Inequalities

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1 AM-GM Inequality

Theorem 1.0.1 (AM-GM Inequality)

For all positive real numbers $a_1, a_2, \cdots, a_n$ where $n \in \mathbb{N}$ and $n \geq 2$ the following inequality holds,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

Proof: We will prove this theorem using a special type of induction know as *Cauchy Induction*. Here's how we'll prove it, (let P_n be the statement for n numbers.)

- We will first show that P_2 is true.
- We will show that $P_n \implies P_{2n}$
- Then we will show that $P_n \implies P_{n-1}$

When these are verified, all the assertions P_n with $n \geq 2$ are shown to be true.
First we need to prove that if a_1, a_2 are two positive reals then

$$\frac{a_1 + a_2}{2} \geq \sqrt[2]{a_1 a_2}$$

This can be easily shown from the fact that $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$.
Next we need show that $P_n \implies P_{2n}$. This is also very easy.

$$a_1 + a_2 + \cdots + a_{2n} \geq n\sqrt[n]{a_1 a_2 \cdots a_n} + n\sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}} \geq 2n\sqrt[2n]{a_1 a_2 \cdots a_{2n}}$$

Now we just need to show that $P_n \implies P_{n-1}$. Let $g = \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}$. Now,

$$\begin{aligned}
& a_1 + \cdots + a_{n-1} + g \geq n\sqrt[n]{a_1 \cdots a_{n-1} \times g} \\
\implies & a_1 + \cdots + a_{n-1} + g \geq n\sqrt[n]{g^{n-1} g} \\
\implies & a_1 + \cdots + a_{n-1} + g \geq ng \\
\implies & a_1 + \cdots + a_{n-1} \geq (n-1)g \\
\implies & a_1 + \cdots + a_{n-1} \geq (n-1)\sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}
\end{aligned}$$

By *Cauchy induction*, the inequality is true for every natural number $n \geq 2$. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

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Theorem 1.0.2 (Weighted AM-GM Inequality)

If $a_1, a_2, \cdots, a_n$ are positive real numbers with $n \geq 2$ and $x_1, x_2, \cdots, x_n$ are n non-negative real numbers such that $\sum_{i=1}^{n} x_i = 1$ then

$$a_1 x_1 + \cdots + a_n x_n \geq a_1^{x_1} \cdots a_n^{x_n}$$

Problem 1.0.3 (BDMO 2019)

Show that if a, b, c are positive real numbers then

$$\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab} \geq 2\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$$

Solution:

$$\begin{aligned}
&(a+b-c)^2 \geq 0 \\
\Longrightarrow\ & a^2+b^2+c^2+2(ab-bc-ca) \geq 0 \\
\Longrightarrow\ & a^2+b^2+c^2 \geq 2(bc+ca-ab) \\
\Longrightarrow\ & \frac{a^2+b^2+c^2}{abc} \geq 2\left(\frac{bc+ca-ab}{abc}\right) \\
\Longrightarrow\ & \frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab} \geq 2\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)
\end{aligned}$$

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Problem 1.0.4

Show that if $a_1, a_2, \cdots, a_n$ are n positive real numbers such that $a_1 a_2 \cdots a_n = 1$ then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$$

Solution: Using the AM-GM Inequality, we have $(1+a_i) \geq 2\sqrt{a_i}$ for all $1 \leq i \leq n$. Now multiplying the inequalities for all values of i we get

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n\sqrt{a_1 a_2 \cdots a_n} = 2^n$$

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Problem 1.0.5

Show that if $x_1, x_2, \cdots, x_n$ are n real numbers then

$$(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) \geq n^2$$

Solution: Using the AM-GM Inequality, we have

$$\begin{aligned}
(x_1+x_2+\cdots+x_n) &\geq n\sqrt[n]{x_1 x_2 \cdots x_n} \\
\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots++\frac{1}{x_n}\right) &\geq n\sqrt[n]{\frac{1}{x_1 x_2 \cdots x_n}}
\end{aligned}$$

Multiplying the two inequalities we get

$$(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) \geq n^2$$

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Problem 1.0.6 (Russia MO 2004)
Let a, b, c be positive real numbers with sum 3. Show that

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \geq ab+bc+ca$$

Solution: We know that

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca \implies 2ab+2bc+2ca = 9-(a^2+b^2+c^2)$$

The inequality is therefore equivalent to

$$a^2+b^2+c^2+2\sqrt{a}+2\sqrt{b}+2\sqrt{c} \geq 9$$

Now using the AM-GM Inequality we have

$$(a^2+\sqrt{a}+\sqrt{a}) \geq 3a$$
$$(b^2+\sqrt{b}+\sqrt{b}) \geq 3b$$
$$(c^2+\sqrt{c}+\sqrt{c}) \geq 3c$$

Adding the 3 inequalities we get

$$a^2+b^2+c^2+2\sqrt{a}+2\sqrt{b}+2\sqrt{c} \geq 9$$

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Problem 1.0.7
Let x, y, z be three positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+x)(1+z)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

2 Jensen's Inequality

§2.1 Convex and Concave Functions

Definition 2.1.1. A function f is said to be **convex** in an interval if and only if for all x and y in the interval and for any $0 < t < 1$

$$(1-t)f(x) + tf(y) \geq f((1-t)x + ty)$$

If the function is **concave** then

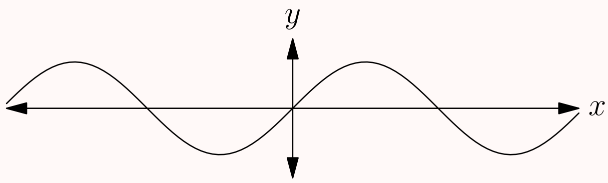
$$(1-t)f(x) + tf(y) \leq f((1-t)x + ty)$$

Theorem 2.1.2

If $f : \mathbb{R} \to \mathbb{R}$ is a function then f is concave if and only if $f''(x) \leq 0$ for all x and similarly f is convex if and only if $f''(x) \geq 0$ for all x.

Example 2.1.3

The function $\sin(x)$ is concave in the interval $[0, \pi]$ and convex in the interval $[\pi, 2\pi]$.



§2.2 Jensen's Inequality

Theorem 2.2.1 (Weak Jensen's Inequality)

If φ is a convex funtion in the interval I and $x_i \in I$ for $i = 1, 2, \cdots, n$ then

$$\varphi\left(\frac{\sum_{i=1}^n x_i}{n}\right) \leq \frac{\sum_{i=1}^n \varphi(x_i)}{n}$$

The reverse inequality holds when φ is concave.

Problem 2.2.2

Show that if ABC is a triangle then

$$\sin(\angle A) + \sin(\angle B) + \sin(\angle C) \leq \frac{3\sqrt{3}}{2}$$

Solution: Since $\sin(x)$ is concave in the interval $[0, \pi]$,

$$\begin{aligned} &\frac{\sin(\angle A) + \sin(\angle B) + \sin(\angle C)}{3} \leq \sin\left(\frac{\angle A + \angle B + \angle C}{3}\right) \\ \Longrightarrow\ &\frac{\sin(\angle A) + \sin(\angle B) + \sin(\angle C)}{3} \leq \sin(60^\circ) \\ \Longrightarrow\ &\sin(\angle A) + \sin(\angle B) + \sin(\angle C) \leq \frac{3\sqrt{3}}{2} \end{aligned}$$

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Problem 2.2.3

Given $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, prove that

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}$$

Solution: We can rewrite the inequality as

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}\,\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{a+b+c}$$

This makes both sides of the inequality homogenous. Hence we can assume $a + b + c = 3$. We can now rewrite the inequality as

$$\sum_{\text{cyc}} \frac{1}{16a} - \frac{1}{(a+3)^2} \geq 0$$

Since the function $f(x) = \frac{1}{16x} - \frac{1}{(x+3)^2}$ is convex in the interval $(0, 3)$,

$$f(a) + f(b) + f(c) \geq 3f$$

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