EGMO:Chpater 1

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Problem 0.1 (JMO 2011/5)

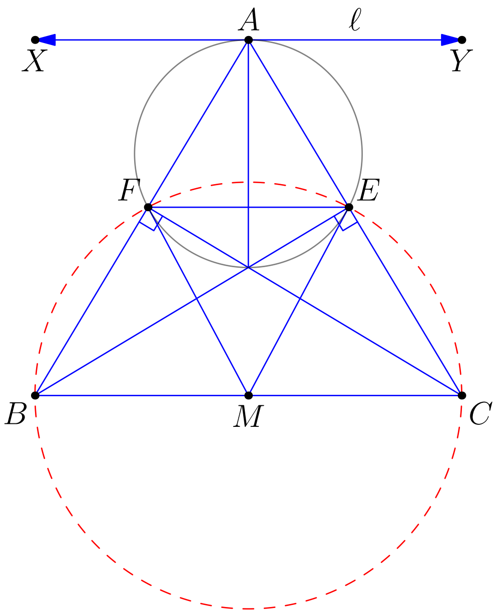
Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that

- lines PB and PD are tangent to ω.
- P, A, C are collinear.
- $DE \parallel AC$.

Show that BE bisects AC.

Problem 0.2 (Lemma 1.44 Three Tangents)

Let ABC be an acute angled triangle. Let BE and CF be the altitudes of $\triangle ABC$ and let M be the midpoint of BC. Prove that ME, MF and the line through A parallel to BC are all tangents to the circle AEF.



Proof: Since $\triangle BEC$ is a right triangle and M is the midpoint of the hypotenuse BC, we have $MB = ME = MC$. Likewise $MB = MF = MC$. Therefore

$$MB = ME = MF = MC$$

Hence B, E, F, C are all lying on the same circle with center M. If we can show that $\angle FAE = \angle FEM$ then it'll be shown that EM is tangent to circle AFE.

$$\angle FEM = \angle FEB + \angle MEB$$

Now since $ME = MB$,

$$\begin{aligned} &\angle FEM = \angle FEB + \angle MEB \\ \implies &\angle FEM = \angle FEB + \angle MBE \end{aligned}$$

Since $CBEF$ is cyclic, $\angle FEB = \angle BCF$.

$$\begin{aligned} &\angle FEM = \angle FEB + \angle MBE \\ \implies &\angle FEM = \angle BCF + \angle MBE \\ \implies &\angle FEM = 180^\circ - \angle BHC \\ \implies &\angle FEM = 180^\circ - \angle FHE \\ \implies &\angle FEM = \angle FAE \end{aligned}$$

Therefore EM is tangent to circle AFE. Likewise FM is also tangent to AFE. Now we need to show that ℓ is tangent to circle AFE. It suffices to show that $\angle XAB = \angle AEF$.

$$\begin{aligned} &\angle AEF = \angle EFC + \angle ECF \\ \implies &\angle AEF = 180^\circ - \angle FEC \\ \implies &\angle AEF = \angle CBF \\ \implies &\angle AEF = \angle XAB \end{aligned}$$

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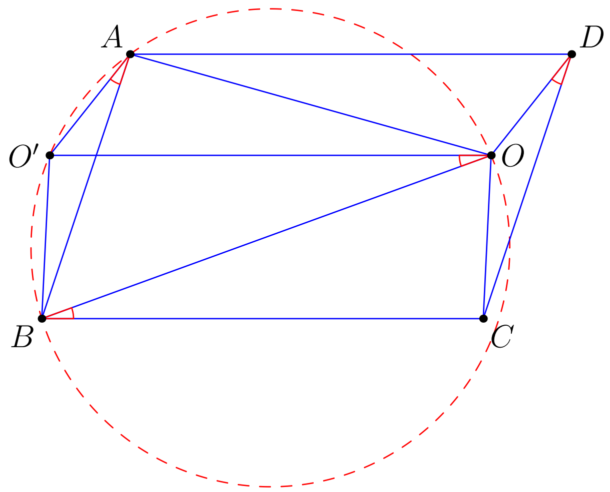
Problem 0.3 (Canada 1997/4)
The point O is inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Show that $\angle OBC = \angle ODC$.



Proof: Apply the translation $T\colon \vec{P} \to \vec{P} + \overrightarrow{CB}$ on points C, O, D. Let $T(O) = O'$. Clearly A, B are the points D, C after applying the translation T.

$$\angle COD + \angle AOB = 180^\circ \implies \angle BO'A + \angle AOB = 180^\circ$$

Therefore $AO'BO$ is cyclic. Hence $\angle O'AB = \angle O'OB$. As $OO' \parallel BC$,

$$\angle O'AB = \angle O'OB = \angle OBC \implies \angle ODC = \angle OBC$$

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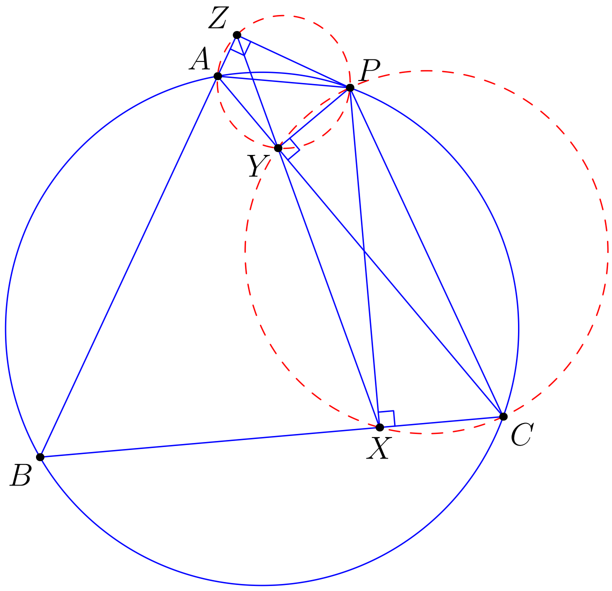
Problem 0.4 (Simson line)

Let ABC be a triangle and let P be any point on the circumcircle of $\triangle ABC$. Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, AB. Prove that X, Y, Z are collinear.



Proof: It suffices to show that $\angle XYC = \angle AYZ$. Since $XYPC$ is cyclic, $\angle XYC = \angle XPC$.

$$\angle XYC = \angle XPC = 90^\circ - \angle PCX$$

Since $ABCP$ is cyclic, $\angle PCX = 180^\circ - \angle PAB$. Therefore

$$\begin{aligned}\angle XYC &= 90^\circ - \angle PCX \\ &= \angle PAB - 90^\circ \\ &= \angle ZPA\end{aligned}$$

Since $AZPY$ is cyclic, $\angle ZPA = \angle AYZ$. Hence

$$\angle XYZ = \angle AYZ$$

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