WORLDQUANT UNIVERSITY

Discrete Time Stochastic Processes Module 3

MSc Financial Engineering

Table of Contents

1. Brief

This document contains the core content for Module 3 of Discrete-time Stochastic Processes, entitled Discrete Martingales. It consists of four video lecture transcripts and five sets of supplementary notes.

2. Course Context

Discrete-time Stochastic Processes is the third course present in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The course introduces derivative pricing in discrete time. It begins with measure-theoretic probability and stochastic processes, with an emphasis on discrete-time martingales. These ideas are then applied to the pricing of derivatives in discrete time, followed by an introduction to interest rate and credit risk modeling.

2.1 Course-level Learning Outcomes

After completing the Discrete-time Stochastic Processes course, you will be able to:

1. Understand the language of measure-theoretic probability.
2. Understand stochastic processes and their applications.
3. Understand the theory of discrete-time martingales.
4. Define trading strategies in discrete time.
5. Create replicating portfolios in discrete time.
6. Model stock price movements on a binomial tree.
7. Price and hedge European derivatives in discrete time.
8. Price and hedge exotic European derivatives in discrete time.
9. Price and hedge American derivatives on a binomial tree.
10. Construct a simple interest rate model on a tree.
11. Price interest rate derivatives on a tree.

2.2 Module Breakdown

The Discrete-time Stochastic Processes course consists of the following one-week modules:

1. Probability Theory
2. Stochastic Processes
3. Discrete Martingales
4. Trading in Discrete Time
5. The Binomial Model
6. American Derivatives
7. An Introduction to Interest Rate Models

3. Module 3:

Discrete Martingales

Module 3 focuses on the theory of Discrete-time Martingales. Martingales are widely used in the modern theory of derivative pricing. In a nutshell, a martingale is a driftless stochastic process, which is just a mathematical analogue of a fair game.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Give examples of discrete-time martingales.
2. State the main results concerning random walks.
3. Apply the optional stopping theorem.
4. Calculate the Doob decomposition of the stochastic process.

3.2 Transcripts and Notes

3.2.1 Notes: Examples of Martingales

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Recall that a discrete-time stochastic process $M = \{M_n : n \in \mathbb{N}\}$ is a martingale with respect to $\mathbb{F}$ if:

1. M is $\mathbb{F}$-adapted.
2. $M_n \in \mathcal{L}^1$ for every $n \in \mathbb{N}$.
3. $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$ for every $n \in \mathbb{N}$.

In this section, we look at some examples of discrete-time martingales. We will look at examples that are based on the random walk defined as follows:

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $X_n \in \mathcal{L}^1$ for each n.

We define the random walk $S = \{S_n : n \in \mathbb{N}\}$ by

$$S_0 = x, \quad S_{n+1} = S_n + X_{n+1} \quad n \in \mathbb{N},$$

for some fixed $x \in \mathbb{R}$. We work with the natural filtration of X defined by

$$\mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_n := \sigma(\{X_1, \dots, X_n\}) \quad n \geq 1.$$

In each case, we will define a new process $W = \{W_n : n \in \mathbb{N}\}$ and show that it is a martingale. In showing that W is a martingale, we are only going to show the third property and leave the first two as an exercise.

Let $W = S$ itself. We stated earlier that S is a martingale with respect to $\mathbb{F}$ if and only if $\mathbb{E}(X_n) = 0$ for every $n \geq 1$. To see this, let $n \in \mathbb{N}$. Then,

$$\mathbb{E}(S_{n+1}| \mathcal{F}_n) = \mathbb{E}(S_n + X_{n+1}|\mathcal{F}_n) = \mathbb{E}(S_n|\mathcal{F}_n) + \mathbb{E}(X_{n+1}|\mathcal{F}_n) = S_n + 0 = S_n.$$

Define $W_n = S_n - n\mu$, where $\mu := \mathbb{E}(X_n)$. Then,

$$\mathbb{E}(W_{n+1}| \mathcal{F}_n) = \mathbb{E}(S_{n+1} - (n+1)\mu|\mathcal{F}_n) = S_n - n\mu \ + \mathbb{E}(X_{n+1} - \mu| \mathcal{F}_n) = W_n.$$

In the special case of a simple random walk, when X_n takes the value 1 with probability p and -1 with probability $q := 1 - p$, then $\mu = p - q$ and $W_n = S_n - (p - q)n$.

Further assume that $\mu = 0$ and $\sigma^2 \coloneqq \mathbb{E}(X_n^2) < \infty$ for every $n \geq 1$. Let $W_n = S_n^2 - n\sigma^2$ for every $n \in \mathbb{N}$. Then, $W = \{W_n : n \in \mathbb{N}\}$ is a martingale with respect to $\mathbb{F}^X$. Indeed, if $n \in \mathbb{N}$, then,

$$\mathbb{E}(W_{n+1}|\mathcal{F}_n) = \mathbb{E}((S_n + X_{n+1})^2 - (n+1)\sigma^2|\mathcal{F}_n) = \mathbb{E}(S_n^2 + 2X_{n+1}S_n + X_{n+1}^2|\mathcal{F}_n) - (n+1)\sigma^2$$
$$= S_n^2 - n\sigma^2 = W_n.$$

For a simple random walk with $\mathbb{P}_{X_n}(\{1\}) = 0.5 = \mathbb{P}_{X_n}(\{-1\})$, we get $\sigma^2 = 1$ and $W_n = S_n^2 - n$.

Let $\theta \in \mathbb{R}$ and define

$$W_n := (M_X(\theta))^{-n} e^{\theta S_n},$$

where M_X is the moment-generating function of X_n. Then,

$$\mathbb{E}(W_{n+1}|\mathcal{F}_n) = \mathbb{E}\big((M_X(\theta))^{-(n+1)} e^{\theta S_{n+1}}|\mathcal{F}_n\big) = (M_X(\theta))^{-n} e^{\theta S_n}\, M_X(\theta)^{-1}\mathbb{E}(e^{\theta X_{n+1}}) = W_n.$$

The case of a simple random walk with $\mathbb{P}_{Xn}(\{1\}) = p = 1 - q = 1 - \mathbb{P}_{Xn}(\{-1\})$, we get:

$$M_X(\theta) = pe^{\theta} + qe^{-\theta} \text{and } W_n = (pe^{\theta} + qe^{-\theta})^{-n} e^{\theta S_n}.$$

If we further assume that $p = q = 0.5$, then $M_X(\theta) = \cosh\theta$ and $W_n = (\cosh\theta)^{-n}\, e^{\theta S_n}$.

Now, assume that X_n's are discrete with $\mathbb{P}_{X_n}(\{1\}) = p = 1 - q = 1 - \mathbb{P}_{X_n}(\{-1\})$ and $p > q$.

Define $W_n = \left(\frac{q}{p}\right)^{S_n}$ for $n \in \mathbb{N}$. Then,

$$\mathbb{E}(W_{n+1}|\mathcal{F}_n) = \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_{n+1}} |\mathcal{F}_n\right) = \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left(\left(\frac{q}{p}\right)^{X_{n+1}}\right) = W_n.$$

3.2.2 Transcript: Martingales in Discrete Time

Hi, in this video we are going to look at martingales in discrete time.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and we are going to assume that this is equipped with a filtration, $\mathbb{F} = \{\mathcal{F}_t : t \in \mathbb{I}\}$, which is an increasing sequence of σ-algebras.
Recall that the stochastic process, $M = \{M_t : t \in \mathbb{I}\}$ that is indexed by the same index set is a martingale with respect to the filtration, $\mathbb{F}$, and the probability measure, $\mathbb{P}$, if it satisfies the following conditions:

1. The first requirement is that M must be $\mathbb{F}$-adapted. This means that at time t we know the value of M_t.
2. The second condition is that M must be integrable for every t. This means that the expected value of the absolute value of M_t is finite for every t. Written in full:

$$E(|M_t|) < \infty \qquad \forall t \in \mathbb{I}.$$

3. The third condition is what we call the defining property of a martingale. This says that the conditional expectation of M_t, given the information at time s, is simply equal to M_s, and this calls almost surely for $s < t$. Written in full:

$$E(M_t \mid \mathcal{F}_s) = M_s \qquad \text{a.s. } s < t.$$

Another way of looking at it is as follows: at time t, the value of M_t is, of course, not necessarily known at time s. However, if we have information contained in $\mathcal{F}_s$, and the best estimator of M_t, given the information at time s, then $\mathcal{F}_S$ will be equal to M_s itself. So, what this intuition means is that a martingale is a driftless process and on average the sample paths look like this. Despite the fact that some of the paths might go up and down, on average they have no drift.

We now look at an example of a martingale.

Consider a probability space, $X = \{X_1, X_2, \dots\}$, where we have defined a sequence of iid random variables that are all in L_1, with the expected value of each of them equal to μ. We define the random walk S_n as $S = \{S_0, S_1, \dots\}$. S_0 is equal to x, where x is a fixed real number. S_{n+1} is simply S_n plus X_{n+1} for every N. What this translates to it that S_n becomes the sum from $k = 1$ up to n of each X_k. Written in full:

$$X = \{X_1, X_2, \dots\}, \quad E(X_i) = \mu$$
$$S = \{S_0, S_1 \dots\} \quad S_0 = x, x \in \mathbb{R}$$
$$S_{n+1} = S_n + X_{n+1}, \quad n \in \mathbb{N}$$
$$S_n = \sum_{k=1}^{n} X_k.$$

Now, since the expected value of X_i is equal to μ, we define W_n as $S_n - n\mu$. So, we are just detrending the random walk S_n.

Let's show that W is indeed a martingale with respect to the σ-algebra, which is the smallest filtration generated by S.

1 First, we must show that W is adapted. So, since $W_n = S_n - n\mu$, where $n\mu$ is a function of S_n, it is implied that W_n is measurable with respect to $\mathcal{F}_n{}^S$.
2 The second property is that W_n is integrable. We have to show that the expected value of the absolute value of W_n is finite and that is equal to the absolute of $S_n - n\mu$, which is less than or equal to, via the triangle inequality, the absolute value of $S_n + n$ times the absolute value of μ. This is equal to the expected value of the absolute value of $S_n + n$ times the absolute value of μ. Again, we can use the triangle inequality to the absolute value of S_n and show that this is less than or equal to the sum of the absolute value of $X_k + n$ times the absolute value of μ. All of this is finite because each X_k is in $L1$.

$$E(|W_n|) = E(|S_n - n\mu|) \leq E(|S_n| + n|\mu|) = E(|S_n|) + n|\mu| \leq \sum_{k=1}^{n} E(|X_k|) + n|\mu| < \infty.$$

3 The third property, which is the main martingale property, is that the conditional expectation of W_{n+1}, given $\mathcal{F}_n^s$, is equal to W_n. If we calculate this, it is equal to the conditional expectation of $S_{n+1}-\ (n+1)\mu$ given $\mathcal{F}_n{}^s$. This can be further simplified to the conditional expectation of $S_n +\ X_{n+1}$, given $\mathcal{F}_n{}^s$ minus $(n+1)\mu$, which is a constant, and, therefore, the conditional expectation is itself.

This simplifies to – when we use the linearity of conditional expectation – the conditional expectation of S_n, given $\mathcal{F}_n{}^s$, is S_n itself because it's measurable, and X_{n+1} is independent of $\mathcal{F}_n{}^s$, so this is $S_n + E\big(X_{n+1}-(n+1)\big)\mu$. This is then equal to the expected value of $S_{n+1} + \mu - (n+1)\mu$, which is equal to $S_n - n\mu$. Therefore, we have shown that W is a martingale. Written in full:

$$
\begin{aligned}
E(W_{n+1}|\mathcal{F}_n{}^s) &= E(S_{n+1} - (n+1)\mu \\
&= E(S_n + X_{n+1}|\mathcal{F}_n{}^s) - (n+1)\mu \\
&= S_n + E(X_{n+1}) - (n+1)\mu \\
&= S_n + \mu - (n+1)\mu \\
&= S_n + n\mu.
\end{aligned}
$$

We now look at another example of a martingale.

Consider the experiment of tossing two coins with the following sample space: $\Omega = \{HH, \dots\ TT\}$ and the following stochastic process:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
HH	1	$\frac{3}{2}$	2
HT	1	$\frac{3}{2}$	1
TH	1	$\frac{1}{2}$	1
TT	1	$\frac{1}{2}$	0

The probability measure $\mathbb{P}$, is equal to $\frac{1}{4}\sum_{\omega\epsilon\Omega}\delta\omega$, meaning that all of the above outcomes are equally likely.

You can easily show, then, the conditional expectation of X_1, given $\mathcal{F}_0{}^X$, because X is the natural filtration of X. In this case, it is equal to the expected value of X_1itself because the information at time 0 generates the trivial σ-algebra. So, the conditional expectation with respect to the trivial σ-algebra is simply equal to the expected value of X itself. In this case, it is equal to $\frac{3}{2}\times\frac{1}{4}\times\frac{3}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}$, which is equal to 1. So, therefore, this is equal to X_0 almost surely. Written in full:

$$
\begin{gathered}
E(X_1|\mathcal{F}_0{}^X = E(X_1) \\
= \frac{3}{2}\times\left(\frac{1}{4}\right)\times\frac{3}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4} \\
= 1 = X_0 \text{ a.s..}
\end{gathered}
$$

Similarly, we can show that the expected value of X_2 given $\mathcal{F}_1{}^X$ is equal to $X_1(\omega)$ almost surely. Written in full:

$$E(X_2 \mid \mathcal{F}_1{}^X)(\omega) = X_1(\omega) \text{ a.s..}$$

This therefore implies that X in indeed a martingale.

Now that we have looked at discrete-time martingales, in the next video, we are going to proceed to martingale transforms.

3.2.3 Notes: Martingale Transforms

In this section, we will continue to assume that $\mathbb{I} \subseteq \mathbb{N}$. For intuition, it will be useful to imagine that you are playing a sequence of games and $X_n - X_{n-1}$ represents your winnings during the n^{th} game if you play unit stakes. Your total winnings by time n is then equal to:

$$\text{cumulative total gains} = (X_1 - X_0) + (X_2 - X_1) + \cdots + (X_n - X_{n-1}) = X_n - X_0.$$

When X is a martingale, then:

$$\mathbb{E}(X_n - X_{n-1} | \mathcal{F}_{n-1}) = 0,$$

implying that the game is fair. When X is a submartingale, the game is biased in your favor; if X is a supermartingale, then the game is unfavorable to you.

Now, suppose that you play a stake of φ_n on the n^{th} game. Then your winnings on the n^{th} game will be $\varphi_n(X_n - X_{n-1})$ and your total gains will be:

$$\text{cumulative total gains} = \varphi_1(X_1 - X_0) + \ldots + \varphi_n(X_n - X_{n-1}) = \sum_{k=1}^{n} \varphi_k(X_k - X_{k-1}).$$

Note that for φ to be a sensible gambling/trading strategy, we need to know the value of φ_n at time $n-1$, i.e. φ_n must be $\mathcal{F}_{n-1}$-measurable.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. A sequence of random variables $\varphi = \{\varphi_n : n = 1,2,3,\ldots\}$ is called previsible or predictable if $\varphi_n \in m\mathcal{F}_{n-1}$ for each $n \in \mathbb{N}^+$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, $X = \{X_n : n \in \mathbb{N}\}$ be a stochastic process and $\varphi = \{\varphi_n : n \in \mathbb{N}^+\}$ be a predictable sequence. We define the martingale transform or discrete stochastic integral of X by φ as the stochastic process $(\varphi \bullet X) = \{(\varphi \bullet X)_n : n \in \mathbb{N}\}$ defined by:

$$(\varphi \bullet X)_0 = 0, \quad (\varphi \bullet X)_n = \sum_{k=1}^{n} \varphi_k (X_k - X_{k-1}),\ n \geq 1.$$

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered space, $\varphi = \{\varphi_n : n \in \mathbb{N}^+\}$ be a bounded ($|\varphi_n| \leq C < \infty$ for every $n \in \mathbb{N}^+$) predictable sequence, and $X = \{X_n : n \in \mathbb{N}\}$ be a stochastic process.

1. If X is a martingale then $(\varphi \bullet X)$ is also a martingale.
2. If φ is non-negative and X is a submartingale (resp. supermartingale), then $(\varphi \bullet X)$ is also a submartingale (resp. supermartingale).

The interpretation of this result is as follows. If you choose a gambling strategy φ, then, provided that the strategy is bounded, it will not turn a fair game into an unfair one (or one that is more favorable towards you). Likewise, if the game is unfavorable towards you (i.e. X is a supermartingale), then it will remain unfavorable if you don't own the casino (i.e. if your stakes stay positive).

There are cases where $(\varphi \bullet X)$ is a martingale even when φ is not bounded. Here is an example. The following trading strategy is the origin of the term "martingale". Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = (X_n)_{n=1}^{\infty}$ be a sequence of i.i.d random variables with $\mathbb{P}(\{X_n = 1\}) = \mathbb{P}\{X = -1\}) = 0.5$. Define the random walk $S = \{S_n : n \in \mathbb{N}\}$ by

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} X_k, \quad n \geq 1\ .$$

Then S is a martingale with respect to $\mathbb{F}^S$. We think of S_n as the cumulative winnings (or losses) when a player invests unit stakes in a game with payoff given by X. Consider the following betting strategy:

$$\varphi_1 = 1, \quad \varphi_{n+1} = 2\varphi_n I_{\{X_n = -1\}}, \quad n \geq 1.$$

That is, you first bet \$1. If you win, you quit playing ($\varphi_2 = 2\varphi_1 I_{X_1} = 0$ if $X_1 = 1$), but if you lose, then you double your next bet. If you win this time, you quit, but if you lose again, you double your bet in the next round, and so on until you win. The idea is that by the first time you win — due to the size of your bet then — you will recover all previous losses and make a profit of \$1. Here is a table that illustrates this:

Game	Stake	Cumulative losses	Profit on winning
1	1	0	1
2	2	1	1
3	4	3	1
4	8	7	1

One can check that $(\varphi \bullet S)$ is a martingale (even though φ is not uniformly bounded).

3.2.4 Transcript: A Worked Example of Martingale Transforms

Hi, in this video we're going to study martingale transforms.

Consider a stochastic process X in discrete time: $X = \{X_n: n = 0, 1, 2 \dots\}$, whereby X_n is the price of a share at time n. We can represent that on a number line, where X_0 is the price now, X_1 is the price in one unit of time, X_2, X_3, and so on.

Let's assume that we invest φ_1 at time 0, φ_2 at time 1, φ_3, φ_4 and so on. Then the gains from using this trading system strategy are given by $(\varphi \bullet X)_n$. That will be $\varphi_1(X_1 - X_0)$, which is the number of shares times the change in share price between time 1 and time 0, plus $\varphi_2(X_2 - X_1)$ and so on, plus, until we get to $\varphi_n(X_n - X_{n-1})$. Written in full:

$$(\varphi \bullet X)_n = \varphi_1(X_1 - X_0) + \varphi_2(X_2 - X_1) + \cdots + \varphi_n(X_n - X_{n-1}).$$

For φ to be a sensible trading strategy, we need to know the value of φ_1 at time 0. We also need to know the value of φ_2 at time 1, φ_3 at time 2, and so on and so forth. We call such a strategy "predictable". So, φ is predictable if $\varphi_n \in m\,\mathcal{F}_{n-1}$-measurable, for every n greater than or equal to 1. In that case, we call this stochastic process $((\varphi \bullet X)_n)$ the martingale transform. In general, however, we call it the transform of X by φ.

These transforms satisfy some interesting properties:

1. If φ_n is uniformly bounded and predictable – "uniformly bounded" means that the absolute value of φ_n is less than some fixed constancy for every n greater than or equal to 1 – and X is a martingale, then the martingale transform is also a martingale.
2. If φ_n is still uniformly bounded and predictable and is also non-negative, then, if X is a sub- or supermartingale, the martingale transform is also a sub- or a supermartingale.

What this means, essentially, is that you can't beat the system, in the sense that if the stock price X_n is a martingale itself to begin with then you cannot turn it into a submartingale or supermartingale by choosing an appropriate strategy – as long as the strategy is bounded.

The second part says that if you're playing a game at a casino, for instance, and your winnings, X or X_n (depending on what the game is), are a submartingale or a supermartingale, you cannot turn them to something else if you are only allowed to invest in positive stakes.

Let's go through an example.

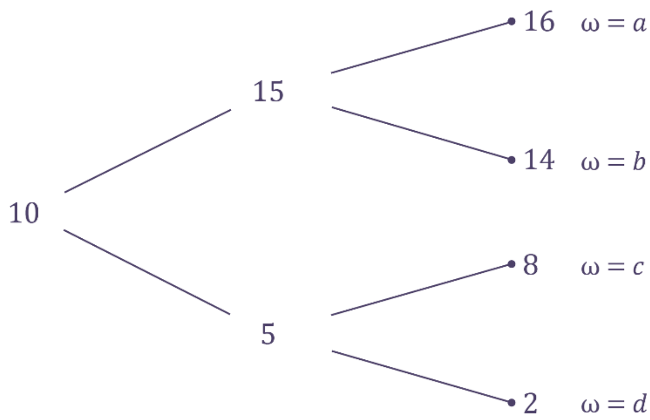
Consider the following probability space: $\Omega = \{a, b, c, d\}$, take $\mathcal{F}$ to be the power set and $\mathbb{P}$ to be Dirac measures at a, b, c and d, all weighted equally. Written in full:

$$\Omega = \{a, b, c, d\}, \qquad \mathcal{F} = 2^{\Omega}, \qquad \mathbb{P} = \frac{1}{4}(S_a + S_b + S_c + S_d).$$

This can be shown as ω and $\mathbb{P}(\{\omega\})$ where a, b, c, and d are all equal to $\frac{1}{4}$. We can then define the stochastic process, such that $X_0(\omega)$ is always equal to 10, $X_1(\omega)$ is equal to 15 or 5, and $X_2(\omega)$ is equal to $16, 14, 8$, or 2. This can be laid out as follows:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
a	$\frac{1}{4}$	10	15	16
b	$\frac{1}{4}$	10	15	14
c	$\frac{1}{4}$	10	5	8
d	$\frac{1}{4}$	10	5	2

We will choose the filtration to be the natural filtration of $X(\mathbb{F} = \mathbb{F}_X)$, which is the smallest filtration that makes X adapted. This can be represented in a tree diagram as follows:



At time 0, the stock is equal to 10, at time 1 the stock is equal to 15 or 5, and at time 2 the stock, and the path branching from 15, is either equal to 16 or 14, and the path branching from 5 is either equal to 8 or 2. The path that ends at 16 is ω corresponding to a, the path that ends at 14 is ω corresponding to b, the path ending at 8 to c, and the path ending at 2, d.

Consider the following trading strategy: φ_1, because φ_1 has to be $\mathcal{F}_0$-measurable and, in this case, $\mathcal{F}_0^X$ will be equal to Ω and the empty set, because this X_0 is constant at time 0. $\mathcal{F}_1^X$ will be equal to the σ-algebra generated by these two blocks, $\{a, b\}$ and $\{c, d\}$. $\mathcal{F}_2^X$ is equal to $\mathcal{F}$, which is the power set of Ω. Therefore, φ_1 must be necessarily constant in order for it be $\mathcal{F}_0$-measurable.

We consider the following trading strategy whereby you invest 2 units of X at time 0, which is φ_1, and then φ_2 will be either 1 on $\{a, b\}$, so it is 1 on the branch that leads from 10 to 15, or it is equal to 3 on the branch $\{c, d\}$. Intuitively, what that means is that if the stock rises to $\$10$, then we decrease our investment to 1, but if it decreases to $\$5$, we increase our investment to 3.

Let's calculate the martingale transform – the gains from trading strategy. This will be calculated for $n = 0, 1, 2$. We know by definition $(\varphi \bullet X)_0 = 0$, meaning there are no gains from trading at time 0. $(\varphi \bullet X)_1$ will be equal to $\varphi_1 \times (X_1 - X_0)$, which will take two possible values. In this case, it will either be $\{2(15-10)\}$ on $\{a, b\}$, or it will be $\{2(5-10)\}$ on $\{c, d\}$. Finally, we have $(\varphi \bullet X)_2$, which is equal to $\varphi_1(X_1 - X_0) + \varphi_2(X_2 - X_1)$ and that will take four possible values.

The first part on a will be 2 times $(15$ minus $10)$, so that is 10, plus φ_2 on a, which is 1, so that will be 1 times $(16$ minus $15)$, so $+1$. On b, it will be 2 times $(15$ minus $10)$ which is 10 (that's the first cumulative gain), plus 1 times $(14$ minus $15)$ which is -1. On c, it will be 2 times $(5$ minus $10)$, which is -10, plus 3 times $(8$ minus $5)$, which is 9. Finally, on d, it will be 2 times $(5$ minus $10)$, which is -10, plus $(2$ minus $5)$ times 3, which is -9.

That is the martingale transform.

3.2.5 Notes: Optional Stopping Theorem

We now look at the interaction between martingales and stopping times.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered space, $X = \{X_n : n \in \mathbb{N}\}$ be a stochastic process, and τ be a stopping time.

1. If X is a martingale, then the stopped process $X^\tau = \{X_n^\tau : n \in \mathbb{N}\}$ is also a martingale.
2. If X is a submartingale (resp. supermartingale), then the stopped process $X^\tau = \{X_n^\tau : n \in \mathbb{N}\}$ is also a submartingale (resp. supermartingale).

This is a remarkable result! It tells us that, if a game is fair, then no matter how we choose to quit, the game will remain fair. If X is a martingale, this implies that

$$\mathbb{E}(X_0) = \mathbb{E}(X_0^\tau) = \mathbb{E}(X_n^\tau) = \mathbb{E}(X_{\tau\wedge n}) \text{ for every } n \in \mathbb{N}.$$

From this equation, can we conclude that

$$\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$$

as well?

Consider a simple random walk. Let $X = \{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d random variables with $\mathbb{P}(\{X_n = 1\}) = \mathbb{P}(\{X_n = -1\}) = 0.5$ for every $n \in \mathbb{N}^+$. Define $S_0 = 0$ and $S_{n+1} = S_n + X_{n+1}$ for $n \in \mathbb{N}$. Let $\tau : \Omega \to \cup \{\infty\}$ be defined as $\tau(\omega) := \inf\{n \in \mathbb{N} : S_n(\omega) = 1\}$. The stopped process S^τ is a martingale (with respect to $\mathbb{F}^S$), so

$$\mathbb{E}(S_{\tau\wedge n}) = \mathbb{E}(S_0) \quad \text{for every } n \in \mathbb{N},$$

but

$$\mathbb{E}(S_\tau) = 1 \neq 0 = \mathbb{E}(S_0).$$

So, in general, $\mathbb{E}(S_\tau) \neq \mathbb{E}(S_0)$.

We now seek to place conditions on τ and X that will allow us to conclude that $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$. Note that if $\tau(\omega) < \infty$, then

$$\lim_{n\to\infty} X_n^\tau(\omega) = \lim_{n\to\infty} X_{\tau(\omega)\wedge n}(\omega) = X_{\tau(\omega)}(\omega) = X_\tau(\omega).$$

Therefore, if $\tau < \infty$ almost surely, then the conclusion that $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$ will hold as soon as we can justify the interchange between limits and expectation in the following:

$$\mathbb{E}(X_0) = \lim_{n\to\infty} \mathbb{E}(X_{\tau\wedge n}) = \mathbb{E}\left(\lim_{n\to\infty} X_{\tau\wedge n}\right) = \mathbb{E}(X_\tau). \tag{1}$$

Recall that the Dominated Convergence Theorem (DCT) allows us to do so, provided that the sequence $(X_{\tau\wedge n})$ is dominated by an integrable random variable $Y \geq 0$. That is, if there exists $Y \in m\mathcal{F}^+ \cap \mathcal{L}^1$ such that $|X_{\tau\wedge n}| \leq Y$, then (1) is permissible. Sufficient (but not necessary) conditions for when this is possible are provided by the following *Optional Stopping Theorem* (OST).

Let $X = \{X_n : n \in \mathbb{N}\}$ be a martingale and τ be a stopping time. Then,

$$\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$$

if any one of the following conditions hold:

1. The stopping time is bounded a.s.; that is, there exists $N \in \mathbb{N}$ such that $\tau \leq N$ a.s..
2. $\tau < \infty$ a.s., and $|X_n^\tau| \leq C$ for some constant $C > 0$.
3. $\mathbb{E}(\tau) < \infty$ and $|X_n^\tau| - X_{n-1}^\tau \leq C$ for some constant $C > 0$.

As an example, consider the symmetric random walk $S = \{S_n : n \in \mathbb{N}\}$ starting at 0 and $\mathbb{F} = \mathbb{F}^S$. Let $a, b \in \mathbb{Z}$ with $a < 0 < b$. Define $\tau = \inf\{n \in \mathbb{N} : S_n = a \text{ or } b\}$. Then τ is a stopping time with $\mathbb{E}(\tau) < \infty$ (therefore $\tau < \infty$ a.s.). Since $S_0 = 0$, we have that $|S_{\tau \wedge n}| \leq \max\{|a|, b\}$. Hence, we can apply the OST to get that

$$0 = \mathbb{E}(S_0) = \mathbb{E}(S_\tau).$$

We now calculate $p = \mathbb{P}(S_\tau = a)$. Since S_τ takes two possible values, we have

$$0 = \mathbb{E}(S_\tau) = pa + (1-p)b,$$

which gives

$$p = \frac{b}{b-a}.$$

In the previous example, we find $\mathbb{E}(\tau)$. Define $W = \{W_n : n \in \mathbb{N}\}$ by $W_n = S_n^2 - n$. We showed that W is also a martingale. Now,

$$\begin{aligned} |W_n^\tau - W_{n-1}^\tau| = |(S_n^\tau)^2 - \tau \wedge n - (S_{n-1}^\tau)^2 + \tau \wedge (n-1)| &\leq (S_n^\tau)^2 + (S_{n-1}^\tau)^2 + 1 \\ &\leq 2\max\{a^2, b^2\} + 1. \end{aligned}$$

It can be shown that $\mathbb{E}(\tau) < \infty$, so we can apply the OST to get

$$0 = \mathbb{E}(W_0) = \mathbb{E}(W_\tau) = \mathbb{E}(S_\tau^2) - \mathbb{E}(\tau) = pa^2 + (1-p)b^2 - \mathbb{E}(\tau),$$

which implies that

$$\mathbb{E}(\tau) = pa^2 + (1-p)b^2 = -ab.$$

3.2.6 Transcript: An Example of the Optional Stopping Theorem

Hi, in this video we show an application of the Optional Stopping Theorem (OST).

Recall that the OST says that if X is a martingale and τ is a stopping time, then we can conclude that the expected value of X at the stopping time is equal to the expected value of X at time 0, provided that any one of the following conditions hold:

1 $\tau \leq N$ a.s.
The stopping time is actually bounded by a constant, N, meaning that τ is less than or equal to N almost surely. This is a very strong condition.

2 $\tau < \infty$ a.s. and $|X_n^\tau| \leq C\ \forall n$
The stopping time, τ, is finite almost surely, and the stopped process is uniformly bounded for every n.

3 $E(\tau) < \infty$ and $|X_n^\tau - X_{n-1}^\tau| \leq C$
The expected value of the stopping time is finite, which also implies that τ is finite almost surely, and the differences in the stopped process or the increments are bounded by some constant, C.

So, if any of these three conditions hold, then we can conclude that $E(X_\tau = E(X_0)$ and this is the OST.

I just want to mention, though, that these are not the most general conditions. The above conclusion still holds under more relaxed assumptions.

Let's look at an example.

Consider a random walk, where $S = \{S_n: n = 0, 1, 2, \dots\}$. Recall that this means that $S_n = S_{n-1} + X_n$, where X_n takes on two values, 1 and -1 with probability $\frac{1}{2}$.

This is what we call a random walk. The sample paths of a random walk look like this – always assume that it starts at 0, so $S_0 = 0$.

Now, let a be a negative integer and let b be a positive integer. So, a is less than 0 and b is greater then 0, and a and b are both integers. We define the following stopping time, whereby τ is the first time that the random walk hits a or b. Written in full:

$$\tau \coloneqq \inf\{n \geq 0 : S_n = a \quad \text{or} \quad b\}.$$

In a diagram, it would look something like this: we have a somewhere here and b at this point. The stopping time τ is the first time that the stopping time τ hits either a or b at this point. So, in this case, τ would be equal to 4.

We want to decide whether or not the OST applies, and in this case we are going to apply condition 2.

Though we are not going to do it in this course, it can be shown that τ is finite almost surely. In fact, a stronger condition holds – the expected value of τ is finite as well, which of course implies that τ is finite almost surely. So, we are just going to show this second part. If we look at the stopped process, S_n^τ, it is always bounded between a and b, meaning that it never goes beyond b and never goes below a. Before the process stops, it is always between a and b, and, once it stops, it is exactly at one of those points.
So, that absolute value of the stopped process is therefore less than or equal to the larger value of the absolute value of a and b, which we are going to define as C, and that is clearly finite. Written in full:

$$|S_n^\tau| \leq \max(|a, |b) = : C < \infty.$$

So, the stopped process is uniformly bounded, the stopping time is finite almost surely, and so condition 2 applies. We can therefore conclude that the expected value of S_τ should be equal to the expected value of S_0, which is 0 in this case.

That will allow us to calculate, for instance, the probability that it hits b before it hits a. So, let's call that probability P.

We can calculate $P(S_\tau = b)$ by using the fact that S_τ takes on two values: a or b. So, $E(S_\tau) = p \bullet b + (1 - p)a$ and because of the equation $E(S_\tau = E(S_0) = 0$, this is equal to 0, which implies that we can solve for p which gives us $\frac{b}{b-a}$. Written in full:

$$E(S_\tau) = E(S_0) = 0$$
$$p = P(S_\tau = b)$$
$$E(S_\tau) = p \bullet b + (1 - p)a = 0$$
$$p = \frac{b}{b - a}.$$

Now that we gone through the OST, in the next notes and video, we are going to look at the Doob decomposition.

3.2.7 Notes: Doob Decomposition

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and $X = \{X_n : n \in \mathbb{N}\}$ be an adapted stochastic process. In this section, we show that if X is integrable, then we can decompose X uniquely as

$$X = X_0 + M + A,$$

where M is a martingale and A is a predictable process, both null at 0 (i.e. $M_0 = A_0 = 0$). We think of A as representing the true "signal" and M representing the "noise", so we can write

$$X = X_0 + \text{signal} + \text{noise}.$$

We begin with a Lemma.

Lemma 1

If M is a predictable martingale with $M_0 = 0$, then $M \equiv 0$ a.s..

Proof 1

For each $n \in \mathbb{N}$,

$$M_n = \mathbb{E}(M_{n+1} | \mathbb{F}_n) = M_{n+1}.$$

Hence, $M_n = M_0 = 0$ for all $n \in \mathbb{N}$.

We now state and prove the Doob decomposition theorem.

Theorem 10 [Doob decomposition]

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space and* $X = \{X_n : n \in \mathbb{N}\}$ *be a stochastic process such that*

1 X *is adapted to* $\mathbb{F}$*,*

2 $X_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \;\; \forall n \in \mathbb{N}$*.*

Then there exist two unique stochastic processes, M *and* A*, both null at zero, such that*

1 M *is a martingale with respect to* $\mathbb{F}$*,*

2 A *is predictable, and*

3 $X_n = X_0 + A_n + M_n$ *for each* $n \in \mathbb{N}$*.*

Furthermore, A is increasing (resp. decreasing) if and only if X is a submartingale (resp. supermartingale). We call (A, M) the *Doob decomposition* of X.

Proof 2

1 We begin by showing uniqueness. Suppose there are two candidates for the Doob decomposition of X: (A, M) and (A', M'). Then, for each $n \in \mathbb{N}$, we have

$$X_n = X_0 + M_n + A_n = X_0 + M'_n + A'_n \Rightarrow \; M_n - M'_n = A_n - A'_n.$$

This implies that $M_n - M'_n$ is a predictable martingale null at zero, which implies that $M_n - M'_n = 0 \Rightarrow \; M_n = M'_n$ and, consequently, $A_n = A'_n$ as well.

2 We now prove existence. With $A_0 = M_0 = 0$, we define (A, M) inductively as

$$A_{n+1} = A_n + \mathbb{E}(X_{n+1} - X_n | \mathbb{F}_n) \text{ and } M_{n+1} = M_n + X_{n+1} - \mathbb{E}(X_{n+1} | \mathbb{F}_n) \quad n \in \mathbb{N}.$$

Solving this gives:

$$A_n = \sum_{k=1}^{n} \mathbb{E}(X_k - X_{k-1} | \mathcal{F}_{k-1}) \text{ and } M_n = \sum_{k=1}^{n} (X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1}))\,.$$

It is an exercise to check that both A and M satisfy the required conditions.

The definitions of A and M have the following interpretations: A_n is the cumulative expected jumps of X, while M_n is the correction term for the actual jumps. Thus, we can write X as,

$$X = \text{Expected Jumps} + \text{Surprises}.$$

The process A is also called the *compensator* of X.

In continuous time, such a decomposition is not always possible. However, P.A. Meyer showed that for a special class of submartingales *(submartingales of class D)*, one can find a similar decomposition. The result is called *The Doob-Meyer Decomposition Theorem* and it is a cornerstone of *'la théorie générale'*.

Let us get our hands dirty. Consider a sequence of i.i.d. random variables $\{X_n : n = 1,2,3, \dots\}$ with $\mathbb{E}(X_n) = \mu$ and the random walk S_n be defined by:

$$S_0 = 0, \quad S_{n+1} = S_n + X_{n+1}, \quad n \in \mathbb{N}.$$

Then it is easy to see that

$$A_n = n\mu \text{ and } M_n = S_n - n\mu,$$

for every $n \in \mathbb{N}$. That is,

$$S_n = 0 + n\mu + (S_n - n\mu) \quad \forall n \in \mathbb{N},$$

where $n\mu$ is the compensator and $S_n - n\mu$ is the martingale part of S.

A martingale X is *square integrable* if $\mathbb{E}(X_n^2) < \infty$ for all $n \in \mathbb{N}$ (i.e. $X_n \in \mathcal{L}^2 \forall n \in \mathbb{N}$). Let X be a square integrable martingale null at 0. By Jensen's inequality, the process $X^2 = \{X_n^2 : n \in \mathbb{N}\}$ is a submartingale, and by the Doob decomposition theorem, there exists a unique *increasing* predictable process A_n such that $X^2 - A$ is a martingale. We call the process A the predictable quadratic variation of X and denote it by $\langle X \rangle$. That is, $\langle X \rangle$ is the unique increasing predictable process such that $X^2 - \langle X \rangle$ is a martingale.

As an exercise, show that

$$\langle X \rangle_n = \sum_{k=1}^{n} \mathbb{E}((X_k^2 - X_{k-1}^2)|\mathcal{F}_{k-1}) = \sum_{k-1}^{n} \mathbb{E}((X_k - X_{k-1})^2|\mathcal{F}_{k-1}), \quad n \in \mathbb{N}^+.$$

Consider a sequence of i.i.d. random variables $\{X_n : n = 1,2,3, \dots\}$ with $\mathbb{P}_{X_n}(\{1\}) = \mathbb{P}_{X_n}(\{-1\}) = 0.5$ and the random walk S_n be defined by:

$$S_0 = 0, \quad S_{n+1} = S_n + X_{n+1}, \quad n \in \mathbb{N}.$$

Since we have shown that $S_n^2 - n$ is a martingale, it follows that $\langle S \rangle_n = n$ for every $n \in \mathbb{N}$.

Here is another useful exercise. Throughout, we will assume that X and Y are square integrable martingales null at 0. Define the *predictable covariation* of X and Y as:

$$\langle X, Y \rangle := \frac{1}{4}(\langle X + Y \rangle - \langle X - Y \rangle).$$

1 Use the Cauchy-Schwartz inequality to prove that the set of square integrable martingales is a vector space.

2 Show that $XY - \langle X, Y \rangle$ is a martingale.

Hint: For $x, y \in \mathbb{R}$, $xy = \frac{1}{4}((x + y)^2 - (x - y)^2)$.

3 Show that XY is a martingale if and only if $\langle X, Y \rangle = 0$.

Let X and Y be adapted integrable processes. (Also assume that $X_0 = Y_0 = 0$ for simplicity.) We define the *quadratic covariation* of X and Y as

$$[X,Y]_n := \sum_{k=1}^{n}(X_k - X_{k-1})(Y_k - Y_{k-1}),$$

and the *quadratic variation* of X as

$$[X]_n := [X,X]_n = \sum_{k=1}^{n}(X_k - X_{k-1})^2.$$

1 Assume that X and Y are square integrable martingales both null at 0. Show that $X^2 - [X]$ is a martingale. Hence conclude that $[X] - \langle X\rangle$ is also a martingale.

2 Let φ be predictable. Show that

$$[\varphi \bullet X] = \varphi^2 \bullet [X].$$

3 [Product rule] For any process Z, define $\Delta Z_n := Z_n - Z_{n-1}$. Show that

$$\Delta(XY)_n = X_{n-1}\Delta Y_n + Y_{n-1}\Delta X_n + \Delta[X,Y]_n.$$

Hence, deduce the following "integration by parts formula":

$$XY = (X_- \bullet Y) + (Y_- \bullet X) + [X,Y],$$

where $X_{-_0} = X_0$, $Y_{-_0} = Y_0$, $X_{-_n} = X_{n-1}$ and $Y_{-_n} = Y_{n-1}$ for $n \geq 1$.

3.2.8 Transcript: Working Through the Doob Decomposition

Hi, in this video we look at the Doob decomposition.

Let X be an adapted stochastic process, where $X = \{X_n: n = 0, 1, 2, 3, \dots\}$. We are going to assume that X is adapted to a specific filtration and that it is also integrable, meaning that the expected value of the absolute value of X_n is finite for every n. Written in full:

$$E(|X_n|) < \infty \forall n.$$

The Doob decomposition theorem says that if those conditions are satisfied, then we can decompose X into three parts:

$$X = X_0 + M + A.$$

The first one is X_0, which is the starting point of X, plus M, where M is a martingale starting at 0, plus A where A is a predictable stochastic process. Both M and A start at 0. Therefore, M_0 and A_0 are both equal to 0. This decomposition is also unique, and we call it the Doob decomposition of the stochastic process X.

If X is a submartingale, then A is increasing. We should actually say "if and only if" – in other words, if A is increasing then X is also a submartingale.

If X is a supermartingale, then A is decreasing.

We can actually calculate A in all cases but it turns out that the formula for calculating A is given by the following recursive relationship:

$$A_0 = 0, \qquad A_{n+1} = A_n + E(X_{n+1} - X_n \mid \mathcal{F}_n).$$

So, what A_n calculates is just the expected jumps, and accumulates them over time. It makes sense, then, when X is a submartingale and A is increasing this will be positive. This is similar to the case of a supermartingale.

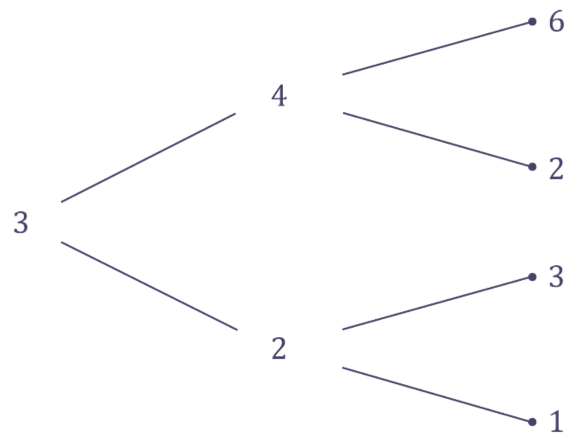
Now, if X is a martingale itself, then X^2 is a submartingale by Jensen's inequality, provided that X is square integrable. We can calculate its Doob decomposition if it is square integrable as X_0^2 plus a martingale, M, plus an increasing predictable process, A, that starts at 0. Written in full:

$$X^2 = X_0^2 + M + A.$$

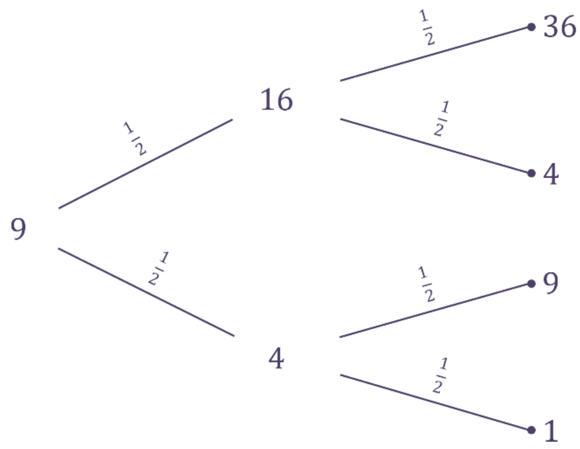
The process, A, is called a predictable quadratic variation of X and we denote it as follows: $< X >$. Therefore $< X >:= A$, which is the increasing predictable process that makes $X^2 – A$ a martingale.

Let's look at an example.

Consider the following stochastic process, which we will show using a tree diagram:



Next, let's calculate the stochastic process X^2, which is a submartingale, using another tree diagram:



What we want to calculate is the quadratic variation of X, which means that we will have to find the Doob decomposition of the stochastic process, X^2. First, let's find what $< X >$ is.

$$< X >_0 = 0$$
$$< X >_1 = E(X_1^2 - X_0^2 | \mathcal{F}_0).$$

Importantly, we are going to be using the natural filtration of X here.

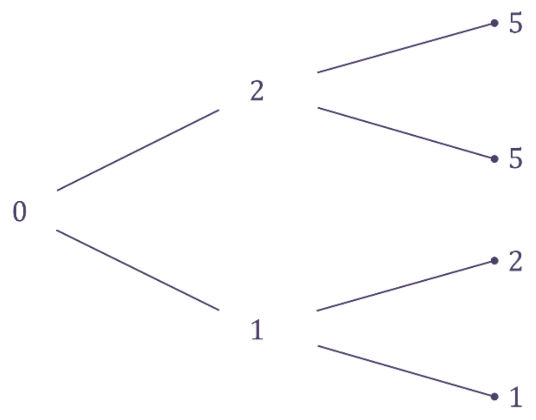
In this case, $\mathcal{F}_0$ is the trivial σ-algebra because X is constant at 0. This means that $< X >_1 = E(X_1^2 - X_0^2 \mid \mathcal{F}_0)$ is simply equal to $E(X_1^2 - X_0^2)$. Therefore, this expectation will be $\frac{1}{2} \times (16 - 9) + \frac{1}{2} \times (4 - 9)$. This simplifies to $\frac{1}{2}(7 - 5)$, which equals 1. This is X_1, which makes sense because the quadratic variation at time 1 should be measurable with respect to the information at time 0. Written in full:

$$< X >_1 = E(X_1^2 - X_0^2 | \mathcal{F}_0)$$
$$= E(X_1^2 - X_0^2 = \frac{1}{2} \times (16 - 9) + \frac{1}{2} \times (4 - 9)$$
$$= \frac{1}{2}(7 - 5) = 1.$$

Next, let's find the quadratic variation at time 2. This should be measurable with respect to the information at time 1 and, therefore, we should say what its value is here at the branch that leads to 16 and the first branch that leads to 9. Therefore, $< X >_2$ is equal to the quadratic variation X at time 1 plus one of two things: so, looking at this branch here, it will be the conditional expectation of the differences, which is $36 - 16 \times \frac{1}{2} + 4 - 16 \times \frac{1}{2}$. Then, on this branch here, it will be $9 - 4 \times \frac{1}{2} + 1 - 4 \times \frac{1}{2}$. We can simplify this to get 5 and 2 as the quadratic variation at that point. Written in full:

$$
\begin{aligned}
< X >_2 =< X >_1 + &\begin{cases} (36-16) \times \frac{1}{2} + (4-16) \times \frac{1}{2} \\ (9-4) \times \frac{1}{2} + (1-4) \times \frac{1}{2} \end{cases} \\
= 1 + &\begin{cases} \frac{1}{2}(20-12) \\ \frac{1}{2}(5-3) \end{cases} \\
= &\begin{cases} 5 \\ 2. \end{cases}
\end{aligned}
$$

We can draw a small tree diagram for the quadratic variation that looks like this:



We have come to the end of Discrete Martingales. In the next module, we are going to talk about trading in discrete time.

3.2.9 Notes: Problem Set

Problem 1

Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = 2^{\Omega}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\{a, b\}, \{c, d\}\})$, $\mathcal{F}_2 = \mathcal{F}$, and $X = \{X_0, X_1, X_2\}$ be a martingale defined as:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
a	$\frac{1}{6}$	κ	24	β_1
b	$\frac{1}{4}$	κ	24	β_2
c	$\frac{1}{4}$	κ	12	β_3
d	$\frac{1}{3}$	κ	12	β_4

Then what is κ?

Solution:

We know the definition of martingale from the theory.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered space and $X = \{X_n : n \in \mathbb{N}\}$ be a stochastic process. Then X is a martingale if and only if X is adapted to $\mathbb{F}$, $X_n \in \mathcal{L}^1$ for each $n \in \mathbb{N}$ and

$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n \quad \mathbb{P} - \text{ a.s..}$$

Moreover, we also know that the condition above is equivalent (more practical) to the following proposition.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered space and $X = \{X_t : t \in \mathbb{I}\}$ be a stochastic process. Then, if X is a martingale, then $\mathbb{E}(X_t) = \mathbb{E}(X_s) = \mathbb{E}(X_0)$ for every $s, t \in \mathbb{I}$. That is, X has a constant mean.

Following the above proposition, we can solve the problem as follows: First, we compute the $\mathbb{E}(X_1)$ which does not depend on κ:

$$\mathbb{E}(X_1) = \frac{1}{6} * 24 + \frac{1}{4} * 24 + \frac{1}{4} * 12 + \frac{1}{3} * 12 \ = \ 17.$$

Now we have to find κ, such that $\mathbb{E}(X_0) = 17$. It is easy to see that $\kappa = 17$ solving the following equation:

$$\frac{1}{6} * \kappa + \frac{1}{4} * \kappa + \frac{1}{4} * \kappa + \frac{1}{3} * \kappa = 17.$$

Thus, the solution is $\kappa = 17$.

Problem 2

Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = \ 2^{\Omega}$, $\mathbb{P} = 0.25(\delta_a + \delta_b + \delta_c + \delta_d)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\{a, b\}, \{c, d\}\})$, $\mathcal{F}_2 = \mathcal{F}$, $X = \{X_0, X_1, X_2\}$, and τ be defined by:

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$\tau(\omega)$
a	$\frac{1}{4}$	10	12	15	2
b	$\frac{1}{4}$	10	12	11	2
c	$\frac{1}{4}$	10	5	7	1
d	$\frac{1}{4}$	10	5	3	1

What is the law of X_τ?

Solution:

First, let's summarize the theory (from the lecture notes) that we are going to need to solve this problem. We know that if τ is a stopping time and $X = \{X_t : t \in \mathbb{I}\}$ is a stochastic process, we can define the *stopped process* $X^\tau = \{X_t^\tau : t \in \mathbb{I}\}$ as $X_t^\tau := X_{\tau\wedge t}$; i.e.,

$$X_t^\tau(\omega) := \begin{cases} X_t(\omega) & t < \tau(\omega) \\ X_{\tau(\omega)}(\omega) & t \geq \tau(\omega) \end{cases} \quad \omega \epsilon \Omega.$$

We also define the random variable X_τ by:

$$X_\tau(\omega) := \begin{cases} X_{\tau(\omega)}(\omega) & \tau(\omega) < \infty \\ 0 & \tau(\omega) = \infty. \end{cases}$$

Thus, the feasible values of X_τ are: $X_2(a) = 15$, $X_2(b) = 11$, and $X_1(c) = X_1(d) = 5$. As we know the law of $X(\omega)$:

$$\mathbb{P} = 0.25(\delta_a + \delta_b + \delta_c + \delta_d).$$

We can easily find the law for X_τ,

$$\mathbb{P} = 0.5\delta_5 + 0.25\delta_{11} + 0.25\delta_{15}.$$

Problem 3

Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = 2^\Omega$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\{a, b\}, \{c, d\}\})$, $\mathcal{F}_2 = \mathcal{F}$, and $\varphi = \{\varphi_1, \varphi_2\}$ be a predictable process defined by

ω	$\varphi_1(\omega)$	$\varphi_2(\omega)$
a	2	3
b	2	α
c	2	1
d	2	β

What is the value of $\alpha + \beta$?

Solution:

From the lecture notes we get:

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. A sequence of random variables $\varphi = \{\varphi_n : n = 1,2,3, \dots\}$ is called previsible or predictable if $\varphi_n \in m\mathcal{F}_{n-1}$ for each $n \in \mathbb{N}^+$.

In our case, as we have α and β in $\varphi_2(\omega)$, we have to find the values of α and β such that $\varphi_2(\omega) \in m\mathcal{F}_1$. As we know (from the problem statement) that $\mathcal{F}_1 = \sigma(\{\{a,b\},\{c,d\}\})$, the value of $\varphi_2(\omega)$ must be the constant for the paths $\{a,b\}$ and $\{c,d\}$, which give us the value for α and β. Thus, the solution is $\alpha = 3$ and $\beta = 1$, and the sum is equal to 4.

Problem 4

Let $\Omega = \{a,b,c,d\}, \mathcal{F} = 2^{\Omega}, \mathbb{P} = 0.25(\delta_a + \delta_b + \delta_c + \delta_d), \mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = \sigma(\{\{a,b\},\{c,d\}\})$, $\mathcal{F}_2 = \mathcal{F}$ and $X = \{X_0, X_1, X_2\}$ and τ be a stopping time (with respect to $\mathbb{F}^X$) defined by

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$\tau(\omega)$
a	$\frac{1}{4}$	10	12	15	1
b	$\frac{1}{4}$	10	12	11	β
c	$\frac{1}{4}$	10	5	7	1
d	$\frac{1}{4}$	10	5	3	α

What are the possible values of α and β?

Solution:

This problem is quite easy. From the table we can infer the stopping time conditions. First, from path a we know that (for sure),

$$\tau(\omega) := \inf\{n \in \{0,1,2\} : X_n(\omega) \geq 12\}.$$

I wrote "for sure" above because with the available data we do not know if the condition is for $X_n(\omega) \geq 12$ or if it also can hold for $X_n(\omega) \geq 11$. We also know that the condition cannot be $X_n(\omega) \geq 10$. (In this case, $\tau(a)$ should be equal to 0 instead of 1.)

From path c we also infer the second condition for $\tau(\omega)$:

$$\tau(\omega) := \inf\{n \in \{0,1,2\} : X_n(\omega) \leq 5\}.$$

Finally, the values of α and β must be given by $\alpha = 1$ and $\beta = 1$.

Problem 5

Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = 2^{\Omega}$, $\mathbb{P} = 0.25(\delta_a + \delta_b + \delta_c + \delta_d)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(\{\{a,b\},\{c,d\}\})$, $\mathcal{F}_2 = \mathcal{F}$ and $X = \{X_0, X_1, X_2\}$ defined by

ω	$\mathbb{P}(\{\omega\})$	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$
a	$\frac{1}{4}$	100	144	196
b	$\frac{1}{4}$	100	144	100
c	$\frac{1}{4}$	100	64	64
d	$\frac{1}{4}$	100	64	36

Let (A, M) be the Doob decomposition of X (i.e. $X = X_0 + M + A$). Then what is $A_2(b)$ equal to?

Solution:

From the lecture notes, we know that

$$A_n = \sum_{k=1}^{n} \mathbb{E}(X_k - X_{k-1} | \mathcal{F}_{k-1}) \text{ and } M_n = \sum_{k=1}^{n} (X_k - \mathbb{E}(X_k | \mathcal{F}_{k-1})).$$

(Note that A does not depend on the path for a and b, so $A(a) = A(b)$.) Thus, the problem consists of computing:

$$A_2(b) = \sum_{k=1}^{2} \mathbb{E}(X_k - X_{k-1}|\mathcal{F}_{k-1} = E(X_1 - X_0|\mathcal{F}_0) + E(X_2 - X_1|\mathcal{F}_1).$$

Let's compute the above expected values step by step (note that we have to use the conditional probability):

$$E(X_1 - X_0|\mathcal{F}_0) = 0.5 * (144 - 100) + 0.5 * (64 - 100) = 4,$$

and

$$E(X_2 - X_1|\mathcal{F}_1) = 0.5 * (196 - 144) + 0.5 * (100 - 144) = 4.$$

Finally, the result must be:

$$A_2(b) = \sum_{k=1}^{2} \mathbb{E}(X_k - X_{k-1}|\mathcal{F}_{k-1}) = E(X_1 - X_0|\mathcal{F}_0) + E(X_2 - X_1|\mathcal{F}_1) = 4 + 4 = 8.$$

3.3 Additional Resources

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