Nonlinear optics

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1 Introduction

During the better part of the course we have studied linear phenomena, that is the interaction between matter and light described by the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2}, \tag{1}$$

where the polarization $\mathbf{P}$ is linear wrt. the electric field of the light $\mathbf{E}$, as described by $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$, where χ is electric suscpetibility of the medium. The course has touched upon nonlinear phenomena, where the polarization can be expanded as $\mathbf{P} = \epsilon_0(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \dots)$.

In this project we describe and discuss two such effects, namely beam self-focusing using the Kerr effect, and wave mixing.

2 The Kerr effect

The Kerr effect is found in media, where the index of refraction is dependent on the electric field of the propagating light. For real life laser beams, where the (time avaraged) amplitude of electric field depends on the position in the beam, this leads to a index of refraction varying over the spacial position within the beam. Under certain and experimentally obtainable conditions, this leads to self-focusing of the beam within a passive medium.

2.1 Mathematical details at a glance

The mathematical formulation of the Kerr effect is derived in Milonni and Eberly [3, sec. 10.3]. For a monochromatic laser field $E = \mathcal{E}e^{-i\omega t}$ with frequency ω in an isotropic and centrosymmetric medium (ie. a medium which quantum states have definite parity), an alternative wave equation for the complex laser field and polarization amplitudes is found. The most important point here is, that the polarization has a linear and a nonlinear contribution $\mathcal{P} = \mathcal{P}^{\mathrm{L}} + \mathcal{P}^{\mathrm{NL}}$:

$$\nabla^2 \mathcal{E} + \frac{\omega^2}{c^2}\mathcal{E} = -\frac{\omega^2}{\epsilon_0 c^2}\mathcal{P} = -\frac{\omega^2}{\epsilon_0 c^2}\left(\epsilon_0\chi_1\mathcal{E} + \epsilon_0\chi_3|\mathcal{E}|^2\mathcal{E}\right). \tag{2}$$

It should be noted, that the succeptibilities here are not simple scalars, but are certain values of the nonlinear succeptibility tensor. For a monochromatic field, χ_1 and χ_3 are constant as ω is constant. Defining

$$-\chi\frac{\omega^2}{c^2}\mathcal{E} \equiv -\chi_1\frac{\omega^2}{c^2}\mathcal{E} - \chi_3\frac{\omega^2}{c^2}|\mathcal{E}|^2\mathcal{E} \tag{3}$$

and using $n^2 = 1 + \chi$ leads to

$$n = \sqrt{1 + \chi_1 + \chi_3|\mathcal{E}|^2} = n_0\sqrt{1 + \frac{\chi_3}{n_0^2}|\mathcal{E}|^2} \tag{4}$$

$$\simeq n_0 + \frac{\chi_3}{2n_0}|\mathcal{E}|^2 \equiv n_0 + \tfrac{1}{2}n_2|\mathcal{E}|^2, \tag{5}$$

where $n_0 = \sqrt{1 + \chi_1}$ is the linear refractive index in the medium. Clearly, the effective refractive index felt by the propagating wave, depends on the form of the electric field.

2.2 Qualitative interpretation and uses

As shown i figure 1, a gaussian beam, which is a beam exhibiting cylindrical symmetry and where the intensity distribution is or nearly is gaussian, feels a larger refractive index closer to the beam axis. Eqn. (5) shows, that this nonlinear correction is proportional to the norm square of the field amplitude, ie. the beam intensity.

The beam size of an gaussian beam is not constant, but varies along the beam axis z, with minimum beam cross section found at the beam waist.

Figure 1: Gaussisk beamprofil og effektivt brydningsindeks.

Using a geometric interpretation of the light, rays moving away from the waist have increasing distance from the axis, ie. radial distance r, leading to a broadening of the beam and a lower intensity. As such, the rays move from a region of high to lower refraction index, leading to a reflection of the ray back towards the beam axis.

This self-focusing competes with diffractive spreading and below a certain critical intensity, self-focusing is not observed. This limit is normally given in terms of total beam power, for example:

$$P_{\text{crit}}^{\text{gauss}} \sim \frac{c\epsilon_0}{8\pi} \frac{\lambda^2}{n_2} \qquad \text{Gaussian beam [3],} \tag{6}$$

$$P_{\text{crit}}^{\text{cyl}} \approx \frac{ca^2}{64} \frac{\lambda^2}{n_2} \qquad \text{Cylindrical beam of uniform intensity [1],} \tag{7}$$

where the angular diffractive divergence of the cylindrical beam is $a\lambda/n_0 2r$. Note here, that the critical power decreases linearly in n_2 but increases quadratically in wavelength.

As can be expected, extreme power past the critical power will lead to undesired effects in the medium. These effects include multiphoton ionisation and other dielectric breakdown effects, potentially leading to destruction of the medium.

In optics, media where this self-focusing is present are termed Kerr lenses, as the passive medium acts as a lens for beams passing through. The dioptric power (inverse focal length) of a Kerr lens of length z for a circular gaussian beam of power P with waist radius w_0 is [4]

$$f_{\text{kerr}}^{-1} = \frac{4}{\pi} \frac{n_2 P}{w_0^4} z. \tag{8}$$

Kerr lenses have the added anvantage over standard optical lenses, that the Kerr lenses do not require alignment of the optical axis of the lens and the beam axis. The is due to the fact, that the optical axis of the Kerr lens is induced by the electric field of the laser beam. This advantage is utilized in Kerr lens mode-locking.

2.3 Numerical work

We wish to confirm the numerical results from Kelley [2], that is we wish to show, that a gaussian beam is focused when propagating in a Kerr lens. Our starting point is the full wave equation and the assumption, that the propagating beam is cylindrical. Per Kelley [2] this leads to a dimensionless form of the wave equation

$$i\frac{\partial \tilde{E}}{\partial \tilde{z}} + \frac{\partial^2 \tilde{E}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{E}}{\partial \tilde{r}} + |\tilde{E}|^2 \tilde{E} = 0, \tag{9}$$

where $\tilde{r} = r/a$, a is a characteristic length of the beam intensity profile, $\tilde{z} = z/2ka^2$ is the characteristic axial length of the beam, $k = n_0\omega/c$ is the wave number of the laser field, and $\tilde{E} = (\chi_1/\chi_3)^{1/2}ka\mathcal{E}$.

The point of solving the dimensionless wave equation is, that we minimize the dependence on physical properties of the beam. This simplifies the numerical approach and makes it easier to obtain a solution exhibiting the correct physical behavior albiet for at generalized monochromatic circular gaussian beam.

The partial differential equation given in Eqn. (9) is categorized as a 1D diffusion problem. To simplify notation, we drop the tildes from the variables in the following discussion, but the varibles remain dimensionless. A method to solve this problem is the Crank-Nicolson method which applies for equations of the form

$$\frac{\partial u}{\partial t} = F\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \tag{10}$$

which is also the case for our equation, since it can be rewritten as

$$i\frac{\partial E}{\partial z} = -\frac{1}{2}\left[\frac{1}{r}\frac{\partial E}{\partial r} + \frac{\partial^2 E}{\partial r^2} + |E|^2 E\right]. \tag{11}$$

In order to evaluate this numerically we discretize z and r in order to make a grid. The first and second order derivatives can then be approximated by the central finite difference

$$f'(x) = \frac{f(x + \frac{1}{2}h) - f(x - \frac{1}{2}x)}{h} \quad f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}. \tag{12}$$

The Crank-Nicolson method then relates the value of E at the $n + 1$ z-step to the value in the n'th step via the equalities of the forward and backward euler method. Note that i denote the step in r

$$\frac{E_i^{n+1} - E_i^n}{\Delta z} = \frac{1}{2}\left[F_i^{n+1} + F_i^n\right]. \tag{13}$$

The right hand side of the can be evaluated using the central difference approximation. This gives a set of equations for E^{n+1} that depends on E^n ie.

the values in the previous step

$$(\alpha + 2\beta^2 - |E_i^{n+1}|^2)E_i^{n+1} - \beta^2(E_{i+2}^{n+1} + E_{i-2}^{n+1}) - \frac{\beta}{r_i}(E_{i+1}^{n+1} - E_{i-1}^{n+1})$$

$$= (\alpha - 2\beta^2 + |E_i^n|^2)E_i^n + \beta^2(E_{i+2}^n + E_{i-2}^n) + \frac{\beta}{r_i}(E_{i+1}^n - E_{i-1}^n), \qquad (14)$$

where $\alpha = -2i/\Delta z$ and $\beta = 1/2\Delta r$.

This is almost a fivediagonal matrix equation except for the term that is cubic in E_i^{n+1}. However with sufficiently small steps one can approximate the cubic term to be $|E_i^{n+1}|^2 E_i^{n+1} \approx |E_i^n|^2 E_i^{n+1}$ ie. take the quadratic part from the last step. In each timestep one must then solve the matrix inversion problem

$$\mathbf{A}E^{n+1} = \mathbf{B}E^n, \qquad (15)$$

where the coefficients that make up these two matricies is the coefficients listed in Eqn. (14).

3 Wave mixing

4 Bibliography

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