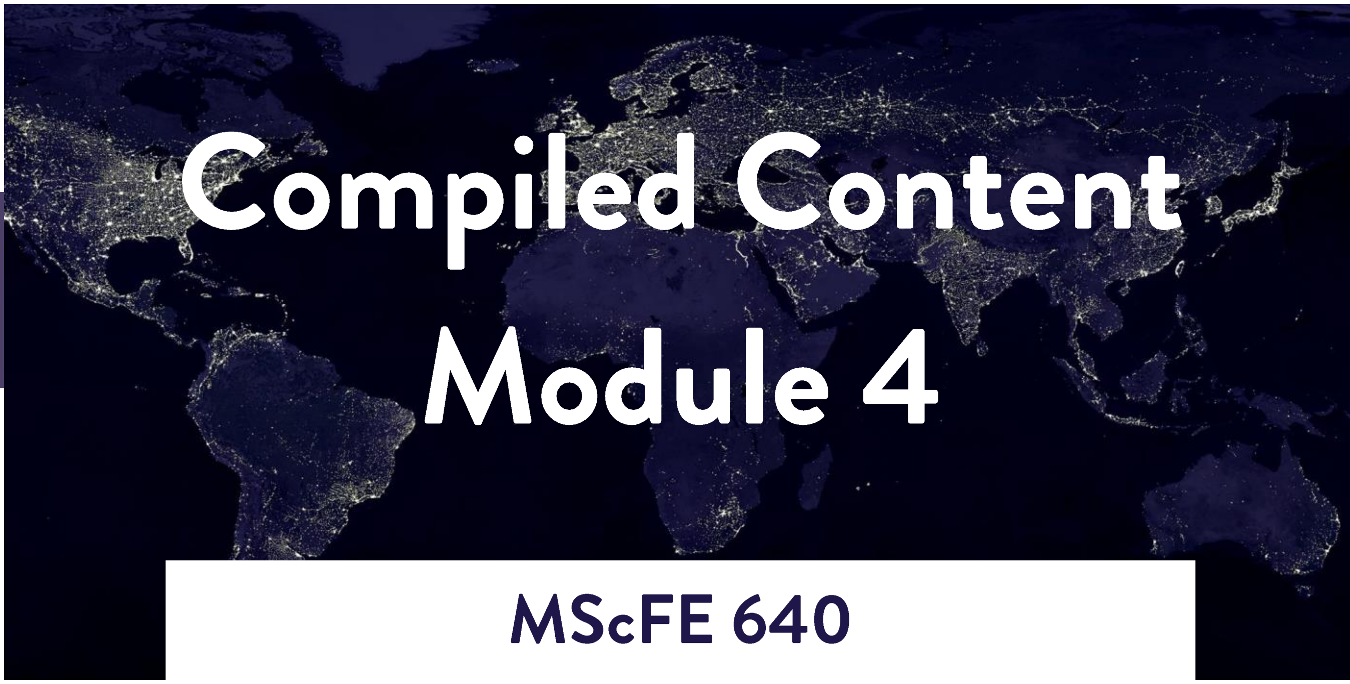
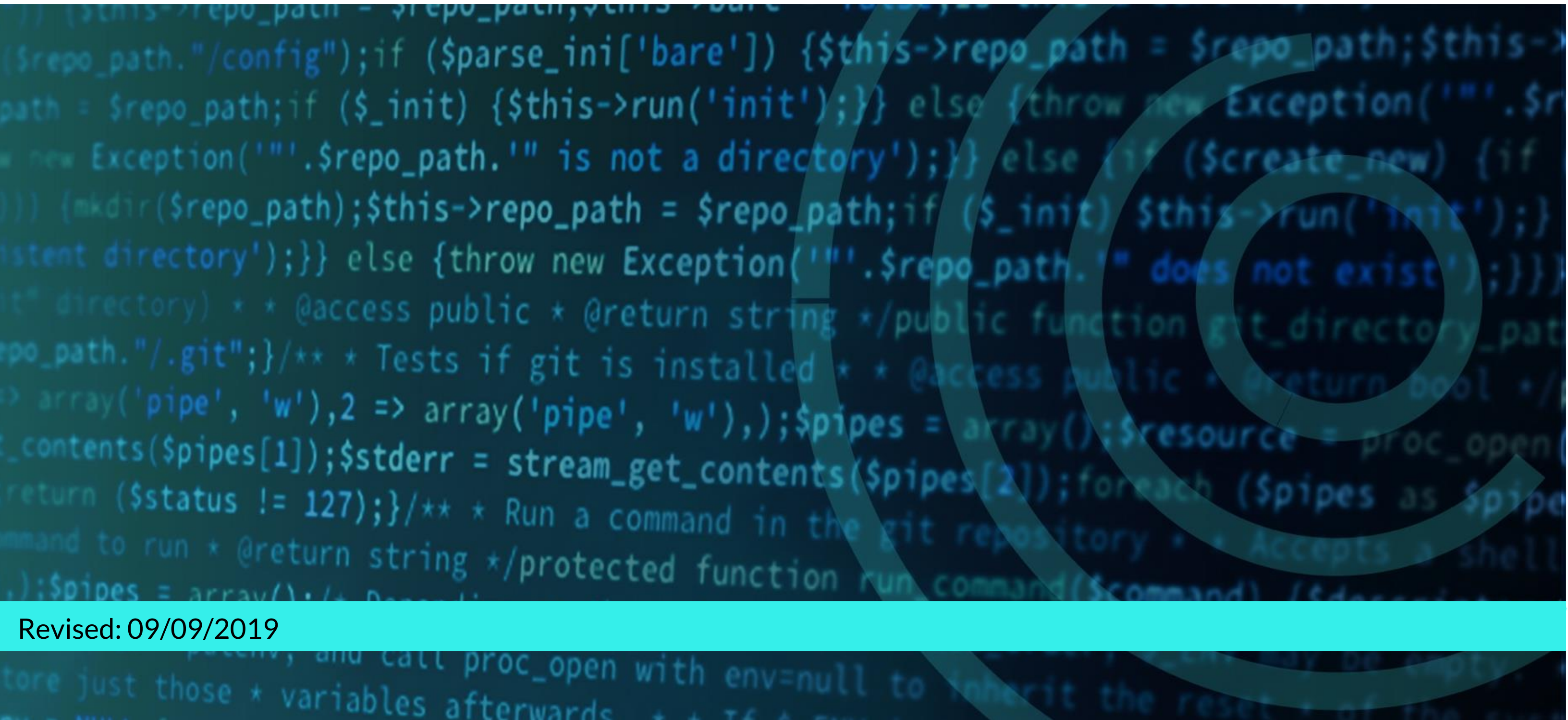


# Compiled Content
# Module 4

## MScFE 640

## Portfolio Theory and Asset Pricing



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## Module 4: Active Frontiers

This module begins by explaining active return and tracking error metrics and how they are used to measure the performance of a portfolio. The module continues by introducing the mean-adjusted tracking and shows how this metric is used to optimize the performance of an active portfolio. Finally, different methods to manage active returns are discussed.

# Unit 1: Tracking Error and Volatility – Part 1 of 2

## Introduction

In this section, we will discuss three issues to consider when assessing a portfolio's return and risk. This foundation will lay the groundwork for tracking error, which is covered in the next section. The first consideration is whether we are looking at return in absolute terms or relative terms. The second addresses whether the portfolio manager is actively managing or just passively managing the fund. The third issue concerns whether we look at risk in *ex ante* or *ex post* terms. Let's now turn to our first consideration: whether we have absolute or relative returns.

## Absolute returns versus relative returns

When it comes to returns – whether it is income, or profits, or anything beneficial – many of us think in relative rather than absolute terms.

If you were to get a certain test score, you might compare your performance in absolute terms (e.g. relative to the minimum and maximum scores allowed), relative terms (e.g. relative to the population taking the test), or some benchmark average. In a famous study done about 20 years ago, 257 faculty, staff, and students at Harvard University responded to a survey. Each question asked which of two states of the world were preferred, one being an absolute case and the other being a relative case. For example, here is one of the questions.

Which of the situations would you prefer to live in?

- One where you have 2 weeks of vacations; others have 1 week.
- One where you have 4 weeks of vacation; others have 8 weeks.

To be sure, there is no "correct" or "best" answer. The responses people give are based on people's tendency to look at the world in relative terms or to view things in absolute terms. The majority of the respondents chose the first option, because they wanted to do better in relative terms than in absolute terms. The idea is that while we may want "more," we also do not want to feel or know that we are underperforming.

## Active investing versus passive investing

When we decide whether to look at the returns on an absolute or relative basis, it is important to consider what type of investment manager we are measuring.

An **active portfolio manager** seeks to produce positive portfolio returns, regardless of the economic environment. The active manager can be evaluated on either absolute or relative terms. There is more than one type of active manager. Let's consider the two extremes: a mutual fund manager and a hedge fund manager.

An **active mutual fund manager** has a stated investment objective, published in a publicly available fund prospectus, that typically states that they will not only meet but beat some relevant index. For this reason, they are likely to be measured on relative terms. Suppose you have a gold mutual fund. The manager is constrained to invest a certain percentage of the fund in gold, with the balance likely being in cash or some other safe investments. If gold were down 20% on the year, but the fund manager is only down 15%, then the manager's performance should be measured on a relative basis – in this case, the active return is 500 basis points.

An **active hedge fund manager**, on the other hand, is likely to be evaluated on absolute terms. While they don't publicly disclose their investment strategies, they have more discretion to choose investments from many asset classes, use leverage, sell securities short, and adopt a variety of other strategies.  These extra freedoms should result in positive performance, even in economic downturns. Suppose you have a hedge fund specializing in commodities. If gold were down 20% on the year, the hedge fund manager could sell gold short and then buy what he or she considers to be a good value. Thus, if they were down 15%, the manger's performance should still be measured on an absolute basis, because there was no requirement that they remain invested in gold.

The second type of portfolio manager is passive. **Passive portfolio managers** seek to match the index they track. Rather than focusing on outperformance, the passive manager seeks to minimize costs and avoid many of the investment biases – e.g. behavioral and cognitive biases – that plague active managers. Since these managers are tracking indexes, they should only be measured in relative terms. Interestingly,

neither active fund managers nor passive fund managers need be human nowadays.

**Passive fund managers** could be very low-cost mutual funds that merely have well-defined rules that track an equity index like the S&P 500. The exact same idea could be executed in a similar investment vehicle called **Exchange-Traded Funds**, or ETFs. Unlike mutual funds, ETFs are securities that trade on the exchange; they can be bought long or sold short; and they can be traded intra-day. Like passive mutual funds, they simply hold baskets of securities that automatically track an equity index like the S&P 500.

Active fund manages could be **robo-advisors** that have rules based on artificial intelligence and data science that guide them to trade systematically. Likewise, there could be ETFs that mimic the behavior of robo-advisors that are driven by rules that were derived from back-testing various trading strategies.

## *Ex ante* versus *ex post*

Return is only one side of the coin. The other is risk. When it comes to risk, we need to distinguish between before the fact and after the fact.

Typically, risk is measured after the fact – this type of risk is called ***ex post* risk**. *Ex post* risk uses historical observations and computes volatility accordingly. It can only be measured once the event has occurred. Sometimes, this is also called realized volatility.

Other times, we would like to measure risk before an event happens. This type of risk is known as ***ex ante* risk**. It is computed using a model to forecast risk.

Some of the factor models you have studied can measure risk from before (*ex-ante*) or from after (*ex-post*). However, a model is only required to measure risk *ex ante* – simply because you do not have the performance. It is not necessary to have a factor model for measuring risk *ex post*, because you can calculate risk from the realized returns.

## Active return

**Active return** is the difference between a portfolio's return and the benchmark's return. Let's formally state the difference between a portfolio's return and a benchmark's return.

Let's define $P_i$ and $P_f$ as the initial and final values of a portfolio, respectively.

Similarly, let's define $B_i$ and $B_f$ as the initial and final values of a benchmark, respectively.

Typically, you would calculate the return of your portfolio as

$$R_p = \frac{P_f - P_i}{P_i}$$

for percent returns or

$$R_p = ln\left(\frac{P_f}{P_i}\right)$$

for log returns.

Similarly, a benchmark has a percent return of

$$R_b = \frac{B_f - B_i}{B_i} \text{ or } R_b = ln\left(\frac{B_f}{B_i}\right)$$

for log returns.

The active return is simply the difference between these two returns:

$$R_{Active} = R_P - R_B.$$

Suppose an actively managed portfolio trails a benchmark by 150 basis points.

What is the active return? -150 basis points, or -1.5%.

Suppose another active portfolio manager exceeds a benchmark by 350 basis points.

What is the active return? +3.5%, or 350 basis points.

In short:

- Active return is negative if we underperform a benchmark's return.
- Active return is positive if we outperform a benchmark's return.
- Active return is zero if we match a benchmark's return.

## Summary

In this section, we discussed the distinction between absolute return and relative return; active fund management and passive fund management; and *ex ante* risk versus *ex post* risk. Active fund managers should be measured using absolute returns. Passive fund managers are most likely measured using relative returns, because they should not be penalized for closely tracking a benchmark that is poorly performing. Most of these measures will be done *ex post*, or after we observe their performance. In the next section, we will build on these competencies to compute tracking error.

# Unit 2: Tracking Error and Volatility – Part 2 of 2

## Introduction

In the previous section, we discussed active return as a method to measure the performance of an actively managed portfolio compared to a benchmark. In this section, we deepen our study by discussing tracking error. Quite often, tracking error is mistakenly understood for active return. Both measure similar ideas – how well one fund matches a benchmark – but each measures differently: active return is a difference, whereas tracking error is a volatility. After defining tracking error, we will discuss a flaw with it that will lead us to a better measure: the mean-adjusted tracking error. We will examine a numerical example that will illustrate these concepts and motivate why we need to think about mean-adjusted tracking error rather than just tracking error.

## Tracking error

Tracking error is often confused with active return. In conversations, people may use them interchangeably; nevertheless, they are not the same mathematically. **Tracking error** is the standard deviation of active returns. Whereas active return can be negative or positive, tracking error is always positive because it is a volatility. In the very rare chance that we have a constant active return, then tracking error would be 0. Regardless, it must be non-negative in all cases.

Measuring risk on an *ex-post* basis, suppose we observe $T$ time periods. The tracking error is defined as:

$$Tracking\ error = \sqrt{\frac{1}{T-1}\sum\nolimits_{i=1}^{T}(R_i - \bar{R})^2}.$$

This formula is simply the standard deviation of the active return. Note the that the two returns in the formula are the active return and the average value of the active return, respectively.

Consider the same actively managed portfolio that trails a benchmark by 150 basis points every day. What is the tracking error? Since the active returns are a constant (albeit a negative number), they have a standard deviation of 0.

Consider the second portfolio manager who leads a benchmark by 350 basis points every day. What is their tracking error? Also 0.

The active return is a sign of how *effective* we are as a portfolio manager. It is measured in relative terms: positive is good; negative is bad. The tracking error tells us how *consistent* we are as a portfolio manager. It is always non-negative, but what it can tell us is that:

- if the tracking error is small, then we have a relatively *stable* active return;
- if the tracking error is large, then we have a relatively *volatile* active return;
- if the tracking error is 0, then we have a *constant* active return.

All these descriptions point back to the active return.

A tracking error of 0 does not imply a riskless situation: it merely indicates a consistent active return. If we could somehow, magically, consistently underperform a benchmark by a constant amount, then we would have no tracking error but we would still have risk.

We could use a better measure of tracking error to distinguish these two portfolio managers whose active returns are 500 basis points apart but who have identical tracking errors. This is called the mean–adjusted tracking error.

## Moving towards mean-adjusted tracking error

Let's review what *ex post* tracking error does and does not measure.

*Ex post* tracking error measures the variability of active returns. *Ex post* tracking error *does not* measure the risk of a fund deviating from a benchmark.

Originally, active risk was applied to passive management where the fund manager's objective is to track an index as closely as possible. Actively managed funds seek to outperform an index. The active risk of actively managed funds cannot be measured by tracking error.

If there is an active return, there is a difference between measuring returns relative to a benchmark and measuring risk relative to a benchmark. If there is no active return, there is no difference between the two.

Let's take a look at the formula for mean-adjusted tracking error (MATE).

$$Mean - adjusted\ tracking\ error\ = \sqrt{\frac{1}{T}\sum\nolimits_{i=1}^{T}(Active\ Return)_i^2}$$

Again, the return we use inside the summation is the active return squared. The mean-adjusted tracking error will no longer be 0 when the mean active return is very different from 0.

Let's take a look at an excellent example given from Carol Alexander's (2008) book.

| **Date** | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Bmark1** | 100 | 90 | 104 | 124 | 161 | 186 | 204 | 235 | 258 | 271 | 339 | 254 | 216 | 216 | 238 | 262 | 275 | |
| **Bmark2** | 100 | 93 | 110 | 136 | 182 | 216 | 245 | 291 | 330 | 360 | 460 | 355 | 311 | 321 | 364 | 413 | 447 | |
| **Fund** | 100 | 91 | 104 | 127 | 167 | 190 | 206 | 234 | 260 | 271 | 346 | 256 | 221 | 223 | 243 | 262 | 273 | |
| | | | | | | | | | | | | | | | | | | |
| **B1 Ret** | -- | -10.0% | 15.6% | 19.2% | 29.8% | 15.5% | 9.7% | 15.2% | 9.8% | 5.0% | 25.1% | -25.1% | -15.0% | 0.0% | 10.2% | 10.1% | 5.0% | |
| **B2 Ret** | -- | -7.0% | 18.3% | 23.6% | 33.8% | 18.7% | 13.4% | 18.8% | 13.4% | 9.0% | 27.9% | -22.8% | -12.4% | 3.2% | 13.4% | 13.5% | 8.2% | |
| **Fund Ret** | -- | -9.0% | 14.3% | 22.1% | 31.5% | 13.8% | 8.4% | 13.6% | 11.1% | 4.2% | 27.7% | -26.0% | -13.7% | 0.9% | 9.0% | 7.8% | 4.2% | |
| | | | | | | | | | | | | | | | | | | **Avg** |
| **ActRet 1** | | 1.0% | -1.3% | 2.9% | 1.7% | -1.8% | -1.3% | -1.6% | 1.3% | -0.8% | 2.6% | -0.9% | 1.3% | 0.9% | -1.2% | -2.3% | -0.8% | -0.01% |
| **ActRet 2** | | -2.0% | -4.0% | -1.5% | -2.3% | -4.9% | -5.0% | -5.2% | -2.3% | -4.8% | -0.2% | -3.2% | -1.3% | -2.3% | -4.4% | -5.6% | -4.0% | -3.32% |
| | | | | | | | | | | | | | | | | | | **TE** |
| **(R1-R1bar)^2** | | 0.01% | 0.02% | 0.08% | 0.03% | 0.03% | 0.02% | 0.03% | 0.02% | 0.01% | 0.07% | 0.01% | 0.02% | 0.01% | 0.01% | 0.05% | 0.01% | **1.59%** |
| **(R2-R2bar)^2** | | 0.02% | 0.00% | 0.03% | 0.01% | 0.03% | 0.03% | 0.03% | 0.01% | 0.02% | 0.10% | 0.00% | 0.04% | 0.01% | 0.01% | 0.05% | 0.01% | **1.59%** |
| | | | | | | | | | | | | | | | | | | MATE |
| **ActRet1^2** | | 0.01% | 0.02% | 0.08% | 0.03% | 0.03% | 0.02% | 0.03% | 0.02% | 0.01% | 0.07% | 0.01% | 0.02% | 0.01% | 0.01% | 0.05% | 0.01% | **1.59%** |
| **ActRet1^2** | | 0.0% | 0.2% | 0.0% | 0.1% | 0.2% | 0.3% | 0.3% | 0.1% | 0.2% | 0.0% | 0.1% | 0.0% | 0.1% | 0.2% | 0.3% | 0.2% | 3.68% |

*Figure 1: Values of a fund and 2 benchmarks*

The value of benchmark 1 is given in row "Bmark1" and is presented in blue. The value of benchmark 2 is given in row "Bmark2" and is presented in orange. The value of our portfolio is given in row "Fund" and remains uncolored.

We can compute percent return in the usual fashion for each of the three securities. Once we have the returns, we compute the active returns for the fund using Benchmark 1 and Benchmark 2. Intuitively, Benchmark 1 should be a better fit. We see that the active return when using Benchmark 1 has an average of nearly 0. Using Benchmark 2, we see that the active return has an average of -3.33%.

In the next two rows, we compute the tracking error using the formula above. It might be surprising to see that they have the same tracking error. Even though the 2nd benchmark has a very different set of returns from the fund, the tracking error is the same because it has a consistent active return. When we examine the standard deviations of each set of active returns, they are the same. The tracking error to each benchmark is the same. However, the mean-adjusted tracking error to the benchmarks are different.

In the last two rows, we compute the mean-adjusted tracking error by squaring the active returns. Now, we see that using Benchmark 1 results in a lower MATE than using Benchmark 2. Let's solve the mean-adjusted tracking error in terms of the tracking error. Note that if we square both sides of the equation for mean-adjusted tracking error, we get:

$$MATE^2 = \frac{1}{T}\sum_{i=1}^{T} R_i^2.$$

Similarly, let's square both sides of the equation for the tracking error. This looks like:

$$TE^2 = \frac{1}{T-1}\sum_{i=1}^{T}(R_i - \bar{R})^2.$$

Expanding, we get:

$$(T-1)TE^2 = \sum_{i=1}^{T} R_p^2 - 2\sum_{i=1}^{T} \bar{R}R_P + \sum_{i=1}^{T} \bar{R}^2.$$

The middle term contains a constant – the average – so it can be moved outside the summation. The middle term then sums all the returns, which is equivalent to $T$ times the average. Thus, the middle term simplifies to:

$$-2 * T\bar{R}^2.$$

Similarly, the last term sums the squared average $T$ times, so we can remove the summation. Thus, the last term simplifies to:

$$T\bar{R}^2.$$

Simplifying, we get:

$$(T-1)TE^2 = \sum_{i=1}^{T} R_p^2 - 2T\bar{R}^2 + T\bar{R}^2 = \sum_{i=1}^{T} R_p^2 - T\bar{R}^2.$$

Rearranging, we get:

$$\sum_{i=1}^{T} R_p^2 = (T-1)TE^2 + T\bar{R}^2.$$

Substituting, we get:

$$MATE^2 = \frac{1}{T}\sum_{i=1}^{T}(T-1)TE^2 + T\bar{R}^2 = \frac{(T-1)}{T}TE^2 + \bar{R}^2$$

$$MATE^2 = \frac{(T-1)}{T} TE^2 + \bar{R}^2.$$

This equation relates mean-adjusted tracking error to tracking error. Notice that each of the terms in the equation is squared, so everything has to be positive. The left-hand side gives the mean-adjusted tracking error. The right-hand side gives the squared tracking error and adds the squared average active return. Therefore, if the average active return is large – e.g. when the fund is not closely following the benchmark – then the MATE will be larger than the tracking error. In our example, the mean-adjusted tracking error is large.

## Summary

In this section, we discussed three metrics for measuring the performance of a portfolio. The first two metrics are the active return and tracking error. The active return simply relates the difference in returns. The tracking error takes that difference and computes its volatility. However, if a fund is not closely matching its benchmark, these two metrics can be misleading. The third metric is the mean-adjusted tracking error. This metric uses both the squared active return and the squared tracking error. Therefore, if a benchmark has a large squared active return or a large squared tracking error, then it will have a large mean-adjusted tracking error. In the next section, we will relate these ideas to our mean-variance portfolio framework.

# Unit 3: Tracking Error Volatility Versus Total Volatility – Part 1 of 2

## Introduction

In the last section, we talked about metrics for a portfolio's tracking error and mean-adjusted tracking error. These reflect the risk associated with a portfolio not closely following its benchmark. In this section, we will discuss how a portfolio can be chosen to lie along the efficient frontier. However, this may not result in the most efficient portfolio with regards to the tracking error.

## Active risk

Originally, active risk was only intended to be applied to a passive fund manager. That manager had a stated investment objective: to track an index as closely as possible. Active risk measures precisely that. The closer the active risk is to zero, the better the manager tracks that index. One could rank the quality of passive funds by the absolute value of the active risk. If the active risk were too large, investors could be suspicious that the manager was not tracking the index, but had some other, undisclosed objectives. In this case, tracking error is synonymous with active risk.

Over time, active managers, whether mutual funds, hedge funds, or other discretionary asset managers, have been compared to indexes. If they are adding value, then their active returns should be well above zero. The active risk of actively managed funds should not be measured by tracking error. If active managers have low active risk, then they are not adding values – but are very likely adding management and performance fees. These results mean that measuring returns relative to a benchmark does not equate to measuring risk relative to a benchmark. The only time this is the case is when the expected active return is zero.

## Optimizing a portfolio using mean-variance methodology

What does it mean, then, to optimize a portfolio?

When we learned classic, mean-variance portfolio theory, we thought in absolute terms (even if this was not explicitly stated). Think of the mean-variance optimization you studied in the previous module as a machine that takes inputs and produces outputs. The three items to input are future returns, standard deviations, and correlation. Sometimes, volatility is measured by variance rather than by standard deviation. **Variance** is simply the standard deviation squared. Also, sometimes you will see **covariance**, which is the individual standard deviations multiplied by the correlation.

Mean-variance optimization should produce the best combination of portfolios, meaning that when we input our best expectations for future returns, standard deviations, and correlations, we will get the highest return for a certain amount of total risk or the lowest total risk for a specified return.

Let's just remind ourselves of a few assumptions.

- First, mean-variance analysis assumes that return distributions are normally distributed.
- Second, Markowitz stated that mean-variance analysis assumes that all investors have quadratic utility.

If these assumptions are met, then Markowitz's model tells us that there are numerous combinations of portfolios that lie along the efficient frontier. We simply have to state our risk tolerance, which would then uniquely describe a set of asset weights. Let's examine each of the assumptions in turn.

Are return distributions normally distributed? This was explained by saying that kurtosis represents uncertainty in the standard deviation, showing up in the fourth moment of a distribution. The existence of excess kurtosis is evidence that a distribution is *non-normal*. One method of dealing with returns that have excess kurtosis is to model their volatility with state-dependence or a stochastic component – this will allow for the existence of more extreme events and therefore imply a higher kurtosis on the distribution of returns.

The existence of excess kurtosis shows that the distribution is no longer only characterized by its mean and variance. Yet mean-variance optimization assumes that the statistical parameters we estimated – not just the standard deviations but the returns and covariances as well – are known and constant. Therefore, using these numbers in a mean-variance framework, we are putting faulty parameter values into the model. Like any calculator, mean-variance optimization follows the GIGO rule: Garbage In, Garbage Out. If we have errors in the inputs, then the outputs will also have errors because we have merely optimized faulty data. Indeed, many optimizations produce poorly-performing portfolios because of errors in producing these estimates. The optimizer can only do as well as the quality of the data going into the optimization.

The second assumption regarding quadratic utility is even more easily refuted. **Quadratic utility** assumes investors have increasing, absolute risk aversion, and that investors are as averse to upside risk as they are to downside risk. Both of these have been shown to not always be true.

## Optimizing a portfolio using mean-tracking error methodology

Markowitz focused on a total portfolio. He did not require a benchmark. By combining assets that have diversification, we can increase the expected return without increasing the necessary risk.

**Mean-tracking error optimization** differs by searching for the highest expected return given a specified tracking error. Implicit in this is the assumption that the investor is indifferent to a portfolio's volatility. Thus, mean-variance optimization seeks to minimize total risk, whereas mean-tracking error optimization seeks to minimize relative risk.

## Quantifying optimizations in the mean-variance approach

Typically, the efficient frontier is identified by maximizing the following quantity with respect to the individual asset weights:

$$Expected\ Return\ minus\ Risk\ Aversion\ times\ Squared\ Standard\ Deviation.$$

The more risk averse we are, the more negative the second term will be. This optimization produces an efficient frontier. As previously discussed, the investor selects a risk tolerance and then finds the optimal portfolio within that risk constraint that maximizes return.

## Quantifying optimization in the mean-tracking error approach

Some investors care only about relative risk – i.e. performance to a benchmark. Remember, we can measure relative risk by the tracking error.

These investors can identify their efficient portfolios by making two substitutions:

1. the aversion to risk becomes the aversion to tracking error;
2. the squared standard deviations become the squared tracking error.

In the mean-tracking error approach, we now want to maximize the following quantity with respect to the individual asset weights:

$$Expected\ Return - Tracking\ Error\ Aversion\ *\ Squared\ Tracking\ Error.$$

Just as in mean-variance theory, this approach produces a 2-dimensional efficient frontier curve. Instead of the x-axis containing absolute risk (the standard deviation of the portfolio's returns), it would display the relative risk – i.e. tracking error, which is the standard deviation of the active returns.

## Quantifying optimization using both absolute and relative risk

If investors are averse to both absolute risk and relative risk, then they can simply include both terms in their optimization. The equation to maximize would then be:

$$Expected\ Return - Risk\ Aversion * Squared\ Standard\ Deviation \\ - Tracking\ Error\ Aversion * Squared\ Tracking\ Error.$$

Unlike the previous two methods, this would provide a 3-dimensional efficient surface.

The three dimensions are:

- the expected return;
- the Portfolio Standard Deviation; and
- the Portfolio Tracking Error.

On the 1st dimension, the efficient surface is bounded by the classic, mean-variance efficient frontier. On the 2nd dimension, the efficient surface is bounded by the

mean-tracking error efficient frontier. On the 3rd dimension, the efficient surface is bounded by combinations of the minimum risk portfolio and the benchmark portfolio.

Although this approach is not much harder mathematically than the other methods, it is much less common. That is because the tendency for investors is to think in absolute terms or in relative terms exclusively. If an optimization were to contain both terms, it would simply mean that the investor has to choose two weights for their risk aversion: one for their absolute risk aversion and one for their relative risk aversion. Keep in mind that all of this is subjective.

## Summary

Active risk, tracking error, and mean-adjusted tracking error can all be applied, but each should be done so thoughtfully. For passive managers, it is suitable to use active risk and tracking error. For active managers, mean-adjusted tracking error could be used. We should use care when optimizing a portfolio as we can and will get different results depending on the approach we adopt.

# Unit 4: Tracking Error Volatility Versus Total Volatility – Part 2 of 2

## Introduction

Understanding total volatility is part of the picture. We also want to examine how to decompose the active return. The field of attributing returns is called **performance attribution**. One of the most common methods in model performance attribution is called the Brinson model. In this section, we will discuss the Brinson model and apply it to a set of returns modeled by an industry-leading, factor model.

## Rationale

If we know there is a portfolio return, then we would like to ask:

- How can the active return be understood due to asset allocation?
- How can the active return be understood due to stock selection?
- Are there interaction effects?

Let's consider one of the most famous equity models in the industry, the MSCI Barra's Global Equity Model II (known as GEM2). According to the website of MSCI (2018), this model includes more significant methodological advances in global equity risk modeling and has a very intuitive model structure.

Our example data contains 48 000 securities on a monthly basis. We are going to select some of those securities as our portfolio.

There are 15 columns in our data set, and we identify each one here:

1. **Barrid**: identifies the security as described by Barra;
2. **Name**: identifies the security's name or description;
3. **Return**: measures the monthly total return;
4. **Date**: starting date of the month;
5. **Sector**: identifies the GICS-classified sector;
6. **Momentum**: measures sustained relative performance;

7 **Value**: measures the extent to which a stock is inexpensively priced;
8 **Size**: measures the difference between large- and small-cap companies;
9 **Growth**: measures the stock's growth potential;
10 **Cap**: measures the capitalization in US dollars;
11 **Yield**: measures the dividend of a security;
12 **Country**: identifies the country where the company is traded;
13 **Currency**: identifies the currency in which the company trades;
14 **Portfolio**: selects top 200 securities based on value scores in January; these are held through the end of December (data comes from 2010);
15 **Benchmark**: the top 1000 securities based on size (cap-weighted) each month.

Our portfolio will consist of 200 equally weighted holdings. We buy in January and hold throughout the year. How well does our portfolio do?

We may outperform due to our chosen sectors; this is the **allocation effect**.

We may outperform due to our specific stock selection; this is the **selection effect**.

We may benefit from an interaction between these two; this is the **interaction effect**.

Our notation is as follows:

- $w_i^B$ is the weight of security $i$ in the benchmark;
- $w_i^P$ is the weight of security $i$ in the portfolio;
- $W_j^B$ is the weight of category $j$ in the benchmark;
- $W_j^P$ is the weight of category $j$ in the portfolio;
- $r_i$ is the return of security $i$;
- $R_j^B$ is the return of category $j$ in the benchmark;
- $R_j^P$ is the return of category $j$ in the portfolio;
- $M$ is the number of securities of category $j$ in the benchmark;
- $N$ is the number of securities of category $j$ in the portfolio;
- $J$ is the total number of categories in the benchmark and portfolio;
- $H$ is the total number of securities in the entire benchmark;
- $K$ is the total number of securities in the entire portfolio.

Here are our formulas:

- $W_j^B = \sum_{i=1}^{M} w_i^B$ gives the total weights in the benchmark's $j$th category;
- $W_j^P = \sum_{i=1}^{N} w_i^P$ gives the total weights in the portfolio's $j$th category;
- $R_j^B = \sum_{i=1}^{M} w_i^B r_i$ gives the total return of the benchmark's $j$th category;
- $W_j^P = \sum_{i=1}^{N} w_i^P r_i$ gives the total return of the portfolio's $j$th category;
- $\sum_{i=1}^{H} w_i^B = \sum_{i=1}^{K} w_i^P = 1$ shows that the weights sum to 1 in the portfolio and the benchmark, respectively;
- $\sum_{i=1}^{M} W_i^B = \sum_{i=1}^{N} W_i^P = 1$ shows that the weights sum to 1 in each category of the portfolio and of the benchmark, respectively;
- $R_P = \sum_{i=1}^{K} w_i^P r_i$ gives the portfolio return by summing individual securities;
- $R_P = \sum_{j=1}^{J} W_j^P R_j^P$ gives the portfolio return by summing each category;
- $R_B = \sum_{i=1}^{H} w_i^B r_i$ gives the benchmark return by summing individual securities;
- $R_B = \sum_{j=1}^{J} W_j^B R_j^B$ gives the benchmark return by summing each category.

As you recall, the active return is computed as the difference between the portfolio return and the benchmark return. We can formalize this as such:

- $R_{active} = R_P - R_B$
- $R_{allocation} = \sum_{j=1}^{k} W_j^P R_j^B - \sum_{j=1}^{k} W_j^B R_j^B$
- $R_{selection} = \sum_{j=1}^{k} W_j^B R_j^P - \sum_{j=1}^{k} W_j^B R_j^B$

Let's interpret these formulas. There is a selection effect if we pick better stocks within sectors. Our stocks in a sector are measured by $R_j^P$. The benchmark's stocks in a sector are measured by $R_j^B$. If $R_j^P > R_j^B$, then we have a positive selection effect. We could set this if we rewrote the selection to be

$$\sum_{j=1}^{k} W_j^B \left(R_j^P - R_j^B\right)$$

and determine if the value in parenthesis is positive.

Similarly, we can rewrite the allocation effect as

$$R_{allocation} = \sum_{j=1}^{k} R_j^B \left(W_j^P - W_j^B\right).$$

Our sectors are measured by $W_j^P$. The benchmark's sectors are measured by $W_j^B$. If we chose good sectors, then our sectors outperform the benchmark's sectors. That is, $W_j^P > W_j^B$.

How do we measure the interaction effect?

$$R_{interaction} = R_{active} - R_{allocation} - R_{selection}$$

All of this is illustrated in R code. First, we have to install a package called pa – short for performance attribution. The first lines of code install the package and then make it accessible to the current R session. Once we attach the package, we use the data function on the built-in data set year. Renaming some of the columns, we see we have the following data:

```
>names(year)
 [1] "barrid"     "name"        "return"     "date"        "sector"
 [6] "momentum"    "value"       "size"        "growth"      "cap.usd"
 [11] "yield"      "country"    "currency"  "portfolio" "benchmark"
```

To relate the column names to the formulas, let's rename the portfolio as wiP, which represents the formula $w_i^P$. Let's also rename the benchmark column as wiB, which represents the formula $w_i^B$. We then examine which columns are categorical data and which are numerical data.[1]

```
# what types of data are in each column?
> ( types = sapply(y, data.class) )
   barrid       name    return       date     sector  momentum
 "factor"  "factor" "numeric"     "Date"  "ordered" "numeric"
    value       size    growth   cap.usd      yield   country
"numeric" "numeric" "numeric" "numeric" "numeric"  "factor"
 currency       wiP       wiB
 "factor" "numeric" "numeric"
```

[1] The R code for this column rename is

```
> y=year;
> names(y)[14]='wiP'
> names(y)[15]='wiB'
```

Now, let's identify the weights of each sector in the benchmark and the weights of each sector in the portfolio.

```
# Identify the weights of the Benchmark by sector ----
> (WjB = tapply(y$wiB, y$sector, sum)/12)
     Energy   Materials Industrials   ConDiscre   ConStaples
0.235344622 0.060288565 0.041291227 0.021284367 0.008506904
 HealthCare  Financials    InfoTech    TeleSvcs    Utilities
0.062887358 0.316589009 0.033287546 0.169755299 0.050765103
> # Identify the weights of the Portfolio by sector ----
> (WjP = tapply(y$wiP, y$sector, sum)/12)
     Energy   Materials Industrials   ConDiscre   ConStaples
      0.085       0.070       0.045       0.050        0.030
 HealthCare  Financials    InfoTech    TeleSvcs    Utilities
      0.015       0.370       0.005       0.300        0.030
```

To normalize the weights, we had to sum them for all 12 months and then divide the total sum by 12. Note that these weights now sum to 1. So, the financial sector is our most concentrated sector (37%) in our portfolio, and information technology is our least concentrated sector (0.5%).

The sector our portfolio weights the least in relative terms, however, is energy. Energy comprises 23.5% of the benchmark but is only 8.5% of our portfolio. The sector our portfolio weights the most in relative terms is telecommunication services. This sector has a weight of 17% in the benchmark but 30% in our portfolio.

```
# Compute the benchmark weighted returns by sector
> (RjB = tapply(y$wiB * y$return, y$sector, sum)/WjB)
      Energy    Materials  Industrials    ConDiscre
 0.061243694 -0.009938344 -0.003522631  0.149452079
  ConStaples   HealthCare   Financials     InfoTech
 0.325327412 -0.006800432  0.002550346 -0.094130917
    TeleSvcs    Utilities
 0.107436004 -0.096279384
> # Compute the portfolio weighted returns by sector
> (RjP = tapply(y$wiP * y$return, y$sector, sum)/WjP)
     Energy   Materials Industrials   ConDiscre  ConStaples
 0.13543176  0.08495214  0.15503333  0.21343900  0.25589333
 HealthCare  Financials    InfoTech    TeleSvcs   Utilities
 0.18245333  0.05626595  0.00000000  0.13836433  0.32584500
```

We now have the formulas to calculate portfolio return in two ways.

```
> #Note these 2 methods of computing portfolio return should be the same ----
> (PRt1 = sum(y$wiP * y$return))
[1] 0.1176235
> (PRt2 = sum(WjP * RjP))
[1] 0.1176235
> #These 2 methods of computing benchmark returns should be the same ----
> (BRt1 = sum(y$wiB * y$return))
[1] 0.03021384
> (BRt2 = sum(WjB * RjB))
[1] 0.03021384
# Let's compute the active return
> (RActive      = PRt1 - BRt1)
[1] 0.08740961
>
> # Allocation is the (out)performance due to SECTOR choice
> (RAllocation  = sum(WjP*RjB) - sum(WjB*RjB))
[1] 0.02108349
>
> # Selection is the (out)performance due to STOCK choice
> (RSelection   = sum(WjB*RjP) - sum(WjB*RjB))
[1] 0.08921906
>
> # Interaction is the (out)performance due to BOTH at same time
> (RInteraction = RActive - RAllocation - RSelection)
[1] -0.02289293
```

This means that we have a positive active return of 8.74%. This is mostly due to selection (8.9%) and some due to allocation (2.1%), but we actually lose a bit due to the interaction between these (-2.3%). In other words, on average we pick better sectors than the benchmark. On average, we also pick better stocks than the benchmark. However, when we tend to pick stocks within winning sectors, they are slightly worse than the average stock in that sector.

Here are the results presented graphically. Each graph represents a particular month; each row represents a specific sector. The benchmark results are shown in orange. The portfolio results are shown in blue.

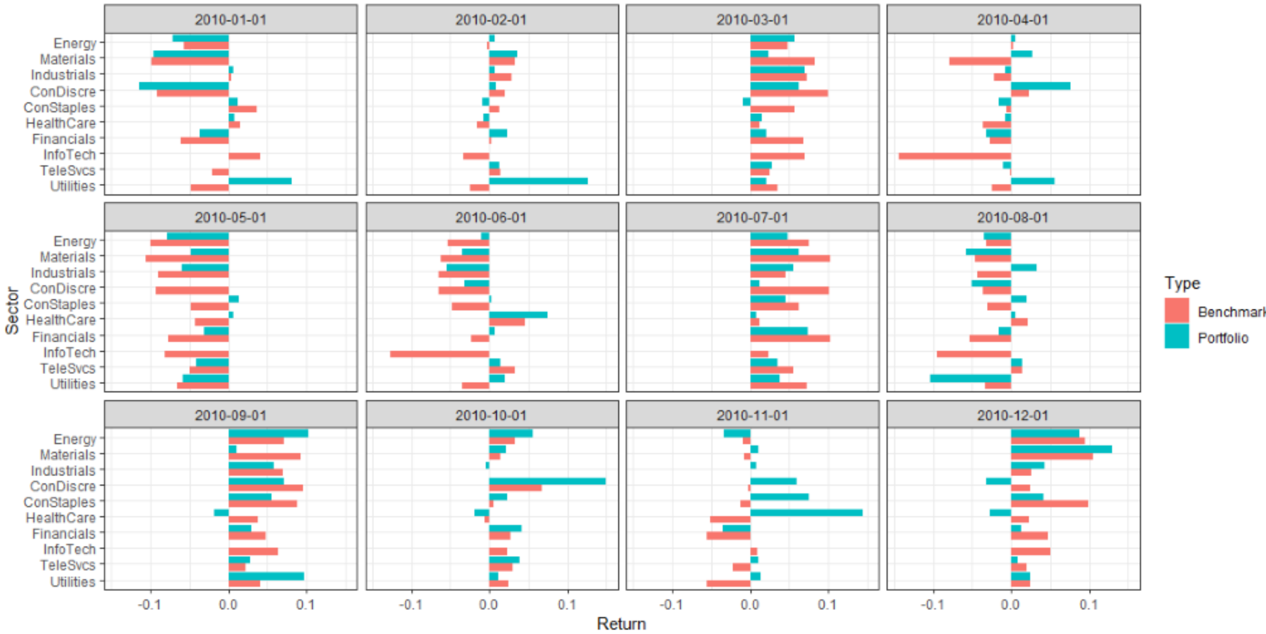


*Figure 2: Return across periods*

## Summary

We have examined how active return can be divided into a selection component, an allocation component, and an interaction component. Good macro hedge fund managers focus on which sectors to pick and when to switch from one sector to another. Macro hedge fund managers focus on sector allocation. Good value hedge fund managers focus on finding the best valued stocks. Their focus is on stock selection. You can tell how each manager is doing by running performance attribution models and assessing the amount of return attributed to sector selection and the amount attributed to stock selection.

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# Collaborative Review Task

In this module, you are required to complete a collaborative review task, which is designed to test your ability to apply and analyze the knowledge you have learned during the week.

## Question

1. Download 1-2 years of SPY. Find two other ETFs that track it.
2. Compute the returns, active returns, and average active return.
3. Compute the tracking error and mean-adjusted tracking error.
4. Which ETF tracks the S&P500 better?
5. Download the Select SPDR funds (tickers = XLB, XLE, XLF, XLI, XLK, XLP, XLRE, XLU, XLV, XLY) over the same time period as you did for SPY.
6. Compute the returns.
7. Write a function that computes active return internally and uses that to compute the mean-adjusted tracking error.
8. Determine which single sector fund best tracks the S&P500.