Discrete-time Stochastic Processes Module 6

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 6 of Discrete-time Stochastic Processes, entitled American Derivatives. It consists of three lecture transcripts, four sets of supplementary notes, and a problem set.

2. Course Context

Discrete-time Stochastic Processes is the third course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The course introduces derivative pricing in discrete time. It begins with measure-theoretic probability and stochastic processes, with an emphasis on discrete-time martingales. These ideas are then applied to the pricing of derivatives in discrete time, followed by an introduction to interest rate and credit risk modeling.

2.1 Course-level Learning Outcomes

After completing the Discrete-time Stochastic Processes course, you will be able to:

1. Understand the language of measure-theoretic probability.
2. Understand stochastic processes and their applications.
3. Understand the theory of discrete-time martingales.
4. Define trading strategies in discrete time.
5. Create replicating portfolios in discrete time.
6. Model stock price movements on a binomial tree.
7. Price and hedge European derivatives in discrete time.
8. Price and hedge exotic European derivatives in discrete time.
9. Price and hedge American derivatives on a binomial tree.
10. Construct a simple interest rate model on a tree.
11. Price interest rate derivatives on a tree.

2.2 Module Breakdown

The Discrete-time Stochastic Processes course consists of the following one-week modules:

1. Probability Theory
2. Stochastic Processes
3. Discrete Martingales
4. Trading in Discrete Time
5. The Binomial Model
6. American Derivatives
7. An Introduction to Interest Rate Models

3. Module 6:

American Derivatives

In finance, the style or family of an option is the class into which the options falls, usually defined by the dates on which the option may be exercised. While a European option can be exercised only at the expiration date of the option, an American option can be exercised at any point before the expiration date. This module will cover the pricing and hedging of American derivatives. We will first introduce exercise strategies and hedging strategies, before defining the Snell envelope of an American option, which is then used to price and hedge these kinds of derivatives.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Understand the difference between European and American options.
2. Price a vanilla American option on a binomial tree.
3. Price an exotic American option on a binomial tree.

3.2 Transcripts and Notes

3.2.1 Transcript: An Introduction to American Derivatives

Hi, in this video we introduce American derivatives.

Recall that a European derivative is simply an $\mathcal{F}_t$-measurable random variable, H, that can only be exercised at time, T. In other words, it only comes to life at time T. In contrast, an American derivative is one that can be exercised at any time, t, before time capital T. So, we can think of it as a stochastic process, instead of a random variable, that takes different values across the entire life of the option. However, it cannot be exercised beyond the maturity time, T – it has to be exercised before time, T.

As an example, we will look at a call option where, for the European case, the call option is simply a random variable, $H = (X_T - K)^+$, where K is the strike price. For the American case, it is a stochastic process, $H_t = (X_t - K)^+$. In other words, if you exercise at time t, the payoff of that option will be $X_t - K$, plus the positive part of that. This is the definition that we are going to use for an American derivative.

The mathematical definition of an *American derivative* is simply a non-negative stochastic process, $H = \{H_t : t \geq 0\}$. This is in contrast to a European derivative, which is simply a random variable that is $\mathcal{F}_t$-measurable.

Now, a European derivative can also be thought of as an American derivative if we make $H_t = 0$ if $t < T$ and $H_T = H$. So, we can convert a European derivative with payoff H into an American derivative because the value of the exercise is 0 since we don't have the option to exercise it at a time prior to time T.

The decision of when to exercise lies with the buyer of the option – in other words, the buyer has the option of choosing the time to exercise, and the time to exercise does not need to be chosen prior to the contract. In other words, the time to exercise can depend on the sample path followed by the stock price. Therefore, it will be a stopping time. We call this an *exercise strategy*. The exercise strategy, τ, which is the decision of when to exercise the American option, is determined by the buyer and is a stopping time. It must be a stopping time, meaning that $\{\tau \leq t\} \in \mathcal{F}_t \forall t \geq 0$. This is the condition that it satisfies.

As an example, we can look at τ to be the first time that the stock is greater than some pre-specified constant. Now, since the exercise cannot happen after time T (and note that the example given above can be infinite), if $(X_t - K)^+$ never exceeds T, we have to hit this with a minimum capital T to ensure that we must exercise by time T if this event never occurs. Written in full:

$$\tau = inf\{t \in \mathbb{I}: (X_t - K)^+ \geq C\} \wedge T.$$

With regards to American options, we have two goals that we want to achieve:

1. Find the price of H. In other words, how much are we willing to pay to receive such an option to exercise at any time between 0 and T? That is pricing the American derivative.
2. The second goal has to do with hedging. In other words, find a trading strategy – trading in the underlying assets of the riskless bank account and the stock price, X, itself, such that the value of the trading strategy, which we will call V_τ, is greater than or equal to H_τ for any exercise strategy, τ. So, the value of the trading strategy dominates H for any trading strategy, τ.

Now that we have introduced American options, in the next video we are going to move on to pricing American options.

3.2.2 Notes: What is an American Derivative?

We continue to work with a financial market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with trading dates $\mathbb{I} = \{0,1, \dots, T\}$.

We have discussed the pricing and hedging of different kinds of European derivatives under various discrete-time asset price models. What makes a derivative European-style is that the exercise date is only at one single time point T.

In this module we relax this constraint and consider derivatives that can be exercised at any time t, where $0 \leq t \leq T$. These are called *American derivative securities*.

Some examples of American derivatives include the following:

1. American call option, whose payoff function at time t is equal to $H_t = (X_t - K)^+$, where K is the strike price.
2. American put option, whose payoff function at time t is equal to $H_t = (K - X_t)^+$, where K is the strike price.

So, we can think of an American derivative as a sequence of non-negative random variables $H = \{H_t : t = 0,1, \dots, T\}$, where H_t is $\mathcal{F}_t$-measurable. This is in contrast to European derivatives, which are represented by only one random variable H, denoting the payoff at maturity.

The decision of the time when to exercise the derivative lies entirely with the buyer and it need not be known at time 0. That is, the exercise time will generally be a random time τ. Since we are working in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we will assume that $\tau : \Omega \to \{0,1, \dots, T\}$ is a stopping time with respect to $\mathbb{F}$. If the buyer does not choose an exercise time, the derivative will automatically be exercised at the maturity time T, which is why τ takes values in $\{0,1, \ldots, T\}$ as opposed to $\{0,1, \ldots, T\} \cup \{\infty\}$.

If the buyer chooses an exercise strategy τ, then they receive a payoff of H_τ at the random time τ. For instance, the buyer may choose to exercise as soon as the payoff reaches or exceeds a pre-specified target $C > 0$, or at maturity time T if that never occurs. This decision strategy can be represented by the stopping time τ defined by

$$\tau(\omega) := inf\{t \in \mathbb{I}: H_t(\omega) \geq C\} \wedge T.$$

Given an American derivative $H = \{H_t: t \in \mathbb{I}\}$, we would like to perform the following:

1. Find the *price* of H at time zero, $\pi(H)$;
2. Using the proceeds from the sale of H at time zero, we want to find a trading strategy φ, with initial capital $\pi(H)$ and value process $V = V(\varphi)$, that *hedges* H, in the sense that

   $$V_\tau \geq H_\tau \quad \mathbb{P} - \text{a.s.},$$

   for any exercise strategy τ. Note that because of the variety of exercise strategies that can be chosen by the buyer, we cannot always guarantee that $V_\tau = H_\tau$ almost surely for every strategy.

Finally, we mention that an ordinary European derivative H can also be viewed as an American derivative if we let $H_T = H$ and $H_t = 0$ for $t < T$.

3.2.3 Notes: Exercise Strategies

As discussed in the previous section, a hedging strategy φ must satisfy

$$V_\tau(\varphi) \geq H_\tau \quad \mathbb{P} - \text{a.s.},$$

for any exercise strategy τ.

We will assume that the market is arbitrage-free and complete, so that there is exactly one equivalent market measure $\mathbb{P}^*$. We will also assume that $H_t \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}^*)$ for each $t \in \mathbb{I}$.

Let $\mathcal{T}$ be the set of all exercise strategies, i.e.

$$\mathcal{T} := \{\tau: \Omega \to \mathbb{I}: \tau \text{ is a stopping time}\}.$$

We call τ^* an *optimal exercise strategy* if

$$\mathbb{E}^*(H_{\tau^*}) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^*(H_\tau).$$

Now define the stochastic process $U = \{U_t: t \in \mathbb{I}\}$ by backward induction as follows:

$$U_T := H_T, \quad U_t := \max\{H_t, \mathbb{E}^*(U_{t+1}|\mathcal{F}_t)\} \quad t = T-1, \dots, 0.$$

The process U is called the *Snell envelope* of H.

The process U is the smallest supermartingale that dominates H. That is, U satisfies the following:

1. U is a supermartingale.
2. $U_t \geq H_t$ a.s. for each $t \in \mathbb{I}$.

3 If $\widetilde{U}$ is another process satisfying **1** and **2**, then $\widetilde{U}_t \geq U_t$ a.s. for all $t \in \mathbb{I}$.

The first part is clear since $U_t \coloneqq \max\{H_t, \mathbb{E}^*(U_{t+1}|\mathcal{F}_t)\} \geq \mathbb{E}^*(U_{t+1}|\mathcal{F}_t)$ for each t. The second part is also clear for a similar reason. For the third part, note that $\widetilde{U}_T \geq H_T = U_T$, and obtain the rest of the inequalities through backward induction.

We will see in the next section that the process U is actually the value (or price) of the American derivative H. The intuition behind this is as follows: at time $t = T$, the price of H is simply equal to the payoff H_T. At time $t = T - 1$, the price of H should be enough to pay for any exercise at the time – i.e. $U_{T-1} \geq H_{T-1}$ – and also be enough to cover subsequent values of H if the derivative is not exercised – i.e. $U_{T-1} \geq \mathbb{E}^*(H_T|\mathcal{F}_{T-1})$. Thus, $U_{T-1} = \max\{ H_{T-1}, \mathbb{E}^*(H_T|\mathcal{F}_{T-1})\}$ is the minimal price required to hedge the derivative.

Now consider the following exercise strategy τ_0:

$$\tau_0 \coloneqq \inf\{ t \in \mathbb{I}: H_t = U_t\}.$$

Since H and U are adapted, τ_0 is a stopping time, and since $H_T = U_T$, we have $\tau_0 \leq T$.

Theorem 2.1

The following hold about τ_0:

1 *The stopped process U^{τ_0} is a martingale with respect to $\mathbb{P}^*$.*

2 *τ_0 is optimal and $U_0 = \mathbb{E}^*\left(H_{\tau_0}\right) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^*(H_\tau)$.*

To prove this result, first note that the second statement follows from the first. Indeed, if **1** holds, then by the OST,

$$U_0 = U_0^{\tau_0} = \mathbb{E}^*\left(U_{\tau_0}\right) = \mathbb{E}^*\left(H_{\tau_0}\right).$$

Since $U_\tau \geq H_\tau$ for every $\tau \in \mathcal{T}$ and U is a supermartingale, then for any $\tau \in \mathcal{T}$, we have

$$U_0 \geq \mathbb{E}^*(U_\tau) \geq \mathbb{E}^*(H_\tau).$$

Hence $\mathbb{E}^*\left(H_{\tau_0}\right) = U_0 \geq \mathbb{E}^*(H_\tau)$ for every $\tau \in \mathcal{T}$. For a proof of **1**, see Bingham & Kiesel (2004).

The following characterizes all optimal times:

Theorem 2.2

An exercise strategy $\tau^* \in \mathcal{T}$ is optimal if and only if $H_{\tau^*} = U_{\tau^*}$ a.s. and the stopped process U^{τ^*} is a martingale with respect to $\mathbb{P}^*$.

To see that the two conditions imply optimality, note that if $H_{\tau^*} = U_{\tau^*}$ a.s. and U^{τ^*} is a martingale, then

$$\mathbb{E}^*(U_{\tau^*}) = \mathbb{E}^*\left(U_T^{\tau^*}\right) = U_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}^*(H_\tau),$$

by the OST.

This theorem implies that if $\tau^* \in \mathcal{T}$ is any optimal strategy, then $\tau^* \geq \tau_0$. Thus, we will call τ_0 the *minimal optimal strategy*.

3.2.4 Transcript: Pricing an American Option

Hi, in this video we outline the steps involved in pricing an American option. We are going to always assume that the market is arbitrage-free and complete. Therefore, there exists a unique EMM, $\mathbb{P}^*$.

Let $H = \{H_t : t = 0, \dots, T\}$, which is a stochastic process, be an American option, or an American derivative in general. We define the *Snell envelope* of H as a new stochastic process, U, which is defined as $U_T \coloneqq H_T$, and $U_t \coloneqq max\big(H_t, E(U_{t+1}|\mathcal{F}_t)\big)$. This is true for $t < T$. So, this is defined by backward induction. This is a very important stochastic process as it will turn out that U_t is actually the price of an American option, H, at time t.

Now, the pricing and hedging of an American option proceeds as can be seen below. The details of these steps and how they are derived can be found in the notes and so we are just going to summarize them here.

1. Find the Snell envelope of H by performing the steps that we looked at earlier.
2. Find the Doob decomposition of the Snell envelope, U. In other words, we write U as follows: $U = U_0 + M - A$. This Doob decomposition is found with respect to the unique EMM, $\mathbb{P}^*$. Here, M is a martingale that starts at 0, $(M_0 = 0)$, with respect to $\mathbb{P}^*$, and A is an increasing process.
3. Next, we use the PRP, or martingale representation theorem, of X since we know that the market is complete and, therefore, X has the PRP and every martingale is a martingale transform with respect to X. We use the PRP to find a predictable process φ such that M is a martingale transform with respect to X by φ. Written in full:

$$M = (\varphi \bullet X).$$

4. Once we have found φ, we are going to choose a trading strategy, φ, with initial capital, U_0. In other words, the trading strategy can be parametrized as (U_0, φ), where U_0 is the initial capital that we use.
5. The strategy (U_0, φ) dominates H. In other words, ($V_\tau \geq H_\tau$ for any exercise strategy τ). This is what we want – in other words, it is a hedge of that, and it is minimal since, if we look at this list carefully, we have it that $V_\tau^* = H_\tau^*$, where τ^* is the minimal optimal trading strategy.

By following these steps, you will be able to price an American option. In the next video, we are going to look at a concrete example of how to apply all of these steps.

3.2.5 Notes: Pricing and Hedging

We now find the price of H and a hedging strategy for H.

Recall that a hedging strategy φ must satisfy the condition that

$$V_t(\varphi) \geq H_t \quad \forall t.$$

A hedging strategy φ is called a *minimal hedging strategy* if (in addition) there exists a stopping time $\tau^* \in \mathcal{T}$ such that

$$V_{\tau^*}(\varphi) = H_{\tau^*}.$$

Since U is a supermartingale, we can write (Doob decomposition)

$$U = U_0 + M - A,$$

where M is a martingale and A is an increasing process, both null at zero. Since the market is complete, by the predictable representation property of X, there exists a predictable process φ such that

$$M_t = \sum_{k=1}^{t} \varphi_k (X_k - X_{k-1}),$$

for every t. We choose the strategy φ with initial capital U_0, so that

$$V_t(\varphi) = U_0 + M_t \geq U_t \geq H_t,$$

for every t. Hence, if the buyer chooses a stopping strategy $\tau \in \mathcal{T}$, then the seller makes a profit of $V_\tau(\varphi) - H_\tau \geq 0$, and this profit is zero if and only if the strategy is optimal.

Thus, we will call U_0 the *price* (or minimal price) of the American derivative H and denote it by $\pi(H)$.

In summary, given an American derivative $H = \{H_t : t \in \mathbb{I}\}$, we perform the following steps:

1. Find the Snell envelope of H, denoted by U.
2. Find the Doob decomposition of U as

$$U = U_0 + M - A,$$

   where M is a martingale and A is increasing.
3. Using PRP, we can find a predictable process φ such that

$$M = (\varphi \bullet X).$$

4. Choose the trading strategy φ with initial capital U_0.
5. Then the value of the strategy dominates H – i.e. φ is a hedging strategy – and it is also minimal since, for instance, $V_{\tau_0} = H_{\tau_0}$.
6. The minimal price to hedge H is therefore given by $\pi(H) \coloneqq U_0$.

Now let us look at some examples. Consider a call option with payoff $H_t = (X_t - K)^+$, where K is the strike price. Since the function $x \mapsto (x - K)^+$ is convex, H is a submartingale with respect to $\mathbb{P}^*$ (since X is a martingale). Thus

$$U_T = H_T, \quad U_{T-1} = \max\big(H_{T-1}, \mathbb{E}^*(H_T | \mathcal{F}_{T-1})\big) = H_{T-1}.$$

Continuing, we see that $U = H$. This implies that

$$\pi(H) = U_0 = \mathbb{E}^*(H_T),$$

which is equal to the corresponding European call option price. Also, $\tau \equiv T$ is an optimal exercise time, implying that for a call option there is no benefit of early exercise.

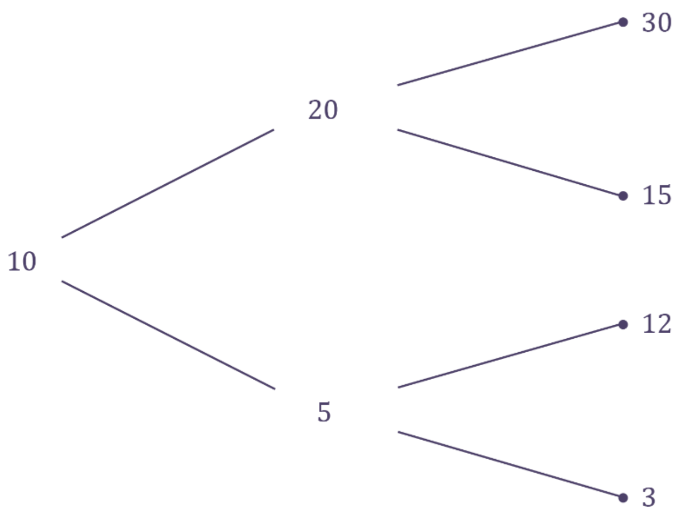
The corresponding situation of a put option will be studied in the next module, where we remove the assumption of zero interest rates.

3.2.6 Transcript: A Worked Example of Pricing an American Option

Hi, in this video we look at an example of pricing an American option.

We will consider a 2-period market that is defined as follows: the sample space, $\Omega = \{a, b, c, d\}$, as usual, $T = 2$, and we only have one risky asset, X, which is a stochastic process whereby $X = \{X_0, X_1, X_2\}$.

The stochastic process, X, evolves according to the following:



Solving the usual simultaneous equation gives the unique EMM, $\mathbb{P}^*$, as follows:

$$\mathbb{P}^* = \frac{1}{9}\delta_a + \frac{2}{9}\delta_b + \frac{4}{27}\delta_c + \frac{14}{27}\delta_d.$$

Now, consider the following stochastic process, which is an American option in this case:

ω	$H_0(\omega)$	$H_1(\omega)$	$H_2(\omega)$
a	$\frac{1}{5}$	1	2
b	$\frac{1}{5}$	1	0
c	$\frac{1}{5}$	0	1
d	$\frac{1}{5}$	0	0

So, this is the American derivative that we want to price and hedge. Therefore, we must first calculate the Snell envelope of H. The Snell envelope is U and so we must calculate $U_0(\omega)$, as well as $U_1(\omega)$ and $U_2(\omega)$. We can add this on to our original table as follows:

ω	$H_0(\omega)$	$H_1(\omega)$	$H_2(\omega)$	$U_0(\omega)$	$U_1(\omega)$	$U_2(\omega)$
a	$\frac{1}{5}$	1	2	$\frac{13}{27}$	1	2
b	$\frac{1}{5}$	1	0	$\frac{13}{27}$	1	0
c	$\frac{1}{5}$	0	1	$\frac{13}{27}$	$\frac{2}{9}$	1
d	$\frac{1}{5}$	0	0	$\frac{13}{27}$	$\frac{2}{9}$	0

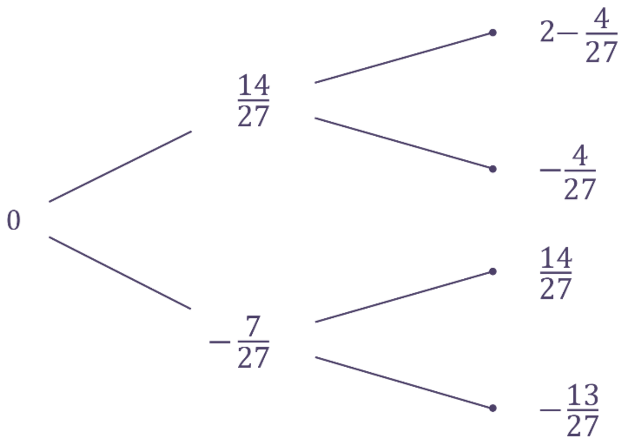
$U_2(\omega)$ is exactly the same as $H_2(\omega)$. For $U_1(\omega)$, we need to compare two things: $H_1(\omega)$ and the conditional expectation of $U_2(\omega)$ given the information at time 1. It turns out that, on the states a and b, $H_1(\omega)$ dominates that conditional expectation. However, on the states c and d, the conditional expectation is much bigger – we can calculate it using a weighted average, which is a normal conditional expectation calculation, and we get $\frac{2}{9}$ for both states. Finally, for $U_0(\omega)$, we compare the conditional expectation of $U_1(\omega)$ given the information at time 0, which, in this case, is just equal to the expectation – so, it's just a weighted average of these numbers, weighting them with these probabilities, and getting one number here. We then compare this number to $\frac{1}{5}$ and it turns out that this conditional expectation is greater than $\frac{1}{5}$. It will be $\frac{13}{27}$ throughout.

So, that is the Snell envelope. Next, let's calculate the minimal optimal exercise strategy, $\tau^*(\omega)$. We can again add it to our original table as follows:

ω	$H_0(\omega)$	$H_1(\omega)$	$H_2(\omega)$	$U_0(\omega)$	$U_1(\omega)$	$U_2(\omega)$	$T^*(\omega)$
a	$\frac{1}{5}$	1	2	$\frac{13}{27}$	1	2	1
b	$\frac{1}{5}$	1	0	$\frac{13}{27}$	1	0	1
c	$\frac{1}{5}$	0	1	$\frac{13}{27}$	$\frac{2}{9}$	1	2
d	$\frac{1}{5}$	0	0	$\frac{13}{27}$	$\frac{2}{9}$	0	2

On the first state, a, we can see that, for the first time, $H = U$. This happens at a at time 1. On the second state, b, $H = U$ at time 1. On state c, they are not equal at time 1 but they are equal at time 2, so that would be 2, and this is the same at state d.

So, that is the minimal optimal exercise time. From there, we calculate the Doob decomposition of U. We write $U = U_0 + M - A$, and it turns out that the martingale part of U, which is M, can be written as follows:



This is the Doob decompostion of U, and what we can see above is the martingale part of M. From here, we must use the PRP to write M as a martingale transform with respect to X, like this: $M = (\varphi \bullet X)$. This will be our hedging strategy, and U_0 will be the price of the American derivative, H. If we do this, we get that $\varphi_1 = \frac{7}{135}$, which is the amount that we should be investing in the asset X at time 1 in order to hedge this American derivative. On the states a and b, $\varphi_2 = \frac{2}{5} I_{\{a,b\}}$ and, on the states c and d, it is $\frac{1}{9}$. Written in full:

$$\varphi_1 = \frac{7}{135}, \quad \varphi_2 = \frac{2}{5} I_{\{a,b\}} + \frac{1}{9} I_{\{c,d\}}.$$

This gives us our hedging strategy for the American option.

That brings us to the end of the module. In the next module, we are going to focus on interest rate modeling.

3.2.7 Notes: A Worked Example

We consider a simple two-period market defined as follows: $\Omega = \{a, b, c, d\}$, $T = 2$, and only one risky asset $X = \{X_0, X_1, X_2\}$ defined as follows:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
a	$\frac{1}{4}$	10	20	30
b	$\frac{1}{4}$	10	20	15
c	$\frac{1}{4}$	10	5	12
d	$\frac{1}{4}$	10	5	3

We also pick $\mathbb{F} = \mathbb{F}^X$.

Solving the usual simultaneous equations, we find the unique EMM to be

$$\mathbb{P}^* = \frac{1}{9}\delta_a + \frac{2}{9}\delta_b + \frac{4}{27}\delta_c + \frac{14}{27}\delta_d.$$

Consider the following American derivative $H = \{H_0, H_1, H_2\}$

ω	$H_0(\omega)$	$H_1(\omega)$	$H_2(\omega)$
a	$\frac{1}{5}$	1	2
b	$\frac{1}{5}$	1	0
c	$\frac{1}{5}$	0	1
d	$\frac{1}{5}$	0	0

We first calculate the Snell envelope U of H. At time $t = 2$, we have $U_2 = H_2$.

Let $t = 1$. Then

$$U_1 = \max\{\mathbb{E}^*(U_2|\mathcal{F}_1), H_1\}.$$

Now,

$$\mathbb{E}^*(U_2|\mathcal{F}_1) = \frac{2\times\frac{1}{9}+0\times\frac{2}{9}}{\frac{3}{9}} I_{\{a,b\}} + \frac{1\times\frac{4}{27}+0\times\frac{14}{27}}{\frac{18}{27}} I_{\{c,d\}}$$
$$= \frac{2}{3} I_{\{a,b\}} + \frac{2}{9} I_{\{c,d\}}.$$

Since $H_1 = I_{\{a,b\}}$, it follows that

$$U_1 = \max\{\mathbb{E}^*(U_2|\mathcal{F}_1), H_1\} = I_{\{a,b\}} + \frac{2}{9} I_{\{c,d\}}.$$

Finally let $t = 0$. Then

$$\mathbb{E}^*(U_1|\mathcal{F}_0) = \mathbb{E}^*(U_1) = 1\times\frac{1}{3}+\frac{2}{9}\times\frac{2}{3} = \frac{13}{27}.$$

Since $H_0 = \frac{1}{5} < \frac{13}{27}$, it follows that

$$U_0 = \max\{\mathbb{E}^*(U_1|\mathcal{F}_0), H_1\} = \mathbb{E}^*(U_1|\mathcal{F}_0) = \frac{13}{27}.$$

The values of U can be summarized in the following table:

ω	$U_0(\omega)$	$U_1(\omega)$	$U_2(\omega)$
a	$\frac{13}{27}$	1	2
b	$\frac{13}{27}$	1	0
c	$\frac{13}{27}$	$\frac{2}{9}$	1
d	$\frac{13}{27}$	$\frac{2}{9}$	0

So, the price of H is $U_0 = \frac{13}{27}$. The price of the corresponding European derivative is

$$2 \times \frac{1}{9} + 1 \times \frac{4}{27} = \frac{10}{27},$$

which is less than its American counterpart.

Let us now calculate τ_0, the smallest optimal strategy. Recall that τ_0 is defined as

$$\tau_0(\omega) := \inf\{ t \in \{0,1,2\} : U_t(\omega) = H_t(\omega)\}.$$

In this example we have

ω	$\tau_0(\omega)$
a	1
b	1
c	2
d	2

Finally, we calculate the hedging strategy φ. We write

$$U = U_0 + M - A,$$

where M is a martingale and A is a positive increasing process. From Module 3, the formula for the Doob decomposition is

$$A_0 = 0, \quad A_1 = A_0 + \mathbb{E}^*(U_1 - U_0|\mathcal{F}_0), \quad \text{and } A_2 = A_1 + \mathbb{E}^*(U_2 - U_1|\mathcal{F}_1).$$

With $M_n = U_n - U_0 - A_n$, it is easy to check that these components are as given below:

ω	$A_0(\omega)$	$A_1(\omega)$	$A_2(\omega)$	$M_0(\omega)$	$M_1(\omega)$	$M_2(\omega)$
a	0	0	$\frac{1}{3}$	0	$\frac{14}{27}$	$2-\frac{4}{27}$
b	0	0	$\frac{1}{3}$	0	$\frac{14}{27}$	$-\frac{4}{27}$
c	0	0	0	0	$-\frac{7}{27}$	$\frac{14}{27}$
d	0	0	0	0	$-\frac{7}{27}$	$-\frac{13}{27}$

We need to find a predictable process φ such that $M = (\varphi \bullet X)$. Note that this is equivalent to finding a hedging strategy for a European derivative with payoff M_2. Hence, we get

$$\varphi_1 = \frac{M_1 - M_0}{X_1 - X_0} = \frac{7}{135},$$

and

$$\varphi_2 = \frac{M_2 - M_1}{X_2 - X_1} = \frac{2}{5} I_{\{a,b\}} + \frac{1}{9} I_{\{c,d\}}.$$

So, a hedging strategy for H begins with an initial capital of $\pi(H) = U_0 = \frac{13}{27}$ and invests φ_1 at time zero and φ_2 at time 1.

3.2.8 Notes: Problem Set

Problem 1

Consider a market $\big((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})X\big)$ with one risky asset $X = \{X_0, X_1, X_2\}$ and random variables τ_1, τ_2, τ_3 defined as follows:

$$\Omega = \{a, b, c, d\}, \quad \mathbb{F} = \mathbb{F}^X, \quad \mathcal{F} = 2^\Omega.$$

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$\tau_1(\omega)$	$\tau_2(\omega)$	$\tau_3(\omega)$
a	$\frac{1}{6}$	15	24	30	1	2	1
b	$\frac{1}{4}$	15	24	20	1	1	1
c	$\frac{1}{4}$	15	12	14	2	2	2
d	$\frac{1}{3}$	15	12	10	2	2	∞

Which of τ_1, τ_2 and τ_3 are valid exercise strategies for an American option?

Solution:

We can discard $\tau_3(\omega)$ because, from the lecture notes, if the buyer does not choose an exercise time, the derivative will automatically be exercised at the maturity time T, which is why τ takes values in $\{0,1,\dots,T\}$ as opposed to $\{0,1,\dots,T\} \cup \{\infty\}$.

Moreover, since we are working in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\tau(\omega)$ can not show different values at $t = 2$ for the same filtration (at $t = 1$). In other words, $\tau_2(\omega)$ cannot have value 2 for path a and value 1 for b. Thus, the only valid exercise strategy for an American option is $\tau_1(\omega)$.

Problem 2

Consider a market $\left((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})X\right)$ with one risky asset $X = \{X_0, X_1, X_2\}$ and random variables H_0, H_1, H_2 defined as follows:

$$\Omega = \{a, b, c, d\}, \quad \mathbb{F} = \mathbb{F}^X, \quad \mathcal{F} = 2^{\Omega}.$$

ω	$\mathbb{P}^*(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H_0(\omega)$	$H_1(\omega)$	$H_2(\omega)$
a	$\frac{1}{4}$	15	20	22	10	4	5
b	$\frac{1}{4}$	15	20	18	10	4	1
c	$\frac{1}{4}$	15	10	14	10	8	7
d	$\frac{1}{4}$	15	10	6	10	8	3

Let U be the Snell envelope of H with respect to $\mathbb{P}^*$. Find $\mathbb{E}^*(U_1)$.

Solution:

We first calculate the Snell envelope U of H. At time $t = 2$, we have $U_2 = H_2$. Let $t = 1$. Then

$$U_1 = max\{\mathbb{E}^*\,(U_2|\mathcal{F}_1), H_1\}.$$

Now,

$$\mathbb{E}^*(U_2|\mathcal{F}_1) = \frac{5 \times \frac{1}{4} + 1 \times \frac{1}{4}}{\frac{2}{4}} I_{\{a,b\}} + \frac{7 \times \frac{1}{4} + 3 \times \frac{1}{4}}{\frac{2}{4}} I_{\{c,d\}}$$
$$= 3I_{\{a,b\}} + 5I_{\{c,d\}}.$$

It follows that

$$U_1 = max\{\,\mathbb{E}^*(U_2|\mathcal{F}_1), H_1\} = 4I_{\{a,b\}} + 8I_{\{c,d\}}.$$

Finally let $t = 0$. Then

$$\mathbb{E}^*(U_1|\mathcal{F}_0) = \mathbb{E}^*(U_1) = 4 \times \frac{1}{2} + 8 \times \frac{1}{2} = 6.$$

Thus, the solution is $\mathbb{E}^*(U_1) = 6$.

3.3 Additional Resources

Bingham, N. & Kiesel, R. (2004). *Risk-Neutral Valuation*. London: Springer-Verlag.

Föllmer, H. & Schleid, A. (2002). *Stochastic Finance: An Introduction to Discrete Time*. De Gruyter.

Shiryaev, A. (1999). *Essentials of Stochastic Finance: Facts, Models, Theory*. World Scientific.

Shreve, S. (2004). *Stochastic Calculus for Finance I*. New York: Springer-Verlag.