Continuous-time Stochastic Processes: Module 4

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 4 of Continuous-time Stochastic Processes, entitled Continuous Trading. It consists of four sets of notes and three lecture videos.

2. Course Context

Continuous-time Stochastic Processes is the fourth course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The aim of the course is to introduce derivative pricing when trading happens in continuous time. It begins by focusing on the stochastic calculus of Brownian motion and its generalization to continuous semimartingales. These ideas are then applied to continuous trading and the pricing of financial derivatives.

2.1 Course-level Learning Outcomes

Upon completion of the Continuous-time Stochastic Processes course, you will be able to:

1. Define and identify Brownian motion processes in multiple dimensions.
2. Solve stochastic differential equations.
3. Apply Ito's Lemma for continuous semimartingales.
4. Apply Girsanov's theorem to construct equivalent local martingale measures.
5. Price and hedge derivatives in various asset price models.
6. Derive the Black-Scholes partial differential equation.
7. Construct asset prices models based on Levy processes.
8. Price interest rate derivatives.

2.2 Module Breakdown

The Continuous-time Stochastic Processes course consists of the following one-week modules:

1. Brownian Motion and Continuous-time Martingales
2. Stochastic Calculus I: Ito Process
3. Stochastic Calculus II: Semimartingales
4. Continuous Trading
5. The Black-Scholes Model
6. An Introduction to Levy Processes
7. An Introduction to Interest Rate Modeling

3. Module 4

Continuous Trading

In this module, we are introduced to the theory of trading in continuous time, including both fundamental theorems of asset pricing. First, attention is given to the notion of a trading strategy in continuous time, together with the notions of admissibility and self-financing strategies. The whole theory is then developed by restricting to admissible and self-financing strategies, and the asset prices are assumed to have non-negative components.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Define the notions of self-financing strategies in continuous time.
2. Define the notions of arbitrage and completeness of a market model in continuous time.
3. Price and hedge derivatives in continuous time.

3.2 Transcripts and Notes

3.2.1 Notes: Introduction to Continuous Trading

In this section we introduce some important notation and terminology that we will be using in the theory of trading in continuous time. This section has many parallels to the trading in discrete time section of the Discrete-time Stochastic Processes module.

The standard theory of continuous trading assumes a fixed finite number of primary assets that are sold in the market. We will denote their prices by $(B, S) = \left(B, S^1, \dots, S^d\right)$, where d is a positive integer.

To be precise, we will assume a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume that (B, S) is a $d+1$-dimensional cadlag semimartingale. In order to avoid having to deal with the concepts like σ-martingales later, we will also assume that the components of (B, S) are positive. We will also assume that $\mathcal{F}_0$ contains events of probability zero and one – i.e. it is *trivial*.

The asset B is a riskless bank account while the assets $S = \left(S^1, \dots, S^d\right)$ are all risky assets. We will assume that $B > 0$.

The acronym cadlag is used for processes with right-continuous sample paths having finite left-hand limits at every time instant.

We will work on the interval $[0, T]$, where $T > 0$ is finite. The whole theory can also be studied over the entire half-line $[0, \infty)$, with some minor modifications.

Sticking with similar conventions to discrete-time trading, it is convenient to work with discount assets and use B as a numeraire. To this end, we define the discounted risky assets $X = \left(X^1, \ldots, X^d\right)$ as

$$X \coloneqq \frac{S}{B} \quad \text{i.e.} \quad X^i \coloneqq \frac{S^i}{B}, \quad i = 1,2, \ldots, d.$$

The discounted bank account has value 1 at all times, hence the discounted assets can be represented as $(1, X) = \left(1, X^1, \ldots, X^d\right)$. From now onwards, we work with the discounted assets.

A *trading strategy* is a pair (η, φ), where η is the holding in the riskless asset and $\varphi = \left(\varphi^1, \ldots, \varphi^d\right)$ is the investment in the risky assets, both being stochastic processes. For reasons that will be clear soon, we insist that φ is a predictable X-integrable process. The *value* of a strategy (η, φ) is the stochastic process $V = V\big((\eta, \varphi)\big)$ defined by

$$V_t \coloneqq \eta_t + \varphi_t X_t,$$

where the second product is a dot product of the vectors $\varphi_t = \left(\varphi_t^1, \ldots, \varphi_t^d\right)$ and $X = \left(X_t^1, \ldots, X_t^d\right)$.

For a trading strategy (η, φ), we define the *gains process* $G = G(\varphi)$ as

$$G_0(\varphi) = 0 \quad G_t(\varphi) \coloneqq \int_0^t \varphi_s \, dX_s \ \text{ for } \ t > 0.$$

Note that there are no gains from trading the riskless asset, which is why the gains process only depends on φ and not η.

We say that a strategy $(\eta.\varphi)$ is *self-financing* if

$$V_t\big((\eta,\varphi)\big) = V_0\big((\eta,\varphi)\big) + G_t(\varphi) \text{ for every } t.$$

A self-financing strategy (η,φ) can also be described by specifying the starting value v_0 and the holding in the risky assets φ. Then η can be recovered from this information. Thus, we will frequently refer to (v_0,φ) as a trading strategy instead of (η,φ) when the strategy is self-financing.

Let's look at an example. Take $d = 1, X = X^1 = W$, where W is a Brownian motion process. Let $v_0 = 1$ and $\varphi_t := 2W_t$, then by Ito's Lemma, we have

$$V_t = 1 + \int_0^t 2W_s \, dW_s = 1 + W_t^2 - t.$$

Hence, we have

$$\eta_t = V_t - \varphi_t X_t = 1 + W_t^2 - t - 2W_t^2 = 1 - W_t^2 - t.$$

Of course, this example does not meet our assumption that the asset prices are positive; it is just for illustrative purposes.

3.2.2 Transcript: Portfolios and Strategies

Hi, in this video we introduce portfolios and strategies in continuous time.

We will consider a general setup where we have $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which is a filtered probability space where the filtration, $\mathcal{F}_t$, and increasing σ-algebras are defined between 0 and time T. All the stochastic processes that we will be dealing with will be defined between 0 and some finite time T. Written in full:

$$\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}.$$

Throughout, we will assume that $\mathcal{F}_0$ is trivial in the sense that it consists of only sets or events of probability 0 and 1.

We will then let X, which consists of $(X^1, \dots, X^d)$, be a d-dimensional cadlag semimartingale. This represents the risky primary assets that are tradable in the market.

We will also have a special asset 1, whose value is always equal to 1, and this is the riskless asset or riskless bank account that is constant and equal to 1.

A trading strategy, or just a strategy, is a pair, which we will write as (η, φ), where η is the holding in the riskless asset – in other words, η is the investment in the riskless bank account – and φ is a d-dimensional, predictable and X-integrable process.

We will define the value of the strategy as $V_t\big((\eta,\varphi)\big) := \eta_t(1) + \varphi_t \bullet X_t$, which is the dot product of vectors. This is equal to $\eta_t + \varphi_t X_t$, as if it's a product of two scalars. Note that this is a dot product of two vectors, φ and X_t. So, that's the value of a trading strategy φ.

We will also define the gains from trading the strategy as $G_0(\varphi) = 0$ and $G_t(\varphi)$ defined as the integral from 0 to t of φ_s with respect to X_s. Written in full:

$$G_0(\varphi) = 0, G_t(\varphi) := \int_0^t \varphi_s dX_s.$$

Of course, this is the stochastic integral with respect to X_s. So, this is analogous to the discrete time case where the gains are simply defined as the martingale transform of φ and X. So, that is the gains from trading.

We say that strategy (η,φ) is self-financing if the value of the strategy $V_t\big((\eta,\varphi)\big) = V_0\big((\eta,\varphi)\big)$ plus the gains from trading, $G_t(\varphi)$. Written in full:

$$V_t\big((\eta,\varphi)\big) = V_0\big((\eta,\varphi)\big) + G_t(\varphi).$$

Note that the gains from trading only depend on φ because there are no gains from trading the riskless asset since its value is always equal to 1.

Going back, we say that the strategy is self-financing if the value, $V_t\big((\eta,\varphi)\big)$, is equal to the starting amount, $V_0\big((\eta,\varphi)\big)$, plus the gains from trading the risky asset X.

If $\big((\eta,\varphi)\big)$ is self-financing, then we can reparametrize it using (v_0,φ), where v_0 is equal to the starting capital. So, as we can see in $V_0\big((\eta,\varphi)\big)$, once we know the starting amount, v_0, and we know what φ is, that completely describes the value of the strategy when the strategy is self-financing. We will sometimes use this notation (v_0,φ) when talking about a self-financing strategy, meaning that we don't need to mention what η is, or what the holding in the bank account is, when we use this notation. We can, of course, recover that because we have the equation $\eta_t + \varphi_t X_t$ that involves η and V_t, so we can solve for η once we know what φX_t and V_t are, and we can get that from this equation: $V_t\big((\eta,\varphi)\big) = V_0\big((\eta,\varphi)\big) + G_t(\varphi)$.

Let's look at an example.

Take, for example, an unrealistic world where we only have one stock and this stock price is a Brownian motion. Consider a case where the trading strategy $v_0 = 1$ and $\varphi_t = 2W_t$, which is 2 times the Brownian motion process. Then, the value of the strategy at time t will be equal to 1 plus – it's self-financing, of course – the integral from 0 to t of $2W_s dW_s$. Written in full:

$$X = W, \qquad v_0 = 1, \qquad \varphi_t = 2W_t$$

$$V_t = 1\int_0^t 2W_s dW_s.$$

This can be evaluated by looking at previous videos in the course.

Now that we've introduced portfolios and strategies, in the next video, we are going to look at arbitrage.

3.2.3 Notes: Arbitrage

In this section we introduce equivalent martingale measures and the notion of arbitrage in continuous time.

We continue to work on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the assets $X = (X^1, \dots, X^d)$ where X is a d-dimensional cadlag semimartingale.

A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is called an *equivalent martingale measure* (EMM) for X if X is a $\mathbb{P}^*$ martingale. Likewise, we say that $\mathbb{P}^*$ is an *equivalent local martingale measure* (ELMM) for X if X is a $\mathbb{P}^*$ local martingale.

We will again use these martingale measures to characterize markets without arbitrage and for pricing derivative contracts.

A self-financing strategy (v_0, φ) is *admissible* if there exists $a \in \mathbb{R}$ such that

$$G_t(\varphi) \geq a \text{ for every } t \geq 0.$$

An admissible strategy is one that has a finite credit line, in the sense that the value strategy is not allowed to be arbitrarily large and negative. This is a realistic assumption in the real world since the amount of credit available is indeed finite. From now onwards, we will make the following assumption:

All trading strategies considered are self-financing and admissible.

A strategy (v_0, φ) is an *arbitrage strategy* or *arbitrage opportunity* if $v_0 = 0, V_T \geq 0\mathbb{P}$-a.s., and $\mathbb{P}(V_T > 0) > 0$. We say that the market satisfies the no-arbitrage condition (NA) if there are no arbitrage strategies in the market.

Let us now illustrate the importance of the admissibility condition. Consider the model with $d = 1$ and $X = W$, where W is a Brownian motion. Define

$$\tau \coloneqq \inf\{t \geq 0 : W_t = 1\}.$$

Then the strategy $(0, \varphi)$ where $\varphi_t = I_{\tau \geq \mathrm{t}}$ is an arbitrage strategy, which is absurd since we expect this model to have no arbitrage strategies (Brownian motion is a martingale). However, this strategy does not satisfy the admissibility condition.

Now assume that there exists an ELMM $\mathbb{P}^*$ for X and let $(0, \varphi)$ be an arbitrage strategy. Then since $V = G$ is bounded below, it is a $\mathbb{P}^*$ local martingale (since it is a stochastic integral with respect to a local martingale and is bounded below because φ is admissible). By applying localization, we get that $\mathbb{P}(V_T > 0) = 0$, which is a contradiction since we assumed that φ is an arbitrage strategy. Hence, the existence of an ELMM guarantees that the market is arbitrage free.

Unfortunately, though, unlike the discrete time case, NA is not enough to guarantee the existence of an EMM or ELMM. We have to strengthen the notion of no arbitrage in order to obtain the reverse implication.

We say that a sequence of (admissible and self-financing) strategies (v_0^k, φ^k) admits a *free lunch with vanishing risk* if

1. $v_0^k = 0$
2. $V_T\left((v_0^k, \varphi^k)\right) \geq -\frac{1}{k}$ for every k
3. There exists $\epsilon > 0$ and $\delta > 0$ such that $\mathbb{P}\left(V_T\left((v_0^k, \varphi^k)\right) > \delta\right) > \epsilon$ for each k.

We can think of this sequence of strategies as approximating an arbitrage.

We say that a market satisfies *no free lunch with vanishing risk* (NFLVR) if there are no sequences of strategies as above.

This notion turns out to be the right strengthening of the NA condition. The result is called the *Fundamental Theorem of Asset Pricing I* (FTAP I).

Theorem 2.1 (FTAP I). *For a financial market with non-negative assets, the following are equivalent:*

1. *The market satisfies the NFLVR condition.*
2. *There exists an ELMM for* X.

Let's look at an example. Consider $d = 1$ and $X = X^1$ with

$$dX_t = X_t(\mu\, dt + \sigma\, dW_t), \;\; X_0 = 1,$$

where μ and $\sigma > 0$ are constants. Notice that $dX_t = X_t dY_t$ where $Y_t = \mu t + \sigma W_t$, hence

$$X_t = \mathcal{E}(Y)_t = e^{Y_t - \frac{1}{2}\langle Y \rangle_t} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

We claim that $\mathbb{P}^*$ with

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T}$$

is an ELMM for X. The theorem that confirms this assertion is the *Girsanov Theorem*, which we will cover in great detail in the next module. For now, the following argument – though not a proof – is reassuring:

$$\mathbb{E}^*(X_t) = \mathbb{E}^*\left(e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}\right) = \mathbb{E}\left(e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T}\right) = 1 \quad \forall t$$

by using the moment-generating function of a normal random variable. So, X has a constant $\mathbb{P}^*$ expectation. Although this does not imply that X is a martingale, it is nevertheless reassuring. (Complete the proof that X is a $\mathbb{P}^*$ martingale as an exercise.) So, we can conclude that this model satisfies the NFLVR condition.

Now consider a market with $d = 2$ and $X = (X^1, X^2)$ with

$$dX_t^1 = \sigma\, dW_t, \qquad X^2 \coloneqq (X^1)^2.$$

Then clearly, X^1 and X^2 cannot both be local martingales under the same measure.

3.2.4 Transcript: Arbitrage

Hi, in this video we introduce arbitrage in continuous time.

So, we consider the same setup $\left((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X\right)$, which is a financial market.

A measure, $\mathbb{P}^*$, on $(\Omega, \mathcal{F})$ is called an **Equivalent Local Martingale Measure** (ELMM) for X if $\mathbb{P}^*$ is equivalent to $\mathbb{P}$, which simply means that $\mathbb{P}^*$and $\mathbb{P}$ have the same null sets, and X is a $\mathbb{P}^*$ local martingale.

So, this is in some ways similar to the notion of an Equivalent Martingale Measure (EMM) that we encountered in discrete time. This time, however, we want X to only be a local martingale.

We are also going to restrict the class of strategies that we are dealing with. First, we will deal only with self-financing trading strategies in this section. We say that (v_0, φ) – where v_0 represents the starting capital and φ represents the holding, which is a stochastic process, when the holding is in X – and this is admissible if the gains from trading are greater than or equal to a for some constant a. That's why the self-financing strategy is admissible if the gains from trading are always greater than or equal to a. Written in full:

$$G_t(\varphi) \geq a \text{ for some } a \in \mathbb{R}.$$

Finally, we say that (v_0, φ) is an arbitrage strategy, or arbitrage opportunity, if the following conditions are satisfied:

1. The starting capital is 0. Therefore, $v_0 = 0$.
2. $V_T \geq 0$.
3. The probability that V_T is greater than 0 is positive. Therefore, $P(V_T > 0) > 0$.

This is exactly the same definition that we had in the discrete-time case.

We say that they market satisfies the **no-arbitrage** (NA) condition if there is no such strategy. In other words, there are no arbitrage opportunities.

Now, in discrete time, we had a nice characterization of the NA condition: it was equivalent to the existence of an EMM. In continuous time, however, the situation is much more complicated.

First, we still have this direction of implication:

If there exists an ELMM for X, then the market satisifies the NA condition.

So, we still have that direction, which we had in discrete time, where we had EMM, instead of ELMM, and if we have this then we are guaranteed that there will be no arbitrage opportunities in the market. However, the converse is no longer true. In other words, just by claiming that there are no arbitrage opportunities, it turns out that we cannot always construct an ELMM for X. What we need is to strengthen the NA condition by using a much stronger condition that we call No Free Lunch with Vanishing Risk (NFLVR), and this is further discussed in the notes.

Let's look at an example.

So, let's take X to have the following stochastic differential equation, assuming that we only have one asset here – X with the following SDE: $\mu X_t dt + \sigma\, X_t dW_t$, where W is of course a Brownian motion, and we will define a new probability measure $\mathbb{P}^*$ as follows: $e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T}$. Clearly, this is strictly positive, and therefore these two measures are equivalent: $\mathbb{P}^*$ is equivalent to $\mathbb{P}$. And by Girsanov's theorem, X is a $\mathbb{P}^*$ local martingale. Written in full:

$$dX_t = \mu X_t dt + \sigma\, X_t dW_t$$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} > 0$$

$$\mathbb{P}^* \equiv \mathbb{P}.$$

Now that we have introduced the notion of arbitrage in continuous time, in the next video we are going to introduce pricing.

3.2.5 Notes: Pricing and Completeness

Let $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ be a financial market. We now consider the pricing of *contingent claims* H, which are simply $\mathcal{F}$ –measurable random variables.

Let H be a contingent claim. A strategy (v_0, φ) *replicates* H (or is a *replicating strategy* for H) if $V_T\big((v_0, \varphi)\big) = H$ $\mathbb{P}$-a.s.. If such a replicating strategy exists, we say that H is *replicable* or *attainable*.

Now assume that the market satisfies the NFLVR condition and let $\mathbb{P}^*$ be an ELMM for X. Let H be a bounded and attainable contingent claim with replicating portfolio (v_0, φ). Then

$$\mathbb{E}^*(H) = v_0,$$

since $G(\varphi)$ is a local martingale. Hence, the starting capital v_0 is the same for every replicating portfolio. We will call this the *price* of H.

The market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ is *complete* if every bounded random variable H is attainable.

A (d-dimensional) stochastic process X is said to have the *predictable representation property* (PRP) with respect to a measure $\mathbb{P}^*$ if every $\mathbb{P}^*$ local martingale M can be written as:

$$M_t = M_0 + \int_0^t \varphi_s \, dX_s,$$

for some predictable X-integrable process φ.

With all the definitions done, we can now state the second fundamental theorem, which links completeness to the uniqueness of an ELMM and the PRP property of X. This theorem is called *The Fundamental Theorem of Asset Pricing II* (FTAP II).

Theorem 3.1 (FTAP II). *Let* $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ *be a market whose components are non-negative and assume that NFLVR holds. Then the following are equivalent:*

1. *The market is complete.*
2. *There is only one ELMM.*
3. X *satisfies PRP with respect to at least one ELMM for* X.

Let us look at a simple example. Consider $d = 1$ and $X = X^1$ with

$$dX_t = \sigma X_t \, dW_t, \quad X_0 = 1, \quad \sigma > 0,$$

where W is a Brownian motion process. We pick the filtration $\mathbb{F}$ to be the natural filtration of W and $\mathcal{F} = \mathcal{F}_T$. Here $\mathbb{P}$ itself is an ELMM since X is a local martingale under $\mathbb{P}$. By the Martingale Representation Theorem, any local martingale M can be written as

$$M_t = M_0 + \int_0^t \varphi_s \, dW_s$$

for some predictable process φ. Using properties of stochastic integration, we see that M can also be written as

$$M_t = M_0 + \int_0^t \psi_s \, dX_s,$$

for some predictable X-integrable process ψ (you can check that $\psi = \varphi/(\sigma X)$). So X satisfies PRP with respect to $\mathbb{P}$, which implies that the market is complete.

Continuing with the previous example, we consider the contingent claim $H = I_{\{X_T > K\}}$ for some constant $K > 0$. Then the price of H is $\mathbb{E}(H) = \mathbb{E}\big(I_{\{X_T > K\}}\big) = \mathbb{P}(X_T > K)$. Now, solving the SDE for X we get

$$X_T = e^{-\frac{1}{2}\sigma^2 T + \sigma W_T},$$

so

$$\mathbb{P}(X_T > K) = \mathbb{P}\left(\sigma W_T > \ln K + \frac{1}{2}\sigma^2 T\right) = 1 - \Phi\left(\frac{\ln K + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right),$$

where Φ is the CDF of a standard normal random variable.

We now show that in this market, every contingent claim in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ can be replicated (not just the bounded ones). Let H be such a contingent claim and define the martingale $M_t \coloneqq \mathbb{E}(H|\mathcal{F}_t\}$ for every t. Since X satisfies PRP, we can find a predictable X-integrable process φ such that

$$H = M_T = M_0 + \int_0^T \varphi_s \, dX_s = \mathbb{E}(H) + \int_0^T \varphi_s \, dX_s.$$

Hence $(\mathbb{E}(H), \varphi)$ is the required replicating portfolio.

The above argument implies that we can replicate and price contingent claims such as call options $(H = (X_T - K)^+)$ and put options $(H = (K - X_T)^+)$. As an exercise, find the prices of these claims using this model.

3.2.6 Transcript: Pricing

Hi, in this video, we introduce the pricing of derivatives in general semimartingale models.

So, first, we will start with a definition: a contingent claim H is attainable if there exists a trading strategy, (v_0, φ), which consists of a starting amount and a trading in the risky asset φ, such that the value of the trading strategy at time T is equal to H almost surely. So, that's what it means for a contingent claim or derivative H to be attainable. In that case, if, in addition, the market has no arbitrage, if NFLVR is satisfied – which of course implies that the set of ELMMs is non-empty, which is the *Fundamental Theorem of Asset Pricing Part I* – then V_0, the starting value of this trading strategy, is equal to the expected value under any ELMM of the derivative H. Written in full:

$$V_0(v_0, \varphi) = v_0 = E^*(H).$$

We can therefore think of $E^*(H)$ as the price of H when H is attainable.

We say that a market is complete if every bounded contingent claim is attainable.

Let's look at an example.

Let's take the same model that we took in the last video. We will assume that the probability space contains a Brownian motion and the stock evolves according to the following stochastic differential equation: $\mu X_t dt + \sigma X_t dW_t$, where dW_t is a Brownian motion. We saw in the last video that the unique EMM $\mathbb{P}^*$ in this market is given by the Radon-Nikodym derivative that looks like this:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T}.$$

Now, we also, again, have a *Fundamental Theorem of Asset Pricing Part II*, which equates completeness with the uniqueness of the ELMM.

By Girsanov's theorem, we know that that the stochastic differential of X_t can be written as: $dX_t = \sigma X_t d\widetilde{W}_t$, where $\widetilde{W}$ is a P^* Brownian motion.

Therefore, to price any derivative of the following form – for instance, if we take H to be equal to X_T^2, which is also called the power option – to calculate its price, we have to evaluate the expected value under $\mathbb{P}^*$ of H, which is the expected value under $\mathbb{P}^*$ of X^2. And that is equal to (when we solve the stochastic differential equation) $E^*\left(e^{-\frac{1}{2}\sigma^{2T}+\sigma\widetilde{W}_T}\right)$. We then have to multiply this by X_0 and take everything to the power of 2. This is then equal to ${X_0}^2$ times the expected value of $\left(e^{-\sigma^2 T+2\sigma\widetilde{W}_T}\right)$, which is equal to ${X_0}^2 e^{-\sigma^{2T}}$ times the expected value of $\left(e^{2\sigma\widetilde{W}T}\right)$, which can be evaluated, using the moment-generating function of a normal – so $\widetilde{W}_T$ under $\mathbb{P}^*$ has a normal distribution with mean 0 and variance T – we get that this is equal to $e^{\frac{1}{2}(2\sigma)^2 T}$.

Written in full:

$$H = {X_T}^2,\ E^*(H) = E^*(X_T^2)$$

$$= E^*\left(\left(X_0 e^{-\frac{1}{2}\sigma^2 T + \sigma \widetilde{W}_T}\right)^2\right) = X_0{}^2 E^*\left(e^{-\sigma^2 T + 2\sigma \widetilde{W}_T}\right)$$

$$= X_0{}^2 e^{-\sigma^2 T} E^*\left(e^{2\sigma \widetilde{W}_T}\right) = X_0{}^2 e^{\sigma 2T}.$$

That's the price of the derivative at time 0 – without discounting, of course. We will see how to use discounting in the next module.

Now that we have covered pricing, we are going to move on to the next module, which covers the Black-Scholes Model.

3.2.7 Notes: Problem Set

Problem 1

Consider a market where X satisfies the following SDE:

$$dX_t = X_t(0.04dt + 0.25\, dW_t), \qquad X_0 = 1.$$

Consider the measure $\mathbb{P}^*$ with

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \mathcal{E}(\alpha W)_1.$$

If $\mathbb{P}^*$ is an ELMM, then what is α?

Solution:

Following the lecture notes, if we have a process following a general SDE,

$$dX_t = X_t(\mu dt + \sigma dW_t).$$

We know that,

$$\frac{dP^*}{dP} = e^{\left(-\frac{\mu}{\sigma}W - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right)}$$

is an ELMM for X. Thus, as we already know that the exponential process given in the question is equal to:

$$\frac{dP^*}{dP} = \varepsilon(\alpha W) = e^{\left(\alpha W - \frac{1}{2}\alpha^2 t\right)}.$$

Finally, if we compare the above equations, we get that alpha must be equal to $-\frac{\mu}{\sigma}$. Taking into account that $\mu = 0.04$ and $\sigma = 0.25$, α is equal to -0.16.

Problem 2

Consider a market with $X = W$, where W is a Brownian motion process. Define the trading strategy (v_0, φ) by

$$v_0 = 1, \qquad \varphi_t \coloneqq 2W_t.$$

Find $G_T(\varphi)$.

Solution:

Following the lecture notes, the gain process is defined as:

$$G_T = \int_0^T \varphi_s dX_s = \int_0^T 2\, W_s dW_s = 2\int_0^T W_s dW_s.$$

Thus, the problem is reduced to find the value of the integral, $2\int_0^T W_s dW_s$. For this, we first need to apply Ito's Lemma to the process W_t^2,

$$dW_t^2 = 2W_t dW_t + dt.$$

If we take integrals in both sides,

$$W_T^2 = 2\int_0^T W_s\, dW_s + \int_0^T ds\,.$$

The first integral in the right side is the one that we are looking for. Thus,

$$2\int_0^T W_s dW_s = W_T^2 - T,$$

which is already the solution of our problem.

Problem 3

Consider a market with $X = W$, where W is a Brownian motion process. Consider the trading strategy (v_0, φ) with value

$$V_t = 1 + W_t^2 - t, \qquad 0 \leq t \leq T.$$

Then what is φ_t equal to?

Solution:

Applying the Ito formula to the function $V_t = 1 + X_t^2 - t$:

$$dV_t \;=\; df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X}\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} d[X]_t,$$

with the following derivatives:

$$\frac{\partial f}{\partial t} = -1, \qquad \frac{\partial f}{\partial X} = 2X_t \;\; and \;\; \frac{\partial^2 f}{\partial X^2} = 2.$$

Thus,

$$dV_t = -dt + 2X_t dX_t + \frac{1}{2} 2dt = 2X_t dX_t.$$

So, as we know from the lecture notes,

$$G_t(\varphi) \coloneqq \int_0^t \varphi_s \;\; dX_s \quad \text{for } t > 0,$$

and also

$$V_t\big((\eta,\varphi)\big) = V_0\big((\eta,\varphi)\big) + G_t(\varphi) \text{ for every } t.$$

Finally, we conclude that

$$\varphi_t = 2X_t = 2W_t.$$