Mathematical Induction

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§1 Preliminaries

Definition 1.1. A **proposition** in mathematics is a statement that is either true or false.

For example, "$2 + 2 = 4$" and "19 is a prime number" both are true mathematical statements. Here are some more examples of propositions.

Proposition 1.2

If $f(n) = n^2 + n + 41$ then $f(n)$ is a prime number for all non-negative integers n.

This is a proposition but the proposition is not true for all non-negative integers. For example, if $n = 40$ then

$$f(40) = 40^2 + 40 + 41 = 40^2 + 2 \times 40 + 1 = 41^2$$

Proposition 1.3 (Goldbach's Conjecture)

Every integer greater than 2 is a sum of two primes.

Goldbach's Conjecture is also a proposition but so far no one has been able to prove that it is true.

Definition 1.4. A **predicate** is a proposition whose truth depends on one or more variables.

For example, "n is a prime number" is a predicate as its truth depends on the value of n. For $n = 3$ the statement is true but for $n = 12$ the statement is false. A function-like notation is used to denote a predicate supplied with specific variable values. For example, we might use the name "P" for the predicate above:

$$P(n) : n \text{ is a prime number}$$

Like before, we can say that $P(3)$ is true and $P(12)$ is false.

Definition 1.5. An **axiom** is a proposition which is accepted as true without any proof.

For example, "$a = b \iff a + c = b + c$" and "two sets are equal if and only if they have the same elements" are examples of axioms.

§2 The Induction Principle

> Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the basis) and that from each rung we can climb up to the next one (the step).
>
> Concrete Mathematics

The **induction principle** claims that if $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0)$ is true where n_0 is some integer and
- $\mathcal{P}(k) \implies \mathcal{P}(k+1)$ where $k \geq n_0$ is an integer

then $\mathcal{P}(n)$ is true for all integers $n \geq n_0$. We will later prove the induction principle but first let's take a look at some examples.

Example 2.1
Show that for all $n \geq 1$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof: We have the predicate

$$P(n)\colon 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

We want show that $P(n)$ is true for all $n \geq 1$. Clearly $P(1)$ is true. Now we just need to show that $P(k) \implies P(k+1)$. Suppose $P(k)$ is true where $k \geq 1$. Now

$$\begin{aligned}
& 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \\
\implies & 1 + 2 + \cdots + k + (k+1) = (k+1) + \frac{k(k+1)}{2} \\
\implies & 1 + 2 + \cdots + (k+1) = (k+1)\left(1 + \frac{k}{2}\right) \\
\implies & 1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}
\end{aligned}$$

And that's it! We just proved that if $P(k)$ is true then $P(k+1)$ is also true. Now from the induction principle, we can say that $P(n)$ is true for all $n \geq 1$.

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There are two main steps in an inductive proof. First we show that $\mathcal{P}(n_0)$ is true where n_0 is an integer. This step is known as the **base step** or the **induction basis**. Next we prove that if $k \geq n_0$ is an integer and $\mathcal{P}(k)$ is true then $\mathcal{P}(k+1)$ is also true. This step is called the **inductive step**. To prove the inductive step we assume that $\mathcal{P}(k)$ is true and then use this assumption to show that $\mathcal{P}(k+1)$ must also be true. The hypothesis that $\mathcal{P}(k)$ is true for some integer $k \geq n_0$ is called the **induction hypothesis**. Here's another example of proof by induction.

Example 2.2
Show that for all $n \in \mathbb{N}$

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

Proof: Earlier we proved that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Therefore it suffices to show that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$. For $n = 1$ the statement is clearly true. We can prove this just by plugging in $n = 1$ and then showing that both sides are equal.

$$1^3 = \frac{1^2(1+1)^2}{4} \implies 1 = 1$$

Now we need to show that if the statement holds true for some integer $k \geq 1$ then it must also hold true for $k + 1$. Suppose the statement is true for $n = k$ for some integer $k \geq 1$.

$$\begin{aligned}
&1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4} \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
\implies &1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}$$

Now from the induction principle, we can say that the statement is true for all $n \geq 1$.

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Okay, enough with examples. We will now prove the induction principle!

Theorem 2.3 (Induction Principle)
If $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0)$ is true where n_0 is some integer and
- $\mathcal{P}(k) \implies \mathcal{P}(k+1)$ where $k \geq n_0$ is an integer

then $\mathcal{P}(n)$ is true for all integers $n \geq n_0$.

Proof: Let $Q(n)$ be the predicate $\mathcal{P}(n_0 + n)$.

$$Q(n) : \mathcal{P}(n_0 + n)$$

We need to show that $Q(n)$ is true for all $n \in \mathbb{N}$ if

- $Q(0)$ is true and
- $Q(n) \implies Q(n+1)$ for all $n \in \mathbb{N}$.

Let T be the set of all non-negative integers for which $Q(n)$ is true and let F be the set of all non-negative integers for which $Q(n)$ is false. It suffices to show that F is an empty set.
For the sake of contradiction, let us assume that F is non-empty. Since F is a non-empty set of non-negative integers, there must exist a minimal element of F. Let k be the smallest element of F. Since $k > 0 \implies k-1 \geq 0$ and $k-1 \notin F$, $k-1$ must be an element of T. Since $Q(k-1) \implies Q(k)$, k must also be an element of T. But that contradicts our assumption that $k \in F$ as $Q(k)$ cannot be both true and false. Therefore F does not have a minimal element which implies F must be an empty set.

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Exercise 2.4. Prove using induction that for all $n \in \mathbb{N}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Exercise 2.5. Prove using induction that if r is a real number not equal to 1 then for all $n \in \mathbb{N}$

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Exercise 2.6. Show that if x, y are real numbers and if $n \geq 2$ is a positive integer then

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)$$

Example 2.7

Show that for all $n \in \mathbb{N}$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Proof: For $n = 1$ the statement is true.

$$\frac{1}{1\times 2} = \frac{1}{1+1} \implies \frac{1}{2} = \frac{1}{2}$$

Assume the statement is true for some integer $k \geq 1$. Now we prove that the statement must also be true for $k+1$.

$$\begin{aligned}
&\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1} \\
\implies &\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
\implies &\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} \\
\implies &\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\
\implies &\frac{1}{1\times 2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}
\end{aligned}$$

Therefore by the induction principle the statement must be true for all $n \in \mathbb{N}$.

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Exercise 2.8. Show that for all $n \in \mathbb{N}$

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n^2 + 3n}{4(n+1)(n+2)}$$

Exercise 2.9. Show that for all $n \in \mathbb{N}$

$$1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$$

Example 2.10 (BDMO)

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and for any $x \in \mathbb{R}$, $f(x+7) \geq f(x) + 7$ and $f(x+1) \leq f(x) + 1$. Find the value of $f(2013)$.

Solution: Notice that

$$f(x+2) \leq f(x+1) + 1 \leq f(x) + 2,$$
$$f(x+3) \leq f(x+2) + 1 \leq f(x) + 3$$

We can generalize and say that if n is a non-negative integer then $f(x+n) \leq f(x) + n$. But how do we prove this? Let's try using induction!
The question already tells us that the statement is true for $n = 1$. We just need to show that

$$f(x+k) \leq f(x) + k \implies f(x+k+1) \leq f(x) + k + 1$$

This is quite trivial.

$$f(x+k+1) \leq f(x+k) + 1 \leq f(x) + k + 1$$

And so we just proved that $f(x+n) \leq f(x) + n$. Now setting $n = 7$, we get

$$f(x+7) \leq f(x) + 7$$

Therefore since $f(x+7) \geq f(x) + 7$ and $f(x+7) \leq f(x) + 7$, we must have $f(x+7) = f(x) + 7$. That's great! But now what? Setting $x = 1$, we get $f(8) = f(1) + 7 = 8$. But how can we find the value of $f(2013)$? We can guess that $f(x) = x$ for all x. Can we prove this? If we can somehow show that $f(x) + 1 = f(x+1)$ for all x then we'll be able to easily show that $f(n) = n$ for all non-negative integer n. So let's try to prove that $f(x) + 1 = f(x+1)$ for all x.
Suppose for the sake of contradiction that there exists some real number r such that $f(r) + 1 \neq f(r+1)$. Therefore $f(r+1)$ must be less than $f(r) + 1$. Now suppose $r = x + 6$ where x is a real number.

$$\begin{aligned} f(r+1) < f(r) + 1 &\implies f(x+7) < f(x+6) + 1 \\ &\implies f(x+7) < f(x) + 7 \end{aligned}$$

But that is impossible as we've shown that $f(x+7) = f(x) + 7$ for all $x \in \mathbb{R}$. Hence such a real number r cannot exist which implies $f(x) + 1 = f(x+1)$ for all $x \in \mathbb{R}$.
We can now use induction to show that $f(n) = n$ for all non-negative integer n. For $n = 1$ the statement is true. We need to prove that if $f(k) = k$ then $f(k+1) = k+1$.

$$f(k+1) = f(k) + 1 \implies f(k+1) = k + 1$$

And we are done! YAY! We not only found the value of $f(2013)$ but also found the value of $f(n)$ for all non-negative integer n. Awesome, right?

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Example 2.11

If n is a positive integer and x, y are two real numbers then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: The statement is evidently true for $n = 1$. Now we need to show that

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k} \implies (x+y)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m-k+1}$$

Okay, let's try to prove it!

$$\begin{aligned}(x+y)^m \times (x+y) &= (x+y)^m x + (x+y)^m y \\ &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1}\end{aligned}$$

Now

$$\begin{aligned}\sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} &= x^{m+1} + \sum_{k=0}^{m-1} \binom{m}{k} x^{k+1} y^{m-k} \\ &= x^{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} x^k y^{m-k+1} \\ \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} &= y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} x^k y^{m-k+1}\end{aligned}$$

Therefore

$$\begin{aligned}(x+y)^{m+1} &= \sum_{k=0}^{m} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k+1} \\ &= x^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} x^k y^{m-k+1} + y^{m+1}\end{aligned}$$

Now

$$\begin{aligned}\binom{m}{k} + \binom{m}{k-1} &= \frac{m!}{(m-k)!k!} + \frac{m!}{(m-k+1)!(k-1)!} \\ &= \frac{m!}{(m-k)!(k-1)!} \left(\frac{1}{k} + \frac{1}{m-k+1} \right) \\ &= \frac{m!}{(m-k)!(k-1)!} \left(\frac{m+1}{k(m-k+1)} \right) \\ &= \frac{(m+1)!}{(m-k+1)!k!} \\ &= \binom{m+1}{k}\end{aligned}$$

Thus we have

$$\begin{aligned}(x+y)^{m+1} &= x^{m+1} + \sum_{k=1}^{m} \left\{ \binom{m}{k} + \binom{m}{k-1} \right\} x^k y^{m-k+1} + y^{m+1} \\ &= x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^k y^{m-k+1} + y^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} x^k y^{m+1-k}\end{aligned}$$

Voila! We just proved the binomial theorem using induction!

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Example 2.12

Show that if H_n is the n-th Harmonic Number, where $n \in \mathbb{N}$, then

$$1 + \frac{\lfloor \log_2 n \rfloor}{2} \le H_n$$

Proof: The problem looks quite complicated! So let's try proving a more simplified version of the problem first. We will show that if n is a positive integer then

$$1 + \frac{n}{2} \le H_{2^n}$$

This looks simpler than the original problem because it does not contain the ugly floor function. But how do we prove this? We are going to use induction on n.
The base case, $n = 1$, is true. We just have to show that

$$1 + \frac{k}{2} \le H_{2^k} \implies 1 + \frac{k+1}{2} \le H_{2^{k+1}}$$

Notice that

$$H_{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}$$

Now since

$$2^{k+1} \ge 2^{k+1} - 1 \ge \cdots \ge 2^k + 2 \ge 2^k + 1$$

we have

$$\frac{1}{2^{k+1}} \le \frac{1}{2^{k+1}-1} \le \cdots \le \frac{1}{2^k+2} \le \frac{1}{2^k+1}$$

Adding the inequalities we get

$$\begin{aligned}\frac{2^k}{2^{k+1}} &\le \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^k+2} + \frac{1}{2^k+1} \\ \implies \frac{1}{2} &\le \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^k+2} + \frac{1}{2^k+1}\end{aligned}$$

Therefore

$$H_{2^k} + \frac{1}{2} \le H_{2^{k+1}} \implies 1 + \frac{k+1}{2} \le H_{2^{k+1}}$$

And we are done! No, not really. We still have to solve the original problem.
Suppose that $\lfloor \log_2 n \rfloor = k$. Now

$$\begin{aligned} 2^k \leq n &\implies H_{2^k} \leq H_n \\ &\implies 1 + \frac{k}{2} \leq H_n \\ &\implies 1 + \frac{\lfloor \log_2 n \rfloor}{2} \leq H_n \end{aligned}$$

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§3 Variants of the Induction Principle

Theorem 3.1

If $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(n_0), \mathcal{P}(n_0+1), \cdots, \mathcal{P}(n_0+m)$ are all true and
- $\mathcal{P}(k) \wedge \mathcal{P}(k+1) \wedge \cdots \wedge \mathcal{P}(k+m)$ implies $\mathcal{P}(k+m+1)$

then $\mathcal{P}(n)$ is true for all integers $n \geq n_0$.

Proof: Let $Q(n)$ be the predicate

$$Q(n) : \mathcal{P}(n) \wedge \mathcal{P}(n+1) \wedge \cdots \wedge \mathcal{P}(n+m)$$

The base case $Q(n_0)$ is true. Assume that $Q(k)$ is true. Now

$$\begin{aligned} &\mathcal{P}(k) \wedge \mathcal{P}(k+1) \wedge \cdots \wedge \mathcal{P}(k+m) \to \mathcal{P}(k+m+1) \\ \implies &\mathcal{P}(k) \wedge \mathcal{P}(k+1) \wedge \cdots \wedge \mathcal{P}(k+m) \to \mathcal{P}(k+1) \wedge \cdots \wedge \mathcal{P}(k+m) \wedge \mathcal{P}(k+1+m) \\ \implies &Q(k) \to Q(k+1) \end{aligned}$$

Therefore by the induction principle $Q(n)$ must be true for all $n \geq n_0$. If $Q(n)$ is true for all $n \geq n_0$ then $\mathcal{P}(n)$ must also be true for all $n \geq n_0$.

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Example 3.2

Find the general term of the sequence defined by $x_0 = 3, x_1 = 4$ and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all $n \geq 1$.

Proof: The first few values of the sequence are

$$3, 4, 5, 6, 7, \cdots$$

We can guess that $x_n = n + 3$. We will use induction on n to show that $x_n = n + 3$ for all $n \geq 0$. Notice that normal induction won't work here because we need the values of both x_k and x_{k-1} to find the value of x_{k+1}. But don't worry, we can use Theorem 3.1.
Let $P(n)$ be the predicate

$$P(n) : x_n = n + 3$$

Since $x_0 = 3, x_1 = 4$, $P(0)$ and $P(1)$ are both true. We need to show that if $k \geq 0$ is an integer then

$$P(k) \wedge P(k+1) \implies P(k+2)$$

Assume that $P(k)$ and $P(k+1)$ are both true. Now

$$\begin{aligned} x_{k+2} &= x_k^2 - (k+1)x_{k+1} \\ \implies x_{k+2} &= (k+3)^2 - (k+1)(k+4)^2 \\ \implies x_{k+2} &= k^2 + 6k + 9 - k^2 - 5k - 4 \\ \implies x_{k+2} &= k + 5 = (k+2) + 3 \end{aligned}$$

Therefore by Theorem 3.1, we can say that $P(n)$ is true for all $n \geq 0$. Hence $x_n = n + 3$ for all $n \geq 0$.

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Example 3.3 (Binet's Formula)

Let F_n be the n-th Fibonacci number. Show that

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the two real roots of the quadratic equation $x^2 - x - 1$.

Proof: We will use induction on n. The base cases $n = 0$ and $n = 1$ are both true.

$$\begin{aligned} F_0 &= \frac{\phi^0 - \psi^0}{\phi - \psi} \implies 0 = 0 \\ F_1 &= \frac{\phi^1 - \psi^1}{\phi - \psi} \implies 1 = 1 \end{aligned}$$

Assume the formula works for k and $k+1$ where $k \geq 0$ is some integer. Now

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ &= \frac{\phi^{k+1} - \psi^{k+1}}{\phi - \psi} + \frac{\phi^k - \psi^k}{\phi - \psi} \\ &= \frac{\left(\phi^{k+1} + \phi^k\right) - \left(\psi^{k+1} + \psi^k\right)}{\phi - \psi} \\ &= \frac{\phi^k\left(\phi + 1\right) - \psi^k(\psi + 1)}{\phi - \psi} \\ &= \frac{\phi^{k+2} - \psi^{k+2}}{\phi - \psi} \end{aligned}$$

Therefore the formula works for all integers $n \geq 0$.

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Theorem 3.4 (Cauchy Induction)
If $\mathcal{P}(n)$ is some predicate and if

- $\mathcal{P}(2)$ is true,
- $\mathcal{P}(k) \implies \mathcal{P}(2k)$ and
- $\mathcal{P}(k) \implies \mathcal{P}(k-1)$

then $\mathcal{P}(n)$ is true for all $n \geq 2$.