WORLDQUANT UNIVERSITY

Continuous-time Stochastic Processes Module 6

MSc Financial Engineering

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1. Brief

This document contains the core content for Continuous-time Stochastic Processes Module 6, entitled An Introduction to Lévy Processes. It consists of four sets of notes and five lecture videos.

2. Course Context

Continuous-time Stochastic Processes is the fourth course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The aim of the course is to introduce derivative pricing when trading happens in continuous time. It begins by focusing on the stochastic calculus of Brownian motion and its generalization to continuous semimartingales. These ideas are then applied to continuous trading and the pricing of financial derivatives.

2.1 Course-level Learning Outcomes

Upon completion of the Continuous-time Stochastic Processes course, you will be able to:

1. Define and identify Brownian motion processes in multiple dimensions.
2. Solve stochastic differential equations.
3. Apply Ito's Lemma for continuous semimartingales.
4. Apply Girsanov's theorem to construct equivalent local martingale measures.
5. Price and hedge derivatives in various asset price models.
6. Derive the Black-Scholes partial differential equation.
7. Construct asset prices models based on Lévy processes.
8. Price interest rate derivatives.

2.2 Module Breakdown

The Continuous-time Stochastic Processes course consists of the following one-week modules:

1. Brownian Motion and Continuous-time Martingales
2. Stochastic Calculus I: Ito Process
3. Stochastic Calculus II: Semimartingales
4. Continuous Trading
5. The Black-Scholes Model
6. An Introduction to Lévy Processes
7. An Introduction to Interest Rate Modeling

3. Module 6

An Introduction to Lévy Processes

In this module we introduce Lévy processes and apply them to modeling stock price returns. Module 6 begins with the Poisson process, before moving on to Lévy processes and a number of important, associated proofs. In the module's later sections, we then discuss applications of Lévy processes to financial modeling, with a particular emphasis on exponential Lévy models.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. State and prove the properties of a Poisson process.
2. State the properties of Lévy processes.
3. Construct exponential Lévy models.

3.2 Transcripts and Notes

3.2.1 Notes: The Poisson Process

Before introducing Lévy processes in general, we will first talk about a very important discontinuous stochastic process: the Poisson process. Throughout this section, we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume that all stochastic processes defined on this space are adapted to $\mathbb{F}$.

A stochastic process $N = \{N_t : t \geq 0\}$ is called a *counting process* if N is cadlag and the sample paths of N are piecewise constant with jumps of size 1. We will also assume that $N_0 = 0$.

So we can think of a counting process N as a stochastic process such that N_t counts the number of events that have occurred up to (and including) time t and the increment $\mathrm{N_t} - \mathrm{N_s}$ (for $0 \leq s < t$) counts the number of events that have occurred in the interval $(s, t]$. Every sample path of a counting process N will move through the states $\mathbb{N} = \{0,1,2,3 \dots\}$ in that order.

A counting process N is called a *homogeneous Poisson process with rate* $\lambda > 0$ if

1. N has independent increments.
2. N has stationary increments.
3. For $s < t$, the increment $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t - s)$; that is,

$$\mathbb{P}_{N_t - N_s} = \sum_{n=0}^{\infty} \frac{\big(\lambda(t-s)\big)^n e^{-\lambda(t-s)}}{n!} \delta_n .$$

For the remainder of this section, let N be a homogeneous Poisson process with rate $\lambda > 0$. Define the random variables (stopping times) $S_0, S_1, \ldots$ as follows:

$$S_0 := 0, \qquad S_n := \inf\{t \geq 0 : N_t = n\}\ n \geq 1.$$

The S_n's are called the *arrival times* of N; that is, S_n is the time of arrival of the nth event. Note that $S_0 \leq S_1 \leq S_2 \leq \ldots$.

We also define the *interarrival times* $T_1, T_2, \ldots$ as

$$T_n := S_n - S_{n-1}, \qquad n \geq 1.$$

These random variables represent the time between successive events of N. For $n \geq 1$, the arrival times can be recovered as

$$S_n = \sum_{i=1}^{n} T_i.$$

Let us now find the distribution of T_n, and, consequently, that of S_n for $n \geq 1$. First note the equivalence of the following events:

$$\{N_t = 0\} = \{T_1 > t\} = \{S_1 > t\}.$$

The intuitive explanation of this relationship is that if no events have occurred by time t (i.e. $N_t = 0$), then the arrival of the first event is after time t (i.e. $S_1 > t$ or $T_1 > t$ since $S_1 = T_1$) and vice versa. Since T_1 is a non-negative random variable and N_t has a Poisson distribution with parameter λt, we have (for $t \geq 0$):

$$1 - F_{T_1}(t) = \mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = e^{-\lambda t}.$$

Hence T_1 (and S_1) has an exponential distribution with parameter $\lambda > 0$.

Let us now find the joint distribution of S_1 and S_2. For $0 \leq w_1 \leq w_2$ we have:

$$F_{S_1S_2}(w_1,w_2) = \mathbb{P}(S_1 \le w_1, S_2 \le w_2) = \mathbb{P}\big(N_{w_1} \ge 1, N_{w_2} \ge 2\big)$$

$$= \mathbb{P}\big(N_{w_1} = 1, N_{w_2} - N_{w_1} \ge 1\big) + \mathbb{P}\big(N_{w_1} \ge 2\big)$$

$$= \lambda w_1 e^{-\lambda w_1}\big(1 - e^{-\lambda(w_2 - w_1)}\big) + \big(1 - \lambda w_1 e^{-\lambda w_1} - e^{-\lambda w_1}\big) = 1 - \lambda w_1 e^{-\lambda w_2} - e^{-\lambda w_1}.$$

Hence the joint density of S_1 and S_2 is:

$$f_{S_1S_2}(w_1,w_2) = \begin{cases} \lambda^2 e^{-\lambda w_2} & 0 \le w_1 \le w_2 \\ 0 & \text{otherwise.} \end{cases}$$

To find the joint density of T_1 and T_2 we note that

$$T_1 = S_1 \text{ and } T_2 = S_2 - S_1.$$

Hence,

$$f_{T_1T_2}(t_1,t_2) = f_{S_1S_2}(t_1, t_1 + t_2) \times 1 = \begin{cases} \lambda^2 e^{-\lambda(t_1+t_2)} & t_1, t_2 \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, T_1 and T_2 are both independent exponential random variables with parameter λ.

It can be shown that, in general, the interarrival times $T_1, T_2, \ldots$ are i.i.d. exponential random variables with parameter $\lambda > 0$. It then follows that the arrival time S_n has a gamma distribution with parameters λ and n; that is:

$$f_{S_n}(w) = \frac{\lambda^n w^{n-1} e^{-\lambda w}}{(n-1)!}, \qquad w > 0.$$

This is easily shown by calculating the moment generating function of S_n:

$$M_{S_n}(\alpha) = \mathbb{E}(e^{\alpha S_n}) = \mathbb{E}\left(exp\left(\alpha \sum_{i=1}^{n} T_i\right)\right) = \mathbb{E}\left(\prod_{i=1}^{n} e^{\alpha T_i}\right) = \prod_{i=1}^{n} \mathbb{E}(e^{\alpha T_i})$$

$$\prod_{i=1}^{n}\left(\frac{\lambda}{\lambda - \alpha}\right) = \left(\frac{\lambda}{\lambda - \alpha}\right)^{n}.$$

We can also define the Poisson process by starting with an i.i.d sequence of exponential random variables $T_1, T_2, \ldots$ with parameter $\lambda > 0$. Then we define the arrival times as

$$S_0 = 0, \qquad S_n = S_{n-1} + T_n \; n \geq 1.$$

Then the counting process N defined by

$$N_t = \sum_{n=1}^{\infty} I_{\{S_n \leq t\}} = \#\{n: S_n \leq t\}$$

is a Poisson process.

3.2.2 Transcript: An Introduction to the Poisson Process

Hi, in this video we introduce the Poisson process.

A stochastic process, N, where $N = \{N_t : t \geq 0\}$, is called a Poisson process if it satisfies the following conditions:

1. N is a **counting process.** This means that $N_0 = 0$ and N_t counts the number of events up to, and including, time t. So, this implies, of course, that the state space of N – in other words, the values that N_t takes – will all be non-negative integers.
2. N has **independent increments**, which means that if we pick two time points, time s and time t, where s is less than t, then the increment N_t minus N_s – in other words, the number of events, using this interpretation that occurs over the interval, between s and t – is independent of the number of events that occur in any other interval that does not overlap with s and t. So, non-overlapping increments are independent.
3. N has **stationary increments**, which means that, again, if we pick two time points, s and t, then the distribution – or the low of $N_t - N_s$, which, in this case, is simply the number of events between time s and time t – does not depend on s. In other words, if we shift this interval a certain length – let's say, we shift it by h units to $s + h$ and $t + h$ – then the distribution of $N_{t+h} - N_{s+h}$ will also be the same as the distribution of $N_t - N_s$. Another way of putting this is that the distribution of the increments $N_t - N_s$ depends only on $t - s$, which is the length of the time interval.
4. The distribution of the increment $N_t - N_s$ is **Poisson**. In other words, $N_t - N_s$ is a Poisson distribution with rate $(\lambda(t-s))$, where λ is a positive constant. We call λ the rate of the Poisson process and we sometimes refer to this as a homogeneous Poisson process because the rate λ is constant.

So, what Poisson distribution means is that the low has the following PMF:

$$\mathbb{P}(N_t - N_s = n) = \frac{\left(\lambda(t-s)\right)^n e^{\lambda(t-s)}}{n!}.$$

This is true for $n = 0, 1, 2 \ldots$ and, of course, $\frac{\left(\lambda(t-s)\right)^n e^{\lambda(t-s)}}{n!} = 0$ otherwise, or elsewhere.

This is a Poisson process and we can call a stochastic process Poisson if it satisfies the conditions listed above.

We can generalize this to what is known as a non-homogeneous or in-homogeneous Poisson process. The difference is that instead of having a constant rate, λ, we have what is known as a transition rate, which is a function of time. However, we also lose property 3, as listed above, whereby we have stationary increments – the increments are no longer stationary as the rate changes over time. We call that a non-homogeneous Poisson process.

We have dealt with the Poisson process briefly before and we showed an important property that, if N is a Poisson process, then the stochastic process $N_t - \lambda_t$, which we call the **compensated Poisson process**, is a martingale. In fact, N itself is a submartingale, so:

$$N = \{N_t : t \leq 0\} \text{ is a martingale.}$$

What do the sample paths of a Poisson process look like?

Quite simply, if we look at a diagram, a point starts at time 0 and remains there for a certain period of time, T_1, until an event occurs, and it jumps up to 1, where it stays for a random amount of time, T_2, until a second event occurs and it jumps up to 2, where it stays for a period of time, T_3, and so on. In the next video, we are going to talk more about these times, which are called **interarrival times.**

Therefore, the sample paths of a Poisson are piecewise constant and have jumps of sizes 1. So, in particular, Poisson processes are an example of stochastic processes that do not have continuous sample paths, unlike Brownian motion.

Now that we have introduced the Poisson process, in the next video we are going to talk about interarrival times.

3.2.3 Transcript: More on the Poisson Process

Hi, in this video we talk more about interarrival times and arrival times of the Poisson process.

Let N be a homogeneous Poisson process, where $N = \{N_t : t \geq 0\}$, with rate $\lambda > 0$.

Remember that the sample paths of N always start at 0 and stay at 0 for a random amount of time, T_1, before jumping up to 1, where it stays for a random amount of time, T_2, before jumping up to 2, where it does the same, staying for a random amount of time, T_3, and so on. So, that's why the state space is the non-negative integers.

We will call these random amounts of time, which are random variables, the interarrival times and we will call the times, up until the n^{th} event, the arrival time of the n^{th} event. So, for example, we will call the time between 0 and the end of T_1, W_1, the arrival time of the first event. The second time interval, between 0 and the end of T_2, will be called W_2, which is the arrival time of the second event, and so on. W_3 will be the arrival time of the third event. So, the obvious relationship between the interarrival times and the arrival times is as follows:

$$T_n = W_n - W_{n-1}, \qquad W_0 = 0$$

$$W_n = \sum_{k=1}^{n} T_k.$$

We can define all of this formally as $W_n = \inf\{t \geq 0 : N_t = n\}$. This is the first time that the Poisson process reaches n, and we then, of course, define T_n as $T_n := W_n - W_{n-1}$. That is the definition of arrival and interarrival times.

Now, the distributional properties of the interarrival times can easily be derived from the properties of a Poisson process. For instance, to find the distribution of T_1, we calculate the probability that T_1 is greater than t. Now, since the W_n's are increasing – in other words W_3 is greater than W_2, which is greater than or equal to W_1 – the T_n's are non-negative random variables. We can calculate the survivor function, in other words, the probability that T_1 is greater than t – by looking at a number line that starts at 0. There are no events that occur in the space between 0 and point t because the first interarrival time, T_1, which is also equal to the arrival time of the first event, is greater than t. In other words, there are no events between 0 and t on the number line. Therefore:

$$Pr(T_1 > t) = Pr(N_t - N_0 = 0).$$

Now, as we saw in the previous video, the distribution of the increment is Poisson, so $N_t - N_0 = 0$ has a Poisson distribution with parameter $\lambda(t)$. Hence, this is equal to $e^{-\lambda t}$.

From there, we can calculate, for $t > 0$, the CDF of T_1, whereby we get $F_{T_1}(t) = 1 - e^{-\lambda t}$. And, of course, it is 0 for $t \leq 0$ because it is a non-negative random variable. Hence, as we can see, this has an exponential distribution with parameter λ.

In fact, we can prove a much more general result that says that all of the interarrival times, $\{T_n\}_{n=1}^{\infty}$, are i.i.d exponentials. In other words, they are independent and identically distributed. All of them have an exponential distribution with parameter $\lambda > 0$. Following on from that, because W_n is the sum of the T_k's, W_n has a Gamma distribution because it is the sum of i.i.d exponentials with parameter $(\alpha = n, \lambda)$ with λ being the same λ that we used in the exponential case.

Now that we have looked at arrival times and interarrival times, in the next video, we are going to introduce Lévy processes.

3.2.4 Notes: Lévy Processes and Their Properties

We now introduce Lévy processes.

Let $X = \{X_t : t \geq 0\}$ be a stochastic processes (adapted to $\mathbb{F}$). We say that X is a Lévy process if:

1. $X_0 = 0$
2. X has independent increments.
3. X has stationary increments.
4. X is *stochastically continuous*, i.e. for every $t \geq 0$ and every $\epsilon > 0$,

$$\lim_{s \to t} \mathbb{P}\left(|X_s - X_t| > \epsilon\right) = 0.$$

So, X_s converges to X_t in probability as s tends to t.

It can be shown that if X is a Lévy process, then X has a cadlag modification, and because of that we will simply assume that X is cadlag.

A trivial example of a Lévy process is the deterministic process $X_t = bt$ where $b \in \mathbb{R}$. It is trivial to show that all the above properties are satisfied.

Another example of a Lévy process is a Brownian motion process. Indeed, if $X = W$ is a Brownian motion, then properties 1, 2, and 3 clearly hold, and property 4 holds due to continuity of the sample paths of X. Combining the two examples, it follows that the so-called *Brownian motion with drift* $X_t = bt + \sigma W_t$ is also a Lévy process. It is the only Lévy process with continuous sample paths.

The (homogeneous) Poisson process discussed above is a Lévy process. Again, the first three properties are clearly satisfied. For the 4th property, we use Markov's inequality (or Chebychev's inequality) to obtain (for $\epsilon > 0$):

$$\mathbb{P}(|N_s - N_t| > \epsilon) \le \frac{\mathbb{E}(|N_s - N_t|)}{\epsilon} = \frac{\lambda|s-t|}{\epsilon} \to 0 \text{ as } s \to t.$$

The paths of N are of course discontinuous, with jumps of size 1 (i.e. $\Delta N \in \{0,1\}$).

One drawback of using the Poisson process to model stock price returns is that the sizes of the jumps are always equal to 1, which is not realistic. We now look at a generalization of this by defining what is called a *compound Poisson process*.

Let N be a homogeneous Poisson process with rate $\lambda > 0$ and $Y_1, Y_2, \dots$ be a sequence of i.i.d. random variables that are also independent of N. A *compound Poisson process* is a stochastic process X defined by

$$X_t := \sum_{n=1}^{\infty} Y_n \, I_{\{N_t \ge n\}} = \begin{cases} 0 & N_t = 0 \\ \sum_{k=1}^{N_t} Y_k & N_t \ge 1. \end{cases}$$

We can think of X as a generalization of the Poisson process, where the sizes of the jumps are random variables, instead of just being all equal to 1.

The compound Poisson process is a Lévy process. The full proof of this fact is left as an exercise, but we will show the stationarity of increments. Consider the times $0 \le s < t$. Then the characteristic function of the increment is

$$\varphi_{X_t - X_s}(u) = \mathbb{E}\left(e^{iu(X_t - X_s)}\right) = \mathbb{E}\left(e^{iu\left(\sum_{k=N_s+1}^{N_t} Y_k\right)}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{iu\left(\sum_{k=N_s+1}^{N_t} Y_k\right)}\middle|\sigma(N_s, N_t)\right)\right)$$

$$= \mathbb{E}\left(\varphi_Y(u)^{N_t - N_s}\right) = \exp\left(\lambda(t-s)(\varphi_Y(u) - 1)\right)$$

since $N_t - N_s$ has a Poisson distribution with rate $\lambda(t-s)$. So, clearly the distribution of the increment only depends on $t - s$.

Thus, so far our most general Lévy process X is

$$X_t = bt + \sigma W_t + \sum_{k=1}^{N_t} Y_k\,.$$

The process consists of the following three terms:

- A deterministic term bt.
- A diffusion term σW_t.
- A pure jump term $\sum_{k=1}^{N_t} Y_k$.

The characteristic function of X_t is

$$\varphi_{X_t}(u) = \mathbb{E}\left(e^{iuX_t}\right) = e^{iubt}\mathbb{E}\left(e^{iu\sigma W_t}\right)\mathbb{E}\left(\exp\left(iu\sum_{k=1}^{N_t} Y_k\right)\right)$$

$$\exp\left(t\left(iub - \frac{1}{2}\sigma^2u^2 + \lambda(\varphi_Y(u) - 1)\right)\right) = \exp\left(t\left(iub - \frac{1}{2}\sigma^2u^2 + \lambda\int_{\mathbb{R}}\left(e^{iuy} - 1\right), d\mathbb{P}_Y(y)\right)\right).$$

It turns out that the characteristic function of any Lévy process is similar to the one above. Let X be a Lévy process and for a fixed $u \in \mathbb{R}$, define $g_u\colon [0,\infty) \to \mathbb{C}$ to be the characteristic function of X_t:

$$g_u(t) \coloneqq \mathbb{E}\left(e^{iuX_t}\right), \qquad t \geq 0.$$

For $s, t > 0$, we have

$$g_u(s + t) = \mathbb{E}\left(e^{iuX_{s+t}}\right) = \mathbb{E}\left(e^{iu(X_t + X_{s+t} - X_t)}\right) = g_u(t)g_u(s).$$

Together with $g_u(0) = 1$, we get that

$$g_u(t) = e^{t\psi(u)}$$

for some function $\psi: \mathbb{R} \to \mathbb{C}$.

The function ψ above is given by the following theorem:

Theorem 2.1 (Lévy-Khintchine). *Let X be a Lévy process. Then the characteristic function of X_t is given by*

$$\varphi_{X_t}(u) = e^{t\psi(u)},$$

where

$$\psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} e^{iuy} - 1 - iuyI_{[-1,1]}(y) d\nu(y)$$

for some $b \in \mathbb{R}, \sigma^2 \geq 0$ and a measure on $\mathcal{B}(\mathbb{R})$ called the Lévy measure *of X, that satisfies*

$$\nu(\{0\}) = 0, \text{ and } \int_{\mathbb{R}} 1 \wedge y^2 \, d\nu(y) < \infty.$$

We call $(b, \sigma^2, \nu)$the *Lévy triplet* of X.

To understand how to find ν, we need to introduce the following *Poisson random measure*. Let $B \in \mathcal{B}(\mathbb{R})$ be a Borel subset of $\mathbb{R}$ such that $0 \notin \bar{B}$, where $\bar{B}$ is the closure of B. For each $t \geq 0$ define the random variable $N_t(B)$ by

$$N_t(B)(\omega) := \sum_{s \leq t} I_B\big(\Delta X_s(\omega)\big) = \#\{s \leq t: \Delta X_s(\omega) \in B\}.$$

That is, $N_t(B)(\omega)$ is the number of jumps of X that are of size B. This random variable is well-defined since X is cadlag, and, therefore, has finitely many jumps of a given size in a finite interval. Now note that

- For fixed $\omega \in \Omega, B \mapsto N_t(B)(\omega)$ is a positive measure (a counting measure).

- For fixed $B \in \mathcal{B}(\mathbb{R})$ with $0 \notin \bar{B}, \omega \mapsto N_t(B)(\omega)$ is a random variable.
- For fixed $B \in \mathcal{B}(\mathbb{R})$ with $0 \notin \bar{B}, (t, \omega) \mapsto N_t(B)(\omega)$ is a counting process.

In fact, $N_{.}(B)$ is a Poisson process. Indeed, define the stopping times $\tau_0 < \tau_1 < \tau_2 < \dots$ by

$$\tau_0 = 0,\ \tau_{n+1} := \inf\{t > \tau_n : \Delta X_t \in B\}.$$

Then $N(B)$ can be written as

$$N_t(B) = \sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}}.$$

Hence all we need to show is that the interarrival times $\tau_{n+1} - \tau_n$ are i.i.d exponentially distributed. This is achieved by showing that the distribution of $\tau_{n+1} - \tau_n$ is memoryless, i.e.

$$\mathbb{P}(\tau_{n+1} - \tau_n > s + t) = \mathbb{P}(\tau_{n+1} - \tau_n > t)\mathbb{P}(\tau_{n+1} - \tau_n > s) \quad s, t > 0.$$

This is left as an exercise.

So, we get that $N(B)$ is a Poisson process with rate $\mathbb{E}\big(N_1(B)\big) =: \nu(B)$. This is how the Lévy measure is obtained.

We now move on to the infinite divisibility property of a Lévy process. Let X be a Lévy process and $t > 0$ be fixed. Then for each $n \geq 1$ we can write X_t as:

$$X_t = \left(X_t - X_{\frac{t(n-1)}{n}}\right) + \dots + \left(X_{\frac{t}{n}} - X_0\right) = \sum_{i=1}^{n} Z_i,$$

where $Z_i := X_{\frac{ti}{n}} - X_{\frac{t(i-1)}{n}}$ for $i = 1, \dots, n$. Since X has stationary and independent increments, it follows that the $Z_i s$ are i.i.d random variables. So we can conclude that for each $n \geq 1$, X_t is equal (in distribution) to the sum of n i.i.d random variables. Such a random variable is said to be *infinitely divisible*.

Here are some examples of distributions that are infinitely divisible:

- If $Y \sim N(\mu, \sigma^2)$, then $Y = \sum_{i=1}^{n} Z_i$ where $Z_i \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$
- If $Y \sim \text{Gamma}\ (\alpha, \lambda)$, then $Y = \sum_{i=1}^{n} Z_i$ where $Z_i \sim \text{Gamma}(\alpha/n, \lambda)$
- If $Y \sim \text{Poisson}\ (\lambda)$,then $Y = \sum_{i=1}^{n} Z_i$ where $Z_i \sim \textit{Poisson}(\lambda/n)$

Let Y be a random variable whose distribution is infinitely divisible. Then the characteristic function of Y is given by

$$\varphi_Y(u) = e^{\psi(u)}$$

where

$$\psi(u) = ibu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} e^{iuy} - 1 - iuyI_{[-1,1]}(y)\, d\nu(y)$$

for some $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and a measure on $\mathcal{B}(\mathbb{R})$ called the *Lévy measure* of X, that satisfies

$$\nu(\{0\}) = 0, \text{and } \int_{\mathbb{R}} 1 \wedge y^2\ d\nu(y) < \infty.$$

The next theorem relates infinitely divisible distributions to Lévy processes.

Theorem 2.2 *Let X be a Lévy process. Then the distribution of X_t is infinitely divisible. Conversely, if Y is a random variable whose distribution is infinitely divisible, then there exists a Lévy process X such that the law of X_1 is the same as the law of Y.*

For example, the Lévy process X such that X_1 has a Gamma distribution is called a *Gamma process.*

3.2.5 Transcript: An Introduction to Lévy Processes

A stochastic process, where $X = \{X_t : t \geq u\}$, is called a Lévy process if it satisfies the following conditions:

1. We will assume that $X_0 = 0$. It is not necessary to make this assumption but it will simplify the calculations that we will be making later on.
2. X has stationary and independent increments, just like it has in Brownian motion and the Poisson process.
3. This is a regulatory condition that assumes that X is stochastically continuous, which means that the limit, as s tends to t for any t positive, of X_s is equal to X_t in probability. So, X_s converges to X_t in probability. If we remember the definition of convergence in probability, what this really says is for any ε positive, the limit, as s tends to t, of the probability absolute value of X_t minus X_s is greater than ε and equal to 0. Written in full:

$$\forall \varepsilon > 0 \quad \lim_{s \to t} \mathbb{P}\left(|X_t - X_s| > \varepsilon\right) = 0.$$

So, those are the three properties that a stochastic process must satisfy in order for it to be a Lévy process.

Now, I just want to mention the third property: this property also implies that X has a cadlag modification. This means that there exists a cadlag stochastic process Y such that Y is a modification of X and, therefore, because of this reason, some authors replace the third condition that we listed with the condition that X is cadlag itself. We will make that assumption here as well. In other words, we are going to assume from the onset that whenever we are dealing with a Lévy process, that Lévy process itself is cadlag.

Here are some examples:

1. The first one is a Brownian motion, W. A Brownian motion easily satisfies the three properties: if we look at the first one, Brownian motion already starts at 0 and, as we have mentioned, it has stationary and independent increments. The third property is automatically satisfied because Brownian motion is continuous itself.

2. In the second example, we take N to be a Poisson process, which also satisfies the three conditions. Let's see why: the first condition requires N_0 to equal 0, which is true, and something that we have stated in previous videos. The second condition agrees that the homogeneous Poisson process has stationary and independent increments. And then we reach the third condition that we need to prove.

   To show that N is stochastically continuous, we have to show that this condition holds for every t. Therefore, let ε be positive, and then we calculate the limit, as s tends to t of the probability of the absolute value of $N_t - N_s$, which is greater than ε. Written in full:

$$\varepsilon > 0, \lim_{s \to t} \mathbb{P}\left(|N_t - N_s| > \varepsilon\right).$$

   Now, for this, we are going to use an inequality known as Markov's inequality that we studied in Probability Theory. This inequality says that if X is a non-negative random variable, then the probability that X is greater than ε, where ε is positive, is less than or equal to the expected value of X over ε for any positive ε. Written in full:

$$\text{If } X \geq 0 \text{ then } \mathbb{P}(X > \varepsilon) \leq \frac{E(X)}{\varepsilon}.$$

   Using Markov's inequality, this probability here is less than or equal to the limit, as s tends to t, which is non-negative (because this is a probability) of the expected value absolute value $|N_t - N_s|$, divided by ε itself, because ε is positive. Written in full:

$$\leq \lim_{s \to t} \frac{E(|N_t - N_s|)}{\varepsilon}.$$

Now, the expected value of this absolute value is either equal to $N_t - N_s$ (when s is less than t) or $N_s - N_t$ (when s is greater than t). So, in any case, this will just be equal to (since this increment is a Poisson distribution) the limit, as s tends to t of λ times the absolute value of $t - s$ (which sorts out the two cases), divided by ε. And this limit is equal to 0 as s tends to t, because the top part goes to 0 and the bottom part is just a constant. Written in full:

$$\leq \lim_{s \to t} \frac{E(|N_t - N_s|)}{\varepsilon} = \lim_{s \to t} \frac{\lambda |t - s|}{\varepsilon} = 0.$$

Therefore, we have shown that N is stochastically continuous and that implies, then, that N, the Poisson process, is a Lévy process.

Now that we have introduced Lévy processes, in the next video we are going to look at more properties of Lévy processes.

3.2.6 Transcript: More Properties of Lévy Processes

Hi, in this video we look at more properties of Lévy processes.

Let X be a Lévy process.

1. The first property that we will look at is the **martingale property**.

   The Lévy process itself is not a martingale all the time; however, if X_t is integrable, then the stochastic process $M_t = X_t - E(X_t)$ is a martingale with respect to the natural filtration of X_t. So, let's show that martingale property.

   The first condition – that it must be adapted – is clear because we are using the natural filtration of X_t, and M_t is a function of X_t. The second property – that it is integrable – has already been stated when we said that X_t itself must be integrable. Therefore, looking at the equation, X_t is integrable and $E(X_t)$ is just a finite constant, which means that M_t will be integrable as well.

   All we need to show, then, is the third martingale property, which says that if $s < t$, we have to calculate the expected value – the conditional expectation – of M_t given $\mathcal{F}_s^x$ for $s < t$. When we calculate that, we get $E(X_t - E(X_t)|\mathcal{F}_s^x)$, which is a constant and can therefore be taken out, to leave us with $E(X_t|\mathcal{F}_s^x) - E(X_t)$.

   The expected value of X_t is equal to $E(X_s + X_t - X_s|\mathcal{F}_s^x) - E(X_t)$. Here, we are doing what is called increment creation, which is then equal to, looking at the first part of the equation (because X_s is $\mathcal{F}_s^x$ measurable), X_s itself, plus the expected value of $X_t - X_s$, because $X_t - X_s$ is independent of $\mathcal{F}_s^x$ due to it being a Lévy process and therefore it has independent increments, minus the expected value of X_t.

This is then equal to $X_s + E(X_t - X_s) - E(X_t)$, and because $X_t - X_s$ has the same distribution as X_{t-s}, this means that we can split the expectation as the expected value of X_t minus the expected value of X_s. We then cancel out $E(X_t)$ in both cases where it occurs, and we get $X_s - E(X_s) = M_s$, which is the martingale property. Written in full:

$$\begin{gathered} s < t, \quad E(M_t | \mathcal{F}_s^x) - E(X_t) \\ = E(X_t | \mathcal{F}_s^x) - E(X_t) \\ = E(X_s + X_t - X_s | \mathcal{F}_s^x) - E(X_t) \\ = X_s + E(X_t - X_s) - E(X_t) \\ = X_s + E(X_t) - E(X_s) - E(X_t) = X_s - E(X_s) = M_s. \end{gathered}$$

2. The second property that we will look at is a distributional property for which we will need to calculate the moment-generating function (MGF) of a Lévy process. So, we will call M_t the MGF of X_t. In other words, this is the expected value of $e^{\alpha X_t}$. Let's try and find the form of this MGF.

   Let's start with the equation:

$$M_{t+s} = E(e^{\alpha X_{t+s}}) = E\left(e^{\alpha(X_t + X_{t+s} - X_t)}\right).$$

   Here, we have performed an increment creation, because this cancels out, and we can write it out as:

$$E\left(e^{\alpha X_t} \cdot e^{\alpha(X_{t+s} - X_t)}\right).$$

   Now, we can use independents of increments to split this expectation to equal the following:

$$E(e^{\alpha X_t}) \cdot E\left(e^{\alpha(X_{t+s} - X_t)}\right).$$

The second part of the equation is equal to $M_t(\alpha)$ times (because X has stationary increments) $X_{t+s} - X_t$, which has the same distribution as $X_s - X_0$, which equals 0 (meaning that it has the same distribution as X_s). Hence, this will be same as the expected value of $e^{\alpha X_s}$, which is $M_s(\alpha)$. Written in full:

$$M_t^{(\alpha)} = M_{X_t}{}^{(\alpha)} = E(e^{\alpha X_t})$$

$$M_{t+s}{}^{(\alpha)} = E(e^{\alpha X_{t+s}}) = E\left(e^{\alpha(X_t + X_{t+s} - X_t)}\right)$$

$$= E\left(e^{\alpha X_t} \cdot e^{\alpha(X_{t+s} - X_t)}\right) = E(e^{\alpha X_t} \cdot E\left(e^{\alpha(X_{t+s} - x_t)}\right)$$

$$= M_t(\alpha) \cdot M_s(\alpha).$$

So, we have found that $M_{t+s}{}^{(\alpha)}$ is equal to $M_t(\alpha)$ times $M_s(\alpha)$. Therefore, we can use this to derive a differential equation satisfied by M_t as follows:

If t and h are positive, and we think of h as being very small, we use this relationship here to get $M_{t+h}{}^{(\alpha)}$, which is equal to $M_t^{(\alpha)} M_h^{(\alpha)}$. If you subtract M_t on both sides, we get:

$$M_{t+h}{}^{(\alpha)} - M_t^{(\alpha)} = M_t^{(\alpha)} \left[M_h^{(\alpha)} - 1\right].$$

We can then divide both sides of the equation by h, and we take the limit as h tends to 0 and assume differentiability, of course. This gives us the derivative, $\frac{d}{dt}$ of $M_t^{(\alpha)}$, which is equal to $M_t^{(\alpha)}$ times something that does not depend on t but depends on α, which we will call ψ of α (some function ψ of α).

We can then solve this differential equation, a separable differential equation, which means we can divide by $M_t^{(\alpha)}$ on both sides, and, of course, get the answer as $M_t^{(\alpha)} = M_0^{(\alpha)} e^{\psi(\alpha)t}$. M_0 is equal to 1, because the Lévy process starts at 0, which means that the final part of the equation will be equal to $e^{\psi(\alpha)t}$. Written in full:

$$t, h > 0 \quad M_{t+h}{}^{(\alpha)} = M_t^{(\alpha)} M_h^{(\alpha)}$$

$$\frac{M_{t+h}(\alpha) - M_t(\alpha)}{h} = \frac{M_t(\alpha)[M_h(\alpha) - 1]}{h}$$

$$\frac{d}{dt} M_t(\alpha) = M_t(\alpha)\psi(\alpha)$$

$$M_t^{(\alpha)} = M_0^{(\alpha)} e^{\psi(\alpha)t} = e^{\psi(\alpha)t}.$$

That is therefore the form of the MGF of the Lévy process X_t. It is just $e^{\psi(\alpha)t}$,depending on what t is.

This is a very useful property of Lévy processes, which is linked to something that we call infinite divisibility of the Lévy process.

Now that we have looked at a few properties of Lévy processes, in the next video, we are going to look at the compound Poisson process and exponential Lévy models.

3.2.7 Notes: Exponential Lévy Models

We now illustrate how to apply Lévy processes to financial modeling. We will consider models for the stock price S that are of the form $S_t = S_0 e^{X_t}$ or $S_t = S_0 \mathcal{E}(X)_t$, where X is a Lévy process. Here $\mathcal{E}(X)$ is the stochastic exponential of X, which we will define below. An important result that we will use here is that every Lévy process is a semimartingale.

We have seen such a model in the previous module (the Black-Scholes model), where S satisfies the following SDE:

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t) = S_t\, dX_t,$$

where $X_t := \mu t + \sigma W_t$ is a Lévy process. We can write S as

$$S_t = S_0 e^{X_t - \frac{1}{2}\sigma^2 t} = S_0 \mathcal{E}(X)_t.$$

To deal with general discontinuous Lévy processes, we need to first introduce the stochastic calculus for general semimartingales. We state only the one-dimensional version of Ito's formula; the multidimensional version can be found in any standard reference for stochastic calculus.

Theorem 3.1 (Ito's Formula). *Let X be a semimartingale and $f: \mathbb{R} \to \mathbb{R}$ be a twice-continuously differentiable function. Then $f(X)$ is also a semimartingale and*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s^-})\, dX_s + \frac{1}{2}\int_0^t f''(X_{s^-})\, d[X]_s + \sum_{s \le t} \left(\Delta f(X_s) - f'(X_{s^-})\Delta X_s - \frac{1}{2} f''(X_{s^-})\Delta X_s^2\right)$$

Here $[X]$ is the (optional) quadratic variation of X, rather than the predictable quadratic variation of X denoted by $\langle X \rangle$. The latter is not even defined for all semimartingales, but the two are equal for continuous semimartingales.

We now introduce the stochastic exponential of a general semimartingale X. Consider the following stochastic differential equation for Y:

$$dY = Y\_dX, \quad Y_0 = 1.$$

That is, we want to find a process Y that satisfies the equation

$$Y_t = 1 + \int_0^t Y_{s^-}\, dX_s.$$

Substituting $U_t = \ln Y_t$ and applying Ito's Lemma gives

$$Y_t = exp(X_t - X_0 - \frac{1}{2}[X]_t) \prod_{s \le t} (1 + \Delta X_s) \exp(-\Delta X_s + \frac{1}{2} \Delta X_s^2).$$

We will denote this process by $\mathcal{E}(X)$.

In general, exponential Lévy models give rise to incomplete markets. Consider the following popular *jump-diffusion* type model for S:

$$dS_t = S_{t^-}(\mu\, dt + \sigma\, dW_t + dJ_t)$$

where

$$J_t = \sum_{k=1}^{N_t} Y_k$$

is a compound Poisson process with $Y_k \succ -1$. The solution to this SDE is

$$S_t = S_0 \mathcal{E}(X)_t$$

where $X_t \coloneqq \mu t + \sigma W_t + J_t$. The solution is

$$S_t = S_0 \exp\left(X_t - \frac{1}{2}(\sigma^2 t + [J]_t)\right) \prod_{s \le t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2} \Delta X_s^2\right)$$

$$= S_0 \exp\left(X_t - \frac{1}{2}\sigma^2 t\right) \prod_{k=1}^{N_t} (1 + Y_k) \exp(-Y_k) = S_0 \exp\left(X_t - J_t - \frac{1}{2}\sigma^2 t\right) \prod_{k=1}^{N_t} (1 + Y_k)$$

$$= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t + \sum_{k=1}^{N_t} \ln(1 + Y_k)\right)$$

since

$$[J]_t = \sum_{s \le t} \Delta J_s^2 = \sum_{s \le t} \Delta X_s^2 = \sum_{k=1}^{N_t} Y_k^2.$$

Transcript: The Compound Poisson Process and Exponential Lévy Models

Hi, in this video we introduce the compound Poisson process and exponential Lévy models.

Let N be a homogeneous Poisson process, where $N = \{N_t : t \geq 0\}$ with constant rate $\lambda > 0$. In addition, let $\{Y_k\}_{k-\wedge}{}^{\infty}$ be a sequence of i.i.d random variables – so, they are independent and have the same distribution.

A continuous-time process X, where $X = \{X_t : t \geq 0\}$, is called a compound Poisson process if:

$$X_t = \begin{cases} 0 & \text{if} \quad N_t = 0 \\ \sum_{k=1}^{N_t} Y_k & \text{if} \quad N_t \geq 1. \end{cases}$$

This is what a compound Poisson process is. So, we are just summing a random number of random variables up to N_t, depending on what N_t is.

Now, it turns out that a compound Poisson process is also a Lévy process. You can check the other conditions as an exercise. What we will do now, however, is check the stationarity of increments.

Let's consider the increment $X_t - X_s$, where s is less than t, and we are going to calculate its MGF.

We are going to calculate $E\left(e^{\alpha(X_t - X_s)}\right)$, which is equal to $E\left(e^{\alpha\left(\sum_{k=N_s+1}^{N_t} Y_k\right)}\right)$. This is what the increment is, if we look at the definition above (assuming, of course, that N_t is greater than or equal to 1). This is equal to, when we use the Law of Total Expectation,

$E(E(e^{\alpha \sum_{k=N_{s+1}}^{N_t} Y_k})$ given the σ-algebra generated by N_t and N_s. If we calculate this conditional expectation, since the Y_k's are independent – we will always assume that the Y_k's are also independent of the Poisson process itself – they are equal to the expectation of the MGF of Y (each one of the Y_k's have the same distribution), all to the power $N_t - N_s$, because this is the sum of $N_t - N_s$ number of variables. $N_t - N_s$ has a Poisson distribution with parameter λ times $t - s$, which is equal to $e^{\lambda(t-s)(M_y(\alpha)-1)}$. As we can see, this only depends on $t - s$, the length of the increment. It doesn't depend on the starting point explicitly. Written in full:

$$\begin{aligned} &= E(E(e^{\alpha \sum_{k=N_{s+1}}^{N_t} Y_k})|\, \sigma(N_t, N_s)) \\ &= E\left(\left(M_y(\alpha)\right)^{N_t - N_s}\right) \\ &= e^{\lambda(t-s)(M_y(\alpha)-1)}. \end{aligned}$$

This shows, therefore, that a compound Poisson process has stationary increments. In addition, you can show that the other properties of Lévy processes are satisfied for X as well.

In finance, we are going to be dealing a lot with what are known as **exponential Lévy models.** When looking at stocks, we say that a stock, S_t, has an exponential Lévy model if S can be written as S_0 times the exponent of X_t, where the stochastic process X_t is a Lévy process.

An example of this that we have dealt with is the case when $X_t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$, where W_t is a Brownian motion. As a reminder, we saw this when we looked at the Black-Scholes model, because the solution of that SDE, the geometric Brownian motion, was $S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$. This is a Lévy process; Brownian motion is a Lévy process; this deterministic process is a Lévy process; and the sum of the two is also a Lévy process.

We are going to see other examples of exponential Lévy models. For instance, we can take X_t to be equal to some constant bt plus σW_t plus a compound Poisson process,

$\sum_{k=1}^{N_t} Y_k$. This is an example of a Lévy process that we can use to construct an exponential Lévy model. The advantage of this is not only that it is continuous as this is, but it also accounts for jumps as well. In other words, this is a combination of a diffusion, which starts at 0 and then, as soon as an event occurs here, at a Poisson rate of λ, this jumps by Y_k. It can jump up or down depending on the sign of Y_k. Y_k can be positive or negative. It then continues as a Brownian diffusion and then jumps again by Y_k, and so on. So, this is an example of a model that has both jumps and diffusion components, and it is very useful in modeling as well.

Now that we have covered the compound Poisson process, we have reached the end of the module.

Notes: Problem Set

Problem 1

Let the number of accidents N occur according to a Poisson process with rate $\lambda = 5$ per day. What is the expected number of accidents between the fifth day and the seventh day?

Solution:

Since N has a Poisson distribution, $E(N) = \lambda t$. That is, we expect λt accidents in t times units. Thus, in our example, the expected number of accidents between the fifth day and the seventh day is equal to $5 * 2 = 10$.

Problem 2

Let the number of accidents N occur according to a Poisson process with rate $\lambda = 2$ per day. What is the probability that the number of accidents between the third day and the fourth day is 3?

Solution:

Let's start with the probability:

$$P(N_4 - N_3 = 3) = P(N_1 = 3).$$

We also know from the lecture notes that

$$P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

So, finally we get

$$P(N_4 - N_3 = 3) = P(N_1 = 3) = \frac{e^{-2}2^3}{3!},$$

which is the solution to the problem.

Problem 3

Let the number of accidents N occur according to a Poisson process with rate $\lambda = 2$ per day. Given that no accidents have occurred in the last 3 days, what is the probability that the next accident occurs within the next day?

Solution:

The probability that the next accident occurs between day four and three is

$$P(N_4 - N_3 > 0) = P(N_1 > 0) = 1 - P(N_1 \leq 0) = 1 - \frac{e^{-\lambda t}(\lambda t)^k}{k!},$$

with $k = 0, \lambda = 2$, and $t = 1$. The solution is equal to

$$P(N_4 - N_3 > 0) = P(N_1 > 0) = 1 - P(N_1 \leq 0) = 1 - \frac{e^{-\lambda t}(\lambda t)^k}{k!} = 1 - e^{-2}.$$

Problem 4

Let N be a Poisson process with rate $\lambda = 2$ and $Y_1, Y_2, \ldots$ be i.i.d. normal random variables with mean $\mu = 2$ and variance $\sigma^2 = 1$. Define the compound Poisson process X by

$$X_t := \sum_{k=1}^{N_t} Y_k.$$

What is $\mathbb{E}(X_2)$?

Solution:

The expected value, considering that $Y_1, Y_2, \ldots$ are i.i.d. normal random variables, can be computed as follows:

$$E[X_t] := E\left[\sum_{k=1}^{N_t} Y_k\right] = E[N_t] * E[Y] = \lambda t * \mu.$$

For $t = 2$, we will have: $E[X_2] = 2 * 2 * 2 = 8$.

Problem 5

Let N be a Poisson process with rate $\lambda = 6$ and $Y_1, Y_2, \ldots$ be i.i.d. uniform random variables between 0 and 2. Define the compound Poisson process X by

$$X_t := \sum_{k=1}^{N_t} Y_k.$$

What is the variance of X_1?

Solution:

We can compute the variance as follows:

$$\begin{aligned} Var\big(X(t)\big) &= \mathbb{E}\big[\operatorname{Var}\big(X(t)\big) \mid N(t)\big] + \operatorname{Var}(\mathbb{E}[X(t) \mid N(t)]) \\ &= \mathbb{E}[N(t)\operatorname{Var}(Y_1)] + \operatorname{Var}(N(t)\mathbb{E}[Y_1]) \\ &= \sigma^2\mathbb{E}[N(t)] + m^2\operatorname{Var}\big(N(t)\big) \\ &= \sigma^2\lambda t + m^2\lambda t \\ &= (\sigma^2 + m^2)\lambda t. \end{aligned}$$

Applying the above expression to our problem, we will get

$$\text{Var}\big(X(t)\big) = (\sigma^2 + m^2)\lambda t = \left(\frac{1}{3} + 1\right) * 6 * 1 = 8.$$

Problem 6

Let X be a Lévy process such that $X_2 \sim \text{Gamma}(\alpha = 20, \lambda = 2)$. If $X_t + bt$ is a martingale, then compute the value of b.

Solution:

We need to compute b such that the expected value of $X_t + bt$, for $t = 2$, is equal to zero. Thus,

$$E[X_2 + 2b] = E[X_2] + 2b = \frac{20}{2} + 2b = 0, \text{ then } b = -5.$$