WORLDQUANT UNIVERSITY

Continuous-time Stochastic Processes: Module 3

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 3 of Continuous-time Stochastic Processes, entitled Stochastic Calculus II: Semimartingales. It consists of three video lecture transcripts and four sets of supplementary notes.

2. Course Context

Continuous-time Stochastic Processes is the fourth course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The aim of the course is to introduce derivative pricing when trading happens in continuous time. It begins by focusing on the stochastic calculus of Brownian motion and its generalization to continuous semimartingales. These ideas are then applied to continuous trading and the pricing of financial derivatives.

2.1 Course-level Learning Outcomes

Upon completion of the Continuous-time Stochastic Processes course, you will be able to:

1. Define and identify Brownian motion processes in multiple dimensions.
2. Solve stochastic differential equations.
3. Apply Ito's Lemma for continuous semimartingales.
4. Apply Girsanov's theorem to construct equivalent local martingale measures.
5. Price and hedge derivatives in various asset price models.
6. Derive the Black-Scholes partial differential equation.
7. Construct asset prices models based on Levy processes.
8. Price interest rate derivatives.

2.2 Module Breakdown

The Continuous-time Stochastic Processes course consists of the following one-week modules:

1. Brownian Motion and Continuous-time Martingales
2. Stochastic Calculus I: Ito Process
3. Stochastic Calculus II: Semimartingales
4. Continuous Trading
5. The Black-Scholes Model
6. An Introduction to Levy Processes
7. An Introduction to Interest Rate Modeling

3. Module 3:

Stochastic Calculus II: Semimartingales

In this module, we extend the stochastic integral of a process when the integrator is a martingale, and we present a more general stochastic process (i.e. more general than the Ito process): the semimartingale. Note that the Ito process presented in the last module is also a semimartingale but semimartingales' stochastic processes go beyond Ito processes. Fortunately, most of the Ito process properties and operations can be easily extended to semimartingales. Thus, as you will see in the problem set, most of the practical applications are similar to the examples provided in Module 2.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Define a semimartingale and give examples of semimartingales.
2. Define the stochastic integral with respect to a square integrable martingale.
3. Define the stochastic integral with respect to a continuous semimartingale.
4. Apply the localization procedure.
5. State and apply Ito's formula for continuous semimartingales.

3.2.1 Notes: Square Integrable Martingales

The first step in the extension of the stochastic integral is to replace Brownian motion W as an integrator with an arbitrary square integrable martingale M. This step is the easiest, and the presentation is very similar to the Brownian case.

We will continue to work on a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \mathcal{F}_t : 0 \leq t \leq T$ satisfies the usual conditions.

Let M be a square integrable martingale null at 0. The last condition $(M_0 = 0)$ is not really a restriction since we will always define $\int_0^t \varphi_s \, dM_s$ to be $\int_0^t \varphi_s \, d(M_s - M_0)$ if $M_0 \neq 0$.

We say that φ is a *simple* process if we can write it as

$$\varphi_t = \sum_{k=1}^{n} H_k \, I_{(\tau_{k-1}, \tau_k]}(t),$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n$ are stopping times and each H_k is bounded and $\mathcal{F}_{\tau_{k-1}}$-measurable for each k. We denote by $\mathbb{S}$ the class of all simple processes.

For a simple process φ, we define the *stochastic integral* of φ with respect to M as the stochastic process $(\varphi \bullet M) = \{(\varphi \bullet M)_t : 0 \leq t \leq T\}$ defined by

$$(\varphi \bullet M)_t := \sum_{k=1}^{n} H_k \left(M_{t \wedge \tau_k} - M_{t \wedge \tau_{k-1}} \right).$$

Just like in the Brownian case, we can extend this definition to a larger class of processes. Define

$$L^2\,(M) \coloneqq \{\varphi : \varphi \text{ is predictable and } \|\varphi\|_M < \infty\},$$

where

$$\|\varphi\|_M \coloneqq \left(\mathbb{E}\left(\int_0^T \varphi_s^2\ d\langle M\rangle_s\right)\right)^{\frac{1}{2}}.$$

It can be shown again that each $\mathbb{S} \subseteq L^2(M)$ and for each $\varphi \in L^2(M)$, there exists a sequence (φ^n) in $\mathbb{S}$ such that $||\varphi^n - \varphi||_M \to 0$ as $n \to \infty$. We then define $(\varphi \bullet M)$ as an appropriate limit of the sequence $(\varphi^n \bullet M)$, and this definition is independent of the chosen sequence. For more details, see *On Square Integrable Martingales* by Kunita and Watanabe (1967).

We will sometimes denote the stochastic integral by

$$\int_0^t \varphi_s\ dM_s.$$

The stochastic integral satisfies the following (for M a square integrable martingale and $\varphi \in L^2(M)$):

1. $\varphi \mapsto (\varphi \bullet M)$ is linear.
2. $(\varphi \bullet M)$ is a square integrable martingale with

$$\langle(\varphi \bullet M)\rangle_t = (\varphi^2 \bullet \langle M\rangle)_t,$$

which can also be written as

$$\left\langle \int_0^{\cdot} \varphi_s \, dM_s \right\rangle_t = \int_0^t \varphi_s^2 \; d\langle M\rangle_s$$

Here is a very neat characterization of the integral:

Theorem 1.1 (Kunita-Watanabe). *Let M be a square integrable martingale and $\varphi \in L^2(M)$. The stochastic integrable $\int_0^{\cdot} \varphi_s \; dM_s$ is the unique square integrable martingale such that*

$$\left\langle \int_0^{\cdot} \varphi_s dM_s, N \right\rangle_t = \int_0^t \varphi_s \, d\langle M, N\rangle_s$$

for every square integrable martingale N.

We will see how to extend the definition to integrands beyond $L^2(M)$ in the next section. Unfortunately, we lose some of the above properties of the integral for these general integrands.

We end this section by mentioning that the same construction can be carried out verbatim using the quadratic variation $[M]$ in place of $\langle M\rangle$.

3.2.2 Transcript: Square Integrable Martingales

Hi, in this video we extend the theory of stochastic integration to the case where the integrator is a square integrable martingale.

Let M be a square integrable martingale. We want to define the stochastic integral of an appropriate stochastic process, φ with respect to M, which is similiar to what we did when M was a Brownian motion:

$$\int_0^t \varphi_s dM_s.$$

We are going to start with what is called a simple process, which is a stochastic process that can be written like this:

$$\varphi_{\mathrm{t}} = \sum_{i=1}^{n} H_i \ I_{(\tau\, i-1,\, \tau\, i]}(t),$$

where, the $\tau_i s$ are stopping times (i.e. when $0 = \tau_0 < \tau_1 < \cdots \tau_n = T$).

φ is a simple process like this:

$$\int_0^t \varphi_s dM_s = \sum_{i=1}^{n} H_i(M_t \wedge \tau_i - M_t \wedge \tau_{i-1}).$$

What we see above is the definition of the stochastic integral, which is similar to the case of a Brownian motion. This means that it also satisfies similar properties to the case of a Brownian motion.

1. $E\left(\int_0^t \varphi_s dM_s\right) = 0.$
2. This stochastic process, $\{\int_0^t \varphi_s dM_s : 0 \leq t \leq T\}$ is a square integrable martingale with the following quadratic variation:

$$\left\langle \int_0^t \varphi_s \, dMs \right\rangle_t = \int_0^t \varphi_s^2 \, d\langle M \rangle_s.$$

3. $E\left(\left(\int_0^t \varphi_s dM_s\right)^2\right)$ is equal to the expected value of the integral of the square of φ_s with respect to the quadratic variation of M at s. Written in full:

$$E\left(\left(\int_0^t \varphi_s dM_s\right)^2\right) = E\left(\int_0^t \varphi_s^2 d\langle M \rangle_s\right).$$

   This is similiar to what we had in Brownian motion, because the quadratic variation of Brownian motion at time s is equal to s itself.

Once we have defined this for simple processes, we again do an extension.

We begin by defining $L^2(M)$ to be a set of predictable processes – meaning that we now restrict this to predictable processes. This restriction was unnecessary in the Brownian case as we just defined it for progressive processes, which are predictable with respect to the Brownian filtration. Here, however, we define it for predictable processes, φ, such that φ is predictable and the norm of φ is finite, where the norm of φ is again defined in a similar way to the Brownian case:

$$\|\varphi\|_M = \left(E\int_0^T \varphi_s^2 d\langle M\rangle_s\right)^{\frac{1}{2}}.$$

Once we have the above, we can show that for every element of $L^2(M)$, there exists a sequence of simple processes that we are going to denote by (φ^n), such that φ^n converges to φ in the following sense:

$$\|\varphi^n - \varphi\|_M \to 0 \; as \; n \to \infty.$$

In that case, we define the stochastic integral of φ to be equal to the limit as n tends to infinity of the stochastic integrals of φ^n, where this limit is taken in L^2. Written in full:

$$\int_0^t \varphi dM_s := \lim_{n\to\infty}\int_0^t \varphi^n dM_s.$$

This limit is independent of the sequence φ^n that we have chosen and that completely defines the stochastic integral with respect to a square integrable martingale.

Now that we have defined a stochastic integral with respect to a square integrable martingale, in the next video, we are going to look at localization.

3.2.3 Notes: Localization

Let $\mathcal{E}$ be a class of stochastic processes (e.g. $\mathcal{E}$ could be the class of square integrable martingales). We say that a process X is locally in $\mathcal{E}$ if there exists an increasing sequence of stopping times (τ_n) such that $\tau_n \to \infty$ as $n \to \infty$, and for each n, the stopped process X^{τ_n} belongs to $\mathcal{E}$. The sequence (τ_n) is called a *localizing sequence* and we will denote by $\mathcal{E}_{\ell o c}$ the class of all processes that are locally in $\mathcal{E}$.

An important example is that of *local martingales*. According to the definition above, a process M is a local martingale if there exists a localizing sequence (τ_n) such that M^{τ_n} is a martingale for each n. Since every stopped martingale is again a martingale, it follows that every martingale is a local martingale. The converse of this result is not true.

Let M be a local martingale and (τ_n) be a localizing sequence, then the sequence $(\tau_n \wedge n)$is also a localizing sequence. Indeed, since $M_t^{\tau_n \wedge n} = M_{n\wedge t}^{\tau_n}$, we have (for $s < t$),

$$\mathbb{E}(M_t^{\tau_n \wedge n} | \mathcal{F}_s) = \mathbb{E}(M_{n\wedge t}^{\tau_n} | \mathcal{F}_s) = M_{n\wedge s}^{\tau_n} = M_s^{\tau_n \wedge n}$$

by the martingale property of $M_{n\wedge \cdot}^{\tau_n}$ for each fixed n (since τ_n is a localizing sequence).

Furthermore, $(\tau_n \wedge n)$ is also increasing and diverges to ∞ as $n \to \infty$. Thus, we can also choose $\tau_n' := \tau_n \wedge n$ as a localizing sequence. This choice has the advantage that $M^{\tau_n'}$ is a *uniformly integrable* martingale. Hence, all local martingales localize to UI martingales.

If M is a continuous local martingale, we can choose a localizing sequence so that it localizes to a bounded martingale. Indeed, if (τ_n) is a localizing sequence for M, then we define (for each n) $\tau_n' \coloneqq \inf\{t \geq 0: |M_t| = n\}$. Then the sequence $\tau_n'' \coloneqq \tau_n \wedge \tau_n'$ is the required localizing sequence.

Recall from the definition of the stochastic integral with respect to a Brownian motion that if $\varphi \in L^2(W)$,then $(\varphi \bullet W)$ is a square integrable martingale. We then extended the definition of the integral to progressive processes φ such that

$$\int_0^t \varphi_s^2 \, ds < \infty \; \forall t \;\; a.s.,$$

but mentioned that the stochastic integral for such processes is no longer a martingale in general. We now show that it is always a local martingale though.

Let W be a Brownian motion and φ be a progressive process such that

$$\int_0^t \varphi_s^2 \, ds < \infty \; \forall t \text{ a.s.}.$$

Define the following sequence of stopping times for each n:

$$\tau_n \coloneqq \inf\{t \geq 0: \int_0^t \varphi_s^2 \, ds > n\}.$$

Then the stopped process φ^{τ_n} belongs to $L^2(W)$, hence the stochastic integral is a local martingale.

There are several examples that show that the stochastic integral can be a strict local martingale – i.e. a local martingale that is not a martingale. One is given by the solution to the following SDE:

$$dX_t = X_t^{\alpha} dW_t, \qquad X_0 = 1$$

for $\alpha > 1$.

The following gives a necessary and sufficient condition for a local martingale to be a martingale:

Theorem 2.1. *A local martingale is a martingale if and only if it is of class DL.*

Now suppose M is a locally square integrable local martingale. That is, there exists a localizing sequence (τ_n) such that M^{τ_n} is a square integrable martingale. Then for each n, the quadratic variation $\langle M^{\tau_n} \rangle$ is well-defined and unique. Now define the process ⟨M⟩ as

$$\langle M \rangle_t := \langle M^{\tau_n} \rangle_t \quad t < \tau_n.$$

This process is unambiguously defined since $\langle M^{\tau_{n+1}} \rangle^{\tau_n} = \langle M^{\tau_n} \rangle$ for every n. We will also call it the (predictable) quadratic variation of the locally square integrable local martingale M. The covariation process and orthogonality are defined similarly.

A typical example of a local martingale that is locally square integrable is a continuous local martingale. So, for the rest of the section we focus on continuous local martingales.

Let M be a continuous local martingale. Define the following set:

$$L^2_{\{loc\}}(M) := \{\varphi \text{ progressive: } \int_0^t \varphi_s^2 \, d\langle M\rangle_s < \infty \ \forall t \geq 0\}.$$

We have the following result:

Theorem 2.2. *Let M be a continuous local martingale and $\varphi \in L^2_{loc}(M)$. Then there exists a unique continuous local martingale $(\varphi \bullet M)$ such that*

$$\langle (\varphi \bullet M), N\rangle = (\varphi \bullet \langle M, N\rangle)$$

for any continuous local martingale N. The process $(\varphi \bullet M)$ is called stochastic integral of φ with respect to M and we will sometimes write it as

$$(\varphi \bullet M)_t = \int_0^t \varphi_s \, dM_s.$$

3.2.4 Transcript: Localization

Hi, in this video we introduce the notion of localization and we use it to extend the theory of stochastic integration.

Let $\mathcal{E}$ be a class of stochastic processes, which is essentially a set of stochastic processes. For an example, we can take $\mathcal{E}$ to be a set of martingales on the same probability space:

$$\mathcal{E} = M.$$

We say that the stochastic process X is locally in $\mathcal{E}$, which we write as $X \in \mathcal{E}_{loc}$, if it satisfies the following conditions:

1. An increasing sequence of stopping times must exist, which we will denote by τ_n, such that $\tau_n \to \infty$ as $n \to \infty$.
2. The stopped process, $X^{\tau_n} \in \mathcal{E} \forall n$. This applies even though X itself need not belong to $\mathcal{E}$.

As a famous example, if we take $\mathcal{E}$ to be a set of all martingales, then $\mathcal{E}_{loc}$, which is the set of all stochastic processes that are locally in $\mathcal{E}$, in this case will simply coincide with the set of all stochastic processes that are local martingales:

$$\mathcal{E} = M, \;\; \mathcal{E}_{loc} = M_{loc}.$$

A popular example of a local martingale that we will see many times in this course, is when W is a Brownian motion and φ is progressive, with the additional condition that

$\int_0^t \varphi_s^2 ds < \infty \forall t$ and this is true almost surely. So, if this condition is satisfied, then we know that the stochastic integral $\int_0^t \varphi_s dW_s$ of the stochastic process φ_s is well-defined.

However, we mentioned in the last module that this is not generally a martingale if φ does not belong to $L^2(W)$. If $\varphi \notin L^2(W)$, then the stochastic integral $\int_0^t \varphi_s dW_s'$ need not be a martingale. This is something important to remember.

However, we will show now that $\int_0^t \varphi_s \, dW_s$ is always a local martingale.

As a sequence, τ_n, which is called a localizing sequence, can be taken as being the first time the infimum, which is $\int_0^t \varphi_s^2 ds$, is greater than or equal to n. Written in full:

$$\tau_n := inf\{t \geq 0: \int_0^t \varphi_s^2 ds \geq n\}.$$

Now, if we look at the stopped process $\varphi_n^{\tau_n}$, it is always bounded above by n and therefore belongs to $L^2(W)$. This means that $\varphi_n^{\tau_n}$ will be a square integrable martingale for all n. So, the stochastic integral $\int_0^t \varphi_s^{\tau_n} dW_s$ will be a square integrable martingale.

$\int_0^t \varphi_s^{\tau_n} dW_s$ is also equal to the stopped process $\left(\int_0^t \varphi_s \, \mathrm{dW_s}\right)^{\tau_n}$. We can check that this is equivalent to stopping the integral process itself, and therefore, the integral process is a local martingale because when we stop it we get a martingale.

Finally, this allows us to extend the theory of stochastic integration to the case where we have $\int_0^t \varphi_s \, dM >$ and this here no longer belongs to L^2 of M, and the martingale M is not necessarily square integrable – in fact, we can extend it even further to the case where it is just a local martingale via localization. Further details are covered in the notes.

Now that we've covered localization, in the next video, we are going to look at semimartingales.

3.2.5 Notes: Semimartingales

In this section, we finally extend the stochastic integral to semimartingales. Luckily, all the hard work has been covered in previous sections, so this last step will be relatively straightforward.

A stochastic process X is called a *semimartingale* if

$$X = X_0 + M + A,$$

where M is a local martingale and A is a finite variation process, both null at zero. Note that this decomposition of X is in general not unique, since there are local martingales of finite variation. It is unique, however, when A is also predictable, and in that case, X is called a *special semimartingale*.

We first assume that X is a *continuous* semimartingale with decomposition $X = X_0 + M + A$ where M and A are both continuous. We have defined the stochastic integral with respect to M in the previous section. We have also defined the stochastic integral with respect to A earlier as a Stieltjes integral. We can then define the stochastic integral of a predictable stochastic process φ with respect to X as

$$(\varphi \bullet X) := (\varphi \bullet M) + (\varphi \bullet A),$$

provided both integrals on the right-hand side exist. If both integrals exist, we will say that φ is X-integrable and we will denote by L(X) the set of all X-integrable predictable processes φ. It is a known result that the space of semimartingales is closed under taking stochastic integrals.

A stochastic process φ is *locally bounded* if there exists an increasing sequence of stopping times (τ_n) such that $\tau_n \to \infty$ and φ^{τ_n} is uniformly bounded. If φ is a locally bounded predictable process, then $\varphi \in L(X)$.

We now move on to Ito's Lemma for continuous semimartingales. First, we extend the definition of the predictable quadratic variation $\langle\rangle$ to continuous semimartingales. If $X = X_0 + M + A$ and $Y = Y_0 + N + B$ are continuous semimartingales, we define $\langle X, Y\rangle := \langle M, N\rangle$ and $\langle X\rangle := \langle X, X\rangle = \langle M\rangle$. Note that for *continuous* semimartingales X and Y, $\langle X, Y\rangle = [X, Y]$, where $[X, Y]$ is the quadratic covariation of X and Y. We also have $\langle \varphi \bullet X, \psi \bullet Y\rangle = \varphi\psi \bullet \langle X, Y\rangle$ for $\varphi \in L(X)$ and $\psi \in L(Y)$.

We can now state Ito's formula.

Theorem 3.1 (Ito's formula). *Let* $X = \left(X^1, \ldots, X^d\right)$ *be a* d*-dimensional continuous semimartingale and* $f: \mathbb{R}^d \to \mathbb{R}$ *be a twice continuously differentiable function. Then* $Y := f(X)$ *is a continuous semimartingale and*

$$Y_t = Y_0 + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}\, dX_s^i + \int_0^t \frac{1}{2} \sum_{j=1}^d \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}\, d\left\langle X^i, X^j\right\rangle_s.$$

In differential notation, this can be written as

$$dY_t = \sum_{i=1}^d \frac{\partial f}{\partial x_i}\, dX_t^i + \frac{1}{2} \sum_{j=1}^d \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}\, d\left\langle X^i, X^j\right\rangle_t.$$

Let X be a continuous semimartingale. We want to define the *stochastic exponential* of X, which is a continuous semimartingale Y satisfying

$$dY_t = Y_t\, dX_t, \quad Y_0 = 1.$$

That is,

$$Y_t = 1 + \int_0^t Y_s \, dX_s.$$

We apply Ito's formula to $\ln Y_t$ to get

$$d \ln Y_t = \frac{1}{Y_t}\, dY_t - \frac{1}{2}\frac{1}{Y_t^2}\, d\langle Y\rangle_t = dX_t - \frac{1}{2}\, d\langle X\rangle_t.$$

Hence

$$\ln Y_t = X_t - \frac{1}{2}\langle X\rangle_t,$$

which implies that

$$Y_t = e^{X_t - \frac{1}{2}\langle X\rangle_t}.$$

We will denote Y by $\mathcal{E}(X)$ and call it the stochastic exponential of X.

If X is a continuous local martingale, then $\mathcal{E}(X)$ is also a continuous local martingale.

Finally, we mention that the stochastic integral can be extended to when X is a semimartingale that is not continuous. This integral is well-defined for locally bounded predictable processes φ and satisfies all the reasonable properties that were satisfied by the other stochastic integrals (linearity, etc). However, the details of the construction are beyond the scope of this course. Interested readers are referred to the reference textbooks.

There is also an extension for vector semimartingales. Let $X = (X^1, \ldots, X^d)$ be a d-dimensional cadlag semimartingale and $\varphi = (\varphi^1, \ldots, \varphi^d)$ be a d-dimensional predictable process. Using the tools of this chapter, we can define the so-called *component-wise stochastic integral* of φ with respect to X as

$$(\mathrm{comp})\varphi \bullet X := \sum_{i=1}^{d} \varphi^i \bullet X^i,$$

which exists if and only if all the component integrals exist. It turns out that this definition is not general enough for applications to mathematical finance. Indeed, let $\tilde{X}$ be a one-dimensional semimartingale and $\tilde{\varphi}$ be a one-dimensional predictable process such that $\tilde{\varphi} \notin L(\tilde{X})$. Define the 2-dimensional processes X and φ as $X := (\tilde{X}, \tilde{X})$ and $\varphi := (\tilde{\varphi}, -\tilde{\varphi})$. Then the component-wise stochastic integral of φ with respect to X does not exist since both component integrals do not exist. On the other hand, however, we expect the integral $\varphi \bullet X$ to be equal to 0 due to the structure of the processes. This is one way in which the component-wise stochastic integral is not sufficient.

The right generalization for vector-valued processes is achieved by defining the so-called *vector stochastic integral* of φ with respect to X. This is a generalization of the component-wise integral since when the latter exists the former will also exist and the two integrals are the same. The details for its construction, together with some applications to mathematical finance can be found in *Vector Stochastic Integrals and The Fundamental Theorems of Asset Pricing* by A.N. Shiryaev and A.S. Cherny.

3.2.6 Transcript: Semimartingales

Hi, in this video we look at semimartingales.

A stochastic process, X, is called a continuous semimartingale if X can be written in the following form:

$$X = X_0 + M + A,$$

where X_0 is of course the starting point of the stochastic process. We are going to insist that M is continuous, but it is also a local martingale. A is a continuous finite variation process, meaning that the sample paths of A have finite variation.

Now, if X is a continuous semimartingale with the above decomposition, we will define the stochastic integral of the stochastic process φ with respect to X in two steps. The first integral will be with respect to M, which is a local martingale as we defined in the previous section, plus the second integral with respect to A, which is the Stieltjes integral. To be clear, the first integral is the stochastic integral, while the second is the ordinary Stieltjes integral. Written in full:

$$\int_0^t \varphi_s dX_s := \int_0^t \varphi_s dM_s + \int_0^t \varphi_s dA_s.$$

Now φ_s will not exist for any arbitrary φ, so, necessarily, we need to make sure that the stochastic integral exists for when φ is locally bounded and predictable so that $\int_0^t \varphi_s dM_s$ and $\int_0^t \varphi_s dA_s$ both exist. Those conditions will both be satisfied if we assume that φ is locally bounded and predictable. However, there are cases where the stochastic integral exists even though φ is not locally bounded and integrable.

We are going to denote by $L(X)$ the set of all processes φ, such that φ is X-integrable in the sense that the stochastic integral exists.

Now that we have introduced the stochastic integral, we can move on to Ito's Lemma for continuous semimartingales.

Let X be a d-dimensional semimartingale, $X = \left(X^1, \ldots, X^d\right)$, which means that each component is a semimartingale, then take $\mathcal{F}$ to be a function, where $\mathcal{F}: \mathbb{R}^d \to \mathbb{R}$, and assume that $\mathcal{F}$ is $C-2$, meaning that it is twice continuously differentiable.

We will define Y_t to be $\mathcal{F}(X_t)$, so this is a new stochastic process and a function of X_t.

Ito's Lemma says that Y_t is also a semimartingale and it gives us the stochastic differential for Y_t with respect to the partial derivatives of the function $\mathcal{F}$ as follows:

$$\mathrm{dY_t} = \sum_{i=1}^{d} \frac{\delta \mathcal{F}}{\delta x_i} dX_t{}^i + \frac{1}{2} \sum_i \sum_j \frac{\delta^2 \mathcal{F}}{\delta x_{i,} \delta x_j} d\left[X^i, X^j\right]_t.$$

Finally, it is important to mention that the notion of a semimartingale can be extended to a case where X is no longer continuous. In general, a stochastic process X is a semimartingale if it can be decomposed into the three parts discussed at the beginning of the video ($X = X_0 + M + A$), where neither M nor A are necessarily continuous, but M is a local martingale and A is a finite variation. Furthermore, we can still define the stochastic integral in the same way for locally bounded and predictable processes when the process X is a just a semimartingale without being continuous. Further details are included in the notes.

That brings us to the end of this module. In the next module, we are going to look at Continuous Trading.

3.2.7 Notes: Problem Set

Problem 1

Consider the processes X and Y satisfying the following SDEs:

$$dX_t = X_t(2\,dt + Y_t\,dW_t^1 + dW_t^2), \qquad dY_t = Y_t\,dt + 5\,dW_t^1 + X_t dW_t^2.$$

Find $\langle X, Y \rangle_t$.

Solution:

In the lecture notes, we have extended the definition of the predictable quadratic variation $\langle\rangle$ to continuous semimartingales. If $X = X_0 + M + A$ and $Y = Y_0 + N + B$ are continuous semimartingales, we define $\langle X, Y \rangle := \langle M, N \rangle$ and $\langle X \rangle := \langle X, X \rangle = \langle M \rangle$. Therefore, we can apply it to our problem as follows:

$$\langle dX, dY \rangle \coloneqq \langle dM, dN \rangle = (Y_t X_t dW_t^1 + X_t dW_t^2)(5dW_t^1 + X_t dW_t^2) = (5Y_t X_t + X_t^2)dt.$$

Above, we have applied the "multiplication rules":

$$dtdt = 0, dtdW = 0, dWdW = dt, dW_i dW_j = 0 \text{ if } i \neq j \text{ (independent)}.$$

Thus, finally we will have,

$$\langle X, Y \rangle_t = \int_0^t (5Y_s X_s + X_s^2) ds.$$

Problem 2

Consider the process X satisfying the following SDE:

$$dX_t = X_t(t\,dt + t^2\,dW_t).$$

Find $\langle X \rangle_t$.

Solution:

In the lecture notes, we have extended the definition of the predictable quadratic variation $\langle\rangle$ to continuous semimartingales. If $X = X_0 + M + A$ and $Y = Y_0 + N + B$ are continuous semimartingales, we define $\langle X, Y\rangle := \langle M, N\rangle$ and $\langle X\rangle := \langle X, X\rangle = \langle M\rangle$. Therefore, we can apply it to our problem as follows,

$$\langle dX\rangle_t = X_t^2 t^4 dt.$$

Thus, the quadratic variation of X is defined as:

$$\langle X\rangle_t = \int_0^t X_s^2 s^4 ds.$$

Problem 3

Consider the process X satisfying the following SDE:

$$dX_t = 2\,dt + 4\,dW_t.$$

Find $\ln \mathcal{E}\,(X)_t$.

Solution:

The stochastic exponential (see lecture notes) is defined as follows:

$$\ln \varepsilon\,(X)_t = X_t - \frac{1}{2}\langle X\rangle_t.$$

On the other hand, to compute $\langle X\rangle_t$ we can apply,

$$\langle X\rangle_t = \int_0^t 4^2 ds = 16t.$$

Therefore, the solution is,

$$\ln \varepsilon\,(X)_t = X_t - \frac{1}{2}\langle X\rangle_t = X_t - \frac{1}{2}16t = X_t - 8t.$$

Solution: $X_t - 8t$.

Problem 4

Consider the process X satisfying the following SDE:

$$dX_t = X_t(2\,dt + 3\,dW_t^1 + dW_t^2).$$

Find $\ln \mathcal{E}\,(X)_t$.

Solution:

Let X be a continuous semimartingale. We can define the *stochastic exponential* of X as:

$$\ln \mathcal{E}\,(X) = X_t - \frac{1}{2}\langle X\rangle_t.$$

Thus, in our example, we will have,

$$\langle X\rangle_t = \int_0^t 10\,X_s^2\,ds.$$

Finally,

$$\ln \mathcal{E}\,(X) = X_t - \frac{1}{2}\langle X\rangle_t = X_t - \frac{1}{2}\int_0^t 10\,X_s^2\,ds = X_t - 5\int_0^t X_s^2\,ds.$$

Problem 5

Consider the processes X and Y satisfying the following SDEs:

$$dX_t = X_t(3\,dt + 4\,dW_t^1 + 3dW_t^2), \qquad dY_t = Y_t\,dt + 2\,dW_t^1.$$

Find $\langle X, Y\rangle_t$.

Solution:

In the lecture notes, we have extended the definition of the predictable quadratic variation $\langle\rangle$ to continuous semimartingales. If $X = X_0 + M + A$ and $Y = Y_0 + N + B$ are continuous semimartingales, we define $\langle X, Y\rangle \coloneqq \langle M, N\rangle$ and $\langle X\rangle \coloneqq \langle X, X\rangle = \langle M\rangle$. Therefore, we can apply it to our problem as follows:

$$\langle dX, dY\rangle \coloneqq \langle dM, dN\rangle = (4X_t dW_t^1 + 3X_t dW_t^2)(2dW_t^1) = 8X_t dt.$$

Above, we have applied the "multiplication rules":

$$dtdt = 0, dtdW = 0, dWdW = dt, dW_i dW_j = 0 \text{ if } i \neq j \text{ (independent)}$$

Thus, finally we will have,

$$\langle X, Y\rangle_t = \int_0^t 8X_s ds.$$

3.2.8 References

Kunita, H. and Watanabe, S. (1967). 'On Square Integrable Martingales'. *Nagoya Mathematical Journal*, 30(30).

Shiryaev, A.N. and Cherny, A.S.. (2002). 'Vector Stochastic Integrals and The Fundamental Theorems of Asset Pricing'*. Proceedings of the Steklov Institute of Mathematics*, 237.