WORLDQUANT UNIVERSITY

Continuous-time Stochastic Processes: Module 2

MSc Financial Engineering

Table of Contents

1. Brief

This document contains the core content for Module 2 of Continuous-time Stochastic Processes entitled Stochastic Calculus I: Ito Process. It consists of five sets of notes and four video lecture scripts.

2. Course Context

Continuous-time Stochastic Processes is the fourth course presented in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The aim of the course is to introduce derivative pricing when trading happens in continuous time. It begins by focusing on the stochastic calculus of Brownian motion and its generalization to continuous semi-martingales. These ideas are then applied to continuous trading and the pricing of financial derivatives.

2.1 Course-level Learning Outcomes

After completing the Continuous-time Stochastic Processes course, you will be able to:

1. Define and identify Brownian motion processes in multiple dimensions.
2. Solve stochastic differential equations.
3. Apply Ito's Lemma for continuous semimartingales.
4. Apply Girsanov's theorem to construct equivalent local martingale measures.
5. Price and hedge derivatives in various asset price models.
6. Derive the Black-Scholes partial differential equation.
7. Construct asset prices models based on Levy processes.
8. Price interest rate derivatives.

2.2 Module Breakdown

The Continuous-time Stochastic Processes course consists of the following one-week modules:

1. Brownian Motion and Continuous-time Martingales
2. Stochastic Calculus I: Ito Process
3. Stochastic Calculus II: Semimartingales
4. Continuous Trading
5. The Black-Scholes Model
6. An Introduction to Levy Processes
7. An Introduction to Interest Rate Modeling

3. Module 2:

Stochastic Calculus I: Ito Process

In this module we introduce the concept of stochastic integrals with respect to a Brownian motion and its properties. Moreover, we define the Ito diffusion process and conclude the module with the martingale representation theorem (MRT). In the first section we review some basic knowledge about Stieltjes integrals in order to provide a smooth transition from deterministic calculus to stochastic integrals.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Define the Stieltjes integral with respect to a finite variation process.
2. Evaluate simple Ito integrals.
3. Apply Ito's Lemma.
4. Solve stochastic differential equations.
5. State and apply the Brownian martingale representation theorem.

3.2 Transcripts and Notes

3.2.1 Notes: Stieltjes Integrals

In this section, we review the theory of Riemann-Stieltjes integration and its generalization to the Lebesgue-Stieltjes integration.

Let $I := [a,b]$ be a closed and bounded interval in $\mathbb{R}$ and $f, g: I \to \mathbb{R}$ be two bounded functions. We want to define the so-called *Riemann-Stieltjes integral* of f with respect to g, denoted by

$$\int_a^b f\ dg \text{ or } \int_a^b f(x)\ dg(x).$$

We will call f the *integrand* and g the *integrator*.

A *partition* of I is a finite set $P = \{x_0, \dots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$. We call $\|P\| := max_i(x_i - x_{i-1})$the *mesh* of P. A partition P' of $I$is called a *refinement* of P if $P \subseteq P'$.A partition P together with a set of points $\{t_1, \dots, t_n\}$ with $t_i \in [x_{i-1}, x_i]$ is called a *tagged partition* of I.

If $P = \{x_0, t_1, x_1, t_2, x_2, \dots, x_{n-1}, t_n, x_n\}$ is a tagged partition of I, we define the *Riemann-Stieltjes sum* of f with respect to g as

$$S(f,g,P) := \sum_{i=1}^{n} f(t_i)\Delta g(x_i),$$

where $\Delta g(x_i) = g(x_i) - g(x_{i-1})$. We say that f is Riemann-Stieltjes (RS) integrable with respect to g on $[a,b]$ if there exists a real number A such that for every $\epsilon > 0$, there exists a partition P_ϵ of $[a,b]$ such that for every tagged refinement P of P_ϵ we have

$$|S(f,g,P) - A| < \epsilon.$$

The unique number A is called the RS integral of f with respect to g over $[a,b]$, and is denoted by

$$A := \int_a^b f \; dg \equiv \int_a^b f(x) \; dg(x).$$

A (slightly) less general definition of RS integration says that f is RS integrable with respect to g if there exists $A \in \mathbb{R}$ such that

$$\lim_{\|P\| \to 0} S(f,g,P) = A,$$

in the sense that for every $\epsilon > 0$, there exists $\delta > 0$ such that if P is a tagged partition with $\|P\| < \delta$, then $|S(f,g,P) - A| < \epsilon$. This definition is still used in many textbooks.

When $g(x) = x$, then RS integration is the same as Riemann integration and the two definitions are in fact equivalent.

If g is continuously differentiable and fg' is Riemann integrable, then f is RS integrable with respect to g and

$$\int_a^b f\, dg = \int_a^b f g'\, dx.$$

Now let $g: [a, b] \to \mathbb{R}$ and $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Define

$$V_g([a, b], P) := \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

and

$$V_g([a, b]) = \sup_P V_g([a, b], P),$$

where the sup is taken over all partitions of $[a, b]$. We call $V_g([a, b])$ the *total variation* of g on $[a, b]$ and we say that g is of *bounded variation* on $[a, b]$ if $V_g([a, b]) < \infty$.

If g is continuously differentiable on $[a, b]$, then it is of bounded variation and the total variation is given by

$$V_g([a, b]) = \int_a^b |g'(x)|\, dx.$$

Let's look at a simple example. Suppose that g is increasing on $[a, b]$. Then for any partition P, we have

$$V_g([a, b], P) = g(b) - g(a),$$

hence

$$V_g([a, b]) = g(b) - g(a).$$

It turns out that all functions of bounded variation are not far from increasing functions. Specifically, $g:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ if and only if $g = g_1 - g_2$ is the difference of two increasing functions g_1 and g_2.

The importance of functions of bounded variation in this section is in the following result:

Theorem 1.1. *Let $f,g:[a,b]\to\mathbb{R}$ be functions. If f is continuous on $[a,b]$ and g has bounded variation on $[a,b]$, then f is RS integrable with respect to g.*

So, a general rule of thumb is that good integrators are functions of bounded variation.

Here is an important scenario that we will encounter in many sections of this module. Suppose that $g:[0,\infty)\to\mathbb{R}$ is such that $V_g([0,b])<\infty$ for each $b>0$. Also assume that g is right-continuous. Then we can show there exist two right-continuous increasing functions g_1 and g_2 such that $g = g_1 - g_2$. Now, each g_i $(i=1,2)$ corresponds to a measure μ_i on $\mathcal{B}([0,\infty))$ such that

$$\mu_i\big((a,b]\big) = g_i(b) - g_i(a),\ i = 1,2.$$

Thus, for appropriate functions $f:[0,\infty)\to\mathbb{R}$ we can define the integrals

$$\int_{[a,b]} f\ d\mu_i.$$

An elementary argument shows that, just like in the case of the Lebesgue measure, these integrals (with respect to μ_i) generalize the RS integral in the sense that if f is RS integrable on $[a, b]$, then f is integrable with respect to μ_i for $i = 1,2$ and

$$\int_a^b f \ dg = \int_{[a,b]} f \ d\mu_1 - \int_{[a,b]} f \ d\mu_2.$$

We will call the integral on the right-hand side the *Lebesgue-Stieltjes integral*, and because of this result, we will sometimes write $\int_a^b f \ dg$ even when referring to the (more general) Lebesgue-Stieltjes integral.

3.2.2 Transcript: The Ito Integral

Hi, in this video we introduce the Ito integral with respect to a Brownian motion.

So, let W be a Brownian motion and φ be a stochastic process. We want to define a new stochastic process, which we are going to denote by $\{(\varphi \bullet X)_t : 0 \leq t \leq T\}$, where T could be infinity, but for now we're going to assume that T is finite. $(\varphi \bullet X)_t$ is called the Ito integral and we will sometimes denote it by:

$$(\varphi \bullet X) = \int_0^t \varphi_s \, dX_s.$$

The intuition is that this should represent the gains from trading if the asset price is given by W and a holding is given by φ. So, this is a continuous-time analog of the martingale transform that we defined in discrete time.

Now, since the parts of Brownian motion do not have finite variation, we cannot define this as a Riemann-Stieltjes or Lebesque-Stieltjes integral. We thus need to consider a new approach, starting with what is called a simple process.

A stochastic process, φ, is simple if φ_t can be represented like this:

$$\sum_{I=1}^{n} HiI_{(ti-1,ti]}(t),$$

where H_i is an F_{ti-1} - measurable random variable and t_1 up to t_n is just a partition of the interval 0 to T. This can be illustrated in a diagram as follows:

We start with a partition. This is t_0, t_1, t_2, and so on, to t_{n-1}; and then, between the interval t_0 and t_1, this random variable takes on one value, which is H_1; and then, between t_1 and t_2, it takes on another value, and so on and so forth. So, that is what the sample path of a simple process looks like. They are piecewise constant and this changes if we change Ω as well.

Now, if φ is a simple process, we are going to define the stochastic integral of this here, $(\int_0^t \varphi_s \; dW_s)$. This will be defined as:

$$\sum_{i=1}^{n} H_i \,(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}).$$

So, that's the stochastic integral and it agrees, of course, with the martingale transform in discrete time, but that is defined only for simple functions.

We will denote the set of all simple processed by $\mathbb{S}$. A stochastic integral satisfies the following properties:

1. $E\left(\int_0^t \varphi_s \; dW_s\right) = 0.$

2. $E\left(\left(\int_0^t \varphi_s \; dW_s\right)^2\right) = E\left(\int_0^t \varphi_s^2 \; ds\right)$, where of course, by Fubini's theorem, we can take the expectation inside.

3. $\{\int_0^t \varphi_s \; dW_s : 0 \leq t \leq T\}$ is a square integrable martingale, because we're still working with φ being a simple process. Its predictable quadratic variation process, $\langle \int_0^{\cdot} \varphi_s \; dW_s \rangle_t = \int_0^t \varphi_s^2 \; ds$. That is the quadratic variation in that case.

4 If φ is deterministic, then this stochastic integral, $\int_0^t \varphi_s \, dW_s$, is a normal distribution with mean 0 and the variance is the integral from 0 to t of ds, which of course coincides with the quadratic variation.

So, those are four properties that are satisfied by the stochastic integral is the integrand is a simple process. Now, we will extend this stochastic integral.

For the first extension, we'll consider a wider class called L^2. We will define $L^2(W)$, with W ofcourse representing Brownian motion, to be the set of all progressive processes. So, φ, such that φ is progressive, and the norm of φ is finite, where this norm with respect to Brownian motion is just simply equal to $(E(\int_0^t \varphi t^2 dt))^{\frac{1}{2}}$. We can show that this is indeed an L^2 space over the product of the Lebesque measure and the original probability measure. So, these are the processes that we are going to define the integral with respect to. The important result is that for every φ in L^2, that allows us to extend the stochastic integral, we can find a sequence of simple processes such that this sequence converges to φ with respect to this L^2 norm. In other words, the distance between φ^n and φ goes to 0 and n tends to infinity.

With that, we will then define the stochastic integral of φ, where φ belongs to L^2, as the limit n tends to infinity of the integrals of the φ^n. We can show that this limit does not depend on the sequence φ^n that we choose to represent φ. We say that this is a layer of extension of the stochastic integral. Again, we can show that with this extension the stochastic integral still satisfies the four properties discussed above in this class L^2 of W.

The final extension is to consider the set of all processes φ such that $\int_0^t \varphi_s^2 \, ds < \infty$ for every t (i.e. $t \geq 0$), and this holds almost surely. Now, in this final extension, unfortunately, we lose the martingale property. In general, the Ito integral, when φ

satisfies only this condition, but not necessarily the stronger condition, will not be a martingale, but it will always be what is known as a local martingale, and we're going to see that in the next module.

Now that we've looked at the Ito integral, in the next video we're going to move on to

3.2.3 Notes: The Ito Integral

Stochastic integral

In this section we introduce - for the first time- the *stochastic integral* of a stochastic process φ with respect to another process X. We will denote this by

$$(\varphi \bullet X) \text{ or } \int_0^t \varphi_s \, dX_s.$$

We call X the *integrator* and φ the *integrand.*

We shall think of the interaction between a simple process $\varphi_s = \Delta(t)$ and, for instance, a Brownian motion $X_s = W(t)$. Regard $W(t)$ as the price per share of an asset at time t. (Of course, it is not the best possible example since Brownian motion can take negative values, but let's ignore that issue for the sake of this illustration.) Now, think of $t_0, t_1, \dots, t_{n-1}$ as the trading dates in the asset, and think of $\Delta(t_0), \Delta(t_1), \dots, \Delta(t_{n-1})$ as the position (number of shares) taking in the asset at each trading date and held to the next trading date. The total gain of this trading strategy is defined by the following stochastic integral:

$$\int_0^t \Delta(t) dW(u) \, ,$$

which is a specific case of the more general stochastic integral introduced above.

Throughout the section, we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an adapted process X, and a positive (extended) number $T \in (0, \infty]$. The stochastic integral will then be regarded as a stochastic process $(\varphi \bullet X)_t$, defined for each $0 \leq t \leq T$.

As the notation suggests, we want this integral to be the continuous-time analogue of the martingale transform from the previous chapter. That is, we want $(\varphi \bullet X)_t$ to represent the cumulative gains when trading (in continuous time) the asset X with stakes equal to φ_t for each $t \in (0, T]$.

First, if the paths of X have finite variation, then we can define $(\varphi \bullet X)$ as a (path-wise) Stieltjesintegral for appropriate processes φ. This is certainly the case, for instance, when $X_t(\omega) = t$ for every $t \in [0, T]$ and $\omega \in \Omega$. In this case, the integral will correspond to the Riemann (or more generally, Lebesgue) integral.

Definition 2.1. *If $X_t(\omega) = t$ for each $t \in [0, T]$, we define the integral of a stochastic process φ with respect to X as the path-wise Riemann integral*

$$(\varphi \bullet X)_t(\omega) \coloneqq \int_0^t \varphi_s(\omega)\ ds, \qquad \omega \in \Omega,\ t \in [0, T],$$

provided the integral exists. In general, if the sample paths of X have finite variation, we define the integral of φ with respect to X as a path-wise Stieltjes integral

$$(\varphi \bullet X)_t(\omega) \coloneqq \int_0^t \varphi_s(\omega)\ dX_s(\omega), \qquad \omega \in \Omega, t \in [0, T],$$

provided the integral exists.

Unfortunately, though, many of the interesting stochastic processes we consider in practice have sample paths that have unbounded variation. A famous example is Brownian motion.

Theorem 2.1. *The sample paths of a Brownian motion have (a.s.) infinite variation on any interval* $[0, t]$*, where* $t > 0$.

This result implies that we cannot define

$$\int_0^t \varphi_s \; dW_s$$

as a (pathwise) Stieltjes integral. We will introduce a new type of integration – called *Ito Integration* – to deal with such problems. Let us first define this integral for *simple processes*.

Simple process

A stochastic process φ is called a **simple process** (or **elementary process**) if there exists a bounded partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ of $[0, T]$ and bounded random variables $H_1, \ldots, H_n$, with $H_i \in m\mathcal{F}_{t_{i-1}}$ for $i = 1,2, \ldots, n$, such that

$$\varphi_t(\omega) = \sum_{i=1}^{n} H_i\,(\omega) I_{(t_{i-1}, t_i]}(t), t \in [0, T], \omega \in \Omega. \tag{1}$$

Ito integral

If φ is a simple process with representation (8.1), we will define the *Ito integral* of φ with respect to a Brownian motion W as the stochastic process $\{(\varphi \bullet W)_t : 0 \leq t \leq T\}$ defined by

$$(\varphi \bullet W)_t := \int_0^t \varphi_s \, dW_s := \sum_{i=1}^{n} H_i \, (W_{t_i \wedge t} - W_{t_{i-1} \wedge t}), \qquad 0 \leq t \leq T.$$

We denote the class of simple processes by $\mathbb{S}$.

If $\varphi \in \mathbb{S}$ is a simple process, then

1 $\mathbb{E}\left(\int_0^t \varphi_s \, dW_s\right) = 0$ for all $0 \leq t \leq T$.

2 Ito isometry

$\mathbb{E}\left(\left(\int_0^t \varphi_s \, dW_s\right)^2\right) = \mathbb{E}\left(\int_0^t \varphi_s^2 \, ds\right) = \int_0^t \mathbb{E}(\varphi_s^2) \, ds$ for all $0 \leq t \leq T$.

3 $\{\int_0^t \varphi_s \, dW_s : 0 \leq t \leq T\}$ is a martingale.

4 If φ is deterministic (i.e. $\varphi_t(\omega)$ does not depend on $\omega \in \Omega$), then

$$\int_0^t \varphi_s \, dW_s \sim N\left(0, \int_0^t \varphi_s^2 \, ds\right) \text{ for all } 0 \leq t \leq T.$$

We now extend the integral to a much wider class of integrands.

First define

$$L^2(W) \coloneqq \{\varphi\colon \varphi \text{is progressive and } \|\varphi\|_W < \infty\},$$

where

$$\|\varphi\|_W \coloneqq \left(\mathbb{E}\left(\int_0^T \varphi_s^2\, ds\right)\right)^{\frac{1}{2}}.$$

It is clear that $L^2(W)$ is simply equal to $L^2([0,T] \times \Omega, Prog, \lambda_1 \otimes \mathbb{P})$, where $Prog$ is the progressive σ-algebra. Also, $\mathbb{S} \subseteq L^2(W)$ is dense in $L^2(W)$ in the sense that for each $\varphi \in L^2(W)$, there exists a sequence (φ^n) of elements of $\mathbb{S}$such that $||\varphi^n - \varphi||_W \to 0$ as $n \to \infty$.

For any $\varphi \in L^2(W)$, we will define the stochastic integral of φ with respect to W as

$$\int_0^t \varphi_s\, dW_s \colon = \lim_{n\to\infty} \int_0^t \varphi^n\, dW_s,$$

where the limit is independent of the chosen sequence.

The general Ito integral also has the same properties as the corresponding integral for elementary processes.

Theorem 2.2. *If* $\varphi \in L^2(W)$, *then*

1. $\mathbb{E}\left(\int_0^t \varphi_s\, dW_s\right) = 0$ *for all* $0 \leq t \leq T$.

2 *Ito isometry*

$$\mathbb{E}((\int_0^t \varphi_s \, dW_s)^2) = \mathbb{E}(\int_0^t \varphi_s^2 \, ds) = \int_0^t \mathbb{E}(\varphi_s^2) \, ds \text{ for all } 0 \leq t \leq T.$$

3 $\{\int_0^t \varphi_s \, dW_s : 0 \leq t \leq T\}$ *is a square integrable martingale.*

4 *If* $\varphi, \psi \in L^2(W)$, *then*

$$\left\langle \int_0^{\cdot} \varphi_s \, dW_s, \int_0^{\cdot} \psi_s \, dW_s \right\rangle_t = \int_0^t \varphi_s \, \psi_s ds,$$

and

$$\left\langle \int_0^{\cdot} \varphi_s \, dW_s \right\rangle_t = \int_0^t \varphi_s^2 \, ds.$$

5 *If* φ *is deterministic (i.e.* $\varphi_t(\omega)$ *does not depend on* $\omega \in \Omega$*), then*

$$\int_0^t \varphi_s \, dW_s \sim N\left(0, \int_0^t \varphi_s^2 \, ds\right) \text{ for all } 0 \leq t \leq T.$$

Here are some examples:

1 The random variable

$$\int_0^1 t^2 \, dW_t$$

has a normal distribution with mean 0 and variance $\int_0^1 t^4 \, dt = 1/5$.

2 For $0 \leq t \leq T < \infty$, the process

$$I_t := \int_0^t W_s^2 \, dW_s$$

is a martingale. Furthermore,

$$\mathbb{E}(I_t) = 0 \text{ and } \mathbb{E}(I_t^2) = \mathbb{E}(\int_0^t W_s{}^4 ds) = \int_0^t \mathbb{E}(W_s{}^4) ds = \int_0^t 3s^2 ds = t^3.$$

The final extension of the stochastic integral is to the class of progressive processes φ such that

$$\int_0^t \varphi_s^2 \; ds < \infty \, \forall t \;\; a.s.$$

Again, these processes can still be approximated – albeit in a weaker manner – by simple processes. Unfortunately, though, the stochastic integral for such processes fails to satisfy some of the properties above. In particular, this stochastic integral is in general not a martingale (but it is always a local martingale – see the next module).

3.2.4 Transcript: Ito's Lemma

Hi, in this video we introduce Ito processes and study **Ito's Lemma**.

An Ito process X is a stochastic process that satisfies the following conditions:

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dWs,$$

is an Ito Integral where W is of course a Brownian motion.

For this to make sense, we of course need $\int_0^t |\mu_s| ds$ to be finite and we need $\int_0^t \sigma_s^2 \, ds$ to be finite for all t. Note that we are working on the interval $0 \le t \le T$ as always.

We sometimes write this in what is called **differential notation**, as follows:

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

(Note that this is just shorthand notation for the full expression of X_t written above).

We call μ_t the drift coefficient and we call σ the diffusion coefficient of this Ito process.

Now, Ito's Lemma tells us that certain functions of Ito processes are themselves Ito processes, and it also gives us the stochastic differential of a function of X_t in terms of the stochastic differential of X_t.

Specifically, it says that if $\mathcal{F}: [0,T] \times \mathbb{R} \to \mathbb{R}$, and is a real-valued function; and that $\mathcal{F} \in C^{1,2}$, which means that $\mathcal{F}$ is once continuously differentiable with respect to the time variable $[0,T]$, and twice continuously differentiable with respect to the spatial variable X. Then, $\mathcal{F}\ (t, X_t)$, where X_t is the Ito process, is an Ito process with the following stochastic differential, given by the formula:

$$d\mathcal{F}(t, X_t) = \frac{\delta\mathcal{F}}{\delta t} dt + \frac{\delta\mathcal{F}}{\delta x} + dX_t \frac{1}{2} \frac{\delta^2\mathcal{F}}{\delta x^2} d < X > t.$$

In other words, take the first derivative with respect to the time variable, dt, plus the first derivative with respect to the spatial variable, dX_t, plus $\frac{1}{2}$, times the second derivative with respect to the spatial variable, d, with respect to the quadratic variation of X at time t. That gives us a stochastic differential of the function of X and t.

Let's look at an example:

Let's consider stochastic process X, which is an Ito process with the following stochastic differential:

$$dX_t = X_t dt + {X_t}^2 dWt.$$

Now, consider the following function: $\mathcal{F}(t, X_t) \ = \ 2t \mid nX_t \ =: \ Y_t$.

We can check that this function, $\mathcal{F}(t,x) = 2tlnx$, satisfies the conditions of Ito's Lemma on the region where x is positive. (Of course, this is only defined for x positive in this case).

So $\mathcal{F}\ (t, X\_t)\ =\ 2t\ |nX_t =: Y_t$ is twice continuously differential with respect to x, and, in fact, the first partial derivative with respect to x is $\frac{2t}{x}$, while the second partial derivative with respect to x is $\frac{2t}{x^2} \times -1$, so it is - $\frac{2t}{x^2}$. The first partial derivative with respect to t, $\frac{\delta \mathcal{F}}{\delta t}$, is equal to $2lnx$.

We can, therefore, apply Ito's Lemma to find a stochastic definition of Y_t. Written in full:

$$dY_t = (2lnX_t)dt + \frac{2t}{xt}dX_t + \frac{1}{2}\left(\frac{-2t}{X_t^2}\right)d < X > t.$$

To simplify that, we take all the dt's together.

Firstly, it will be $2lnX_t$, as that has a dt in it. dX_t is given by $X_t dt$, so we will multiply this and substitute this term, so we get plus $2t$, which also has a dt. Remember that the quadratic variation, $d < X > t = X_t^4 dt$. Written in full:

$$\left[2lnX_t + 2t - \frac{2t}{2}X_t^2\right]dt.$$

Secondly,

$$[2tX_t]dWt.$$

We can further simplify this if we want to.

Now that we have shown an application of Ito's Lemma, in the next video we're going to move on to stochastic differential equations.

3.2.5 Notes: Ito Diffusion Processes

Ito process

An adapted stochastic process $X = \{X_t : 0 \leq t \leq T\}$ is an *Ito process* if there exists stochastic process $\mu = \{\mu_t : 0 \leq t \leq T\}$ and $\sigma = \{\sigma_t : 0 \leq t \leq T\}$ such that

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s \text{ for all } 0 \leq t \leq T,$$

where

$$\int_0^t \sigma_s^2 ds < \infty \text{ and } \int_0^t |\mu_s| ds < \infty \text{ for all } 0 \leq t \leq T. a.s.$$

Stochastic differential

We often write this equation in differential notation as

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and call it the *stochastic differential* of X. The process μ is called the *drift term* and σ is called the *diffusion term* of X. Note that if $\sigma \in L^2(W)$, then X is a martingale if and only if $\mu \equiv 0$; i.e. if and only if X is driftless.

Here are some examples:

1 A Brownian motion is an Ito process since

$$W_t = W_0 + \int_0^t 1 \, dW_s.$$

Thus,

$$dW_t = 0dt + 1dW_t.$$

2 The process

$$X_t = t^3 + \int_0^t W_s \, dW_s = 0 + \int_0^t 3s^2 \, ds + \int_0^t W_s \, dW_s$$

is an Ito process. We have

$$dX_t = 3t^2 \, dt + W_t \, dW_t.$$

3 An Ito process can also be given implicitly. Define

$$X_t = X_0 + \int_0^t \mu X_s \, ds + \int_0^t \sigma X_s \, dW_s, \qquad \mu \in \mathbb{R}, \sigma \in (0, \infty).$$

Then X is an Ito process called geometric Brownian motion. Its stochastic differential is

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t.$$

We will later see how to "solve" this stochastic differential equation to get X explicitly.

If X is an Ito process with differential

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

we will define the integral of a stochastic process φ with respect to X as

$$\int_0^t \varphi_s \, dX_s \coloneqq \int_0^t \varphi_s \, \mu_s \, ds + \int_0^t \varphi_s \, \sigma_s \, dW_s,$$

provided the integrals on the right-hand side exist.

One of the most important results in stochastic calculus is the Ito's lemma. Before we state it, we need to introduce the quadratic variation of a stochastic process.

Quadratic variation

Let X and Y be stochastic processes. We define the *quadratic covariation* of X and Y as the stochastic process $[X, Y]$ defined by

$$[X, Y]_t \coloneqq \lim_{||P|| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}),$$

where $P = \{t_0, t_1, \dots, t_n\}$ is a partition of $[0, t]$, provided the limit exists (in probability). We call $[X] \coloneqq [X, X]$ the *quadratic variation* of X.

If W is a Brownian motion, then $[W]_t = \langle W \rangle_t = t$. In general, for any continuous martingale M, $[M] = \langle M \rangle$, which is why $[.]$ and $\langle . \rangle$ are often used interchangeably (in the case of a continuous martingale).

The quadratic variation behaves in a similar way to the predictable quadratic variation, $\langle . \rangle$. If X is an Ito process with stochastic differential

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

then

$$[X]_t = \int_0^t \sigma_s^2 \, ds.$$

In general, the quadratic variation of a process of finite variation is zero.

We can now state Ito's lemma:

Theorem 3.1.(Ito's lemma.) *Let* $f: [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *be a function of two variables* $f = f(t,x)$ *such that the partial derivatives*

$$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x} \text{ and } \frac{\partial^2 f}{\partial x^2}$$

all exist and are continuous. Suppose that X *is an Ito process with*

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

Then $Y_t = f(t, X_t)$ *is also an Ito process, and*

$$dY_t = df(t, X_t) = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2} + \mu_t \frac{\partial f}{\partial x}\right] dt + \sigma_t \frac{\partial f}{\partial x} dW_t.$$

In integral notation, we have

$$Y_t = Y_0 + \int_0^t \left[\frac{\partial f}{\partial s} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2} + \mu_s \frac{\partial f}{\partial x}\right] ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} dW_s.$$

An easier way to remember this formula is to write it as follows:

$$dY_t = df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2} d[X]_t.$$

Consider the following:

$$dX_t = \mu dt + \sigma dW_t, \ \ \mu \in \mathbb{R}, \ \ \sigma \in (0, \infty).$$

We find the stochastic differential equation of $Y_t = 2t^2 e^{2X_t}$.Note that $f(t, x) = 2t^2 e^{2x}$ satisfies all the hypotheses of Ito's lemma with

$$\frac{\partial f}{\partial t} = 4te^{2x}, \ \frac{\partial f}{\partial x} = 4t^2 e^{2x} \ and \ \frac{\partial^2 f}{\partial x^2} = 8t^2 e^{2x}.$$

Thus,

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 \mathrm{f}}{\partial \mathrm{x}^2} (dX_t)^2 = 4te^{2X_t} dt + 4t^2 e^{2X_t} dX_t + \frac{1}{2} 8t^2 e^{2X_t} (dX_t)^2 \\ &= (4te^{2X_t} + 4t^2 e^{2X_t} \mu + 4t^2 e^{2X_t} \sigma^2)\, dt + 4t^2 e^{2X_t} \sigma dW_t. \end{aligned}$$

An *Ito diffusion process* X is a solution to the following Stochastic Differential Equation (SDE)

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

for some real-valued functions μ and σ. That is, X is given implicitly by

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(t, X_s)\, dW_s, \qquad t \geq 0.$$

Our first example is *geometric Brownian motion*. Consider the following SDE:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = \text{given},$$

where μ and σ are constants, with $\sigma > 0$. We want to solve this equation by finding X explicitly in terms of Brownian motion W. We apply Ito's lemma to $Y_t = \ln X_t$ to get

$$dY_t = d \ln X_t = 0 \, dt + \frac{1}{X_t} \, dX_t - \frac{1}{2X_t^2} d[X]_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

Hence

$$Y_t = Y_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s \;=\; Y_0 \;+\; (\mu - \frac{1}{2}\sigma^2)t \;+ \sigma W_t.$$

Therefore,

$$X_t = e^{Y_t} = exp\left(lnX_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right) = X_0 exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right).$$

Notice that since

$$ln \, X_t = Y_t = ln \, X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \sim N\left(ln \, X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right),$$

Xt has a log-normal distribution. As an exercise, find the mean and variance of X_t.

For our second example, we look at the *Ornstein-Uhlenbeck process*. Assume that X_0 is fixed and known, and let

$$dX_t = \alpha(\mu - X_t)\,dt + \sigma\,dW_t, \quad \alpha, \sigma > 0, \quad \mu \in \mathbb{R}.$$

We apply Ito's lemma to $Y_t = e^{\alpha t} X_t$:

$$dY_t = d(e^{\alpha t} X_t) = \alpha e^{\alpha t} X_t\,dt + e^{\alpha t} dX_t + 0 = \alpha\mu e^{\alpha t}\,dt + \sigma e^{\alpha t}\,dW_t,$$

which gives

$$Y_t = Y_0 + \int_0^t \alpha\,\mu e^{\alpha s} ds + \int_0^t \sigma e^{\alpha s} dW_s = Y_0 + \mu(e^{\alpha t} - 1) + \int_0^t \sigma e^{\alpha s}\,dW_s.$$

Therefore (note that $Y_0 = X_0$),

$$X_t = e^{-\alpha t} Y_t = X_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + e^{-\alpha t} \int_0^t \sigma e^{\alpha s} dW_s.$$

So, X_t has a normal distribution with

$$\mathbb{E}(X_t) = X_{0e^{-\alpha t}} + \mu(1 - e^{-\alpha t})$$

and

$$Var(X_t) = Var(e^{-\alpha t} \int_0^t \sigma e^{\alpha s} dW_s) = e^{-2\alpha t} \int_0^t \sigma^2 e^{2\alpha s} ds = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}).$$

3.2.6 Transcript: Stochastic Differential Equations

Hi, in this video we go through an example of solving a stochastic differential equation.

We are going to consider what is called geometric Brownian motion. This is the stochastic differential equation that looks like this: $dX_t = X_t\mu_t dt + X_t\sigma \mathrm{d}W_t$, where μ and σ are constants. We are going to assume, of course, that σ is strictly positive.

What this means is that $X_t = X_0 + \int_0^t X_s \, \mu \mathrm{ds} + \int_0^t X_s \sigma dW_s$ (i.e. with respect to Brownian motion). As you can see, X_t is given implicitly, in terms of itself.

So, what we want to do by solving this stochastic differential equation is express X_t explicitly in terms of Brownian motion and other stochastic processes for whose distribution we know.

We are also going to assume that $X_0 = 1$. This is not very important, we can assume anything, but just to be sure, we are going to assume that $X_0 = 1$.

So, the trick in solving stochastic differential equations of this form is to try and find a transformation of X_t. In other words, we want to find a new stochastic process Y_t that is a function of X_t, such that the stochastic differential of Y_t can be solved easily, and then you invert that transformation to get back X_t. Put differently, we have to find Y_t, which is a function of t and X_t, such that Y_t is easy to solve. In other words, we can solve for Y_t explicitly, and this function is invertible, so that we get back X_t and calculate the distribution of X_t. The motivation for that is obtained from looking at this $(dX_t = X_t\mu \mathrm{d}t)$, at least initially, as an ordinary differential equation. So, without the stochastic term then the solution would be the log of X_t.

Let's try to make the following transformation:

$$Y_t = \mathcal{F}(t_1, X_t) = lnX_t.$$

So, we have made Y_t the log of X_t and now let's find the stochastic differential equation of Y_t using Ito's lemma

.

By taking dY_t, using Ito's lemma, we have to find the first derivative with respect to time, which is 0 in this case. We add dY_t to the first derivative with respect to X, which is $\frac{1}{X_t}dX_t + \frac{1}{2}$ times the second derivative with respect to the spatial variable, which is $\frac{-1}{X_t^2}$, and that is multiplied by d of the quadratic variation of X_t. So, that's the stochastic differential of Y_t. Written in full:

$$dY_t = \frac{1}{X_t}dX_t + \frac{1}{2}\left(\frac{-1}{{X_t}^2}\right)d\langle X\rangle_t$$

Let's simplify that.

We will change the dt term to μ, and then this part here, (minus $\frac{1}{2}$ times d and the quadratic variation), will be $X_t\sigma^2$. So, X_t cancels with this part here and we get minus

$\frac{1}{2}\sigma^2$, which is the dt term, plus the dW terms, which is equal to σdW_t. That's a stochastic differential of Y_t. Written in full:

$$(\mu - \frac{1}{2}\sigma^2)dt \; + \sigma dW_t.$$

Since we assume that X_0 is equal to 1, this means that Y_0 is equal to 0 and, therefore, Y_t is equal to $\int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s$. This is something that we can actually evaluate explicitly as it doesn't depend on Y_t or any other unknown processes. These are all known stochastic processes and that is what we wanted.

So, this will be equal to μ minus $\frac{1}{2}\sigma^2$ times t, because this is a constant, plus, again since this is a constant, this will be σ times W_t minus W_0, which is 0. That is the expression for Y_t. And as we can see, Y_t has a normal distribution with a mean of itself as it is non-stochastic. So, that's μ minus $\frac{1}{2}\sigma^2 t$. This part has mean 0, so that is 0. Then the variance: as this does not have variance it equals σ^2 times t. This gives us the distribution of Y_t. Written in full:

$$Y_t = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma \mathrm{W_t} \sim \mathbb{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right).$$

Now we go back and find X_t by inverting this. So, we have to exponentiate on both sides: X_t is equal to e to the power Y_t, which in this case is e to the power μ minus $\frac{1}{2}\sigma^2 t$ plus σW_t. That is the expression for X_t and we've solved it explicitly in terms of Brownian motion. Written in full:

$$X_t = e^{Yt} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \quad .$$

We must now ask: what is the distribution of X_t?

Since the log of X_t, which is Y_t, has a normal distribution with these parameters here, that implies that X_t has a log normal distribution and we apply, of course, the same parameters of that normal distribution. Written in full:

$$lnX_t = Y_t \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$$

$$\Rightarrow X_t \sim logN\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, r^2 t\right)$$

So, X_t has a log normal distribution and this is a very popular model for stock price returns. As an exercise you can calculate the expected value of X_t and the variance of X_t using the moment-generating function (MGF) of Y_t.

Now that we've illustrated how to solve the stochastic differential equation, in the next video we're going to move onto the Martingale representation theorem.

3.2.7 Notes: Martingale Representation and Multidimensional Processes

We now turn to a result that is of great importance in hedging derivative securities.

Martingale representation theorem

Let W be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We know that if $\varphi \in L^2(W)$, then the process X such that $dX_t = \varphi_t dW_t$ is a martingale. The martingale representation theorem (MRT) says that, under certain conditions, the converse is also true – that is, all martingales are just stochastic integrals with respect to W.

Theorem 4.1 (MRT). *Let M be a martingale that is adapted to the natural filtration of W. Then there exists a predictable process φ such that*

$$M_t = M_0 + \int_0^t \varphi_s \, dW_s, \quad \textit{for every } t.$$

If M *is square integrable then* $\varphi \in L^2(W)$.

Let us apply this result to the martingale $X_t = W_t^2 - t$. We want to find φ such that

$$W_t^2 - t = \int_0^t \varphi_s \, dW_s, \quad \text{for every } t.$$

Let us apply Ito's lemma to find the stochastic differential of X. We have

$$dX_t = -dt + 2W_t \, dW_t + dt = 2W_t \, dW_t.$$

Hence

$$X_t = \int_0^t 2W_s \, dW_s,$$

giving

$$\varphi_t = 2W_t.$$

Note that by rearranging terms we get

$$W_T^2 = T + \int_0^T 2W_s \, dW_s = \mathbb{E}(W_T^2) + \int_0^T 2W_s \, dW_s.$$

This is an example of the *Ito representation theorem.*

Theorem 4.2. *Let* H *be an* $\mathcal{F}_T^W$ *-measurable random variable with* $\mathbb{E}(|H|) < \infty$*. Then there exists a predictable process* φ *such that.*

$$H = \mathbb{E}(H) + \int_0^T \varphi_s \, dW_s.$$

If $\mathbb{E}(H^2) < \infty$ *then* $\varphi \in L^2(W)$.

The proof just applies the MRT to $M_t = \mathbb{E}(H|\mathcal{F}_t)$. (Remember that we assume that $\mathcal{F}_0$ is trivial.)

We now consider extending the results of the previous sections to multidimensional processes.

An m-dimensional Brownian motion $W = (W^1, \dots, W^m)$ is a process such that each W^i is a Brownian motion process and $[W^i, W^j]_t = \delta_{ij} t$ for every t.

Likewise, we will define a d-dimensional Ito process $X = (X^1, \dots, X^d)$ as a process such that for each $i = 1, \dots, d$,

$$dX_t^i = \mu_t^i dt + \sum_{j=1}^{m} \sigma_t^{ij} dW_t^j$$

for some processes μ^i and σ^{ij}. We will sometimes write this in matrix/vector notation as

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

Where $\mu = (\mu^1, \dots, \mu^d)$ and $\sigma = (\sigma^{ij})$ is a $d \times m$ matrix.

We can then state a general version of Ito's lemma

Theorem 4.3 [Ito]. *Let X be a d-dimensional Ito process and $f: \mathbb{R}^d \to \mathbb{R}$ be a function that is twice continuously differentiable. Then the prices Y defined by $Y_t := f(X_t)$ is also an Ito process and*

$$dY_t = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} d\left[X^i, X^j\right]_t.$$

As an example, let $X = (X^1, X^2)$ satisfy the following SDEs:

$$dX^1 = \mu_1 X_t^1 dt + \sigma_1 \, dW_t^1 + \sigma_1 \, dW_t^2, \quad dX_t^2 = \mu_2 dt + \sigma_2 \, dW_t^1.$$

Let $Y_t = X_t^1 X_t^2$. The SDE of Y is

$$dY_t = X_t^2 dX_t^1 + X_t^1 dX_t^2 + d[X^1, X^2]_t,$$

where $d[X^1, X^2]_t = \sigma_1 \sigma_2 \, dt$.

There is also a multidimensional version of the MRT. As an exercise, formulate

3.2.8 Transcript: The Martingale Representation Theorem

Hi, in this video we look at an application of the martingale representation theorem.

The martingale representation theorem says that if, in continuous time, $M = \{M_t: 0 \leq t \leq T\}$, is a martingale that is adapted to the Brownian motion filtration, which we will denote by $\mathbb{F}_w$, then there exists a predictable process, φ, such that $M_t = M_0$ plus the stochastic integral with respect to this process φ. Written in full:

$$M_t = M_0 + \int_0^t \varphi_s dW_s$$

Of course,φ has to satisfy the usual condition that $\int_0^t \varphi_s^2 ds$ must be finite for all t, almost surely.

Another formulation, or an extension of this result, says that if H is a random variablethat is $\mathcal{F}_T^W$-measurable (with F^W of course being the Brownian filtration and we are evaluating the time T), and it is in L^2, in the sense that the expected value of H^2 is finite, then H can be written as the expected value of H plus a stochastic integral with respect to Brownian motion, where φ belongs to L^2 of W. This is called the **Ito representation theorem**. Written in full:

$$H = E(H) + \int_0^T \varphi_t dW_s, \varphi \in L^2(W).$$

Let's look at an example.

Let's take H to be equal to ${W_T}^2$ and find out what this stochastic process is. So, since this is a Brownian motion, the expected value of H, or the expected value of ${W_T}^2$, will be equal to T.

Now, we have to calculate what φ is and for that we will define the following martingale: $F(t, W_t)$. This is defined to be the expected value of H given $\mathcal{F}_t^W$, which is the expected value of ${W_T}^2$, given $\mathcal{F}_t^W$, which is equal to the expected value of ${W_T}^2$ given W_t, because W is a Markov process, so conditioning on the filtration up to time t is equivalent to just conditioning on ${W_T}^2$.

So, we have to calculate this and for that we will rewrite this by creating an increment as W_T minus W_t plus W_t all squared, given W_T, which is equal to the expected value of W_T minus ${W_T}^2$ plus 2 times W_T times W_T minus W_T plus ${W_T}^2$ all given W_T, which is equal to: now this part here, this expectation conditional with W_T is independent of this section here, so this would just be the variance, which is T minus t, and then this part here, we can take out and get this part, which is independent of W_T; and, therefore, the expectation of this will be equal to 0.

Finally, this part here, since this is ${W_T}^2$ given W_t, this will just be ${W_T}^2$. So, this is plus ${W_T}^2$. And here we have the martingale that we are dealing with.

Written in full:

$$H \ = {W_T}^2$$

$$E(H) = E(W_T^2) = T.$$

$$F(t, W_t) := E\left(H \middle| \mathcal{F}_t^{\mathcal{W}}\right) = E\left(W_T^2 \middle| \mathcal{F}_t^{\mathcal{W}}\right)$$

$$= E(W_t^2 | W_t) = E((W_T - W_t + W_t)^2 | W_t)$$

$$= E((W_T - W_t)^2 + 2W_t(W_T - W_t) + W_t^2 | W_t)$$

$$= T - t + W_t^2.$$

Now, since this is a martingale, we can therefore notice that this is actually equal to $F_t W_t$ where $F(t, x)$ is this function here: $F(t, x) = T - t + x^2$. If you define this function, then this is just F applied to Brownian motion. We can therefore apply Ito's lemma, since this satisfies all the conditions of Ito's lemma, to get the following: $dF(t, W_t)$ is equal to the first derivative with respect to time, dt, plus the first derivative with respect to x, dW_t, plus $\frac{1}{2}$ times the second derivative with respect to the spatial variable, the quadratic variation of Brownian motion, which is t itself.

This simplifies to the first derivative with respect to time plus $\frac{1}{2}$ second derivative with respect to the spatial variable, and that's dt, plus this part here: the partial derivative with respect to x, dW_t, and this is equal to 0, because of the Martingale property. This means that the drift should be equal to 0. This derivative here will be $2x$, and then we substitute Brownian motion inside there, and we get $2W_t dW_t$.

Therefore, we have the following: $F(t, W_t)$ is simply equal to the integral, which is a stochastic differential. So, it's $F(0, W_0)$ plus the integral from 0 to t of $2W_s dW_s$. And, therefore, if we substitute T, F of T, W_T will just be H itself. So, this implies that H is equal to $F(0, W_0)$ and when we substitute 0 here, we get T plus the integral from 0 to T of $2W_t dW_t$. And therefore, this is our integrand. So that's what φ is.

Written in full:

$$dF(t,W_t) = \frac{\delta F}{\delta t}dt + \frac{\delta F}{\delta x}dW_t + \frac{1}{2}\frac{\delta^2 F}{\delta x^2}dt$$

$$= \left[\frac{\delta F}{\delta t} + \frac{1}{2}\frac{\delta^2 F}{\delta x^2}\right]dt + \frac{\delta F}{\delta x}dW_t = 2W_t dW_t$$

$$F(t,W_t) = F(0,W_0) + \int_0^t 2W_s dW_s$$

$$\Rightarrow H = T + \int_0^T 2W_t dW_t.$$

That brings us to the end of the module. In the next module we're going to look at Stochastic Calculus II: Semimartingales.

3.2.9 Problem Set

Problem 1

If $dX_t = 2dt + 3dW_t$ and $Y_t = X_t^2$, then dY_t is equal to...?

Solution:

We need to apply Ito's rule considering $Y_t = F(t, X_t) = X_t^2$ as follows:

$$dY_t = dF(t, X_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}dX_t^2.$$

In our case, the first derivative, $\frac{\partial F}{\partial t}$ is zero, and the other terms give us:

$$dY_t = dF(t, X_t) = 4X_t dt + 6X_t dW_t + 9dt.$$

Take into account that in order to get the last term above we make use of $dX_t^2 = 3^2 dW_t^2 = 9dt$.

The solution will be: $dY_t = (4X_t + 9)dt + 6X_t dW_t$.

Problem 2

Compute the variance of:

$$\int_0^2 t^2 dW_t.$$

Solution:

We need to apply the following property (see in lecture notes).

If φ is deterministic, then:

$$\int_0^t \varphi_s dW_t \backsim N\left(0, \int_0^t \varphi_s^2 ds\right).$$

Thus, we can compute the variance as:

$$\int_0^t \varphi_s^2 ds = \int_0^2 s^4 ds = \frac{s^5}{5}|_0^2 = \frac{32}{5}.$$

The solution is equal to 32/5.

Problem 3

If $dX_t = X_t dt + 2X_t dW_t$ and $dY_t = 3dt - 3Y_t^2 dW_t$, then what is $d[X, Y]_t$?

Solution:

Remember the following result from the theory. Let $X = (X^1, X^2)$satisfy the following SDEs:

$$dX_t^1 = \mu_1 X_t^1 dt + \sigma_1 \, dW_t^1 + \sigma_1 \, dW_t^2, \quad dX_t^2 = \mu_2 dt + \sigma_2 \, dW_t^1.$$

Let $Y_t = X_t^1 X_t^2$. The SDE of Y is

$$dY_t = X_t^2 dX_t^1 + X_t^1 dX_t^2 + d[X^1, X^2]_t,$$

Where $d[X^1, X^2]_t = \sigma_1 \sigma_2 \, dt$. Thus, the result should be: $-6X_t Y_t^2 dt$. You can derive this by yourself, taking into account the "multiplication rules":

$dtdt = 0, dtdW = 0, dWdW = dt, dW_i dW_j = 0$ if $i \neq j (independent)$

and

$$d[X, Y]_t = 3X_t dt^2 - 3X_t Y_t^2 dt dW_t + 6X_t dW_t dt - 6X_t Y_t^2 dW_t dW_t = -6X_t Y_t^2 dt.$$

Problem 4

The martingale $M_t = W_t^3 - 3tW_t$ can be represented as$M_t = \int_0^t \varphi_s \, dW_s$ where...?

Solution:

Applying the Ito formula to the function $f(t, X_t) = X_t^3 - 3tX_t$:

$$dM_t = df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} d[X]_t.$$

With the following derivatives,

$$\frac{\partial f}{\partial t} = -3X_t, \;\; \frac{\partial f}{\partial X} = 3X_t^2 - 3t \text{ and } \frac{\partial^2 f}{\partial X^2} = 6X_t.$$

Thus,

$$dM_t = -3X_t dt + (3X_t^2 - 3t) dX_t + \frac{1}{2} 6X_t dt = 3(X_t^2 - t) dXt.$$

So, we can conclude that:

$$\varphi_t = 3(X_t^2 - t).$$

Problem 5

If $dX_t = 2dt + 3dW_t$ and $Y_t = X_t^2$, then what is $d[Y]_t$ equal to?

Solution:

Applying the Ito formula to the function $f(t, X_t) = X_t^2$:

$$dM_t = df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} d[X]_t.$$

With the following derivatives,

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial X} = 2X_t \text{ and } \frac{\partial^2 f}{\partial X^2} = 2.$$

The process for dY_t is defined by:

$$dY_t = 2X_t dX_t + 9dt = 6X_t dW_t + (4X_t + 9)dt.$$

Thus, the differential quadratic variation of Y_t will be:

$$d[Y]_t = 36X_t^2 dt.$$

Problem 6

Let $\varphi_t = -2I_{(0,4]}(t) + 5W_4 I_{(4,7]}(t)$. Then $\int_0^7 \varphi_t \, dW_t$ is...?

Solution:

The above integral is defined by parts as:

$$\int_0^7 \varphi_t \, dW_t = \int_0^7 \left(-2I_{(0,4]}(t) + 5W_4 I_{(4,7]}(t)\right) dW_t =$$

$$= -2\int_0^4 dW_t + 5\int_4^7 W_4 dW_t = -2W_4 + 5W_4(W_7 - W_4).$$

Problem 7

If $dX_t = dt + 2t^4 dW_t$ then what is $[X]_t$ equal to?

Solution:

The quadratic variation behaves in a similar way to the predictable quadratic variation, $\langle . \rangle$. If X is an Ito process with stochastic differential

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

then

$$[X]_t = \int_0^t \sigma_s^2 \; ds.$$

In our example, we just have to compute the following integral,

$$[X]_t = \int_0^t \sigma_s^2 \; ds = \int_0^t 4 \, t^8 \, ds = \frac{4}{9} t^9.$$