Compiled Content Module 3

MScFE 640

Portfolio Theory and Asset Pricing

Revised: 09/09/2019

Table of Contents

Module 3: Multi-period Portfolio Theory and Asset Pricing

Module 3 begins by discussing strategic and tactical asset allocation and continues with discussion on portfolio and asset characteristics that affect the rebalancing decision. The multi-stage equity valuation and common valuation are then discussed, and a detailed description of bond features provided. At the end of the module yield curves, spreads, and the concept of yield to maturity are discussed with various examples on how to value bonds.

Unit 1: Asset Allocation and the Rebalancing Decision

In Module 1 we gained a good introductory understanding of modern portfolio theory. To recap: we learnt that the portfolio on the efficient frontier will have the highest return amongst all portfolios that have the same volatility. It is considered efficient because it will have the highest return per unit of volatility taken on – i.e. the highest Sharpe Ratio.

In these notes we will move on to a discussion of an investor's strategic asset allocation and portfolio rebalancing.

Strategic asset allocation

Strategic asset allocation refers to selecting the portfolio that is appropriate for a unique investor from the universe of available portfolios. The strategic asset allocation choice will be influenced by the following factors (amongst others):

- The investor's **return requirement**. This refers to the rate of return the investor must achieve or wishes to achieve. If an investor is planning for retirement, they can reasonably accurately calculate a rate of return they will require on their assets to meet retirement goals after considering future income, spending needs (pre- and post-retirement), and a reserve for surprises (e.g. medical expenses).

- The investor's **risk tolerance**. This refers to both their *willingness* (an emotional issue) and their *ability* (a practical issue) to take risk. For example, some people will simply not be comfortable with the possibility of their portfolio declining in value (indicating low willingness). Others may not be able to, due to having a low level of wealth and fixed, non-negotiable future expenses (indicating low ability).

- An investor may have ethical or religious considerations which prohibit them from investing in certain assets.

Example: Strategic asset allocation for an individual

A wealth manager is meeting with his client, Mr Avis. They are deciding which of the five portfolios listed in the table below is most appropriate for Mr Avis. Mr Avis has the following constraints:

- He must earn an annual rate of return of 9.7% to meet his retirement goals.

- He has a moderate risk tolerance level. In consultation with his wealth manager, they have decided that portfolio risk should not exceed 11%.

Portfolio	Asset class				Expected return	Expected risk
	Cash & equivalents	AAA-rated corporate bonds	Large-cap US equity	Private equity		
A	30%	40%	25%	5%	8.2%	7.4%
B	25%	40%	25%	10%	9.1%	9.2%
C	20%	40%	30%	10%	9.8%	10.4%
D	15%	40%	30%	15%	10.5%	11.3%
E	5%	35%	35%	25%	11.6%	18.9%

The most appropriate portfolio for Mr Avis is Portfolio C. This is arrived at by a process of elimination:

- Portfolio A and B's return is too low.
- Portfolio D and E's risk is too high.

Portfolio C would be Mr Avis's strategic asset allocation. Mr Avis's portfolio will be invested in cash, AAA-rated corporate bonds, large-cap US equity, and private equity in the proportions indicated in the example.

However, as time passes the following might happen:

- Return, volatility, and correlation expectations for asset classes will change. This will impact the expected return and risk of Portfolio C.
- Some asset classes will increase in value and some will decrease. The asset classes that increase in value with make up a greater proportion of the portfolio, at the expense of those that have decreased in value. Thus, the portfolio will become **unbalanced** – i.e. not

invested in the asset classes in the proportions recommended by the strategic asset allocation.

Portfolio rebalancing refers to the process of returning the portfolio to the strategic asset allocation as time passes.

A portfolio manager may be required to invest a client's assets strictly according to the strategic asset allocation. Or, the client may authorize the manager to deviate from the strategic asset allocation for short time periods in order to make investments which the manager believes will add value to the portfolio. This freedom to act is known as **tactical asset allocation**.

Portfolio rebalancing

Two questions are relevant to portfolio rebalancing: *when* and *how*. With regards to *when*, two options are available:

- **Calendar rebalancing**. The portfolio is rebalanced on a predetermined, regular basis. The main advantage of this method is that it is a disciplined system and makes it unnecessary to monitor the portfolio between rebalancing dates. The main disadvantage is that the portfolio may drift far from the strategic asset allocation between rebalancing dates.

- **Percentage-of-portfolio rebalancing**. A "band" is created around each asset class. If the weight of an asset class exceeds the band, the need to rebalance is triggered. For example, if a band of 10% were set around large-cap US equity in Portfolio C, then the need to rebalance would be triggered if portfolio allocation to this asset class was < 36% or > 44% (10% of 40% is 4%, so the corridor of acceptable allocations is 40%±4%).

With regards to *how*, three rebalancing strategies are available:

Buy-and-hold

The **buy-and-hold strategy**, as the name implies, does not involve any rebalancing at all. In this strategy, the high return assets will come to dominate the portfolio over time. The high return assets will also have higher risk. As a result, the portfolio will become riskier with the passing of time. It is also possible that the asset classes with a recent period of good returns will become fully valued or overvalued, leading to a decline in the rate of return or negative return in subsequent periods. This rebalancing strategy would be appropriate for investors whose risk tolerance increases as their level of wealth increases.

Constant mix

The **constant mix strategy** requires that the portfolio be returned to the strategic asset allocation when it is rebalanced.

Example: Portfolio rebalancing according to the constant mix strategy

A portfolio consists of two asset classes: equity and cash. For this example, we'll assume cash returns 0%. The portfolio is worth $100 initially, with 60% allocation to equity and 40% to cash. The portfolio follows a calendar rebalancing strategy with monthly rebalancing. One month later, equity has returned -16.7% and is worth $50.

The unbalanced portfolio's asset class weights are 55.6% for equity and 44.4% for cash. To rebalance the portfolio, $3.96 of equity will be purchased. The balanced portfolio will be 60% ($53.96) invested in equity and 40% ($36.04) invested in cash.

The important feature of the constant mix strategy is that implementing it results in the selling of assets that have increased in value, and the purchase of assets that have decreased in value. Implementing this strategy is more likely than the other strategies to result in the selling of fully valued or over-valued assets and the purchase of undervalued assets. This rebalancing strategy would be appropriate for investors whose risk tolerance does not change as their level of wealth changes.

Constant proportion portfolio insurance

The **constant proportion portfolio insurance (CPPI)** strategy sets an asset class's allocation with the following formula:

$$Target\ asset\ class\ investment = M(portfolio\ value - floor\ value\ for\ asset\ class)$$

M is a constant proportion, such as 1.2, which is set initially. The floor value is also set initially.

Example: Asset class investment according to the CPPI

We'll use the same portfolio that was used to demonstrate the constant mix strategy. The floor value of equity is $50 and M=1.2.

One month later, equity has returned -16.7%. The unbalanced portfolio is now worth $50 (equity) + $40 (cash) = $90. According to the CPPI strategy, the new allocation to equity should be:

$$Target\ asset\ class\ investment = 1.2(\$90 - \$50) = \$48$$

To rebalance the portfolio: $2 of equity will be sold, leaving the balanced portfolio with a $48 investment in equity and $42 investment in cash.

The important feature of the CPPI strategy is that implementing it results in the selling of assets that have declined in value, and the purchase of assets that have increased in value. Similar to the buy-and-hold strategy, this results in increased allocation to fairly- or over-valued assets, and decreased allocation to under-valued assets. The high-return assets will have higher risk, leading to the portfolio becoming riskier with time.

Band width in percentage-of-portfolio rebalancing

The percentage-of-portfolio rebalancing strategy requires that a band or corridor be set around each asset class in the portfolio. How wide should this band be? This is dependent on four factors:

- **Correlation between asset classes.** The higher the correlation between asset classes, the wider the band should be. Why? Imagine a portfolio that is 60% invested in equity and 40% invested in bonds. The correlation between them is 1 – i.e. perfectly positively correlated. With this correlation, the portfolio will always remain invested 60%/40% regardless of the asset class returns. The portfolio will not stray from the strategic asset allocation. Therefore, if correlation between asset classes is low, the band should be narrower because the portfolio is more likely to become unbalanced. Since being unbalanced is undesirable, the bands should be narrower to ensure the portfolio is rebalanced to the strategic asset allocation before large deviations arise.

- **Transaction costs**. The higher the transaction costs, the wider the band should be. This is because larger deviations from the strategic asset allocation are required to justify the cost of trading.

- **Volatility of asset classes.** The higher the volatility of asset classes, the narrower the band should be. High volatility means that the portfolio is more likely to become unbalanced.

- **Investor's risk tolerance**. The higher an investor's risk tolerance, the wider the band should be. Larger increases in riskiness of the portfolio are required to justify the cost of trading.

Unit 2: Multi-stage Equity Valuation and Common Valuation Multiples

In Module 1, we introduced the Gordon Growth Model (GGM). In those notes, it was applied in single-stage valuation, but it was stated that most businesses have a multi-stage growth trajectory. We will study how the GGM can be applied in these situations.

Another common method of valuation is to use price multiples. This method is best used to compare similar businesses. For example, comparing the price-to-book (P/B) ratio of a mining company and a technology company would not make sense. Most of a mining company's value will be captured in its book value: the deposits and infrastructure it owns. A technology company does not have many fixed assets, therefore its book value will not capture much of its true value. A price multiple tells the investor how much they'll be paying per unit of something: earnings, cash flow, book value, etc.

Equity valuation (multiple periods)

Multi-period Gordon Growth Model

The GGM was given in Module 1 as follows:

$$V_0 = \frac{D_0(1+g)}{r-g},$$

where:

V_0 = the present value (at time 0) of the share

D_0 = the most recently paid dividend per share

g = the periodic growth rate of dividends

r = the periodic equity discount rate.

(Note that we do not necessarily have to use dividends as the cash flow input into the GGM. We can use other cash flow inputs, depending on our perspective when performing the valuation. This topic was discussed in more detail in Module 1.)

The following example will describe the implementation of the GGM to a multiple-stage growth path.

Example: Multiple-period valuation using the GGM

It is the end of 2007 and a company, Nkosi Company, has just reported free cash flow to equity (FCFE) of $2 per share. An analyst is valuing the company on behalf of a private equity investor who intends to purchase a majority stake in the company. Therefore, the analyst decides to use FCFE to value the company. The analyst uses an equity discount rate of 12.5%.

FCFE is expected to grow at the following rates in the next 6 years:

Year	FCFE growth rate
2008	18%
2009	18%
2010	16%
2011	12%
2012	11%
2013	6%

In 2014 and into the future, FCFE is expected to grow at 6%. What is the present value of a share of the Nkosi Company?

Solution:

Future FCFE has been modeled in the table below.

Year	2008	2009	2010	2011	2012	2013
FCFE growth	18%	18%	16%	12%	11%	6%
FCFE	$2.36	$2.78	$3.23	$3.62	$4.02	$4.26
PV of FCFE	$2.10	$2.20	$2.27	$2.26	$2.23	$2.10
Terminal value						$69.47

The **PV of FCFE** is the Present Value of FCFE. It is each year's FCFE discounted by the number of years into the future it is. For example, the end of 2008 is one year away from the present moment so the present value of 2008's FCFE of $2.36 is:

$$\frac{\$2.36}{(1+12.5\%)^1} = \$2.10.$$

Similarly, the end of 2013 is six years away from the present moment so the present value of 2013's FCFE of $4.26 is:

$$\frac{\$4.26}{(1+12.5\%)^6} = \$2.10.$$

The total present value of Nkosi Company is the sum of two components:

1. The sum of the present values of the FCFEs from 2008 to 2013; and
2. The present value of the terminal value.

Component 1 is: $\$2.10 + \$2.20 + \$2.27 + \$2.26 + \$2.23 + \$2.10 = \$13.16.$

Calculating the terminal value simply requires applying the single-stage GGM at the end of 2013. At the end of 2013:

- FCFE of $4.26 has just been earned $\therefore FCFE_{2013} = \$4.26$
- g is 6%
- r continues to be 12.5%.

So, applying the single stage GGM:

$$V_{2013} = \frac{FCFE_{2013}(1+g)}{(r-g)} = \frac{\$4.26(1+6\%)}{(12.5\% - 6\%)} = \$69.47.$$

To calculate component 2, the terminal value must now be discounted to its 2007 value:

$$V_{2007} = \frac{V_{2013}}{(1+r)^{2013-2007}} = \frac{\$69.47}{(1+12.5\%)^6} = \$34.27.$$

Summing component 1 and 2: $\$13.16 + \$34.27 = \$47.43$ is the present value per share of Nkosi Company.

H-model

The H-model is also a multiple-stage model which assumes a linear decrease in the growth rate, from an initial high value to a lower, sustainable value. The valuation of a company that is an early entrant to a profitable industry that attracts competition over time would be the most relevant application of this model. Initially, the company is very profitable, and earnings grow rapidly. The profitability of the industry attracts competition, which lowers profits. By the end, an equilibrium has been reached, with the survivors of the competitive phase each maintaining a roughly constant market share of an industry that has saturated the market.

The formula for the H-model is the following:

$$V_0 = \frac{D_0(1+g_L) + D_0 H(g_S - g_L)}{(r - g_L)},$$

where:

V_0 = the present value

D_0 = the most recently paid dividend (or other cash flow)

g_L = sustainable long-term growth rate after period 2H

H = the number of periods over which growth is expected to linearly decline from the short-term high growth rate to the sustainable long-term growth rate, divided by 2

g_S = short-term high growth rate

r = equity discount rate.

Example: Application of the H-model

Germiquet Motor Company (GMC) is a designer and manufacturer of electric vehicles. Most existing vehicle manufacturers are set up for the manufacture of petroleum-fueled vehicles. They are struggling to adjust their operations to the manufacture of electric vehicles. Government regulations, subsidies, and lower input costs have made electric vehicles more affordable and as a result demand is growing rapidly. GMC is solely focused on the manufacture of electric vehicles and is currently very profitable. Growth in dividends is expected to be 15% in the coming year. The most recent dividend was $2.50. As competitors enter the market, GMC's growth rate is expected to decline linearly over a 15 year period, to 6%. The equity discount rate is 10%. The value of GMC based on the H-model is:

$$V_0 = \frac{D_0(1+g_L) + D_0H(g_S - g_L)}{(r - g_L)} = \frac{\$2.50(1+6\%) + (\$2.50)(^{15}/_{2})(15\% - 6\%)}{(10\% - 6\%)}$$

$$V_0 = \$108.44$$

Multiples

Price multiples allow you to compare the relative value of similar assets. It answers the question: how much am I paying per unit of something? For example, in grocery stores a product – e.g. milk – that is sold in different quantities – e.g. 500 mL, 1 L, 2 L – will quote each product's price per liter, to allow consumers to make comparisons between brands and quantities quickly. Price multiples in asset valuation perform a similar function for investors.

Commonly used price multiples include:

- Price-to-earnings (P/E) ratio, or its inverse: earnings yield
- Price-to-book (P/B) ratio
- Price-to-sales (P/S) ratio
- Dividend yield.

Example: Using price multiples in valuation

Excerpts from Whitton Company's recent financial statements appear below:

Balance sheet	
	'000s
Current assets	$1 000
Non-current assets	$2 500
Current liabilities	$750
Non-current liabilities	$350

Income statement	
	'000s
Revenue	$800
Cost of sales	$250
Operating expenses	$300
Gross income	$250
Tax paid	$50
Net income	$200

At financial year end Whitton Company had 10 million shares outstanding. The company follows a policy of paying out 25% of net income as dividends. The market price of Whitton Company is 24 cents per share.

The price-to-earnings ratio of Whitton is:

$$\frac{\textit{Price per share}}{\textit{Earnings per share}} = \frac{\$0.24}{\$200\ 000 / 10\ 000\ 000} = \frac{\$0.24}{\$0.02} = 12.$$

Or alternatively the earnings yield is:

$$\frac{\$0.02}{\$0.24} = 8.33\%.$$

The price-to-book ratio is:

$$\frac{\textit{Price per share}}{\textit{Book value per share}} = \frac{\$0.24}{(\$1\ 000\ 000 + \$2\ 500\ 000 - 750\ 000 - \$350\ 000) / 10\ 000\ 000}$$

$$= \frac{\$0.24}{\$2\ 400\ 000 / 10\ 000\ 000} = \frac{\$0.24}{\$0.24} = 1.$$

The price-to-sales ratio is:

$$\frac{\textit{Price per share}}{\textit{Sales per share}} = \frac{\$0.24}{\$800\ 000 / 10\ 000\ 000} = \frac{\$0.24}{\$0.08} = 3$$

The dividend yield will be:

$$\frac{\textit{Dividend per share}}{\textit{Price per share}} = \frac{25\% \times \$200\ 000 / 10\ 000\ 000}{\$0.24} = 2.08\%$$

An important point to be aware of when implementing price multiples in asset valuation is the distinction between relative value and intrinsic value. Price multiples should only be used to compare similar companies. For example, a price multiple might tell you that company A is cheap compared to company B, but that does not necessarily mean that company A is cheap compared to its intrinsic value. Both companies A and B may be overvalued compared to their intrinsic value. For this reason, both relative value and intrinsic value should be calculated in a thorough analysis of company. The terms relatively undervalued/overvalued and undervalued/overvalued should be used to refer to a company's relative value and intrinsic value respectively.

Unit 3: Bond Features

At its very simplest, a bond is an "I Owe You". An investor who purchases a bond gets a commitment from the issuer of the bond to repay the par value (also known as **face value**) at a future date. In effect, the investor has lent money to the bond issuer, who is the borrower. Bonds have unique features and terminology which will be introduced in these notes.

Par value and maturity

The **par value** of a bond is the loan amount that must be repaid to bondholders at maturity. The **maturity** of a bond (usually quoted in years) is the time until the par value of a bond is due.

Example A: Bond par value and maturity

Martin Corporation needs $50 million of additional capital for expansion. It approaches an investment bank to assist it with a bond offering. The investment bank assists Martin Corporation to issue 525 000 bonds, each with a par value of $100, which will be repaid in 5 years. The investment bank's fees are $2.5 million, leaving Martin Corporation with the remaining $50 million. Martin Corporation now has an obligation to repay bondholders $52.5 million in 5 years' time when the bond matures.

Zero-coupon bonds

The $52.5 million, 5-year Martin Corporation bond described in Example A is known as a **zero-coupon bond**. This is because it does not pay **coupons**, or interest payments, during the life of the bond. Bondholders earn their return by purchasing the bonds at a discount to their par value. The example will illustrate this.

Example B: Zero-coupon bonds

In Example A, Martin Corp. was able to borrow money from investors for free. In the real world, investors are not that generous. In order to attract investors, Martin Corp. would need to offer them a return on their investment. For a zero-coupon bond, this would be achieved by issuing the bonds at a discount to their par value. For example, each bond promised to pay $100 at maturity in 5 years' time. If Martin Corp. wanted to offer a return of 10% it could issue these bonds in exchange for $62.09 each.

$$\frac{\$100}{(1+10\%)^5} = \$62.09$$

By paying $62.09 today and receiving a bond that pays $100 at maturity in 5 years' time, a bondholder would earn a 10% annual return.

Bonds

Bonds frequently make interest payments, called coupons, during their life. The two main reasons for this are:

1. It provides investors with income while their capital is being used by the borrower,
2. It reduces investors' risk by providing a return in small increments over the life of the investment, instead of as one single payment at the end.

The coupon rate that a borrower must offer to attract investors depends broadly on two factors:

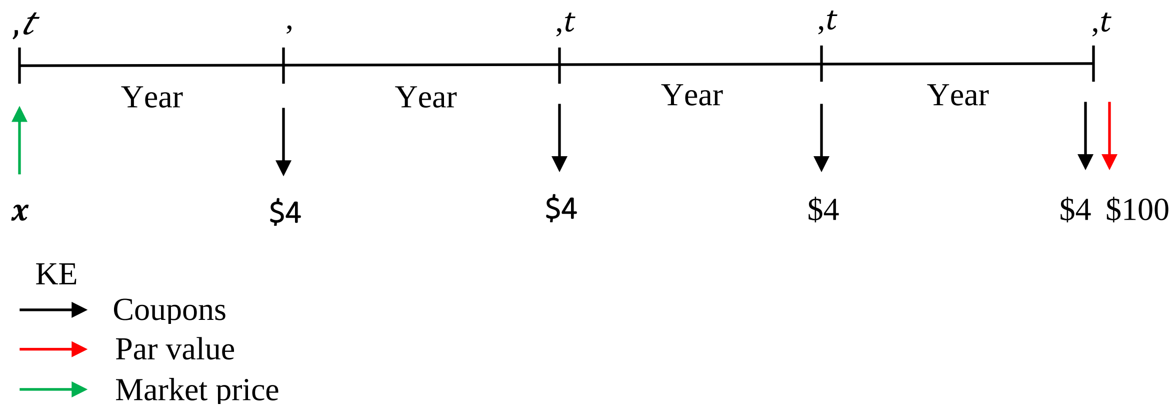
1. The supply of, and demand for, money in the market,
2. The borrower's credit worthiness.

The supply and demand for money in the market will drive up coupon rates when supply is low/demand is high and drive them down when supply is high/demand is low. A borrower's **credit worthiness** is an assessment by external parties of how likely they are to honor their obligations. Borrowers with high credit worthiness will be able to borrow at lower rates as lenders assess them to be less risky, while the opposite will be true for a borrower with low credit worthiness. A borrower with a higher credit worthiness can offer a lower coupon rate compared to a borrower with lower credit worthiness when seeking to borrow the same amount of money.

Example C: Coupon rates and credit worthiness

Martin Corp. has been rated Aaa by Moody's, an international credit rating agency. This is the highest-quality credit rating that can be assigned. Instead of offering a zero-coupon bond, Martin Corp. decides to offer a coupon rate of 4%, paid annually at the end of the year.

The bonds cash flows will look like this:



The illustration above should allow you to visualize when cash flows occur, as this is very important in bond valuation. Notice that at maturity both the par value and a final coupon are paid, for a total payment of $104.

Note that the green arrow is orientated to indicate that this is a cash outflow for the investor – i.e. this is a payment that is made to entitle the bond investor to receive the bond's cash flows.

Coupons can be paid at frequencies other than annually. However, the coupon rate is always quoted in annual terms. For example, a 4% semi-annual pay bond will pay 4% of par value annually, or 2% of par value every 6 months. A 4% quarterly-pay bond will pay 1% of par value quarterly. It is important to remember that interest rates are always quoted in annualized terms, therefore when working with bonds that have cash flows other than annually, appropriate adjustments to interest rates need to be made when discounting these cash flows to their present value – this will be more fully explained in the next set of notes.

Term structure of spot interest rates

A **spot interest rate** is the annualized interest rate used to discount a single risk-free payment that will occur at a future date. For example, if the 1-year spot interest rate is 6% then the present value of a $1 payment that will be received in 1 year is:

$$= \frac{\$1}{(1+6\%)^1} = \$0.9434.$$

If the 2-year spot interest rate is 7% then the present value of a $1 payment that will be received in 2 years is:

$$= \frac{\$1}{(1+7\%)^2} = \$0.8734.$$

A spot interest rate is denoted $r(x)$, where x is the time to maturity. So, the 1-year and 2-year spot interest rates are denoted $r(1)$ and $r(2)$ respectively. The **term structure of spot interest rates** refers to the spot interest rates for a range of time periods, or **maturities**. We will use the following term structure of spot interest rates throughout these notes:

Time to maturity	Spot interest rate
1	6%
2	7%
3	8%
4	9%
5	10%

Table 1: Term structure of spot interest rates

Conclusion

You now understand par value, coupons, and maturity, which are the basic features of bonds. Bonds can become a lot more complex, for example:

- They may have an embedded put option. A **put option** allows the bondholder to *put* the bond back to the issuer. This means that the bondholder, at their discretion, may exercise their right to redeem the bond and receive its par value from the issuer.
- They may have an embedded call option. A **call option** allows the issuer to *call* the bond. This means that the issuer, at their discretion, may exercise their right to redeem the bond by paying par value to bondholders.
- They may be convertible. The bond may be convertible into other forms of capital, for example equity.

A bond without these complex features, such as we have discussed in these notes, is known as a **straight bond**. A bond with an embedded put option would be called a **putable bond**, a bond with an embedded call option a **callable bond**, and a bond with a conversion option a **convertible bond**.

These notes will not cover the valuation of such bonds. However, the fundamental introduction to bond valuation provided in these notes will allow the student to progress into valuing more complex bonds.

Unit 4: Yield Curves, Spreads, and Valuing Bonds

In the third set of notes, the basic features of bonds were described, and the term structure of spot interest rates was introduced. This is reproduced here:

Time to maturity	Spot interest rate
1	6%
2	7%
3	8%
4	9%
5	10%

Table A: Term structure of spot interest rates

In these notes we will learn how the yield to maturity of coupon-paying and zero-coupon bonds is calculated from the term structure of spot interest rates. We will also learn why the term structure of spot interest rates is more frequently and simply known as the **spot yield curve**. We will cover valuation of both risky and risk-free bonds.

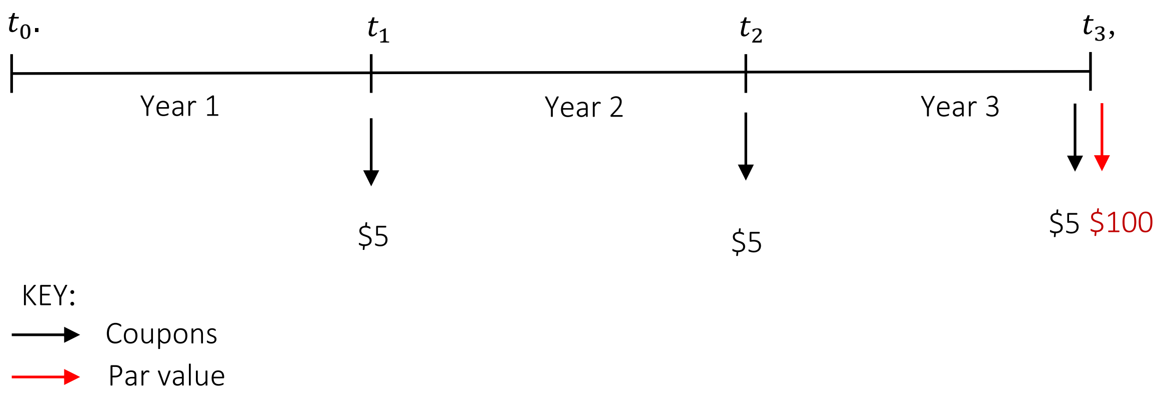
The concept of yield to maturity

The purpose of this section is to distinguish **yield to maturity** from **spot interest rate**. Yield to maturity is denoted $y(x)$, where x is the time to maturity. To demonstrate the difference, let us use the example of a risk-free, 3-year, zero-coupon bond with $100 par value. In three years' time, this bond will pay $100 to its owner, so its present value (using $r(3)$ from Table A) is:

$$= \frac{\$100}{(1+8\%)^3} = \$79.3832.$$

We could calculate this bond's yield to maturity by entering N=3; PV=-$79.3822; PMT=$0; FV=$100 into a financial calculator (or Excel) and solving for I/Y. The I/Y calculated will be 8%. (Note that the 3-year yield to maturity [$\mathrm{y}(3)$] and the 3-year spot interest rate [$\mathrm{r}(3)$] are the same.)

Now let's modify the bond a bit. Let us pretend the bond pays a coupon annually of \$5 (at the end of the year). The following illustration will illustrate the bond's cash flows:



We can go about valuing this bond by considering each cash flow individually. There are three coupon payments of \$5 each and the par value of \$100. To value the bond, we should value each cash flow as though it were a zero-coupon bond.

We value each zero-coupon bond in the same way we have done it previously: by discounting the cash flow by the relevant spot interest rate. In this case, we'll use the spot interest rates of $r(1)$, $r(2)$ and $r(3)$:

$$Present\ value\ of\ bond = \frac{\$5}{(1+6\%)^1} + \frac{\$5}{(1+7\%)^2} + \frac{\$105}{(1+8\%)^3} = \$92.4366.$$

We could calculate this bond's yield to maturity by entering N=3; PV=-\$92.4366; PMT=\$5; FV=\$100 into a financial calculator and solving for I/Y. The I/Y calculated will be 7.9312%. The financial calculator solves for I/Y in the following equation:

$$\frac{\$5}{(1+I/Y)^1} + \frac{\$5}{(1+I/Y)^2} + \frac{\$105}{(1+I/Y)^3} = \$92.4366.$$

(Note that the 3-year yield to maturity [y(3)] and the spot interest rate [r(3)] are different. This is due to the reinvestment of coupons received before maturity.) The \$5 received at time t_1 must be reinvested for 2 years at the 2-year spot interest rate existing at time t_1. Similarly, the \$5 received at time t_2 must be reinvested for 1 year at the 1-year spot interest rate existing at time t_2. These rates are called **forward rates** and are denoted $f(T^*, T)$, where T^* is the time until initiation of the rate and T is the time to maturity from T^*. For example, the 2-year spot interest rate beginning one year from now is denoted $f(1, 2)$.

The forward rate can be implied from spot rates based on the following logical assumption: the forward rate is a rate which makes a zero-coupon bond investor indifferent between two investment options.

For example, the 2-year spot interest rate, one year from now must be the rate that makes an investor indifferent between:

- Purchasing a 3-year zero-coupon bond; or
- Purchasing a 1-year zero-coupon bond, then reinvesting the proceeds for 2 years in a 2-year zero-coupon bond.

Example: Calculating forward rates

The following spot interest rates exist:

Time to maturity	Spot interest rate
1	6%
2	7%
3	8%
4	9%
5	10%

The present value of a 3-year zero coupon bond is:

$$= \frac{\$100}{(1+8\%)^3} = \$79.3832.$$

We could instead:

- Purchase a 1-year zero-coupon bond;
- Reinvest the proceeds received in one year in a 2-year zero-coupon bond.

The value at maturity of \$79.3832 invested for one year is $\$79.3832 * (1+6\%)^1 = \$84.1462$. Now we need to find the yield on \$84.1462 reinvested and paying \$100 at maturity.

$$\$84.1462 * \left(1 + y(2)\right)^2 = \$100$$

$$\therefore y(2) = 9.01\% = f(T^*, T).$$

The implied 2-year spot rate one year from now is 9.01%.

The formula for calculating forward rates from spot rates is the following:

$$[1 + r(T^* + T)]^{(T^*+T)} = [1 + r(T^*)]^{T^*}[1 + f(T^*, T)]^T.$$

Applying this formula to the problem we have just solved:

$$[1 + r(1 + 2)]^{(1+2)} = [1 + r(1)]^1[1 + f(1,2)]^2$$

$$\therefore [1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$

$$\therefore [1 + 8\%]^3 = [1 + 6\%]^1[1 + f(1,2)]^2$$

$$\therefore f(1,2) = \sqrt{\frac{(1 + 8\%)^3}{(1 + 6\%)^1}} - 1 = 9.01\%$$

Due to the fact that spot interest rates are the same as their yield to maturity, the term structure of interest rates is frequently called the **spot yield curve**.

The par curve

The **par curve** is very important in bond valuation. The par curve is the yield to maturity (or **par rate**) of risk-free sovereign bonds issued at par, for a range of maturities. If a bond is issued at par, the present value of future cash flows is equal to par value. Recently issued "on the run" bonds are typically used to create the par curve because these are typically priced at or close to par. The following example will illustrate the calculation of the par curve from the spot yield curve.

Example: Calculating the par curve from the spot yield curve

The following spot interest rates exist:

Time to maturity	Spot yield
1	6%
2	7%
3	8%
4	9%
5	10%

In order for a 1-year, coupon paying, annual-pay bond to be issued at par, it must pay a coupon of 6% annually:

$$Present\ value\ of\ bond = \frac{\$6 + 100}{(1+6\%)^1} = \$100.$$

To calculate the 2-year par rate we must calculate the answer to the following question: what must the coupon rate on a 2-year, coupon paying, annual pay bond be in order for its present value to be equal to par value? We solve for x in the following equation:

$$\$100 = \frac{\$x}{(1+6\%)^1} + \frac{\$x + \$100}{(1+7\%)^2}$$

$$\therefore x = \$6.9673.$$

Therefore, to be priced at par, a risk-free bond must pay coupon rate of 6.9673%.

To calculate the 3-year par rate, we solve for x in the following equation:

$$\$100 = \frac{\$x}{(1+6\%)^1} + \frac{\$x}{(1+7\%)^2} + \frac{\$x + \$100}{(1+8\%)^3}$$

$$\therefore x = \$7.9050.$$

Therefore, to be priced at par, a risk-free bond must pay coupon rate of 7.9050%.

Following the same methodology, we can calculate the remainder of the par rates.

Time to maturity	Spot yield	Par rate
1	6.0%	6.0000%
2	7.0%	6.9673%
3	8.0%	7.9050%
4	9.0%	8.8110%
5	10.0%	9.6855%

The par curve values sovereign bonds, which are considered to be risk-free. The par curve therefore serves as the benchmark for the valuation of all other bonds in the market (Ho, Lee and Wilcox, 2016). If a par curve is not available or unreliable, an alternative benchmark is the **swap rate curve**, which is the fixed-rate leg in the interest rate swap derivative market. The par curve and swap curve may differ from each other for the following reasons:

- The cash flows from swaps are subject to higher default risk than sovereign bonds.
- Liquidity for specific maturities may differ.

Bootstrapping

If we are provided with the par curve, we can calculate the spot yield curve by a process known as **bootstrapping**. The following example will illustrate this.

Example: Bootstrapping

The following par curve rates exist for annual coupon government debt:

One-year par rate: 6.0000%

Two-year par rate: 6.9673%

Three-year par rate: 7.9050%

The following steps will describe the calculation of the 1-year, 2-year and 3-year spot yields from the given par rates. We start by calculating the shortest spot rate (which in this example is 1 year) then move to the next longest maturity, and so on.

Step 1: Derive the one-year spot rate from the one-year par rate.

If the government decided to issue a 1-year, annual-coupon bond priced at par, they would have to issue a bond paying a coupon of 6%. Therefore, the one-year spot rate $[r(1)]$ must be:

$$\$100 = \frac{\$6 + \$100}{(1 + r(1))^1}$$

$$\therefore r(1) = 6\%.$$

Step 2: Derive the two-year spot rate $[r(2)]$ from the two-year par rate and $r(1)$.

If the government decided to issue a 2-year, annual-coupon bond priced at par, they would have to issue a bond paying a coupon of 6.9673%. Therefore, the two-year spot rate $[r(2)]$ must be:

$$\$100 = \frac{\$6.9673}{(1 + 6\%)^1} + \frac{\$6.9673 + \$100}{(1 + r(2))^2}$$

$$\therefore (1 + r(2))^2 = \frac{\$106.9673}{\$93.4271}$$

$$\therefore r(2) = 7\%.$$

Step 3: Derive the three-year spot rate $[r(3)]$ from the three-year par rate, $r(1)$, and $r(2)$.

If the government decided to issue a 3-year, annual-coupon bond priced at par, they would have to issue a bond paying a coupon of 7.9050%. Therefore, the three-year spot rate $[r(3)]$ must be:

$$\$100 = \frac{\$7.9050}{(1 + 6\%)^1} + \frac{\$7.9050}{(1 + 7\%)^2} + \frac{\$7.9050 + \$100}{(1 + r(3))^3}$$

$$\therefore (1 + r(3))^3 = \frac{\$107.9050}{\$85.6379}$$

$$\therefore r(3) = 8\%.$$

Spreads

The par curve is the yield-to-maturity of risk-free sovereign bonds issued at par. Not all bonds are risk-free, and bond investors may want additional return to compensate them for risks that they face. This additional return is normally measured in terms of a **spread**, referring to a spread, or difference, measured in basis points between the yield to maturity of a bond on the par curve and the bond referred to.

For example, if a risk-free, 5-year, annual-pay, coupon-paying, recently issued sovereign bond has a yield to maturity of 9.69% and a similar corporate – i.e. risky – bond has a yield to maturity of 12%, then the spread is 2.31% or 231 basis points. Common spread measures used are the swap spread, the z-spread and the TED spread.

Swap spread

The swap spread is a popular way to indicate credit spreads in a market. The **swap spread** is the difference between the fixed rate of an interest rate swap and the yield-to-maturity of a bond on the par curve with the same maturity as the swap. The swap spread reveals the premium charged by bond investors for credit and liquidity risks.

Z-spread

Another method for measuring risk is the zero-spread, or z-spread. The **z-spread** is the constant spread that needs to be added to the spot yield curve to make the discounted cash flows of a risky bond equal to its market price.

TED spread

Another common spread used in fixed income valuation is the TED spread. (The name is derived from the "T" in US T-bill and ED, which is the ticker symbol for the Eurodollar futures contract.) The **TED spread** is the difference between the USD-denominated LIBOR rate and the yield on a T-bill of matching maturity. Since US T-bills are considered risk-free, and LIBOR is the rate at which leading London banks will loan money to each other on a short-term basis (up to 1 year), the TED spread is an indicator of perceived credit risk in the general economy. For example, the TED spread will rise when investors think there is a higher likelihood of banks defaulting on loans to each other.

Conclusion

Note that if the market price of risk-free bonds does not match the prices determined by the spot yield curve there is a possibility for arbitrage profits. This would be done via a process called stripping and reconstitution.

Example: Stripping, reconstitution, and arbitrage

From a previous example we know that the market price of a risk-free, 3-year, $100 par value, annual pay, 7.9050% coupon bond should be $100. How could an arbitrage profit be made if the market price is $98? The following steps will describe this.

Step 1: Purchase the bond for $98.

Step 2: Sell the zero-coupon bonds (at their present value) corresponding to the payment time and amount of the 3-year coupon-paying bond:

$$Present\ value\ of\ bond = \frac{\$7.9050}{(1+6\%)^1} + \frac{\$7.9050}{(1+7\%)^2} + \frac{\$7.9050 + \$100}{(1+8\%)^3} = \$100.$$

A $2 profit is made per bond. The process of separating a bond's coupon and par value payments is known as **stripping**. The opposite process is known as **reconstitution**.

The actual return on a bond will only equal the yield to maturity when:

- Spot interest rates (or spot yields) do not change;
- Coupons are received exactly when due and immediately reinvested;
- Principal is received exactly when due;
- The bond is held all the way to maturity.

Appendix: Why does the yield curve invert?

The yield curve is normally upward-sloping. The yield curve becomes inverted when bond market investors predict difficult economic times ahead. Bond market investors expect that in response to these difficult economic times, the central bank will lower interest rates to lower the cost of borrowing and boost the economy. If interest rates are lowered, the prices of existing bonds go up. Long-maturity bonds' prices will increase by a greater amount than short-maturity bonds' prices. As a result, demand amongst bond market investors for long-maturity bonds increases. As demand increases, the price of long-maturity bonds increases. As their price increases, the yield of long-maturity bonds lowers. In some cases, the yield on long-maturity bonds falls below the yield on short-maturity bonds, and the yield curve has become inverted.

Bibliography

Ho, T., Lee S., and Wilcox, S. (2016). The Term Structure and Interest Rate Dynamics. *Alternative Investments and Fixed Income, CFA Program Curriculum, Level II*, 5.