Cevians

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§1 Cevians

Definition. Let $\triangle ABC$ be a triangle and let X be a point on BC. The segment AX is called a **cevian** of triangle ABC.

Theorem 1.1

If ABC is a triangle and X is a point on BC then

$$\frac{BX}{XC} = \frac{[ABX]}{[ACX]}$$

Corollary 1.1.1

$$\frac{BX}{XC} = \frac{AB \times \sin\angle BAX}{AC \times \sin\angle CAX}$$

Proof: Since

$$[ABX] = \frac{1}{2}AB \times AX \times \sin\angle BAX$$
$$[ACX] = \frac{1}{2}AC \times AX \times \sin\angle CAX$$

we have

$$\frac{BX}{XC} = \frac{[ABX]}{[ACX]} = \frac{AB \times \sin\angle BAX}{AC \times \sin\angle CAX}$$

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Corollary 1.1.2

$$\frac{AB}{AC} = \frac{\sin\angle ACB}{\sin\angle ABC}$$

Proof:

$$\frac{BX}{XC} = \frac{AB \times BX \times \sin\angle ABC}{AC \times XC \times \sin\angle ACB} \implies \frac{AB}{AC} = \frac{\sin\angle ACB}{\sin\angle ABC}$$

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Corollary 1.1.3 (Angle Bisector Theorem)
If AX is the angle bisector of $\angle BAC$ then

$$\frac{BX}{CX} = \frac{BA}{CA}$$

Problem 1.2
$ABCD$ is a square with $AB = 1$ and M, N are the midpoints of BC and CD. P, Q are the intersection of BD, AM and BD, AN. Find the length of PQ.

Solution: Since AP is a cevian of triangle ABD, we have

$$\frac{PD}{BP} = \frac{\sin\angle PAD}{\sin\angle PAB} = \frac{\sin 90^\circ - \angle PAB}{\sin\angle PAB} = \frac{\cos\angle PAB}{\sin\angle PAB} = \cot\angle PAB$$

Since $\triangle ABM$ is a right triangle, we have

$$\cot\angle PAB = \frac{AB}{BM} = 2$$

Therefore

$$\frac{PD}{BP} = 2 \implies \frac{QD + PQ}{PB} = 2 \implies \frac{PQ}{PB} = 1 \implies BP = PQ = QD$$

Hence

$$PQ = \frac{\sqrt{2}}{3}$$

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Problem 1.3 (All-Russian Olympiad 1995/2)
A chord CD of a circle with center O is perpendicular to a diameter AB. A chord AE bisects the radius OC. Show that the line DE bisects the chord BC.

Solution: Let M be the midpoint of OC and let DE intersect BC at N.

Let $\angle BAE = \alpha$ and $\angle EAC = \beta$. Since $OM = MC$,

$$\begin{aligned}
&\frac{MC}{OM} = \frac{AC \times \sin\beta}{OA \times \sin\alpha} \\
\implies &\frac{AC \times \sin\beta}{OA \times \sin\alpha} = 1 \\
\implies &\left(\frac{AC}{2 \times OA}\right) \times \left(\frac{\sin\beta}{\sin\alpha}\right) = \frac{1}{2} \\
\implies &\frac{\cos\angle BAC \times \sin\beta}{\sin\alpha} = \frac{1}{2} \\
\implies &\boxed{\frac{\cos(\alpha+\beta) \times \sin\beta}{\sin\alpha} = \frac{1}{2}}
\end{aligned}$$

If we can somehow show that
$$\frac{EC \times \sin NEC}{EB \times \sin NEB} = 1$$
then we'll be done. Now
$$\angle NEB = \angle DEB = \angle DAB = \angle BAC = \alpha + \beta$$
Since $DACE$ is cyclic
$$\angle DEC = 180^\circ - \angle DAC = 180^\circ - 2(\alpha+\beta) \implies \angle NEC = 180^\circ - 2(\alpha+\beta)$$
Using the sine law we get
$$\frac{EB}{\sin\alpha} = \frac{EC}{\sin\beta} = 2r \implies \frac{EC}{EB} = \frac{\sin\beta}{\sin\alpha}$$
Hence
$$\begin{aligned}
\frac{EC \times \sin NEC}{EB \times \sin NEB} &= \left(\frac{EC}{EB}\right) \times \left(\frac{\sin(180^\circ - 2\alpha - 2\beta)}{\sin(\alpha+\beta)}\right)\\
&= \left(\frac{\sin\beta}{\sin\alpha}\right) \times \left(\frac{\sin 2(\alpha+\beta)}{\sin(\alpha+\beta)}\right)\\
&= \left(\frac{\sin\beta}{\sin\alpha}\right) \times \left(\frac{2\sin(\alpha+\beta)\cos(\alpha+\beta)}{\sin(\alpha+\beta)}\right)\\
&=2 \times \left(\frac{\sin\beta \times \cos(\alpha+\beta)}{\sin\alpha}\right) = 1
\end{aligned}$$
Therefore since
$$\frac{NC}{BN} = \frac{EC \times \sin NEC}{EB \times \sin NEB} = 1 \implies NC = NB$$
N must be the midpoint of BC.

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§2 Stewart's Theorem

Theorem 2.1 (Stewart's Theorem)

Let $\triangle ABC$ be a triangle and let AX be a cevian of $\triangle ABC$. If $BC = a, AC = b, AB = c, AX = x, BX = m$ and $CX = n$ then
$$b^2m + c^2n = a(x^2 + mn)$$

Proof: See https://en.wikipedia.org/wiki/Stewart's_theorem

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