
KTU NOTES

12/05/21

Module - II

Eulerian and Hamiltonian Graph

Euler Graph

Defn : Euler line (Euler Tour)

It is a closed walk in a graph that contains all the edges of the graph exactly once.

Euler Graph

A graph G is called an Euler graph if it contains Euler line, otherwise it is non-Euler.

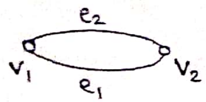
Note :- Euler graph is always a connected graph.

eg.

1) 

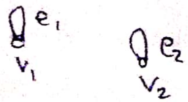
- Euler graph

$v_1 e_1 v_1 \rightarrow$ Euler Tour

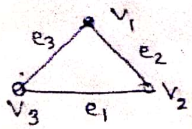
2) 

← Euler graph

$v_2 e_2 v_1 e_1 v_2 \rightarrow$ Euler Tour

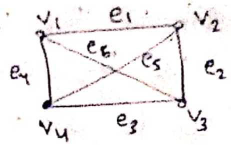
3) 

→ Not Euler graph (Disconnected)

4) 

$v_3 e_2 v_1 e_3 v_2 e_1 v_3 \rightarrow$ Euler Tour

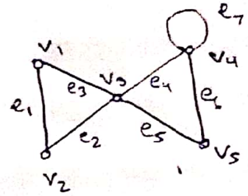
$K_3 \rightarrow$ Euler graph

5) 

$K_4 \rightarrow$ Not Euler graph

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1 e_6 v_3$

↳ not an Euler Tour

6) 

→ Euler graph

$v_1 e_3 v_3 e_4 v_4 e_7 v_4 e_6 v_5 e_5 v_3$

$e_2 v_2 e_1 v_1 \rightarrow$ Euler Tour

Theorem

* IMP

A given connected graph G is an Euler graph if and only if all the vertices of G are of even degree.

Proof of Part I

Let G be an Euler graph

ie- G contains an Euler line say W

ie- W contains all the edges in G exactly once

Let u be the vertex where 'W' starts and ends.

Let v be an internal vertex different from u.

Each time the tour W comes to v by one edge leaves v by another edge.

ie.. Each visit of 'W' to v contribute '2' to d(v).

Hence d(v) is an even no.

For vertex u, d(u) gets one from the first edge ...

If w makes an intermediate visit u, then w comes to u by one edge and leaves u by another edge. i.e. Every such visit of w to u contribute 2 to d(u). Finally when w returns to u, d(u) is increased by one.

∴ d(u) = 1 + Even no. + 1

= Even no.

∴ Degree of Every vertex is Even.

Proof of Part 2

Assume that all the vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G, such that no edge is traced more than once. we continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; tracing cannot stop at any vertex but v, since v is also of even degree we shall eventually reach v when the tracing comes to an end.

If this closed walk w includes all the edges of G, G is an Euler graph.

otherwise we remove from G all the edges in w and obtain a subgraph H of G formed by the remaining edges.

Since both 'G' and 'w' have all the vertices of Even of degree, the degrees of vertices of H are also even. More over H must touch 'w' atleast one vertex say 'a', since G is connected.

Starting from 'a', we can construct a new walk in H. since all the vertices in H are also even, This walk must terminate at vertex a.

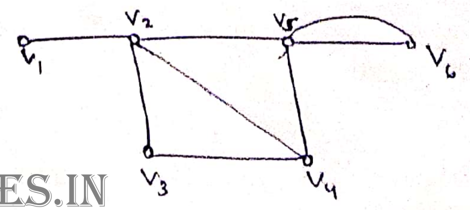
We can combine this walk in H with w, form a new walk, which starts and ends at vertex v and has more edges than w.

This process can be repeated until we obtain a closed walk that contains all edges in G.

∴ The Graph G contains an Euler Tour,

∴ G is an Euler graph

Q Check whether the following graph is Euler or not. If so find an Euler Tour.



Soln. A graph G is Eulerian if and only if degree of every vertex is even.

Here $d(v_1) = 1$ an odd no.

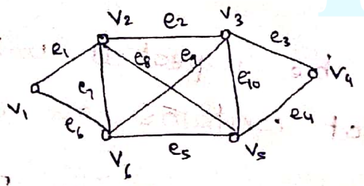
$\therefore$ v_1 is an odd vertex

$\therefore$ By the theorem, the given graph is not Euler.

Q. Is complete graph K_n Euler always. Justify your answer

Ans Not always. But a complete graph K_n is Euler if n is an odd number. (K_1, K_3, K_5 ... etc)

Q Check whether a given graph is Euler or not. If so find an Euler Tour.



Ans Yes the graph is Euler since degree of every vertex is even.

Euler Tour : $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_{10}\ v_3\ e_9\ v_6$
$e_7\ v_2\ e_8\ v_5\ e_5\ v_6\ e_6\ v_1$

Fleury's Algorithm

It describes a procedure which ~~converts~~ construct an Euler Tour in a Euler graph.

Step 1 : Choose any vertex v_1 in the Euler graph and set $W_0 = \{v_1\}$

Step 2 : If the walk $W_i = \{v_1 e_1 v_2 \dots e_{i-1} v_i\}$ has been chosen, such that all the edges selected are different, then choose another edge e_i such that,

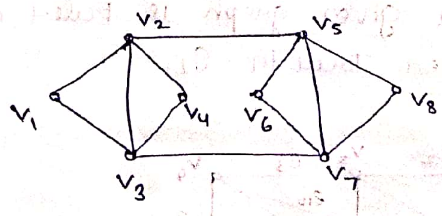
a) e_i is different from all previously selected edges $e_1, e_2 \dots e_{i-1}$

b) e_i is incident with v_i.

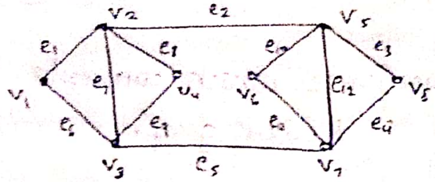
c) Unless there is no alternative, e_i is not a cut edge of the edge deleted graph.

Step 3 : If W_i covers every edge of G, stop otherwise goto step 2.

Q Find an Euler Tour in,



Ans)

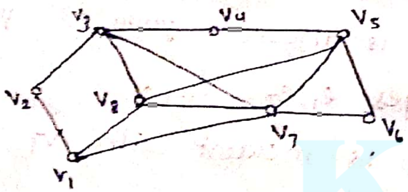


e_3 is a cut edge (If cycle then graph gets disconnected)

Euler Tour -: $V_1\ e_1\ V_2\ e_2\ V_5\ e_3\ V_8\ e_4\ V_7\ e_{11}\ V_6\ e_{10}\ V_5$
$e_{12}\ V_7\ e_5\ V_3\ e_9\ V_4\ e_8\ V_2\ e_7\ V_3\ e_6\ V_1$

(If there is another option dont choose cut edge.)

Q Check whether the graph is Euler or not. Justify

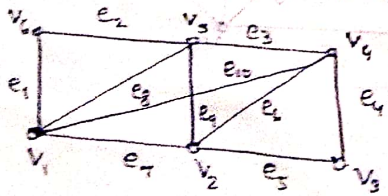


Ans Here $d(v_1) = 3$ is odd

A graph G is Euler if and only if degree of every vertex is even.

$\therefore$ The graph is not Euler

Q Check whether the given graph is Euler or not. If so find an Euler Tour in G.



A The given graph is Euler since degree of every vertex is Even.

Euler Tour -: $V_1\ e_1\ V_6\ e_2\ V_5\ e_3\ V_4\ e_4\ V_3\ e_5\ V_2\ e_7\ V_1$
$e_8\ V_5\ e_9\ V_2\ e_6\ V_4\ e_{10}\ V_1$

Theorem

A connected graph G is Eulerian if and only if graph G can be decomposed into edge-disjoint cycles [G is the union of edge-disjoint cycles].

Proof of Part 1

Assume that a connected graph G is Euler.

$\therefore \exists$ an Euler Tour with all the edges exactly once.

$\therefore$ G contains a cycle C_1 (If G be a graph with degree of every vertex is atleast 2, then G contains a cycle)

Let $G_1 = G - E(C_1)$ $\qquad E(C_1)$ - Edge set of cycle C_1

If G_1 is empty, we get $G = C_1$ a cycle. (Graph itself a cycle)

Otherwise $d(v)$ in G_1 is $d(v)$ in $G - 2$ (Again more than 2)

Discard all the isolated vertices, we get another cycle C_2 in G_1.

Now $G_2 = G_1 - E(C_2)$ where $E(C_2)$ - Edge set of cycle C_2

If G_2 is empty, we get $G = C_1 \cup C_2$ $[E(C_1) \cap E(C_2) = \phi]$

Otherwise, repeat the same process, till we get an empty graph (Null graph)

Finally $G = C_1 \cup C_2 \cup \dots \cup C_r$

$\therefore$ G is the edge disjoint union of cycles.

Proof of Part 2

Assume G is the union of edge disjoint cycle. Since degree of every vertex in a cycle is 2, the degree of every vertex in G is a multiple of 2.

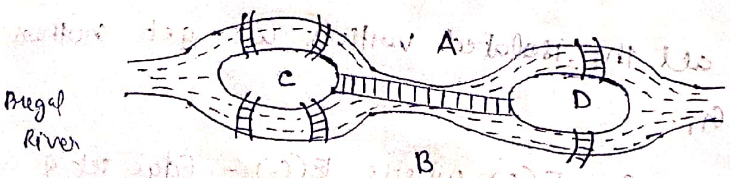
ie. Every vertex in G is Even.

$\therefore$ G is Euler.

* Konigsberg Bridge Problem

Graph Theory originated, when Euler showed that the famous Konigsberg Bridge Problem has no solution. It was in 1736 and Euler is known as the father of graph Theory.

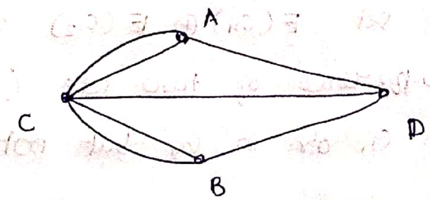
The city of Konigsberg has 7 bridges connecting 4 lands.



The problem is starting from any one of the 4 lands, and crossing all the bridges exactly once, can a person return to his starting point.

Let us draw the graph in which each land is represented by a vertex and each bridge is represented by an edge in a graph.

The problem become - can we find an Euler Tour in G.



A graph is Euler if and only if degree of every vertex is Even.

Here in this graph, Degree of every vertex is odd.

$\therefore$ This graph is not Euler

ie.. we cannot find such walk in this graph.

* Operations on Graph

1) Union

The union of two graphs G_1 and G_2 is a Graph G, written by $G = G_1 \cup G_2$, with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$

2) Intersection

The intersection of two graphs G_1 and G_2 is a Graph G, written by $G = G_1 \cap G_2$, with vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

3) Ringsum of Graphs

Ringsum of two graphs G_1 and G_2 is a graph G, written by $G = G_1 \oplus G_2$ with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \oplus E(G_2)$ where $\oplus$ is the symmetric difference of two sets. (contain edges that are either in G_1 or in G_2 but not in both).

Note

1) $G_1 \cup G_2 = G_2 \cup G_1$
2) $G_1 \cap G_2 = G_2 \cap G_1$ } Commutative
3) $G_1 \oplus G_2 = G_2 \oplus G_1$

4) If G_1 & G_2 are edge disjoint, Then $G_1 \cap G_2$ is a null graph and $G_1 \cup G_2 = G_1 \oplus G_2$

5) $G \cup G = G \cap G = G$

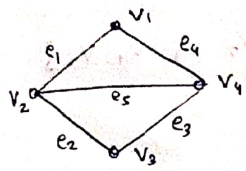
6) $G \oplus G$ = a null graph

7) G_1 & G_2 are vertex disjoint, Then $G_1 \cap G_2$ is empty.

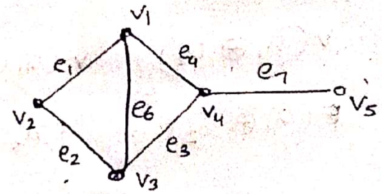
4) Decomposition

A graph G is said to have be decomposed into two subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ & $G_1 \cap G_2$ = a null graph

Examples

1)


G_1



G_2

Find $G_1 \cup G_2$, $G_1 \cap G_2$ & $G_1 \oplus G_2$

Ans) $G_1 \cup G_2 \Rightarrow$

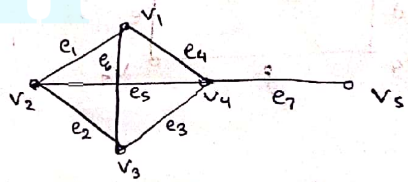
$V(G_1) = \{v_1, v_2, v_3, v_4\}$ $V(G_2) = \{v_1, v_2, v_3, v_4, v_5\}$

$V(G_1) \cup V(G_2) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G_1) = \{e_1, e_2, e_3, e_4, e_5\}$ $E(G_2) = \{e_1, e_2, e_3, e_4, e_6, e_7\}$

$E(G_1) \cup E(G_2) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

$\therefore$

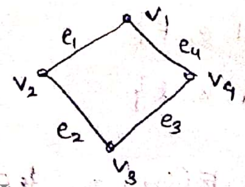
$G_1 \cup G_2 \Rightarrow$


$G_1 \cap G_2 \Rightarrow$

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2) = \{v_1, v_2, v_3, v_4\}$

$E(G_1 \cap G_2) = E(G_1) \cap E(G_2) = \{e_1, e_2, e_3, e_4\}$

$\therefore$

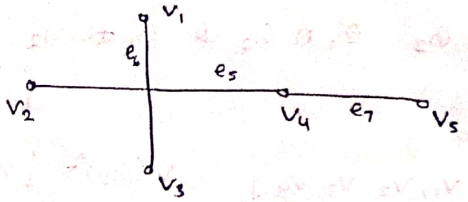
$G_1 \cap G_2 \Rightarrow$


$G_1 \oplus G_2 \Rightarrow$

$V(G_1 \oplus G_2) = V(G_1 \cup G_2) = \{v_1, v_2, v_3, v_4, v_5\}$

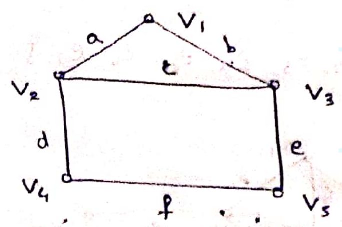
$E(G_1 \oplus G_2) = \{e_5, e_6, e_7\}$

$G_1 \oplus G_2 \rightarrow$

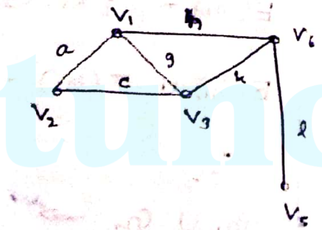


2) Find $G_1 \cup G_2$, $G_1 \cap G_2$, $G_1 \oplus G_2$ where

$G_1 \rightarrow$



$G_2 \rightarrow$



Ans) $G_1 \cup G_2$

$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$

$V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$

$V(G_2) = \{v_1, v_2, v_3, v_5, v_6\}$

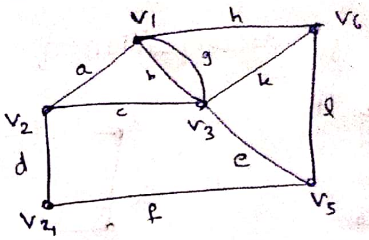
$V(G_1) \cup V(G_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E(G_1) = \{a, b, c, d, e, f\}$

$E(G_2) = \{a, c, g, h, k, l\}$

$E(G_1 \cup G_2) = E(G_1) \cup E(G_2) = \{a, b,$ [illegible] $h, k, l\}$

$G_1 \cup G_2 \Rightarrow$

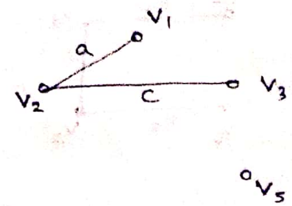


$G_1 \cap G_2$

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2) = \{v_1, v_2, v_3, v_5\}$

$E(G_1 \cap G_2) = E(G_1) \cap E(G_2) = \{a, c\}$

$G_1 \cap G_2 \rightarrow$

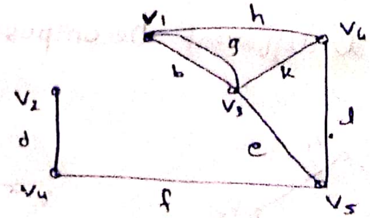


$G_1 \oplus G_2$

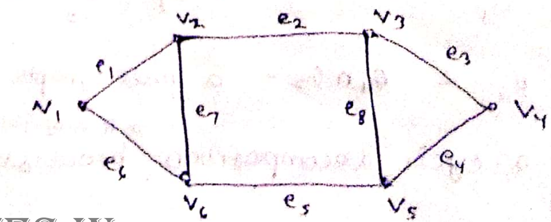
$V(G_1 \oplus G_2) = V(G_1 \cup G_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E(G_1 \oplus G_2) = \{b, d, e, f, g, h, k, l\}$

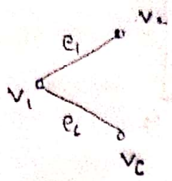
$G_1 \oplus G_2 \Rightarrow$



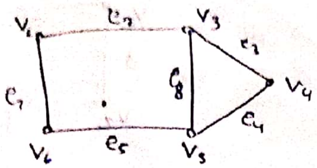
3) Draw a decomposition of the graph given



Ans An example of decomposition of G is given below,



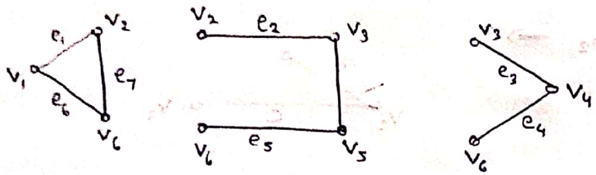
G_1



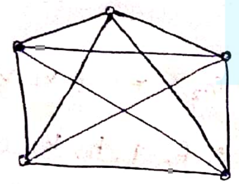
G_2

$G_1 \cup G_2 = G$ & $G_1 \cap G_2 =$ a null graph.

Another example :-

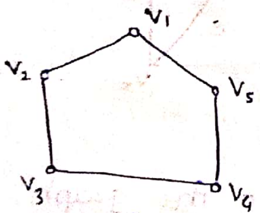


Q $K_5 \Rightarrow$

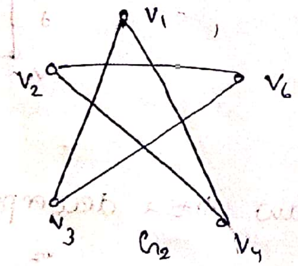


Write any 2 different Decomposition of K_5

Ans) 1)



G_1

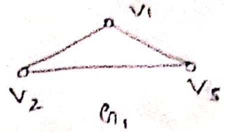


G_2

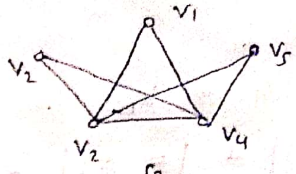
$G_1 \cup G_2 = K_5$ & $G_1 \cap G_2 =$ a null graph

(This is a cycle decomposition. It is only possible because its Euler graph)

2)



G_1



G_2

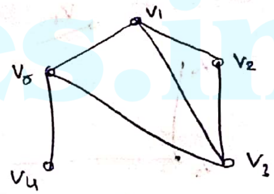
$G_1 \cup G_2 = K_5$ $G_1 \cap G_2$: a null graph

Complement of a Graph

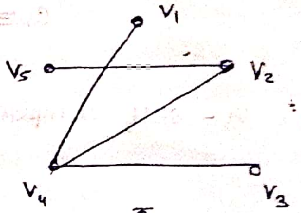
The complement of a graph G, denoted by $\bar{G}$ is a graph with $V(G) = V(\bar{G})$ such that two distinct vertices of $\bar{G}$ are adjacent if and only if they are not adjoint in G.

e.g

1)

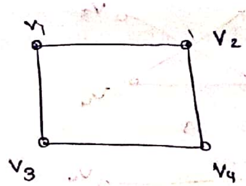


G

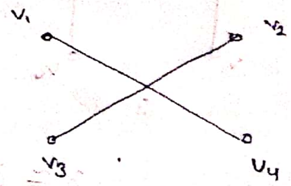


$\bar{G}$

2)

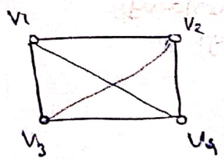


G



$\bar{G}$

$G \cup \bar{G} \Rightarrow$



$\Rightarrow K_4$

Note

1) $G \cup \bar{G} = K_n$ → a complete graph

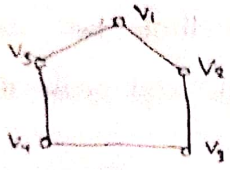
2) $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$ → Number of edges in K_n

* Self complementary graph

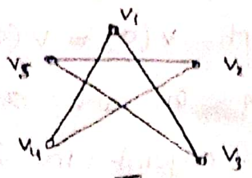
A graph G is said to be self complementary if G is isomorphic to its complement

eg.

i)

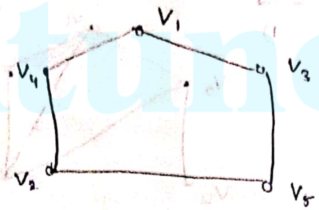


G



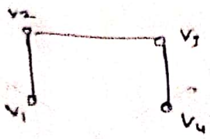
$\bar{G}$

$G \cong \bar{G}$

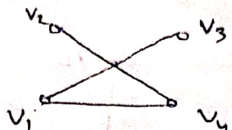


∴ G - self complementary graph.

2)

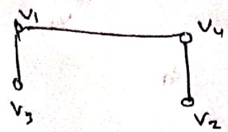


G



$\bar{G}$

$G \cong \bar{G}$



∴ G - self complementary graph.

Deletion & Fusion

Edge deletion in Graphs

If e is an edge of G, then G − e is the graph obtained by removing the edge of G. The sub-graph of G thus obtained is called an edge deleted sub-graph of G.

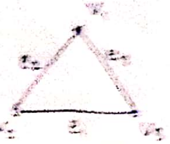
Vertex deletion in Graphs

If v is a vertex of G, then G − v is the graph obtained by removing the vertex v and all edges of G that are incident on v. The sub graph of G thus obtained is called a vertex deleted sub-graph of G.

Fusion of vertices

A pair of vertices u & v are said to be fused or merged if the two vertices u & v are replaced by a single vertex w, such that every edge incident with either u or v is incident with the new vertex w.

Eg: 1)

 → G

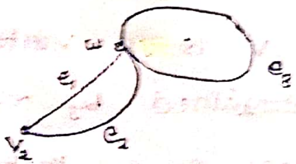
G - {e_3}

 — An edge deleted sub-graph

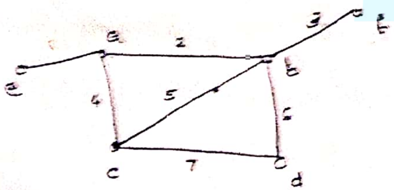
G - {v_3}

 → vertex deleted sub graph

Fusion of v_1 & v_2 ⇒

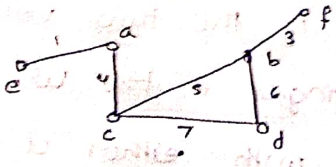


2)

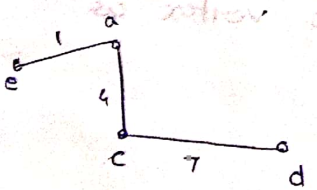


Draw (i) G - {2}
(ii) G - {b}
(iii) Fusion of vertices a & b.

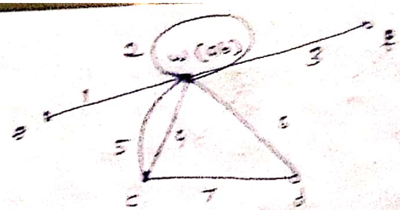
Ans) (i) G - {2}



(ii) G - {b}



(iii) Fusion of a & b



Cut Edges & Cut vertices

Cut Edge

An edge 'e' of a graph G is said to be a cut-edge of G if G - e is disconnected.

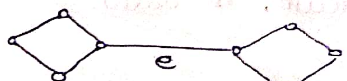
Cut Vertex

A vertex v of a graph G is said to be a cut-vertex if G - v is disconnected.

Theorem

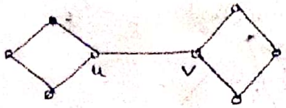
An edge 'e' of a graph G is a cut-edge of G if and only if it is not contained in any cycle of G.

Example:

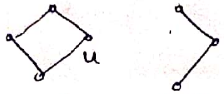


G - {e} ⇒ 

∴ e is called cut edge of G



$G - \{v\} \rightarrow$



disconnected

$\therefore$ v is called cut vertex of G.

* Hamiltonian Graph

Definition -:

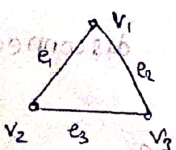
Hamiltonian Path - It is a path that visits each vertex of a graph G exactly once. A graph that contain Hamiltonian Path is called a traceable graph.

Hamiltonian circuit (or Hamiltonian cycle) - It is a cycle that visits each vertex exactly once. (Except for the vertex that is both start and end which is visited twice).

Hamiltonian Graph : It is a graph that contains a Hamiltonian circuit.

Examples -:

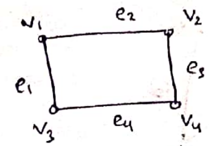
1)



K_3 → $v_2\ e_2\ v_3\ e_3\ v_1\ e_1\ v_2$ → Hamiltonian circuit

$\therefore K_3$ is Hamiltonian Graph

2)



$v_4\ e_4\ v_3\ e_1\ v_1\ e_2\ v_2\ e_3\ v_4$ → Hamiltonian circuit

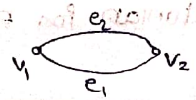
$\therefore$ It is a Hamiltonian Graph.

3)

v e v → Hamiltonian circuit

$\therefore$ It is a Hamiltonian Graph.

4)



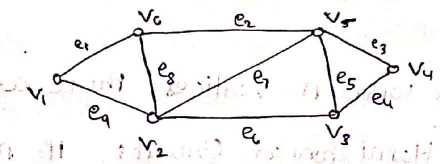
$v_1\ e_1\ v_2\ e_2\ v_1$ → Hamiltonian circuit

$\therefore$ It is a Hamiltonian graph.

Problems

Q Check whether the following graphs are Hamiltonian or not. If so find a hamiltonian circuit in it.

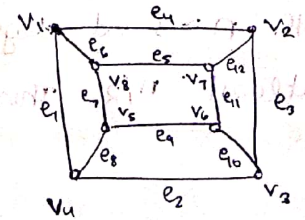
1)



$v_1\ e_9\ v_2\ e_6\ v_3\ e_4\ v_4\ e_3\ v_5\ e_2\ v_6\ e_1\ v_1$ - Hamiltonian circuit

$\therefore$ The given graph is Hamiltonian.

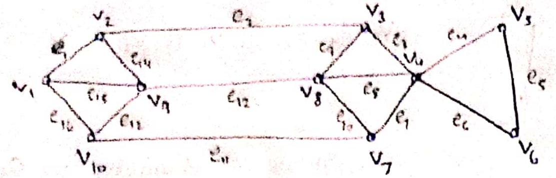
2)



$v_1\ e_1\ v_4\ e_2\ v_3\ e_3\ v_2\ e_{12}\ v_7\ e_{11}\ v_6\ e_9\ v_5\ e_7\ v_8\ e_6\ v_1$ → Hamiltonian circuit

$\therefore$ The given graph is Hamiltonian

3)



Not Hamiltonian Graph. Not hamiltonian circuit exist

Note

1) Every complete graph is Hamiltonian for $n \geq 3$
2) All ~~gr~~ cycles are Hamiltonian.

Results

1) Let G be a simple graph with $n \geq 3$ vertices. If every pair of non adjacent vertices u, v of G, $d(u) + d(v) \geq n$, Then G is Hamiltonian.

2) In a complete graph with n vertices, there are $(n-1)/2$ edge disjoint Hamiltonian Circuits, if n is an odd number ≥ 3.

Theorem

If G is a simple graph with $n \geq 3$ and degree of every vertex in G be atleast $n/2$. Then G is Hamiltonian.

Proof

Let G be a simple graph with $n \geq 3$, given that $d(v) \geq n/2 \quad \forall \; v \in V(G)$

Let u & v are two non-adjacent vertices of G (if possible)

Then $d(u) \geq n/2$ & $d(v) \geq n/2$

$\therefore d(u) + d(v) \geq n$

$\therefore$ By Result 1, The graph G is Hamiltonian.

Theorem (Proof of Result 2)

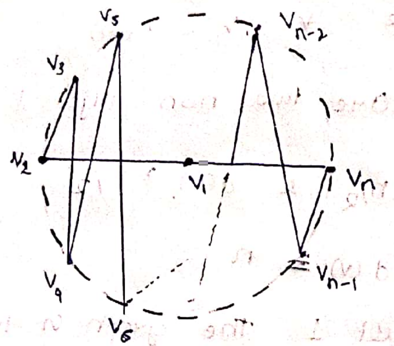
In a complete graph K_n, where $n \geq 3$ is odd, there are $(n-1)/2$ edge disjoint Hamiltonian cycles

Proof

Note that a complete graph has $\frac{n(n-1)}{2}$ edges and a Hamiltonian cycle in K_n contains only n edges.

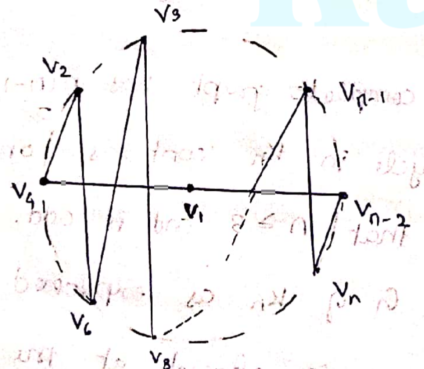
Now, assume that $n \geq 3$ and is odd. Construct a sub-graph G of K_n as explained below.

The vertex v_1 is placed at the centre of the circle and the remaining $(n-1)$ vertices are placed on the circle, at equal distance along the circle such that angle made at the centre by two points is $\frac{360}{n-1}$ degrees. The vertices with odd suffixes are placed along the upper half of the circle and vertices with even suffixes are placed along lower half circle.



Consider a Hamiltonian circuit $v_1 v_2 v_3 v_4 v_5 \cdots v_n v_1$

Now we rotate the vertices along the curve for $\frac{360}{n-1}$ degrees (clock wise). we will get another Hamiltonian circuit,



$v_1 v_4 v_2 v_6 v_3 v_8 \cdots v_{n-1} v_1$

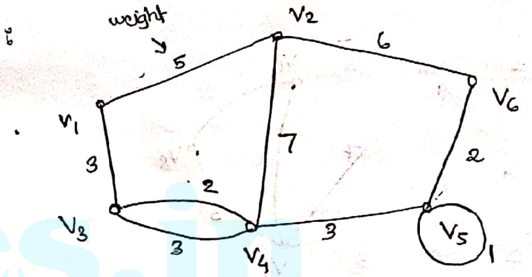
Clearly these two Hamiltonian circuits have no edge in common. There such $(n-1)$ rotations possible. Out of these $(n-1)$ rotations, half of the rotations are isomorphic (all the vertices in the upper half will be in the lower half and vice versa)

$\therefore$ There are $\frac{n-1}{2}$ distinct Hamiltonian edge disjoint cycle in a complete graph K_n when n is odd.

Weighted Graph

A weighted graph is a graph in which each edge 'e' has been assigned a real no: $w(e)$ called the weight or length of the edge e. The weight of a graph G is the sum of weights of all its edges.

e.g:



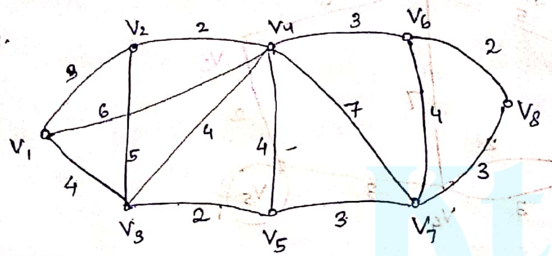
Travelling Salesman's Problem

Suppose a travelling salesman's territory includes several towns with roads connecting certain pair of these towns. As a part of his work, he has to visit each town. For this, he needs to plan a round trip in such a way that that he can visit each of the towns exactly once.

We represent the salesman territory by a weighted graph G where the vertices correspond to the towns and two vertices are joined by a weighted edge if and only if a road connecting the corresponding towns which does not pass through any of the other towns, the edge's weight representing the length of the road between the towns.

Then the problem reduces to check whether the graph G is a Hamiltonian graph and to construct a Hamiltonian cycle of minimum weight (length) of G is Hamiltonian. This problem is known as travelling salesman problem.

eg.



Hamiltonian circuit -:

(1) $V_1 V_2 V_4 V_6 V_8 V_7 V_5 V_3 V_1$

length of the circuit = 22 km

(2) $V_1 V_4 V_6 V_8 V_7 V_5 V_3 V_2 V_1$

length = 27 km

(3) $V_1 V_2 V_3 V_5 V_7 V_8 V_6 V_4 V_1$

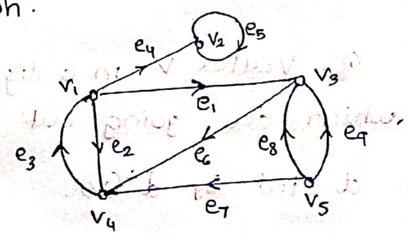
length = 27 km

$\therefore$ (1) is the Hamiltonian circuit with minium weight (length)

Directed Graph (oriented graph)

A directed graph or digraph G consist of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices. In other words, if each edge of the graph G has a direction, then the graph is called a directed graph or digraph.

eg.



$e_1 = (V_1, V_3)$
$e_6 = (V_3, V_4)$
$e_2 = (V_1, V_4)$
$e_3 = (V_4, V_1)$

The vertex v_i, which edge e_k is incident out of, is called initial vertex e_k and the vertex v_j, which edge e_k is incident into is called terminal vertex of e_k.

In above graph G, V_5 is the intial vertex and V_4 is the terminal vertex of edge e_7.

Loop -: An edge for which the initial and terminal vertex are same is called a loop. eg. e_5 in G.

Parallel edges -: Two directed edges are said to be parallel if they are mapped onto same ordered pair of vertices ie. both initial and terminal vertices are same for both. eg. e_8 and e_9 are parallel but e_2 and e_3 are not parallel

A digraph that has no loop and parallel edge is called a simple digraph.

Degrees in Digraph

The in-degree of a vertex v in a digraph is the number of edges which are coming into the vertex v and is denoted by $d^-(v)$.

The out-degree of vertex v in a digraph is the number of edges which are going out from the vertex v and is denoted by $d^+(v)$.

Isolated Vertex

A vertex in which the in-degree and out degree are both equal to zero is called isolated vertex.

Pendant Vertex

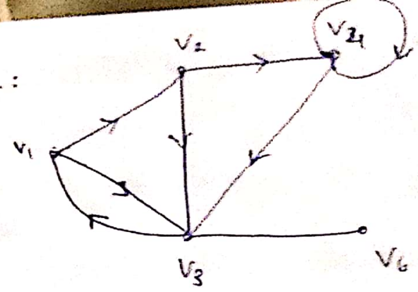
A vertex v in digraph is called pendant vertex if it is of degree one.

i.e. $d^+(v) + d^-(v) = 1$

Source and Sink

A vertex with zero in-degree is called a source and a vertex with zero out-degree is called sink.

Example -:



V_1	V_2	V_3
$d^-(V_1) = 1$	$d^-(V_2) = 1$	$d^-(V_3) = 3$
$d^+(V_1) = 2$	$d^+(V_2) = 2$	$d^+(V_3) = 2$

V_4	V_5	V_6
$d^-(V_4) = 2$	$d^+(V_5) = 0$	$d^-(V_6) = 1$
$d^+(V_4) = 2$	$d^-(V_5) = 0$	$d^+(V_6) = 0$

V_5 is an example of isolated vertex.

V_6 is an example of pendant vertex.

V_6 - sink V_5 (not source or sink)

Theorem

In a digraph G, the sum of the in-degrees and the sum of the out-degrees of the vertices are equal to the number of edges

i.e. $\Sigma d^+(v) = \Sigma d^-(v) = e$ where e denotes the no. of edges.

Proof -: Let $S^+ = \sum_{v \in V(G)} d^+(v)$ and $S^- = \sum_{v \in V(G)} d^-(v)$

Every edge of G contributes 1 to S^- and 1 to S^- each edge exactly once is S^+.

and once in s^-

$\therefore s^+ = s^- = e$

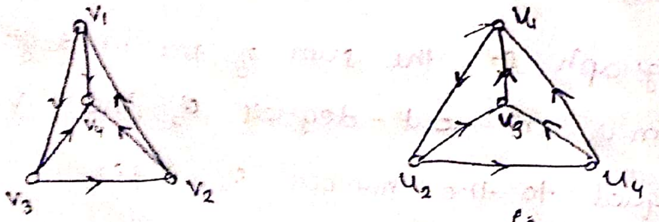
Note

Many properties of directed graphs are the same as those of un directed ones, the graph obtained from a Digraph by disregarding the orientation is called the undirected graph.

Isomorphic Digraph

Two digraphs are said to be isomorphic not only must their corresponding undirected graphs be isomorphic but the direction of the corresponding edges must be same.

Eg.

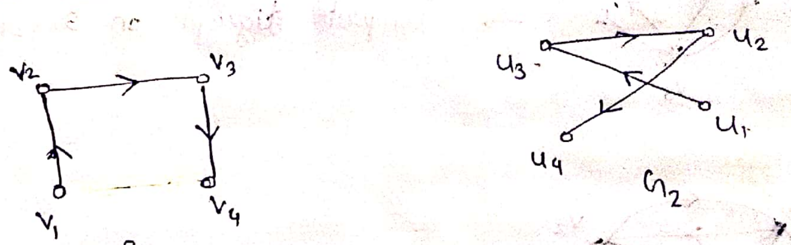


G_1 G_2

Here G_1 and G_2 are not isomorphic, $\because$ all corresponding edges are not having same direction.

Q. Check whether the following graphs are isomorphic or not.

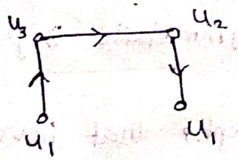
Ans



G_1 G_2

Ans Yes the two graphs are isomorphic $\because$ they both have same no. of edges and vertices and direction of corresponding edges are also similar.

G_2 can be drawn as



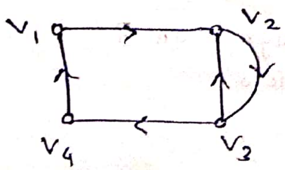
$G_1 \cong G_2$

Types of Digraphs

1) Simple Digraph

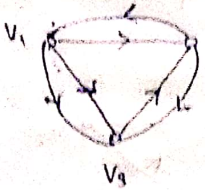
A digraph that has no loops and parallel edges is a simple digraph.

Eg.

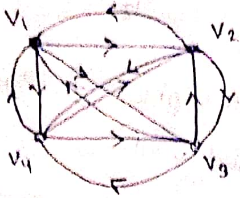


2) Complete Digraph

A complete digraph is a directed graph in which every pair of distinct vertices is connected by of unique edges (one in each direction)



→ complete digraph on 3 vertices

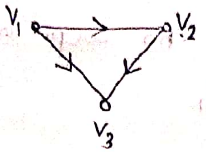


→ complete digraph on 4 vertices.

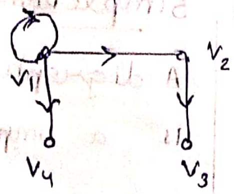
3) Asymmetric Digraphs

Digraphs that have almost one directed edge between a pair of vertices but are allowed to have loops are called asymmetric digraph.

Eg.



Asymmetric digraph on 3 vertices

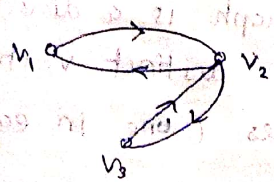


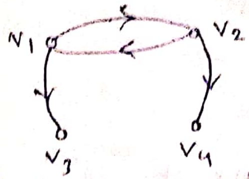
(V_1, V_2) ✓, (V_2, V_1) ✗

4) Symmetric Digraph

A digraph in which for every edge (a,b). There is also an edge (b,a).

Eg.

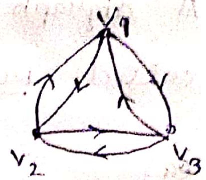




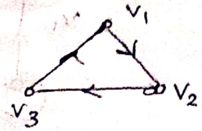
Not symmetric
Not Asymmetric

Note

1) complete symmetric digraph



complete Asymmetric digraph



2) A complete symmetric digraph has $n(n-1)$ edges and a complete digraph has $\frac{n(n-1)}{2}$ edges.

3) A complete asymmetric digraph is called a Tournament.

Balanced digraph – A graph is said to be balanced if $d^+(v_i) = d^-(v_i)$ $\forall$ $v_i \in V(G)$.

A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

Eg.

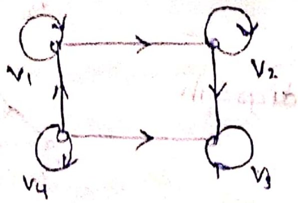
$d^+(V_1) = 1$ $d^-(V_1) = 1$
$d^+(V_2) = 2$ $d^-(V_2) = 2$
$d^+(V_3) = 1$ $d^-(V_3) = 1$
$d^+(V_4) = 0$ $d^-(V_4) = 0$

Digraphs and Binary Relation

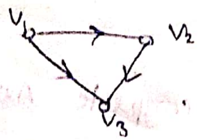
Reflexive Digraph

They are the digraphs with loops at all vertices. A digraph in which no vertex has loops, is called irreflexive digraphs.

eg.



Reflexive

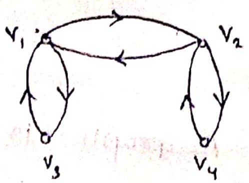


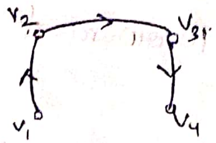
irreflexive

Symmetric Digraph

They are digraphs in which every edge (a,b), there exist an edge (b,a) also. (ie, $aRb \rightarrow bRa$)

eg.



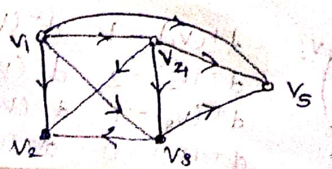


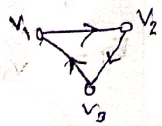
Not symmetric

Transitive Digraph

A digraph G is said to be transitive if any three vertex (x,y,z) show that edges (x,y), and (y,z) in G imply (x,z) in D

eg.



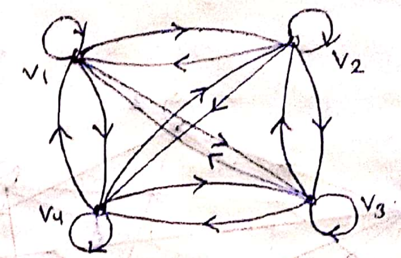


Not transitive

Equivalence Digraph

An equivalence digraph is a directed graph which represents an equivalence relation (Reflexive, Symmetric and Transitive)

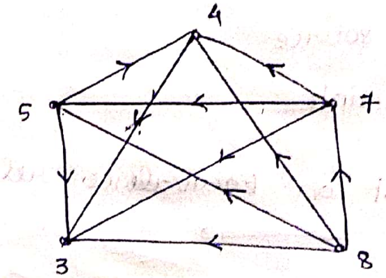
eg.



Examples

1) Consider $S = \{3,4,7,5,8\}$. Define a relation R on S by $R = \{(a,b) \mid a \text{ is greater than } b\}$. Draw the digraph corresponding to R.

Ans)



8 - source

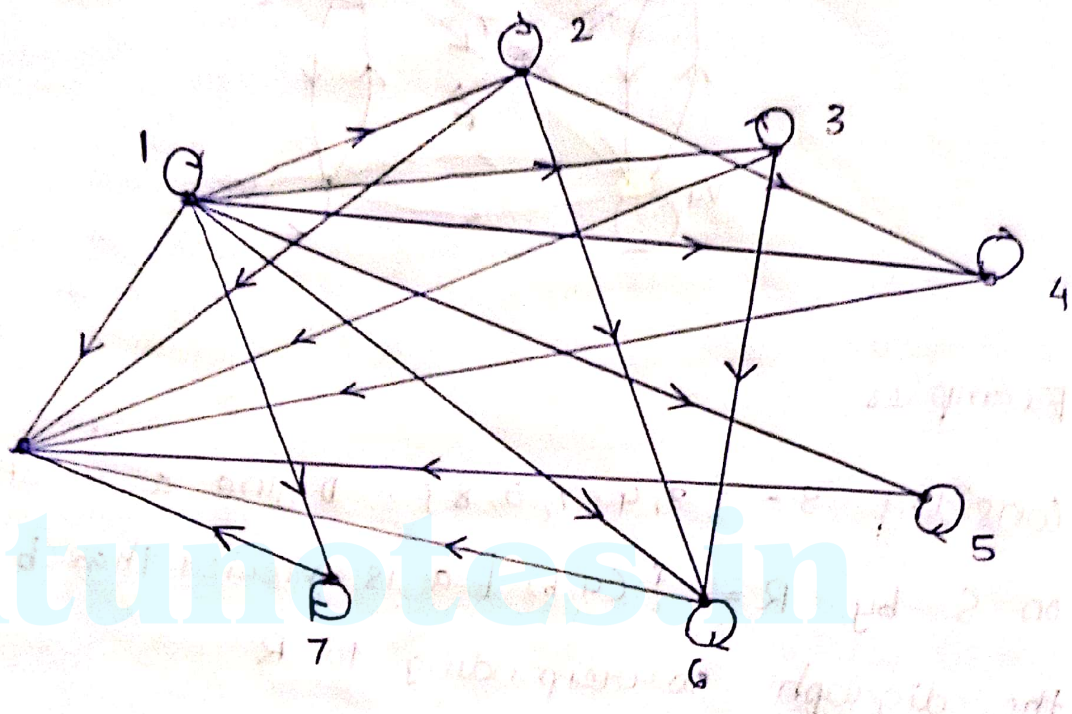
3 - sink

2) Draw a digraph corresponding to the relation R on Z_8 defined by $aRb \Rightarrow a/b$; $a,b \in Z_8$. Check whether R is an equivalence relation.

Ans) $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

(Z_n = set of all integers modulo $n = \{0, 1, 2, \ldots, n-1\}$)

$a/b \rightarrow \frac{b}{a} = I$ I - Integer



1 - source

0 - sink

R is not an Equivalence relation