# Inequalities

Munir Uz Zaman

Date: May 9, 2022

# Contents

# 1 AM-GM Inequality

> **Theorem 1.0.1** (AM-GM Inequality)
>
> For all positive real numbers $a_1, a_2, \cdots, a_n$ where $n \in \mathbb{N}$ and $n \geq 2$ the following inequality holds,
> $$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$
> Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

**Proof:** We will prove this theorem using a special type of induction know as *Cauchy Induction*. Here's how we'll prove it, (let $P_n$ be the statement for $n$ numbers.)

- We will first show that $P_2$ is true.

- We will show that $P_n \implies P_{2n}$

- Then we will show that $P_n \implies P_{n-1}$

When these are verified, all the assertions $P_n$ with $n \geq 2$ are shown to be true.

First we need to prove that if $a_1, a_2$ are two positive reals then

$$\frac{a_1 + a_2}{2} \geq \sqrt[2]{a_1 a_2}$$

This can be easily shown from the fact that $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$.

Next we need show that $P_n \implies P_{2n}$. This is also very easy.

$$a_1 + a_2 + \cdots + a_{2n} \geq n\sqrt[n]{a_1 a_2 \cdots a_n} + n\sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}} \geq 2n\sqrt[2n]{a_1 a_2 \cdots a_{2n}}$$

Now we just need to show that $P_n \implies P_{n-1}$. Let $g = \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}$. Now,

$$a_1 + \cdots + a_{n-1} + g \geq n\sqrt[n]{a_1 \cdots a_{n-1} \times g}$$
$$\implies a_1 + \cdots + a_{n-1} + g \geq n\sqrt[n]{g^{n-1}g}$$
$$\implies a_1 + \cdots + a_{n-1} + g \geq ng$$
$$\implies a_1 + \cdots + a_{n-1} \geq (n-1)g$$
$$\implies a_1 + \cdots + a_{n-1} \geq (n-1)\sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}$$

By *Cauchy induction*, the inequality is true for every natural number $n \geq 2$. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

♣

> **Theorem 1.0.2** (Weighted AM-GM Inequality)
>
> If $a_1, a_2, \cdots, a_n$ are positive real numbers with $n \geq 2$ and $x_1, x_2, \cdots, x_n$ are $n$ non-negative real numbers such that $\sum_{i=1}^{n} x_i = 1$ then
> $$a_1 x_1 + \cdots + a_n x_n \geq a_1^{x_1} \cdots a_n^{x_n}$$

**Problem 1.0.3** (BDMO 2019)

Show that if $a, b, c$ are positive real numbers then

$$\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \geq 2\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)$$

**Solution:**

$$
\begin{aligned}
&(a + b - c)^2 \geq 0 \\
\implies\ & a^2 + b^2 + c^2 + 2(ab - bc - ca) \geq 0 \\
\implies\ & a^2 + b^2 + c^2 \geq 2(bc + ca - ab) \\
\implies\ & \frac{a^2 + b^2 + c^2}{abc} \geq 2\left(\frac{bc + ca - ab}{abc}\right) \\
\implies\ & \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \geq 2\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)
\end{aligned}
$$

♣

**Problem 1.0.4**

Show that if $a_1, a_2, \cdots, a_n$ are $n$ positive real numbers such that $a_1 a_2 \cdots a_n = 1$ then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n$$

**Solution:** Using the AM-GM Inequality, we have $(1 + a_i) \geq 2\sqrt{a_i}$ for all $1 \leq i \leq n$. Now multiplying the inequalities for all values of $i$ we get

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n \sqrt{a_1 a_2 \cdots a_n} = 2^n$$

♣

**Problem 1.0.5**

Show that if $x_1, x_2, \cdots, x_n$ are $n$ real numbers then

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$$

**Solution:** Using the AM-GM Inequality, we have

$$
\begin{aligned}
(x_1 + x_2 + \cdots + x_n) &\geq n \sqrt[n]{x_1 x_2 \cdots x_n} \\
\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + +\frac{1}{x_n}\right) &\geq n \sqrt[n]{\frac{1}{x_1 x_2 \cdots x_n}}
\end{aligned}
$$

Multiplying the two inequalities we get

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2$$

♣

---

**Problem 1.0.6** (Russia MO 2004)

Let $a, b, c$ be positive real numbers with sum 3. Show that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

---

**Solution:** We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \implies 2ab + 2bc + 2ca = 9 - (a^2 + b^2 + c^2)$$

The inequality is therefore equivalent to

$$a^2 + b^2 + c^2 + 2\sqrt{a} + 2\sqrt{b} + 2\sqrt{c} \geq 9$$

Now using the AM-GM Inequality we have

$$(a^2 + \sqrt{a} + \sqrt{a}) \geq 3a$$
$$(b^2 + \sqrt{b} + \sqrt{b}) \geq 3b$$
$$(c^2 + \sqrt{c} + \sqrt{c}) \geq 3c$$

Adding the 3 inequalities we get

$$a^2 + b^2 + c^2 + 2\sqrt{a} + 2\sqrt{b} + 2\sqrt{c} \geq 9$$

♣

---

**Problem 1.0.7**

Let $x, y, z$ be three positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

---