WORLDQUANT UNIVERSITY

Discrete-time Stochastic Processes Module 7

MSc Financial Engineering

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1. Brief

This document contains the core content for Module 7 of Discrete-time Stochastic Processes, entitled An Introduction to Interest Rate Models. It consists of four video lecture transcripts and four sets of supplementary notes.

2. Course Context

Discrete-time Stochastic Processes is the third course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The course begins with measure-theoretic probability and stochastic processes, with an emphasis on discrete-time martingales. These ideas are then applied to the pricing of derivatives in discrete time, followed by an introduction to interest rate and credit risk modeling.

2.1 Course-level Learning Outcomes

After completing the Discrete-time Stochastic Processes course, you will be able to:

1. Understand the language of measure-theoretic probability.
2. Understand stochastic processes and their applications.
3. Understand the theory of discrete-time martingales.
4. Define trading strategies in discrete time.
5. Create replicating portfolios in discrete time.
6. Model stock price movements on a binomial tree.
7. Price and hedge European derivatives in discrete time.
8. Price and hedge exotic European derivatives in discrete time.
9. Price and hedge American derivatives on a binomial tree.
10. Construct a simple interest rate model on a tree.
11. Price interest rate derivatives on a tree.

2.2 Module Breakdown

The Discrete-time Stochastic Processes course consists of the following one-week modules:

1. Probability Theory
2. Stochastic Processes
3. Discrete Martingales
4. Trading in Discrete Time
5. The Binomial Model
6. American Derivatives
7. An Introduction to Interest Rate Models

3. Module 7:

Interest Rate Models

An interest rate is the amount of interest owed per period as a proportion of the principal sum. While these rates are constantly moving, we can analyze them in discrete time using the models that are presented in this module to understand their movement at any given moment in time. This is useful as it can affect future financial decisions that we make. In this module, we will pay close attention to the pricing and hedging of interest rate assets.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Understand the meaning of "term structure of interest rates".
2. Price simple interest rate products such as bonds and bond options.
3. Construct a simple, tree-based discrete-time model for interest rates.

3.2 Transcripts and Notes

3.2.1 Notes: Equity Derivatives with Non-zero Interest Rates

Consider a market with discrete-time trading at times $t \in \mathbb{I} := \mathbb{N} = \{0,1,2,3,\dots\}$.

In the past few modules, we have dealt with the pricing and hedging of financial instruments by simply working with discounted prices (or assuming that the interest rate is zero). This "essentially" covers most of the mathematical technicalities associated with derivative pricing, without the notational burden of introducing interest rates.

In this section, we complete the discussion of equity derivative pricing by relaxing the zero interest rate assumption. Specifically, we will assume that the bank account is not equal to 1 at all times but it is a predictable stochastic process. We are going to assume that the market consists of d assets, the risky primary assets $S^1, \dots, S^d$ and a locally riskless bank account B, which evolves according to the equations

$$B_0 = 1, \quad B_{n+1} = (1 + R_n)B_n \quad n = 0,1,2,\dots,$$

where R_n is the effective interest rate applicable between period n and $n + 1$. We assume that the prices of all these assets and interest rates are stochastic processes, all defined on a common filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

To say that B is locally riskless, we mean that the interest rate process $R = \{R_n : n = 1,2,3, \dots\}$ is adapted to $\mathbb{F}$. We will also assume that $R_n > -1$ for each n; in most cases, $R_n > 0$.

To compare asset prices at different time points, we will define the *discount factor* $D = \{D_0, D_1, \dots\}$ and the discounted asset prices $X = \{X^1, \dots, X^d\}$ by

$$D_n \coloneqq \frac{1}{B_n} \text{ and } X_n \coloneqq D_n S_n, \quad n \in \mathbb{I}.$$

If H is a European derivative that expires at time $T > 0$, consider its discounted value $D_T H$. If $D_T H$ is attainable, we can apply the theory of trading in discrete time to X and the discounted bank account 1 to find a predictable process φ such that

$$D_T H = \mathbb{E}^*(D_T H) + \sum_{k=1}^{T} \varphi k \cdot \Delta X_k.$$

This trading strategy has a holding in the (discounted) bank account equal to

$$\eta_n = \bar{V}_n - \varphi_n \cdot X_n \,,$$

where $\bar{V}$ is the (discounted) value of the strategy given by

$$\bar{V}_n = \mathbb{E}^*(D_T H | \mathcal{F}_n) = \mathbb{E}^*(D_T H) + \sum_{k=1}^{n} \varphi k \cdot \Delta X_k \,,$$

for each $n = 0,1,2, \dots, T$.

Now, we want to show that we can find a trading strategy $(\tilde{\eta}, \tilde{\varphi})$, where $\tilde{\eta}$ is the investment in B and $\tilde{\varphi}$ is the investment in S, such that the value of this strategy V satisfies

$$V_T\big((\tilde{\eta},\tilde{\varphi}\,)\big) = V_0\big((\tilde{\eta},\tilde{\varphi}\,)\big) + \sum_{k=1}^{T} \tilde{\varphi} k \cdot \Delta S_k + \sum_{k=1}^{T} \tilde{\eta}_k\, \Delta B_k = H.$$

Choose $\tilde{\eta} = \eta$ and $\tilde{\varphi} = \varphi$. Then,

$$V_0\left((\eta,\varphi)\right) = \varphi 1 \cdot S_0 + \eta_1 B_0 = \varphi_1 \cdot X_0 + (\bar{V}_0\big((\eta,\varphi)\big) - \varphi 1 \cdot X_0) = (\bar{V}_0\big((\eta,\varphi)\big) = \mathbb{E}^*(D_T H),$$

and

$$V_T\big((\eta,\varphi)\big) = \mathbb{E}^*(D_T H) + \sum_{k=1}^{T} \varphi_k\, \Delta S_k + \sum_{k=1}^{T} \eta_k\, \Delta B_k$$

$$= \mathbb{E}^*(D_T H) + \sum_{k=1}^{T} \varphi_k\, \Delta S_k + \sum_{k=1}^{T} (\bar{V}_k - \varphi_k \cdot X_k)\Delta B_k$$

$$= \mathbb{E}^*(D_T H) + \sum_{k=1}^{T} \varphi_k\, B_{k-1}\Delta X_k + \sum_{k=1}^{T} (\bar{V}_k B_k - \bar{V}_k B_{k-1})$$

$$= \mathbb{E}^*(D_T H) + \sum_{k=1}^{T} (\bar{V}_k B_k - B_{k-1}\bar{V}_{k-1}) = \mathbb{E}^*(D_T H) + B_T \bar{V}_T - B_0 \bar{V}_0 = B_T \bar{V}_T = B_T D_T H = H.$$

Thus, the strategy (φ,η) replicates H, and the price of H is given by:

$$\pi(H) = V_0\left((\eta,\varphi)\right) = \mathbb{E}^*(D_T H).$$

Furthermore, for any time $0 \le n \le T$, the price of H is given by:

$$V_n\big((\eta,\varphi)\big) = B_n \mathbb{E}^*(D_T H|\mathcal{F}_n) = \mathbb{E}^*\left(\tfrac{D_T}{D_n} H \middle| \mathcal{F}_n\right).$$

Let's look at an example. Consider the binomial model, where the interest rate R_n is equal to the constant r for all n, where $r > 0$. The stock S evolves according to

$$S_0 = \text{fixed}, \quad S_n = S_{n-1} u^{Z_n} d^{1-Z_n},$$

where $0 < d < u$ are fixed constants and Z_n are i.i.d Bernoulli random variables.

An EMM, $\mathbb{P}^*$, should make the discounted assets $X_n \coloneqq S_n(1+r)^{-nr}$ martingales. Hence, if we let $p^* \coloneqq \mathbb{P}^*(\{Z_n = 1\})$, then,

$$(1+r)^{-1}(up^* + (1-p^*)d) = 1,$$

which implies that

$$p^* = \frac{(1+r) - d}{u - d}.$$

Therefore, an EMM exists (and is unique) if and only if $d < 1 + r < u$. It follows that the price of a European contract H that expires at time T is

$$\pi(H) = \mathbb{E}^*((1+r)^{-T} H) = (1+r)^{-T} \mathbb{E}^*(H).$$

Replicating portfolios are obtained in a similar manner.

3.2.2 Transcript: Pricing Equity Derivatives with Non-Zero Interest Rates

Hi, in the past few modules we made the simplifying assumption that interest rates were 0 when pricing derivatives. In this module, we will relax that assumption and assume that interest rates are, in general, not equal to 0, and stochastic.

We will consider the same model where we have a vector of stock prices, $S = (S^1, \dots, S^d)$, which is a d-dimensional stochastic process. We also have B, which is a *locally riskless* bank account. B evolves according to the following formula:

$$B_{n+1} = B_n(1 + R_n),$$

where R_n is the interest rate between the periods n and $n+1$.

If we draw this on a timeline, we have, at time n, the interest rate R_n, which applies between n and $n+1$. n is B_n and $n+1$ is B_{n+1}. So, this is the set up when the interest rates are, in general, not equal to 0, and stochastic. At this point, time 0, the interest rate that applies between time 0 and time 1 is denoted by R_0, and is known at time 0, and that is what we mean by the bank account being "locally riskless" because, between time 0 and time 1, we know exactly what the return will be. The same applies to time 1 and time 2 — the interest rate there is R_1, which is known at time 1, and, therefore, we know what the return will be between periods 1 and 2, etc. This is what is meant by B being locally riskless.

We will also make the assumption throughout that $R_n > -1$. In fact, in most cases, $R_n > 0$, but there are cases where the interest rate can be negative. In this course, however, interest rates will always be positive.

We also define the discount factor as $D_n = \frac{1}{B_n}$. What this means is that the discount factor is simply the present value of $\$1$ that is paid at time n. So, it is the value of $\$1$ at time 0 that is payable at time n.

Now, how do we price derivatives in this environment? Consider, again, a financial derivative that expires at time T — let's call it H. The pricing of derivatives when the interest rates are not equal to 0 is very similar to the hedging of derivatives when the interest rates are equal to 0, and the setup follows the steps below.

1 Define the discounted derivative or claim,

$$\widetilde{H} := D_T H.$$

Since D_T is the discount factor to time 0, this will be the discounted derivative.

2 Replicate $\widetilde{H}$ using $(1, X)$. What we mean by this is that we trade in the discounted riskless asset, whose value is always equal to 1, and the discounted risky assets — the stocks, $(S^1, \dots, S^d)$, whose value is X. If you recall from the previous section, we defined $X_n = D_n S_n$, which is the discounted stock price. We have used a replicating portfolio before and seen how this is obtained — in particular, if the market is complete, then we know that such a replicating portfolio will exist. Let's denote that replicating portfolio by (v_0, φ), where v_0 is equal to the expected value under any risk-neutral measure, assuming that D_T is replicable, of $\widetilde{H}$. Written in full:

$$(v_0, \varphi), v_0 = E^*(\widetilde{H}).$$

So, that is the starting capital, which is also equal to the price of the derivative. φ is the trading in the risky asset, X.

3 The same portfolio, (v_0, φ), applied to (B, S), (not to $(1, X)$ anymore, but to the bank account itself and the stock, (B, S) — in other words, we hold the same amount, φ, in the stock instead of holding φ in X, the discounted stock), replicates the derivative H itself. So, we just use the same replicating portfolio from above to replicate the derivative, H.

4 From that, we get that the price of H — not the discounted derivative but the price of the derivative itself — is actually equal to v_0, which is the expected value of the discounted derivative. Written in full:

$$\pi(H) = v_0 = E^*(D_T H).$$

So, the prices of the two replicating portfolios are exactly the same.

In general, if t is any time between 0 and T, $(0 < t < T)$, then,

$$\pi_t = \frac{1}{Dt} E(D_T H \,|\mathcal{F}_t).$$

This gives us the pricing formula for any intermediate time between 0 and T.
Now that we have looked at pricing equity derivatives in a stochastic interest rate environment, in the next video we are going to price basic interest rate assets.

3.2.3 Notes: Basic Interest Rate Assets

We now turn towards interest-rate-related assets.

In addition to the money market bank account B, the primary interest rate assets are *zero-coupon bonds*. A zero-coupon bond with expiry date T is an instrument that pays 1 at time T. We will denote its price at time $0 \leq t \leq T$ by $P(t, T)$, so that $P(T, T) = 1$.

Consider a zero-coupon bond that expires at time m. We define the *yield* of this bond to be the single period interest rate y_m over the period between 0 and m, such that the bond price is equal to the 1 discounted using this interest rate. That is,

$$(1 + y_m)^{-m} = P(0, m),$$

or

$$y_m = \left(\frac{1}{P(0, m)}\right)^{\frac{1}{m}} - 1.$$

The function $m \mapsto y_m$, $m = 1,2, \ldots$, is called the *yield curve*.

$$\frac{1}{1 + R_{m,n}} = P(m, n),$$

which implies that

$$R_{m,n} = \frac{1}{P(m, n)} - 1.$$

Consider a *forward rate agreement* (FRA), which is an asset H that pays the difference between a floating interest rate and a fixed interest rate. To be precise, let $0 \leq m < n \leq T$ be two time points, then the payoff at time m of a FRA is given by

$$H = N\left(R_{m,n} - R\right),$$

where R is a pre-specified fixed interest rate and N is a fixed positive constant, known as the *nominal*. We will ignore any day-count conventions and simplify the notation by working with $N = 1$ all the time. For this contract, R is normally chosen so that the price of H at time 0 is equal to 0, and this value of R is known as the *par FRA rate*. It has been shown in previous courses that the par FRA rate is given by:

$$R = \frac{P(0, m)}{P(0, n)} - 1.$$

Another common but simple interest rate related asset is the *interest rate swap*. Let $0 \leq m_1 < m_2 < \ldots < m_{k+1}$ be integers, then the payoff of an interest rate swap is

$$H = \sum_{i=1}^{k} N\left(R_{m_i, m_{i+1}} - R\right).$$

Again, the par swap rate, or fair swap rate, is the value of R making the price of H at time 0 equal to 0. This is equal to

$$R = \frac{1 - P(0, m_{k+1})}{\sum_{i=1}^{k} P\left(0, m_i + 1\right)}.$$

3.2.4 Transcript: Zero Coupon Bonds

Hi, in this video we introduce zero coupon bonds, or ZCB's.

ZCB's are the basic interest rate assets and they are defined as follows.
If you consider a time horizon, T, a ZCB that expires at time T is simply an asset that pays the value $\$1$ at time T. It makes no other payments between time 0 and time T.

We will denote the price at time t of a ZCB that pays $\$1$ at time T by $P(t,T)$. This is also called the *expiry date* of the ZCB.

We define the yield of a ZCB as the singular interest rate that satisfies the following equation:

$$y_m \text{ s.t. } (1+y_m)^{-m} = P(0,m).$$

In this equation, $(1+y_m)^{-m}$ is just discounting the payment of 1 that occurs at time m, and $P(0,m)$ is the ZCB price.

If we illustrate this in a number line, we have time 0 and time m, and a ZCB that expires at time m. At time 0, we have its price, $P(0,m)$, and the yield is simply the single period interest rate constant over all periods. This means that we have y_m at all periods, $1,2,3$ and so on, that we can use to discount the payment of 1 to time 0. It must also satisfy the equation above. This is known as the *yield of the bond.*

If we solve the equation, we will see that:

$$y_m = \left(\frac{1}{P(0,m)}\right)^{\frac{1}{m}} - 1.$$

We can sketch this as a graph. On the x-axis, we have the maturity, m, and on the y-axis, we have the price, $P(0, m)$. Typically, if the interest rates are positive, the ZCB prices will look like this: at time 0, the ZCB price will be 1, and then it will decrease and so on and so forth as the maturity increases. On the other hand, the yield, because it is the inverse of that, will typically increase inversely to the ZCB price.

Now, if $\mathbb{P}^*$ is an EMM for the ZCB prices, we have the following formula that must be satisfied to show that the discounted ZCB price must be a martingale:

$$D_t P(t,T) = E^*(D_T 1|\mathcal{F}_t).$$

In this formula, $D_T 1$ is the price of the ZCB at time T (in other words, this is $P(T,T)$, which is always equal to 1 because we know that the ZCB pays the value 1 at time T, meaning that it must be equal to its price, and then we discount that price at time 0 and that's what we get). We then take the conditional expectation of that discounted price at time T, and we must get the discounted ZCB price at time T.

In general, if we have a series of cashflows at different times — for example at time 0, time T_1, time T_2, and so on up to time T_K, denoted by cashflow 1, or C_{T_1}, up to C_{T_K} — then the price of a contract that pays this series of cashflows (let's call it H) will be:

$$\pi(H) = \sum_{I=1}^{K} E^*(C_{T_i} \cdot D_{T_i}).$$

Now, if the cash flows are deterministic, meaning that they are not random at all, we can take C_{T_i} outside of the expectation and we get the following:

$$\pi(H) = \sum_{I=1}^{K} E^*(C_{T_i} \cdot D_{T_i}) = \sum_{I=1}^{K} C_{T_i} E^*(D_{T_i}) = \sum_{I=1}^{K} C_{T_i} \cdot P(0, T_i).$$

In the above equation, the expected value is under the EMM of D_{T_i}. If we substitute 0 for D_t in this equation, $D_t P(t,T) = E^*(D_T 1|\mathcal{F}_t)$, we get D_0, which is 1, $P(0,T) = E^*(D_T|\mathcal{F}_0)$, which is just the expectation of D_T. Therefore, this is just a ZCB price that expires, or has maturity T_i, evaluated at time 0.

What this means is that we can use the ZCB prices to price other cashflow instruments like this one.

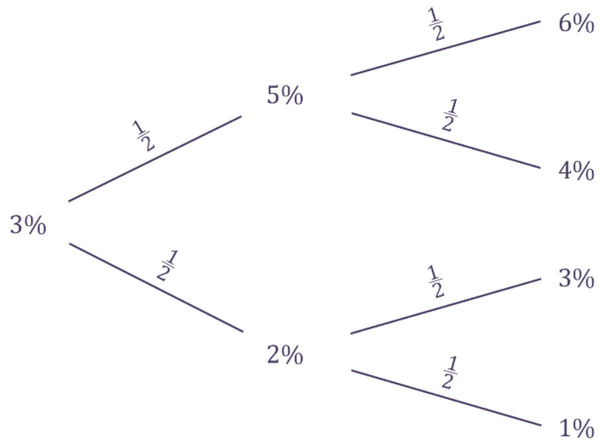
Now that we have introduced ZCB's, in the next video we are going to introduce a simple model for interest rates.

3.2.5 Transcript: A Simple Model for Interest Rates

Hi, in this video we study a simple interest rate model and illustrate how to price ZCB's in this model.

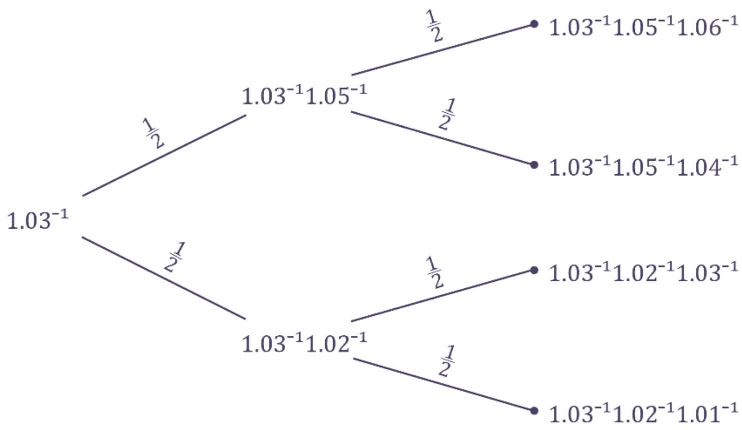
We are going to draw this model on a binomial tree. First though, let's take the interest rate R, which, in this case, will consist of $\{R_0, R_1, R_2\}$. Remember that R_0 is the interest between time 0 and time 1, R_1 runs between time 1 and time 2, and R_2 runs between time 2 and time 3. Our time horizon, then, consists of the following points: $0, 1, 2, 3$, with R_0, R_1 and R_2 stretching between them appropriately.

Our interest rate evolves as follows:



It starts at 3%, and that's R_0, and then, one step ahead, R_1 takes on two possible values: 5% or 2%, with the probability $\frac{1}{2}$. Then, taking another step ahead to R_2, with equal probabilities, we get 6% or 4%, and on the branch of the tree leading on from 2% we get, with the same equal probabilities, 3% and 1%. So, that's the binomial tree for the evolution of interest rates.

Now, we can calculate the discount factors. Remember that $D_0 = 1$ (there is no discounting at time 0). $D_1 = \frac{1}{1+R_0}$, $D_2 = \frac{1}{1+R_1} \cdot D_1$, and, finally, $D_3 = D_2 \cdot \frac{1}{1+R_{2.}}$. So, that's how the discount factors evolve, and we can calculate it based on the following tree:



So, that is the evolution of the discount factors, D_1, D_2 and D_3.

Now, let's work through an example of how we can calculate the ZCB prices. If we start with $P(0,0)$, we will get 1, as this is always equal to 1. Looking at $P(0,1)$, which is a bond that expires at time 1, we get the expected value under the risk-neutral measure of D_1, which we have assumed is equal to ½ — in other words, we have assigned equal probabilities to all of them. This is finally equal to 1.03^{-1}. Written in full:

$$P(0,1) = E^*(D_1) = 1.03^{-1}.$$

Turning to $P(0,2)$, we calculate the ZCB price at time 2 using the following equation:

$$P(0,2) = E^*(D_2) = \frac{1}{2}\left(1.03^{-1}(1.05^{-1})\right) = \frac{1}{2}(1.03^{-1})(1.02^{-1}).$$

We can do the same for $P(0,3)$ by calculating the expected value of D_3. The equations given at the end of the last branches of the binomial tree are all of the possibilities for D_3, which we must multiply by $\frac{1}{2}$, of course, and we get $\frac{1}{4}$ times this, $\frac{1}{4}$ times this, $\frac{1}{4}$ times that, plus $\frac{1}{4}$ times that. And that is how we calculate the ZCB price with maturity time 3 at time 0.

Now that we have covered a simple interest rate model, that brings us to the end of the final module.

3.2.6 Notes: A Simple Model for Interest Rates

We now consider an interest rate model and use it to price more complicated interest-rate-related assets.

The prices of the assets considered in the previous unit were model-independent. That is, to price them we did not need a specific model for interest rates – their prices are the same under any model. However, there are many interest rate assets whose prices are model-dependent. These include interest rate derivatives such as *caplets, floorlets, caps* and *floors.*

The aim of this section is to give an example of a simple interest rate model and illustrate how we can price interest rate assets using this model.

We work with a time horizon $T = 3$. Let $\Omega = \{a, b, c, d, e, f, g, h\}$, $\mathcal{F} = 2^{\Omega}$ and assume that

$$\mathbb{P}^* = \frac{1}{8} \sum_{\omega \in \Omega} \delta_{\omega}.$$

We assume that the interest process $R = \{R_0, R_1, R_2\}$ evolves as follows:

ω	$\mathbb{P}^*(\{\omega\})$	$R_0(\omega)$	$R_1(\omega)$	$R_2(\omega)$
a	$\frac{1}{8}$	0.03	0.05	0.06
b	$\frac{1}{8}$	0.03	0.05	0.06
c	$\frac{1}{8}$	0.03	0.05	0.04
d	$\frac{1}{8}$	0.03	0.05	0.04
e	$\frac{1}{8}$	0.03	0.02	0.03
f	$\frac{1}{8}$	0.03	0.02	0.03
g	$\frac{1}{8}$	0.03	0.02	0.01
h	$\frac{1}{8}$	0.03	0.02	0.01

We let $\mathcal{F}_0 = \sigma(R_0)$, $\mathcal{F}_1 = \sigma(R_0, R_1)$, $\mathcal{F}_2 = \sigma(R_0, R_1, R_2)$ and $\mathcal{F}_3 = \mathcal{F}$.

We now calculate the discount factors. First, $D_0 = 1$ since $B_0 = 1$. Now,

$$D_1 = \frac{1}{1 + R_0} = 1.03^{-1}.$$

For D_2, we have

$$D_2(\omega) = D_1(\omega) \times \frac{1}{1 + R_1(\omega)} = \begin{cases} (1.03 \times 1.05)^{-1} & \omega \in \{a, b, c, d\} \\ (1.03 \times 1.02)^{-1} & \omega \in \{e, f, g, h\}. \end{cases}$$

Finally, for D_3 we have

$$D_3(\omega) = D_2(\omega) \times \frac{1}{1+R_2(\omega)} = \begin{cases} (1.03 \times 1.05 \times 1.06)^{-1} & \omega \in \{a, b\} \\ (1.03 \times 1.05 \times 1.04)^{-1} & \omega \in \{c, d\} \\ (1.03 \times 1.02 \times 1.03)^{-1} & \omega \in \{e, f\} \\ (1.03 \times 1.02 \times 1.01)^{-1} & \omega \in \{g, h\}. \end{cases}$$

We summarize these values in a table:

ω	$D_0(\omega)$	$D_1(\omega)$	$D_2(\omega)$	$D_3(\omega)$
a	1	0.03^{-1}	$(1.03 \times 1.05)^{-1}$	$(1.03 \times 1.05 \times 1.06)^{-1}$
b	1	0.03^{-1}	$(1.03 \times 1.05)^{-1}$	$(1.03 \times 1.05 \times 1.06)^{-1}$
c	1	0.03^{-1}	$(1.03 \times 1.05)^{-1}$	$(1.03 \times 1.05 \times 1.04)^{-1}$
d	1	0.03^{-1}	$(1.03 \times 1.05)^{-1}$	$(1.03 \times 1.05 \times 1.04)^{-1}$
e	1	0.03^{-1}	$(1.03 \times 1.02)^{-1}$	$(1.03 \times 1.02 \times 1.03)^{-1}$
f	1	0.03^{-1}	$(1.03 \times 1.02)^{-1}$	$(1.03 \times 1.02 \times 1.03)^{-1}$
g	1	0.03^{-1}	$(1.03 \times 1.02)^{-1}$	$(1.03 \times 1.02 \times 1.01)^{-1}$
h	1	0.03^{-1}	$(1.03 \times 1.02)^{-1}$	$(1.03 \times 1.02 \times 1.01)^{-1}$

We now calculate the zero-coupon bond prices. At time 0, we have the following bond prices to calculate:

$$P(0,1), \quad P(0,2), \quad and\ P(0,3).$$

First,

$$P(0,1) = \mathbb{E}^*(D_1) = D_1 = 1.03^{-1} \approx 0.971.$$

Also,

$$P(0,2) = \mathbb{E}^*(D_2) = \frac{1}{2}((1.03 \times 1.05)^{-1} + (1.03 \times 1.02)^{-1}) \approx 0.938.$$

Finally,

$$P(0,3) = \mathbb{E}^*(D_3) = \frac{1}{4}((1.03 \times 1.05 \times 1.06)^{-1} + (1.03 \times 1.05 \times 1.04)^{-1})$$

$$+\frac{1}{4}((1.03 \times 1.02 \times 1.03)^{-1} + (1.03 \times 1.02 \times 1.01)^{-1}) \approx 0.907.$$

We can get the yields as:

$$y_1 = \frac{1}{0.971} - 1 = 0.03, \quad y_2 = \left(\frac{1}{0.938}\right)^{\frac{1}{2}} - 1 = 0.0325, \quad \text{and } y_3 = \left(\frac{1}{0.907}\right)^{\frac{1}{3}} - 1 = 0.033.$$

The yield curve is tabulated below:

m	y_m
1	3.00%
2	3.25%
3	3.30%

We can move on to time 1. Here we need to calculate the following bond prices:

$$P(1,1), \quad P(1,2), \quad \text{and } P(1,3).$$

First, $P(1,1) = 1$.

For $P(1,2)$, we have

$$P(1,2) = \mathbb{E}^*\left(\frac{D_2}{D_1}\middle|\mathcal{F}_1\right) = \frac{D_2}{D_1} = 1.05^{-1}I_{\{a,b,c,d\}} + 1.02^{-1}I_{\{e,f,g,h\}}.$$

Finally, for $P(1,3)$ we have

$$P(1,3) = \mathbb{E}^*\left(\frac{D_3}{D_1}\middle|\mathcal{F}_1\right) = \frac{1}{2}((1.05 \times 1.06)^{-1} + (1.05 \times 1.04)^{-1})I_{\{a,b,c,d\}}$$

$$+\frac{1}{2}((1.02 \times 1.03)^{-1} + (1.02 \times 1.03)^{-1} + (1.02 \times 1.01)^{-1})I_{\{e,f,g,h\}}$$

$$= \ 0.907 I_{\{a,b,c,d\}} = 0.961 I_{\{e,f,g,h\}}.$$

We now move on to time 2. Here we need to calculate the following bond prices:

$$P(2,2) \text{ and } P(2,3).$$

Again, $P(2,2) = 1$. For $P(2,3)$, we have

$$P(2,3) = \mathbb{E}^* \left(\frac{D_3}{D_2} \middle| \mathcal{F}_2 \right) = 1.06^{-1} I_{\{a,b\}} + 1.04^{-1} I_{\{c,d\}} + 1.03^{-1} I_{\{e,f\}} + 1.01^{-1} I_{\{g,h\}}.$$

Finally, we move on to time 3. Here, there's only one bond price to consider: $P(3,3)$, whose value is 1.

3.2.7 Notes: Interest Rate Derivatives

We now look at pricing interest rate derivatives using the model developed in the previous chapter.

The first derivative that we consider is the interest rate *caplet*. A caplet is a call option on an interest rate, and its payoff is

$$H_{caplet_t} = (R_t - K)^+,$$

where K is the strike and R_t is the reference interest rate. Again, this payoff is multiplied by a nominal amount and a day count convention fraction, both of which we will ignore for simplicity. Its price (at time 0) is therefore given by:

$$\pi\left(H_{caplet_t}\right) = \mathbb{E}^*(D_t(R_t - K)^+).$$

Similarly, an interest rate *floorlet* is a put option on an interest rate with payoff equal to

$$H_{floorlet_t} = (K - R_t)^+,$$

where K is the strike and t is the reference interest rate. Its price is calculated as

$$\pi\left(H_{floorlet_t}\right) = \mathbb{E}^*(D_t(K - R_t)^+).$$

Let's now look at a concrete example to price these contracts. We consider the interest rate model discussed in the previous section. Consider a caplet and a floorlet both with strike price $K = 0.025$ on the rate R_1. Their prices are given by

$$\pi\left(H_{caplet_1}\right) = \mathbb{E}^*(D_1(R_1 - 0.025)^+) = 1.03^{-1}(0.05 - 0.025) \times \frac{1}{2} \approx 0.012$$

and

$$\pi\left(H_{floorlet_1}\right) = \mathbb{E}^*(D_1(0.025 - R_1)^+) = 1.03^{-1}(0.025 - 0.02) \times \frac{1}{2} \approx 0.002427.$$

Similarly, if the reference rate is R_2 instead, then the prices are

$$\pi\left(H_{caplet_2}\right) = \mathbb{E}^*(D_2(R_2 - 0.025)^+)$$
$$= 1.03^{-1}\left(1.05^{-1}(0.06 - 0.025)\tfrac{1}{4} + 1.05^{-1}(0.04 - 00.25)\tfrac{1}{4} + 1.02^{-1}(0.03 - 0.025)\tfrac{1}{4}\right) \approx 0.01275$$

and

$$\pi\left(H_{floorlet_2}\right) = \mathbb{E}^*(D_2(0.025 - R_2)^+) = 1.03^{-1} \times 1.02^{-1}(0.025 - 0.01) \times \frac{1}{4} \approx 0.003569.$$

An interest rate *cap* is a derivative whose payoff is a sum of caplets on different reference rates. Similarly, an interest rate *floor* is a derivative whose payoff is the sum of floorlets. It is clear then that their prices are simply sums of the corresponding caplets/floorlets.

Continuing with the previous example, consider a cap and a floor, both on R_0, R_1, R_2 and with a strike of $K = 0.025$. Then,

$$\pi_{cap} = (0.03 - 0.025) + 0.012 + 0.01275$$

and

$$\pi_{floor} = 0 + 0.002427 + 0.003569.$$

3.2.8 Transcript: Concluding Video

Congratulations on finishing Discrete-time Stochastic Processes, the third course in the WorldQuant University Master of Science in Financial Engineering.

In this course, you were introduced to derivative pricing in discrete time. In the next course, Continuous-time Stochastic Processes, we will explore stochastic processes in continuous time, and apply that knowledge to the pricing of derivatives in continuous-time asset price models.

Now that you've completed the Discrete-time Stochastic Processes course, you should be able to:

- Understand the language of measure-theoretic probability,
- Understand stochastic processes and their applications,
- Understand the theory of discrete-time martingales,
- Define trading strategies in discrete time,
- Create replicating portfolios in discrete time,
- Model stock price movements on a binomial tree,
- Price and hedge European derivatives in discrete time,
- Price and hedge exotic European derivatives in discrete time,
- Price and hedge American derivatives on a binomial tree,
- Construct a simple interest rate model on a tree, and finally,
- Price interest rate derivatives on a tree.

Thank you for your engagement throughout this course, and we hope you enjoy the rest of the program. Good luck!