Mathematical Induction

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§1 Preliminaries

Definition 1.1. A **proposition** in mathematics is a statement that is either true or false.

For example, "$2 + 2 = 4$" and "19 is a prime number" both are true mathematical statements. Here are some more examples of propositions.

Proposition 1.2
If $f(n) = n^2 + n + 41$ then $f(n)$ is a prime number for all non-negative integers n.

This is a proposition but sadly this statement is not true for all non-negative integers. For example, if $n = 40$ then

$$f(40) = 40^2 + 40 + 41 = 40^2 + 2 \times 40 + 1 = 41^2$$

Proposition 1.3 (Goldbach's Conjecture)
Every integer greater than 2 is a sum of two primes.

Goldbach's Conjecture is also a proposition but so far no one has been able to prove that it is true.

Definition 1.4. A **predicate** is a proposition whose truth depends on one or more variables.

For example, "p is a prime number" is a predicate as its truth depends on the value of p. For $p = 3$ the statement is true but for $p = 132$ the statement is false. A function-like notation is used to denote a predicate supplied with specific variable values. For example, we might use the name "P" for the predicate above:

$$P(n) = \text{“}p \text{ is a prime number”}$$

Like before, we can say that $P(3)$ is true and $P(132)$ is false.

Definition 1.5. An **axiom** is a proposition which is accepted as true without any proof.

For example, "$a = b \iff a + c = b + c$" and "two sets are equal if and only if they have the same elements" are examples of axioms.

§2 The Induction Principle

Suppose we have the predicate

$$P(n) = \text{“}\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\text{”}$$

We want prove that $P(n)$ is true for all non-negative integers. The induction principle claims that if $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0)$ is true where n_0 is some integer and
- $\mathcal{P}(k) \implies \mathcal{P}(k+1)$ where $k \geq n_0$

then $\mathcal{P}(n)$ is true for all $n \geq n_0$. We will later prove the induction principle but for now let's use this principle and try to prove the fact that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

We want show that $P(n)$ is true for all $n \geq 1$. Clearly $P(1)$ is true. Now we just need to show that $P(k) \implies P(k+1)$. Suppose $P(k)$ is true where $k \geq 1$. Now

$$\begin{aligned}
&1 + 2 + \cdots + k = \frac{k(k+1)}{2} \\
\implies &1 + 2 + \cdots + k + (k+1) = (k+1) + \frac{k(k+1)}{2} \\
\implies &1 + 2 + \cdots + (k+1) = (k+1)\left(1 + \frac{k}{2}\right) \\
\implies &1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}
\end{aligned}$$

And that's it! We just proved that if $P(k)$ is true then $P(k+1)$ is also true. Now from the induction principle, we can say that $P(n)$ is true for all $n \geq 1$.
There are two main steps in a proof involving induction. First we show that $\mathcal{P}(n_0)$ is true where n_0 is an integer. This step is known as the **base step** or the **induction basis**. Next we prove that if k is an integer greater that or equal to n_0 and $\mathcal{P}(k)$ is true then $\mathcal{P}(k+1)$ is also true. This step is called the **inductive step**. Here are some more examples of proof by induction.

Example 2.1 (BDMO)
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and for any $x \in \mathbb{R}$, $f(x+7) \geq f(x) + 7$ and $f(x+1) \leq f(x) + 1$. Find the value of $f(2013)$.

Solution: Notice that

$$\begin{aligned}
f(x+2) &\leq f(x+1) + 1 \leq f(x) + 2, \\
f(x+3) &\leq f(x+2) + 1 \leq f(x) + 3
\end{aligned}$$

We can generalize and say that if n is a non-negative integer then $f(x+n) \leq f(x) + n$. But how do we prove this? Let's try using induction!
The question already tells us that the statement is true for $n = 1$. We just need to show that

$$f(x+k) \leq f(x) + k \implies f(x+k+1) \leq f(x) + k + 1$$

This is quite trivial.

$$f(x+k+1) \leq f(x+k) + 1 \leq f(x) + k + 1$$

And so we just proved that $f(x+n) \leq f(x) + n$. Now setting $n = 7$, we get

$$f(x+7) \leq f(x) + 7$$

Therefore since $f(x+7) \geq f(x) + 7$ and $f(x+7) \leq f(x) + 7$, we must have $f(x+7) = f(x) + 7$. That's great! But now what? Setting $x = 1$, we get $f(8) = f(1) + 7 = 8$. But how can we find the

value of $f(2013)$? We can guess that $f(x) = x$ for all x. Can we prove this? If we can somehow show that $f(x) + 1 = f(x+1)$ for all x then we'll be able to easily show that $f(n) = n$ for all non-negative integer n. So let's try to prove that $f(x) + 1 = f(x+1)$ for all x.
Suppose that there exists some real number r such that $f(r) + 1 \neq f(r+1)$. Therefore $f(r+1)$ must be less than $f(r) + 1$. Now suppose $r = x + 6$ where x is a real number.

$$f(r+1) < f(r) + 1 \implies f(x+7) < f(x+6) + 1$$

Now notice that

$$f(x+6) + 1 \leq f(x+5) + 2 \leq f(x+4) + 2 \leq \cdots \leq f(x) + 7$$

Therefore we have

$$f(x+7) < f(x) + 7$$

But that is impossible as we've shown that $f(x+7) = f(x) + 7$ for all $x \in \mathbb{R}$. Hence such a real number r cannot exist which implies $f(x) + 1 = f(x+1)$ for all $x \in \mathbb{R}$.
We can now use induction to show that $f(n) = n$ for all non-negative integer n. For $n = 1$ the statement is obviously true. We need to prove that if $f(k) = k$ then $f(k+1) = k+1$. This is also quite trivial.

$$f(k+1) = f(k) + 1 \implies f(k+1) = k+1$$

And we are done! YAY! We not only found the value of $f(2013)$ but also found the value of $f(n)$ for all non-negative integer n. Awesome, right?

♣

Example 2.2

If n is a non-negative integer and x, y are two real numbers then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: The statement is evidently true for $n = 1$. Now we need to show that

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \implies (x+y)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m-k+1}$$

Okay, let's try to prove it!

$$\begin{aligned}(x+y)^m \times (x+y) &= (x+y)^m x + (x+y)^m y \\ &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1}\end{aligned}$$

Now

$$\begin{aligned}\sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} &= x^{m+1} + \sum_{k=0}^{m-1} \binom{m}{k} x^{k+1} y^{m-k} \\ &= x^{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} x^k y^{m-k+1} \\ \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} &= y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} x^k y^{m-k+1}\end{aligned}$$

Therefore

$$
\begin{aligned}
(x+y)^{m+1} &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^{k} y^{m-k+1} \\
&= x^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} x^k y^{m-k+1} + y^{m+1}
\end{aligned}
$$

Now

$$
\begin{aligned}
\binom{m}{k} + \binom{m}{k-1} &= \frac{m!}{(m-k)!k!} + \frac{m!}{(m-k+1)!(k-1)!} \\
&= \frac{m!}{(m-k)!(k-1)!} \left(\frac{1}{k} + \frac{1}{m-k+1} \right) \\
&= \frac{m!}{(m-k)!(k-1)!} \left(\frac{m+1}{k(m-k+1)} \right) \\
&= \frac{(m+1)!}{(m-k+1)!k!} \\
&= \binom{m+1}{k}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(x+y)^{m+1} &= x^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} x^k y^{m-k+1} + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^k y^{m-k+1} + y^{m+1} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m+1-k}
\end{aligned}
$$

Voila! We just proved the binomial theorem using induction!

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Example 2.3

Show that if H_n is the n-th Harmonic Number, where $n \geq 4$, then

$$1 + \frac{\lfloor \log_2 n \rfloor}{2} < H_n$$

Proof: The problem looks quite complicated! So let's try proving a more simplified version of the problem first. We will show that if n is an integer greater than 1 then

$$1 + \frac{n}{2} < H_{2^n}$$

This looks more simpler than the original problem because it does not contain the ugly floor function. But how do we prove this? We are going to use induction on n.
The base case, $n = 2$, is trivial as usual. We just have to show that

$$1 + \frac{k}{2} < H_{2^k} \implies 1 + \frac{k+1}{2} < H_{2^{k+1}}$$

Notice that

$$H_{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}$$

Now since

$$2^{k+1} > 2^{k+1} - 1 > \cdots > 2^k + 2 > 2^k + 1$$

we have

$$\frac{1}{2^{k+1}} < \frac{1}{2^{k+1}-1} < \cdots < \frac{1}{2^k+2} < \frac{1}{2^k+1}$$

Adding the inequalities we get

$$\begin{aligned} \frac{2^k}{2^{k+1}} &< \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^k+2} + \frac{1}{2^k+1} \\ \implies \frac{1}{2} &< \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^k+2} + \frac{1}{2^k+1} \end{aligned}$$

Therefore

$$H_{2^k} + \frac{1}{2} < H_{2^{k+1}} \implies 1 + \frac{k+1}{2} < H_{2^{k+1}}$$

And we are done! No, not really. We still have to solve the original problem. Suppose that $\lfloor \log_2 n \rfloor = k$. Now

$$\begin{aligned} 2^k \le n &\implies H_{2^k} \le H_n \\ &\implies 1 + \frac{k}{2} \le H_n \\ &\implies 1 + \frac{\lfloor \log_2 n \rfloor}{2} \le H_n \end{aligned}$$

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When you complete a mathematical proof by induction correctly