# Mathematical Induction

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## Contents

1 Preliminaries 1

2 The Induction Principle 2

3 Variants of the Induction Principle 8

## §1 Preliminaries

**Definition 1.1.** A **proposition** in mathematics is a statement that is either true or false.

For example, "2 + 2 = 4" and "19 is a prime number" both are true mathematical statements. Here are some more examples of propositions.

**Proposition 1.2**
If $f(n) = n^2 + n + 41$ then $f(n)$ is a prime number for all non-negative integers $n$.

This is a proposition but the proposition is not true for all non-negative integers. For example, if $n = 40$ then

$$f(40) = 40^2 + 40 + 41 = 40^2 + 2 \times 40 + 1 = 41^2$$

**Proposition 1.3** (Goldbach's Conjecture)
Every integer greater than 2 is a sum of two primes.

Goldbach's Conjecture is also a proposition but so far no one has been able to prove that it is true.

**Definition 1.4.** A **predicate** is a proposition whose truth depends on one or more variables.

For example, "$n$ is a prime number" is a predicate as its truth depends on the value of $n$. For $n = 3$ the statement is true but for $n = 12$ the statement is false. A function-like notation is used to denote a predicate supplied with specific variable values. For example, we might use the name "P" for the predicate above:

$$P(n) : n \text{ is a prime number}$$

Like before, we can say that $P(3)$ is true and $P(12)$ is false.

**Definition 1.5.** An **axiom** is a proposition which is accepted as true without any proof.

For example, "$a = b \iff a + c = b + c$" and "two sets are equal if and only if they have the same elements" are examples of axioms.

## §2 The Induction Principle

*Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the basis) and that from each rung we can climb up to the next one (the step).*

- Concrete Mathematics

The **induction principle** claims that if $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0)$ is true where $n_0$ is some integer and
- $\mathcal{P}(k) \implies \mathcal{P}(k+1)$ where $k \geq n_0$ is an integer

then $\mathcal{P}(n)$ is true for all integers $n \geq n_0$. We will later prove the induction principle but first let's take a look at some examples.

**Example 2.1**

Show that for all $n \geq 1$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

**Proof:** We have the predicate

$$P(n)\colon 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

We want show that $P(n)$ is true for all $n \geq 1$. Clearly $P(1)$ is true. Now we just need to show that $P(k) \implies P(k+1)$. Suppose $P(k)$ is true where $k \geq 1$. Now

$$\begin{aligned} 1 + 2 + \cdots + k &= \frac{k(k+1)}{2} \\ \implies 1 + 2 + \cdots + k + (k+1) &= (k+1) + \frac{k(k+1)}{2} \\ \implies 1 + 2 + \cdots + (k+1) &= (k+1)\left(1 + \frac{k}{2}\right) \\ \implies 1 + 2 + \cdots + (k+1) &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

And that's it! We just proved that if $P(k)$ is true then $P(k+1)$ is also true. Now from the induction principle, we can say that $P(n)$ is true for all $n \geq 1$.

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There are two main steps in an inductive proof. First we show that $\mathcal{P}(n_0)$ is true where $n_0$ is an integer. This step is known as the **base step** or the **induction basis**. Next we prove that if $k \geq n_0$ is an integer and $\mathcal{P}(k)$ is true then $\mathcal{P}(k+1)$ is also true. This step is called the **inductive step**. To prove the inductive step we assume that $\mathcal{P}(k)$ is true and then use this assumption to show that $\mathcal{P}(k+1)$ must also be true. The hypothesis that $\mathcal{P}(k)$ is true for some integer $k \geq n_0$ is called the **induction hypothesis**. Here's another example of proof by induction.

**Example 2.2**
Show that for all $n \in \mathbb{N}$

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

**Proof:** Earlier we proved that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Therefore it suffices to show that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$. For $n = 1$ the statement is clearly true. We can prove this just by plugging in $n = 1$ and then showing that both sides are equal.

$$1^3 = \frac{1^2(1+1)^2}{4} \implies 1 = 1$$

Now we need to show that if the statement holds true for some integer $k \geq 1$ then it must also hold true for $k + 1$. Suppose the statement is true for $n = k$ for some integer $k \geq 1$.

$$\begin{aligned}
&1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4} \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

Now from the induction principle, we can say that the statement is true for all $n \geq 1$.

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Okay, enough with examples. We will now prove the induction principle!

**Theorem 2.3** (Induction Principle)
If $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0)$ is true where $n_0$ is some integer and
- $\mathcal{P}(k) \implies \mathcal{P}(k+1)$ where $k \geq n_0$ is an integer

then $\mathcal{P}(n)$ is true for all integers $n \geq n_0$.

**Proof:** Let $Q(n)$ be the predicate $\mathcal{P}(n_0 + n)$.

$$Q(n) : \mathcal{P}(n_0 + n)$$

We need to show that $Q(n)$ is true for all $n \in \mathbb{N}$ if

- $Q(0)$ is true and
- $Q(n) \implies Q(n+1)$ for all $n \in \mathbb{N}$.

Let $T$ be the set of all non-negative integers for which $Q(n)$ is true and let $F$ be the set of all non-negative integers for which $Q(n)$ is false. It suffices to show that $F$ is an empty set.
For the sake of contradiction, let us assume that $F$ is non-empty. Since $F$ is a non-empty set of non-negative integers, there must exist a minimal element of $F$. Let $k$ be the smallest element of $F$. Since $k > 0 \implies k-1 \geq 0$ and $k-1 \notin F$, $k-1$ must be an element of $T$. Since $Q(k-1) \implies Q(k)$, $k$ must also be an element of $T$. But that contradicts our assumption that $k \in F$ as $Q(k)$ cannot be both true and false. Therefore $F$ does not have a minimal element which implies $F$ must be an empty set.

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**Exercise 2.4.** Prove using induction that for all $n \in \mathbb{N}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**Exercise 2.5.** Prove using induction that if $r$ is a real number not equal to 1 then for all $n \in \mathbb{N}$

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

**Exercise 2.6.** Show that if $x, y$ are real numbers and if $n \geq 2$ is a positive integer then

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)$$

**Example 2.7**

Show that for all $n \in \mathbb{N}$

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

**Proof:** For $n = 1$ the statement is true.

$$\frac{1}{1 \times 2} = \frac{1}{1+1} \implies \frac{1}{2} = \frac{1}{2}$$

Assume the statement is true for some integer $k \geq 1$. Now we prove that the statement must also be true for $k+1$.

$$\begin{aligned}
&\frac{1}{1 \times 2} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1} \\
\implies &\frac{1}{1 \times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
\implies &\frac{1}{1 \times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} \\
\implies &\frac{1}{1 \times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\
\implies &\frac{1}{1 \times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}
\end{aligned}$$

Therefore by the induction principle the statement must be true for all $n \in \mathbb{N}$.

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**Exercise 2.8.** Show that for all $n \in \mathbb{N}$

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n^2+3n}{4(n+1)(n+2)}$$

**Exercise 2.9.** Show that for all $n \in \mathbb{N}$

$$1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$$

**Example 2.10** (BDMO)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and for any $x \in \mathbb{R}$, $f(x+7) \geq f(x) + 7$ and $f(x+1) \leq f(x) + 1$. Find the value of $f(2013)$.

**Solution:** Notice that

$$f(x+2) \leq f(x+1) + 1 \leq f(x) + 2,$$
$$f(x+3) \leq f(x+2) + 1 \leq f(x) + 3$$

We can generalize and say that if $n$ is a non-negative integer then $f(x+n) \leq f(x) + n$. But how do we prove this? Let's try using induction!
The question already tells us that the statement is true for $n = 1$. We just need to show that

$$f(x+k) \leq f(x) + k \implies f(x+k+1) \leq f(x) + k + 1$$

This is quite trivial.

$$f(x+k+1) \leq f(x+k) + 1 \leq f(x) + k + 1$$

And so we just proved that $f(x+n) \leq f(x) + n$. Now setting $n = 7$, we get

$$f(x+7) \leq f(x) + 7$$

Therefore since $f(x+7) \geq f(x)+7$ and $f(x+7) \leq f(x)+7$, we must have $f(x+7) = f(x)+7$. That's great! But now what? Setting $x = 1$, we get $f(8) = f(1) + 7 = 8$. But how can we find the value of $f(2013)$? We can guess that $f(x) = x$ for all $x$. Can we prove this? If we can somehow show that $f(x) + 1 = f(x+1)$ for all $x$ then we'll be able to easily show that $f(n) = n$ for all non-negative integer $n$. So let's try to prove that $f(x) + 1 = f(x+1)$ for all $x$.
Suppose for the sake of contradiction that there exists some real number $r$ such that $f(r) + 1 \neq f(r+1)$. Therefore $f(r+1)$ must be less than $f(r) + 1$. Now suppose $r = x + 6$ where $x$ is a real number.

$$\begin{aligned} f(r+1) < f(r) + 1 &\implies f(x+7) < f(x+6) + 1 \\ &\implies f(x+7) < f(x) + 7 \end{aligned}$$

But that is impossible as we've shown that $f(x+7) = f(x) + 7$ for all $x \in \mathbb{R}$. Hence such a real number $r$ cannot exist which implies $f(x) + 1 = f(x+1)$ for all $x \in \mathbb{R}$.
We can now use induction to show that $f(n) = n$ for all non-negative integer $n$. For $n = 1$ the statement is true. We need to prove that if $f(k) = k$ then $f(k+1) = k+1$.

$$f(k+1) = f(k) + 1 \implies f(k+1) = k + 1$$

And we are done! YAY! We not only found the value of $f(2013)$ but also found the value of $f(n)$ for all non-negative integer $n$. Awesome, right?

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**Example 2.11**

If $n$ is a non-negative integer and $x, y$ are two real numbers then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Proof:** The statement is evidently true for $n = 1$. Now we need to show that

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \implies (x+y)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m-k+1}$$

Okay, let's try to prove it!

$$\begin{aligned}(x+y)^m \times (x+y) &= (x+y)^m x + (x+y)^m y \\ &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1}\end{aligned}$$

Now

$$\begin{aligned}\sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} &= x^{m+1} + \sum_{k=0}^{m-1} \binom{m}{k} x^{k+1} y^{m-k} \\ &= x^{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} x^k y^{m-k+1} \\ \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} &= y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} x^k y^{m-k+1}\end{aligned}$$

Therefore

$$\begin{aligned}(x+y)^{m+1} &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\ &= x^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} x^k y^{m-k+1} + y^{m+1}\end{aligned}$$

Now

$$\begin{aligned}\binom{m}{k} + \binom{m}{k-1} &= \frac{m!}{(m-k)!k!} + \frac{m!}{(m-k+1)!(k-1)!} \\ &= \frac{m!}{(m-k)!(k-1)!} \left( \frac{1}{k} + \frac{1}{m-k+1} \right) \\ &= \frac{m!}{(m-k)!(k-1)!} \left( \frac{m+1}{k(m-k+1)} \right) \\ &= \frac{(m+1)!}{(m-k+1)!k!} \\ &= \binom{m+1}{k}\end{aligned}$$

Thus we have

$$\begin{aligned}(x+y)^{m+1} &= x^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} x^k y^{m-k+1} + y^{m+1} \\ &= x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^k y^{m-k+1} + y^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m+1-k}\end{aligned}$$

Voila! We just proved the binomial theorem using induction!

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**Example 2.12**

Show that if $H_n$ is the $n$-th Harmonic Number, where $n \in \mathbb{N}$, then

$$1 + \frac{\lfloor \log_2 n \rfloor}{2} \leq H_n$$

**Proof:** The problem looks quite complicated! So let's try proving a more simplified version of the problem first. We will show that if $n$ is a positive integer then

$$1 + \frac{n}{2} \leq H_{2^n}$$

This looks simpler than the original problem because it does not contain the ugly floor function. But how do we prove this? We are going to use induction on $n$.
The base case, $n = 1$, is true. We just have to show that

$$1 + \frac{k}{2} \leq H_{2^k} \implies 1 + \frac{k+1}{2} \leq H_{2^{k+1}}$$

Notice that

$$H_{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}$$

Now since

$$2^{k+1} \geq 2^{k+1} - 1 \geq \cdots \geq 2^k + 2 \geq 2^k + 1$$

we have

$$\frac{1}{2^{k+1}} \leq \frac{1}{2^{k+1}-1} \leq \cdots \leq \frac{1}{2^k+2} \leq \frac{1}{2^k+1}$$

Adding the inequalities we get

$$\begin{aligned}\frac{2^k}{2^{k+1}} &\leq \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^k+2} + \frac{1}{2^k+1} \\ \implies \frac{1}{2} &\leq \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^k+2} + \frac{1}{2^k+1}\end{aligned}$$

Therefore

$$H_{2^k} + \frac{1}{2} \leq H_{2^{k+1}} \implies 1 + \frac{k+1}{2} \leq H_{2^{k+1}}$$

And we are done! No, not really. We still have to solve the original problem.
Suppose that $\lfloor \log_2 n \rfloor = k$. Now

$$\begin{aligned} 2^k \leq n &\implies H_{2^k} \leq H_n \\ &\implies 1 + \frac{k}{2} \leq H_n \\ &\implies 1 + \frac{\lfloor \log_2 n \rfloor}{2} \leq H_n \end{aligned}$$

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## §3 Variants of the Induction Principle

**Theorem 3.1**

If $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0), \mathcal{P}(n_0+1), \cdots, \mathcal{P}(n_0+m)$ are all true and
- $\mathcal{P}(k) \land \mathcal{P}(k+1) \land \cdots \land \mathcal{P}(k+m)$ implies $\mathcal{P}(k+m+1)$

then $\mathcal{P}(n)$ is true for all integers $n \geq n_0$.

**Proof:** Let $Q(n)$ be the predicate

$$Q(n) : \mathcal{P}(n) \land \mathcal{P}(n+1) \land \cdots \land \mathcal{P}(n+m)$$

The base case $Q(n_0)$ is true. Assume that $Q(k)$ is true. Now

$$\begin{aligned} &\mathcal{P}(k) \land \mathcal{P}(k+1) \land \cdots \land \mathcal{P}(k+m) \to \mathcal{P}(k+m+1) \\ \implies &\mathcal{P}(k) \land \mathcal{P}(k+1) \land \cdots \land \mathcal{P}(k+m) \to \mathcal{P}(k+1) \land \cdots \land \mathcal{P}(k+m) \land \mathcal{P}(k+1+m) \\ \implies &Q(k) \to Q(k+1) \end{aligned}$$

Therefore by the induction principle $Q(n)$ must be true for all $n \geq n_0$. If $Q(n)$ is true for all $n \geq n_0$ then $\mathcal{P}(n)$ must also be true for all $n \geq n_0$.

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**Example 3.2**

Find the general term of the sequence defined by $x_0 = 3, x_1 = 4$ and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all $n \geq 1$.

**Proof:** The first few values of the sequence are

$$3, 4, 5, 6, 7, \cdots$$

We can guess that $x_n = n + 3$. We will use induction on $n$ to show that $x_n = n + 3$ for all $n \geq 0$. Notice that normal induction won't work here because we need the values of both $x_k$ and $x_{k-1}$ to find the value of $x_{k+1}$. But don't worry, we can use Theorem 3.1.
Let $P(n)$ be the predicate

$$P(n) : x_n = n + 3$$

Since $x_0 = 3, x_1 = 4$, $P(0)$ and $P(1)$ are both true. We need to show that if $k \geq 0$ is an integer then

$$P(k) \wedge P(k+1) \implies P(k+2)$$

Assume that $P(k)$ and $P(k+1)$ are both true. Now

$$\begin{aligned} x_{k+2} &= x_k^2 - (k+1)x_{k+1} \\ \implies x_{k+2} &= (k+3)^2 - (k+1)(k+4)^2 \\ \implies x_{k+2} &= k^2 + 6k + 9 - k^2 - 5k - 4 \\ \implies x_{k+2} &= k + 5 = (k+2) + 3 \end{aligned}$$

Therefore by Theorem 3.1, we can say that $P(n)$ is true for all $n \geq 0$. Hence $x_n = n + 3$ for all $n \geq 0$.

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**Example 3.3** (Binet's Formula)

Let $F_n$ be the $n$-th Fibonacci number. Show that

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the two real roots of the quadratic equation $x^2 - x - 1$.

**Proof:** We will use induction on $n$. The base cases $n = 0$ and $n = 1$ are both true.

$$\begin{aligned} F_0 &= \frac{\phi^0 - \psi^0}{\phi - \psi} \implies 0 = 0 \\ F_1 &= \frac{\phi^1 - \psi^1}{\phi - \psi} \implies 1 = 1 \end{aligned}$$

Assume the formula works for $k$ and $k+1$ where $k \geq 0$ is some integer. Now

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ &= \frac{\phi^{k+1} - \psi^{k+1}}{\phi - \psi} + \frac{\phi^k - \psi^k}{\phi - \psi} \\ &= \frac{\left(\phi^{k+1} + \phi^k\right) - \left(\psi^{k+1} + \psi^k\right)}{\phi - \psi} \\ &= \frac{\phi^k\left(\phi + 1\right) - \psi^k(\psi + 1)}{\phi - \psi} \\ &= \frac{\phi^{k+2} - \psi^{k+2}}{\phi - \psi} \end{aligned}$$

Therefore the formula works for all integers $n \geq 0$.

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