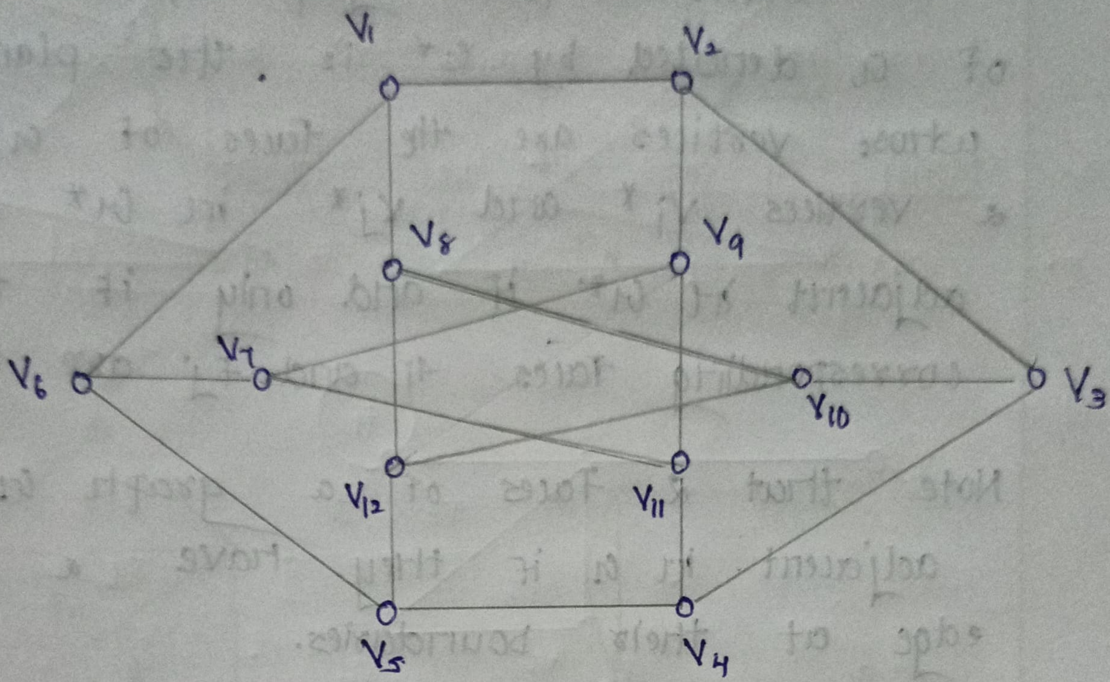
$\therefore 25 \leq 24$.

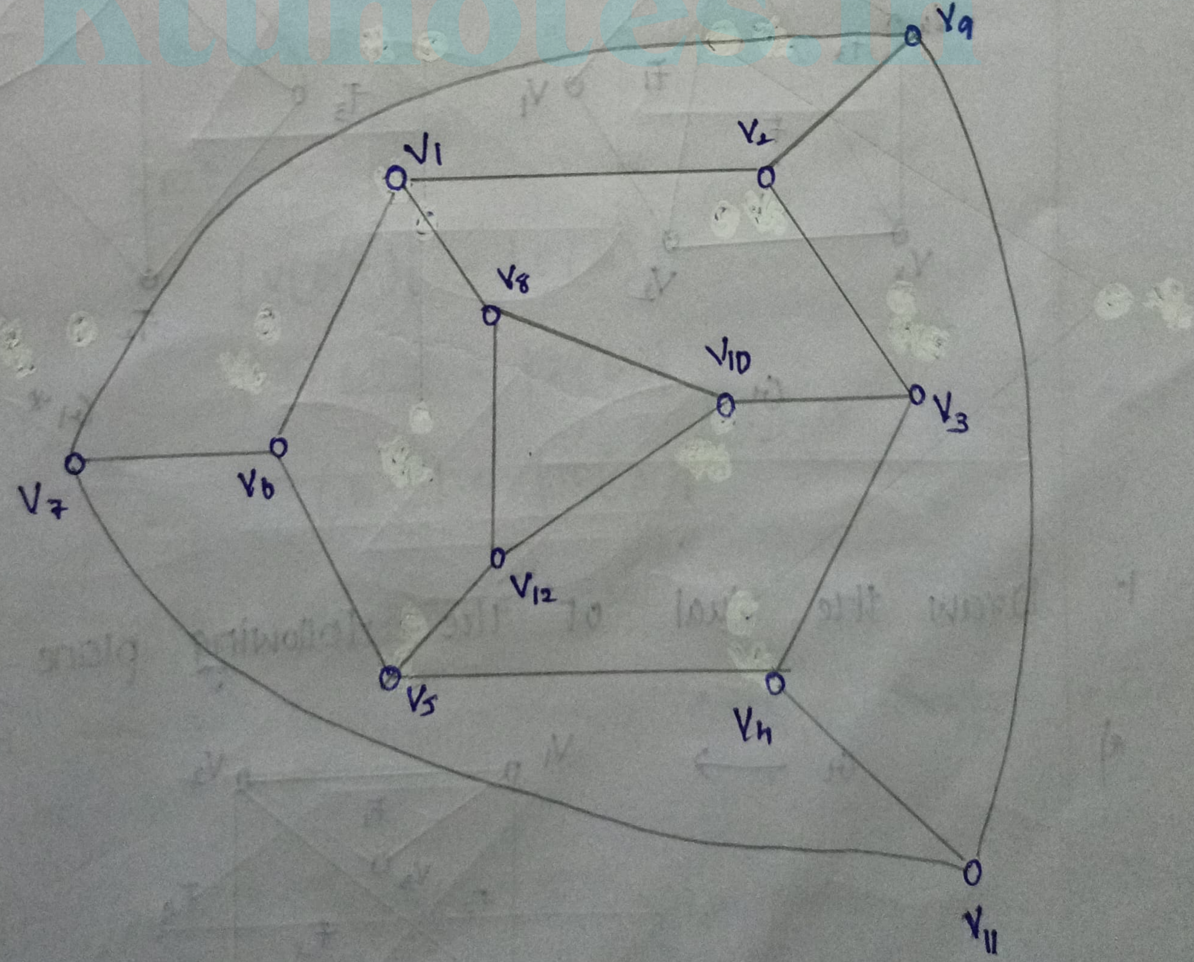
This is a contradiction $25 > 24$

ie our Assumption is wrong.

$\therefore$ Given Graph is non-planar

Check whether the graph is planar or not.

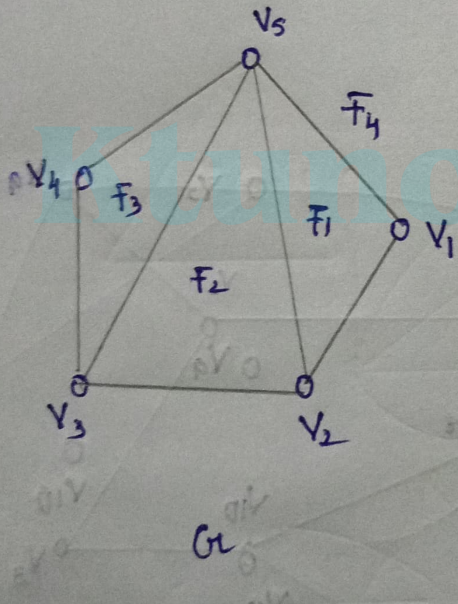


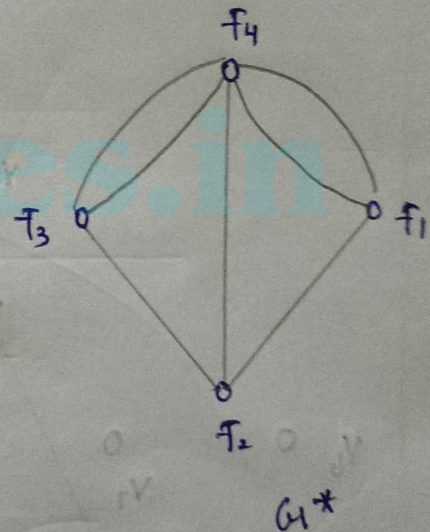


Geometric Dual of A Graph

Given a plane graph G, the dual graph of G, denoted by G^* is the plane graph whose vertices are the faces of G such that 2 vertices v_i^* and v_j^* in G^* are adjacent in G^* if and only if the corresponding faces f_i and f_j are adjacent in G.

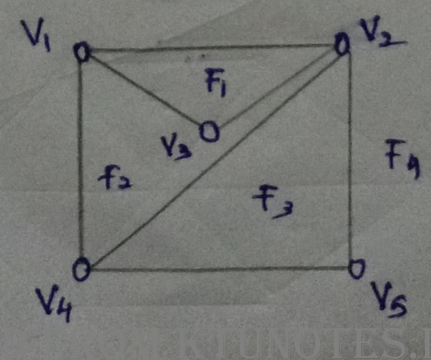
Note that 2 faces of a graph G are adjacent in G if they have a common edge at their boundaries.

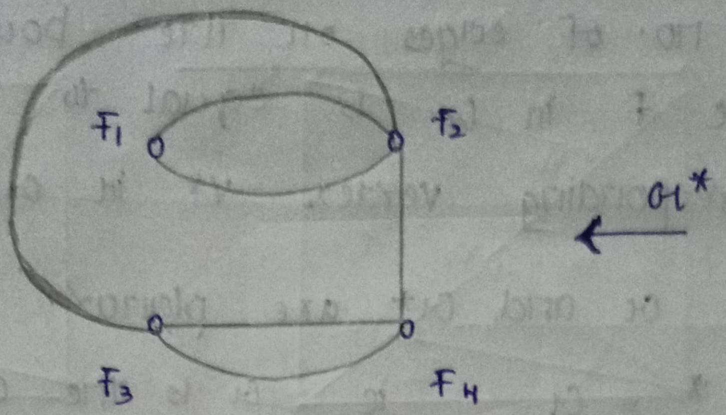




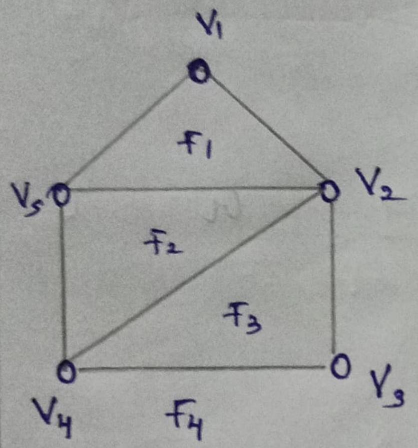
1. Draw the dual of the following plane graph

a) G →

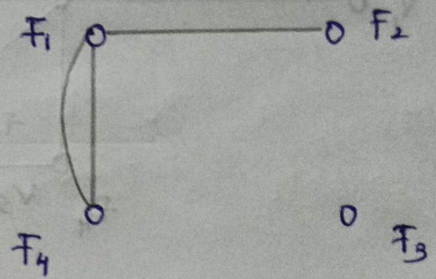




b)



G



G*

Note that :-

$$|V(G^*)| = |F(G)|$$

$$|E(G^*)| = |E(G)|$$

$$|F(G^*)| = |V(G)|$$

Relationship b/w planar graph and its dual

1. A loop in G corresponds to a Pendant edge in G*.

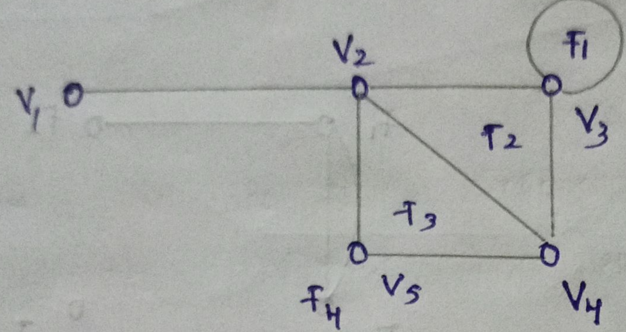
2. A Pendant edge in G corresponds to a loop in G*.

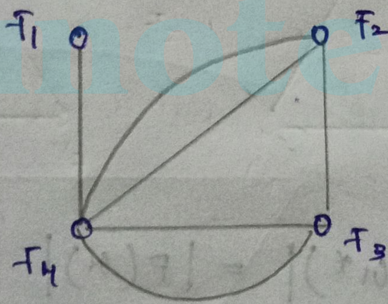
3. The no. of edges on the boundary of a face f in G is equal to the degree of the corresponding vertex v^* in G^*.

4. Both G and G^* are planar

5. $G^{**} = G$ ie G is the dual of G^*

c)

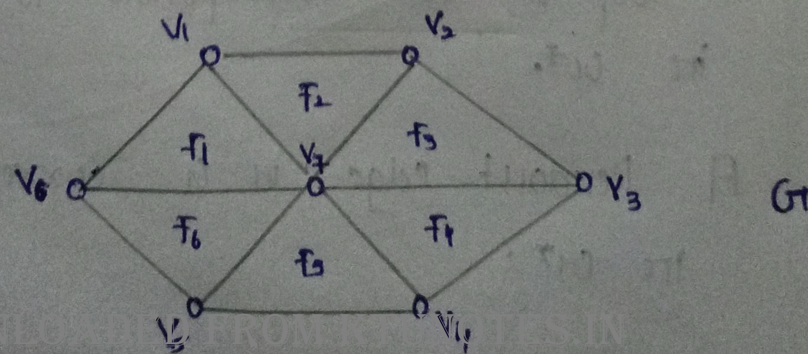


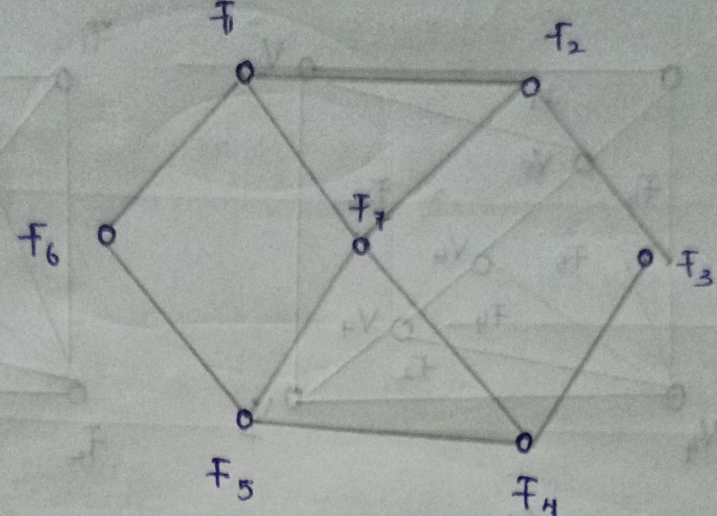


Self Dual Graphs

A Graph G is said to be a self dual if it is isomorphic to its Geometric dual G^*

eg:-

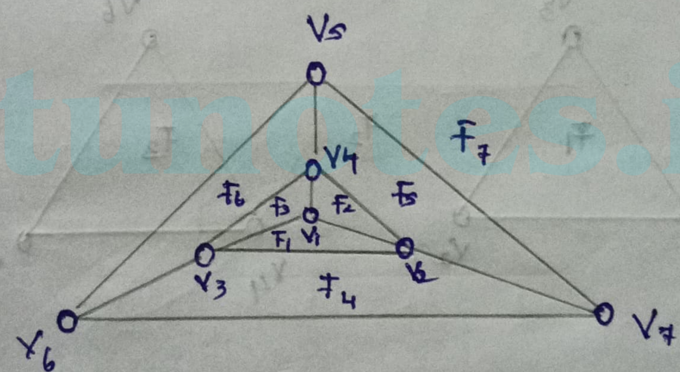




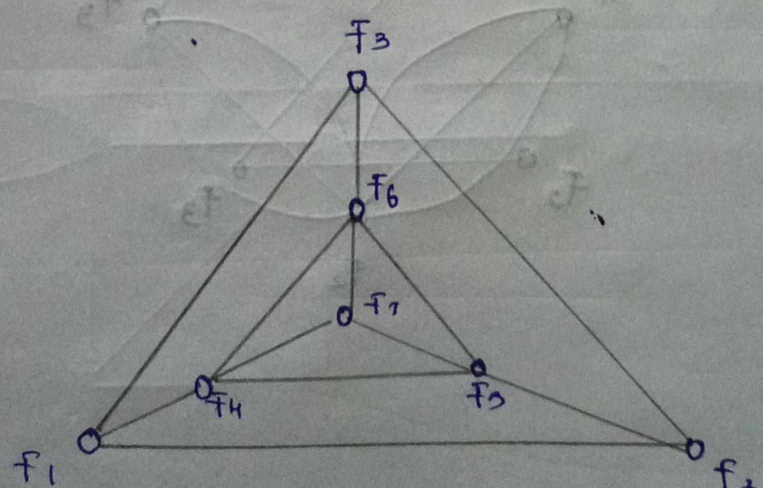
G is isomorphic to G* (G ≅ G*)

∴ Given Graph is Self Dual

Check whether the given graphs are self dual or not.



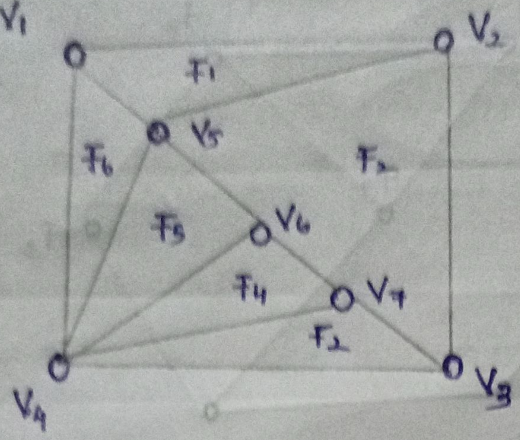
G



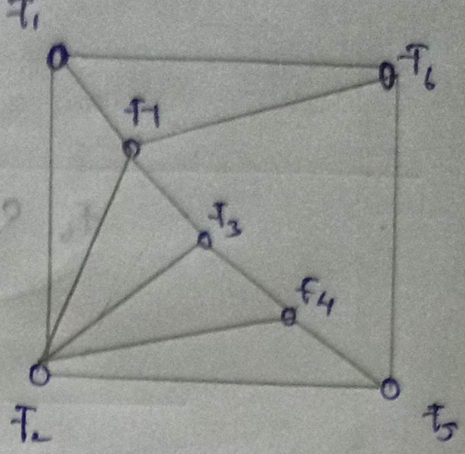
G*

G ≅ G*

G is a self Dual Graph



G



G*

Let G be the following disconnected planar graph. Draw the dual G*