Discrete-time Stochastic Processes
Module 4

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 4 of Discrete-time Stochastic Processes, entitled Trading in Discrete Time. It consists of four sets of notes, four lecture transcripts, a problem set, and a collaborative review task.

2. Course Context

Discrete-time Stochastic Processes is the third course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The course introduces derivative pricing in discrete time. It begins with measure-theoretic probability and stochastic processes, with an emphasis on discrete-time martingales. These ideas are then applied to the pricing of derivatives in discrete time, followed by an introduction to interest rate and credit risk modeling.

2.1 Course-level Learning Outcomes

Upon completion of the Discrete-time Stochastic Processes course, you will be able to:

1. Understand the language of measure-theoretic probability.
2. Understand stochastic processes and their applications.
3. Understand the theory of discrete-time martingales.
4. Define trading strategies in discrete time.
5. Create replicating portfolios in discrete time.
6. Model stock price movements on a binomial tree.
7. Price and hedge European derivatives in discrete time.
8. Price and hedge exotic European derivatives in discrete time.
9. Price and hedge American derivatives on a binomial tree.
10. Construct a simple interest rate model on a tree.
11. Price interest rate derivatives on a tree.

2.2 Module Breakdown

The Discrete-time Stochastic Processes course consists of the following one-week modules:

1. Probability Theory
2. Stochastic Processes
3. Discrete Martingales
4. Trading in Discrete Time
5. The Binomial Model
6. American Derivatives
7. An Introduction to Interest Rate Models

3. Module 4:

Trading in Discrete Time

By viewing the values of variables that occur at distinct, individual points in time, we have a useful lens through which the frequently erratic practice of trading can be studied. This module introduces the concept of a trading strategy in discrete time, as well as how it can be used to define concepts like no-arbitrage and completeness of a financial market. It also discusses the two fundamental theorems of asset pricing in discrete time.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Define the notion of trading, and the value and gains of a strategy, in discrete time.
2. Find replicating portfolios in discrete-time trading.
3. Define the notions of arbitrage and market completeness in discrete time.
4. Find an equivalent martingale measure.
5. State both fundamental theorems of asset pricing.

3.2 Transcripts and Notes

3.2.1 Transcript: Portfolios and Strategies - Definitions

Hi, in this video we introduce *portfolios* and *strategies*. The definitions that we are going to introduce are very important for the rest of the course.

Consider a market which consists of $d+1$ primary assets. So, $S = \left(S^0, S^1, S^2, \dots S^d\right)$. S^0 is what we call the *riskless asset* and it represents the bank account. The assets, S^1 up to S^d, are what we call the *risky assets*. You can think of them as shares in a company, for instance. We will assume that the riskless bank account evolves as follows: S_t^0 at time t is equal to the previous value, $S_{t-1}^0(1+r_t)$, where r_t is the interest between time $t-1$ and time t. Therefore, it must be predictable – so, r is a predictable stochastic process.

Throughout this video, it will be useful to work with the discounted assets, which we will define as X. So, $X^i = \frac{S^i}{S^0}$. What this means is that X^i at time t is just S_t^i divided by S^0. Written in full:

$$X^i = \frac{S^i}{S^0}, \quad X_t^i = \frac{S_t^i}{S_t^0}.$$

This represents the units in terms of the riskless asset – the price of X in terms of the price of S_0, which is the riskless bank account. It is very convenient to work with these units, and we will write them as follows: $X = \left(X^0, X^1, \dots, X^d\right) = \left(1, X^1, \dots, X^d\right)$ since $\frac{X^0}{X^0}$ is just equal to 1.

We then introduced the notion of a *trading strategy*, which is a predictable process, φ, which, in this case, is in many dimensions. It consists of the following components: $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$, where φ_t^i is the number of units of asset i that were invested in. As an example of a trading strategy, φ^0 will be the number of units in the riskless asset that were invested in at time t.

We also define the value of a trading strategy φ as $V(\varphi)$, which is a stochastic process consisting of $\{V_t(\varphi): t = 0,1, \dots, T\}$, where T is some time horizon. The stochastic process can be defined as follows: $V_0(\varphi) = \varphi_1^0 X_0^0 + \varphi_1^1 X_0^1 + \dots + \varphi_1^d X_0^d$.

Basically, this is number of shares that are invested in asset 0 at time 0, times the price, plus the number of shares that are invested in asset 1 at time 0, times the price of asset 1 at time 0, and so on. We can simply write this as $\varphi_1 \cdot X_0$, which are all the stochastic processes at time 0. (Note that this is a dot product, not a martingale transform yet.) Then, for any other time, $V^t(\varphi)$ will simply be equal to $\varphi_t \cdot X_t$, which we can write as $\varphi_t^0 \times X_t^0$ and so on up to $\varphi_t^d \times X_t^d$. That, therefore, is the value of the trading strategy at time t.

We also define the gains from trading using the strategy $G(\varphi)$, where $\{G_t(\varphi): t = 0,1, \dots, T\}$. As $G_0(\varphi)$ is equal to 0 (meaning that there are no gains from trading at time 0) and $G_t(\varphi)$for times t greater than or equal to 1, $G_t(\varphi)$ will be equal to the sum of the martingale transform. This will be:

$$\sum_{k=1}^{t} \varphi_k . \Delta X_k,$$

which is the dot product because we want to calculate it for each component of X.

Note that the gains from trading the riskless asset — in other words, the first component of the dot product — will be equal to 0, since the riskless asset is constant and equal to 1. That is the gains process.

We call the strategy *self-financing*. We say that φ is self-financing if the following happens:

$$\varphi_t \cdot X_t = \varphi_{t+1} \cdot X_t.$$

What this means, intuitively, is that at time t you will have the strategy φ_t, which moves us from time $t-1$ to time t. φ_t is therefore the number of shares that we buy at time $t-1$ and sell at time t. As a result, φ_t times X_t is the value of this strategy at time t. The amount that we have invested in total at time t is $\varphi_t \cdot X_t$. (That's the left-hand side of the equation.) On the right-hand side, $\varphi_{t+1} \cdot X_t$, would be the result if we changed to a new strategy (namely, φ_{t+1}), which would be the number of shares that we hold between time t and time $t+1$. If we make that change, the value of the trading strategy does not change between that transition. Another way of putting this is that there is no injection of cash and there is no withdrawal from the trading strategy — and that's what we call a self-financing trading strategy.

This is also equivalent to the value of the process $V_t(\varphi)$, which is equal to $V_0(\varphi)$ plus the gains from trading. Written in full:

$$V_t(\varphi) = V_0(\varphi) + G_t(\varphi).$$

Since the gains from trading the riskless asset are equal to 0, another way of interpreting this is that for a self-financing trading strategy, all we need to specify is the starting amount, V_0, and the holding in the risky assets. So, we do not necessarily need to specify the holding in the riskless asset if a strategy is self-financing.

Now that we've completed the definitions of portfolios and strategies, in the next notes and video we're going to focus on arbitrage.

3.2.2 Notes: Portfolios and Strategies

We consider a financial market with a finite time horizon and trading in discrete time. Specifically, we will assume that the market consists of $d+1$ assets whose prices are discrete-time stochastic processes. We let $T \in \mathbb{N}^+$ be the time horizon and assume that the trading dates are $t = 0,1,2,\ldots,T$. We let $(B,S) = (B,S^1,\ldots,S^d)$ be a stochastic process that represents that prices of all the $d+1$ assets. This means that for each $i = 1,\ldots,d$, $S^i = \{S_t^i : t = 0,1,\ldots,T\}$ is a discrete-time stochastic process.

To properly talk about randomness and the flow of information, we will work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume that (B,S) is adapted to $\mathbb{F}$, where $\mathbb{F} = \{\mathcal{F}_t : t = 0,1,\ldots,T\}$. We will also assume that Ω has finitely many elements.

We call the tuple $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), (B,S))$ a market.

We will assume that the asset B is a riskless bank account with a predictable interest rate process $r = \{r_t : t = 1,2,\ldots,T\}$. That is,

$$B_t = B_{t-1}(1 + r_t), \quad t = 1,2,\ldots,T,$$

where $r_t \succ -1$ for all t.

To simplify the presentation – and without loss of generality – we will assume that B has a strictly positive price process (i.e. $B > 0$) and express the prices of all the assets in units of B. Asset B is said to be a *numeraire*. To this end, we define the discounted prices $X = (X^1,\ldots,X^d)$ by

$$X^i := \frac{S^i}{\mathrm{B}}, \quad i = 1,\ldots,d.$$

Thus, the discounted prices are $(1, X) = \left(1, X^1, \dots, X^d\right)$. From now onwards, we will work with the discounted prices.

A *trading strategy* is a predictable process $(\eta, \varphi) = \left(\eta, \varphi^1, \dots, \varphi^d\right)$, where φ_t^i is the number of units of asset i held from time $t-1$ to time t and η_t is the amount invested in the riskless asset over the same period. The *value* of a trading strategy (η, φ) is the stochastic process $V\big((\eta, \varphi)\big) = \big\{V_t\big((\eta, \varphi)\big): t = 0,1, \dots, T\big\}$ defined by

$$V_0\big((\eta, \varphi)\big) = (\eta_1, \varphi_1) \cdot (1, X_0) = \eta_1 + \sum_{i=1}^{d} \varphi_1^i X_0^i$$

and

$$V_t\big((\eta, \varphi)\big) := (\eta_t, \varphi_t) \cdot (1, X_t) = \eta_t + \sum_{i=1}^{d} \varphi_t^i X_t^i \quad t \geq 1.$$

The *gains process* associated with (η, φ) is the stochastic process $G(\varphi) = \{G_t(\varphi): t = 0,1, \dots, T\}$ defined by

$$G_0(\varphi) := 0, \quad G_t(\varphi) := \sum_{k=1}^{t} \varphi_k \cdot (X_k - X_{k-1}) \text{ for } t = 1,2, \dots, T.$$

(Note that there are no gains from trading the riskless asset.)

A trading strategy φ is self-financing if

$$(\eta_t, \varphi_t) \cdot (1, X_t) = (\eta_{t+1}, \varphi_{t+1}) \cdot (1, X_t) \text{ for all } t = 1, \dots, T.$$

The interpretation of this condition is that the value of the strategy only changes through the gains from trading – there is no influx or outgo of funds from investment at any point in time.

Equivalently, φ is self-financing if and only if

$$V_t\big((\eta,\varphi)\big) = V_0\big((\eta,\varphi)\big) + G_t(\varphi) \text{ for } t = 0,1,\ldots,T.$$

Since there are no gains from trading asset 0, we see that for a self-financing strategy knowledge of the initial value of the strategy and $\varphi^1,\ldots,\varphi^d$ allows us to recover uniquely the amount invested in the riskless asset via the formula:

$$V_0\big((\eta,\varphi)\big) + G_t(\varphi) = \eta_t + \sum_{i=1}^{d} \varphi_t^i X_t^i.$$

Thus, we can (and will) represent a trading strategy by specifying an initial value v_0 and predictable holdings in the other assets $\varphi = \big(\varphi^1,\ldots,\varphi^d\big)$. We will write this as (v_0,φ).

3.2.3 Transcript: A Worked Example of Arbitrage

Hi, in this video we introduce the notion of an arbitrage.

Consider a market which has the following components: $\left((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X\right)$ – where Ω is the sample space, $\mathcal{F}$ is the σ-algebra, $\mathbb{F}$ is the filtration, and $\mathbb{P}$ is the probability measure – together with X which, remember, is a d-dimensional process.

The first component, which represents the riskless asset, is always equal to 1. A strategy φ is called an *arbitrage strategy* if it satisfies the following conditions:

1. $V_0(\varphi) = 0$. In other words, the starting capital is 0.
2. $V_T(\varphi) \geq 0$ with probability 1. So, almost surely, this strategy never makes a loss.
3. There is a positive probability that this strategy makes a positive profit: $\mathbb{P}(V_T(\varphi) > 0) > 0$. So, the probability that V_T is greater than 0 is also positive. We call such a strategy is an arbitrage strategy because, intuitively, it's just an open opportunity whereby one can make a riskless profit.

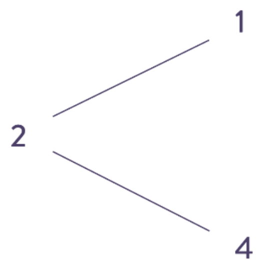
A market is arbitrage-free if there are no arbitrage strategies. Let's look at an example.

Consider the sample space Ω which consists of two points, $\{a, b\}$. We will take T to be equal to 1, making it a one time horizon and a filtration, $\mathcal{F}_1$, which is equal to the trivial σ-algebra. $\mathcal{F}_2 = \mathcal{F} = 2^{\Omega}$, which is just the powerset of Ω. And finally, we will define the probability measure $\mathbb{P}$ to weight these two equally: $\mathbb{P} = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$. Written in full:

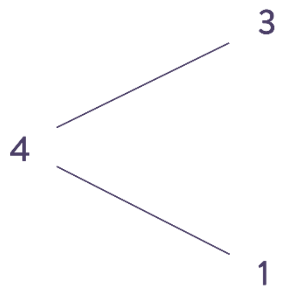
$$\Omega = \{a, b\}, \quad T = 1, \quad \mathcal{F}_1 = \{\emptyset, \Omega\}$$
$$\mathcal{F}_2 = \mathcal{F} = 2^{\Omega}, \quad \mathbb{P} = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b.$$

Next, we must define the following stochastic processes, or asset prices, which consist of two risky assets: $(1, X^1, X^2)$ – that means that $d = 2$ in our model.

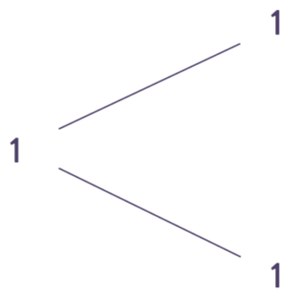
X^1 is given by the following tree diagram:



Similarly, X^2 is given by:



Of course, the riskless asset, 1, is always equal to 1 under all states:



Now, consider the following trading strategy: φ_1 (we only need to specify φ_1 because the time horizon is equal to 1 in this case), $\varphi_1 = (10,1,-3)$, which is $\mathcal{F}_0$-measurable.

Let's now calculate $V_0(\varphi)$, which will be the dot product of $(10,1,-3) \cdot \ldots$ we must dot it with the values of the process at time 0, which is $(1,2,4)$. That gives us $10 + 2$, which is $12, -12$, which is equal to 0. Written in full:

$$V_0(\varphi) = (10,1,-3) \cdot (1,2,4) = 0.$$

Now let's calculate $V_1(\varphi)$. Here, we will have to calculate the dot product of $(10, 1, -3)$, with whatever X^1 is. So, that is equal to... on the two states of the world, the top one will be a dot product between $(10, 1, -3)$ $\cdot$, on the first state this is $(1, 4$ (as this is what X^1 is), 3 (as this is what X^2 is)), or, on the second state, $(10,1 - 3) \cdot (1,1,1)$. That, therefore, gives us two possibilities. The first one will be $10 + 4$, which is $14, 9 = 5$. The second one will be $10 + 1$, which is $11, -3 = 8$.

$$\begin{aligned} V_1(\varphi) = (10,1,-3) \cdot X^1 &= \begin{cases} (10,1,-3) \cdot (1,4,3) \\ (10,1,-3) \cdot (1,1,1) \end{cases} \\ &= \begin{cases} 5 \geq 0 \\ 8 \end{cases}. \end{aligned}$$

As you can see, the random variable V_1 is always non-negative and is positive with positive probability. In fact, probability that it is positive is equal to 1 in this case. Therefore, φ is an arbitrage opportunity.

3.2.4 Notes: Arbitrage

From now until the end of the course, we assume that all trading strategies are self-financing. Also, to simplify notation, we will sometimes write $\varphi^0 := \eta$ and talk about a strategy $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$. We will also denote the discounted riskless asset by X^0, i.e. $X^0 \equiv 1$.

A trading strategy φ is an arbitrage strategy if

$$V_0(\varphi) = 0, \quad V_T(\varphi) \geq 0 \ \mathbb{P} - \text{a.s. and } \mathbb{P}(V_T(\varphi) > 0) > 0.$$

In words, an arbitrage strategy is a strategy that requires no investment to enter into ($V_0 = 0$), yet the strategy poses no risk of loss ($V_T \geq 0$) and a possibility of making a positive gain ($\mathbb{P}(V_T(\varphi) > 0) > 0$). A market model is arbitrage-free if it does not allow for any arbitrage strategies/opportunities.

Let $T = 1$, $d = 2$ and $\Omega = \{a, b\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = 2^\Omega$, $\mathbb{P} = 0.5\delta_a + 0.5\delta_b$. Define $(X^0, X^1, X^2) = (1, X^1, X^2)$ as follows:

ω	$(X_0^0(\omega), X_0^1(\omega), X_0^2(\omega))$	$(X_1^0(\omega), X_1^1(\omega), X_1^2(\omega))$
a	(1,2,4)	(1,4,3)
b	(1,2,4)	(1,1,2)

Define the strategy φ by

$$\varphi_1 = (\varphi_1^0, \varphi_1^1, \varphi_1^2) = (10, 1, -3).$$

Then

$$V_0(\varphi) = (10, 1, -3) \cdot (1, 2, 4) = 10 + 2 - 12 = 0,$$

and

$$V_1(\varphi) = \varphi_1 \cdot X_1 = \begin{cases} 10 + 4 - 9 = 5 & \omega = a \\ 10 + 1 - 6 = 5 & \omega = b. \end{cases}$$

Hence φ is an arbitrage strategy.

In general, it is much more difficult to show that a market has no arbitrage opportunities. We will now find conditions on the market that are equivalent to the no arbitrage condition.

Let $\mathbb{P}^*$ be a probability measure on $(\Omega, \mathcal{F})$. We call $\mathbb{P}^*$ an *equivalent martingale measure* (EMM) for X if $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ (i.e. $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{P}^*(A) = 0 \quad \forall A \in \mathcal{F}$) and X is an $(\mathbb{F}, \mathbb{P}^*)$-martingale. We will denote by $\mathcal{P}$ the (possibly empty) set of EMM's for X. We will also write $\mathbb{E}^*$ to denote the expectation with respect to $\mathbb{P}^*$.

Let $T = 2$, $d = 1$ and $\Omega = \{a, b, c, d\}$, $\mathbb{P} = \frac{1}{4}(\delta_a + \delta_b + \delta_c + \delta_d)$ and

ω	$\mathbb{P}^*(\{\omega\})$	$(1, X_0(\omega))$	$(1, X_1(\omega))$	$(1, X_2(\omega))$
a	$\frac{1}{9}$	(1,12)	(1,18)	(1,36)
b	$\frac{2}{9}$	(1,12)	(1,18)	(1,9)
c	$\frac{1}{6}$	(1,12)	(1,9)	(1,12)
d	$\frac{1}{2}$	(1,12)	(1,9)	(1,8)

Then $\mathbb{P}^*$ is an EMM for $(1, X)$. (1 is trivially a martingale, so we will only concentrate on X.) We choose the filtration to be the natural filtration of X given by

$$\mathcal{F}_0^X = \{\emptyset, \Omega\},\ \mathcal{F}_1^X = \big\{\emptyset, \Omega, \{a, b\}, \{c, d\}\big\} \text{ and } \mathcal{F}_2 = \sigma(\{X_0, X_1, X_2\}) = 2^\Omega.$$

The equivalence of $\mathbb{P}$ and $\mathbb{P}^*$ is clear since the only null set is the empty set for both of them. Now we show that X is a $\mathbb{P}^*$-martingale.

First note that $\mathbb{E}(|X_n|) \leq 36 < \infty$ for all $n \in \mathbb{I}$.

$X_n \in m\mathcal{F}_n^X \ \forall n \in \mathbb{I} = \{0,1,2\}$ is trivial since we are using the natural filtration of X.

$$\mathbb{E}^*\left(X_1 \middle| \mathcal{F}_0^X\right) = \mathbb{E}^*(X_1|\{\emptyset, \Omega\}) = \mathbb{E}^*(X_1) = \frac{1}{9} \times 18 + \frac{2}{9} \times 18 + \frac{1}{6} \times 9 + \frac{1}{2} \times 9 = 12 = X_0$$

$$\mathbb{E}^*\left(X_2 \middle| \mathcal{F}_1^X\right) = \alpha I_{\{a,b\}} + \beta I_{\{c,d\}} = \frac{\frac{1}{9}X_2(a) + \frac{2}{9}X_2(b)}{\frac{1}{9} + \frac{2}{9}} I_{\{a,b\}} + \frac{\frac{1}{6}X_2(c) + \frac{1}{2}X_2(d)}{\frac{1}{6} + \frac{1}{2}} I_{\{c,d\}}$$
$$= 18 I_{\{a,b\}} + 9 I_{\{c,d\}}.$$

Clearly, $\mathbb{E}^*\left(X_2 \middle| \mathcal{F}_1^X\right) = X_1$, so X is a martingale.

Now suppose that $\mathbb{P}^*$ is an EMM for X and let φ be a strategy with $V_0(\varphi) = 0$ and $V_T(\varphi) \geq 0$. Since $V_t(\varphi) = V_0(\varphi) + \sum_{k=1}^{t} \varphi_k (X_k - X_{k-1})$ is a martingale transform with $V_T(\varphi) \geq 0$, it follows that $V(\varphi)$ is a martingale too. (See Shiryaev's "Essentials of Stochastic Finance".) Hence $\mathbb{E}^*\big(V_T(\varphi)\big) = 0$, which implies that $\mathbb{P}(V_T(\varphi) > 0) = 0$. So, there are no arbitrage opportunities.

The converse of the previous also holds, i.e. absence of arbitrage opportunities also implies the existence of an EMM. This remarkable result is called *The Fundamental Theorem of Asset Pricing I* (FTAP I).

Theorem [FTAP I]

For a financial market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, the following are equivalent:*

1. *There are no arbitrage opportunities.*
2. $\mathcal{P} \neq 0$.

This theorem gives us an alternative way of deciding whether or not a market admits arbitrage opportunities.

3.2.5 Transcript: Equivalent Martingale Measures

Hi, in this video we introduce martingale measures.

Consider a market that consists of the following components: $\big((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X\big)$, where X is the vector of discounted assets.

A measure, $\mathbb{P}^*$, which is defined on the same σ-algebra, $\mathcal{F} \to [0,1]$, is called an *Equivalent Martingale Measure*, denoted by EMM, if it satisfies the following conditions:

1. $\mathbb{P}^* \equiv \mathbb{P}$, in the sense that they have the same null set. Another way of writing this is that $\mathbb{P}^*$ is absolutely continuous with respect to $\mathbb{P}$ and $\mathbb{P}$ is also absolutely continuous with respect to $\mathbb{P}^*(\mathbb{P}^* \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{P}^*)$.
2. All of the components of X are an $(\mathbb{F}, \mathbb{P}^*)$ martingale. So, X is a martingale with respect to this new probability measure. We call it an EMM.

We will denote by P the set of all EMM's for X. Now, this is very important in classifying markets with arbitrage opportunities. The theorem that covers all of that is called *The Fundamental Theorem of Asset Pricing I* (FTAP I).

So, FTAP I says that the following two notions are equivalent:

1. The market has no arbitrage strategies.
2. The set P, of EMM's for X is non-empty.

In a nutshell, what the theorem says is that the absence of arbitrage opportunities is equivalent to the existence of at least one EMM for X.
Let's look at an example. Consider $\Omega = \{a, b, c, d\}, \mathcal{F} = 2^{\Omega}$ and $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$.

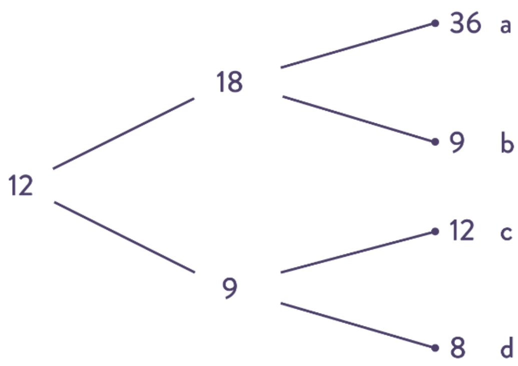
$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \sigma\big(\{\{a, b\}, \{c, d\}\}\big), \quad \mathcal{F}_2 = \mathcal{F}.$$

We are going to define $\mathbb{P}$ to be $\frac{1}{4}$ times the sum of the Dirac measures. Written in full:

$$\mathbb{P} = \frac{1}{4}(\delta_a + \delta_b + \delta_c + \delta_d).$$

We are also going to assume that $d = 1$, so there is one risky asset and that the time horizon $T = 2$.

The risky asset evolves as follows:

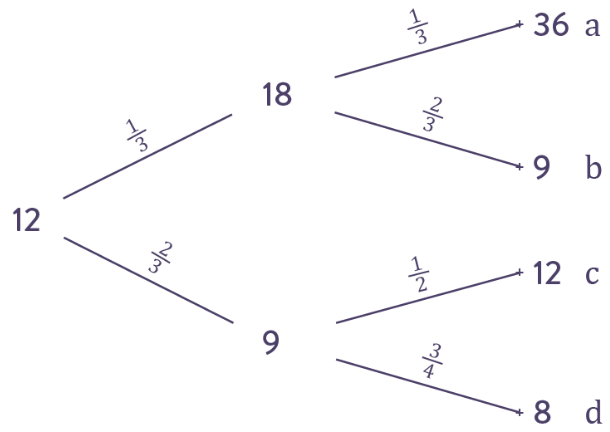


This is the stock X^1, where 12 is $t = 0$, 18 is $t = 1$, and 36 is $t = 2$.

So, we want to find an EMM for X. If we define the following probability measure, we can show that $\mathbb{P}^*$ is an EMM for X:

ω	$\mathbb{P}^*(\{\omega\})$
a	$\frac{1}{9}$
b	$\frac{2}{9}$
c	$\frac{1}{6}$
d	$\frac{1}{2}$

Let's show this. So, first of all, on the branches, or corresponding conditional probabilities, we get $\frac{1}{3}$ on the branch leading from 12 to 18 and $\frac{2}{3}$ on the branch leading from 12 to 9. On the branch leading from 18 to 36, we get $\frac{1}{3}$, and $\frac{2}{3}$ on the branch leading from 18 to 9. Similarly, we will get $\frac{1}{4}$ on the branch leading from 9 to 12 and $\frac{3}{4}$ on the branch leading from 9 to 8.



To check that X is a martingale with respect to this probability measure, we have to calculate a weighted average of these. So, if we take $\frac{1}{3} \times 36 + \frac{2}{3} \times 9$, for instance, we will get 18. Similarly, $\frac{1}{4} \times 12 + \frac{3}{4} \times 8 = 9$. As we can see, they correspond to the two values in the middle of the tree, 18 and 9. The same applies to the first section of the tree: if we take $\frac{1}{3} \times 18$, which equals 6, plus $\frac{2}{3} \times 9$, which also equals 6, we get 12.

Therefore, X, the second component at least, is a martingale with respect to this probability measure. The riskless asset, since it is always constant and equal to 1, will always be a martingale.

Now that we have covered martingale measures, in the next notes and video, we are going to look at pricing and completeness.

3.2.6 Notes: Pricing and Replication

We will now assume that $\mathcal{F}_T = \mathcal{F}$.

A *contingent claim* H is an $\mathcal{F}_T$-measurable random variable. We call a contingent claim H a *derivative* of the underlying assets X if H is measurable with respect to $\mathcal{F}_T^X$.

We think of H as a liability that expires at time T. Such a liability is called a *European contingent claim*. In a later module we will meet contingent claims whose expiry date can be at any time before time T. Such contingent claims are called *American contingent claims*.

Here are some common examples of European derivatives H (with $d = 1$ and $X = X^1$):

- Call option: $H = (X_T - K)^+$, where $K > 0$ is called the strike price.
- Put option: $H = (K - X_T)^+$, where $K > 0$ is called the strike price.
- Asian call option: $H = \left(\frac{1}{T+1}\sum_{t=0}^{T} X_t - K\right\}^+$, where K is the strike price.
- Digital option: $H = I_{\left\{\max\limits_{0 \le t \le T} X_t \ge K\right\}}$, where $K > 0$ is the barrier.

Given a contingent claim H, we want to answer the following questions:

1. What is the price of H at time 0? That is, how much should one pay at time 0 in return for the random payment H at time T?
2. Having sold the contingent claim at time 0, how can the seller protect himself against the random outflow H by investing in the primary assets X?

The first question has to do with *pricing*, while the second one is to do with *replication*.

A strategy/portfolio φ such that $V_T(\varphi) = H$ $\mathbb{P}$-a.s. is called a *replicating portfolio* or *replicating strategy*. If there exists a replicating strategy for H, we say that H is *attainable*.

It turns out the two questions (of pricing and replication) are related. Indeed, if $\mathcal{P} \neq \emptyset$ and H can be replicated by a portfolio φ (i.e. $V_T(\varphi) = H$), then for any $\mathbb{P}^* \in \mathcal{P}$,

$$V_0(\varphi) = \mathbb{E}^*\big(V_T(\varphi)\big) = \mathbb{E}^*(H).$$

That is, the initial capital required to replicate H is always $\mathbb{E}^*(H)$, and this quantity is independent of the EMM $\mathbb{P}^*$. It makes sense then to call this the no-arbitrage price of H and we will denote it by $\pi(H)$.

Thus, an attainable contingent claim H has a unique no-arbitrage price $\pi(H)$, and that price is equal to $\mathbb{E}^*(H)$ for any EMM $\mathbb{P}^*$.

Let us consider an example. Let $T = 1$, $d = 1$ and $\Omega = \{a, b, c\}$, $\mathbb{P} = \frac{1}{3}(\delta_a + \delta_b + \delta_c)$ and

ω	$(X_0^0(\omega), X_0^1(\omega))$	$(X_1^0(\omega), X_1^1(\omega))$
a	$(1,2)$	$(1,1)$
b	$(1,2)$	$(1,2)$
c	$(1,2)$	$(1,4)$

An equivalent martingale measure $\mathbb{P}^*$ is characterized by numbers $\mathbb{P}^*(a) =: \alpha$, $\mathbb{P}^*(b) =: \beta$ and $\mathbb{P}^*(c) =: \gamma$, such that

$$\alpha, \beta, \gamma > 0, \quad \alpha + \beta + \gamma = 1$$

and the martingale property on X^1:

$$\alpha + 2\beta + 4\gamma = 2.$$

Solving this system of equations gives

$$\mathcal{P} = \left\{\mathbb{P}_p^* = (2p, 1-3p, p) : 0 < p < \frac{1}{3}\right\}.$$

.

So, the market admits no arbitrage opportunities.

Continuing with the previous example, consider the contingent claim H defined as follows:

ω	$H(\omega)$
a	1
b	3
c	7

Then H is attained by the portfolio $\varphi_1 = (\varphi_1^0, \varphi_1^1) = (-1,2)$ with $V_0(\varphi) = (-1,2) \cdot (1,2) = 3$. Also, if $\mathbb{P}_p^*$ is an EMM, then

$$\pi(H) = \mathbb{E}^*(H) = 2p + 3(1-3p) + 7p = 3 = V_0(\varphi),$$

as expected.

What happens if H is not attainable? Does it still have a unique no-arbitrage price? Consider the following contingent claim H':

ω	$H'(\omega)$
a	2
b	2
c	6

To find a replicating strategy $\varphi = (\varphi^0, \varphi^1)$, we need to solve the following system of equations:

$$\varphi_1^0 + \varphi_1^1 = 2$$
$$\varphi_1^0 + 2\varphi_1^1 = 2$$

$$\varphi_1^0 + 4\varphi_1^1 = 6,$$

which clearly does not have a solution. So H' is not attainable.

Also, if $\mathbb{P}_p^* = (2p, 1-3p, p) : 0 < p < \frac{1}{3}$ is an EMM, then

$$\mathbb{E}^{*p}(H') = 2(2p) + 2(1-3p) + 6p = 4p + 2,$$

which depends on p. So H' has no unique no-arbitrage price.

So how do we price a non-attainable claim H? It turns out that the (infinite) set of no-arbitrage prices for H is given by

$$\{\mathbb{E}^*(H): \mathbb{P}^* \in \mathcal{P}\}$$

.

In the previous example, this is

$$\left\{4p + 2 : 0 < p < \frac{1}{3}\right\} = \left(2, 2 + \frac{4}{3}\right).$$

One can check that if the price of H' is greater than $2 + \frac{4}{3}$ or less than 2, then arbitrage opportunities will exist.

3.2.7 Transcript: An Example of Completeness

Hi, in this video we are going to introduce pricing derivatives and market completeness.

Consider a market that consists of all the usual components: $M = \left((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X\right)$, where X is the primary asset. A contingent claim, H, is an $\mathcal{F}_T$-measurable random variable. We call H a derivative of the primary asset X if H is $\mathcal{F}_T^X$-measurable. In other words, it is measurable with respect to the null set of filtration of $\mathcal{F}$ at time T .

So, given a contingent claim, H, we want to do the following:

1. Find the price of H, which we are going to call $\pi(H)$. In other words, what is the value of H at time 0, because we know what it is at time capital T.
2. Replicate the contingent claim, H. What this means is that we want to find a portfolio or strategy, φ, such that $V_T(\varphi) = H$. What this means is that we are going to trade in the primary assets and the value of the trading strategy will be equal to H under all states of the world. Or, we can say $\mathbb{P}$ almost surely in that case.

Now, such a trading strategy, if it does exist, is called a *replicating strategy*, and — if such a strategy exists — we say that H is *attainable* in that case. Then the market, M, is complete if every contingent claim H is attainable. Completeness essentially means that every contingent claim is attainable. The *Fundamental Theorem of Asset Pricing II* (FTAP II) again relates market completeness to the number of EMM's and says that the following are equivalent (under the important assumption that the market M is arbitrage-free):

1. M is complete. In other words, every contingent claim is attainable.
2. The set P of EMM's only has one element. Remember that the assumption is already that M is arbitrage-free, which that P has at least one element. However, completeness says that it must have exactly one element.

3 The third property is something called the *Predictable Representation Property,* which basically says that every martingale, M, which is a martingale that adapted to the same filtration of the market, with $M_0 = 0$ is a martingale transform of X. So, every martingale that starts at 0 is a martingale transform with respect to X.

That is the FTAP II, which says that all of the above conditions are equivalent.

Now, if H is attainable then the starting value of any replicating portfolio is the same. Then, if φ is a replicating portfolio strategy, $V_0(\varphi)$ is constant. That constant is equal to the expected value under any EMM of H. Written in full:

$$V_0(\varphi) = E^*(H).$$

What that is saying is that no matter which EMM you pick, and no matter which replicating strategy you pick, these two will always be the same. We will therefore define this to be the price of the derivative, H, and call it $\pi(H)$. So, the price of an attainable claim H is simply equal to the expectation under any EMM of H or, equivalently, it is equal to the starting value of any replicating portfolio, φ. Written in full:

$$V_0(\varphi) = E^*(H) =: \pi(H).$$

Let's look at an example.

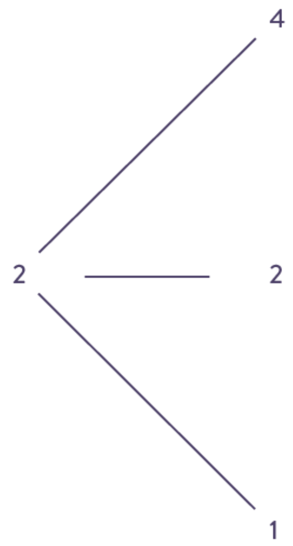
Let $T = 1 = d$, meaning that we have one risky asset. We will take Ω to consist of three points, $\{a, b, c\}$, and the probability measure, $\mathbb{P}$, to be an equal weighting of the points a, b, c as well. Written in full:

$$T = 1 = d, \quad \Omega = \{a, b, c\}, \quad \mathbb{P} = \frac{1}{3}(\delta_a, \delta_b, \delta_c).$$

Then, we will consider the evolution of the risky asset, X_1, as follows:

ω	$X_0(\omega)$	$X_1(\omega)$
a	2	1
b	2	2
c	2	4

So, that is the evolution of the stock price. We can draw it in a tree diagram format as follows:



The stock price moves to 1, 2, or 4 at time 1, $t = 1$. The originating point, 2, is $t = 0$.

Consider the following claim: $H(\omega)$ takes on the value of 1, 3, and 4, which we can add to our table:

ω	$X_0(\omega)$	$X_1(\omega)$	$H(\omega)$
a	2	1	1
b	2	2	3
c	2	4	7

Now, first let us show that this market has no arbitrage by finding an EMM. To do this, we have to specify the probability of a, which we will call α, the probability of b, which we'll call β and the probability of c, which we will call γ. α, β, and γ must satisfy the following equations:

- $\alpha + \beta + \gamma = 1$, since this is a probability measure.
- The martingale condition on X_1 will be $\alpha + 2\beta + 4\gamma = 2$, which is the starting point.

Now, we can see that this is two equations and three unknowns, and it turns out that it has an infinite number of solutions. These solutions are given by the following:

$$\mathbb{P} = \left\{\mathbb{P}_p^* = (2p, 1 - 3p, p) : 0 < \mathrm{p} < \frac{1}{3}\right\}.$$

So, this is the set of all EMM's. $2p$ represents the probability of a, $1 - 3$ represents the probability of b, and p represents the probability of c. So, there are no arbitrage opportunities. However, as you can see, as the above equation has infinitely many martingale measures, the market is not complete. This means that there are some claims that will not be attainable.

It turns out, however, that H is not one of those. H is, in fact, attainable by the following strategy. If we let φ_1 be $(-1, 2)$, then $V_0(\varphi)$ is equal to the dot product of $(-1, 2) \cdot (1,$ which is the risky asset, and 2, which is the riskless asset) and this gives us 3. So, any replicating strategy should start at 3. Written in full:

$$\varphi_1 = (-1{,}2), \quad V_0(\varphi) = (-1{,}2) \cdot (1{,}2) = 3.$$

Then, we can take $V_1(\varphi)$ to check that this does indeed replicate the derivative under all of the different states. So, under the state a, for instance, it is equal to the dot product of $(-1, 2)$ and $(1, 1)$, which gives us 1. This corresponds to a. We can check it again on b and c and see that it gives us the same value there.

Now, let's calculate the expectation of H under any EMM.

$$E^*(H) = 2p + 3(1 - 3p) + 7p = 3.$$

The final result, 3, is exactly equal to the value of this replicating portfolio and, therefore, the price of this attainable claim is 3, and for all replicating portfolios the initial value will be equal to 3.

We have come to the end of trading in discrete time. In the next module, we are going to look at the binomial model.

3.2.8 Notes: Completeness

A market is *complete* if every contingent claim is attainable.

Consider the market from the previous section. Let $T = 1$, $d = 1$ and $\Omega = \{a, b, c\}$, $\mathbb{P} = \frac{1}{3}(\delta_a + \delta_b + \delta_c)$ and

ω	$(X_0^0(\omega), X_0^1(\omega))$	$(X_1^0(\omega), X_1^1(\omega))$
a	(1,2)	(1,1)
b	(1,2)	(1,2)
c	(1,2)	(1,4)

The contingent claim H' defined by

ω	$H'(\omega)$
a	2
b	2
c	6

is not attainable, so the market is not complete.

The Fundamental Theorem of Asset Pricing II (FTAP II) characterizes completeness of a market.

Theorem [FTAP II]

For a financial market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ *with no arbitrage opportunities, the following are equivalent:*

1. *The market is complete.*
2. $\mathcal{P}$ *has exactly one element.*

3 *X has the predictable representation property (PRP) with respect to every $\mathbb{P}^* \in \mathcal{P}$: every $(\mathbb{F}, \mathbb{P}^*)$-martingale M with $M_0 = 0$ can be written as a martingale transform with respect to X.*

The third part of the theorem gives us a replicating strategy for each claim H in a complete market. Indeed, let H be a contingent claim with $\mathbb{E}^*(|H|) < \infty$. Then the process $M = \{M_t : t = 0,1,\dots,T\}$ defined by $M_t \coloneqq \mathbb{E}^*(H|\mathcal{F}_t)$ is a martingale with $M_T = H$. By PRP, there exists a predictable process φ^H such that

$$M_t = \mathbb{E}^*(H) + \sum_{k=1}^{t} \varphi_k^H \cdot (X_k - X_{k-1}), \quad t = 1,2,\dots,T.$$

Substituting $t = T$, we see that the process φ^H is a replicating strategy for H. Consider

$$\Omega = \{\omega = (\omega_1,\dots,\omega_T) : \omega_i \in \{-1,1\}, \quad i = 1,2,\dots,T\}$$

and let $Y_t(\omega) \coloneqq \omega_t$ for $\omega = (\omega_1,\dots,\omega_T)$. Define $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \sigma(\{Y_1,\dots,Y_t\})$ for $t \geq 1$. Consider two assets X^0 and X^1 whose discounted assets are $X^0 \equiv 1, X_0^1 =$ constant, and

$$X_t^1 = X_{t-1}^1 \, exp(\sigma_t Y_t + \mu_t), \quad t \geq 1,$$

for some predictable processes μ and σ with $0 \leq |\mu_t| < \sigma_t$ for every t. Assume that $\mathbb{P}$ is such that $\mathbb{P}(\{\omega\}) > 0$ for every $\omega \in \Omega$. Then there is a unique EMM $\mathbb{P}^*$ for X. Furthermore, if M is a $\mathbb{P}^*$-martingale, then

$$M_t = M_0 + \sum_{k=1}^{t} \varphi_k^M (X_k - X_{k-1}),$$

where

$$\varphi_k^M = \frac{M_k - M_{k-1}}{X_k - X_{k-1}}.$$

This also gives a replicating portfolio for any contingent claim H as

$$\varphi_t^M = \frac{\mathbb{E}^*(H|\mathcal{F}_t) - \mathbb{E}^*(H|\mathcal{F}_{t-1})}{X_t - X_{t-1}}.$$

3.2.9 Notes: Problem Set

Problem 1

Consider a market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with one risky asset $X = \{X_0, X_1, X_2\}$ defined as follows:

$$\Omega = \{a, b, c, d\}, \quad \mathbb{F} = \mathbb{F}^X, \quad \mathcal{F} = 2^\Omega.$$

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
a	$\frac{1}{6}$	15	26	30
b	$\frac{1}{4}$	15	26	20
c	$\frac{1}{4}$	15	12	14
d	$\frac{1}{3}$	15	12	10

Consider the self-financing trading strategy (v_0, φ) defined by

$$v_0 = 10, \quad \varphi_1 = 2, \quad \varphi_2 = I_{\{X_0 > X_1\}} - I_{\{X_1 \geq X_0\}}.$$

Here v_0 is the initial capital and φ is the investment in X. Find $G_1(\varphi)(a)$, the gains from trading at time 1.

Solution:

We know that the *gains process* associated with (η, φ) is the stochastic process $G(\varphi) = \{G_t(\varphi): t = 0,1, \dots, T\}$ defined by

$$G_0(\varphi) := 0, \quad G_t(\varphi) := \sum_{k=1}^{t} \varphi k \cdot (X_k - X_{k-1}) \quad \text{for} \quad t = 1,2, \dots, T.$$

(Note that there are no gains from trading the riskless asset.)

In this case, we have to focus on $G_1(\varphi)(a)$, thus

$$G_1(\varphi)(a) := \sum_{k=1}^{t} \varphi_k \cdot (X_k - X_{k-1}) = \varphi_1 \cdot (X_1(a) - X_0(a)) = 2 * (26 - 15) = 22.$$

The solution is 22.

Problem 2

Consider a market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with one risky asset $X = \{X_0, X_1\}$ defined as follows:

$$\Omega = a, b, c, d, \quad \mathbb{F} = \mathbb{F}^x, \quad \mathcal{F} = 2^{\Omega}$$

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$
a	$\frac{1}{3}$	4	8
b	$\frac{1}{3}$	4	5
c	$\frac{1}{3}$	4	2

Which one of these measures is an EMM for X?

1 $\mathbb{P}^* = \frac{1}{5}\delta_a + \frac{1}{2}\delta_b + \frac{3}{10}\delta_c$

2 $\mathbb{P}^* = \frac{1}{4}\delta_a + \frac{1}{3}\delta_b + \frac{5}{12}\delta_c$

3 $\mathbb{P}^* = \frac{1}{4}\delta_a + \frac{1}{6}\delta_b + \frac{7}{12}\delta_c$

4 $\mathbb{P}^* = \delta_a + \delta_b + \delta_c$

Solution:

Let $\mathbb{P}^*$ be a probability measure on $(\Omega, \mathcal{F})$. We call $\mathbb{P}^*$ an *equivalent martingale measure* (EMM) for X if $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ (i.e. $\mathbb{P}(A) = 0 \iff \mathbb{P}^*(A) = 0 \quad \forall A \in \mathcal{F}$) and X is an $(\mathbb{F}, \mathbb{P}^*)$-martingale.

The only option that holds the above conditions is

$$\mathbb{P}^* = \frac{1}{4}\delta_a + \frac{1}{6}\delta_b + \frac{7}{12}\delta_c\,.$$

The equivalence of $\mathbb{P}$ and $\mathbb{P}^*$ is clear since the only null set is the empty set for both of them. Now we need also to show that X is a $\mathbb{P}^*$-martingale.

First note that $\mathbb{E}(|X_n|) \leq 8 < \infty$ for all $n \in \mathbb{I}$. $X_n \in m\mathcal{F}_n^X \; \forall n \in \mathbb{I} = \{0,1\}$ is trivial since we are using the natural filtration of X.

$$\mathbb{E}^*(X_1 | \mathcal{F}_0^X) = \mathbb{E}^*(X_1 | \{\emptyset, \Omega\}) = \mathbb{E}^*(X_1) = \frac{1}{4} \times 8 + \frac{1}{6} \times 5 + \frac{7}{12} \times 3 = 4 = X_0.$$

Problem 3

Consider a financial market with the following components:

$$\Omega = \{a, b, c\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \text{and } \mathcal{F}_1 = \mathcal{F}.$$

Let $\mathbb{P}$, $X = \{X_0, X_1\}$, and H be defined as follows:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$H(\omega)$
a	$\frac{1}{3}$	4	8	4
b	$\frac{1}{3}$	4	5	10
c	$\frac{1}{3}$	4	2	16

Find the price of H.

Solution:

We need to find a replicating strategy $\varphi = (\varphi^0, \varphi^1)$, thus we need to solve the following system of equations:

$$1\varphi^0 + 8\varphi^1 = 4$$
$$1\varphi^0 + 5\varphi^1 = 10$$
$$1\varphi^0 + 2\varphi^1 = 16.$$

The solution is $\varphi^0 = 20$ and $\varphi^1 = -2$. The initial price of X is 4, and we assume that the risk free asset cost is 1, thus the strategy cost is $20 * 1 - 2 * 4 = 12$, which should also be the price of H.

Problem 4

Consider a financial market with the following components:

$$\Omega = \{a, b, c\}, \quad \mathcal{F} = 2^\Omega, \quad \mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \text{and } \mathcal{F}_1 = \mathcal{F}.$$

Let $\mathbb{P}$, $X = \{X_0, X_1\}$, and H be defined as follows:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$H(\omega)$
a	$\frac{1}{3}$	100	120	-20
b	$\frac{1}{3}$	100	112	γ
c	$\frac{1}{3}$	100	90	10

If H is attainable and has a unique no arbitrage price of 0, then what is γ?

Solution:

We need to find a replicating strategy $\varphi = (\varphi^0, \varphi^1)$, thus we need to solve the following system of equations:

$$\begin{aligned} 1\varphi^0 + 120\varphi^1 &= -20 \\ 1\varphi^0 + 112\varphi^1 &= \gamma \\ 1\varphi^0 + 90\varphi^1 &= 10. \end{aligned}$$

From the first and third equation we get that $\varphi^1 = -1$. If we substitute the value of φ^1 in the second equation, we get

$$\varphi_0 = \frac{\gamma + 112}{1}.$$

As the statement says that the value of H is zero, we can get γ from the following condition $H = 0$,

$$1 * \varphi_0 + 100 * \varphi_1 = 0, \quad \gamma = -12.$$

Thus, the solution is $\gamma = -12$.

Problem 5

Consider a market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with one risky asset $X = \{X_0, X_1, X_2\}$ and a contingent claim H defined as follows:

$$\Omega = \{a, b, c, d\}, \quad \mathbb{F} = \mathbb{F}^X, \quad \mathcal{F} = 2^\Omega.$$

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega)$
a	$\frac{1}{6}$	15	24	30	12
b	$\frac{1}{4}$	15	24	20	0
c	$\frac{1}{4}$	15	12	14	10
d	$\frac{1}{3}$	15	12	10	0

Compute the price of H at time 1.

Solution:

It is important to remark that the problem is asking about the price at $t = 1$. Thus, we need to focus on the filtration on $t = 1$ or, in other words, we will have to compute two different prices at $t = 1$, $H_1\{a, b\}$ and $H_1\{c, d\}$.

Let's start for the path $\{a, b\}$. In this case, we have to solve the following system of equations:

$$1\varphi^1 + 30\varphi^2 = 12$$
$$1\varphi^1 + 20\varphi^2 = 0.$$

The solution of the above system is $\varphi_1 = -24$ and $\varphi_2 = \frac{12}{10}$. Thus the price $H_1\{a,b\}$ is equal to

$$H_1\{a,b\} = -24 * 1 + 24 * \frac{12}{10} = \frac{24}{5}.$$

If we follow the same steps for $\{c,d\}$, we first solve the system:

$$1\varphi^1 + 14\varphi^2 = 10$$
$$1\varphi^1 + 10\varphi^2 = 0.$$

The solution is $\varphi_1 = -\frac{100}{4}$ and $\varphi_2 = \frac{10}{4}$. The price, $H_1\{c,d\}$, will be,

$$H_1\{c,d\} = -1 * \frac{100}{4} + 12 * \frac{10}{4} = 5.$$

Finally, we just can express the solution as

$$H_1 = \frac{24}{5} I_{\{a,b\}} + 5\, I_{\{c,d\}}.$$

3.3 Additional Resources

Bingham, N. and Kiesel, R. (2004). *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*. London: Springer.

Föllmer, H. and Schied, A. (2004). *Stochastic finance*. Berlin: Walter de Gruyter.

Shiryaev, A. (2001). *Essentials of Stochastic Finance: Facts, Models, Theory*. Translated by N. Kruzhilin. Singapore: World Scientific.

Shreve, S. (2005). *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*. New York: Springer.

3.4 Collaborative Review Task

In this module, you are required to complete a collaborative review task, which is designed to test your ability to apply and analyze the knowledge you have learned during the week.

Question

Let $T = 2$, $d = 1$ and $\Omega = \{a, b, c, d\}, \mathbb{P} = \frac{1}{4}(\delta_a + \delta_b + \delta_c + \delta_d)$ and

ω	$(X_0^0(\omega), X_0^1(\omega))$	$(X_1^0(\omega), X_1^1(\omega))$	$(X_2^0, X_2^1(\omega))$	$H(\omega)$
a	(1,12)	(1,18)	(1,36)	6
b	(1,12)	(1,18)	(1,9)	1
c	(1,12)	(1,9)	(1,12)	8
d	(1,12)	(1,9)	(1,8)	4

Assume that $\mathbb{F}$ is the natural filtration of X given by

$$F_0^X = \{\emptyset, \Omega\}, F_1^X = \{\emptyset, \Omega, \{a, b\}, \{c, d\}\}, \text{ and } F2 = \sigma(\{X_0, X_1, X_2\}) = 2^{\Omega}.$$

1. Show that the market is complete.
2. Find the unique no-arbitrage price of H.
3. Construct a replicating strategy for H.

Remember to show all your calculations, otherwise marks will be deducted.