Combinatorics

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Date: December 3, 2021

Contents

1 Counting

§1.1 Basics

Theorem 1.1.1 (Addition Principle)

If $E_1, E_2, \cdots, E_k$ are pairwise disjoint events that can happen in $n_1, n_2, \cdots, n_k$ different ways respectively then the number of ways E_1, E_2, $\cdots$ or E_k can happen is $\sum_{j=1}^{k} n_j$.
More formally stated in terms of set theory, if $E_1, E_2, \cdots, E_k$ are pairwise disjoint sets, that is $E_i \cap E_j = \phi$ for all $1 \leq i, j \leq k$ and $i \neq j$, then,

$$\left| \bigcup_{j=1}^{k} E_j \right| = \sum_{j=1}^{k} |E_j|$$

For example, suppose there are 3 ways by air, 2 ways by sea and 4 ways by land to get from city A to B. Then the number of ways we can go to B from A in total is $3 + 2 + 4 = 9$.

Theorem 1.1.2 (Multiplication Principle)

If an event E can be decomposed into k ordered events $E_1, E_2, \cdots, E_k$ which can occur in $n_1, n_2, \cdots, n_k$ ways respectively then the total number of ways the event E can happen is $\prod_{j=1}^{k} n_j$.
More formally stated using set-theoretic terminology, if $E_1, E_2, \cdots, E_k$ are non-empty sets then,

$$\left| \prod_{j=1}^{k} E_k \right| = \prod_{j=1}^{k} |E_k|$$

Problem 1.1.1

Find the number of binary sequences of length n.

Solution: The set of all binary sequences of length n is,

$$\{(a_1, a_2, \cdots, a_n) \mid a_i \in \{0, 1\}, 1 \leq i \leq n\} = \prod_{i=1}^{n} \{0, 1\}$$

Therefore,

$$\left| \prod_{i=1}^{n} \{0, 1\} \right| = \prod_{i=1}^{n} |\{0, 1\}| = \prod_{i=1}^{n} 2 = 2^n$$

Thus the number of binary sequences of length n is 2^n.

♣

Problem 1.1.2

Let $X = \{0, 1, 2, \cdots, 100\}$ and let, $S = \{(a, b, c) \mid a, b, c \in X \text{ and } a < b, c\}$. Find $|S|$.

Solution: The problem may be divided into disjoint cases by considering $a = 1, 2, \cdots, 99$. For $a = k \in \{1, 2, \cdots, 99\}$ the number of choices for b and c are $100 - k$. Thus the number of required ordered triples (k, b, c) is $(100 - k)^2$. Therefore,

$$|S| = \sum_{k=1}^{99} (100 - k)^2 = \sum_{j=1}^{99} j^2$$

Using the formula $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n + 1)(2n + 1)$, we obtain

$$|S| = \sum_{j=1}^{99} j^2 = \frac{99 \times 100 \times 199}{6} = 328350$$

♣

Definition 1.1.1. A sequence of numbers $a_1 a_2 \cdots a_n$ is called a *k-ary sequence*, where $n, k \in \mathbb{N}$ and $a_i \in \{0, 1, 2, \cdots, k - 1\}$ for all $1 \leq i \leq n$. n is said to be the *length* of the sequence. If $k = 2$ then we call it a *binary sequence*.

For example the set of all binary sequences of length 3 is $\{000, 001, 011, 111, 101, 110, 010, 100\}$