Continuous-time Stochastic Processes
Module 7

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 7 of Continuous-time Stochastic Processes, entitled An Introduction to Interest Rate Modeling. It consists of four video lecture scripts and four sets of supplementary notes.

2. Course Context

Continuous-time Stochastic Processes is the fourth course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The aim of the course is to introduce derivative pricing when trading happens in continuous-time. It begins by focusing on the stochastic calculus of Brownian motion and its generalization to continuous semimartingales. These ideas are then applied to continuous trading and the pricing of financial derivatives.

2.1 Course-level Learning Outcomes

Upon completion of the Continuous-time Stochastic Processes course, you will be able to:

1. Define and identify Brownian motion processes in multiple dimensions.
2. Solve stochastic differential equations.
3. Apply Ito's Lemma for continuous semimartingales.
4. Apply Girsanov's theorem to construct equivalent local martingale measures.
5. Price and hedge derivatives in various asset price models.
6. Derive the Black-Scholes partial differential equation.
7. Construct asset prices models based on Levy processes.
8. Price interest rate derivatives.

2.2 Module Breakdown

The Continuous-time Stochastic Processes course consists of the following one-week modules:

1. Brownian Motion and Continuous-time Martingales
2. Stochastic Calculus I: Ito Process
3. Stochastic Calculus II: Semimartingales
4. Continuous Trading
5. The Black-Scholes Model
6. An Introduction to Levy Processes
7. An Introduction to Interest Rate Modeling

3. Module 7:

An Introduction to Interest Rate Modeling

In this, the final module of Continuous-time Stochastic Processes, we turn our attention to a few commonly used interest rate models. Module 7 begins by describing interest rate assets, before introducing the Ho-Lee, Vasicek, Cox-Ingersoll-Ross, and Hull-White models. It concludes with a derivation of the bond pricing equation, as well as some examples of bond prices in these models.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Describe the basic interest rate assets.
2. Derive the most common interest rate models.
3. Solve the bond pricing equation to calculate the bond prices in these models.

3.2 Transcripts and Notes

3.2.1 Notes: Interest Rate Assets

In the last six modules we have assumed that interest rates are constant. This assumption is not too harmful when dealing with equity derivatives, since the impact of changes in interest rates is minimal to equity derivatives, as compared to changes in the underlying asset.

This module is concerned with the pricing of interest rate related assets, so it is necessary to consider the more realistic scenario of stochastic interest rates.

There are various frameworks to model interest rates. In this course we are going to focus on models for the so-called *short rate*. In these short rate models, the bank account B is assumed to satisfy the following equation:

$$dB_t = r_t B_t dt, \quad B_0 = 1,$$

where r_r is the instantaneous short rate, which is now assumed to be a stochastic process.
Solving this equation gives

$$B_t = e^{\int_0^t r_s \, ds}$$

for $t \geq 0$, and

$$B_t = B_s e^{\int_s^t r_u \, du}$$

for $0 \leq s \leq t$. We will therefore spend time constructing models for the short rate r.

We will fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume that all stochastic processes are defined on this space and adapted to $\mathbb{F}$.

For $0 \leq t \leq T$ we define the stochastic discount factor between t and T as

$$D(t,T) := \frac{B_t}{B_T} = e^{-\int_t^T r_s \, ds}.$$

This represents the present value at time t of one unit of currency payable at time T. We call T the maturity time and we have $D(T,T) = 1$. This is again stochastic, since the interest rate is also stochastic.

The first non-trivial interest rate asset is the zero-coupon bond. A zero-coupon bond is an asset that pays one unit of currency at a future time $T > 0$, called the maturity. The time t price of a zero-coupon bond with maturity time T will be denoted by $P(t,T)$. Note that by definition we have $P(T,T) = 1$.

Do not confuse $D(t,T)$ and $P(t,T)$. The discount factor, $D(t,T)$, is a random unknown quantity at time t. It is calculated from the "true" but unknown evolution of r_s for $t \leq s \leq T$. On the other hand, the zero-coupon bond price, $P(t,T)$, is an asset price that is known at time t. W will see later that it is calculated by averaging $D(t,T)$ over different paths or evolutions of r between t and T. That is,

$$P(t,T) = \mathbb{E}^*[D(t,T)|\mathcal{F}_t]$$

for some pricing measure $\mathbb{P}^*$.

For a zero-coupon bond with price $P(t,T)$ we define the nominal continuously compounded annual (NACC) yield $R(t,T)$ as the constant NACC interest rate such that $P(t,T)$ is equal to the discount value of one unit at time T, discounted using the constant rate $R(t,T)$. In symbols:

$$e^{-R(t,T)(T-t)} = P(t,T) \Rightarrow R(t,T) = \frac{-\ln P(t,T)}{T-t}.$$

Similarly, we define $L(t,T)$ to be the corresponding simple yield:

$$\frac{1}{(1+L(t,T)(T-t))} = P(t,T) \Rightarrow L(t,T) = \frac{1- \ P(t,T)}{(T-t)(P(t,T)}.$$

For fixed t, the function $T \mapsto \ L(t,T)$ is called the yield curve.

In general, a coupon-bearing bond that pays coupons of $C_1, \dots, C_n$ at times $t \ \leq T_1 < \cdots < T_n$ and a nominal of N at maturity time T_n has a price of

$$\sum_{k=1}^{n} C_k P(t,T_k) + NP(t,T_n)$$

at time t. This is a result of linearity of expectation. So we can use $P(t,T)$ to discount payments.

3.2.2 Transcript: Term Structure

Hi, in this video we introduce stochastic interest rates in continuous time.

So, all along, we have assumed that r, the interest rate, is constant for all maturities. This then implies, of course, that the bank account evolves according to the following SDE:

$$dB_t = rB_t dt, \quad B_0 = 1.$$

Solving this SDE gives us:

$$B_t = e^{rt}, \quad D_t = e^{-rt},$$

where D_t is the discount factor.

Now, we are going to assume that the interest rate, r, is not constant, and, therefore, the bank account evolves according to the following SDE:

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

If we solve this SDE, we get:

$$B_t = e^{\int_0^t r_s ds}.$$

So, instead of just simply multiplying r by t when it was constant, we now have to evaluate an integral as we do above. Of course, the discount factor, D_t, is equal to $e^{-\int_0^t r_s ds}$.

We are also going to generalize the discount factor and define $D(t,T)$ as the discount factor between periods t and T. Using the formula $D_t = e^{-\int_0^t r_s ds}$, we get $e^{-\int_t^T r_s ds}$.

This interest rate, r_t, that we are going to be using now is called a **short rate** and in the next video we will see how to model it as a stochastic process.

We also introduce the zero-coupon bond, which is an asset that has the following payoff:

A zero-coupon bond that expires at time T is an asset that pays one unit of currency at time T. What we are interested in, however, is the price of that zero-coupon bond at an intermediate time, t, between 0 and T. We will denote this price by $P(t,T)$. So, this is the price of a zero-coupon bond with maturity, or, as it is sometimes called, expiry time, T. We are only evaluating it at time t.

Now, if $\mathbb{P}^*$ is an ELMM, then, since the discounted prices must be martingales, we have the following formula, where the discounted zero-coupon bond price at time t must be equal to the conditional expectation of the dicounted zero-coupon bond price at time T, meaning that we have to discount it by T, given $\mathcal{F}_t$:

$$D_t P(t,T) = E^*(P(T,T)D_T|\mathcal{F}_t).$$

That is the risk-neutral pricing formula that we have been using all along and we are now applying it to zero-coupon bond prices.

Now, since the zero-coupon bond at time T pays \$1, (which means that its value is equal to 1), we get $E^*(D_T|\mathcal{F}_t)$, and, if we rearrange terms, we get the pricing formula for bonds, which says that $P(t,T)$, the price of a zero-coupon bond at time t that has maturity time T, is simply equal to the expected value under the pricing measure, or the ELMM we are using. We then divide by D_t and, since D_T is adapted, we can take it inside and write it as:

$$P(t,T) = E^* \left(\frac{D_T}{D_t} \middle| \mathcal{F}_t \right).$$

So, this is the bond-pricing formula when we have stochastic interest rates.

Now that we have introduced stochastic interest rates in continuous time, in the next video we are going to look at a short rate model called the Vasicek model.

3.2.3 Notes: Short Rate Models

We now discuss a few widely known short rate models. For all the models, the dynamics of r are presented in a risk neutral/pricing measure.

The Ho-Lee model

In the Ho-Lee Model the risk neutral dynamics of r are:

$$dr_t = \theta(t)dt + \sigma dW_t,$$

where W is a Brownian motion, $\theta(t)$ is deterministic, and σ is a constant.

We can easily solve the SDE to get

$$r_t = r_0 + \int_0^t \theta(s)ds + \sigma W_t.$$

Hence r_t is normally distributed with

$$\mathbb{E}(r_t) = r_0 + \int_0^t \theta(s)ds \text{ and } \mathrm{Var}\,(r_t) = \sigma^2 t.$$

The fact that r_t is normally distributed implies that there is a positive probability of r_t being negative, no matter what the parameters are. This is not a desirable property of the model if we assume that interest rates are positive.

Vasicek model

In this model the short rate has the following risk neutral dynamics:

$$dr_t = (\theta - \alpha r_t)dt + \sigma dW_t,$$

where $\theta, \alpha,$ and σ are constants and W is a Brownian motion. Another common formulation of this model (and the one used in the lecture videos) is

$$dr_t = \alpha(\theta|-r_t)dt + \sigma dW_t.$$

Students are expected to be familiar with both expressions – that is why both formulations are introduced. However, we will always refer to the first formulation of the model in the rest of the notes and exercises.

To solve for r_t we apply Ito's formula to $f(t, r_t) \;=\; e^{\alpha t} r_t$ to get

$$df(t, r_t) = \alpha e^{\alpha t} r_t \, dr + e^{\alpha t} dr_t = \theta e^{\alpha t} dt + \sigma e^{\alpha t} dW_t,$$

which implies that

$$e^{\alpha t} r_t = r_0 \int_0^t \theta e^{\alpha s} ds + \int_0^t \sigma e^{\alpha s} \, dW_s = r_0 + \frac{\theta}{\alpha} \left(e^{\alpha t} - 1\right) + \int_0^t \sigma e^{\alpha s} \, dW_s.$$

Therefore,

$$r_t = r_0 e^{-\alpha t} + \frac{\theta}{\alpha} \left(1 - e^{-\alpha t}\right) + \int_0^t \sigma e^{\alpha(s-t)} \, dW_s.$$

We can see again that r_t is normally distributed with parameters:

$$\mathbb{E}(r_t) = r_0 e^{-\alpha t} + \frac{\theta}{\alpha} \left(1 - e^{-\alpha t}\right) \text{ and Var } (r_t) = \int_0^t \sigma^2 e^{2\alpha(s-t)} \, dW_s.$$

Again, the rate can go negative.

In this model, the rate r is mean-reverting with a long run mean of θ/α. This is a desirable property since it is in line with empirical observations.

Cox-Ingersoll-Ross model

In the Cox-Ingersoll-Ross (CIR) model, the risk neutral dynamics of r are

$$dr_t = (\theta - \alpha r_t)\, dt + \sigma\sqrt{r_t}\, dW_t,$$

where $\alpha, \theta,$ and σ are constants. This process is often called the square root process. Again, as with the Vasicek model, an alternative formulation is

$$dr_t = \alpha(\theta - r_t)\, dt + \sigma\sqrt{r_t}\, dW_t.$$

However, the first one will be the default.

This model can be considered an improvement to the Vasicek model, and it avoids negative rates. If $2\theta \geq \sigma^2$, then r_t never touches zero.

The distribution of rt is given by the non-central chi-squared distribution. We will not show the derivation here, but we note that

$$\mathbb{E}(r_t) = r_0 e^{-\alpha t} + \frac{\theta}{\alpha}\,(1 - e^{-\alpha t}) \text{ and } \mathrm{Var}\,(r_t) = r_0 \frac{\sigma^2}{\alpha}\,(e^{-\alpha t} - e^{-2\alpha t}) + \frac{\theta\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2.$$

Hull-White model

One of the major drawbacks of the last two models presented is that, due to only having three parameters, they fail to reproduce observed bond prices – i.e. they fail to "fit the market".

The Hull-White model is an extension to the Vasicek model that allows θ to be a deterministic function of time, rather than a constant. This flexibility ensures that the model can fit the market-observed bond prices.

The dynamics of r in the Hull-White model are

$$dr_t = \alpha(\theta(t) - \alpha r_t)dt + \sigma\, dW_t,$$

where $\theta(t)$ is a deterministic function of time and α and σ are constants. In some specifications of the model, α is also a deterministic function of t. There is also an alternative formulation as

$$dr_t = \alpha(\theta(t) - r_t)dt + \sigma dW_t,$$

which is sometimes used in the lecture videos but not in the notes and quizzes. You are expected to be familiar with both.

The Hull-White model is similar to the Vasicek model and the derivation of the distributions are almost identical. To solve for r_t, first we apply Ito's formula to $f(t, r_t) = e^{\alpha t} r_t$ to get

$$df(t, r_t) = \alpha e^{\alpha t} r_t\, dr + e^{\alpha t}\, dr_t = \theta(t) e^{\alpha t}\, dt + \sigma e^{\alpha t}\, dW_t,$$

which implies that

$$e^{\alpha t} r_t = r_0 + \int_0^t \theta(s)\, e^{\alpha s} ds + \int_0^t \sigma e^{\alpha s}\, dW_s.$$

Therefore,

$$r_t = r_0 e^{-\alpha t} + \int_0^t \theta(s)\, e^{\alpha(s-t)} ds + \int_0^t \sigma e^{\alpha(s-t)}\, dW_s.$$

We can see again that rt is normally distributed with parameters:

$$\mathbb{E}(r_t) = r_0 e^{-\alpha t} + \int_0^t \theta(s)\, e^{\alpha(s-t)} ds \text{ and } \mathrm{Var}\,(r_t) = \int_0^t \sigma^2 e^{2\alpha(s-t)}\, ds.$$

Again, the rate can go negative.

In this model, the rate r is mean-reverting with a long run mean of $\theta(t)/\alpha$ that keeps changing. This is a desirable property since it is in line with empirical observations.

3.2.4 Transcript: The Vasicek Model

Hi, in this video we introduce a short rate model known as the Vasicek model.

In the Vasicek model, the short rate, r, satisfies the following SDE:

$$dr_t = \alpha(\theta - r_t)\,dt + \sigma dW_t,$$

where α, θ and σ are all positive constants. So, it is mean-reverting in this form. This is actually called an **Ornstein-Uhlenbeck process**, which you should remember from the second module.

To solve this SDE, we apply Ito's Lemma to r_t times $e^{\alpha t}$. If we find the find the stochastic differential of $d(r_t e^{\alpha t})$, we get $e^{\alpha t} dr_t$, by taking the derivative with respect to r_t, plus, taking this part here, we get $\alpha e^{\alpha t} r_t dt$. Written in full:

$$d(r_t e^{\alpha t}) = e^{\alpha t} dr_t + \alpha e^{\alpha t} r_t dt.$$

We can then simplify this to get:

$$(\alpha e^{\alpha t}\theta - \alpha e^{\alpha t} r_t)dt + e^{\alpha t}\sigma dW_t + \alpha e^{\alpha t} r_t dt.$$

We can further simplify this, when we realize that we can cancel out two parts of the equation ($-\alpha e^{\alpha t} r_t$ and $\alpha e^{\alpha t} r_t dt$), to get:

$$\alpha e^{\alpha t}\theta dt + \sigma e^{\alpha t} dW_t.$$

So, this is the stochastic differential, and it doesn't depend on any unknown processes so we can integrate it to solve for r_t.

Doing this, on the left-hand side, we get:

$$r_t e^{\alpha t} - r_0 = \int_0^t \alpha e^{\alpha s} \theta ds + \int_0^t \sigma e^{\alpha s} dW_s.$$

So, this is the solution for r_t, which we can further simplify to get:

$$r_t = r_0 e^{-\alpha t} + e^{-\alpha t} \int_0^t \alpha e^{\alpha s} \sigma ds + e^{-\alpha t} \int_0^t \sigma e^{\alpha s} dW_s.$$

So, this is the solution for r_t. As you can see, this part, $r_0 e^{-\alpha t} + e^{-\alpha t} \int_0^t \alpha e^{\alpha s} \sigma ds$, is deterministic, meaning that it doesn't have any randomness in it. The only random part is the end of the equation, $e^{-\alpha t} \int_0^t \sigma e^{\alpha s} dW_s$, and, since the integrand, $\sigma e^{\alpha s}$, in this case is also deterministic, it follows that r_t has a normal distribution.

When looking to calculate the mean of the equation, we look at the first part of the equation only, as the mean of the second part of the equation is 0. If we evaluate the integral in the first part of the equation, we get:

$$E(r_t) = r_t e^{-\alpha t} + \theta(1 - e^{-\alpha t}).$$

So, that is the mean of the first part of the equation. If we add it to the variance of the second part of the equation, we get:

$$\mathrm{Var}(r_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

So, this is the mean and the variance of the normal distribution.

Of course, a huge disadvantage of this is that it implies that interest rate can in fact be negative with a positive probability since the normal distribution does go negative.

Now that we have introduced the Vasicek model, in the next video we are going to look at other interest rate models.

3.2.5 Notes: Bond Pricing

We now consider the pricing of bonds under each of the four models presented above.

First, if H is a derivative that expires at time T, then its price at time $t < T$ is given by $\mathbb{E}^*(D(t,T)H|\mathcal{F}_t)$, where $\mathbb{P}^*$ is an ELMM for the discounted bond prices. This implies that the bond prices must satisfy the equation

$$P(t,T) = \mathbb{E}^*(D(t,T)|\mathcal{F}_t)$$

for $t < T$ since $P(T,T) = 1$. Indeed, since $P(t,T)/B_t$ is a martingale and $P(T,T) = 1$, we have

$$\frac{P(t,T)}{B_t} = \mathbb{E}^*\left(\frac{P(T,T)}{B_T}|\mathcal{F}_t\right) = \mathbb{E}^*\left(\frac{1}{B_T}|\mathcal{F}_t\right).$$

Multiplying both sides by B_t and noting that B_t is $\mathcal{F}_t$ −measurable gives the result.

Now suppose that the risk-neutral dynamics of the short rate r are as follows:

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t.$$

Also suppose that the zero-coupon bond prices are of the form

$$P(t,T) = F^T(t,r_t),$$

for some sufficiently smooth function F^T. Then through some no-arbitrage considerations, we can show that F^T satisfies the following PDE:

$$\frac{\partial F^T}{\partial t} + \mu(t,r)\frac{\partial F^T}{\partial r} + \frac{1}{2}\,\sigma^2(t,r)\frac{\partial^2 F^T}{\partial r^2} - rF^T = 0, \quad F^T(T,r) = 1.$$

This PDE is called the *fundamental term structure PDE*.

The short rate models we considered above have the special property that the bond prices are of the form

$$P(t,T) = F^T(t,r_t) = e^{A(t,T)-B(t,T)r_t}$$

for some sufficiently smooth functions A and B. Such models are called *affine term-structure models.*

Let us now substitute the new expression for F^T into the PDE to obtain the corresponding equations satisfied by A and B. We get

$$\frac{\partial A}{\partial t} - \mu(t,r)B + \frac{1}{2}\sigma^2(t,r)B^2 - \left(1 + \frac{\partial B}{\partial t}\right)r = 0.$$

Now suppose that μ and σ^2 can be expressed as

$$\mu(t,r) = a(t)r + b(t) \text{ and } \sigma^2(t,r) = c(t)r + d(t).$$

Then the equation becomes

$$\frac{\partial A}{\partial t} - \ b(t)B + \frac{1}{2}d(t)B^2 = \left(1 + \frac{\partial B}{\partial t} + a(t)B - \frac{1}{2}d(t)B^2\ \right)r.$$

Since the LHS is independent of r, while the RHS is dependent on r, both sides must be equal to 0. This gives the following system of differential equations for A and B:

$$
\begin{cases}
\dfrac{\partial A}{\partial t} = b(t)B - \dfrac{1}{2}d(t)B^2 & A(T,T) = 0 \\
\dfrac{\partial B}{\partial t} = -a\,(t)B + \dfrac{1}{2}c(t)B^2 - 1 & B(T,T) = 0.
\end{cases}
$$

Let us now solve the differential equations to find the bond price for some of the four models.

First, we start with the Ho-Lee model. Note that for this model we have

$$
a(t) = c(t) = 0, \text{ and } b(t) = \theta(t), \; d(t) = \sigma^2.
$$

Thus, the system of equations is

$$
\begin{cases}
\dfrac{\partial A}{\partial t} = \theta(t)B - \dfrac{1}{2}\sigma^2 B^2 & A(T,T) = 0 \\
\dfrac{\partial B}{\partial t} = \; -1 & B(T,T) = 0.
\end{cases}
$$

We can easily solve the second equation for B to get $B(t,T) = \; T - \; t$. Substituting this to the first equation and integrating we get

$$
A(t,T) = -\int_t^T \theta(s)(T-s)ds + \frac{1}{6}\sigma^2(T-t)^3.
$$

Next is the Vasicek model. We have

$$
a(t) = -\alpha, \; b(t) = \theta, \; c(t) = 0, \text{ and } d\,(t) = \sigma^2,
$$

hence the differential equations are:

$$
\begin{cases}
\dfrac{\partial A}{\partial t} = \theta B - \dfrac{1}{2}\sigma^2 B^2 & A(T,T) = 0 \\
\dfrac{\partial B}{\partial t} = \alpha B - 1 & B(T,T) = 0.
\end{cases}
$$

The second equation is linear with constant coefficients, hence it can solved by multiplying both sides by an integrating factor. The solutions are:

$$B(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right), \text{and } A(t,T) = \frac{\left(B(t,T) - (T-t)\right)\left(\alpha\theta - \frac{1}{2}\sigma^2\right)}{\alpha^2} - \frac{\sigma^2 B^2(t,T)}{4\alpha}.$$

Finally, we look at the CIR model. We have:

$$\begin{cases} \dfrac{\partial A}{\partial t} = \theta B & A(T,T) = 0 \\ \dfrac{\partial B}{\partial t} = \alpha B + \dfrac{1}{2}\sigma^2 B^2 - 1 & B(T,T) = 0. \end{cases}$$

The solution to these equations is lengthy and we will not show it here. Refer to Brigo and Mercurio's *Interest Rate Models: Theory and Practice* for a discussion.

3.2.6 Transcript: Other Short Rate Models

Hi, in this video we discuss some other popular interest rate models.

The Ho-Lee model

This model states that the short rate, r, evolves according to the following SDE:

$$dr_t = \theta_t dt + \sigma dW_t.$$

As you can see, θ is a function of time and σ is a positive constant.

The solution to the SDE is simple: it just says that r_t has a normal distribution.

The Vasicek model

As a reminder, we looked at this model in the previous video. This model states that the short rate evolves according to the following SDE:

$$dr_t = \alpha(\theta - r_t)dt + \sigma dW_t.$$

A huge advantage of the Vasicek model, compared to the Ho-Lee Model, is that it has the property of mean reversion, $\alpha(\theta - r_t)$. If r_t is greater than θ, then the drift becomes negative and it pulls the process down towards θ and, if r_t is less than θ, the drift becomes positive and pulls it towards θ.

We can look at it on a graph: if we have θ on the y-axis (which is sometimes called **the level of mean reversion**) then, if r_t is less than θ, the drift is positive, meaning that this is pulled towards θ; and, as soon as r_t is greater than θ, the drift becomes negative and the interest rate r_t is pulled downwards towards θ itself.

However, r_t still has a normal distribution, which implies that we can have negative interest rates with positive probability.

The Hull-White model

This model's SDE for r_t is similar to the Vasicek model, except that the parameters are now time-dependent rather than constant:

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dW_t.$$

The advantage of this model is that the level of mean reversion, θ, can be varied. So, instead of having one level of mean reversion on our graph, we can have a level of mean reversion that changes with time. This has huge advantages in real-life modeling because this model fits the market data better than the Vasicek model.

The Cox-Ingersoll-Ross (CIR) model

In this model, the short rate, dr_t, evolves according to the following SDE:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t.$$

An important assumption that we make is that $2ab \geq \sigma^2$. This assumption ensures that the interest rates that we get from this model are always positive, which is a huge advantage compared to the previous three models because those three models can give us negative interest rates with positive probability. With the CIR model, on the other hand, the interest rates are always positive.

Now that we have covered these interest rate models, we have reached the end of our final module.

3.2.8 Notes: Problem set

Problem 1

Assume that r evolves according to the Vasicek model with

$$r_0 = 0.1, \quad \theta = 0.15, \quad \alpha = 0.2, \quad \text{and } \sigma = 0.5.$$

What is $\mathbb{E}(r_5)$?

Solution:

From the lecture notes, we know that:

$$\mathbb{E}(r_t) = r_0 e^{-\alpha t} + \frac{\theta}{\alpha}(1 - e^{-\alpha t}) = 0.1e^{-0.2*5} + \frac{0.15}{0.2}(1 - e^{-0.2*5}) = 0.511.$$

Problem 2

Assume that r evolves according to the Vasicek model with

$$r_0 = 0.1, \; \theta = 0.15, \; \alpha = 0.2, \text{ and } \sigma = 0.5.$$

If the zero-coupon bond price $P(1,3)$ under this model is $e^{A(1,3)-B(1,3)r_1}$, then what is $B(1,3)$ equal to?

Solution:

Next is the Vasicek model. We have

$$a(t) = -\alpha, \; b(t) = \theta, \; c(t) = 0, \text{ and } d(t) = \sigma^2,$$

hence the differential equations are

$$\begin{cases} \dfrac{\partial A}{\partial t} = \theta B - \dfrac{1}{2}\sigma^2 B^2 & A(T,T) = 0 \\ \dfrac{\partial B}{\partial t} = \alpha B - 1 & B(T,T) = 0. \end{cases}$$

The second equation is linear with constant coefficients, hence it can solved by multiplying both sides by an integrating factor. The solutions are:

$$B(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right), \text{ and } A(t,T) = \frac{\left(B(t,T) - (T-t)\right)\left(\alpha\theta - \frac{1}{2}\sigma^2\right)}{\alpha^2} - \frac{\sigma^2 B^2(t,T)}{4\alpha}.$$

In our case, we only care about B, and then,

$$B(1{,}3) = \frac{1}{0.2}\left(1 - e^{-0.2(3-1)}\right) = 1.6484.$$

Problem 3

Assuming that $\int_0^t r_s\, ds \sim N\,(0.01t, 0.2t)$, then compute the value of $\mathbb{E}\big(D(0,1)\big)$.

Solution:

We know from the lecture notes that:

$$D(t,T) \coloneqq \frac{B_t}{B_T} = e^{-\int_t^T r_s\, ds}.$$

And, as the exponent is a normal distribution, we can apply the following,

$$\mathbb{E}\left(D(0{,}1)\right) = e^{\mu + \sigma^2/2} = e^{-0.01+0.2/2} = e^{0.09}.$$

Problem 4

Assume that r evolves according to the Ho-Lee model with

$$r_0 = 0.06, \quad \theta(t) = 0.003, \quad \text{and } \sigma = 0.25.$$

If the zero-coupon bond price $P(1,3)$ under this model is $e^{A(1,3)-B(1,3)r_1}$, then compute the value of $B(1,3)$.

Solution:

From the lecture notes we know that in the Ho-Lee model we have

$$a(t) = c(t) = 0, \quad \text{and } b(t) = \theta(t), \quad d(t) = \sigma^2.$$

Thus, the system of equations is

$$\begin{cases} \dfrac{\partial A}{\partial t} = \theta(t)B - \dfrac{1}{2}\sigma^2 B^2 & A(T,T) = 0 \\ \dfrac{\partial B}{\partial t} = -1 & B(T,T) = 0. \end{cases}$$

We can easily solve the second equation for B to get $B(t,T) = T - t$. Substituting this to the first equation and integrating we also can get

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2(T-t)^3.$$

In our case the solution is straightforward: $B(1,3) = 3 - 1 = 2$.

Problem 5

Assume that r evolves according to the Vasicek model with

$$r_0 = 0.06, \qquad \theta = 0.09, \qquad \alpha = 0.11, \text{ and } \sigma = 0.25.$$

If the zero-coupon bond price $P(2,3)$ under this model is $e^{A(2,3)-B(2,3)r_2}$, then what is $B(2,3)$ equal to?

Solution:

Next is the Vasicek model. We have

$$a(t) = -\alpha, \;\; b(t) = \theta, \;\; c(t) = 0, \text{ and } d(t) = \sigma^2.$$

Hence the differential equations are:

$$\begin{cases} \dfrac{\partial A}{\partial t} = \theta B - \dfrac{1}{2}\sigma^2 B^2 & \qquad A(T,T) = 0 \\ \dfrac{\partial B}{\partial t} = \alpha B - 1 & \qquad B(T,T) = 0. \end{cases}$$

The second equation is linear with constant coefficients, hence it can solved by multiplying both sides by an integrating factor. The solutions are:

$$B(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right), \text{ and } A(t,T) = \frac{\big(B(t,T) - (T-t)\big)\left(\alpha\theta - \frac{1}{2}\sigma^2\right)}{\alpha^2} - \frac{\sigma^2 B^2(t,T)}{4\alpha}.$$

In our case, we only care about B, and then

$$B(2{,}3) = \frac{1}{0.11}\left(1 - e^{-0.11(3-2)}\right) = 0.94696.$$

3.2.7 Transcript: Concluding Video

Congratulations on finishing Continuous-time Stochastic Processes, the fourth course in the WorldQuant University Master of Science in Financial Engineering.

In this course, we explored stochastic processes in continuous time and applied that knowledge to the pricing of derivatives in continuous-time asset price models.

In the next course, Computational Finance, you will be introduced to the Python programming language, Monte Carlo methods, as well as applications to option pricing and risk measurement.

Now that you have completed the Continuous-time Stochastic Processes course, you should be able to:

- Define and identify Brownian motion processes in multiple dimensions.
- Solve stochastic differential equations.
- Apply Ito's Lemma for continuous semimartingales.
- Apply Girsanov's Theorem to construct equivalent local martingale measures.
- Price and hedge derivatives in various asset price models.
- Derive the Black-Scholes partial differential equation.
- Construct asset price models based on Lévy processes.
- Price interest rate derivatives.

Thank you for your engagement throughout this course. We hope that you will continue to enjoy the rest of the program.

Good luck!

3.4 References

Brigo, D. & Mercurio, F. (2001) *Interest Rate Models: Theory and Practice*. Springer.