# EGMO:Chpater 1

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**Problem 0.1** (JMO 2011/5)

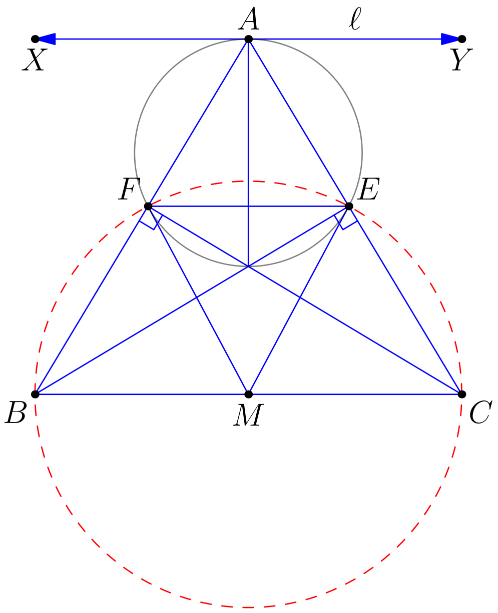
Points $A, B, C, D, E$ lie on a circle $\omega$ and point $P$ lies outside the circle. The given points are such that

- lines $PB$ and $PD$ are tangent to $\omega$.
- $P, A, C$ are collinear.
- $DE \parallel AC$.

Show that $BE$ bisects $AC$.

**Problem 0.2** (Lemma 1.44 Three Tangents)

Let $ABC$ be an acute angled triangle. Let $BE$ and $CF$ be the altitudes of $\triangle ABC$ and let $M$ be the midpoint of $BC$. Prove that $ME$, $MF$ and the line through $A$ parallel to $BC$ are all tangents to the circle $AEF$.




**Proof:** Since $\triangle BEC$ is a right triangle and $M$ is the midpoint of the hypotenuse $BC$, we have $MB = ME = MC$. Likewise $MB = MF = MC$. Therefore

$$MB = ME = MF = MC$$

Hence $B, E, F, C$ are all lying on the same circle with center $M$. If we can show that $\angle FAE = \angle FEM$ then it'll be shown that $EM$ is tangent to circle $AFE$.

$$\angle FEM = \angle FEB + \angle MEB$$

Now since $ME = MB$,

$$\begin{aligned} &\angle FEM = \angle FEB + \angle MEB \\ \implies &\angle FEM = \angle FEB + \angle MBE \end{aligned}$$

Since $CBEF$ is cyclic, $\angle FEB = \angle BCF$.

$$\begin{aligned} &\angle FEM = \angle FEB + \angle MBE \\ \implies &\angle FEM = \angle BCF + \angle MBE \\ \implies &\angle FEM = 180^\circ - \angle BHC \\ \implies &\angle FEM = 180^\circ - \angle FHE \\ \implies &\angle FEM = \angle FAE \end{aligned}$$

Therefore $EM$ is tangent to circle $AFE$. Likewise $FM$ is also tangent to $AFE$. Now we need to show that $\ell$ is tangent to circle $AFE$. It suffices to show that $\angle XAB = \angle AEF$.

$$\begin{aligned} &\angle AEF = \angle EFC + \angle ECF \\ \implies &\angle AEF = 180^\circ - \angle FEC \\ \implies &\angle AEF = \angle CBF \\ \implies &\angle AEF = \angle XAB \end{aligned}$$

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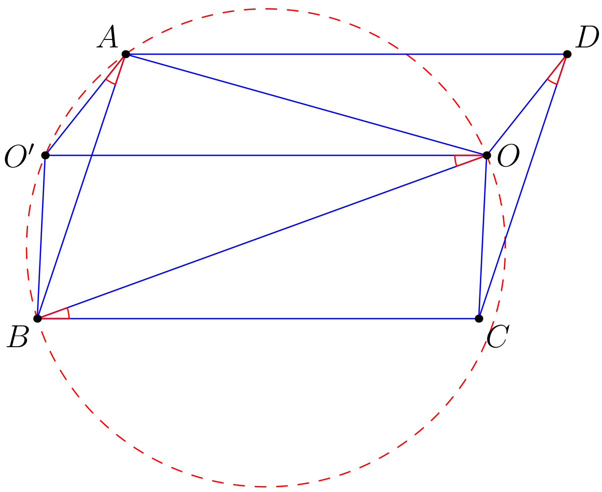
**Problem 0.3** (Canada 1997/4)
The point $O$ is inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Show that $\angle OBC = \angle ODC$.




**Proof:** Apply the translation $T\colon \vec{P} \to \vec{P} + \overrightarrow{CB}$ on points $C, O, D$. Let $T(O) = O'$. Clearly $A, B$ are the points $D, C$ after applying the translation $T$.

$$\angle COD + \angle AOB = 180^\circ \implies \angle BO'A + \angle AOB = 180^\circ$$

Therefore $AO'BO$ is cyclic. Hence $\angle O'AB = \angle O'OB$. As $OO' \parallel BC$,

$$\angle O'AB = \angle O'OB = \angle OBC \implies \angle ODC = \angle OBC$$

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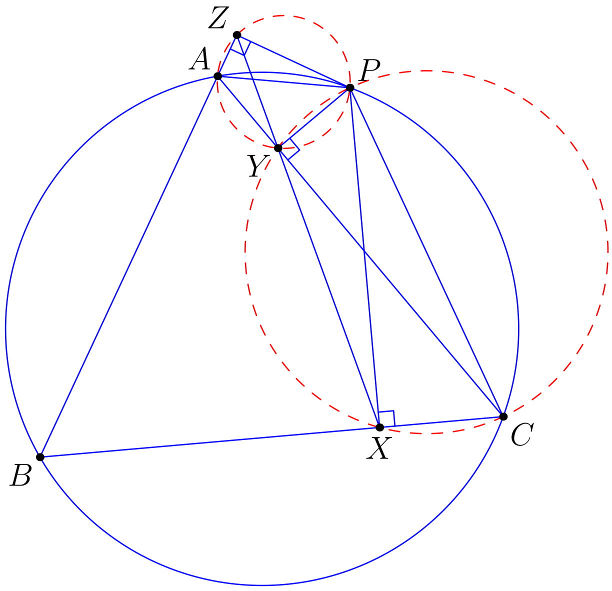
**Problem 0.4** (Simson line)

Let $ABC$ be a triangle and let $P$ be any point on the circumcircle of $\triangle ABC$. Let $X, Y, Z$ be the feet of the perpendiculars from $P$ onto lines $BC, CA, AB$. Prove that $X, Y, Z$ are collinear.




**Proof:** It suffices to show that $\angle XYC = \angle AYZ$. Since $XYPC$ is cyclic, $\angle XYC = \angle XPC$.

$$\angle XYC = \angle XPC = 90^\circ - \angle PCX$$

Since $ABCP$ is cyclic, $\angle PCX = 180^\circ - \angle PAB$. Therefore

$$\begin{aligned} \angle XYC &= 90^\circ - \angle PCX \\ &= \angle PAB - 90^\circ \\ &= \angle ZPA \end{aligned}$$

Since $AZPY$ is cyclic, $\angle ZPA = \angle AYZ$. Hence

$$\angle XYZ = \angle AYZ$$

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