WORLDQUANT UNIVERSITY

Discrete Time Stochastic Processes Module 5

MSc Financial Engineering

Table of Contents

1. Brief

This document contains the core content for Module 5 of Discrete-time Stochastic Processes, entitled The Binomial Model. It consists of four sets of notes, four video lecture transcripts, and a problem set.

2. Course Context

Discrete-time Stochastic Processes is the third course present in the WorldQuant University (WQU) Master of Science in Financial Engineering (MScFE) program. The course introduces derivative pricing in discrete time. It begins with measure-theoretic probability and stochastic processes, with an emphasis on discrete-time martingales. These ideas are then applied to the pricing of derivatives in discrete time, followed by an introduction to interest rate and credit risk modeling.

2.1 Course-level Learning Outcomes

After completing the Discrete-time Stochastic Processes course, you will be able to::

1. Understand the language of measure-theoretic probability.
2. Understand stochastic processes and their applications.
3. Understand the theory of discrete-time martingales.
4. Define trading strategies in discrete time.
5. Create replicating portfolios in discrete time.
6. Model stock price movements on a binomial tree.
7. Price and hedge European derivatives in discrete time.
8. Price and hedge exotic European derivatives in discrete time.
9. Price and hedge American derivatives on a binomial tree.
10. Construct a simple interest rate model on a tree.
11. Price interest rate derivatives on a tree.

2.2 Module Breakdown

The Discrete-time Stochastic Processes course consists of the following one-week modules:

1. Probability Theory
2. Stochastic Processes
3. Discrete Martingales
4. Trading in Discrete Time
5. The Binomial Model
6. American Derivatives
7. An Introduction to Interest Rate Models

3. Module 5:

The Binomial Model

This module is about a popular discrete-time model for asset prices – the binomial model – which is used to represent asset dynamics for discrete processes, as the asset values change randomly. The module begins by defining the binomial model in detail and then proceeds to price different kinds of derivatives in this model.

3.1 Module-level Learning Outcomes

After completing this module, you will be able to:

1. Construct a binomial tree.
2. Price and hedge vanilla European derivatives on a tree.
3. Price and hedge path-dependent and exotic European options on a tree.

3.2 Transcripts and Notes

3.2.1 Notes: Modeling on a Tree

We describe a special market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$ with trading occurring at times $t = 0,1, \dots, T$, where T is a positive integer. First, we will assume that the market has only one risky asset, X (i.e. $d = 1$). Thus, the primary tradeable assets can be represented as a vector $(1, X)$ where 1 is the risk-free asset.

The *binomial model* or *binomial tree* assumes a very simple form: at each time t, the price of X at the next time step $(t+1)$ can take only two values, both of which are multiples of X_t. To be precise, let u and d be positive real numbers with $d < u$. We will assume that for every $t \in \{1, \dots T\}$,

$$X_t = X_{t-1} u^{Z_t} d^{(1-Z_t)},$$

where the Z_t's are i.i.d Bernoulli random variables. Thus,

$$X_t = \begin{cases} uX_{t-1} & \text{if } Z_t = 1 \\ dX_{t-1} & \text{if } Z_t = 0. \end{cases}$$

We can interpret this as follows: if we know the value of X_{t-1}, the value of X_t can either jump "up" to uX_{t-1} or jump "down" to dX_{t-1}. We have not yet assigned probabilities to these jumps.

If we proceed inductively, we can write (for each $t \geq 1$):

$$X_t = X_0 \prod_{i=1}^{t} u^{Z_i} d^{(1-Z_i)} = X_0 \left(\frac{u}{d}\right)^{\sum_{i=1}^{t} Z_i} d^t = X_0 \left(\frac{u}{d}\right)^{Y_t} d^t,$$

where $Y_t \coloneqq \sum_{i=1}^{t} Z_i$ has a binomial distribution.

With these heuristics, we can now formally define all the components of the market $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), X)$. First, let

$$\Omega \coloneqq \{\omega = (\omega_1, \dots, \omega_T) : \omega_i \in \{0,1\}\}.$$

Define $\mathcal{F} \coloneqq 2^{\Omega}$ and assume that $\mathbb{P}(\{\omega\}) > 0$ for each $\omega \in \Omega$. Define the sequence $\{Z_t : t = 1, \dots, T\}$ as

$$Z_t(\omega) \coloneqq \omega_t, \quad \omega \in \Omega, \quad 1 \le t \le T.$$

Define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$ by:

$$\mathcal{F}_0 \coloneqq \{\emptyset, \Omega\}, \quad \text{and } \mathcal{F}_t \coloneqq \sigma(\{Z_1, \dots, Z_t\}) \quad t \ge 1.$$

We can then define the price process $X = \{X_t : t = 0.1, \dots, T\}$ as

$$X_0 = \text{constant, and } X_t = X_0 \prod_{i=1}^{t} u^{Z_i} d^{(1-Z_i)} = X_0 \left(\frac{u}{d}\right)^{\sum_{i=1}^{t} Z_i} d^t, \ \ t \ge 1,$$

where $0 < d < u$ are fixed constants.

We now consider martingale measures on this market. If $\mathbb{P}^*$ is a martingale measure, then for every $t \ge 1$,

$$X_{t-1} = \mathbb{E}^*(X_t | \mathcal{F}_{t-1}) = X_{t-1} \mathbb{E}^*\left(u^{Z_t} d^{(1-Z_t)} | \mathcal{F}_{t-1}\right),$$

which implies that

$$\mathbb{E}^*\left(u^{Z_t} d^{(1-Z_t)} \middle| \mathcal{F}_{t-1}\right) = 1.$$

If we let $p^*(Z_1, \dots, Z_{t-1}) \coloneqq \mathbb{P}(Z_t = 1|\mathcal{F}_{t-1})$, then

$$\mathrm{up}^*(Z_1, \dots, Z_{t-1}) + \left(1 - p^*(Z_1, \dots, Z_{t-1})\right)d = 1,$$

which implies that

$$p^*(Z_1, \dots, Z_{t-1}) = \frac{1-d}{u-d} =: p^*.$$

It is left as an exercise to show that this implies that the random variables Z_t are i.i.d Bernoulli random variables with parameter p^*, provided $d < 1 < u$. Thus, the market has a unique EMM if and only if $d < 1 < u$. We shall henceforth make this assumption.

3.2.2 Transcript: Looking at the Binomial Model

Hi, in this video, we study a simple but important example of a complete market model called *the binomial model*.

In this model we will assume that the market consists of only one risky asset, X, and the riskless asset 1.

The risky asset is assumed to evolve as follows: X_0 is fixed, and for every $t \in \{0, 1, 2, \dots, T-1\}$, X_{t+1} is a function of X_t:

$$X_{t+1} = \begin{cases} uX_t \\ dX_t \end{cases},$$

where $0 < d < u$.

This model has a unique EMM if and only if $d < 1 < u$ and, in this case, the risk neutral conditional probability of an upward movement is:

$$p^* = \frac{1-d}{u-d}.$$

Let Y_t be the number of up movements up to time t (with $Y_0 := 0$). Then Y has a binomial distribution with parameters t and p^*. That is,

$$\mathbb{P}(Y_t = y) = \binom{t}{y} p^{*y}(1-p^*)^{t-y} \quad y = 0, 1, \dots, t.$$

The random variables $Z_t := Y_t - Y_{t-1}$ are i.i.d Bernoulli random variables.

If H is a contingent claim, then the price of H is given by:

$$\pi(H) = \mathbb{E}^*(H) = \sum_{z \in Z} H(z) \prod_{t=1}^{T} p^{*z_t} (1 - p^*)^{1 - z_t},$$

where $Z = \{z = (z_1, \dots, z_T) : z_t \in \{0, 1\}\}$.

If H is a derivative that depends only on the terminal value of $X, H = h(X_T)$, then we can write the price as:

$$\pi(H) = \mathbb{E}^*(H) = \mathbb{E}^*\left(h(X_0 u^{Y_T} d^{T - Y_T})\right) = \sum_{y=0}^{T} h(X_0 u^{Y_T} d^{T - Y_T}) \binom{T}{y} p^{*y} (1 - p^*)^{T - y}.$$

3.2.3 Notes: Pricing on a Tree

We now want to price contingent claims in the market constructed in the previous section.

Let H be a contingent claim. Since the market is complete, the unique no-arbitrage price of H is given by $\pi(H) = \mathbb{E}^*(H)$. From the previous section, we can write $\mathbb{P}^*$ as:

$$\mathbb{P}^*(\{\omega\}) = \textstyle\prod_{t=1}^{T} p^{*\omega_t}(1-p^*)^{1-\omega_t}, \quad \omega = (\omega_1, \dots, \omega_T) \in \Omega.$$

Hence,

$$\pi(H) = \mathbb{E}^*(H) = \sum_{\omega \epsilon \Omega} H(\omega) \prod_{t=1}^{T} p^{*\omega t}(1-p^*)^{1-\omega t}.$$

If H is a derivative that depends only on the terminal value of X, $H = h(X_T)$, then we can write the price as

$$\pi(H) = \mathbb{E}^*(H) = \mathbb{E}^*\left(h(X_0 u^{Y_T} d^{T-Y_T})\right) = \sum_{y=0}^{T} h\left(X_0 u^y d^{T-y}\right) \binom{T}{y} p^*(1-p^*)^{T-y},$$

where $Y_T := \sum_{i=1}^{T} Z_i$ has a binomial distribution with parameters $n = T$ and $p = p^*$.

Let us look at a concrete example of a call option. A call option is a derivative H with payoff function $H = h(X_T) = (X_T - K)^+$, where K is the strike price. The price is therefore given by:

$$\pi_C := \pi(H) = \mathbb{E}^*(H) = \sum_{y=0}^{T} (X_0 u^y d^{T-y} - K)^+ \binom{T}{y} p^{*y}(1-p^*)^{T-y}.$$

With the following values $X_0 = 100$, $K = 110$, $u = 1/d = 1.2$, $T = 2$, we get

$$p^* = \frac{1-d}{u-d} = \frac{45}{99} = \frac{5}{11}$$

and the price is

$$\pi_C = (100 \times 1.2^{-2} - 110)^+ \left(\frac{6}{11}\right)^2 + 2(100 - 110)^+ \left(\frac{5}{11}\right)\left(\frac{6}{11}\right) + (100 \times 1.2^2 - 110)^+ \left(\frac{5}{11}\right)^2$$
$$= 34 \left(\frac{5}{11}\right)^2 = \frac{850}{121}.$$

Another popular example is the put option with a payoff function of $H = (K - X_T)^+$, where K is again called the strike price. The price is given by

$$\pi_P \coloneqq \pi(H) = \mathbb{E}^*(H) = \sum_{y=0}^{T} (K - X_0 u^y d^{T-y})^+ \binom{T}{y} p^{*y} (1-p^*)^{T-y}.$$

Using the same parameters as above ($X_0 = 100$, $K = 110$, $u = 1/d = 1.2$, $T = 2$), we get:

$$\pi_P = (110 - 100 \times 1.2^{-2})^+ \left(\frac{6}{11}\right)^2 + 2(110 - 100)^+ \left(\frac{5}{11}\right)\left(\frac{6}{11}\right) + (110 - 100 \times 1.2^2)^+ \left(\frac{5}{11}\right)^2$$
$$= 10 + \frac{850}{121}.$$

Now, consider the derivative H which is the difference between the call option and the put option. Then the payoff of H is:

$$H = (X_T - K)^+ - (K - X_T)^+ = \begin{cases} X_T - K & \text{if } X_T > K \\ X_T - K & \text{if } X_T \leq K \end{cases} = X_T - K.$$

Hence,

$$\mathbb{E}^*((X_T - K)^+ - (K - X_T)^+) = \mathbb{E}^*(X_T - K),$$

which implies that

$$\pi_C - \pi_P = X_0 - K.$$

This relationship between the call price π_C and the put price π_P is called the *put-call parity*. We can also verify for the prices calculated above as

$$\frac{850}{121} - \left(10 + \frac{850}{121}\right) = -10 = 100 - 110.$$

3.2.4 Transcript: Pricing an Option in the Binomial Model

Hi, in this video we look at an example of pricing an option in the binomial model.

Let us look at a concrete example of a call option. A call option is a derivative H with payoff function $H = h(X_T) = (X_T - K)^+$, where K is the strike price. The price is therefore given by:

$$\pi_C := \pi(H) = \mathbb{E}^*(H) = \sum_{y=0}^{T} (X_0 u^y d^{T-y} - K)^+ \binom{T}{y} p^{*y}(1-p^*)^{T-y}.$$

With the following values $X_0 = 100, K = 110, u = 1/d = 1.2, T = 2$, we get:

$$p^* = \frac{1-d}{u-d} = \frac{45}{99} = \frac{5}{11},$$

and the price is

$$\begin{aligned}\pi_C &= (100 \times 1.2^{-2} - 110)^+ \left(\frac{6}{11}\right)^2 + 2(100-110)^+ \left(\frac{5}{11}\right)\left(\frac{6}{11}\right) + (100 \times 1.2^2 - 110)^+ \left(\frac{5}{11}\right)^2 \\ &= 34\left(\frac{5}{11}\right)^2 = \frac{850}{121}.\end{aligned}$$

Another popular example is the put option with a payoff function of $H = (K - X_T)^+$, where K is again called the strike price. The price is given by:

$$\pi_P := \pi(H) = \mathbb{E}^*(H) = \sum_{y=0}^{T} (K - X_0 u^y d^{T-y})^+ \binom{T}{y} p^{*y}(1-p^*)^{T-y}.$$

Using the same parameters as above ($X_0 = 100, K = 110, u = 1/d = 1.2, T = 2$), we get:

$$\pi_P = (110 - 100 \times 1.2^{-2})^+ \left(\frac{6}{11}\right)^2 + 2(110 - 100)^+ \left(\frac{5}{11}\right)\left(\frac{6}{11}\right) + (110 - 100 \times 1.2^2)^+ \left(\frac{5}{11}\right)^2$$
$$= 10 + \frac{850}{121}.$$

Now consider the derivative H which is the difference between the call option and the put option. Then the payoff of H is:

$$H = (X_T - K)^+ - (K - X_T)^+ = \begin{cases} X_T - K & \text{if} \quad X_T > K \\ X_T - K & \text{if} \quad X_T \leq K \end{cases} = X_T - K.$$

Hence,

$$\mathbb{E}^*((X_T - K)^+ - (K - X_T)^+) = \mathbb{E}^*\,(X_T - K),$$

which implies that

$$\pi_C - \pi_P = X_0 - K.$$

This relationship between the call price π_C and the put price π_P is called the *put-call parity*. We can also verify for the prices calculated above as:

$$\frac{850}{121} - \left(10 + \frac{850}{121}\right) = -10 = 100 - 110.$$

3.2.5 Notes: Hedging

We now look at the problem of replication in a binomial tree.

Since the market is complete, we know that the unique EMM $\mathbb{P}^*$ has PRP, in the sense that for every $(\mathbb{F}, \mathbb{P}^*)$-martingale M, there exists a predictable process, φ^M, such that for every $1 \leq t \leq T$,

$$M_t - M_0 = \sum_{k=1}^{t} \varphi_k^M (X_k - X_{k-1}).$$

The process φ^M advertized above can be found as follows. First note that for each $t \geq 1$,

$$M_t = M_{t-1} + \varphi_t^M (X_t - X_{t-1}),$$

which implies that

$$\varphi_t^M = \frac{M_t - M_{t-1}}{X_t - X_{t-1}}$$

since $X_t \neq X_{t-1}$.

Now, let H be a contingent claim. Define the martingale $V^H = (V_t^H : 0 \leq t \leq T)$ by:

$$V_t^H \coloneqq \mathbb{E}^*(H|\mathcal{F}_t), \qquad 0 \leq t \leq T.$$

Note that $V_0^H = \mathbb{E}^*(H) = \pi(H)$.

Now since V^H is an $(\mathbb{F}, \mathbb{P}^*)$-martingale, it follows that we can find a predictable process φ^H such that

$$V_t^H - V_0^H = \sum_{k=1}^{t} \varphi_k^H (X_k - X_{k-1}),$$

which implies that

$$V_t^H = \pi(H) + \sum_{k=1}^{t} \varphi_k^H (X_k - X_{k-1}).$$

Furthermore,

$$\varphi_t^H = \frac{V_t^H - V_{t-1}^H}{X_t - X_{t-1}}.$$

Notice that since

$$V_t^H = \pi(H) + \sum_{k=1}^{t} \varphi_k^H (X_k - X_{k-1}),$$

it follows that V^H is the value of the replicating strategy φ^H, and $V_T^H = \mathbb{E}^*(H|\mathcal{F}_T) = H$.

Let us look at an example. Consider the example discussed in the previous section with the following values: $X_0 = 100, u = 1/d = 1.2, T = 2$. Consider a call option with strike price $K = 110$. We want to calculate φ^H, where $H = (X_T - 110)^+$.

Since $T = 2$, the sample space is:

$$\Omega = \{(1,1), (1,0), (0,1), (0,0)\}.$$

For convenience, we will replace 1 with u and 0 with d and write

$$\Omega = \{(u,u), (u,d), (d,u), (d,d)\}.$$

The values of V can be summarized in the following table:

ω	$V_0^H(\omega)$	$V_1^H(\omega)$	$V_2^H(\omega)$	$X_T(\omega)$	$H(\omega) = (X_T(\omega) - 110)^+$
(u, u)	$\frac{850}{121}$	$\frac{170}{11}$	34	$100u^2$	34
(u, d)	$\frac{850}{121}$	$\frac{170}{11}$	0	100	0
(d, u)	$\frac{850}{121}$	0	0	100	0
(d, d)	$\frac{850}{121}$	0	0	$100d^2$	0

The hedging strategy φ^H is given by:

$$\varphi_t^H = \frac{V_t^H - V_{t-1}^H}{X_t - X_{t-1}}.$$

The values are summarized below:

ω	$\varphi_1^H(\omega)$	$\varphi_2^H(\omega)$
(u, u)	$\frac{51}{121}$	$\frac{17}{22}$
(u, d)	$\frac{51}{121}$	$\frac{17}{22}$
(d, u)	$\frac{51}{121}$	0
(d, d)	$\frac{51}{121}$	0

The interpretation of the strategy is as follows. At time 0, starting with an initial capital of $V_0^H = \frac{850}{121}$, buy $\varphi_1^H = \frac{51}{121}$ units of X and invest the remainder $\eta_1^H = V_0^H - \varphi_1^H X_0 = \frac{-4250}{121}$. So, the transaction involves borrowing $\frac{4250}{121}$ from the bank. Hold these quantities until time 1.

At time 1, if the share goes up from 100 to 120, then the value of the portfolio will be $V_1^H((u,u)) = V_1^H((u,d)) = \frac{-4250}{121} + \frac{51}{121} \times 120 = \frac{170}{11}$, and, in that case, we will change our holding in X to be $\varphi_2^H\big((u,u)\big) = \varphi_2^H\big((u,d)\big) = \frac{17}{22}$ and the bank account investment to $\eta_2^H = \frac{170}{11} - \frac{17}{22} \times 120 = \frac{-850}{11}$.

Likewise, if the stock falls to $1.2^{-1} \times 100$, the value of the portfolio will be $V_1^H\big((d,u)\big) = V_1^H\big((d,d)\big) = \frac{-4250}{121} + \frac{51}{121} \times (1.2^{-1} \times 100) = 0$, and there will be no investment in the stock ($\varphi_2^H\big((d,u)\big) = \varphi_2^H\big((d,d)\big) = 0$), and no investment in the bank account $\eta_2 = V_2 - \varphi_2^H X_2 = 0$.

At time 2, if the share goes up again (to 144), the value of the portfolio will be $V_2\big((u,u)\big) = \frac{-850}{11} + \frac{17}{22} \times 144 = 34$. For all other cases, V^H is equal to 0, which corresponds to the value of the derivative.

3.2.6 Transcript: Hedging an Option in the Binomial Model

Hi, in this video we go through an example of hedging an option in the binomial model.

Consider the example discussed in the previous section, with the following values: $X_0 = 100, u = 1/d = 1.2, T = 2$. Consider a call option with strike price $K = 110$. We want to calculate φ^H, where $H = (X_T - 110)^+$.

Since $T = 2$, the sample space is:

$$\Omega = \{(1,1),(1,0),(0,1),(0,0)\}.$$

For convenience, we will replace 1 with u and 0 with d, and write

$$\Omega = \{(u,u),(u,d),(d,u),(d,d)\}.$$

The values of V can be summarized in the following table:

ω	$V_0^H(\omega)$	$V_1^H(\omega)$	$V_2^H(\omega)$	$X_T(\omega)$	$H(\omega) = (X_T(\omega) - 110)^+$
(u,u)	$\frac{850}{121}$	$\frac{170}{11}$	34	$100u^2$	34
(u,d)	$\frac{850}{121}$	$\frac{170}{11}$	0	100	0
(d,u)	$\frac{850}{121}$	0	0	100	0
(d,d)	$\frac{850}{121}$	0	0	$100d^2$	0

The hedging strategy φ^H is given by:

$$\varphi_t^H = \frac{V_t^H - V_{t-1}^H}{X_t - X_{t-1}}.$$

The values are summarized below:

ω	$\varphi_1^H(\omega)$	$\varphi_2^H(\omega)$
(u, u)	$\frac{51}{121}$	$\frac{17}{22}$
(u, d)	$\frac{51}{121}$	$\frac{17}{22}$
(d, u)	$\frac{51}{121}$	0
(d, d)	$\frac{51}{121}$	0

3.2.7 Notes: Exotic Derivatives

In this section we will give examples of some common exotic derivatives and show how to price them. We will continue with the market used in the previous section. Recall that this market has the following parameters:

$$T = 2,\ X_0 = 100, u = 1/d = 1.2 \text{ and } p^* = 5/11.$$

The first example we consider is an *Asian option.* The payoff of an *arithmetic average Asian call option* with strike price K is

$$H := \max\left(\frac{1}{T+1}\sum_{t=0}^{T} X_t - K, 0\right) = \left(\frac{1}{T+1}\sum_{t=0}^{T} X_t - K\right)^+.$$

Its payoff is similar to that of a vanilla (ordinary) call option, but instead of using the terminal value of the stock price X_T, one uses the average of X over the lifetime of the option. The are variations on the time points used to calculate the average.

In our example, the payoff looks like this:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = (\frac{1}{3}\Sigma_{t=0}^{2} X_t(\omega) - 110\ )^+$
(u, u)	100	120	144	34/3
(u, d)	100	120	100	0
(d, u)	100	100×1.2^{-1}	100	0
(d, d)	100	100×1.2^{-1}	100×1.2^{-2}	0

The price can then be calculated as

$$\pi(H) = \frac{34}{3} \times p^{*2} = \frac{850}{363}.$$

The price of an Asian put option can be calculated in a similar way. One can also replace the arithmetic average with a geometric average, and the corresponding option is called a *geometric average Asian option*.

Next, we look at a *lookback option*. The payout of a lookback put option is:

$$H \coloneqq \underset{0\leq t\leq T}{\text{MAX}} X_t - X_T.$$

For our example, the payoff values are tabulated below:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = \max_0 \leq_t \leq_2 X_t(\omega) - X_2(\omega)$
(u, u)	100	120	144	0
(u, d)	100	120	100	20
(d, u)	100	100×1.2^{-1}	100	0
(d, d)	100	100×1.2^{-1}	100×1.2^{-2}	$\frac{275}{9}$

The price of H is calculated as follows:

$$\pi(H) = 20 \times \frac{5}{11} \times \frac{6}{11} + \frac{275}{9} \times \left(\frac{6}{11}\right)^2 \approx 14.0496.$$

The last example we look at is a *barrier option*. A barrier option makes a payment if the stock price hits or misses a barrier. As an example, we consider an *up-and-out call option* whose payoff is

$$H := (X_T - K)^+ I_{\{max_{0\leq t\leq T} X_t \geq B\}} = \begin{cases} 0 & \text{if } max_{0\leq t\leq T} X_t \geq B \\ (X_T - K)^+ & \text{otherwise.} \end{cases}$$

Here, K is the strike price and $B > \max(K, X_0)$ is the barrier. So, the option becomes void if X crosses the barrier B; otherwise it behaves like an ordinary call option, hence the name "up and out".

For parameters $B = 115$ and $K = 95$ we get:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = (X_2(\omega) - 95)^+ I_{\{\max_0 \leq_t \leq_2 X_t(\omega) > 115\}}$
(u,u)	100	120	144	0
(u,d)	100	120	100	0
(d,u)	100	100×1.2^{-1}	100	5
(d,d)	100	100×1.2^{-1}	100×1.2^{-2}	0

The price of H is calculated as follows:

$$\pi(H) = 5 \times \frac{6}{11} \times \frac{5}{11} = \frac{150}{121} \approx 1.23967.$$

3.2.8 Transcript: How to Price Some Exotic Options

Hi, in this video, we go through examples of how to price some exotic options.

The first example we consider is an *Asian option*. The payoff of an *arithmetic average Asian call option* with strike price K is:

$$H := max\left(\frac{1}{T+1}\sum_{t=0}^{T} X_t - K, 0\right) = \left(\frac{1}{T+1}\sum_{t=0}^{T} X_t - K\right)^{+}.$$

Its payoff is similar to that of a vanilla (ordinary) call option, but instead of using the terminal value of the stock price X_T, one uses the average of X over the lifetime of the option. There are variations on the time points used to calculate the average.

In our example, the payoff looks like this:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = (\frac{1}{3}\Sigma_{t=0}^{2} X_t(\omega) - 110\ )^{+}$
(u, u)	100	120	144	34/3
(u, d)	100	120	100	0
(d, u)	100	100×1.2^{-1}	100	0
(d, d)	100	100×1.2^{-1}	100×1.2^{-2}	0

The price can then be calculated as:

$$\pi(H) = \frac{34}{3} \times p^{*2} = \frac{850}{363}.$$

Next, we look at the *lookback option*. The payout of a lookback put option is:

$$H := \max_{0 \le t \le T} X_t - X_T.$$

For our example, the payoff values are tabulated below:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = \max_{0 \leq t \leq 2} X_t(\omega) - X_2(\omega)$
(u, u)	100	120	144	0
(u, d)	100	120	100	20
(d, u)	100	100×1.2^{-1}	100	0
(d, d)	100	100×1.2^{-1}	100×1.2^{-2}	$\frac{275}{9}$

The price of H is calculated as follows:

$$\pi(H) = 20 \times \frac{5}{11} \times \frac{6}{11} + \frac{275}{9} \times \left(\frac{6}{11}\right)^2 \approx 14.0496.$$

3.2.9 Notes: Problem Set

Problem 1

Consider a binomial tree with the following parameters:

$$u = 1.4, \ \ d = 0.7, \ \ T = 2, \ \ X_0 = 100.$$

What is the price of a call option with strike price $K = 100$ and expiring at time $T = 2$?

Solution:

First of all, remember the payoff at maturity for a call option,

$$H := \max\{X_T - K, 0\}.$$

The payoff values are tabulated below:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = \max\{X_2(\omega) - K, 0\}$
(u, u)	100	140	196	96
(u, d)	100	140	98	0
(d, u)	100	70	98	0
(d, d)	100	70	49	0

The second step is always to compute the P^*. From the lecture notes:

$$p^* = \frac{1-d}{u-d} = \frac{1-0.7}{1.4-0.7} \approx 0.4286.$$

From the table above we know that, $H(\{u, d\}) = H(\{d, u\}) = H(\{d, d\}) = 0$. The price of H is calculated as follows:

$$\pi(H) = \mathbb{E}^*(H) = P^*(\{u,u\})H(\{u,u\}) + P^*(\{u,d\})H(\{u,d\}) + P^*(\{d,u\})H(\{d,u\}) + P^*(\{d,d\})H(\{d,d\}) = p^2 \times 96 \approx 17.6327.$$

Problem 2

Consider a binomial tree with the following parameters:

$$u = 1.25, \;\; d = 1/u, \;\; T = 2, \;\; X_0 = 400.$$

Consider a hedging strategy (η, φ) for the derivative H with payoff $H = X_2 - 319$, where φ is the investment in X. Compute the value of φ_1.

Solution:

The payoff values are tabulated below:

ω	$X_0(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$H(\omega) = \max\{X_2(\omega) - 319, 0\}$
(u,u)	400	500	625	306
(u,d)	400	500	400	81
(d,u)	400	320	400	81
(d,d)	400	320	256	-63

The second step is always to compute the P^*. From the lecture notes:

$$p^* = \frac{1-d}{u-d} = \frac{1-4/5}{5/4-4/5} = \frac{4}{9} \approx 0.444.$$

Notice that since

$$V_t^H = \pi(H) + \sum_{k=1}^{t} \varphi_k^H (X_k - X_{k-1}),$$

it follows that V^H is the value of the replicating strategy φ^H, and $V_T^H = \mathbb{E}^*(H|\mathcal{F}_T) = H$.

For instance, we can compute $V_1^H(\{u,.\})$ as follows:

$$V_1^H(\{u,.\}) = V_1^H(\{u,u\}) = V_1^H(\{u,d\}) = \frac{4}{9} * 306 + \frac{5}{9} * 81 = 181.$$

Following the same procedure,

$$V_1^H(\{d,.\}) = V_1^H(\{d,u\}) = V_1^H(\{d,d\}) = \frac{4}{9} * 81 - \frac{5}{9} * 63 = 1.$$

Last step is to compute V_0^H:

$$V_0^H = \left(\frac{4}{9}\right)^2 * 81 + 2 * 81 * \frac{4}{9} * \frac{5}{9} - \left(\frac{5}{9}\right)^2 * 63 = 81.$$

The values of V can be summarized in the following table:

ω	$V_0^H(\omega)$	$V_1^H(\omega)$	$V_2^H(\omega)$	$X_2(\omega)$	$H(\omega) = X_2(\omega) - 319$
(u,u)	81	181	306	625	306
(u,d)	81	181	81	400	81
(d,u)	81	1	81	400	81
(d,d)	81	1	-63	256	-63

The hedging strategy φ^H is given by

$$\varphi_t^H = \frac{V_t^H - V_{t-1}^H}{X_t - X_{t-1}}.$$

The problem only asks about φ_1^H, thus,

$$\varphi_1^H = \frac{V_1^H - V_0^H}{X_1 - X_0} = \frac{181 - 81}{500 - 400} = 1.$$

The solution is $\varphi_1^H = 1$.

3.3 Additional Resources

Follmer, H. and Schleid, A. 2002. *Stochastic Finance: An Introduction in Discrete Time*.

Shreve, S. 2004. *Stochastic Calculus for Finance I: The Binomial Model.*

Shiryaev, A. 1999. *Essentials of Stochastic Finance.*

BK. *Risk Neutral Valuation*.