1 Polynomials

Definition 1.0.1. A Polynomial $P(x)$ is an one variable expression or function of the form

$$P(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \cdots, a_n$ are constants and $n \in \mathbb{N}$. The constants a_i are called the *coefficients* of the polynomial. We will denote $A[x]$ as the set of all polynomials with $a_i \in A$. If $n \neq 0$ then n is called the *degree* of the polynomial $P(x)$ and write $\deg P(x) = n$. If $a_n = 1$ then we say that the polynomial is *monic*.
r is called a *root* of the polynomial $P(x)$ if and only if $P(r) = 0$.

§1.1 Division Algorithm

Theorem 1.1.1 (The Division Algorithm)
Given two polynomial $A(x)$ and $B(x)$ there exists unique polynomials $Q(x)$ and $R(x)$ with $\deg R(x) < \deg B(x)$ such that,

$$A(x) = Q(x)B(x) + R(x)$$

The polynomials $Q(x)$ and $R(x)$ are known as the *quotient* and the *remainder*, respectively. If the remainder $R(x) = 0$ then we say that $B(x)$ divides $A(x)$ and write $B(x) \mid A(x)$.

For example, if $B(x) = x^2 - x + 1$ and $A(x) = x^5 + x^3 + 2x$ then,

$$x^5 + x^3 + 2x = \left(x^3 + x^2 + x\right)\left(x^2 - x + 1\right) + x$$

In this example, the remainder $R(x) = x$ and the quotient $Q(x) = x^3 + x^2 + x$.
Suppose $B(x)$ is a polynomial of degree $n \geq 1$ and let $A(x) = x - z$ be linear polynomial. Now from Theorem 1.1.1 we know that there exists polynomials $Q(x)$ and $R(x)$ with $\deg R(x) < 1$ such that,

$$B(x) = A(x)Q(x) + R(x)$$

Since $0 \leq \deg R(x) < 1$, $R(x)$ must be a constant polynomial, we can assume $R(x) = r$ where $r \in \mathbb{R}$. Therefore,

$$\begin{aligned} B(x) &= A(x)Q(x) + R(x) \\ &= (x - z)Q(x) + r \end{aligned}$$

Now if $r = 0$ then,

$$B(x) = (x - z)Q(x) \implies B(z) = 0$$

Now if $B(z) = 0$ that is if z is a root of the polynomial $B(x)$ then,

$$B(z) = (z - z)\, Q(x) + r \implies B(z) = r \implies r = 0$$

Therefore we have proved the following theorem.

Theorem 1.1.2 (Factor Theorem)
The real number z will be a root of the polynomial $P(x)$ if and only if $P(x)$ is divisible by $x - z$.

Corollary 1.1.2.1
The number $-\frac{b}{a}$ where $a, b \in \mathbb{R}$ will be a root of the polynomial $P(x)$ if and only if the polynomial $P(x)$ is divisible by $ax + b$.

If $P(x)$ has the root z then the Factor Theorem guarantees that there exists a polynomial $Q_0(x)$ such that,

$$P(x) = (x - z)\, Q_0(x)$$

Now if,

$$P(x) = (x - z)^m\, Q(x)$$

then we say that z is root of $P(x)$ of *multiplicity* m.

§1.2 The Fundamental Theorem of Algebra

Theorem 1.2.1 (The Fundamental Theorem of Algebra)
The Fundamental Theorem of Algebra states that, every polynomial $P(x)$ in $\mathbb{C}[x]$ has at least one root in $\mathbb{C}$

Corollary 1.2.1.1
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n then,

$$P(x) = k(x - z_1)(x - z_2)\cdots(x - z_n)$$

where, $k = a_n$ and $z_i \in \mathbb{C}$. The numbers $z_1, z_2 \cdots z_n$ are not necessarily distinct.

§1.3 Roots of Cubic Polynomials

Finding the roots of a cubic polynomial is quite hard. So, we are going to first try to solve the cubic polynomial,

$$f(x) = x^3 + px + q$$

where $p, q \in \mathbb{R}$. Setting $p = -3ab$ and $q = a^3 + b^3$ we get,

$$f(x) = x^3 + a^3 + b^3 - 3abx$$

Now using the formula,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

we get,

$$f(x) = (x + a + b)(x^2 - (a + b)x + a^2 + b^2 - ab)$$

Therefore the 3 roots of f are,

$$\begin{aligned}
x_1 &= -a-b \\
x_2, x_3 &= \frac{a+b \pm \sqrt{a^2+b^2+2ab-4a^2-4b^2+4ab}}{2} \\
&= \frac{a+b \pm \sqrt{-3a^2-3b^2+6ab}}{2} \\
&= \frac{a+b \pm \sqrt{-3(a-b)^2}}{2} \\
&= \frac{a+b \pm \sqrt{3}i(a-b)}{2} \\
&= \frac{(1+\sqrt{3}i)a \pm (1-\sqrt{3}i)b}{2} \\
&= \frac{1+\sqrt{3}i}{2}a \pm \frac{1-\sqrt{3}i}{2}b
\end{aligned}$$

Now we have express the root in terms of p, q that is, we have to express a, b in terms of p, q. Now,

$$\begin{aligned}
q &= a^3+b^3 \\
p &= -3ab \\
\implies a^3b^3 &= -\frac{p^3}{27}
\end{aligned}$$

Let $u = a^3$ and $v = b^3$. Notice that u and v are the roots of the quadratic polynomial,

$$P(x) = x^2-(u+v)x+uv = x^2-q^3x-\frac{p^3}{27}$$

Using the quadratic equation we get,

$$\begin{aligned}
u, v &= \frac{q^3 \pm \sqrt{q^6+\frac{4}{27}p^3}}{2} \\
a, b &= \sqrt[3]{\frac{q^3}{2} \pm \frac{\sqrt{q^6+\frac{4}{27}p^3}}{2}} \\
&= \sqrt[3]{\frac{q^3}{2} \pm \sqrt{\frac{q^6}{4}+\frac{p^3}{27}}}
\end{aligned}$$

So let's now try to solve the cubic equation using the results we've got so far,

$$f(x) = x^3-x^2-2x+1$$

First we have to use substitution to transform the polynomial into another polynomial of the form,

$$x^3+px+q$$

Since every polynomial can be uniquely defined by its coefficients, we can associate or express a polynomial of degree n by a unique point in $n+1$ dimensional space. For example we can express the polynomial,

$$(1)\cdot x^3+(-1)\cdot x^2+(-2)\cdot x+(1)$$

as,

$$(1)\cdot x^3+(-1)\cdot x^2+(-2)\cdot x+(1)\to(1,-1,-2,1)$$

Likewise, the point $(5,-1,0,1)$ can be used to represent the polynomial,

$$(5,-1,0,1)\to 5x^3-x^2+1$$

Notice that,

$$(x_3,x_2,\cdots,x_0)+(y_3,y_2,\cdots,y_0)=(x_3+y_3,x_2+y_2,\cdots,x_0+y_0)$$

Say that there exists a polynomial $u(x)=(x+n)$ where $n\in\mathbb{C}$ such that,

$$x^3-x^2-2x+1=u(x)^3+pu(x)+q$$

This equation can also be represented as,

$$(1,-1,-2,1)=(1,3n,3n^2,n^3)+(0,0,p,pn)+(0,0,0,q)\implies(1,-1,-2,1)=(1,3n,3n^2+p,n^3+pn+q)$$

This gives us the system of equation,

$$\begin{aligned}3n&=-1\\3n^2+p&=-2\\n^3+pn+q&=1\end{aligned}$$

Therefore,

$$\begin{aligned}n&=-\frac{1}{3}\\p&=-\frac{7}{3}\\q&=\frac{7}{27}\end{aligned}$$

Thus,

$$x^3-x^2-2x+1=\left(x-\frac{1}{3}\right)^3-\frac{7}{3}\left(x-\frac{1}{3}\right)+\frac{7}{27}\implies f(x)=g\left(x-\frac{1}{3}\right)$$

where

$$g(x)=x^3-\frac{7}{3}x+\frac{7}{27}$$

Now we can use the results we've proved earlier to solve the polynomial $g(x)$. After that we add $\frac{1}{3}$ to the 3 roots of $g(x)$. The 3 numbers we will get by adding $\frac{1}{3}$ are the 3 roots of $f(x)$.

§1.4 Lagrange Interpolation

Theorem 1.4.1 (Lagrange Interpolation)

Let $\alpha_0,\alpha_1,\cdots,\alpha_n$ be distinct real numbers and $\beta_0,\beta_1,\cdots,\beta_n$ be another set of $n+1$ real numbers. Then there exists a unique polynomial,

$$P(x)=\sum_{i=0}^{n}\left(\prod_{\substack{j=0\\j\neq i}}^{n}\frac{x-\alpha_j}{\alpha_i-\alpha_j}\right)\beta_i$$

with $\deg P(x)\le n$ such that $P(\alpha_k)=\beta_k$ for all $0\le k\le n$.

Proof: Let,

$$D_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - \alpha_j}{\alpha_k - \alpha_j} = \frac{(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{k-1})(x-\alpha_{k+1})\cdots(x-\alpha_n)}{(\alpha_k-\alpha_0)(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n)}$$

If $x = \alpha_k$ then $D_k(x) = 1$ else if $x = \alpha_i$ where $i \neq k$ then $D_k(x) = 0$. Thus the polynomial,

$$P(x) = \sum_{k=0}^{n} D_k(x)\beta_k$$

will be equal to β_k for all $x = \alpha_k$. It is also clear that the polynomial $P(x)$ has degree at most n since $\deg D_k(x) = n$ for all $0 \leq k \leq n$.
Now suppose that there exists two polynomials $P_1(x)$ and $P_2(x)$, with degree at most n, such that,

$$P_1(\alpha_k) = P_2(\alpha_k) = \beta_k,\ 0 \leq k \leq n$$

Therefore the polynomial $Q(x) = P_1(x) - P_2(x)$ has $n+1$ distinct roots. But that is impossible since we know that $\deg Q(x) \leq n$ and a polynomial of degree n has at most n distinct roots. This proves that the polynomial $P(x)$ must be unique, that is, $P(x)$ is the only polynomial, with degree at most n, such that, $P(\alpha_k) = \beta_k$ for all $0 \leq k \leq n$

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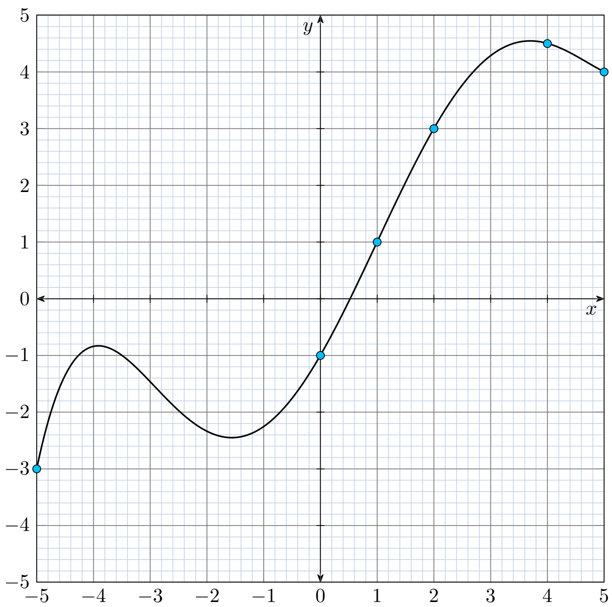

Figure 1.1: Plot of a Lagrange Polynomial

Figure 1.1 shows the Lagrange polynomial going through the points,

$$\{(1,1),(2,3),(0,-1),(5,4),(-5,-3),(4,4.5)\}$$

We can easily compute Lagrange polynomials in python using `sympy`.

```
>>> import sympy
>>> x = sympy.symbols('x')
>>> points = [(1,1), (2,3), (0, -1), (5, 4), (-5, -3), (4, 4.5)]
>>> expr = sympy.interpolate(points, x)
>>> print(expr)
0.00281746031746032*x**5 - 0.0129761904761905*x**4 -
↪   0.111944444444445*x**3 + 0.384404761904762*x**2 +
↪   1.73769841269841*x - 1
```

Problem 1.4.1

Let $P(x)$ be a polynomial of degree n such that, $P(k) = 2^k$ for all $0 \le k \le n$. Find $P(n+1)$.

Solution: From Theorem 1.4.1 we have,

$$P(x) = \sum_{k=0}^{n} 2^k D_k(x)$$

where,

$$\begin{aligned} D_k(x) &= \frac{x(x-1)\cdots(x-k+1)(x-k-1)(x-k-2)\cdots(x-n+1)(x-n)}{(k)(k-1)\cdots(1)(-1)(-2)\cdots(k-n+1)(k-n)} \\ &= (-1)^{n-k}\frac{x(x-1)\cdots(x-k+1)(x-k-1)(x-k-2)\cdots(x-n+1)(x-n)}{k!(n-k)!} \end{aligned}$$

Therefore,

$$\begin{aligned} P(n+1) &= \sum_{k=0}^{n}(-1)^{n-k}2^k\frac{(n+1)n(n-1)\cdots(n-k+2)(n-k)(n-k-1)\cdots 1}{k!(n-k)!} \\ &= \sum_{k=0}^{n}(-1)^{n-k}2^k\frac{(n+1)!}{k!(n-k)!(n-k+1)} \\ &= \sum_{k=0}^{n}(-1)^{n-k}2^k\binom{n+1}{k} \\ &= (-1)\left(\sum_{k=0}^{n+1}\binom{n+1}{k}2^k(-1)^{n-k+1}\right) + 2^{n+1} \\ &= (-1)\left(2-1\right)^{n+1} + 2^{n+1} \\ &= 2^{n+1} - 1 \end{aligned}$$

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