# Algebra

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# Contents

# 1 Polynomials

**Definition 1.0.1.** A Polynomial $P(x)$ is an one variable expression or function of the form

$$P(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \cdots, a_n$ are constants and $n \in \mathbb{N}$. The constants $a_i$ are called the *coefficients* of the polynomial. We will denote $A[x]$ as the set of all polynomials with $a_i \in A$. If $n \neq 0$ then $n$ is called the *degree* of the polynomial $P(x)$ and write $\deg P(x) = n$. If $a_n = 1$ then we say that the polynomial is *monic*. $r$ is called a *root* of the polynomial $P(x)$ if and only if $P(r) = 0$.

## §1.1 Division Algorithm

**Theorem 1.1.1** (The Division Algorithm)
Given two polynomial $A(x)$ and $B(x)$ there exists unique polynomials $Q(x)$ and $R(x)$ with $\deg R(x) < \deg B(x)$ such that,

$$A(x) = Q(x)B(x) + R(x)$$

The polynomials $Q(x)$ and $R(x)$ are known as the *quotient* and the *remainder*, respectively. If the remainder $R(x) = 0$ then we say that $B(x)$ divides $A(x)$ and write $B(x) \mid A(x)$.

**Proof:** We will first prove the uniqueness of the polynomials $Q(x)$ and $R(x)$. Assume,

$$A(x) = Q_1(x)B(x) + R_1(x), \quad \deg R_1(x) < \deg B(x)$$
$$A(x) = Q_2(x)B(x) + R_2(x), \quad \deg R_2(x) < \deg B(x)$$

Now,

$$(Q_1(x) - Q_2(x))\,B(x) + (R_1(x) - R_2(x)) = 0$$

Let $q(x) = Q_1(x) - Q_2(x)$ and $r(x) = R_1(x) - R_2(x)$. Now,

$$q(x)B(x) + r(x) = 0$$
$$\implies q(x)B(x) = -r(x)$$

If $q(x) \neq 0$ then $\deg r(x) = \deg q(x) + \deg B(x) \geq \deg B(x)$. But that is impossible since $\deg R_2(x) < \deg B(x) \implies \deg(R_2(x) - R_1(x)) < \deg B(x)$. Thus $q(x)$ must be zero. Consequently $r(x)$ will also be zero. Therefore $R_1(x) = R_2(x)$ and $Q_1(x) = Q_2(x)$.
Now we will prove the existence of the polynomials $Q(x)$ and $R(x)$. Notice the following algorithm,

1: $A(x) \leftarrow a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$
2: $B(x) \leftarrow b_n x^n + b_{n-1}b^{n-1} + \cdots + b_0$
3: $R(x) \leftarrow A(x)$
4: **while** $\deg R(x) \geq \deg B(x)$ **do**
5: $\quad a \leftarrow$ leading coefficient of $R(x)$
6: $\quad b \leftarrow$ leading coefficient of $B(x)$

$d \leftarrow \deg R(x) - \deg B(x)$
$Q(x) = Q(x) + \left(\frac{a}{b}\right) x^d$
$R(x) \leftarrow R(x) - \left(\frac{a}{b}\right) x^d B(x)$
**output** $Q(x)$ and $R(x)$

In each iteration of the while loop, $\deg R(x)$ is decreasing (mono-variant) and the polynomial $Q(x)B(x) + R(x)$ always stays equal to $A(x)$ (invariant). At some point we will eventually get $\deg R(x) \leq \deg B(x)$ which proves the existence of $Q(x)$ and $R(x)$.

**Remark.** Notice that, if $A(x), B(x) \in \mathbb{R}[x]$ then $Q(x), R(x) \in \mathbb{R}[x]$. This implies that, if $A(x), B(x) \in \mathbb{R}[x]$ and $B(x) \mid A(x)$ then $\frac{A(x)}{B(x)} \in \mathbb{R}[x]$

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For example, if $B(x) = x^2 - x + 1$ and $A(x) = x^5 + x^3 + 2x$ then,

$$x^5 + x^3 + 2x = \left(x^3 + x^2 + x\right)\left(x^2 - x + 1\right) + x$$

In this example, the remainder $R(x) = x$ and the quotient $Q(x) = x^3 + x^2 + x$.

**Theorem 1.1.2** (Remainder Theorem)

If $P(x)$ is a polynomial and $a$ is a constant then the remainder upon dividing $P(x)$ by the linear polynomial $x - a$ is equal to $P(a)$.

**Proof:** From the Division Algorithm we know that there exists polynomials $Q(x)$ and $R(x)$ such that,

$$P(x) = Q(x)\,(x - a) + R(x)$$

Since $\deg R(x) < \deg(x-a) = 1$, $R(x)$ must be a constant polynomial. Let us assume, $R(x) = r$. Now letting $x = a$ we get,

$$P(a) = Q(a) \times (a - a) + r \implies P(a) = r$$

Therefore $P(a)$ is the remainder upon dividing $P(x)$ by $x - a$. QED

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**Theorem 1.1.3** (Factor Theorem)

The number $z$ will be a root of the polynomial $P(x)$ if and only if $P(x)$ is divisible by $x - z$.

**Proof:** We will first prove that, $P(z) = 0 \implies (x - z) \mid P(x)$. Let us assume that $r$ is the remainder upon dividing $P(x)$ by $x - z$. Now we know from the Remainder Theorem that, $P(z) = r$. But since $z$ is a root of $P(x)$, $P(z) = r = 0$. Therefore since $r = 0$, we must have $(x - z) \mid P(x)$. Using similar arguments one can also prove the converse.

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**Corollary 1.1.3.1**

The number $-\frac{b}{a}$ where $a, b \in \mathbb{R}$ will be a root of the polynomial $P(x)$ if and only if the polynomial $P(x)$ is divisible by $ax + b$.

If $P(x)$ has the root $z$ then the Factor Theorem guarantees that there exists a polynomial $Q(x)$ such that,

$$P(x) = (x - z)\,Q(x)$$

Now if,

$$P(x) = (x - z)^m\,Q'(x), \quad Q'(z) \neq 0$$

then we say that $z$ is root of $P(x)$ of *multiplicity* $m$.
For example, in the polynomial $P(x) = (x-2)^2\,(x-3)$ the root 2 has multiplicity 2 and the root 3 has multiplicity 1.

## §1.2 The Fundamental Theorem of Algebra

**Theorem 1.2.1** (The Fundamental Theorem of Algebra)
The Fundamental Theorem of Algebra states that, every polynomial $P(x)$ in $\mathbb{C}[x]$ has at least one root in $\mathbb{C}$

**Corollary 1.2.1.1**
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree $n$ then,

$$P(x) = k(x - z_1)(x - z_2)\cdots(x - z_n)$$

where, $k = a_n$ and $z_i \in \mathbb{C}$. The numbers $z_1, z_2 \cdots z_n$ are not necessarily distinct.

**Proof:** This is an immediate consequence of The Fundamental Theorem of Algebra and Factor Theorem.

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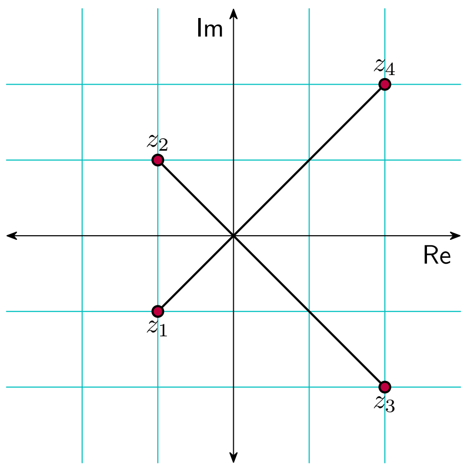


Figure 1.1: The 4 complex roots of the polynomial $x^4 - 2x^3 + 2x^2 + 8x + 16$

**Theorem 1.2.2** (Complex Conjugate Root Theorem)
If $P(x) \in \mathbb{R}[x]$ and $z = a + bi$ where $a, b \in \mathbb{R}$ is a complex root of the polynomial $P(x)$ then $\overline{z} = a - bi$ is also a root of the polynomial $P(x)$.

**Proof 1:** We have to show that, $P(z)=0 \implies P(\overline{z})=0$. Let $\mathbb{C}'=\{ki : k\in\mathbb{R}\}$ and let $\mathbb{R}$ be the set of real numbers. Now,

$$\begin{aligned} z^k+\overline{z}^k &= (a+bi)^k+(a-bi)^k \\ &= \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j} i^j + \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j}(-i)^j \\ &= \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j}\left\{i^j+(-i)^j\right\} \end{aligned}$$

Notice, $i^j+(-i)^j$ will be zero if $j$ is odd. If $j$ is even then $i^j+(-i)^j=2i^j=2(-1)^{\frac{j}{2}}$. Therefore,

$$\begin{aligned} z^k+\overline{z}^k &= \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j}\left\{i^j+(-i)^j\right\} \\ &= \sum_{l=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2l}b^{2l}a^{k-2l}\left\{2(-1)^l\right\} \\ &= \sum_{l=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2l}2(-1)^l b^{2l}a^{k-2l} \end{aligned}$$

**Remark.** The set, $\left\{2l : 0\le l\le \left\lfloor\frac{k}{2}\right\rfloor\right\}$, contains all even integers (including zero) less than or equal to $k$.

Therefore, $z^k+\overline{z}^k\in\mathbb{R}$ for all $0\le k\in\mathbb{Z}$. This implies that,

$$P(z)+P(\overline{z})=\sum_{i=0}^{n}a_i\left(z^i+\overline{z}^i\right)\in\mathbb{R}$$

But since $P(z)=0$, $P(z)+P(\overline{z})\in\mathbb{R}$ implies $P(\overline{z})\in\mathbb{R}$. Now,

$$\begin{aligned} z^k-\overline{z}^k &= (a+bi)^k-(a-bi)^k \\ &= \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j} i^j - \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j}(-i)^j \\ &= \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j}\left\{i^j-(-i)^j\right\} \end{aligned}$$

If $j$ is even then $i^j-(-i)^j$ will be equal to zero. If $j$ is odd that is $j=2l-1$ for some $l\in\mathbb{N}$ then $i^j-(-i)^j=i^{2l-1}\left(1-(-1)^{2l-1}\right)=2i^{2l-1}=2(-1)^l i^{-1}=2(-1)^l i^3=2(-1)^{l+1}i$.
Therefore,

$$\begin{aligned} z^k-\overline{z}^k &= \sum_{j=0}^{k}\binom{k}{j}b^j a^{k-j}\left\{i^j-(-i)^j\right\} \\ &= \sum_{l=1}^{\lfloor\frac{k+1}{2}\rfloor}\binom{k}{2l-1}b^{2l-1}a^{k-2l+1}2(-1)^{l+1}i \\ &= \left(\sum_{l=1}^{\lfloor\frac{k+1}{2}\rfloor}\binom{k}{2l-1}b^{2l-1}a^{k-2l+1}2(-1)^{l+1}\right)i \end{aligned}$$

**Remark.** The set $\left\{2l-1 : 1 \le l \le \left\lfloor \frac{k+1}{2} \right\rfloor\right\}$ contains all odd positive integers less than or equal to $k$.

Thus, $z^k - \overline{z}^k \in \mathbb{C}'$ for all $k \in \mathbb{N}$. Now,

$$\begin{aligned} P(z) - P(\overline{z}) &= \sum_{i=0}^{n} a_i \left(z^i - \overline{z}^i\right) \\ \implies P(z) - P(\overline{z}) &= \sum_{i=1}^{n} a_i \left(z^i - \overline{z}^i\right) + a_0 \left(z^0 - \overline{z}^0\right) \\ \implies P(z) - P(\overline{z}) &= \sum_{i=1}^{n} a_i \left(z^i - \overline{z}^i\right) \in \mathbb{C}' \end{aligned}$$

But since $P(z) = 0$, $P(z) - P(\overline{z}) \in \mathbb{C}'$ implies $P(\overline{z}) \in \mathbb{C}'$. And so, $P(\overline{z}) \in \mathbb{R} \cup \mathbb{C}' \implies P(\overline{z}) \in \{0\} \implies P(\overline{z}) = 0$. QED

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**Proof 2(wiki):** Since $P(z) = 0$,

$$P(z) = \sum_{k=0}^{n} a_k z^k = 0$$

Now using the properties of complex conjugates,

$$P(\overline{z}) = \sum_{k=0}^{n} a_k \overline{z}^k = \sum_{k=0}^{n} a_k \overline{z^k} = \sum_{k=0}^{n} \overline{a_k z^k} = \overline{\sum_{k=0}^{n} a_k z^k} = \overline{P(z)} = \overline{0} = 0$$

Therefore, $P(\overline{z}) = 0$.

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**Corollary 1.2.2.1**

If $z$ is a complex root of the polynomial $P(x)$ of multiplicity $m$ then $\bar{z}$ is also a complex root of the polynomial $P(x)$ of multiplicity $m$. That is, complex conjugate roots have the same multiplicity.

**Proof:** If $z \in \mathbb{R}$ then obviously $z$ and $\bar{z}$ will have the same multiplicity as $z = \bar{z}$. Let us assume $z \notin \mathbb{R}$ and let $m$ and $n$ be the multiplicity of $z$ and $\bar{z}$ respectively. Without loss of generality, we can assume $n < m$. Now, let

$$P(x) = (x-z)^m (x-\bar{z})^n Q(x)$$

Now,

$$\begin{aligned} P(x) &= (x-z)^n (x-\bar{z})^n (x-z)^{m-n} Q(x) \\ \implies \frac{P(x)}{(x-z)^n (x-\bar{z})^n} &= (x-z)^{m-n} Q(x) \end{aligned}$$

Let, $R(x) = \frac{P(x)}{(x-z)^n (x-\bar{z})^n}$. Since $P(x) \in \mathbb{R}[x]$ and $(x-z)^n (x-\bar{z})^n \in \mathbb{R}[x]$, $R(x) \in \mathbb{R}[x]$. Therefore, $R(x) = (x-z)^{m-n} Q(x) \in \mathbb{R}[x]$. As $z$ is a root of $R(x)$ and $R(x) \in \mathbb{R}[x]$, $\bar{z}$ must also be a root of $R(x)$ which implies the multiplicity of $\bar{z} > n$. But that contradicts our assumption that $\bar{z}$ has multiplicity $n$. Therefore, $m$ and $n$ must be equal.

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**Corollary 1.2.2.2**
Every polynomial $P(x)$ in $\mathbb{R}[x]$ can be expressed in the form,

$$P(x) = f_1^{e_1}(x) f_2^{e_2}(x) \cdots f_n^{e_n}(x)$$

where the polynomials $f_i(x)$ are either linear or quadratic polynomials in $\mathbb{R}[x]$ and $e_i \in \mathbb{N}$

**Corollary 1.2.2.3**
If $P(x) \in \mathbb{R}[x]$ and $\deg P(x)$ is odd then $P(x)$ has at least on real root.

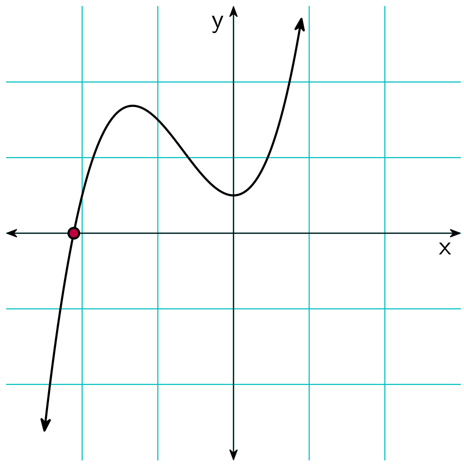


Figure 1.2: The real root of the cubic polynomial $f(x) = x^3 + 2x^2 + 0.5$

## §1.3 Quadratic Polynomials

**Definition 1.3.1.** A *quadratic polynomial* is a polynomial of the form,

$$P(x) = ax^2 + bx + c$$

where $a, b, c$ are constants and $a \neq 0$.

One can find the roots of a quadratic polynomial using the well known *quadratic formula*,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value $\Delta = b^2 - 4ac$ is called the *discriminant* of the quadratic polynomial. The discriminant gives us the following informations about the roots of the quadratic polynomial,

- $\Delta > 0 \iff x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$
- $\Delta = 0 \iff x_1, x_2 \in \mathbb{R}$ and $x_1 = x_2$
- $\Delta < 0 \iff x_1, x_2 \in \mathbb{C}$ and $x_1 \neq x_2$

**Theorem 1.3.1**

The value $P\left(-\frac{b}{2a}\right)$ is either the maximum (if $a > 0$) or the minimum value (if $a < 0$) of the quadratic polynomial, $P(x) = ax^2 + bx + c$

**Proof:**

$$\begin{aligned} & P(x) = ax^2 + bx + c \\ \implies & P(x) = a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ \implies & P(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \end{aligned}$$

If $a < 0$ then,

$$P(x) = \left(c - \frac{b^2}{4a}\right) - |a|\left(x + \frac{b}{2a}\right)^2$$

$P(x)$ will reach its maximum when $x + \frac{b}{2a} = 0 \implies x = -\frac{b}{2a}$.
If $a > 0$ then,

$$P(x) = \left(c - \frac{b^2}{4a}\right) + a\left(x + \frac{b}{2a}\right)^2$$

$P(x)$ will reach its minimum when $x + \frac{b}{2a} = 0 \implies x = -\frac{b}{2a}$.

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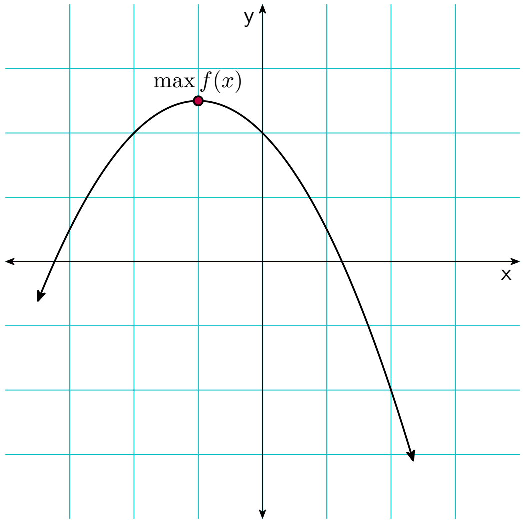
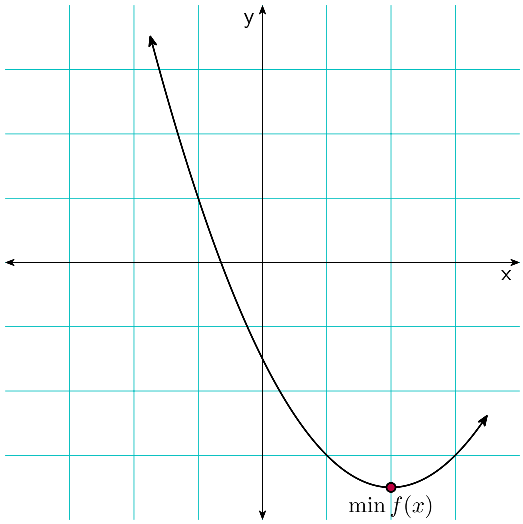



(a) Maximum of $f(x) = -0.5x^2 - x + 2$




(b) Minimum of $f(x) = 0.5x^2 - 2x - 1.5$

## §1.4 Roots of Cubic Polynomials

Finding the roots of a cubic polynomial is quite hard. So, we are going to first try to solve the cubic polynomial,

$$f(x) = x^3 + px + q$$

where $p, q \in \mathbb{R}$. Setting $p = -3ab$ and $q = a^3 + b^3$ we get,

$$f(x) = x^3 + a^3 + b^3 - 3abx$$

Now using the formula,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

we get,

$$f(x) = (x + a + b)(x^2 - (a + b)x + a^2 + b^2 - ab)$$

Therefore the 3 roots of $f$ are,

$$\begin{aligned}
x_1 &= -a-b \\
x_2, x_3 &= \frac{a+b \pm \sqrt{a^2+b^2+2ab-4a^2-4b^2+4ab}}{2} \\
&= \frac{a+b \pm \sqrt{-3a^2-3b^2+6ab}}{2} \\
&= \frac{a+b \pm \sqrt{-3(a-b)^2}}{2} \\
&= \frac{a+b \pm \sqrt{3}i(a-b)}{2} \\
&= \frac{(1+\sqrt{3}i)a \pm (1-\sqrt{3}i)b}{2} \\
&= \frac{1+\sqrt{3}i}{2}a \pm \frac{1-\sqrt{3}i}{2}b
\end{aligned}$$

Now we have express the root in terms of $p, q$ that is, we have to express $a, b$ in terms of $p, q$. Now,

$$\begin{aligned}
q &= a^3+b^3 \\
p &= -3ab \\
\implies a^3b^3 &= -\frac{p^3}{27}
\end{aligned}$$

Let $u = a^3$ and $v = b^3$. Notice that $u$ and $v$ are the roots of the quadratic polynomial,

$$P(x) = x^2-(u+v)x+uv = x^2-q^3x-\frac{p^3}{27}$$

Using the quadratic equation we get,

$$\begin{aligned}
u, v &= \frac{q^3 \pm \sqrt{q^6+\frac{4}{27}p^3}}{2} \\
a, b &= \sqrt[3]{\frac{q^3}{2} \pm \frac{\sqrt{q^6+\frac{4}{27}p^3}}{2}} \\
&= \sqrt[3]{\frac{q^3}{2} \pm \sqrt{\frac{q^6}{4}+\frac{p^3}{27}}}
\end{aligned}$$

So let's now try to solve the cubic equation using the results we've got so far,

$$f(x) = x^3-x^2-2x+1$$

First we have to use substitution to transform the polynomial into another polynomial of the form,

$$x^3+px+q$$

Since every polynomial can be uniquely defined by its coefficients, we can associate or express a polynomial of degree $n$ by a unique point in $n+1$ dimensional space. For example we can express the polynomial,

$$(1)\cdot x^3+(-1)\cdot x^2+(-2)\cdot x+(1)$$

as,

$$(1) \cdot x^3 + (-1) \cdot x^2 + (-2) \cdot x + (1) \to (1, -1, -2, 1)$$

Likewise, the point $(5, -1, 0, 1)$ can be used to represent the polynomial,

$$(5, -1, 0, 1) \to 5x^3 - x^2 + 1$$

Notice that,

$$(x_3, x_2, \cdots, x_0) + (y_3, y_2, \cdots, y_0) = (x_3 + y_3, x_2 + y_2, \cdots, x_0 + y_0)$$

Say that there exists a polynomial $u(x) = (x + n)$ where $n \in \mathbb{C}$ such that,

$$x^3 - x^2 - 2x + 1 = u(x)^3 + pu(x) + q$$

This equation can also be represented as,

$$(1, -1, -2, 1) = (1, 3n, 3n^2, n^3) + (0, 0, p, pn) + (0, 0, 0, q) \implies (1, -1, -2, 1) = (1, 3n, 3n^2 + p, n^3 + pn + q)$$

This gives us the system of equation,

$$\begin{aligned} 3n &= -1 \\ 3n^2 + p &= -2 \\ n^3 + pn + q &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} n &= -\frac{1}{3} \\ p &= -\frac{7}{3} \\ q &= \frac{7}{27} \end{aligned}$$

Thus,

$$x^3 - x^2 - 2x + 1 = \left(x - \frac{1}{3}\right)^3 - \frac{7}{3}\left(x - \frac{1}{3}\right) + \frac{7}{27} \implies f(x) = g\left(x - \frac{1}{3}\right)$$

where

$$g(x) = x^3 - \frac{7}{3}x + \frac{7}{27}$$

Now we can use the results we've proved earlier to solve the polynomial $g(x)$. After that we add $\frac{1}{3}$ to the 3 roots of $g(x)$. The 3 numbers we will get by adding $\frac{1}{3}$ are the 3 roots of $f(x)$.

## §1.5 Lagrange Interpolation

**Theorem 1.5.1** (Lagrange Interpolation)

Let $\alpha_0, \alpha_1, \cdots, \alpha_n$ be distinct real numbers and $\beta_0, \beta_1, \cdots, \beta_n$ be another set of $n + 1$ real numbers. Then there exists a unique polynomial,

$$P(x) = \sum_{i=0}^{n} \left( \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - \alpha_j}{\alpha_i - \alpha_j} \right) \beta_i$$

with $\deg P(x) \le n$ such that $P(\alpha_k) = \beta_k$ for all $0 \le k \le n$.

**Proof:** Let,

$$D_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-\alpha_j}{\alpha_k-\alpha_j} = \frac{(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{k-1})(x-\alpha_{k+1})\cdots(x-\alpha_n)}{(\alpha_k-\alpha_0)(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n)}$$

If $x = \alpha_k$ then $D_k(x) = 1$ else if $x = \alpha_i$ where $i \neq k$ then $D_k(x) = 0$. Thus the polynomial,

$$P(x) = \sum_{k=0}^{n} D_k(x)\beta_k$$

will be equal to $\beta_k$ for all $x = \alpha_k$. It is also clear that the polynomial $P(x)$ has degree at most $n$ since $\deg D_k(x) = n$ for all $0 \leq k \leq n$.
Now suppose that there exists two polynomials $P_1(x)$ and $P_2(x)$, with degree at most $n$, such that,

$$P_1(\alpha_k) = P_2(\alpha_k) = \beta_k,\ 0 \leq k \leq n$$

Therefore the polynomial $Q(x) = P_1(x) - P_2(x)$ has $n+1$ distinct roots. But that is impossible since we know that $\deg Q(x) \leq n$ and a polynomial of degree $n$ has at most $n$ distinct roots. This proves that the polynomial $P(x)$ must be unique, that is, $P(x)$ is the only polynomial, with degree at most $n$, such that, $P(\alpha_k) = \beta_k$ for all $0 \leq k \leq n$

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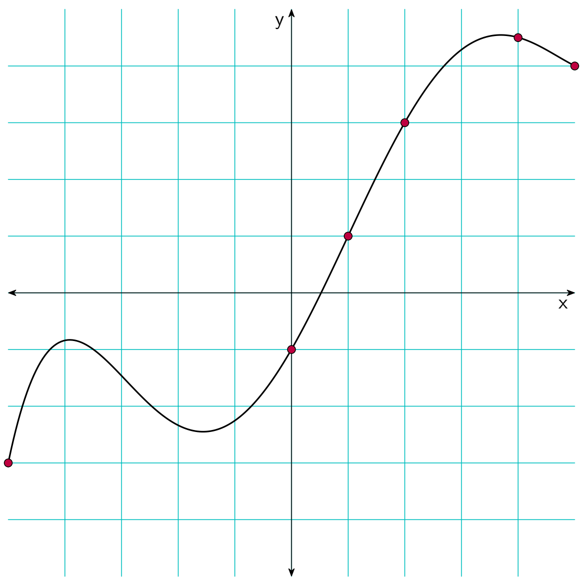


Figure 1.4: Plot of a Lagrange Polynomial

Figure 1.4 shows the Lagrange polynomial going through the points,

$$\{(1,1),(2,3),(0,-1),(5,4),(-5,-3),(4,4.5)\}$$

We can easily compute Lagrange polynomials in python using `sympy`.

```
>>> import sympy
>>> x = sympy.symbols('x')
>>> points = [(1,1), (2,3), (0, -1), (5, 4), (-5, -3), (4, 4.5)]
>>> expr = sympy.interpolate(points, x)
>>> print(expr)
```

```
0.00281746031746032*x**5 - 0.0129761904761905*x**4 -
↪  0.111944444444445*x**3 + 0.384404761904762*x**2 +
↪  1.73769841269841*x - 1
```

**Problem 1.5.1**

Let $P(x)$ be a polynomial of degree $n$ such that, $P(k) = 2^k$ for all $0 \leq k \leq n$. Find $P(n+1)$.

**Solution:** From Theorem 1.5.1 we have,

$$P(x) = \sum_{k=0}^{n} 2^k D_k(x)$$

where,

$$\begin{aligned} D_k(x) &= \frac{x(x-1)\cdots(x-k+1)(x-k-1)(x-k-2)\cdots(x-n+1)(x-n)}{(k)(k-1)\cdots(1)(-1)(-2)\cdots(k-n+1)(k-n)} \\ &= (-1)^{n-k}\frac{x(x-1)\cdots(x-k+1)(x-k-1)(x-k-2)\cdots(x-n+1)(x-n)}{k!(n-k)!} \end{aligned}$$

Therefore,

$$\begin{aligned} P(n+1) &= \sum_{k=0}^{n}(-1)^{n-k}2^k\frac{(n+1)n(n-1)\cdots(n-k+2)(n-k)(n-k-1)\cdots 1}{k!(n-k)!} \\ &= \sum_{k=0}^{n}(-1)^{n-k}2^k\frac{(n+1)!}{k!(n-k)!(n-k+1)} \\ &= \sum_{k=0}^{n}(-1)^{n-k}2^k\binom{n+1}{k} \\ &= (-1)\left(\sum_{k=0}^{n+1}\binom{n+1}{k}2^k(-1)^{n-k+1}\right) + 2^{n+1} \\ &= (-1)\left(2-1\right)^{n+1} + 2^{n+1} \\ &= 2^{n+1} - 1 \end{aligned}$$

♣