Continuous-time Stochastic Processes
Module 5

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 5 of Continuous-time Stochastic Processes, entitled The Black-Scholes Model. It consists of four lecture video transcripts and five sets of supplementary notes.

2. Course Context

Continuous-time Stochastic Processes is the fourth course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The aim of the course is to introduce derivative pricing when trading happens in continuous-time. It begins by focusing on the stochastic calculus of Brownian motion and its generalization to continuous semimartingales. These ideas are then applied to continuous trading and the pricing of financial derivatives.

2.1 Course-level Learning Outcomes

Upon completion of the Continuous-time Stochastic Processes course, you will be able to:

1. Define and identify Brownian motion processes in multiple dimensions.
2. Solve stochastic differential equations.
3. Apply Ito's Lemma for continuous semimartingales.
4. Apply Girsanov's theorem to construct equivalent local martingale measures.
5. Price and hedge derivatives in various asset price models.
6. Derive the Black-Scholes partial differential equation.
7. Construct asset prices models based on Levy processes.
8. Price interest rate derivatives.

2.2 Module Breakdown

The Continuous-time Stochastic Processes course consists of the following one-week modules:

1. Brownian Motion and Continuous-time Martingales
2. Stochastic Calculus I: Ito Process
3. Stochastic Calculus II: Semimartingales
4. Continuous Trading
5. The Black-Scholes Model
6. An Introduction to Levy Processes
7. An Introduction to Interest Rate Modeling

3. Module 5:

The Black-Scholes Model

In this module, we introduce an important example of a complete market model: the Black-Scholes Model. First, we consider the model with one risky asset and a riskless bank account that grows at a constant, continuously compounded rate of interest. We then show how to price both vanilla and exotic derivatives in this simple model, as well as how to derive the celebrated Black-Scholes equation, before concluding with the model's multi-asset generalization.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. State the stock price dynamics of the classical Black-Scholes model.
2. Construct local equivalent martingale measures in Black-Scholes-type models.
3. Derive the Black-Scholes partial differential equation.
4. Price options in the generalized Black-Scholes model.

3.2.1 Notes: An Introduction to the Black-Scholes Model

The Black-Scholes Model is an important example of a complete market model. We will begin with the simple model consisting of one stock S and a riskless bank account B. We will define all stochastic processes on a fixed time horizon $[0,T]$, where $T > 0$.

First, fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a Brownian motion W. We assume that $\mathbb{F} = \mathbb{F}^W$ and $\mathcal{F} = \mathcal{F}_T$. In this model, the stochastic processes S and B satisfy the following SDEs:

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \qquad dB_t = rB_t dt,$$

where $\mu \in \mathbb{R},\ \sigma > 0$ and $r > 0$ are constants. Here r is the constant continuously compounded risk-free rate.

Solving both SDEs, we obtain:

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}, \;\; B_t = B_0 e^{rt}.$$

We now turn to the discounted assets $(1, X)$, where $X = S/B$. By Ito's Lemma, the SDE for X is

$$dX_t = X_t\big((\mu - r)\, dt + \sigma dW_t\big).$$

To find an ELMM for this model, we will need a powerful result known as Girsanov's theorem. To motivate this result, let Z be a standard normal random variable $(Z \sim N(0,1))$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and μ be a real number. Define a new probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}^*} := e^{\mu Z - \frac{1}{2}\mu^2}.$$

Then $\mathbb{P}$ is a probability measure that is equivalent to $\mathbb{P}$. Furthermore, the moment-generating function of Z under $\mathbb{P}^*$ is given by

$$\mathbb{E}^*(e^{\alpha Z}) = \mathbb{E}\left(e^{\alpha Z} e^{\mu Z - \frac{1}{2}\mu^2}\right) = e^{-\frac{1}{2}\mu^2}\mathbb{E}\left(e^{(\alpha+\mu)Z}\right) = e^{\alpha\mu + \frac{1}{2}\alpha^2},$$

which is the moment-generating function of a $N(\mu, 1)$ random variable. Thus, by choosing μ we can create a probability measure that “shifts" the mean of Z (from 0 to μ) but keeps the variance the same.

Girsanov's Theorem does something similar with Brownian motions. We will let $\|x\|$ denote the Euclidean norm of a vector x in $\mathbb{R}^d$.

Theorem 1.1 (Girsanov's Theorem) *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered space and* $W = W^1, \dots, W^d)$ *be a* d*-dimensional Brownian motion on this space. Let* $\gamma = (\gamma_1, \dots, \gamma_d)$ *be a vector of progressive processes satisfying*

$$\mathbb{P}\left(\int_0^T \left(\gamma_s^i\right)^2 ds < \infty\right) = 1, \quad i = 1, \dots, d.$$

Define the continuous local martingale Z *by*

$$Z_t := \exp\left(\sum_{i=1}^{d}\int_0^t \gamma_s^i dW_s^i - \frac{1}{2}\int_0^t \|\gamma_s\|^2 ds\right).$$

Assume that Z *is a martingale and define a new probability measure* $\mathbb{P}^*$ *by*

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z_T.$$

Then the process $\widetilde{W} = \widetilde{W}^1, \ldots, \widetilde{W}^d)$ *defined by*

$$\widetilde{W}_t^i := W_t^i - \int_0^t \gamma_s^i ds$$

is a d *-dimensional Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^*)$.

We can now apply Girsanov's theorem to find an ELMM for X above. Let γ be a one-dimensional process such that

$$\mathbb{P}\left(\int_0^T \gamma_s^2\, ds < \infty\right) = 1.$$

Define Z by

$$Z_t := \exp\left(\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right).$$

Then $Z = \{Z_t : 0 \leq t \leq T\}$ is a UI martingale with $\mathbb{E}(Z_t) = 1$ for every $0 \leq t \leq T$ and we can therefore define a measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = Z_T.$$

Then the process $\widetilde{W}$ defined by

$$\widetilde{W}_t \coloneqq W_t - \int_0^t \gamma_s ds$$

is a Brownian motion with respect to $\mathbb{P}^*$. Now we can rewrite the SDE for X in terms of $\widetilde{W}$ as

$$dX_t = X_t\big((\mu - r)\,dt + \sigma\,dW_t\big) = X_t\left((\mu - r)dt + \sigma d\widetilde{W}_t + \sigma\gamma_t dt\right)$$
$$= X_t\left((\mu - r + \sigma\gamma_t)dt + \sigma d\widetilde{W}_t\right).$$

Hence, for X to be a $\mathbb{P}^*$ local martingale, we need to choose

$$\gamma_t = -\frac{\mu - r}{\sigma}.$$

The quantity $\frac{(\mu-r)}{\sigma}$ is often called the **market price of risk.**

So, let us pick $\gamma_t = -\frac{\mu-r}{\sigma}$. Then the SDE for X is

$$dX_t = X_t\sigma\,d\widetilde{W}_t,$$

a local martingale under $\mathbb{P}^*$. (In fact, X is a true martingale, so $\mathbb{P}^*$ is a martingale measure.) We can therefore conclude that this model satisfies NFLVR by the first fundamental theorem of asset pricing (FTAP I).

Now we move on to uniqueness of the measure. By the martingale representation of Brownian motion, we observe that X satisfies PRP with respect to $\mathbb{P}^*$, hence the market is complete and the ELMM measure is unique.

Using these results, we can then price any contingent claim H using the formula

$$\pi(H) = \mathbb{E}^*(e^{-rT}H).$$

We will go through some examples in the next sections.

3.2.2 Transcript: Girsanov's Theorem

Hi, in this video we introduce the Black-Scholes model and illustrate how to apply Girsanov's theorem to finding an ELMM in this model.

So, the one-dimensional Black-Scholes model says that the undiscounted stock price, S, evolves according to the following stochastic differential equation:

$$dS_t = S_t(Mdt + \sigma dW_t).$$

If we take the discounted stock price, which is $\frac{S_t}{e^{rt}}$, where r is the constant risk-free rate, then the stochastic differential of X_t will be equal to $X_t((\mu - r)\,dt + \sigma dW_t)$. So, we are just removing r from the drift itself. W is the Brownian motion and we will assume that the probability space contains this Brownian motion, W. Written in full:

$$dS_t = S_t(Mdt + \sigma dW_t)$$

$$X_t = \frac{S_t}{B_t} = \frac{S_t}{e^{rt}}$$

$$dX_t = X_t\big((\mu - r)dt + \sigma\, dW_t\big).$$

Now, we want to find a ELMM for this model and we are going to apply Girsanov's theorem to do that.

Girsanov's theorem says that if we have a new probability measure, let's call it $\mathbb{P}^*$, that is equivalent to $\mathbb{P}$ and has the following Radon-Nikodym derivative, or density with respect to $\mathbb{P}$, $e^{-\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds}$, then, this new stochastic process, $\widetilde{W}_t$, which equals W_t plus $\int_0^t \theta_s ds$, is a $\mathbb{P}^*$ Brownian motion. Written in full:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\int_0^T \theta_S dW_S - \frac{1}{2}\int_O^T \theta_S^2 ds}$$

$$\widetilde{W}_t = W_t + \int_0^t \theta_s ds.$$

So, W itself, the original Brownian motion, need not be a Brownian motion under this new probability measure, $\mathbb{P}^*$, but Girsanov's theorem gives us a way of transforming the old Brownian motion into a new Brownian motion, $\widetilde{W}_t$.

Now, we want to try and choose this stochastic process, θ, such that this new measure that we get, $\mathbb{P}^*$, makes the discounted stock price, X, a local martingale. So, in other words, it does not have any drift in its term.

So, let's express this Brownian motion, W, in terms of $\widetilde{W}_t$. We will get:

$$dX_t = X_t\left((\mu - r)dt + \sigma\left(d\widetilde{W}_t - \theta_t dt\right)\right),$$

which simplifies to

$$X_t\left((\mu - r - \sigma\theta_t)dt + \sigma d\widetilde{W}_t\right).$$

We want to choose θ_t so that the equation above is driftless and it is clear that we have to make sure that the above is 0. So, we need $\mu - r - \sigma\theta_t = 0$, which implies that θ_t should be equal to $\frac{\mu-r}{\sigma}$. In other words, it's independent of t itself.

Therefore, the Radon-Nikodym derivative corresponding to any ELMM is given by:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T},$$

after evaluating the corresponding integral. So, that is the Radon-Nikodym derivative and, with that probability measure, the discounted stock price is a local martingale. Hence, this is an ELMM. As we can see, it is unique — there is no other value of θ_t that will make this a local martingale.

Now that we have looked at Girsanov's theorem, in the next video, we are going to price a call option in the Black-Scholes model.

3.2.3 Notes: Pricing Options

In this section we illustrate how to price a call option and a put option using the Black-Scholes model.

First, we consider a call option on S with a strike price K. This is a derivative whose payoff is

$$H = (S_T - K)^+.$$

Under the ELMM $\mathbb{P}^*$, the dynamics of X are

$$dX_t = X_t \sigma \, d\widetilde{W}_t,$$

where $\widetilde{W}$ is a $\mathbb{P}^*$-Brownian motion. Hence the dynamics of S are

$$dS_t = S_t\big(r \, dt + \sigma \, d\widetilde{W}_t\big).$$

Solving this yields

$$S_t = S_0 e \left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{W}_t \ .$$

We now calculate the price of H.

$$\begin{aligned}\pi(H) &= \mathbb{E}^*(e^{-rT}\,(S_T - K)^+) = e^{-rT}\mathbb{E}^*\,((S_T - K)^+) = e^{-rT}\mathbb{E}^*\left(\left(S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma \widetilde{W}_T} - K\right)^+\right)\\ &= e^{-rT}\int_{-\infty}^{\infty}\left(S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z} - K\right)^+ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, dz,\end{aligned}$$

where we have used the fact that $\mathbb{P}^*{}_{\widetilde{W}_T} = \mathbb{P}^*_{\sqrt{T}z}$, where $Z \sim N(0,1)$. To evaluate this integral, we note that the integrand is only non-zero when

$$S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z} > K,$$

which is equivalent to

$$z = \frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} := -d_2.$$

Hence,

$$\begin{aligned}\pi(H) &= e^{-rT}\int_{-d_2}^{\infty}\left(S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z} - K\right)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, dz \\ &= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)\end{aligned}$$

after completing the square, where

$$d_1 \coloneqq d_2 + \sigma\sqrt{T}$$

and Φ is (as usual) the standard normal CDF.

In general, if $0 \leq t \leq T$, then the price of H at time t is

$$\mathbb{E}^*\left(e^{-r(T-t)}(S_T - K)^+ \middle| \mathcal{F}_t\right) = S_t\Phi\big(d_1(t)\big) - Ke^{-r(T-t)}\Phi\big(d_2(t)\big),$$

where

$$d_1(t) = \frac{\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2(t) = d_1(t) - \sigma\sqrt{T-t}.$$

Note that $d_i(0) = d_i$ for $i = 1,2$.

We now price a put option. For that, note that

$$(S_T - K)^+ - (K - S_T)^+ = \begin{cases} S_T - K & S_T > K \\ S_T - K & S_T \leq K \end{cases} = S_T - K.$$

Therefore, the put option price (πp) is related to the call option price (πC) by the equation

$$\pi_C - \pi_P = \mathbb{E}^*\big(e^{-rT}(S_T - K)\big) = S_0 - Ke^{-rT}.$$

This relationship is called the *put-call parity*.

Now let us consider some non-vanilla options. First we price a digital (binary) option that pays 1 if the terminal stock price value exceeds a pre-specified threshold $K > 0$ and 0 otherwise. This option is sometimes called a *cash-or-nothing call* and has a payoff H_C that can be represented as

$$H_C = I_{\{S_T > K\}}.$$

The price of this derivative is given by

$$\mathbb{E}^*\big(e^{-rT}\, I_{\{S_T>K\}}\big) = e^{-rT}\mathbb{P}^*\,(S_T > K) = e^{-rT}\big(1 - \Phi(-d_2)\big) = e^{-rT}\Phi(\mathrm{d}_2).$$

Now we consider an *asset-or-nothing call*, whose payoff is

$$H_A = S_T I_{\{S_T>K\}}.$$

This is a derivative whose payoff at maturity is equal to the value of the stock price (S_T) if the stock price is greater than K and zero otherwise. Its price is given by

$$\mathbb{E}^*(e^{-rT}\, H_A) = e^{-rT}\,\mathbb{E}^*\left(S_T I_{\{S_T>K\}}\right) = S_0\Phi(d_1),$$

through a similar calculation to the call option price.

Note that the payoff of a call option H is related to the digital options above by

$$H = H_A - KH_C,$$

hence the relationship between the prices too.

Finally, we mention that there are put option equivalents of these. Define them yourself and find their prices.

3.2.4 Transcript: Pricing a Call Option

Hi, in this video we derive the price of a call option in the Black-Scholes model.

Consider a European call option whose payoff, H, is given by $(S_T - K)^+$, so the positive part of $S_T - K$.

We are assuming the one-dimensional version of the Black-Scholes model, in the sense that the stock price, under the real-world probability measure, evolves according to the following stochastic differential equation when W is a Brownian motion:

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

In the last video, we showed how to find an ELMM, and, in that case, the stock price will have the following stochastic differential equation:

$$dS_t = S_t(rdt + \sigma d\widetilde{W}_t)$$

where, $\widetilde{W}$ is a Brownian motion under the ELMM $\mathbb{P}^*$.

To calculate the price of H, we have to find the expected value under $\mathbb{P}^*$of the discounted value of H, which is shown by:

$$E^* = \left(e^{-rT}\left(S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\widetilde{W}_T} - K\right)^+\right).$$

Now, the above is an expectation and the only random variable is $\widetilde{W}_T$, which has a normal distribution with mean 0 and variance T. So, we can express this in terms of a standard normal random variable, which we write as:

$$e^{-rT}E^*\left(\left(S_0e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z}-K\right)^+\right)$$

where Z has a standard normal distribution.

This can be rewritten as:

$$e^{-rT}\int_{-\infty}^{\infty}(S_0e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z}-K)^+,$$

which we multiply by the density of z, (z is standard normal), so, written in full, that will give us:

$$e^{-rT}\int_{-\infty}^{\infty}(S_0e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z}-K)^+\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.$$

To evaluate this expectation, we must just find the region where this is positive, because this is 0 when K is greater than this quantity here, and evaluate the integral over that region, which, as shown in the notes, gives us:

$$S_0\emptyset(d_1)-e^{rT}K\emptyset(d_2).$$

In the above equation:

- $\emptyset$ is the CDF of a standard normal,
- $d_1=\frac{In\left(\frac{S_0}{K}\right)+\left(r+\frac{1}{2}^{\sigma^2}\right)T}{\sigma\sqrt{T}}$, and
- $d_2=d_1-\sigma\sqrt{T}.$

So, that's the expression for d_1 and d_2, which gives us the price of the call option at time 0.

Now that we have priced a call option, in the next video, we are going to look at how to price a digital option.

3.2.5 Notes: Hedging

In this section we derive the hedging formula in the Black-Scholes model and also derive the Black-Scholes PDE.

Consider an option whose payoff H is of the form $H = h(S_T)$ for some Borel measurable function $h: \mathbb{R} \to \mathbb{R}$. We want to find a trading strategy (v_0, φ) such that $V_T\,((v_0; \varphi)) \;=\; H$.

We will first find a trading strategy in the discounted assets $(1, X)$ that replicates the discounted derivative $\widetilde{H} := e^{-rT}H = e^{-rT}h(e^{rT}X_T) = g(X_T)$ and then show that the same strategy applied to the original assets (B, S) replicates H.

Consider the martingale $M \;= \{M_t :\; 0 \leq t \leq T\}$ defined by $M_t := \mathbb{E}^*(g(X_T)|\mathcal{F}_t)$. Since X is a Markov process, we have

$$M_t = \mathbb{E}^*(g(X_T)|\mathcal{F}_t) = \mathbb{E}^*(g(X_T)|X_t),$$

hence by the Doob-Dynkin theorem, there exists a function $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$M_t = F(t, X_t).$$

By Ito's Lemma, we have

$$dM_t = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x}\, dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\, d\langle X\rangle_t = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 F}{\partial x^2}\right) dt + \sigma X_t \frac{\partial F}{\partial x} dW_t,$$

hence

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 F}{\partial x^2} = 0$$

since M is a martingale.

Thus, since $M_T = F(T, X_T) = g(X_T) = \widetilde{H}$, we have

$$\widetilde{H} = \mathbb{E}^*(\widetilde{H}) + \int_0^T \frac{\partial F}{\partial x} dX_t,$$

which means that $\left(\mathbb{E}^*(\widetilde{H}), \varphi\right)$ where

$$\varphi_t \coloneqq \frac{\partial F}{\partial x}(t, X_t)$$

is a replicating strategy for $\widetilde{H}$. The corresponding holding in the riskless asset B is

$$\eta_t = F(t, X_t) - \varphi_t X_t.$$

Now we show that the same strategy – applied to (B, S) instead of $(1, X)$ – replicates H as well. Let V be the value of this strategy. Then

$$V_T = \eta_T B_T + \varphi_T S_T = (F(T, X_T) - \varphi_T X_T)\, B_T + \varphi_T S_T = B_T F(T, X_T) = \widetilde{H} B_T = H.$$

It also follows that $e^{-rt} V_t = F(t, X_t)$, hence $V_t = e^{rt} F(t, X_t) = e^{rt} F(t, S_t e^{-rt}) =: V(t, S_t)$ for some function $V: [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Using the chain rule we see that

$$\frac{\partial F}{\partial X_t} = \frac{\partial V}{\partial S_t}.$$

Hence, since F satisfies the PDE

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 F}{\partial X_t^2} = 0,$$

then $V(t, S_t)$ satisfies the PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + rS_t \frac{\partial V}{\partial S_t} - rV(t, S_t) = 0,$$

together with the boundary condition $V(T, S_T) = h(S_T)$. This equation is the celebrated *Black-Scholes* PDE.

Let H be a call option with strike K. We know that for this derivative,

$$V(t, S_T) = S_t \Phi\big(d_1(t)\big) - Ke^{-r(T-t)}\Phi\big(d_2(t)\big),$$

where

$$d_1(t) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} \text{ and } d_2(t) = d_{1(t)} - \sigma\sqrt{T - t}\,.$$

The hedging strategy φ is given by

$$\varphi_t = \frac{\partial V}{\partial S_t} = \Phi\big(d_1(t)\big).$$

Check this as an exercise.

3.2.6 Transcript: Pricing a Digital Option

Hi, in this video we introduce a digital option and show how to price it in the Black-Scholes model.

A digital option is a derivative whose payoff is of the following form: $H = I_{\{S_T > K\}}$. So it pays the value 1 if S_T is greater than K and 0 otherwise. This is the indicator.

We can draw it in a diagram, where the x-axis is S_T and the y-axis is H. When S_T is less than K, the option pays the value 0 and, as soon as S_T is greater than K, it pays the value 1. In comparison to a call option on the other hand, it pays the value 0 when S_T is less than K and it pays S_T minus K when S_T is greater than K.

To find the price of H, we have to find the expected value under a risk-neutral measure of the discounted derivative H; or the discounted payoff of H, which is equal to:

$$E^*(e^{-rT}H) = e^{-rT}E^*(H).$$

We can simplify this to: $e^{-rT}E^*(I_{\{S_T>K\}})$ under the risk-neutral measure.

Note that the expected value of an indicator is just the probability of the set itself. So, the risk-neutral probability when S_T is greater than K is equal to:

$$e^{-rT}\,\mathbb{P}^*(S_T > K).$$

Now, S_T is equal to S_0 under the risk-neutral measure, as follows:

$$S_T = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\widetilde{W}_T}$$

Therefore, if you substitute that, we get:

$$S_T > K \Leftrightarrow S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\widetilde{W}_T} > K,$$

which is equal to, taking the second part of the equation, dividing by S_0 and taking the logarithms,

$$\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\widetilde{W}_T > In\left(\frac{K}{S_0}\right).$$

This can be further simplified to:

$$\sigma\widetilde{W}_T > In\left(\frac{K}{S_0}\right) - \left(r-\frac{1}{2}\sigma^2\right)T.$$

We then divide by σ root T, because this is a normal random variable. So, if we replace this with this, we get:

$$\mathbb{P}^*(S_T > K) = \mathbb{P}^*\left(Z > \frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right),$$

because of the standard deviation of the beginning point.

This is equal to:

$$1 - \mathbb{P}^*\left(Z < \frac{In\left(\frac{K}{S_0}\right) - \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

And, using the properties of the standard normal distribution, we see that this is equal to:

$$1 - \emptyset(-d_2) = \emptyset(d_2),$$

where d_2 was calculated in the previous video.

Now that we have looked at the price of a digital option, in the next video we're going to look at the general Black-Scholes model.

3.2.7 Notes: Generalized Black-Scholes

We now extend the Black-Scholes model to multiple assets.

Consider $d > 1$ assets $S = (S^1, \dots, S^d)$ whose prices evolve according to the following SDEs:

$$dS_t^i = S_t^i \left(\mu_i \, dt + \sum_{j=1}^{m} \sigma_{ij} dW_t^j \right) \quad i = 1, \dots, d,$$

where μ_i, σ_{ij} are constants and $W = (W^1, \dots, W^m)$ is an m-dimensional Brownian motion process ($\langle W^i, W^j \rangle_t = \delta_{ij} t$).

These SDEs can also be written succinctly as

$$dS_t = S_t(\mu \, dt + \sigma \, dW_t)$$

for appropriately defined matrices μ and σ.

We will assume that $\mathbb{F} = \mathbb{F}^W$ and $\mathcal{F} = \mathcal{F}_T$.

We now attempt to find an ELMM in this model by applying Girsanov's theorem. Let $\gamma = (\gamma^1, \dots, \gamma^m)$ be a W-integrable vector process so that the positive local martingale $\varepsilon\big((\gamma \bullet W)\big)$is a true martingale. Define the measure $\mathbb{P}^*$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} := \varepsilon\big((\gamma \bullet W)\big)_T = \exp\left(\sum_{i=1}^{m} \int_0^T \gamma_s^i \, dW_s^i - \frac{1}{2} \int_0^T \|\gamma_s\|^2 \, ds \right).$$

Then $\widetilde{W} = (\widetilde{W}^1, \dots, \widetilde{W}^m)$ is an m-dimensional Brownian motion with respect to $\mathbb{P}^*$, where

$$\widetilde{W}_t^i := W_t^i - \int_0^t \gamma_s^i \, ds, \quad i = 1, \dots, m.$$

Substituting these equations to the SDEs for the discounted asset $X_t = e^{-rt} S_t$, we get

$$dX_t^i = X_t^i \left((\mu_i - r) + \sum_{j=1}^{m} \sigma_{ij} dW_t^j \right) = S_t^i \left(\left(\mu_i - r + \sum_{j=1}^{m} \gamma_t^j \sigma_{ij} \right) dt + \sum_{j=1}^{m} \sigma_{ij} d\widetilde{W}_t^j \right).$$

So, for $\mathbb{P}^*$ to be an ELMM, we need to choose γ so that

$$\mu_i - r + \sum_{j=1}^{m} \gamma_t^j \sigma_{ij} = 0 \qquad i = 1, \dots, d.$$

This system of equations can have no solutions, a unique solution, or infinitely many solutions, depending on the relationship between m, d, r, σ_{ij}, and μ_i.

Let us look at a concrete example. Consider two stocks S and U whose prices have the following dynamics:

$$dS_t = S_t \left(\mu_S dt + \sigma_S \left(\rho dW_t^1 + \sqrt{1 - \rho^2} \, dW_t^2 \right) \right)$$

and

$$dU_t = U_t (\mu_U \, dt + \sigma_U \, dW_t^1),$$

where W^1 and W^2 are independent Brownian motion processes and $\mu_S, \mu_U, \sigma_S, \sigma_U, \rho$ are all constants with $|\rho| < 1$. Define a new process W^3 by

$$W_t^3 := \int_0^t \rho dW_s^1 + \int_0^t \sqrt{1-\rho^2}\, dW_s^2 = \rho W_t^1 + \sqrt{1-\rho^2}\, W_t^2.$$

Then W^3 is also a Brownian motion process, with $\langle W^3, W^1\rangle_t = \rho t$, and the pair of SDEs can be rewritten as

$$dS_t = S_t(\mu_S\, dt + \sigma_S\, dW_t^3) \text{ and } dU_t = U_t(\mu_U\, dt + \sigma_U\, dW_t^1).$$

To find an ELMM for (S, U), we have to solve the following system of equations

$$\mu_S - r + \gamma_t^1 \sigma_S \rho + \gamma_t^2 \sigma_S \sqrt{1-\rho^2} = 0$$
$$\mu_U - r + \gamma_t^1 \sigma_U = 0$$

for γ_t^1 and γ_t^2, which can easily be seen to have a unique solution.

Under the unique ELMM $\mathbb{P}^*$, the dynamics of S and U are given by

$$dS_t = S_t\left(r\, dt + \ \sigma_S\left(\rho d\widetilde{W}_t^1 + \ \sqrt{1-\rho^2}\, d\widetilde{W}_t^2\right)\right)$$

and

$$dU_t = U_t\left(r\, dt + \sigma_U\, d\widetilde{W}_t^1\right),$$

Where $\widetilde{W}^i i = 1{,}2$ are $\mathbb{P}^*$ Brownian motions.

We again define a third $\mathbb{P}^*$ Brownian motion $\widetilde{W}^3$ as

$$\widetilde{W}_t^3 := \int_0^t \rho\, d\widetilde{W}_s^1 + \int_0^t \sqrt{1-\rho^2}\, d\widetilde{W}_s^2 = \rho \widetilde{W}_t^1 + \sqrt{1-\rho^2}\, \widetilde{W}_t^2,$$

and rewrite the SDE for S as

$$dS_t = S_t\big(r\,dt + \sigma_S\,d\widetilde{W}_t^3\big).$$

Now, let us price options on this model. First, options that depend only on one asset can be priced using exactly the same formulae from the one-dimensional model. We consider the following *exchange option* whose payoff H is

$$H = \max\,\{S_T - U_T, 0\}.$$

This is an option to exchange one asset for another at maturity. To price this option, we first solve the SDEs for S and U to get

$$S_T = S_0 e^{\left(r - \frac{1}{2}\sigma_S^2\right)T + \sigma_S \widetilde{W}_T^3} \; and \; U_T = U_0 e^{\left(r - \frac{1}{2}\sigma_U^2\right)T + \sigma_U \widetilde{W}_T^1}.$$

Then by writing $H = \max\left\{\frac{S_T}{U_T} - 1{,}0\right\}$ and considering the process $V = S/U$, one can use a modified version of the one-dimensional Black-Scholes call option pricing formula derived in the previous sections to get

$$\pi(H) = S_0\Phi(d_1) - U_0\Phi(d_2),$$

where

$$d_1 = \frac{ln\left(\frac{S_0}{U_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \; and \; \sigma \coloneqq \sqrt{\sigma_S^2 + \sigma_U^2 - 2\rho\sigma_S\sigma_U}.$$

3.2.8 Transcript: Generalized Black-Scholes Model

Hi, in this video we introduce a multidimensional version of Black-Scholes model.

Recall that the one-dimensional Black-Scholes model says that the stock price evolves according to the following geometric Brownian motion:

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

where μ and σ are constants and W is a Brownian motion.

We now want to generalize this to the case where, instead of one stock or risky asset, s, we have d of them, where d is finite but greater than or equal to 2, in this case. Written in full:

$$s\left(s^1, \dots, s^d\right) \quad d \geq 2.$$

The Black-Scholes analog of that model still uses something similar to geometric Brownian motion, then. What we need is a vector of Brownian motion. In other words, we need m-dimensional Brownian motion processes from W^1 up to W^m, which are independent. The covariation between them is as follows:

$$\langle W^i, W^j \rangle_t = \delta_{ijt}.$$

This means that it is t if i is equal to j, and 0 otherwise.

The multidimensional Black-Scholes model says that the stock prices of S^1 up to S^d all evolve according to the following SDEs: dS_t^i is equal to S_t^i times $\mu_i dt$ –

where each stock has its own drift – plus, and then in volatility terms, just a combination of the Brownian motion processes, which are the sum from $j = 1$ up to m of $\sigma_{ij} dW^j{}_t$. That is the stochastic differential equation when $i = 1$ up to d. Written in full:

$$dS_t{}^i = S_t{}^i\left(\mu_i dt + \sum_{j=1}^{m} \sigma_{ij}\, dW^j{}_t\right).$$

We normally write this in vector notation as:

$$dS_t = S_t \mu dt + \sigma dW_t,$$

where these are now matrices instead of vectors themselves.

This model is guaranteed to have an ELMM if m is greater than or equal to d. There are cases when m is less than d and there is no ELMM.

The model is complete – in other words, there is only one ELMM that is guaranteed if $m = d$. Of course, there are cases where this doesn't hold and the model is still complete, but we won't explore those cases.

3.2.9 Notes: Problem set

Problem 1

Consider the BS model with $\mu = 0.1, r = 0.06$, and $\sigma = 0.25$. Then what is B_{12} equal to?

Solution:

First, fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a Brownian motion W. We assume that $\mathbb{F} = \mathbb{F}^W$ and $\mathcal{F} = \mathcal{F}_T$. In this model, the stochastic processes S and B satisfy the following SDEs:

$$dS_t = S_t(\mu\, dt + \sigma dW_t), \quad dB_t = rB_t dt,$$

where $\mu \in \mathbb{R}, \sigma > 0$, and $r > 0$ are constants. Here r is the constant continuously compounded risk-free rate.

Solving both SDEs, we obtain:

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}, \quad B_t = B_0 e^{rt}.$$

Thus, we just have to substitute in B_t as follows:

$$B_{12} = B_0 e^{r*12} = B_0 e^{0.06*12} = B_0 e^{0.72}.$$

Problem 2

Consider the BS model with $S_0 = 100, \mu = 0.1, r = 0.06, T = 1$, and $\sigma = 0.25$. What is the price of a call option with strike price $K = 110$?

Solution:

Following the lecture notes, the call price is defined as:

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t) = S_t \Phi(d_1(t)) - Ke^{-r(T-t)} \Phi(d_2(t)),$$

where

$$d_1(t) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2(t) = d_1(t) - \sigma\sqrt{T-t}.$$

Note that $d_i(0) = d_i$ for $i = 1, 2$. Applying the above equations to our case,

$$d_1(t) = \frac{ln\left(\frac{100}{110}\right) + \left(0.06 + \frac{1}{2}0.25^2\right)(1)}{0.25\sqrt{1}} = -0.0162407 \text{ and } d_2(t) = d_1(t) - 0.25\sqrt{1}$$
$$= -0.2662407.$$

Finally, we apply (for $t = 0$)

$$\mathbb{E}^*(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t) = S_t \Phi(d_1(t)) - Ke^{-r(T-t)} \Phi(d_2(t)) = 8.42966.$$

Problem 3

Consider the BS model with $S_{0.5} = 112, \mu = 0.2, r = 0.04, T = 1$, and $\sigma = 0.30$. Consider a call option with a strike price of $K = 100$ and its corresponding hedging strategy (v_0, φ). Then what is $\varphi_{0.5}$ equal to?

Solution:

Let H be a call option with strike K. We know that for this derivative,

$$V(t, S_t) = S_t\Phi\big(d_1(t)\big) - Ke^{-r(T-t)}\sigma\big(d_2(t)\big),$$

where

$$d_1(t) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2(t) - \sigma\sqrt{T-t}.$$

The hedging strategy φ is given by

$$\varphi_t = \frac{\partial V}{\partial S_t} = \Phi\big(d_1(t)\big).$$

In our case, d_1 is equal to:

$$d_1(0.5) = \frac{\text{In}\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\text{In}\left(\frac{112}{100}\right) + \left(0.04 + \frac{1}{2}0.3^2\right)(0.5)}{0.3\sqrt{0.5}} = 0.734583,$$

and

$$\varphi_{0.5} = \frac{\partial V}{\partial S_t} = \Phi\big(d_1(0.5)\big) = 0.76870.$$

Problem 4

Consider the BS model with $S_{0.5} = 108, \mu = 0.1, r = 0.06, T = 1$, and $\sigma = 0.25$. What is the price of an asset-or-nothing call option with strike price $K = 110$ at time 0.5?

Solution:

This is a derivative whose payoff at maturity is equal to the value of the stock price (S_T) if the stock price is greater than K and zero otherwise. Its price is given by

$$C = S_0 * \Phi(d_1),$$

where

$$d_1(0.5) = \frac{\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{118}{110}\right)+\left(0.06+\frac{1}{2}0.25^2\right)(0.5)}{0.25\sqrt{0.5}} = 0.154296,$$

and

$$C_{0.5} = S_{0.5} * \Phi(d_1) = 108 * \Phi(0.154296) = 60.6217.$$

Problem 5

Consider the BS model with $S_0 = 120$, $\mu = 0.2$, $r = 0{,}04$, $T = 1$, and $\sigma = 0.30$. Compute the price of a derivative with payoff $H = \max\{S_T, 80\}$.

Solution:

First of all, we have to notice that the payoff $H = \max\{S_T. 80\}$ is equivalent to:

$$H = \max\{S_T, 80\} = 80 + \max\{S_T - 80{,}0\},$$

where the payoff $\max\{S_T - 80{,}0\}$ represent a call option with strike 80. The call option price today is equal to (applying Black-Scholes): 44. Therefore, the price of the payoff should be equal to:

$$Price = 80 * e^{-0.04*1} + 44 = 120.86,$$

which is the solution that we were looking for.